

Cubic curves and totally geodesic subvarieties of moduli space

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Abstract

In this paper we present the first example of a primitive, totally geodesic subvariety $F \subset \mathcal{M}_{g,n}$ with $\dim(F) > 1$. The variety we consider is a surface $F \subset \mathcal{M}_{1,3}$ defined using the projective geometry of plane cubic curves. We also obtain a new series of Teichmüller curves in \mathcal{M}_4 , and new $\mathrm{SL}_2(\mathbb{R})$ -invariant varieties in the moduli spaces of quadratic differentials and holomorphic 1-forms.

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1. Introduction

Let $\mathcal{M}_{g,n}$ denote the moduli space of compact Riemann surfaces of genus g with n marked points. A *complex geodesic* is a holomorphic immersion $f : \mathbb{H} \rightarrow \mathcal{M}_{g,n}$ that is a local isometry for the Kobayashi metrics on its domain and range. It is known that $\mathcal{M}_{g,n}$ contains a complex geodesic through every point and in every possible direction.

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We say a subvariety $V \subset \mathcal{M}_{g,n}$ is *totally geodesic* if every complex geodesic tangent to V is contained in V . It is *primitive* if it does not arise from a simpler moduli space via a covering construction.

A *Teichmüller curve* is a totally geodesic subvariety of $\mathcal{M}_{g,n}$ of dimension one. These rare and remarkable objects are closely related to billiards in polygons, Jacobians with real multiplication, and dynamical rigidity. They are uniformized by Fuchsian groups defined over number fields, but they are generally not arithmetic.

This paper gives the first example of a primitive, totally geodesic *Teichmüller surface* in moduli space. We also obtain a new infinite series of Teichmüller curves in \mathcal{M}_4 , and new $\mathrm{SL}_2(\mathbb{R})$ -invariant subvarieties in moduli spaces of quadratic differentials and holomorphic 1-forms.

Our constructions depend in a fundamental way on the classical subject of cubic curves in the plane (Section 2) and space curves of genus four (Section 4), giving an unexpected connection between algebraic geometry and Teichmüller theory.

The *flex locus* $F \subset \mathcal{M}_{1,3}$. A point in $\mathcal{M}_{1,3}$ is specified by a pair (A, P) consisting of a compact Riemann surface A of genus one, and an unordered set $P \subset A$ with $|P| = 3$. We can also regard P as an effective divisor of degree three.

The *flex locus* $F \subset \mathcal{M}_{1,3}$ is defined by:

$$F = \left\{ (A, P) : \begin{array}{l} \exists \text{ a degree three rational map } \pi : A \rightarrow \mathbb{P}^1 \text{ such that} \\ \text{(i) the fibers of } \pi \text{ are linearly equivalent to } P, \text{ and} \\ \text{(ii) } P \text{ is a subset of the cocritical points of } \pi. \end{array} \right\}$$

Here $x' \in A$ is a *cocritical point* of π if

$$(1.1) \quad \{x, x'\} = (\text{a fiber of } \pi)$$

for some critical point x of π . (We allow $x' = x$.)

We refer to F as the flex locus because, when A is defined by a homogeneous cubic polynomial f , π is given by projection from a point where $\det D^2 f = 0$. See Section 3.

The main result of this paper, proved in Section 5, is:

THEOREM 1.1. *The flex locus F is a totally geodesic, irreducible complex surface in $\mathcal{M}_{1,3}$.*

Let $\mathcal{T}_F \rightarrow F$ denote an irreducible component of the preimage of F in $\mathcal{T}_{1,3}$. Since \mathcal{T}_F is totally geodesic, it is homeomorphic to a ball. In Section 6 and Section 7 we will show:

THEOREM 1.2. *The complex manifold \mathcal{T}_F is not isomorphic to any traditional Teichmüller space $\mathcal{T}_{g,n}$.*

COROLLARY 1.3. *The Teichmüller surface $F \subset \mathcal{M}_{1,3}$ is primitive, i.e., it does not arise from a covering construction.*

Strata. The surface F is closely related to an algebraic threefold $G \subset \mathcal{M}_4$, which is abundantly populated by new Teichmüller curves. Our proof of Theorem 1.1 depends on this relation.

To define G , we first need notation for strata. Let $\Omega\mathcal{M}_g \rightarrow \mathcal{M}_g$ denote the moduli space of holomorphic 1-forms of genus g . As usual, given $a_i > 0$ with $\sum_1^n a_i = 2g - 2$, we let $\Omega\mathcal{M}_g(a_1, \dots, a_n) \subset \Omega\mathcal{M}_g$ denote the stratum consisting of 1-forms (X, ω) such that

$$(\omega) = \sum_1^n a_i p_i$$

for some distinct points $p_1, \dots, p_n \in X$. We use exponential notation for repeated indices; for example, $\Omega\mathcal{M}_g(2, 2, \dots, 2) = \Omega\mathcal{M}_g(2^{g-1})$.

Within a given stratum, we can impose the additional condition that there exists an involution $J : X \rightarrow X$, with fixed points (p_1, \dots, p_n) , such that $J^*(\omega) = -\omega$. The resulting locus is a *Prym stratum*

$$\Omega\mathcal{M}_g(a_1, \dots, a_n)^- \subset \Omega\mathcal{M}_g.$$

Note that J is uniquely determined by ω , provided $g > 1$. We allow $a_i = 0$, to account for fixed points that are not zeros. Since the number of fixed points of J is preserved under limits, each Prym stratum is a closed subvariety of $\Omega\mathcal{M}_g$.

There is a natural action of $SL_2(\mathbb{R})$ on $\Omega\mathcal{M}_g$ that preserves both types of strata, and whose orbits project to complex geodesics in \mathcal{M}_g .

The gothic locus $G \subset \mathcal{M}_4$. Given a Riemann surface X with a distinguished involution J , we say a holomorphic map $p : X \rightarrow B$ is *odd* if there exists an involution $j \in \text{Aut}(B)$ such that $p(J(x)) = j(p(x))$ for all $x \in X$. Let

$$\Omega G = \left\{ (X, \omega) \in \Omega\mathcal{M}_4(2^3, 0^3)^- : \begin{array}{l} \exists \text{ a curve } B \in \mathcal{M}_1 \text{ and an odd,} \\ \text{degree three rational map } p : X \rightarrow B \\ \text{such that } |p(Z(\omega))| = 1. \end{array} \right\}$$

Here $Z(\omega)$ denotes the zero set of ω . The condition that p sends the three zeros of ω to a single point implies that $p_*(\omega) = 0$.

We refer to the variety G obtained by projecting ΩG to \mathcal{M}_4 as the *gothic locus*. (The terminology is inspired by Figure 1.)

The relationship between F and G can be summarized as follows: given any form $(X, \omega) \in \Omega G$ with involution J , we obtain a point $(A, P) \in F$ by setting $(A, q) = (X, \omega^2)/J$, and marking the poles of q .

Using this natural map $\Omega G \rightarrow F$, in Section 4 and Section 5 we will show:

THEOREM 1.4. *The space $\Omega G \subset \Omega \mathcal{M}_4$ is a closed, irreducible variety of dimension four, locally defined by real linear equations in period coordinates.*

In particular, the variety ΩG is locally isomorphic to a finite union of four-dimensional subspaces of \mathbb{C}^{10} .

COROLLARY 1.5. *The locus ΩG is invariant under the natural action of $SL_2(\mathbb{R})$.*

The crux of the proof of Theorem 1.4 is a lower bound on $\dim \Omega G$ coming from our study of F . The surprise is that a small number of conditions on the periods of ω produce an elliptic curve B and a map $p : X \rightarrow B$.

The fact that F is totally geodesic follows readily from Corollary 1.5, by transporting $SL_2(\mathbb{R})$ -orbits in ΩG to complex geodesics in F .

Teichmüller curves and real multiplication. Let $\mathcal{O}_D \cong \mathbb{Z}[(D + \sqrt{D})/2]$ denote the real quadratic order of discriminant D , where $D > 0$ and $D \equiv 0$ or $1 \pmod{4}$.

Let (X, ω) be a form whose membership in ΩG is ratified by an involution J and a map $p : X \rightarrow B$. Then we also have a natural map $\phi : X \rightarrow A = X/J$. Taking the quotient of $\text{Jac}(X)$ by divisors pulled back from A and B , we obtain the polarized Abelian surface

$$(1.2) \quad C = C(X|A, B) = \text{Jac}(X) / \text{Im}(\text{Jac}(A) \times \text{Jac}(B)).$$

Let $\Omega G_D \subset \Omega G$ denote the locus where C admits real-multiplication by \mathcal{O}_D with ω as an eigenform. Its projection to \mathcal{M}_4 will be denoted by G_D . In Section 8 we will show:

THEOREM 1.6. *For every discriminant $D > 0$, the locus $G_D \subset \mathcal{M}_4$ is a finite union of Teichmüller curves.*

THEOREM 1.7. *If D is not a square, then every component of G_D is geometrically primitive.*

THEOREM 1.8. *If the stabilizer of a form in ΩG contains a hyperbolic element γ , then $SL(X, \omega)$ is a lattice and $(X, \omega) \in \Omega G_D$ for some D with $\mathbb{Q}(\sqrt{D}) = \mathbb{Q}(\text{tr } \gamma)$.*

Polygon models for gothic forms. To conclude, we will describe an explicit construction of forms in the locus ΩG .

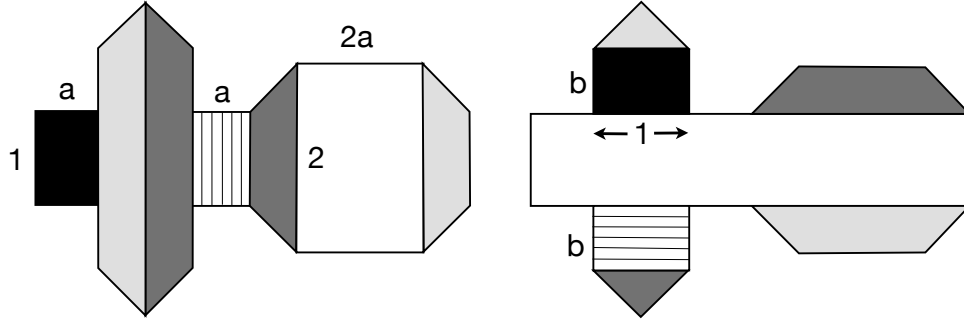


Figure 1. The cathedral polygon $P(a, b)$.

Figure 1 illustrates two copies of a polygon $P(a, b) \subset \mathbb{C}$. This polygon is symmetric about the x -axis, and each of its edges has slope $0, \infty$ or ± 1 . Pairs of parallel edges are glued together by translation to form a compact Riemann surface $X = P(a, b)/\sim$ of genus four. The edge pairings for $P(a, b)$ can be read off from the condition that regions with the same shade on the right or the left cover cylinders on X . The form $dz|P(a, b)$ descends to a form $\omega \in \Omega(X)$ with three double zeros, coming from the vertices of $P(a, b)$.

In Section 9 we will show:

THEOREM 1.9. *For any $a, b > 0$, the holomorphic 1-form*

$$(X, \omega) = (P(a, b), dz)/\sim$$

lies in ΩG .

THEOREM 1.10. *If, in addition, there are rational numbers x, y and $d \geq 0$ such that*

$$(1.3) \quad a = x + y\sqrt{d}, \quad b = -3x - 3/2 + 3y\sqrt{d},$$

then (X, ω) generates a Teichmüller curve. In fact $(X, \omega) \in G_D$ for some D with $\mathbb{Q}(\sqrt{D}) = \mathbb{Q}(\sqrt{d})$.

COROLLARY 1.11. *Every real quadratic field K arises as $K = \mathbb{Q}(\sqrt{D})$ for some D with $G_D \neq \emptyset$.*

Outline of the paper.

1. In Section 2 and Section 3 we describe the surface F from the perspective of classical projective geometry.

Every pair $(A, P) \in \mathcal{M}_{1,3}$ can be presented as a triple of collinear points on a smooth cubic curve in the plane,

$$P = L \cap A \subset \mathbb{P}^2.$$

Similarly, every degree three rational map $\pi_S : A \rightarrow \mathbb{P}^1$, with fibers linearly equivalent to P , is obtained by projection from a point $S \in \mathbb{P}^2$. We find that π_S has a triple of collinear cocritical points if and only if S itself lies on a related cubic curve, the *Hessian* $HA \subset \mathbb{P}^2$.

Since the moduli space \mathcal{M}_1 of smooth cubics is one-dimensional, this shows that F itself is two-dimensional. In fact, F is naturally swept out by an open subset \widetilde{F} of the *universal Cayleyan*, a smooth surface discussed in Section 3.

2. In Section 4 we use the fact that $\dim F = 2$ to show $\dim \Omega G = 4$; while in Section 5, we show that in period coordinates, ΩG is contained in a finite union of four-dimensional linear spaces defined over \mathbb{R} . It follows that ΩG is $\mathrm{SL}_2(\mathbb{R})$ -invariant and that F is totally geodesic.

The algebraic formula for G given in equation (4.7) below also provides a direct proof that $\dim \Omega G \geq 4$.

3. In Section 6 we review the theory of covering constructions, and exhibit another totally geodesic surface $S_{11} \subset \mathcal{M}_{1,3}$ which arises in this way. We then give a topological proof that F is primitive.
4. In Section 7 we show, via an analysis of the Kobayashi metric, that \mathcal{T}_F is not isomorphic to any traditional Teichmüller space $\mathcal{T}_{g,n}$. This result gives a geometric proof that F is primitive, and suggest that one might regard \mathcal{T}_F itself as a new type of Teichmüller space, on an equal footing with $\mathcal{T}_{g,n}$.
5. In Section 8 we show that the loci $G_D \subset \mathcal{M}_4$ are finite unions of Teichmüller curves; the proof is similar to the case of the Weierstrass curves in \mathcal{M}_2 [McM03a]. Finally, in Section 9 we show that Figure 1 defines 1-forms in ΩG and, for suitable parameters, these forms generate Teichmüller curves in $\bigcup G_D$.

Notes and references. The components of G_D with \sqrt{D} irrational give a new, infinite series of geometrically primitive Teichmüller curves.

The previously known examples consist of four infinite series and two sporadic cases. The first three series come from the Weierstrass curves $W_D \subset \mathcal{M}_g$, defined for $g = 2, 3$ and 4 [Cal04], [McM03a], [McM06]. The fourth is the Bouw–Möller series, which gives finitely many more examples in \mathcal{M}_g for every $g > 1$ [BM10]; see also [Vee89], [Hoo13], [Wri13]. Finally, there are two sporadic examples associated to the Coxeter diagrams E_7 and E_8 ; see [KS00], [Vor96] and [Lei04].

The locus ΩG itself has many interesting properties. For example, it is the first known primitive, $\mathrm{SL}_2(\mathbb{R})$ -invariant subvariety of $\Omega \mathcal{M}_g$ defined over \mathbb{Q} (in period coordinates), aside from the obvious examples like strata. It is also an example of an affine invariant manifold of rank 2. For more on general properties of affine invariant manifolds, see [Wri14] and [Wri15].

A program provided by A. Eskin led us to focus on the cathedral forms and provided evidence that they should lie in a new invariant subvariety of $\Omega\mathcal{M}_4$. A special case of Theorem 1.10 was first proved using the algorithm described in [Muk17a], which showed directly that $\mathrm{SL}(X, \omega)$ is a lattice for $(x, y, d) = (0, 1/2, 2)$.

Further results and useful background can be found in the surveys [Mas06], [Mö9] and [Zor06].

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2. Cubic curves

In this section we recall some classical constructions from projective geometry. These constructions associate, to any smooth plane cubic curve $A \subset \mathbb{P}^2$, three other curves: the *Hessian* $HA \subset \mathbb{P}^2$, the *Cayleyan* CA , and the *satellite Cayleyan* SA . The last two reside in the dual projective plane \mathbb{P}^2 . In the next section, we will see that the points in the flex locus lying over A are naturally parametrized by SA .

Useful references for this material include [Cay57], [Cre62], [Sal79] and [Dol12].

Plane cubics. Let A be a *plane cubic curve*, given as the zero set

$$A = Z(f) \subset \mathbb{P}^2$$

of a homogeneous polynomial $f : \mathbb{C}^3 \rightarrow \mathbb{C}$ of degree three. We say A is a *triangle* if it is projectively equivalent to the cubic $Z(XYZ)$, and a *Fermat cubic* if it is equivalent to $Z(X^3 + Y^3 + Z^3)$. We will be mostly interested in the case where f is irreducible and A is smooth.

Polars, satellites and projections. Let $S = [s]$ denote the point in \mathbb{P}^2 determined by a nonzero vector $s = (s_0, s_1, s_2) \in \mathbb{C}^3$. The *polar conic* of A with respect to S is defined by

$$(2.1) \quad \mathrm{Pol}(A, S) = Z(\langle s, \nabla f(x) \rangle) = Z\left(\sum s_i \frac{df}{dx_i}\right).$$

The *satellite conic* of A (cf. [Sal79, p. 62]) is defined by

$$(2.2) \quad \mathrm{Sat}(A, S) = Z(\langle x, \nabla f(s) \rangle^2 - 4f(s)\langle s, \nabla f(x) \rangle).$$

Projection from S defines a rational map

$$\pi_S : A \rightarrow \mathbb{P}^1.$$

(Intrinsically, the target is the linear system of hyperplanes through S .) If A is smooth and $S \notin A$, then projection from S to \mathbb{P}^1 is a rational map of degree

three, and one can readily check that

$$(2.3) \quad A \cap \text{Pol}(A, S) = \{\text{critical points of } \pi_S\}.$$

Moreover, the cocritical points of π_S (defined by equation (1.1)) come from its satellite: we have

$$(2.4) \quad A \cap \text{Sat}(A, S) = \{\text{cocritical points of } \pi_S\}.$$

Note. Relation (2.3) holds, more generally, for any smooth hypersurface $A \subset \mathbb{P}^n$, and relation (2.4) holds whenever A is cubic, as does the alternative formula:

$$(2.5) \quad \text{Pol}(A, S) = Z(\langle D^2 f(s)x, x \rangle).$$

Lattès maps. For the remainder of this section, we assume that the cubic curve $A = Z(f)$ is smooth. The tangent line to A at x will be denoted by $T_x A \subset \mathbb{P}^2$. The space of tangent lines forms the dual sextic

$$\check{A} = \{T_x A : x \in A\} \subset \check{\mathbb{P}}^2$$

in the dual projective plane. Let $\{x, x'\}$ denote the points where $T_x A$ meets A . If a line L meets A at (a, b, c) , then the three points (a', b', c') also lie on a line $L' = \delta_A(L)$. This construction defines the holomorphic *Lattès* map

$$(2.6) \quad \delta_A : \check{\mathbb{P}}^2 \rightarrow \check{\mathbb{P}}^2$$

associated to A , of interest in complex dynamics (see, e.g., [DJ10], [Ron10], [BL01]). Its algebraic degree is four. The critical values of A coincide with the dual sextic \check{A} .

The Hessian. The *Hessian* of A is the cubic curve defined by $HA = Z(\det D^2 f)$. The nine flexes of A are given by $HA \cap A$.

The Hessian can be described geometrically in terms of the polars and satellites of A ; namely,

$$(2.7) \quad HA = \{S \in \mathbb{P}^2 : \text{the polar conic } \text{Pol}(A, S) \text{ is singular}\},$$

and

$$(2.8) \quad A \cup HA = \{S \in \mathbb{P}^2 : \text{the satellite conic } \text{Sat}(A, S) \text{ is singular}\}.$$

These statements follow directly from equations (2.5) and (2.2), using the fact that a conic $Z(\sum a_{ij}x_i x_j)$ is singular if and only if $\det(a_{ij}) = 0$. When the conic is singular, it becomes a pair of lines; thus (2.8) implies:

$$(2.9) \quad \text{For } S \notin A, \text{ the projection } \pi_S : A \rightarrow \mathbb{P}^1 \text{ has three collinear cocritical points} \iff S \in HA.$$

From cocritical to critical. Since it plays an important role in the sequel, we sketch a more geometric proof of (2.9). Suppose $S \notin A$ and $\pi_S : A \rightarrow \mathbb{P}^1$ has three collinear cocritical points $P_i, i = 1, 2, 3$, with corresponding critical points $Q_i \in A$. Let L be the line through $\{P_1, P_2, P_3\}$, and let R_i be the line joining S to $P_i, i = 1, 2, 3$. Then $\sum_1^3 P_i + 2Q_i$ forms the base locus of the pencil of cubics $|A + \lambda R_1 R_2 R_3|$. Within this pencil one can find a reducible cubic of the form $L + C$. Since the conic C has three tangent lines R_i that pass through a single point, it is singular; and since $C \cap A = \sum 2Q_i$, there is a line M through $\{Q_1, Q_2, Q_3\}$ such that $C = 2M$. We then have $M \subset \text{Pol}(A, S)$, so $S \in HA$ by (2.7). The converse is immediate, as we will see below.

The Cayleyan. For each $S \in HA$, we have a pair of distinct lines such that

$$\text{Pol}(A, S) = L_1 \cup L_2.$$

The *Cayleyan* $CA \subset \check{\mathbb{P}}^2$ is the set of all lines that arise in this way; that is,

$$CA = \{L \in \check{\mathbb{P}}^2 : L \subset \text{Pol}(A, S) \text{ for some } S \in HA\}.$$

The point S is uniquely determined by L , since it lies on $T_x A$ for all $x \in L \cap A$. Thus we have a natural degree two covering map, $CA \rightarrow HA$. The curve CA is also cubic; see equation (2.14) below.

The satellite Cayleyan. Note that if $S \notin A$ and $x \in A$ is a critical point of π_S , then x' is a cocritical point. As a consequence, if $S \in HA$ and $\text{Pol}(A, S) = L_1 \cup L_2$, then

$$\text{Sat}(A, S) = L'_1 \cup L'_2.$$

We refer to the set of lines that arise in this way as the *satellite Cayleyan*,

$$SA = \delta_A(CA) = \{L \in \check{\mathbb{P}}^2 : L \subset \text{Sat}(A, S) \text{ for some } S \in HA\}.$$

Since CA is irreducible, so is SA . It is generically a curve of degree 12 and genus one, with interesting singularities.

Normal form. Here is an explicit description of the polar and satellite lines for an arbitrary smooth cubic $A \subset \mathbb{P}^2$ as seen from a point $S \in HA - A$.

Let $\text{Pol}(A, S) = L_1 \cup L_2$ and $\text{Sat}(A, S) = L'_1 \cup L'_2$. It is easy to see that $S \notin L_1$. Choose affine coordinates (x, y) on $\mathbb{C}^2 \subset \mathbb{P}^2$ so that $S = [0 : 1 : 0]$ is the vertical point at infinity, and L_1 is the x -axis. Then, it is readily verified that A is defined by a cubic equation of the form

$$(2.10) \quad f(x, y) = y^3 + b(x)y^2 + c(x) = 0,$$

where $b, c \in \mathbb{C}[x]$ are polynomials of degrees (at most) one and three respectively; and that the polar and satellite lines for any such cubic are defined by

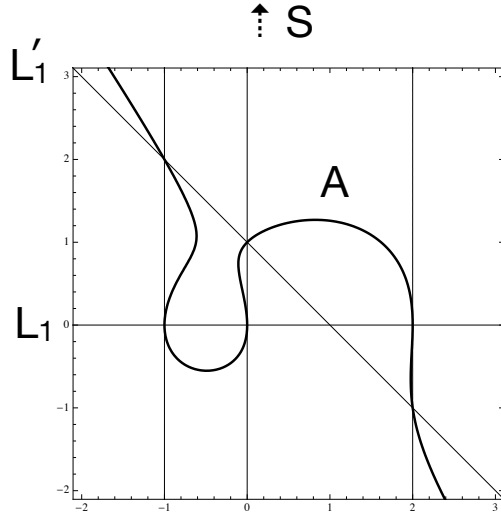


Figure 2. We have $L_1 \subset \text{Pol}(A, S)$ and $L'_1 \subset \text{Sat}(A, S)$.

the vanishing of the linear forms:

$$(2.11) \quad \begin{aligned} L_1(x, y) &= y; & L_2(x, y) &= y + (2/3)b(x); \quad \text{and} \\ L'_1(x, y) &= y + b(x); & L'_2(x, y) &= y - (1/3)b(x). \end{aligned}$$

In particular, all four lines pass through a single point in \mathbb{P}^2 .

Figure 2 shows the lines L_1 and L'_1 for the cubic defined by equation (2.10) with $b(x) = x - 1$ and $c(x) = x(x + 1)(x - 2)$.

The Hesse pencil. We conclude with a discussion of the Hesse family of cubics, which will be used in our description of the flex locus. A useful reference for this topic is [AD09].

The Hesse pencil is the family of cubic curves $A_t \subset \mathbb{P}^2$, defined for $t \in \mathbb{P}^1$ by

$$(2.12) \quad X^3 + Y^3 + Z^3 - 3tXYZ = 0.$$

The curve A_0 is a Fermat cubic, while A_∞ is the triangle defined by $XYZ = 0$. The cubic A_t is smooth over the points

$$(2.13) \quad \widetilde{\mathcal{M}}_1 = \{t \in \mathbb{C} : t^3 \neq 1\} \subset \mathbb{P}^1;$$

otherwise it is a triangle.

The base locus $E = A_0 \cap A_\infty$ of the Hesse pencil coincides with the nine flexes of A_0 , as well as the flexes of every other smooth curve in the family. Thus, we can regard an element of the Hesse pencil as an elliptic curve with

marked 3-torsion. More precisely, we have a natural isomorphism

$$\widetilde{\mathcal{M}}_1 \cong \mathbb{H}/\Gamma(3);$$

the modular j -invariant of A_t is given by

$$j(t) = \frac{27(t^4 + 8t)^3}{(t^3 - 1)^3};$$

and j defines a covering map of orbifolds

$$j : \widetilde{\mathcal{M}}_1 \rightarrow \mathcal{M}_1 = \mathbb{H}/\mathrm{SL}_2(\mathbb{Z}) \cong \mathbb{C},$$

with deck group $\mathrm{PSL}_2(\mathbb{F}_3) \cong A_4$. Note that A_t is a Fermat cubic if and only if $j(t) = 0$, which gives $t = 0, -2$ or $1 \pm \sqrt{-3}$.

When A belongs to the Hesse pencil, so do the cubics HA and CA (using the natural dual basis to identify \mathbb{P}^2 and \mathbb{P}^2). In fact, $HA_t = A_h$ and $CA_t = A_c$ for the values

$$(2.14) \quad h(t) = \frac{4 - t^3}{3t^2} \quad \text{and} \quad c(t) = \frac{2 + t^3}{3t}$$

(see [AD09, §3]). Using these formulas one can verify that, for a smooth cubic curve A :

Either A is a Fermat cubic and HA and CA are triangles, or HA and CA are also smooth cubics.

3. The flex locus

In this section we discuss the flex locus from the perspective of plane cubics. We begin by establishing an alternative definition of F , using the language of Section 2.

THEOREM 3.1. *A point $(A, P) \in \mathcal{M}_{1,3}$ lies in F if and only if there is a plane cubic model for A and a line L in the satellite Cayleyan SA such that $P = A \cap L$.*

Using the universal Cayleyan, we then show:

THEOREM 3.2. *The flex locus $F \subset \mathcal{M}_{1,3}$ is the image of a smooth, irreducible surface under a proper immersion.*

Finally we define a four-dimensional bundle of quadratic differentials $QF \rightarrow F$, analogous to $\Omega G \rightarrow G$, and show:

THEOREM 3.3. *The locus $QF \subset Q\mathcal{M}_{1,3}$ is a closed, irreducible subvariety of dimension four.*

Markings. To prove Theorems 3.2 and 3.3 we will explicitly construct a smooth, irreducible surface \widetilde{F} , a finite manifold cover of moduli space

$$u : \widetilde{\mathcal{M}}_{1,3} \rightarrow \mathcal{M}_{1,3},$$

and a proper immersion

$$\delta : \widetilde{F} \rightarrow \widetilde{\mathcal{M}}_{1,3},$$

such that $u \circ \delta$ sends \widetilde{F} to F .

The smooth surface \widetilde{F} is of interest in its own right. A point in \widetilde{F} corresponds to triple (A, P, π_S) , with $S \in HA$, satisfying the definition of F in Section 1, together with a marking of the 3-torsion of A . The surface F itself is *not* smooth, and these choices serve to separate its sheets and resolve its orbifold points.

Cubic models. To make the connection to Section 2, recall that any Riemann surface $A \in \mathcal{M}_1$ can be presented as a smooth cubic curve $A \subset \mathbb{P}^2$. This *plane cubic model* for A is unique up to automorphisms of A and \mathbb{P}^2 .

Proof of Theorem 3.1. Suppose $(A, P) \in F$. Let $A \subset \mathbb{P}^2$ be the plane cubic model determined by the complete linear system $|P|$. Then $P = A \cap L$ for some line $L \in \check{\mathbb{P}}^2$. Since $(A, P) \in F$, there is a degree three rational map $\pi : A \rightarrow \mathbb{P}^1$ such that (i) the fibers of π are linearly equivalent to P , and (ii) P is contained in the cocritical points of π . Property (i) implies that π is given by projection from some point $S \in \mathbb{P}^2 - A$; and (ii) implies that P is contained in the satellite conic $\text{Sat}(A, S)$ (see assertion (2.4)). Since P consists of three distinct points, this implies we have $L \subset \text{Sat}(A, S)$; hence the satellite conic is singular, and we have $L \in SA$. The converse is similar. \square

Cubics and lines. We now turn to the proof of Theorem 3.2. For concreteness, we will work with the family of Hesse cubics $A_t \subset \mathbb{P}^2$ defined by (2.12). Consider the Zariski open subset of $\mathbb{P}^1 \times \check{\mathbb{P}}^2$ defined by

$$\widetilde{\mathcal{M}}_{1,3} = \{(t, L) : A_t \text{ is smooth and } |L \cap A_t| = 3\}.$$

Since every cubic occurs, up to isomorphism, in the Hesse family, we have a natural covering map of orbifolds $u : \widetilde{\mathcal{M}}_{1,3} \rightarrow \mathcal{M}_{1,3}$ given by

$$u(t, L) = (A_t, L \cap A_t).$$

We remark that the deck group Γ of $\widetilde{\mathcal{M}}_{1,3}/\mathcal{M}_{1,3}$ has order 216; it satisfies

$$(3.1) \quad \Gamma \cong \text{Aut}(\mathbb{P}^2)^E \cong \mathbb{F}_3^2 \rtimes \text{SL}_2(\mathbb{F}_3),$$

where $\text{Aut}(\mathbb{P}^2)^E$ denotes the group of projective transformations stabilizing nine basepoints $E = A_0 \cap A_\infty$ of the Hesse pencil [AD09]. The fibers of u correspond to different markings of the flexes of A .

The universal Cayleyan. Next we define the *universal Cayleyan* over the Hesse family by

$$CA_{\mathbb{P}^1} = \{(t, L) \in \mathbb{P}^1 \times \check{\mathbb{P}}^2 : L \in CA_t\}.$$

The locus $CA_{\mathbb{P}^1} \rightarrow \mathbb{P}^1$ is a smooth elliptic surface, even though some of its fibers, such as CA_0 , are triangles. In particular, $CA_{\mathbb{P}^1}$ is irreducible. Smoothness follows from the fact that $c(t)$ has a simple pole in equation (2.14), so that near a triangular fiber $CA_{\mathbb{P}^1}$ is locally isomorphic to the surface $xy = t$.

Lattès maps. Recall from equation (2.13) that $A = A_t$ is smooth if and only if $t \in \widetilde{\mathcal{M}}_1 \subset \mathbb{P}^1$. Let

$$(3.2) \quad \delta : \widetilde{\mathcal{M}}_1 \times \check{\mathbb{P}}^2 \rightarrow \widetilde{\mathcal{M}}_1 \times \check{\mathbb{P}}^2$$

be the proper map defined by $\delta(t, L) = (t, \delta_{A_t}(L))$ using the Lattès construction (2.6).

Normalization of a cover of F. Finally we define a Zariski open subset of the universal Cayleyan by

$$\widetilde{F} = CA_{\mathbb{P}^1} \cap \delta^{-1}(\widetilde{\mathcal{M}}_{1,3}).$$

It is easy to see, e.g., from the example in Figure 2, that \widetilde{F} is nonempty; and since $CA_{\mathbb{P}^1}$ is a smooth, irreducible surface, so is \widetilde{F} .

Proof of Theorem 3.2. Since the map δ in equation (3.2) is proper, so is its restriction $\delta|_{\widetilde{F}}$. It is also an immersion, since the critical values of δ_A correspond to lines with $|L \cap A| < 3$, and these configurations are excluded from $\widetilde{\mathcal{M}}_{1,3}$. Since u is a covering map of orbifolds, the composition $u \circ \delta$ is a proper immersion; and since $SA = \delta_A(CA)$, its image is F by Theorem 3.1. \square

COROLLARY 3.4. *The surface F is birational to \mathbb{P}^2 .*

Proof. The hyperelliptic involution $-I \in \text{SL}_2(\mathbb{F}_3)$ belongs to the group Γ in equation (3.1), so the map $u \circ \delta : \widetilde{F} \rightarrow F$ factors through a rational quotient of the elliptic surface $CA_{\mathbb{P}^1}$. Thus F itself is rational. \square

One can also check that in the example of Figure 2, S is uniquely determined by $L'_1 \in SA$, and hence:

The map $\delta : \widetilde{F} \rightarrow \widetilde{\mathcal{M}}_{1,3}$ is generically 1-to-1.

The space of quadratic differentials $QF \rightarrow F$. We conclude by defining a bundle of quadratic differentials to record the directions of Teichmüller geodesics in F .

Recall that the cotangent space to a point $(Y, P) \in \mathcal{M}_{g,n}$ is naturally identified with the vector space $Q(Y, P)$ of meromorphic quadratic differentials

q on Y with $(q) + P \geq 0$. A point in the moduli space of quadratic differentials, $Q\mathcal{M}_{g,n} \rightarrow \mathcal{M}_{g,n}$, is specified by a triple (Y, P, q) as above with $q \neq 0$.

Now let $(A, P) \in \mathcal{M}_{1,3}$ be an elliptic curve with marked points whose membership in F is ratified by a rational map $\pi : A \rightarrow \mathbb{P}^1$ of degree three. Let

$$QF(A, P, \pi) = \{q \in Q(A, P) : (q) = Z - P \text{ for some fiber } Z \text{ of } \pi\}.$$

To take into account of the possibility that π is not unique, let $QF(A, P) = \bigcup_{\pi} Q(A, P, \pi)$. Finally, let $QF \rightarrow F$ denote the subspace of $Q\mathcal{M}_{1,3} \rightarrow \mathcal{M}_{1,3}$ whose fiber over $(A, P) \in F$ is $QF(A, P)$.

Proof of Theorem 3.3. Recall that for $(t, L) \in \tilde{F}$, there is a unique $S \in \mathbb{P}^2$ such that $L \subset \text{Pol}(A_t, S)$, and a unique $L' \subset \text{Sat}(A_t, S)$ such that $L' = \delta_{A_t}(L)$. Define a bundle $Q\tilde{F} \rightarrow \tilde{F}$ with fibers

$$Q\tilde{F}(t, L) = QF(A_t, L' \cap A_t, \pi_S) \cong \mathbb{C}^2 - \{0\}.$$

We then have a natural proper map immersion $Q\delta : Q\tilde{F} \rightarrow Q\mathcal{M}_{1,3}$ covering the map $\delta : \tilde{F} \rightarrow \mathcal{M}_{1,3}$, and its image is QF . Since \tilde{F} is irreducible, so is QF and clearly $\dim QF = \dim Q\tilde{F} = 4$. □

Sheets of F. The proof shows that, given $(A, P) \in F$, there are only finitely many possibilities for the associated map $\pi : A \rightarrow \mathbb{P}^1$, and the different choices of π index the different sheets of the immersed surface F passing through (A, P) .

4. The gothic locus

In this section we discuss the correspondence between quadratic differentials and Prym forms, and use it to relate the flex locus F to the threefold $G \subset \mathcal{M}_4$ defined in Section 1. We will show:

THEOREM 4.1. *The squaring map gives a natural algebraic isomorphism*

$$\text{sq} : \Omega G \rightarrow QF(-1^3, 1^3).$$

Here $QF(-1^3, 1^3)$ denotes the intersection of QF with the principal stratum of $Q\mathcal{M}_{1,3}$.

COROLLARY 4.2. *The locus ΩG is a closed, irreducible four-dimensional subvariety of $\Omega\mathcal{M}_4$.*

Strata. We begin by reviewing notation for strata of quadratic differentials with marked points. Recall that a point of $Q\mathcal{M}_{g,n}$ is specified by a triple (Y, P, q) . The stratum

$$Q\mathcal{M}_{g,n}(a_1, \dots, a_s) \subset Q\mathcal{M}_{g,n}$$

is defined by the requirement that there exist distinct points (p_1, \dots, p_s) in Y such that $P = \{p_1, \dots, p_n\}$ and the divisor of q satisfies

$$(q) = \sum_1^s a_i p_i.$$

Here $\sum a_i = 4g - 4$, $a_i \geq -1$ for all i , and $a_i \geq 1$ if $i > n$.

Genus 4. Now consider a 1-form $(X, \omega) \in \Omega\mathcal{M}_4(2^3, 0^3)^-$, with associated involution J . Since $|\text{Fix}(J)| = 6 = |\chi(X)|$, the quotient Riemann surface $A = X/J$ has genus one. Moreover, the form ω^2 is J -invariant, so it descends to a meromorphic quadratic differential q on X , with three simple poles and three simple zeros. Marking the poles by P , we obtain a form $(A, P, q) \in Q\mathcal{M}_{1,3}(-1^3, 1^3)$. Conversely, given a quadratic differential in this stratum, passage to the Riemann surface X where $\omega = \sqrt{q}$ becomes single-valued defines a point $(X, \omega) \in \Omega\mathcal{M}_4(2^3, 0^3)^-$.

Summing up, we have a natural algebraic isomorphism

$$(4.1) \quad \text{sq} : \Omega\mathcal{M}_4(2^3, 0^3)^- \cong Q\mathcal{M}_{1,3}(-1^3, 1^3),$$

respecting the action of $\text{SL}_2(\mathbb{R})$.

Proof of Theorem 4.1. Let (X, ω) be a form in $\Omega\mathcal{M}_4(2^3, 0^3)^-$, with

$$\text{sq}(X, \omega) = (A, P, q).$$

Let J be the unique involution such that $J^*(\omega) = -\omega$, and let $\text{Fix}(J) = Z' \cup P'$ where $|Z'| = |P'| = 3$ and $Z' = Z(\omega)$. We have a natural degree two map $\phi : X \rightarrow A$, injective on $\text{Fix}(J)$, such that

$$(4.2) \quad (q) = Z - P = \phi(Z') - \phi(P').$$

We will show that $(X, \omega) \in \Omega G \iff (A, P, q) \in QF$.

From gothic to flex. First assume that $(X, \omega) \in \Omega G$. We then have a degree three map to an elliptic curve, $p : X \rightarrow B$, and an involution $j \in \text{Aut}(B)$, such that

$$(4.3) \quad p(J(x)) = j(p(x)).$$

By the definition of ΩG , Z' is a fiber of p .

Choose the origin in B so that $j(x) = -x$, and let $r : B \rightarrow B/j \cong \mathbb{P}^1$ be the quotient map. Then $r \circ p : X \rightarrow \mathbb{P}^1$ is a J -invariant map of degree six. Consequently we have a unique degree three rational map $\pi : A \rightarrow \mathbb{P}^1$ making

the diagram:

$$(4.4) \quad \begin{array}{ccc} & X & \\ \phi \swarrow & & \searrow p \\ A & & B \\ \pi \searrow & & \swarrow r \\ & \mathbb{P}^1 & \end{array}$$

(Note: The arrows from X to A and X to B are labeled with a '3' below them.)

commute. We will show that:

- (i) the fibers of π are linearly equivalent to P , and
- (ii) P is contained in the cocritical points of π .

To see (i), simply note that Z is a fiber of π since Z' is a fiber of $r \circ p$, and equation (4.2) shows that Z is linearly equivalent to P because the canonical bundle of A is trivial.

To prove (ii), note that p maps $\text{Fix}(J)$ into $\text{Fix}(j)$ by (4.3), and that $\text{Fix}(j) = B[2]$ coincides with the set of 2-torsion points in B . Let us denote the points of $B[2]$ by $\{e'_1, e'_2, e'_3, e'_4\}$, with $p(Z') = (e'_4)$. Since $p^{-1}(e'_i)$ is also J -invariant for $i = 1, 2, 3$, we can find $P'_i, Q'_i \in X$ such that, as divisors, we have

$$p^{-1}(e'_i) = P'_i + Q'_i + J(Q'_i)$$

and $P_i = \phi(P'_i) \in P$. Let $Q_i = \phi(Q'_i)$, and let $e_i = r(e'_i)$. Then again as divisors, we have

$$(4.5) \quad \pi^{-1}(e_i) = P_i + 2Q_i$$

for $i = 1, 2, 3$, and hence P_i is a cocritical point of π . This proves (ii) and shows that $(A, P, q) \in QF$.

From flex to gothic. Now suppose $(A, P, q) \in QF(-1^3, 1^3)$. Let $\pi : A \rightarrow \mathbb{P}^1$ be a rational map of degree three verifying conditions (i) and (ii) above. Let $P = \{P_1, P_2, P_3\}$, let $\pi(P_i) = e_i$ for $i = 1, 2, 3$, and let $\pi(Z) = e_4$. Then we can find $Q_i \in A$ such that (4.5) holds. Let $Q = Q_1 + Q_2 + Q_3$.

By the definition of the flex locus (Section 3), there is a plane cubic model for A and a pair of lines $L \in CA$ and $L' \in SA$ such that $P = A \cap L'$ and $Q = A \cap L$.

Let $r : B \rightarrow \mathbb{P}^1$ be an elliptic curve, presented as a 2-fold covering of \mathbb{P}^1 branched over $E = \{e_1, e_2, e_3, e_4\}$. One can also regard B/\mathbb{P}^1 as the Riemann surface of the function \sqrt{f} , where

$$(f) = e_1 + e_2 + e_3 - 3e_4.$$

Now note that, because the divisors Z, P and Q are all linearly equivalent,

$$D = (f \circ \pi) - (q) = (P + 2Q - 3Z) - (Z - P) = 2Q + 2P - 4Z$$

is the divisor of the *square* of a rational function on A . Thus \sqrt{q} and $\sqrt{f \circ \pi}$ define the same Riemann surface X/A , and hence the map $\pi : A \rightarrow \mathbb{P}^1$ lifts to a map $p : X \rightarrow B$ making diagram (4.4) commute. This lift intertwines the $\mathbb{Z}/2$ Galois groups of X/A and B/\mathbb{P}^1 , so p is odd. Then, since Z' is a fiber of p , the form (X, ω) belongs to ΩG . \square

Proof of Corollary 4.2. By Theorem 3.3, QF is a closed, irreducible subvariety of $QM_{1,3}$ dimension four. Thus $\Omega G \subset \Omega \mathcal{M}_4$ inherits these properties from QF , by Theorem 4.1 and the fact (see Section 1) that the Prym stratum $\Omega \mathcal{M}_4(2^3, 0^3)^-$ is closed in $\Omega \mathcal{M}_4$. \square

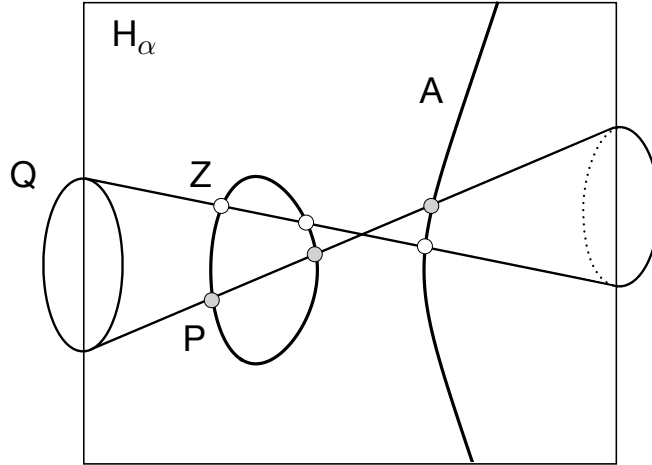


Figure 3. The canonical model for $X \subset \mathbb{P}^3$ gives six distinguished points $Z \cup P$ on the cubic curve A .

Remarks on the canonical model. The canonical embedding gives an illuminating geometric picture of the relationship between X and A .

Let (X, ω) be a 1-form in $\Omega \mathcal{M}_4(2^3, 0^3)^-$ with involution J and quotient curve $A = X/J$ of genus one. The eigenspaces of J determine a splitting

$$(4.6) \quad \Omega(X) = \Omega(A) \oplus \Omega(A)^\perp,$$

where we have identified $\Omega(A) = \mathbb{C}\alpha$ with the span of a J -invariant form $\alpha \in \Omega(X)$.

Let $\text{Fix}(J) = Z \cup P$, where $Z = Z(\omega)$. We have $|Z| = |P| = 3$. An involution of a hyperelliptic curve has at most four fixed points, provided it is not the hyperelliptic involution itself. Since $|\text{Fix}(J)| = 6$, the curve X is not hyperelliptic. Thus its canonical linear system provides an embedding

$$X \subset \mathbb{P}\Omega(X)^* \cong \mathbb{P}^3.$$

Every nonzero 1-form η on X determines a plane $H_\eta \subset \mathbb{P}^3$.

It is classical that the space curve X , of degree six, is the transverse intersection $Q \cap C$ of an irreducible quadric and a cubic surface in \mathbb{P}^3 [GH78, p. 258]. The surface Q is uniquely determined by X , but C is not.

The automorphism J acts naturally on $\Omega(X)$, and hence on \mathbb{P}^3 . Its fixed-point set in \mathbb{P}^3 is the union of the plane H_α and the point H_α^\perp dual to $\Omega(A)^\perp$. Projection from this point yields a J -invariant map

$$\Phi : X \rightarrow H_\alpha.$$

The fibers of Φ consist of pairs of points that are interchanged by J . Hence $A = \Phi(X) \subset H_\alpha$ is a cubic plane curve, naturally isomorphic to X/J ; and the fixed-points of $J|X$ are given simply by

$$\text{Fix}(J) = Z \cup P = X \cap H_\alpha = A \cap Q.$$

We can now make the cubic surface defining X canonical, by letting C be the cone over A with vertex H_α^\perp .

The conic $Q \cap H_\alpha$ is singular, since its intersection with the line $H_\omega \cap H_\alpha$ consists of the three points Z . Since $J(Q) = Q$, this implies that Q itself is singular. Thus $Q \cap H_\alpha$ is a pair of lines through the singular point of Q — one of the form $H_\omega \cap H_\alpha$, containing $Z = Z(\omega)$ — and the other containing the remaining fixed-points P . See Figure 3.

We remark that the fact that Q is singular implies:

$$X \text{ lies on the } \theta\text{-null divisor } \mathcal{M}'_4 \subset \mathcal{M}_4.$$

See [TiB88], [Gen15, Th. 1.3] and [ACGH85, Ex. A-3, p.196] for more details.

Equations for $X \in G$. We can now give an explicit formula for the canonical model of any curve X in the gothic locus $G \subset \mathcal{M}_4$. Namely, we find that $X = C \cap Q$ can be defined in affine coordinates by the two equations

$$(4.7) \quad y^3 + b(x)y^2 + c(x) = 0 \quad \text{and} \quad z^2 = (y + b(x))(x + a).$$

Here $a \in \mathbb{C}$, $\deg(b) \leq 1$, $\deg(c) \leq 3$, and $H_\omega = Z(x + a)$.

To see this, recall that Theorem 4.1 associates to any form $(X, \omega) \in \Omega G$ a point (A, P) in the flex locus and a quadratic differential $q \in Q(A, P)$. As we saw in Section 2 (see equations (2.10) and (2.11)), we can choose affine coordinates on $\mathbb{P}^2 \cong H_\alpha$ so that A is given by the first equation in (4.7), and $P = A \cap L'_1$, where L'_1 is the satellite line defined by $y + b(x) = 0$. In these coordinates $\pi_S(x, y) = x$, so the zeros $Z(q) \subset A$ are cut out by the equation $x + a = 0$ for some $a \in \mathbb{C}$. The zeros of q become the zeros of ω after passing to the double cover $X \rightarrow A$. Since the quadric Q is characterized by the fact that it is symmetric under J and passes through $P \cup Z(\omega)$, it is given by the second equation in (4.7) above. Moreover, the associated elliptic curve B is defined in coordinates (t, x) by

$$t^2 + c(x)(x + a) = 0,$$

and the degree three map $p : X \rightarrow B$ is given by the substitution $t = yz$.

One can also use equation (4.7) to show directly that $\dim G \geq 3$. Namely, if we fix $c(x) = x^3 - 1$ and vary a and b , then equation (4.7) determines a rational map $\Phi : \mathbb{C}^3 \dashrightarrow G$; and one can verify that Φ is finite-to-one by considering the seven critical values of $f : X \rightarrow B \rightarrow \mathbb{P}^1$.

Linear systems and rational maps. We conclude by describing some of the previously considered maps and spaces more intrinsically.

Let (X, ω) be a form in ΩG , with associated genus one quotients A and B . Let (A, P, q) be the corresponding point in QF , with associated map $\pi : A \rightarrow \mathbb{P}^1$. Let $(\alpha, \beta, \omega, \omega')$ be an orthogonal basis for $\Omega(X)$, where α and β are pulled back from A and B respectively. Then:

1. the degree six map $f : X \rightarrow \mathbb{P}^1$, factoring through A or B as in the commutative diagram (4.4), can be presented in the form $f = \omega/\omega'$;
2. a basis for $QF(A, P, \pi)$ is given by the J -invariant quadratic differentials $q = \omega^2$ and $q' = \omega'\omega$;
3. the degree three map $\pi : A \rightarrow \mathbb{P}^1$ can be expressed as $\pi = q'/q$; and
4. the tangent space to the sheet of F defined by π is the kernel of the cotangent vector $\beta\omega \in Q(A, P)$.

To see (1), note that the fibers of f as originally defined are divisors of canonical forms that push forward to 0 on A and B . Points (2) and (3) follow easily. For (4), to show, e.g., that

$$\int_A \beta\omega \frac{\bar{q}'}{|q|} = \frac{1}{2} \int_X \beta\bar{\omega}'|f| = 0,$$

observe that $\beta|f|$ is pulled back from B , while the pushforward of ω' to B is zero.

Note that $(\beta\omega) = W - P$ only depends on π : the zeros W of $\beta\omega$ are the three critical points of π satisfying $\pi(W) \cap \pi(P) = \emptyset$.

5. F is totally geodesic

In this section we will prove our main results:

- (I) The locus ΩG is locally defined by real linear equations in period coordinates;
- (II) The loci ΩG and QF are $SL_2(\mathbb{R})$ -invariant; and
- (III) The flex locus $F \subset \mathcal{M}_{1,3}$ is a totally geodesic surface.

These assertions correspond to Theorem 1.4, Corollary 1.5 and Theorem 1.1 respectively.

Here and in the sequel, cohomology is taken with complex coefficients unless otherwise specified.

Invariant varieties and period coordinates. We first remark that (I) implies (II) by general principles.

Recall that for any point $(X_0, \omega_0) \in \Omega\mathcal{M}_g$, there exists a neighborhood U of (X_0, ω_0) (or an orbifold chart) in its stratum and a natural open analytic embedding

$$\iota : U \hookrightarrow H^1(X_0, Z(\omega_0)) \cong \mathbb{C}^{2g+|Z(\omega_0)|-1},$$

called the *period map*, that sends (X, ω) to the unique relative cohomology class $[\omega]$ such that $\langle [\omega], \gamma \rangle = \int_\gamma \omega$. Using this map, we implicitly identify U with an open subset of the vector space on the right.

A closed subset $M \subset \Omega\mathcal{M}_g(a_1, \dots, a_n)$ is *locally defined by real linear equations* if, for each point $(X_0, \omega_0) \in M$ and U as above, there exists a finite set of complex subspaces $S_i \subset H^1(X_0, Z(\omega_0))$, each invariant under $v \mapsto \bar{v}$, such that

$$U \cap M = U \cap (S_1 \cup \dots \cup S_k).$$

The conclusion of (I) means that ΩG has this form.

There is a natural action of the connected group $\mathrm{GL}_2^+(\mathbb{R})$ on any complex vector space V with a real structure, defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot v = \begin{pmatrix} 1 & i \\ & \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \mathrm{Re} v \\ \mathrm{Im} v \end{pmatrix}.$$

The action of $\mathrm{SL}_2(\mathbb{R})$ on $\Omega\mathcal{M}_g$ has this form in period coordinates (see, e.g., [McM03a, §3]). Evidently any subspace of V defined over \mathbb{R} is $\mathrm{SL}_2(\mathbb{R})$ -invariant, and so:

(5.1) *Any closed set $M \subset \Omega\mathcal{M}_g$ locally defined by real linear equations is $\mathrm{SL}_2(\mathbb{R})$ invariant.*

(The assumption that M is closed is used to prove the group action is global.) Thus (I) implies (II) for ΩG , and invariance is inherited by QF via Theorem 4.1, using the fact that the squaring map in equation (4.1) respects the action of $\mathrm{SL}_2(\mathbb{R})$.

A criterion for linearity. For the proof of (I) we will use the following general result.

THEOREM 5.1. *Let $M \subset \Omega\mathcal{M}_g(a_1, \dots, a_n)$ be an algebraic variety whose irreducible components have dimension $\geq d$. Suppose that for every $(X, \omega) \in M$, there is a d -dimensional subspace S , defined over a real number field, such that*

$$[\omega] \in S \subset H^1(X, Z(\omega)).$$

Then M is locally defined by real linear equations in period coordinates, and $\dim(M) = d$.

Proof. Fix a point $(X, \omega) \in M$. Let \mathcal{S} be the collection of all d -dimensional subspaces S as above. Then there exists a ball U about (X, ω) in period coordinates such that $M \cap U$ is an analytic variety of dimension $\geq d$, contained in $\bigcup \mathcal{S}$. Since each $S \in \mathcal{S}$ is defined over a number field, the collection \mathcal{S} is countable. Consequently $\dim(M \cap U) = d$.

By Noetherian properties of analytic sets, $S \cap M \cap U$ has nonempty interior in S for only finitely many subspaces $\{S_1, \dots, S_n\} \subset \mathcal{S}$. Since $S_i \cap U$ is connected, it is contained in M for all i . By the Baire category theorem, $\bigcup_1^n (S_i \cap U)$ is dense in $M \cap U$. Since $\bigcup_1^n S_i$ is also closed, this implies that $M \cap U = \bigcup_1^n S_i \cap U$. □

The Abelian surface $C(X|A, B)$. Recall from equation (1.2) that any $(X, \omega) \in \Omega G$, with associated maps $X \rightarrow A$ and $X \rightarrow B$ of degrees two and three respectively, determines a polarized Abelian surface

$$C = C(X|A, B) = \text{Jac}(X) / \text{Im}(\text{Jac}(A) \times \text{Jac}(B)).$$

The submersion $\text{Jac}(X) \rightarrow C$ determines an orthogonal splitting,

$$\Omega(X) \cong \Omega(A) \oplus \Omega(B) \oplus \Omega(C),$$

with $\omega \in \Omega(C)$. Similarly, on the level of relative cohomology, we obtain an exact sequence

$$H^0(Z(\omega)) \rightarrow H^1(X, Z(\omega)) \rightarrow H^1(A) \oplus H^1(B) \oplus H^1(C) \rightarrow 0.$$

Since $J|Z(\omega) = \text{Id}$ and $H^1(X)^J = H^1(A)$, we have a natural isomorphism

$$H^1(B) \oplus H^1(C) \cong \text{Ker}(J + \text{Id}) \subset H^1(X, Z(\omega))$$

defined over \mathbb{Q} .

The class $[\omega]$ lies in the $H^1(C)$ factor above, since $J^*\omega = -\omega$ and ω pushes forward to zero on B .

Proof of (I). By Theorems 3.3 and 4.1, ΩG is an irreducible four-dimensional algebraic subvariety of $\Omega \mathcal{M}_4(2, 2, 2)$. As we have just seen, the relative periods of any form $(X, \omega) \in \Omega G$ lie in a four-dimensional subspace $H^1(C) \subset H^1(X, Z(\omega))$ defined over \mathbb{Q} . By Theorem 5.1, these two facts imply that ΩG is locally defined by real linear equations in period coordinates. □

Proof of (III). Recall that the Teichmüller norm on the cotangent space $Q(A, P)$ is given by $\|q\| = \int_X |q|$, and that the Beltrami coefficient $\bar{q}/|q|$ represents a unit vector tangent to the Teichmüller geodesic generated by q , which is itself the projection to $\mathcal{M}_{1,3}$ of the orbit $\text{SL}_2(\mathbb{R}) \cdot q$.

By Theorem 3.2, F is an immersed smooth surface. Let (A, P) be a point on a sheet of F with associated degree three map $\pi : A \rightarrow \mathbb{P}^1$. Let T_1 be the unit tangent space to this sheet at (A, P) , and let Q_1 be the unit ball in $QF(A, P, \pi)$. Note that T_1 and Q_1 are both homeomorphic to S^3 .

By (II), we have a continuous map $\mu : Q_1 \rightarrow T_1$ given by

$$\mu : q \mapsto \frac{\bar{q}}{|q|}.$$

By uniqueness of the Teichmüller mapping, μ is injective, so by invariance of domain, it is a homeomorphism. Thus every complex geodesic tangent to F at (A, P) is contained in F , because it is the projection of an $SL_2(\mathbb{R})$ -orbit in QF . \square

Defining equations for ΩG . The proof of (I) also yields the following more precise result:

THEOREM 5.2. *Given $(X, \omega) \in \Omega G$ with associated Abelian surface $C = C(X|A, B)$, there exists a neighborhood U of $[\omega]$ in period coordinates such that*

$$[\omega] \in U \cap H^1(C) \subset \Omega G.$$

Proof. Use the bundle $Q\tilde{F} \rightarrow \tilde{F}$ from the proof of Theorem 3.3, along with Theorem 4.1, to construct a four-dimensional sheet of ΩG through (X, ω) along which $H^1(C)$ is constant, and use local injectivity of the period map. \square

Remarks on linear and totally geodesic varieties.

1. Statement (5.1) has a converse [M08, Prop. 1.2]: any $GL_2^+(\mathbb{R})$ -invariant analytic subvariety $V \subset \Omega\mathcal{M}_g$ is locally defined by real linear equations. Remarkably, the closure of any $GL_2^+(\mathbb{R})$ orbit is also locally linear [EMM15].
2. In contrast to the case of Teichmüller curves, whenever $V \subset \mathcal{M}_{g,n}$ is a totally geodesic variety of dimension two or more, a typical complex geodesic in V is generated by a *truly quadratic* differential q . That is, q is not the square of a 1-form. Thus it is useful to work directly with $Q\mathcal{M}_{g,n}$ when studying such V .
3. There are also interesting higher-dimensional $SL_2(\mathbb{R})$ -invariant varieties $M \subset \Omega\mathcal{M}_g$ which, unlike ΩG , are unrelated to totally geodesic varieties. The prime examples are the eigenform loci in genus $g = 2, 3$ and 4 [McM06].
4. It is known that the only complex *symmetric space* that can occur as an immersed totally geodesic submanifold of $\mathcal{T}_{g,n}$ is the unit disk [Ant15].

6. F is primitive

A holomorphic 1-form $(X, \omega) \in \Omega\mathcal{M}_g$ is *primitive* unless there is a form $(Y, \eta) \in \Omega\mathcal{M}_h$, $h < g$, and a holomorphic map $p : X \rightarrow Y$ such that $\omega = p^*(\eta)$. A similar definition applies to quadratic differentials.

In this section we will show:

- (I) a typical form in QF or ΩG is primitive; and therefore
- (II) the surface F does not arise via a covering construction.

Another proof of primitivity appears in the next section.

Covering constructions. Let (Σ_g, Π_n) denote a smooth, oriented topological surface of genus g , with a set of n marked points $\Pi_n \subset \Sigma_g$. Consider a branched covering map

$$\pi : (\Sigma_g, \Pi_n) \rightarrow (\Sigma_h, \Pi_m)$$

whose critical points $C(\pi)$ satisfy

$$(6.1) \quad C(\pi) \cup \Pi_n = \pi^{-1}(\Pi_m).$$

Condition (6.1) can be weakened to $C(\pi) \subset \pi^{-1}(\Pi_m)$ if $(h, m) = (1, 1)$.

By pulling back complex structures, we then have an induced map

$$F_\pi : \mathcal{T}_{h,m} \rightarrow \mathcal{M}_{g,n}.$$

Let $\mathcal{M}'_{h,m} = \mathcal{T}_{h,m}/\Gamma$ be the quotient of Teichmüller space by the subgroup of finite index in $\text{Mod}_{h,m}$ consisting of mapping classes that lift to (Σ_g, Π_n) . Then F_π descends to give a holomorphic map

$$f_\pi : \mathcal{M}'_{h,m} \rightarrow \mathcal{M}_{g,n}.$$

Condition (6.1) insures that the lift of a Teichmüller mapping remains a Teichmüller mapping, and thus f_π is a local isometry. The same condition implies that all the cocritical points of π are marked.

Let V be a proper subvariety of $\mathcal{M}_{g,n}$ that is totally geodesic. We say V arises via a *covering construction* if there exists a map π as above, with $\dim \mathcal{M}_{h,m} < \dim \mathcal{M}_{g,n}$, and a totally geodesic variety $V_0 \subset \mathcal{M}'_{h,m}$, such that

$$V = f_\pi(V_0) \subset \mathcal{M}_{g,n}.$$

(We allow $V_0 = \mathcal{M}'_{h,m}$.)

We say V is *primitive* if it does not arise via a covering construction.

Example. Given integers $a, b > 0$, consider the surface $S_{ab} \subset \mathcal{M}_{1,3}$ defined by

$$S_{ab} = \left\{ (A, P) : \begin{array}{l} \text{for some ordering of the points of } P, \\ \text{we have } [aP_1 + bP_2] = [(a+b)P_3] \in \text{Pic}(A) \end{array} \right\}.$$

We claim that:

$S_{11} \subset \mathcal{M}_{1,3}$ is a totally geodesic surface defined by a covering construction.

In fact, every $(A, P) \in S_{11}$ admits an involution swapping P_1 and P_2 . The degree two quotient map $\pi : (A, P) \rightarrow (\mathbb{P}^1, Q)$ gives a sphere with 5 marked points — the critical values of π , and the common image of P_1 and P_2 . Applying the covering construction, we obtain a totally geodesic immersion

$$\mathcal{M}'_{0,5} \rightarrow \mathcal{M}_{1,3}$$

with image S_{11} .

Proof of (I) for QF. Since $\dim Q\mathcal{M}_{0,4} = 2 < \dim QF = 4$, the following statement makes precise the fact that most forms in QF are primitive.

LEMMA 6.1. *Every form in $Q\mathcal{M}_{1,3}(-1^3, 1^3)$ is either primitive or the pullback of a form in $Q\mathcal{M}_{0,4}$ via a covering construction.*

Proof. Let (A, P, q) be a form in $Q\mathcal{M}_{1,3}(-1^3, 1^3)$. Suppose $q = p^*(q')$ with $q' \in Q(A', P')$ and $\deg(p) > 1$. If A' has genus one, then p must be a covering map of degree three; but then q' has just one pole, which is impossible. Thus $A' \cong \mathbb{P}^1$. Any zero of q' must be simple, and have $\deg(p)$ simple pre-images; thus in the presence of a zero, we must have $\deg(p) \leq 3$ and q' must have at least five poles. This contradicts the fact that q only has three poles. Thus q' has four poles and no zeros, i.e., $q' \in Q\mathcal{M}_{0,4}$; and since q only has simple zeros, p can only be branched over the poles of q' . \square

Proof of (II). Suppose that $F \subset \mathcal{M}_{1,3}$ arises via a covering construction $f_\pi : \mathcal{M}'_{h,m} \rightarrow \mathcal{M}_{1,3}$. Then every complex geodesic contained in F is generated by the lift of a quadratic differential in $Q\mathcal{M}_{h,m}$. Thus QF contains no primitive form, contrary to statement (I) above. \square

Proof of (I) for ΩG . For later applications to Teichmüller curves, we conclude by proving a similar primitivity theorem for forms in ΩG .

LEMMA 6.2. *Every form in $\Omega\mathcal{M}_4(2^3, 0^3)^-$ is either primitive or the pullback of a form in $\Omega\mathcal{M}_1$.*

Proof. Let (X, ω) be a form in $\Omega\mathcal{M}_4(2^3, 0^3)^-$. Suppose $\omega = p^*(\eta)$, with $(Y, \eta) \in \Omega\mathcal{M}_g$, $1 < g < 4$. By the Riemann–Hurwitz formula, we must have $g = 2$ and $\deg(p) = 2$ or 3 . But if $\deg(p) = 2$, then p has two simple critical points, contradicting the fact that ω has no simple zeros. Thus p is a covering map of degree three, and η has a single zero of multiplicity two.

Let J be the unique involution of X fixing $Z(\omega)$, and let j be the hyperelliptic involution of Y ; it is the unique involution fixing $Z(\eta)$. Since $J^*(\omega) = -\omega$, and $j^*(\eta) = -\eta$, the map p is odd; that is, $p(J(x)) = j(p(x))$ for all $x \in X$. Thus $(X, \omega^2)/J$ is the pullback of the quadratic differential $(Y, \eta^2)/j$. Since the latter is not in $Q\mathcal{M}_{0,4}$, this contradicts Lemma 6.1. \square

7. The Kobayashi metric on F

In this section we give a more geometric proof that F is primitive, by showing that \mathcal{T}_F is not isomorphic to any traditional Teichmüller space. The proof is a variation on a theme of Royden; it is based on an analysis of the Kobayashi metric.

The Kobayashi metric. Recall that \mathcal{T}_F denotes an irreducible component of the preimage of F in $\mathcal{T}_{1,3}$. Since \mathcal{T}_F is totally geodesic, it is a smooth, contractible complex manifold; in fact \mathcal{T}_F is homeomorphic, via the exponential

map, to its tangent space at any point. We can regard \mathcal{T}_F as the universal cover of the surface \widetilde{F} introduced in Section 3.

It is well known that the Kobayashi and Teichmüller metrics agree on $\mathcal{T}_{g,n}$ [Gar87, §7]. Similarly:

The Kobayashi and Teichmüller metrics agree on \mathcal{T}_F , and the inclusion $\mathcal{T}_F \subset \mathcal{T}_{1,3}$ is an isometry.

In fact, any two distinct points $x, y \in \mathcal{T}_F$ lie on a unique complex geodesic $D \subset \mathcal{T}_{1,3}$; since \mathcal{T}_F is totally geodesic, we have $D \subset \mathcal{T}_F \subset \mathcal{T}_{1,3}$; and since inclusions are contractions, D is also a Kobayashi geodesic in \mathcal{T}_F .

The vertices of a norm. Let V be a two-dimensional normed complex vector space. We say V has n vertices if the set of $v \in V$ where the function $\|v\|$ is real-analytic is the complement of n distinct lines through the origin. To distinguish \mathcal{T}_F from $\mathcal{T}_{g,n}$, we will show:

THEOREM 7.1. *The cotangent space at any point in $\mathcal{T}_{0,5}$ has five vertices in the Teichmüller norm, while the cotangent space at a typical point in \mathcal{T}_F has six.*

Pairs of quadratic differentials. The following general result describes points where the Teichmüller norm is *not* smooth. Consider quadratic differentials $q_0, q_1 \in Q(X, P)$, where $(X, P) \in \mathcal{M}_{g,n}$. Fix $x \in X$, and let $m_i = \text{ord}_x(q_i)$ denote the order of vanishing of q_i at x (or -1 if q_i has a simple pole there).

LEMMA 7.2. *If $m_1 < (m_0 - 1)/2$, then $f(t) = \|q_0 + tq_1\|$ is not C^2 at $t = 0$.*

Proof. Note that $|q_0 + tq_1|$ is a convex function of $t \in \mathbb{R}$, and if a sum of convex functions is C^2 , then each function has a bounded (distributional) second derivative. Thus it suffices to show that $\int_U |q_0 + tq_1|$ has an unbounded second derivative in some neighborhood U of x ; this is done in [Roy71, §2] and, allowing simple poles, in [Gar87, §9.4]. □

Proof of Theorem 7.1. First note that the function $\|q\|$ is real-analytic on any stratum of $Q\mathcal{M}_{g,n}$, since it can be expressed as a polynomial in the absolute periods of \sqrt{q} and their complex conjugates. In particular, $\|q\|$ is real-analytic on the generic stratum $Q\mathcal{M}_{0,5}(-1^5, 1)$. Thus for any $(X, P) \in \mathcal{M}_{0,5}$, the norm is real-analytic except possibly on the five lines $V_i \subset Q(X, P)$ where $q = q_0$ has only four poles. For any such q_0 , we can find an $x \in P$ and a $q_1 \in Q(X, P)$ such that q_1 has a simple pole at x , but q_0 does not. Applying Lemma 7.2 with $(m_0, m_1) = (0, -1)$, we find that $\|q\|$ is not C^2 at q_0 and hence $Q(X, P)$ has five vertices.

Now consider the cotangent space $QF(A, P, \pi)$ to a point $(A, P) \in \mathcal{T}_F$. Note that for $S \in HA$, the projection $\pi_S : A \rightarrow \mathbb{P}^1$ has six simple critical

values except in the finitely many cases where S lies on the tangent line to a flex of A . Thus π also has six critical values for a typical point in \mathcal{T}_F .

Since \mathcal{T}_F is totally geodesic, the Teichmüller norm on its cotangent space at (A, P) is the restriction of the Teichmüller norm on $Q(A, P)$. Recall that for any $q \neq 0$ in $Q(A, P, \pi)$, we have $(q) = Z - P$ where Z is a fiber of π , and that the generic stratum here is $QM_{1,3}(-1^3, 1^3)$. Thus the norm is real-analytic except possibly along the six lines $V_i \subset QF(A, P, \pi)$ where $(q_0) = Z - P$ and the support of Z contains a simple critical point x of π . In this case $x \notin P$, and hence $m_0 = \text{ord}_x(q_0) = 2$, while $m_1 = \text{ord}_x(q_1) = 0$ for most other $q_1 \in QF(A, P, \pi)$; thus $\|q\|$ fails to be C^2 at q_0 by Lemma 7.2. Therefore the normed cotangent space to \mathcal{T}_F typically has six vertices. \square

Proof of Theorem 1.2. Suppose \mathcal{T}_F is isomorphic to a traditional Teichmüller space. Then we have $\mathcal{T}_F \cong \mathcal{T}_{0,5} \cong \mathcal{T}_{1,2}$ as a complex manifold, since there is only one two-dimensional Teichmüller space up to isomorphism. Since the Kobayashi metric on a space depends only on its complex structure, and agrees with the Teichmüller metric in these cases, this implies that the cotangent bundles of $\mathcal{T}_{0,5}$ and \mathcal{T}_F are isomorphic as bundles of normed vector spaces, contradicting Theorem 7.1. \square

Proof of Corollary 1.3. Since F has codimension one, if it arises from a totally geodesic surface $V_0 \subset \mathcal{M}'_{h,m}$ via a covering construction, then we must have $V_0 = \mathcal{M}'_{h,m}$ and hence $\mathcal{T}_F \cong \mathcal{T}_{h,m}$, contrary to Theorem 1.2. \square

Remark. For a visualization of the Teichmüller norm on the cotangent space to $\mathcal{M}_{0,5}$, see [Muk17b].

8. Teichmüller curves in \mathcal{M}_4

A Teichmüller curve is *geometrically primitive* if it is generated by a primitive 1-form (in the sense of Section 6). Every Teichmüller curve arises from a unique geometrically primitive Teichmüller curve via a covering construction [Mö6, Th. 2.5]. It is useful to work with 1-forms, rather than quadratic differentials, so the class $[\omega] \in H^{1,0}(X)$ can be discussed from the point of view of Hodge theory.

In this section we will show:

- (I) the locus $G_D \subset \mathcal{M}_4$ is a finite union of Teichmüller curves;
- (II) if D is not a square, then every component of G_D is geometrically primitive;
- (III) if the stabilizer of a form in ΩG contains a hyperbolic element γ , then $SL(X, \omega)$ is a lattice and $(X, \omega) \in \Omega G_D$ for some D with $\mathbb{Q}(\text{tr } \gamma) = \mathbb{Q}(\sqrt{D})$.

These assertions are restatements of Theorems 1.6, 1.7 and 1.8. Aside from (II), the proofs follow the same lines as the proofs in genus two given in [McM03a].

We remark that the curves $G_D \subset \mathcal{M}_4$ also descend to give isometrically immersed curves $F_D \subset F$.

Eigenforms. As in Section 1, given a discriminant $D > 0$, we let ΩG_D denote the set of $(X, \omega) \in \Omega G$ with associated A and B such that:

1. there exists a proper, self-adjoint action of \mathcal{O}_D by endomorphisms of $C(X|A, B)$; and
2. $T^*\omega \in \mathbb{C}\omega$ for all $T \in \mathcal{O}_D$.

(Here *proper* means \mathcal{O}_D is a maximal quadratic subring of $\text{End}(C)$.) In brief, (X, ω) is an *eigenform* for real multiplication by \mathcal{O}_D . The projection of ΩG_D to \mathcal{M}_4 is denoted by G_D .

Proof of (I). We refine the discussion of Section 5. By the theory of Hilbert modular surfaces (cf. [vdG88]), the eigenforms for real multiplication by \mathcal{O}_D form an algebraic subvariety of ΩG of codimension ≤ 2 . Thus every irreducible component of ΩG_D has dimension ≥ 2 . On the other hand, for any $(X, \omega) \in \Omega G_D$, we have an endomorphism $T \in \text{End}(C)$ and a $\lambda \in K = \mathbb{Q}(\sqrt{D})$ such that $[\omega]$ belongs to the two-dimensional subspace

$$S = \text{Ker}(T - \lambda I) \subset H^1(C) \subset H^1(X, Z(\omega)).$$

Applying Theorem 5.1 we find that ΩG_D is $\text{SL}_2(\mathbb{R})$ -invariant, and therefore G_D is finite union of irreducible curves, each of which is totally geodesic. \square

Proof of (II). By Lemma 6.2, if $(X, \omega) \in \Omega G_D$ is not primitive, then it is the pullback of a form of genus $h = 1$. But then the \mathcal{O}_D -invariant subspace of $H^1(X, \mathbb{R})$ spanned by $[\text{Re}(\omega)]$ and $[\text{Im}(\omega)]$ is defined over \mathbb{Q} , so D is a square. \square

Proof of (III). Let $\text{Aff}(X, \omega)$ denote the group of *affine automorphisms* of (X, ω) . The derivative $D\psi \in \text{SL}_2(\mathbb{R})$ of any affine automorphism is constant in charts on $X - Z(\omega)$ where $\omega = dz$. The map $\psi \mapsto D\psi$ sends $\text{Aff}(X, \omega)$ to $\text{SL}(X, \omega)$, the stabilizer of (X, ω) in $\text{SL}_2(\mathbb{R})$.

Suppose we have a hyperbolic element $\gamma \in \text{SL}(X, \omega)$. Replacing γ with γ^2 , we can assume that $\text{tr}(\gamma) > 0$. Choose a element $\psi \in \text{Aff}(X, \omega)$ with $D\psi = \gamma$.

Let $C = C(X|A, B)$ as above. Given X , there are only finitely many choices for A and B , and they are locally preserved along an $\text{SL}_2(\mathbb{R})$ -orbit, so after replacing ψ with an iterate we can assume it stabilizes $H^1(C) \subset H^1(X)$.

Since $\psi^*|_{H^1(C; \mathbb{R})}$ preserves a symplectic form, its eigenvalues come in reciprocal pairs, and its action is unitary for the Hermitian form $\langle \alpha, \beta \rangle = (i/2) \int_X \alpha \wedge \bar{\beta}$ on $H^1(C)$. Thus the map

$$T = \psi^* + (\psi^{-1})^*|_{H^1(C)}$$

is self-adjoint, its eigenspaces are orthogonal and their dimensions are even.

After replacing (X, ω) by another point in its $SL_2(\mathbb{R})$ orbit, we can assume that

$$D\psi = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}.$$

Then ψ is a pseudo-Anosov mapping, with invariant foliations defined by the harmonic forms $\operatorname{Re}(\omega)$ and $\operatorname{Im}(\omega)$. These forms give a ψ^* -eigenbasis for a two-dimensional subspace $V \subset H^1(C)$, with eigenvalues λ and λ^{-1} . By the theory of pseudo-Anosov mappings, these eigenvalues are simple and all other eigenvalues of $\psi^*|_{H^1(C)}$ have modulus less than λ .

In particular, the eigenvalues of $T|_{H^1(C)}$ are given by t and t' , where $t = \lambda + \lambda^{-1}$ and $|t'| < t$; The corresponding eigenspaces V and V' satisfy $V' = V^\perp$.

Extend $\{\omega\}$ to an orthogonal basis $\{\omega, \omega'\}$ for $\Omega(C)$. Then the cohomology classes $[\omega], [\omega']$ give a basis for $H^{1,0}(C)$ consisting of eigenvectors for T . Consequently T preserves $H^{1,0}(C)$. Since T also preserves $H^1(C, \mathbb{Q})$, by replacing T with nT for some $n > 0$, we can insure that $T \in \operatorname{End}(C)$.

The ring $\mathbb{Z}[T]$ is quadratic since T has exactly two eigenvalues. Let D be the discriminant of the maximal quadratic order in $\operatorname{End}(C)$ containing $\mathbb{Z}[T]$. Then $(X, \omega) \in \Omega G_D$, and hence $SL(X, \omega)$ is a lattice by (I). The field $K = \mathbb{Q}(\operatorname{tr} \gamma)$ is the same as the field generated by the eigenvalues of T , so $K = \mathbb{Q}(\sqrt{D})$. □

9. Explicit polygonal constructions

In this section we will prove Theorems 1.9 and 1.10. The latter gives new examples of Teichmüller curves in \mathcal{M}_4 . In the course of the proof we will give explicit equations for ΩG in period coordinates.

Cylinder deformations. We begin with some notation. We write the usual action of $\gamma \in GL_2^+(\mathbb{R})$ on $\Omega\mathcal{M}_g$ by

$$(X, \omega) \mapsto \gamma \cdot (X, \omega).$$

Let $C \subset X$ be a collection of parallel cylinders, and let $\phi : (X, \omega) \rightarrow (X', \omega')$ be a PL (piecewise linear) map for the flat metrics $|\omega|$ and $|\omega'|$. We say (X', ω') is a *cylinder deformation* of (X, ω) , and write

$$(9.1) \quad (X', \omega') = \gamma_C \cdot (X, \omega),$$

if $D\phi = \gamma$ on C and $D\phi = \operatorname{Id}$ on the rest of X (cf. [Wri15]).

Note that we must have $\gamma(v) = v$ on vectors v parallel to ∂C ; in particular, if $\det(\gamma) = 1$, then γ is a shear. Conversely, any linear map γ that fixes vectors parallel to ∂C determines a new form $\gamma_C \cdot (X, \omega) \in \Omega\mathcal{M}_g$.

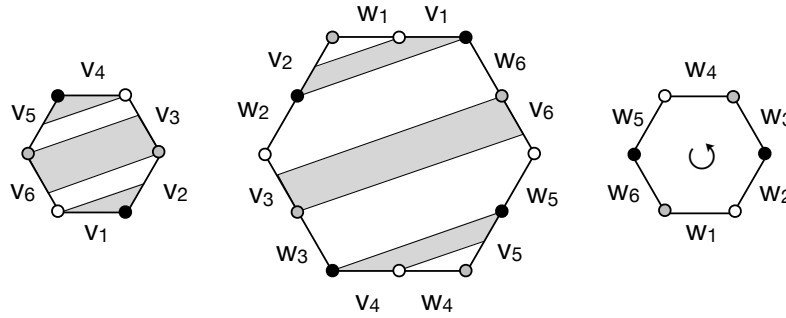


Figure 4. Hexagon form (X_0, ω_0) in ΩG , and parallel cylinders $C \subset X_0$.

A symmetric form in ΩG . Now consider the holomorphic 1-form $(X_0, \omega_0) \in \Omega \mathcal{M}_4(2, 2, 2)$ obtained by gluing together three regular hexagons along parallel edges as indicated in Figure 4. The three zeros of ω are indicated by white, black and gray dots. The edges are oriented to run counter-clockwise around the right and left hexagons, in the complex directions $v_i, w_i \in \mathbb{C}$. We claim:

The form (X_0, ω_0) lies in the gothic locus ΩG .

In fact, rotation of each hexagon by 60° preserves the gluing pattern, and thus descends to an automorphism $T : X_0 \rightarrow X_0$ that cyclically permutes the three zeros of ω_0 . If we let $J = T^3$ and define an elliptic curve by $B = X_0 / \langle T^2 \rangle$, then $J^* \omega_0 = -\omega_0$ and the degree three quotient map $\pi : X_0 \rightarrow B$ sends $Z(\omega_0)$ to a single point. Thus (X_0, ω_0) belongs to ΩG by the definition in Section 1.

Period coordinates. We can regard v_i and w_i as linear coordinates on the cohomology group $H^1(X_0, Z(\omega_0))$, since each oriented edge connects two zeros of ω_0 .

In fact, these 12 vectors span $H_1(X_0; Z(\omega_0)) \cong \mathbb{C}^{10}$, and are subject to the two relations $\sum v_i = \sum w_i = 0$. For the picture at hand, we have $v_i(\omega_0) = w_i(\omega_0) = \zeta^{i-1}$, where $\zeta = \exp(\pi i/3)$.

In these coordinates, a sheet of ΩG is locally defined by the linear relations:

$$(9.2) \quad \begin{aligned} v_{i+3} &= -v_i \quad \text{and} \quad w_{i+3} = -w_i, \quad i = 1, 2, 3; \quad \text{and} \\ v_1 + v_3 + v_5 &= w_1 + w_3 + w_5 = 0. \end{aligned}$$

The first set of equations insures that $J^*(\omega) = \omega$, while the second insures that $\omega \in H^1(B)^\perp$. To see that the second condition is correct, just note that $v_1 + v_3 + v_5$ and $w_1 + w_3 + w_5$ span the T^2 -invariant subspace of $H_1(X_0)$, and hence represent a basis for $H_1(B)$. These equations define an open subset of a sheet of ΩG by Theorem 5.2.

Cylinders and $H_1(B)$. The shaded region in Figure 4 covers a collection of three parallel cylinders, $C = C_1 \cup C_2 \cup C_3 \subset X$. We claim:

Any cylinder deformation of the form $(X, \omega) = \gamma_C \cdot (X_0, \omega_0)$ also lies in ΩG .

To see this, it suffices to show that shearing C preserves the period conditions in (9.2). The first set is preserved because $J(C) = C$. As for the second set, by examining the left hexagon in Figure 4, we see that the relative cycle $v = 2v_1 - (v_1 + v_2 + v_6)$ can be represented by two arcs lying entirely in C . But modulo the first set of equations, we have $v = v_1 + v_3 + v_5$; thus $v(\omega_0) = 0$, and hence $v(\omega) = 0$. Similarly, the cycle $w_1 + w_3 + w_5$ can be represented by arcs outside of C , so its vanishing is also preserved by γ_C .

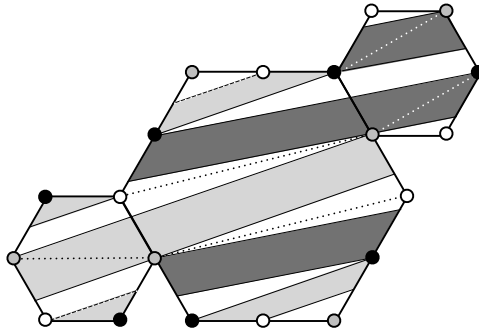


Figure 5. Preparation for cut and paste.

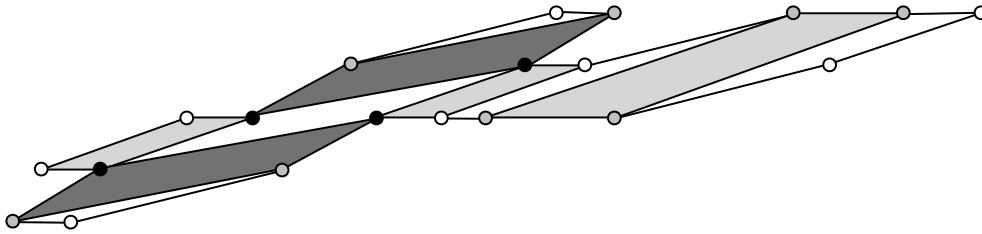


Figure 6. Result of cut and paste is a new polygon P' .

PL maps. Figure 5 gives another presentation of the surface (X_0, ω_0) , in which the dark shaded regions cover two more parallel cylinders $D = D_1 \cup D_2 \subset X_0$. Since $J(D) = D$, and the cycles $v_1 + v_3 + v_5$ and $w_1 + w_3 + w_5$ can be represented by arcs outside of D , deformations of the form $\gamma'_D \cdot (X_0, \omega_0)$ also lie in ΩG . In fact, by combining γ_C , γ'_D and the action of $GL_2^+(\mathbb{R})$, we can conclude:

(9.3) *If there exists a PL map $\phi : (X_0, \omega_0) \rightarrow (X, \omega)$ such that $D\phi$ is constant on C , D and their complement, then $(X, \omega) \in \Omega G$.*

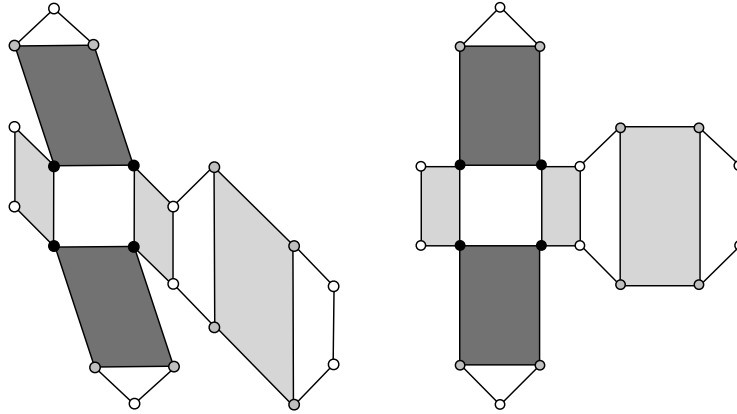


Figure 7. Linear image $\gamma(P')$, followed by shearing.

With this information in hand, we may complete the:

Proof of Theorem 1.9. We first show that $(X, \omega) = P(a, b)/\sim$ is in ΩG for any $a, b > 0$. Start with $(X_0, \omega_0) \in \Omega G$ as presented in Figure 5. Cutting along the dotted lines and reassembling, we obtain a new polygon P' representing the same 1-form, shown in Figure 6. Every dotted edge is now part of the boundary of P' .

Let $\gamma \in \text{GL}_2^+(\mathbb{R})$ be the unique linear map that sends the pair of symmetric white triangles in Figure 6 to isosceles right triangles with horizontal bases of unit length. The result is a new 1-form $(X_1, \omega_1) = \gamma \cdot (X_0, \omega_0)$, which is the quotient of the polygon $\gamma(P')$ shown in Figure 7 at the left. (In fact, if we allow complex values of a, b , then $P' = P((1 - i)/2, (3 + i)/2)$).

Finally, by applying cylinder deformations, we can transform the light and dark regions into rectangles (as shown at the right) to produce the form $(X_2, \omega_2) = P(a, b)/\sim$ for any $a, b > 0$. By observation (9.3), the resulting form is still in ΩG . \square

Proof of Theorem 1.10. Now suppose $a, b \in K = \mathbb{Q}(\sqrt{d})$ satisfy equation (1.3). We may assume that \sqrt{d} is irrational, since (X, ω) is square-tiled in the rational case. Note that the horizontal cylinders in (X, ω) have moduli $(m_1, m_2) = (b, (4a + 2)^{-1})$, while the vertical cylinders have moduli $(m'_1, m'_2) = (a, (4b + 6)^{-1})$. As is easily verified, equation (1.3) is exactly the condition needed to insure that m_1/m_2 and m'_1/m'_2 are rational.

This rationality implies that $\text{Aff}^+(X, \omega)$ contains a vertical and a horizontal Dehn twist, whose product produces a hyperbolic element $\gamma \in \text{SL}(X, \omega)$ (cf. [McM06, §4]). The relative periods $\text{Per}(\omega) \otimes \mathbb{Q}$ form a vector space of rank 2

over $K' = \mathbb{Q}(\text{tr } \gamma)$ [McM03b, Th. 9.5], and thus $K' = K$. By Theorem 1.8, this implies there is a $D > 0$ with $\mathbb{Q}(\sqrt{D}) = \mathbb{Q}(\sqrt{d})$ such that $(X, \omega) \in G_D$. In particular, (X, ω) generates a Teichmüller curve. \square

Remark: An open subset of ΩG . Since ΩG is linear in period coordinates, Theorem 1.9 also holds for the open set of complex parameters (a, b) that determine embedded polygons. In fact, if we apply the action of $\text{GL}_2^+(\mathbb{R})$ to the resulting forms, then we obtain a dense open subset of ΩG . Figures 6 and 7 show cathedral polygons of this more general type.

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