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Higher ramification and the local Langlands correspondence

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Abstract

Let F be a non-Archimedean locally compact field. We show that the local Langlands correspondence over F has a property generalizing the higher ramification theorem of local class field theory. If π is an irreducible cuspidal representation of a general linear group $\mathrm{GL}_n(F)$ and σ the corresponding irreducible representation of the Weil group $\mathcal{W}_F$ of F, the restriction of σ to a ramification subgroup of $\mathcal{W}_F$ is determined by a truncation of the simple character θ_π contained in π, and conversely. Numerical aspects of the relation are governed by an Herbrand-like function Ψ_Θ depending on the endo-class Θ of θ_π. We give a method for calculating Ψ_Θ directly from Θ. Consequently, the ramification-theoretic structure of σ can be predicted from the simple character θ_π alone.

1. We examine the local Langlands correspondence [16], [19], [21], [23] for general linear groups over a non-Archimedean locally compact field F. We obtain striking new results connecting the fine structure of cuspidal representations of $\mathrm{GL}_n(F)$, as in the classification scheme of [11], and the ramification-theoretic structure of Galois representations.

Our main theorem generalizes the higher ramification theorem of local class field theory. It gives rise to a function analogous to the classical Herbrand function of a field extension. Our second theorem is an algorithm for calculating that function. Taken together, the results offer an unprecedented opportunity to transmit detailed structure across the correspondence, pointing a new direction for the subject. Here, we only indicate very first steps.

2. Let $\mathcal{W}_F$ be the Weil group of a separable algebraic closure $\bar{F}/F$. Let $\widehat{\mathcal{W}}_F$ be the set of equivalence classes of irreducible, smooth, complex representations of the locally profinite group $\mathcal{W}_F$. (From now on, when speaking of a representation of a locally profinite group, we will *always* assume it to be smooth and complex.) For each integer $n \geqslant 1$, let $\mathcal{A}_n^0(F)$ be the set of equivalence classes of irreducible cuspidal representations of the general

linear group $\mathrm{GL}_n(F)$. To work in a dimension-free manner, we set $\widehat{\mathrm{GL}}_F = \bigcup_{n\geqslant 1} \mathcal{A}_n^0(F)$: given $\pi \in \widehat{\mathrm{GL}}_F$, there is a unique integer $\mathrm{gr}(\pi) = m \geqslant 1$ such that $\pi \in \mathcal{A}_m^0(F)$.

The local Langlands correspondence for F provides a canonical bijection

$$\begin{aligned} \widehat{\mathrm{GL}}_F &\longrightarrow \widehat{\mathcal{W}}_F, \\ \pi &\longmapsto {}^L\pi, \end{aligned} \tag{1}$$

such that $\dim {}^L\pi = \mathrm{gr}(\pi)$. The correspondence truly embodies a vast generalization of local class field theory. However, there is more to local class field theory than the existence of the Artin reciprocity map $\boldsymbol{a}_F : \mathcal{W}_F \to F^\times$. A mere existence statement falls short of revealing many useful properties and applications. So too for the Langlands correspondence: knowledge of its existence, or even a construction, does not automatically yield significant new insight.

3. An instance suggests itself. If ϵ is a real parameter, $\epsilon \geqslant 0$, let $\mathcal{W}_F^\epsilon$ be the corresponding ramification subgroup of $\mathcal{W}_F$ in the upper numbering convention of [24]. In particular, $\mathcal{W}_F^0$ is the inertia subgroup $\mathcal{I}_F$ of $\mathcal{W}_F$. Let $\mathcal{W}_F^{\epsilon+}$ be the closure of the subgroup $\bigcup_{\delta>\epsilon} \mathcal{W}_F^\delta$. Thus $\mathcal{W}_F^{0+}$ is the wild inertia subgroup $\mathcal{P}_F$ of $\mathcal{W}_F$.

The *first ramification theorem* of local class field theory asserts that $\boldsymbol{a}_F(\mathcal{P}_F)$ is the group $U_F^1 = 1+\mathfrak{p}_F$ of principal units in F. More generally, let $k \geqslant 1$ be an integer and write $U_F^k = 1+\mathfrak{p}_F^k$. The *higher ramification theorem* asserts that $\boldsymbol{a}_F(\mathcal{W}_F^k) = U_F^k$ and $\boldsymbol{a}_F(\mathcal{W}_F^{k+}) = U_F^{1+k}$. It therefore yields an isomorphism between the group of characters of U_F^k and the group of characters of $\mathcal{W}_F^k$ trivial on $\mathcal{W}_F^k \cap \mathcal{W}_F^{\mathrm{der}}$, where $\mathcal{W}_F^{\mathrm{der}} = \mathrm{Ker}\, \boldsymbol{a}_F$ is the (closed) derived subgroup of $\mathcal{W}_F$. Consequently, the fine structure of characters of U_F^k is reflected in characters of $\mathcal{W}_F^k$. Of course, $\mathcal{W}_F^k$ admits characters that are not trivial on $\mathcal{W}_F^k \cap \mathcal{W}_F^{\mathrm{der}}$.

4. The Ramification Theorem of [5, 8.2 Theorem], [9, 6.1] provides a generalization of the first ramification theorem of local class field theory. It is written in terms of endo-classes of *simple characters* in $\mathrm{GL}_n(F)$, in the sense of [11] (and the background notes below). Simple characters are very special characters of specific compact open subgroups of $\mathrm{GL}_n(F)$, with a multitude of extraordinary properties. Not least is the ability to *transfer* simple characters between general linear groups of differing dimensions in a way that preserves relations of intertwining and conjugacy. This leads to the notion of *endo-equivalence* of simple characters, developed in [2]. It provides an equivalence relation on the class of all simple characters in all general linear groups over F, the equivalence classes being called *endo-classes*.

A representation $\pi \in \mathcal{A}_n^0(F)$ contains a unique conjugacy class of simple characters (Corollary 1 of [7]). These necessarily lie in the same endo-class Θ_π.

If $\sigma = {}^L\pi$, the Ramification Theorem asserts that the restriction $\sigma \mid \mathcal{P}_F$ of σ to $\mathcal{P}_F$ depends only on Θ_π, and conversely. More precisely, if $\sigma \in \widehat{\mathcal{W}}_F$, then $\sigma \mid \mathcal{P}_F$ is a direct sum of irreducible representations of $\mathcal{P}_F$, all of which are $\mathcal{W}_F$-conjugate and occur with the same multiplicity. So, writing $\widehat{\mathcal{P}}_F$ for the set of equivalence classes of irreducible representations of the profinite group $\mathcal{P}_F$, the representation σ yields a unique element of $\mathcal{W}_F\backslash\widehat{\mathcal{P}}_F$ that we choose to denote $[\sigma;0]^+$. On the other hand, let $\mathcal{E}(F)$ be the set of endo-classes of simple characters over F. Given $\Theta \in \mathcal{E}(F)$, there exists $\pi \in \widehat{\mathrm{GL}}_F$ so that $\Theta = \Theta_\pi$. If $\sigma = {}^L\pi$, the orbit $[\sigma;0]^+$ depends only on Θ rather than the choice of π. We therefore denote it ${}^L\Theta$. The map

$$\begin{aligned} \mathcal{E}(F) &\longrightarrow \mathcal{W}_F\backslash\widehat{\mathcal{P}}_F, \\ \Theta &\longmapsto {}^L\Theta, \end{aligned} \tag{2}$$

is then a bijection. Results of [9], [3] show that the Langlands correspondence can, in essence, be re-constructed from the bijection (2) via an explicit process.

5. Our main result here shows how (2) may be refined into a *family* of bijections generalizing the higher ramification theorem of local class field theory. It is based on the fact that the Langlands correspondence preserves conductors of pairs.

If σ is a finite-dimensional, semisimple representation of $\mathcal{W}_F$, let $\mathrm{sw}(\sigma)$ be the Swan conductor of σ and write $\varsigma(\sigma) = \mathrm{sw}(\sigma)/\dim\sigma$. For $\pi_1, \pi_2 \in \widehat{\mathrm{GL}}_F$, let $\mathrm{sw}(\pi_1 \times \pi_2)$ be the Swan conductor of the pair (π_1, π_2). This is defined via the local constant $\varepsilon(\pi_1 \times \pi_2, s, \psi)$ of [20], [25]. Setting

$$\varsigma(\pi_1 \times \pi_2) = \frac{\mathrm{sw}(\pi_1 \times \pi_2)}{\mathrm{gr}(\pi_1)\,\mathrm{gr}(\pi_2)},$$

the correspondence (1) has the property

$$\varsigma(\pi_1 \times \pi_2) = \varsigma({}^L\pi_1 \otimes {}^L\pi_2), \quad \pi_i \in \widehat{\mathrm{GL}}_F. \tag{3}$$

6. We exploit parallel structures carried by the sets $\mathcal{E}(F)$ and $\mathcal{W}_F\backslash\widehat{\mathcal{P}}_F$. On the Galois side, one defines a pairing Δ on $\widehat{\mathcal{W}}_F$ by

$$\Delta(\sigma,\tau) = \inf\{\epsilon > 0 : \mathrm{Hom}_{\mathcal{W}_F^\epsilon}(\sigma,\tau) \neq 0\}, \quad \sigma,\tau \in \widehat{\mathcal{W}}_F.$$

This is symmetric and satisfies an ultrametric inequality, but does not separate points. The value $\Delta(\sigma,\tau)$ depends only on the orbits $[\sigma;0]^+, [\tau;0]^+ \in \mathcal{W}_F\backslash\widehat{\mathcal{P}}_F$, so Δ induces a pairing, again denoted Δ, on $\mathcal{W}_F\backslash\widehat{\mathcal{P}}_F$. The second version of Δ separates points and is an ultrametric on $\mathcal{W}_F\backslash\widehat{\mathcal{P}}_F$. The following result derives from [17].

Proposition A. *Let $\sigma \in \widehat{\mathcal{W}}_F$. There exists a unique continuous function $\varSigma_\sigma(x)$, $x \geqslant 0$, such that $\varsigma(\check\sigma \otimes \tau) = \varSigma_\sigma\big(\Delta(\sigma,\tau)\big)$ for all $\tau \in \widehat{\mathcal{W}}_F$.*

Here, $\check{\sigma}$ is the *contragredient* of σ. The *decomposition function* $\Sigma_\sigma(x)$ is given by a formula (3.1.2) expressing the way σ decomposes when restricted to the ramification subgroups $\mathcal{W}_F^x$, $x > 0$. Consequently, one needs detailed knowledge of the inner workings of σ in order to write it down. It depends only on $[\sigma; 0]^+$, so we sometimes write $\Sigma_\sigma = \Sigma_{[\sigma;0]^+}$.

This material is covered in Sections 1–3 and is mostly familiar, but we have taken care to ensure that the narrative is complete. A couple of deeper results have exact analogues on the GL-side. We have chosen to prove the GL-versions, in the appropriate place, and then deduce the Galois versions via the Langlands correspondence.

7. Rather more surprising is the existence of exact analogues on the GL-side, developed in Sections 4 and 5. That $\mathcal{E}(F)$ carries a canonical ultrametric $(\Theta, \Upsilon) \mapsto \mathbb{A}(\Theta, \Upsilon)$ is already implicit in [2]. It is given by an explicit formula (5.1.1) in terms of transfers of simple characters. However, the conductor formula of [10] can be reformulated in terms of $\mathbb{A}$ to yield

Proposition B. *Let $\Theta \in \mathcal{E}(F)$. There exists a unique continuous function $\Phi_\Theta(x)$, $x \geqslant 0$, such that $\varsigma(\check{\pi} \times \rho) = \Phi_\Theta\big(\mathbb{A}(\Theta, \Theta_\rho)\big)$ for any $\pi \in \widehat{\mathrm{GL}}_F$ satisfying $\Theta_\pi = \Theta$ and any $\rho \in \widehat{\mathrm{GL}}_F$.*

Again, $\check{\pi}$ is the contragredient of π. The *structure function* Φ_Θ can be written down completely in terms of Θ (4.4.1). Throughout Sections 4 and 5, we have to pay attention to the behavior of $\mathbb{A}$ and Φ_Θ relative to tamely ramified extensions of the base field F. This prepares the way for later results.

Propositions A and B are results of rather different kinds. Proposition A, while not trivial, has no claim to great depth. Proposition B, on the other hand, emerges on combining two deep and highly developed theories, the complete account of the smooth dual of $\mathrm{GL}_n(F)$ from [11], [14], [15], [2], and Shahidi's analysis of the Rankin–Selberg local constant in terms of intertwining operators and Plancherel measure [25]. This comparison is an instance of a common phenomenon: it is usually easier to access matters of depth via the GL-side.

8. In Section 6, we use (3) to combine the propositions and get the first of our main results.

Higher Ramification Theorem. *Let $\Theta \in \mathcal{E}(F)$. For $\epsilon > 0$, define $\delta > 0$ by*

$$\Phi_\Theta(\delta) = \Sigma_{{}^L\Theta}(\epsilon). \tag{4}$$

If $\Upsilon \in \mathcal{E}(F)$, then $\mathbb{A}(\Theta, \Upsilon) < \delta$ if and only if $\Delta({}^L\Theta, {}^L\Upsilon) < \epsilon$.

The result holds equally with nonstrict inequalities. This form is easy to prove and contains everything of substance. However, working back through the definitions, one finds a more concrete version (6.5 Corollary). For representations $\sigma, \tau \in \widehat{\mathcal{W}}_F$, the condition $\Delta(\sigma,\tau) < \epsilon$ is equivalent to σ and τ having a common irreducible component on restriction to $\mathcal{W}_F^{\epsilon}$. On the other side, take $\pi, \rho \in \widehat{\mathrm{GL}}_F$. The condition $\mathbb{A}(\Theta_\pi, \Theta_\rho) < \delta$ is equivalent to π, ρ each containing a representative of the same endo-class of *truncated* simple characters, in the more general sense of [2] and 5.2 below. The severity of the truncation is measured by δ. The theorem thus implies a parametrization of conjugacy classes of representations of ramification groups by endo-classes of truncated simple characters, the Langlands correspondence inducing a bijection between the set of $\pi \in \widehat{\mathrm{GL}}_F$ containing a given truncated endo-class and the set of $\sigma \in \widehat{\mathcal{W}}_F$ containing the corresponding representation of a ramification subgroup.

Example. Let $k \geqslant 1$ be an integer, and let ϕ be a character of $\mathcal{W}_F^k$ trivial on commutators: equivalently, $\phi = \tilde{\phi} \mid \mathcal{W}_F^k$ for some character $\tilde{\phi}$ of $\mathcal{W}_F$. Thus $\tilde{\phi} = \chi \circ \boldsymbol{a}_F$ for a character χ of $F^\times$. The restriction $\chi_1 = \chi \mid U_F^1$ is a simple character in $\mathrm{GL}_1(F)$. The restriction $\chi_k = \chi \mid U_F^k$ is a truncation of χ_1 and gives the endo-class corresponding to ϕ under the main theorem.

9. Our second main result concerns the change of scale $\epsilon \mapsto \delta$ in the Higher Ramification Theorem. Define a function $\Psi_\Theta(x)$, $x \geqslant 0$, $\Theta \in \mathcal{E}(F)$, by $\Psi_\Theta = \Phi_\Theta^{-1} \circ \Sigma_{L_\Theta}$. Thus, in the theorem, $\delta = \Psi_\Theta(\epsilon)$. The function Ψ_Θ is continuous, positive, strictly increasing, piecewise linear and smooth outside of a finite set. It plays a role analogous to the classical Herbrand functions, so we appropriate the name. Our second result, the *Interpolation Theorem* of Section 7, gives a procedure for calculating Ψ_Θ directly from Θ, without the recourse to the Langlands correspondence implicit in its definition. Since Θ determines Φ_Θ explicitly, the theorem yields the Galois-theoretic decomposition function Σ_{L_Θ}, *with no reference to Galois theory*!

The Herbrand function Ψ_Θ has simple behavior relative to tamely ramified base field extension (7.1). Using this, we show that Ψ_Θ can be calculated from the values of $\mathbb{A}(\Theta, \chi\Theta)$, as χ ranges over a certain set of characters of $F^\times$, along with the corresponding result relative to tame base field extensions. The final statement 7.5 is very simple, but the extraction of explicit formulas promises to be a challenging task. Here we examine only the easiest example of *essentially tame representations* 7.7.

For our concluding Section 8, we change to the Galois side to broach a related question: if we are given a decomposition function Σ_σ, what does it tell us about σ? We show that the first discontinuity of the derivative Σ_σ' gives a canonical family of presentations of σ as an induced representation in a manner respecting ramification structures. Recent work suggests this approach provides a useful complement to the Interpolation Theorem. We finish Section 8 with a few specific examples, without full proofs, to give the reader some perspective on phenomena beyond the scope of this paper.

Background and notation. Throughout, F is a non-Archimedean local field with finite residue field of characteristic p. The symbols $\mathfrak{o}_F$, $\mathfrak{p}_F$, $\Bbbk_F = \mathfrak{o}_F/\mathfrak{p}_F$, $U_F = \mathfrak{o}_F^\times$, $U_F^k = 1{+}\mathfrak{p}_F^k$, $k \geqslant 1$, and $\upsilon_F : F^\times \twoheadrightarrow \mathbb{Z}$ all have their customary meaning.

Let $\bar{F}/F$ be a separable algebraic closure of F and $\mathcal{W}_F = \mathcal{W}(\bar{F}/F)$ the Weil group of $\bar{F}/F$. Let E/F be a finite separable extension of F. When working in the Galois-theoretic context, we generally assume E to be a subfield of $\bar{F}$ and write $\mathcal{W}_E$ for the Weil group $\mathcal{W}(\bar{F}/E)$ of $\bar{F}/E$. We identify $\mathcal{W}_E$ with the open subgroup of $\mathcal{W}_F$ that fixes all elements of E under the natural action of $\mathcal{W}_F$ on $\bar{F}$.

We make extensive use of the theory of simple characters [11], along with endo-classes and tame lifting [2]. An overview, containing what we need, can be found in [1]: we give the barest summary here.

Let $\mathfrak{a}$ be a hereditary $\mathfrak{o}_F$-order in $A = \mathrm{End}_F(V)$, where V is an F-vector space of finite dimension. We set $U_\mathfrak{a} = \mathfrak{a}^\times$. If $\mathfrak{p}$ is the Jacobson radical $\mathrm{rad}\,\mathfrak{a}$ of $\mathfrak{a}$, then $U_\mathfrak{a}^k = 1{+}\mathfrak{p}^k$, $k \geqslant 1$. We define the positive integer $e_\mathfrak{a}$ by $\mathfrak{p}_F\mathfrak{a} = \mathfrak{p}^{e_\mathfrak{a}}$: this is the *$\mathfrak{o}_F$-period of* $\mathfrak{a}$. If E/F is a subfield of A, we say $\mathfrak{a}$ is *E-pure* if $x\mathfrak{a}x^{-1} = \mathfrak{a}$ for all $x \in E^\times$.

Let $[\mathfrak{a}, n, 0, \beta]$ be a simple stratum in A ([11, 1.5.5]): in particular, the algebra $E = F[\beta]$ is a field and $\mathfrak{a}$ is E-pure. As in [11, 3.1], one attaches to this stratum an open subgroup $H^1(\beta, \mathfrak{a})$ of $U_\mathfrak{a}^1$ and writes $H^k(\beta, \mathfrak{a}) = H^1(\beta, \mathfrak{a}) \cap U_\mathfrak{a}^k$, $k \geqslant 1$.

Take a character ψ_F of F of level one (to use the terminology of [11]). This means that ψ_F is trivial on $\mathfrak{p}_F$, but not trivial on $\mathfrak{o}_F$. Following Chapter 3 of [11], one attaches to $[\mathfrak{a}, n, 0, \beta]$ and ψ_F a specific nonempty, finite set $\mathcal{C}(\mathfrak{a}, \beta, \psi_F)$ of characters of the compact group $H^1(\beta, \mathfrak{a})$. These are the *simple characters in* $\mathrm{Aut}_F(V)$ defined by $[\mathfrak{a}, n, 0, \beta]$. The dependence on ψ_F is rather trivial, so we usually regard it as permanently fixed and omit it from the notation: thus $\mathcal{C}(\mathfrak{a}, \beta, \psi_F) = \mathcal{C}(\mathfrak{a}, \beta)$.

In the same situation, let m be an integer, $0 \leqslant m < n$. The symbol $\mathcal{C}(\mathfrak{a}, m, \beta)$ means the set of characters of $H^{m+1}(\beta, \mathfrak{a})$ obtained by restricting the characters in $\mathcal{C}(\mathfrak{a}, \beta)$: thus $\mathcal{C}(\mathfrak{a}, 0, \beta) = \mathcal{C}(\mathfrak{a}, \beta)$. We refer to the elements of sets $\mathcal{C}(\mathfrak{a}, m, \beta)$ as *truncated* simple characters. The general theory of endo-equivalence in [2] applies equally to truncated simple characters.

1. **Ramification groups**

We start with a sequence of three sections on the Weil group and its representations. This one provides a brief *aide mémoire* for basic ramification theory and introduces some nonstandard notation.

1.1. Let $\mathcal{I}_F$ be the inertia subgroup of $\mathcal{W}_F$ and $\mathcal{P}_F$ the wild inertia subgroup. Attached to a real number $\epsilon \geqslant -1$ is the ramification subgroup $\mathcal{W}_F^\epsilon$ of $\mathcal{W}_F$. We use the upper numbering convention of [24, Chap. IV], so that $\mathcal{W}_F^{-1} = \mathcal{W}_F$ and $\mathcal{W}_F^0 = \mathcal{I}_F$. This traditional notation is typographically inconvenient so, from now on, we use

$$\mathcal{R}_F(\epsilon) = \mathcal{W}_F^\epsilon, \quad \epsilon \geqslant 0. \tag{1.1.1}$$

The definition of the ramification sequence gives the semi-continuity property

$$\mathcal{R}_F(\epsilon) = \bigcap_{\delta<\epsilon} \mathcal{R}_F(\delta), \quad \epsilon > 0.$$

One also forms the subgroup $\bigcup_{\delta>\epsilon} \mathcal{R}_F(\delta)$ and its closure

$$\mathcal{R}_F^+(\epsilon) = \mathrm{cl}\Big(\bigcup_{\delta>\epsilon} \mathcal{R}_F(\delta)\Big)$$

in $\mathcal{W}_F$. This need not equal $\mathcal{R}_F(\epsilon)$: one says that ϵ is a *jump of* $\bar{F}/F$ if $\mathcal{R}_F^+(\epsilon) \neq \mathcal{R}_F(\epsilon)$. In particular,

$$\mathcal{R}_F(0) = \mathcal{I}_F, \quad \mathcal{R}_F^+(0) = \mathcal{P}_F.$$

Each of the groups $\mathcal{R}_F(x)$, $\mathcal{R}_F^+(x)$, $x \geqslant 0$, is profinite, closed and normal in $\mathcal{W}_F$. We summarize the main properties of the ramification groups, relative to finite quotients of $\mathcal{W}_F$, in the form we shall use them. We use the conventions of [24] for numbering the ramification subgroups of a finite Galois group.

LEMMA. *Let $x \geqslant 0$. Let E/F be a finite Galois extension with $\Gamma = \mathrm{Gal}(E/F)$.*

(1) *The canonical image of $\mathcal{R}_F(x)$ in Γ is the ramification group Γ^x.*

(2) *Suppose x is not a jump in the ramification sequence for E/F, that is, $\Gamma^x = \Gamma^{x+\epsilon}$ for some $\epsilon > 0$. The image of $\mathcal{R}_F^+(x)$ in Γ is then Γ^x.*

(3) *Suppose x is a jump in the ramification sequence for E/F, that is, $\Gamma^x \neq \Gamma^{x+\epsilon}$, $\epsilon > 0$.*
   (a) *If x is the largest jump for E/F, then the image of $\mathcal{R}_F^+(x)$ in Γ is trivial.*
   (b) *Otherwise, the image of $\mathcal{R}_F^+(x)$ in Γ is Γ^y, where y is the least jump such that $y > x$.*

In the context of the lemma, it is often useful to have the notation $\Gamma^{x+} = \bigcup_{y>x} \Gamma^y$. Thus x is a jump for E/F if $\Gamma^x \neq \Gamma^{x+}$. In all cases, Γ^{x+} is the image of $\mathcal{R}_F^+(x)$.

1.2. We make frequent use of the following facts.

LEMMA 1. *If K/F is a finite, tamely ramified field extension with $e = e(K|F)$, then $\mathcal{P}_K = \mathcal{P}_F$ and*

$$\mathcal{R}_F(x) = \mathcal{R}_K(ex), \quad x > 0,$$
$$\mathcal{R}_F^+(x) = \mathcal{R}_K^+(ex), \quad x \geqslant 0.$$

Proof. This follows from the definition of the upper numbering of ramification groups. □

LEMMA 2. *If $0 < x \leqslant y$, the commutator group $[\mathcal{R}_F(x), \mathcal{R}_F(y)]$ is contained in $\mathcal{R}_F^+(y)$. Moreover,*

$$[\mathcal{R}_F^+(0), \mathcal{R}_F(x)] \subset \mathcal{R}_F^+(x), \quad x \geqslant 0.$$

In particular, the group $\mathcal{R}_F(x)/\mathcal{R}_F^+(x)$, $x > 0$, is central in $\mathcal{R}_F^+(0)/\mathcal{R}_F^+(x)$.

Proof. The first assertion is implied by IV, Section 2, Proposition 10 of [24]. The second then follows from the definition of $\mathcal{R}_F^+(x)$. □

2. Representations and ramification

Let $\widehat{\mathcal{W}}_F$ be the set of isomorphism classes of irreducible representations of $\mathcal{W}_F$. Let $\widehat{\mathcal{W}}_F^{\mathrm{ss}}$ be the set of isomorphism classes of finite-dimensional semisimple representations of $\mathcal{W}_F$ (cf. [6, 28.7 Proposition]).

Let $\widehat{\mathcal{R}}_F(\epsilon)$ be the set of isomorphism classes of irreducible representations of the profinite group $\mathcal{R}_F(\epsilon)$, $\epsilon > 0$, and define $\widehat{\mathcal{R}}_F^+(\epsilon)$, $\epsilon \geqslant 0$, analogously. The group $\mathcal{W}_F$ acts on both $\widehat{\mathcal{R}}_F(\epsilon)$ and $\widehat{\mathcal{R}}_F^+(\epsilon)$ by conjugation.

We investigate interactions between representations of $\mathcal{W}_F$ and the filtration by ramification groups. We identify the jumps in the ramification sequence for $\bar{F}/F$ and define a canonical pairing on $\widehat{\mathcal{W}}_F$.

2.1. We start at a general level.

PROPOSITION 1. *Let H be an open subgroup of $\mathcal{P}_F$. There exists $\epsilon > 0$ such that H contains $\mathcal{R}_F(\epsilon)$. For any such ϵ, there exists $\epsilon' < \epsilon$ such that H contains $\mathcal{R}_F(\epsilon')$.*

Proof. The group H is of the form $\mathcal{P}_F \cap \mathcal{W}_K$ for a finite extension K/F. Since $\mathcal{R}_F(\epsilon)$ is normal in $\mathcal{W}_F$, we may replace H by the intersection of its $\mathcal{W}_F$-conjugates and assume K/F is a Galois extension. Enlarging K if necessary, we may further assume that K/F is not tamely ramified.

Let δ be the largest ramification jump for K/F. Thus $\delta > 0$ and H contains $\mathcal{R}_F^+(\delta)$. Thus H contains $\mathcal{R}_F(\epsilon)$ if and only if $\epsilon > \delta$, and the result follows. □

Proposition 2. *Let* $\xi \in \widehat{\mathcal{R}}_F(\epsilon)$, $\epsilon > 0$.

(1) *The kernel of* ξ *contains* $\mathcal{R}_F(\delta)$ *for some* $\delta > \epsilon$.

(2) *There exists* $\sigma \in \widehat{\mathcal{W}}_F$ *such that* ξ *is equivalent to an irreducible component of the restriction* $\sigma \mid \mathcal{R}_F(\epsilon)$.

Proof. Since $\mathcal{R}_F(\epsilon)$ is profinite, the kernel of ξ is open, hence of the form $H \cap \mathcal{R}_F(\epsilon)$ for an open subgroup H of $\mathcal{W}_F$. Part (1) now follows from Proposition 1.

If E/F is a finite extension, set $\mathcal{G}_E = \mathrm{Gal}(\bar{F}/E)$. In part (2), it is enough to find an irreducible representation of $\mathcal{G}_F$ containing ξ on restriction to $\mathcal{R}_F(\epsilon)$. We form the representation $I = \mathrm{Ind}_{\mathcal{R}_F(\epsilon)}^{\mathcal{G}_F}\,\xi$ of $\mathcal{G}_F$ smoothly induced from ξ. Thus I is the union $\bigcup_{E/F} I^{\mathcal{G}_E}$ of its spaces of $\mathcal{G}_E$-fixed points, as E/F ranges over finite Galois extensions contained in $\bar{F}$. The space $I^{\mathcal{G}_E}$ provides a representation of the finite group $\mathrm{Gal}(E/F)$. Consequently, I has an irreducible $\mathcal{G}_F$-subspace σ, and any such σ has the desired property. □

Complement. Proposition 2 holds, with the same proof, on replacing $\widehat{\mathcal{R}}_F(\epsilon)$, $\epsilon > 0$ with $\widehat{\mathcal{R}}^+_F(\epsilon)$, $\epsilon \geqslant 0$.

Apology. Proposition 2, applied to $\mathcal{P}_F = \mathcal{R}^+_F(0)$, replaces the incorrect proof of [9, 1.2 Proposition]. It also plugs a gap inadvertently left in the proof of [5, 8.2 Theorem]: we thank A. Kılıç for drawing our attention to the problem.

2.2. Let $\sigma \in \widehat{\mathcal{W}}_F$ and $\epsilon > 0$. The restriction $\sigma \mid \mathcal{R}_F(\epsilon)$ of σ to $\mathcal{R}_F(\epsilon)$ is semisimple. Its irreducible components are all $\mathcal{W}_F$-conjugate and occur with the same multiplicity. Thus σ determines a unique conjugacy class $[\sigma;\epsilon] \in \mathcal{W}_F\backslash\widehat{\mathcal{R}}_F(\epsilon)$. Similarly, for $\epsilon \geqslant 0$, σ determines a unique conjugacy class $[\sigma;\epsilon]^+ \in \mathcal{W}_F\backslash\widehat{\mathcal{R}}^+_F(\epsilon)$.

Proposition. *The orbit maps*

$$\begin{aligned} \widehat{\mathcal{W}}_F &\longrightarrow \mathcal{W}_F\backslash\widehat{\mathcal{R}}_F(\epsilon), \\ \sigma &\longmapsto [\sigma;\epsilon], \end{aligned} \qquad \epsilon > 0$$

and

$$\begin{aligned} \widehat{\mathcal{W}}_F &\longrightarrow \mathcal{W}_F\backslash\widehat{\mathcal{R}}^+_F(\epsilon), \\ \sigma &\longmapsto [\sigma;\epsilon]^+, \end{aligned} \qquad \epsilon \geqslant 0$$

are surjective.

Proof. The assertion re-states 2.1 Proposition 2 and its complement. □

2.3. Let $\sigma \in \widehat{\mathcal{W}}_F$. By 2.1 Proposition 1, $\operatorname{Ker}\sigma$ contains $\mathcal{R}_F(\epsilon)$ for ϵ sufficiently large. One defines the *slope* $\mathrm{sl}(\sigma)$ of σ by

$$\mathrm{sl}(\sigma) = \inf\{\epsilon > 0 : \mathcal{R}_F(\epsilon) \subset \operatorname{Ker}\sigma\}. \tag{2.3.1}$$

Thus $\mathrm{sl}(\sigma) = 0$ if and only if σ is trivial on $\mathcal{P}_F$: one says that σ is *tamely ramified.*

Proposition. *Let $\sigma \in \widehat{\mathcal{W}}_F$ and suppose that $\mathrm{sl}(\sigma) = s > 0$. The group $\mathcal{R}_F^+(s)$ is then contained in $\operatorname{Ker}\sigma$ while $\sigma \mid \mathcal{R}_F(s)$ is a direct sum of nontrivial characters of $\mathcal{R}_F(s)$.*

Proof. The first assertion follows from the definition of $\mathcal{R}_F^+(s)$. The group $\mathcal{R}_F(s)/\mathcal{R}_F^+(s)$ is abelian by 1.2 Lemma 2, so $\sigma \mid \mathcal{R}_F(s)$ is a direct sum of $\mathcal{W}_F$-conjugate characters. If these characters were trivial, $\mathcal{R}_F(s)$ would be contained in $\operatorname{Ker}\sigma$. Since $\operatorname{Ker}\sigma \cap \mathcal{P}_F$ is open in $\mathcal{P}_F$, it would contain $\mathcal{R}_F(t)$ for some $t < s$, by 2.1 Proposition 1, contrary to the definition of s. □

Corollary.

(1) *If $s > 0$ is the slope of a representation $\sigma \in \widehat{\mathcal{W}}_F$, then $\mathcal{R}_F(s) \neq \mathcal{R}_F^+(s)$. In particular, s is a jump in the ramification sequence for $\bar{F}/F$.*

(2) *If $s > 0$ is a jump in the ramification sequence for $\bar{F}/F$, there exists $\sigma \in \widehat{\mathcal{W}}_F$ with slope s.*

Proof. Assertion (1) follows directly from the proposition. The profinite group $\mathcal{R}_F(s)$ admits a nontrivial smooth character ξ that is trivial on the closed subgroup $\mathcal{R}_F^+(s)$. Assertion (2) is therefore given by 2.2 Proposition. □

2.4. We can now identify the jumps in the ramification sequence, knowing that they all arise as slopes of irreducible representations.

If $\rho \in \widehat{\mathcal{W}}_F^{\mathrm{ss}}$, let $\mathrm{sw}(\rho)$ denote the exponential *Swan conductor* of ρ. Thus $\mathrm{sw}(\rho)$ is a nonnegative integer and, if we write $\rho = \bigoplus_{i=1}^r \tau_i$, with $\tau_i \in \widehat{\mathcal{W}}_F$, then $\mathrm{sw}(\rho) = \sum_{i=1}^r \mathrm{sw}(\tau_i)$.

Basic connection. *If $\sigma \in \widehat{\mathcal{W}}_F$, then $\mathrm{sl}(\sigma) = \mathrm{sw}(\sigma)/\dim\sigma$. In particular, $\mathrm{sl}(\sigma) \in \mathbb{Q}$.*

Proof. See Théorème 3.5 of [18]. □

We complete the argument with a sharp result, which seems to lie rather deep.

Proposition. *Let $x > 0$, $x \in \mathbb{Q}$, and write $x = a/b$ for relatively prime, positive integers a, b. There exists $\sigma \in \widehat{\mathcal{W}}_F$ such that $\mathrm{sw}(\sigma) = a$ and $\dim\sigma = b$.*

We defer the proof to 6.3 below. We deduce

Corollary. *If $x \in \mathbb{R}$, $x > 0$, then $\mathcal{R}_F(x) \neq \mathcal{R}_F^+(x)$ if and only if $x \in \mathbb{Q}$.*

2.5. The orbit maps $\widehat{\mathcal{W}}_F \to \mathcal{W}_F\backslash\widehat{\mathcal{R}}_F(\epsilon)$ and $\widehat{\mathcal{W}}_F \to \mathcal{W}_F\backslash\widehat{\mathcal{R}}_F^+(\epsilon)$ of 2.2 factor through the orbit map $\widehat{\mathcal{W}}_F \to \mathcal{W}_F\backslash\widehat{\mathcal{P}}_F$. We use the same notation for the implied maps $\mathcal{W}_F\backslash\widehat{\mathcal{P}}_F \to \mathcal{W}_F\backslash\widehat{\mathcal{R}}_F(\epsilon)$ and $\mathcal{W}_F\backslash\widehat{\mathcal{P}}_F \to \mathcal{W}_F\backslash\widehat{\mathcal{R}}_F^+(\epsilon)$. We set

$$\Delta(\xi,\zeta) = \inf\Big\{\epsilon > 0 : [\xi;\epsilon] = [\zeta;\epsilon]\Big\}, \quad \xi,\zeta \in \mathcal{W}_F\backslash\widehat{\mathcal{P}}_F. \tag{2.5.1}$$

The pairing Δ is clearly symmetric: $\Delta(\xi,\zeta) = \Delta(\zeta,\xi)$. Its values are nonnegative rational numbers (2.4 Corollary).

Proposition.

(1) *If $\xi,\zeta \in \mathcal{W}_F\backslash\widehat{\mathcal{P}}_F$, then $\Delta(\xi,\zeta) = 0$ if and only if $\xi = \zeta$.*
(2) *If $\xi,\zeta \in \mathcal{W}_F\backslash\widehat{\mathcal{P}}_F$ and $\delta = \Delta(\xi,\zeta) > 0$, then $[\xi;\delta]^+ = [\zeta;\delta]^+$ while $[\xi;\delta] \neq [\zeta;\delta]$.*
(3) *If $\xi,\zeta,\psi \in \mathcal{W}_F\backslash\widehat{\mathcal{P}}_F$, then*

$$\Delta(\xi,\zeta) \leqslant \max\Big\{\Delta(\xi,\psi), \Delta(\psi,\zeta)\Big\}. \tag{2.5.2}$$

The pairing Δ is an ultrametric on $\mathcal{W}_F\backslash\widehat{\mathcal{P}}_F$.

Proof. In part (1), one implication is trivial, so suppose $\xi \neq \zeta$. As in 2.2, there exists an irreducible representation $\tilde{\xi}$ of $\mathrm{Gal}(\bar{F}/F)$ containing ξ on restriction to $\mathcal{P}_F$. Choose $\tilde{\zeta}$ similarly. There exists a finite Galois extension K/F such that both $\tilde{\xi}$ and $\tilde{\zeta}$ are inflated from representations of $\Gamma = \mathrm{Gal}(K/F)$. The extension K/F is not tamely ramified: otherwise, ξ and ζ would be the orbit of the trivial character of $\mathcal{P}_F$. So, K/F has a least positive ramification jump ϕ. If $0 < \epsilon < \phi$, then $\Gamma^\epsilon = \Gamma$ whence $[\xi;\epsilon] \neq [\zeta;\epsilon]$. This implies $\Delta(\xi,\zeta) > \epsilon > 0$, contrary to hypothesis.

In part (2), write $\xi = \{\xi_i : i \in I\}$, where $\xi_i \in \widehat{\mathcal{P}}_F$ and I is a finite set. Likewise set $\zeta = \{\zeta_j : j \in J\}$. For $\epsilon > 0$, the condition $\Delta(\xi,\zeta) \leqslant \epsilon$ is equivalent to $\mathrm{Hom}_{\mathcal{R}_F(\epsilon)}(\xi_i,\zeta_j)$ being nonzero for some choice of i and j. This, in turn, is equivalent to $\check{\xi}_i \otimes \zeta_j$ containing the trivial character of $\mathcal{R}_F(\epsilon)$. When this condition holds, 2.1 Proposition 1 implies that $\check{\xi}_i \otimes \zeta_j$ contains the trivial character of $\mathcal{R}_F(\epsilon')$ for some $\epsilon' < \epsilon$. Applying this observation to the case $\epsilon = \delta$, the assertion follows.

Part (3) follows directly from the definition. $\square$

It is often more convenient to view Δ as a pairing on $\widehat{\mathcal{W}}_F$, setting

$$\Delta(\sigma,\tau) = \Delta\big([\sigma;0]^+, [\tau;0]^+\big), \quad \sigma,\tau \in \widehat{\mathcal{W}}_F.$$

This version is again symmetric and has the ultrametric property (2.5.2), but it does not separate points. In this form,

$$\Delta(\sigma,\tau) = \inf\Big\{\epsilon > 0 : \mathrm{Hom}_{\mathcal{R}_F(\epsilon)}(\sigma,\tau) \neq 0\Big\}. \tag{2.5.3}$$

2.6. We consider the behavior of Δ under tamely ramified base field extension. Temporarily write Δ_F for the pairing (2.5.1). Let K/F be a finite tame extension, and let Δ_K be the analogue of Δ_F relative to the base field K. Thus $\mathcal{P}_K = \mathcal{P}_F$ and $\mathcal{R}_K(\epsilon) = \mathcal{R}_F(\epsilon/e)$, where $e = e(K|F)$ (1.2 Lemma 1). Consequently,

PROPOSITION. *If $\xi, \zeta \in \widehat{\mathcal{P}}_F = \widehat{\mathcal{P}}_K$, then*

$$e\,\Delta_F(\mathcal{W}_F\xi, \mathcal{W}_F\zeta) = \min\left\{\Delta_K(\mathcal{W}_K\xi, \mathcal{W}_K g\zeta) : g \in \mathcal{W}_K\backslash\mathcal{W}_F\right\}.$$

3. **Ultrametric and conductors**

We link the ultrametric Δ on $\mathcal{W}_F\backslash\widehat{\mathcal{P}}_F$ to conductors of tensor products of representations of $\mathcal{W}_F$. The basic ideas come from Heiermann's note [17].

3.1. Set

$$\varsigma(\sigma) = \mathrm{sw}(\sigma)/\dim\sigma, \quad \sigma \in \widehat{\mathcal{W}}_F^{\mathrm{ss}}.$$

If $\sigma \in \widehat{\mathcal{W}}_F$, this reduces to $\varsigma(\sigma) = \mathrm{sl}(\sigma)$, as in 2.4.

Let $\sigma \in \widehat{\mathcal{W}}_F$, say $\sigma : \mathcal{W}_F \to \mathrm{Aut}_{\mathbb{C}}(V)$, for a finite dimensional complex vector space V. The semisimple representation $\check\sigma \otimes \sigma$ thus acts on the space $X_\sigma = \check V \otimes V$. Write

$$X_\sigma(\delta) = X_\sigma^{\mathcal{R}_F^+(\delta)} = \bigcap_{\epsilon>\delta} X_\sigma^{\mathcal{R}_F(\epsilon)}, \quad \delta \geqslant 0$$

for the space of $\mathcal{R}_F^+(\delta)$-fixed points in X_σ. Let $X'_\sigma(\delta)$ be the unique $\mathcal{R}_F^+(\delta)$-complement of $X_\sigma(\delta)$ in X_σ. Since $\mathcal{R}_F^+(\delta)$ is a normal subgroup of $\mathcal{W}_F$, the spaces $X_\sigma(\delta)$, $X'_\sigma(\delta)$ are $\mathcal{W}_F$-stable and provide semisimple representations of $\mathcal{W}_F$.

LEMMA. *Let $\sigma, \tau \in \widehat{\mathcal{W}}_F$. If $\delta = \Delta(\sigma, \tau)$, then*

$$\varsigma(\check\sigma \otimes \tau) = (\dim\sigma)^{-2}\Big(\delta\dim X_\sigma(\delta) + \mathrm{sw}\, X'_\sigma(\delta)\Big). \tag{3.1.1}$$

This formulation is to be found on p. 572 of [17]; cf. (3.1.3) below. We need a slightly different emphasis.

PROPOSITION. *For $\sigma \in \widehat{\mathcal{W}}_F$ and $\delta \geqslant 0$, define*

$$\varSigma_\sigma(\delta) = (\dim\sigma)^{-2}\Big(\delta\dim X_\sigma(\delta) + \mathrm{sw}\, X'_\sigma(\delta)\Big). \tag{3.1.2}$$

The function $\varSigma_\sigma$ is continuous, strictly increasing, piecewise linear and convex. Its derivative $\varSigma'_\sigma$ is continuous outside of a finite set.

Proof. Write $\check\sigma \otimes \sigma = \sum_i \psi_i$, where the ψ_i are irreducible. We prove that

$$\delta\dim X_\sigma(\delta) + \mathrm{sw}\, X'_\sigma(\delta) = \sum_i \max\{\delta\dim\psi_i, \mathrm{sw}\,\psi_i\}. \tag{3.1.3}$$

Let ψ be an irreducible component of $\check\sigma\otimes\sigma$. If $\delta\dim\psi \geqslant \operatorname{sw}\psi$, then $\delta\geqslant \varsigma(\psi)$, by the basic connection of 2.4. Thus ψ is trivial on $\mathcal{R}_F^+(\delta)$ (2.3 Proposition), whence ψ is a subspace of $X_\sigma(\delta)$. If, on the other hand, $\delta\dim\psi < \operatorname{sw}\psi$ then $\delta<\varsigma(\psi)$ and ψ is a subspace of $X'_\sigma(\delta)$. The desired relation now follows.

Each term in the sum (3.1.3) is a continuous, nondecreasing, function. One factor ψ_i is the trivial representation, and that contributes a strictly increasing term. All assertions are now immediate. □

Comparing with (3.1.1), we have

$$\Sigma_\sigma\big(\Delta(\sigma,\tau)\big) = \varsigma(\check\sigma\otimes\tau), \quad \sigma,\tau\in\widehat{\mathcal{W}}_F. \tag{3.1.4}$$

There is a consequence, useful in more general applications, although we do not need it here.

Corollary. *The pairing $(\sigma,\tau)\mapsto\varsigma(\check\sigma\otimes\tau)$ of* (3.1.4) *satisfies the ultrametric inequality*

$$\varsigma(\check\sigma_1\otimes\sigma_2)\leqslant\max\{\varsigma(\check\sigma_1\otimes\sigma_3),\varsigma(\check\sigma_3\otimes\sigma_2)\},\quad \sigma_i\in\widehat{\mathcal{W}}_F.$$

Proof. The proof is identical to that of 5.4 Corollary below. We choose to give the details there. □

Notation. The function Σ_σ, as defined in (3.1.2), depends only on the class $[\sigma;0]^+\in\mathcal{W}_F\backslash\widehat{\mathcal{P}}_F$. It is sometimes necessary to reflect this via the notation

$$\Sigma_\sigma(x) = \Sigma_{[\sigma;0]^+}(x). \tag{3.1.5}$$

3.2. Say that $\sigma\in\widehat{\mathcal{W}}_F$ is *totally wild* if the restriction $\sigma\mid\mathcal{P}_F$ is irreducible. If σ is such a representation, and if K/F is a finite, tamely ramified field extension, the restriction $\sigma^K=\sigma\mid\mathcal{W}_K$ is irreducible and totally wild.

Proposition. *If $\sigma\in\widehat{\mathcal{W}}_F$ is totally wild and K/F is a finite tame extension, then*

$$\Sigma_\sigma(x) = e^{-1}\Sigma_{\sigma^K}(ex),\quad x\geqslant 0,$$

where $e=e(K|F)$.

Proof. This follows from 1.2 Lemma 1. □

3.3. The function Σ_σ has a strong uniqueness property, although we do not need it at this stage.

Proposition. *The function Σ_σ, defined by* (3.1.2), *is the unique continuous function satisfying* (3.1.4).

The proposition is an immediate consequence of the following, proved in 6.3 below.

Density Lemma. *Let $\sigma \in \widehat{\mathcal{W}}_F$. The set $\{\Delta(\sigma,\tau) : \tau \in \widehat{\mathcal{W}}_F\}$ is dense on the half-line $x \geqslant 0$, $x \in \mathbb{R}$.*

3.4. We record a continuity property of the function $\sigma \mapsto \varSigma_\sigma$.

Proposition. *If $\sigma, \tau \in \widehat{\mathcal{W}}_F$ and $\delta = \Delta(\sigma,\tau)$, then $\varSigma_\sigma(x) = \varSigma_\tau(x)$, $x \geqslant \delta$.*

Proof. By definition, the condition $\delta = \Delta(\sigma,\tau)$ is equivalent to $[\sigma;\epsilon] = [\tau;\epsilon]$ for all $\epsilon > \delta$. If $\dim\sigma = a$, $\dim\tau = b$, this condition is equivalent to $b\sigma \mid \mathcal{R}_F(\epsilon) \cong a\tau \mid \mathcal{R}_F(\epsilon)$ for $\epsilon > \delta$. Comparing first trivial components and then nontrivial ones, we get

$$b^2 X_\sigma(\epsilon) \cong a^2 X_\tau(\epsilon), \quad b^2 X'_\sigma(\epsilon) \cong a^2 X'_\tau(\epsilon).$$

The assertion now follows from the definition (3.1.2). □

4. **Invariants of simple characters**

We pass to the GL-side. In this section, we recall and develop some features of the theory of simple characters using mainly [11] and [2].

4.1. We start with a detail from [10, 6.4]. Let E/F be a finite field extension, and let $A = \mathrm{End}_F(E)$. Let $\mathfrak{a}$ be the unique E-pure hereditary $\mathfrak{o}_F$-order in A. Let $\beta \in E^\times$ satisfy $E = F[\beta]$ and $m = -\upsilon_E(\beta) > 0$. We assume that the quadruple $[\mathfrak{a}, m, 0, \beta]$ is a *simple stratum*, in the sense of [11, (1.5.5)].

Let a_β denote the adjoint map $A \to A$, $x \mapsto \beta x - x\beta$, and $s_{E/F} : A \to E$ a tame corestriction relative to E/F, [11, 1.3]. The sequence

$$0 \to E \longrightarrow A \xrightarrow{a_\beta} A \xrightarrow{s_{E/F}} E \to 0$$

is then exact. There exist $\mathfrak{o}_F$-lattices $\mathfrak{l}$, $\mathfrak{l}'$ in E and $\mathfrak{m}$, $\mathfrak{m}'$ in A such that the sequence

$$0 \to \mathfrak{l} \longrightarrow \mathfrak{m} \xrightarrow{a_\beta} \mathfrak{m}' \xrightarrow{s_{E/F}} \mathfrak{l}' \to 0$$

is exact. For Haar measures μ_E, μ_A on E and A respectively, the quantity

$$C(\beta) = \frac{\mu_E(\mathfrak{l})\,\mu_A(\mathfrak{m}')}{\mu_E(\mathfrak{l}')\,\mu_A(\mathfrak{m})}$$

is independent of all these choices. If $q = |\Bbbk_F|$, there is an integer $\mathfrak{c}(\beta)$ such that

$$C(\beta) = q^{\mathfrak{c}(\beta)}. \tag{4.1.1}$$

As an example, consider the case where β is *minimal over F*. In concrete terms, this means that $m = -\upsilon_E(\beta)$ is relatively prime to $e = e(E|F)$ and, for a prime element ϖ of F, the coset $\beta^e\varpi^m + \mathfrak{p}_E$ generates the residue field extension $\Bbbk_E/\Bbbk_F$.

Proposition. *Set* $e = e(E|F)$, $f = f(E|F)$. *If* β *is minimal over* F, *then* $\mathfrak{c}(\beta) = mf(ef-1)$.

Proof. In this situation, the sequence

$$0 \to \mathfrak{o}_E \longrightarrow \mathfrak{a} \xrightarrow{a_\beta} \beta\mathfrak{a} \xrightarrow{s_{E/F}} \beta\mathfrak{o}_E \to 0$$

is exact [11, (1.4.15)]. The result then follows from a short calculation. □

4.2. Let $\mathcal{E}(F)$ be the set of endo-classes of simple characters over F, including the trivial element $\mathbf{0}$. Let $\Theta \in \mathcal{E}(F)$, $\Theta \neq \mathbf{0}$. There is a finite-dimensional F-vector space V, a simple stratum $[\mathfrak{a}, m, 0, \beta]$ in $\mathrm{End}_F(V)$ and a simple character $\theta \in \mathcal{C}(\mathfrak{a}, \beta, \psi_F)$ such that θ has endo-class Θ. One says that θ is a *realization of* Θ (on $\mathfrak{a}$, relative to ψ_F). Let $e_\mathfrak{a}$ be the $\mathfrak{o}_F$-period of $\mathfrak{a}$. We recall [2, (8.11)] that the quantities

$$\begin{aligned} \deg\Theta &= [F[\beta]{:}F], \qquad m_\Theta = m/e_\mathfrak{a}, \\ e_\Theta &= e(F[\beta]|F), \qquad f_\Theta = f(F[\beta]|F) \end{aligned} \tag{4.2.1}$$

depend only on Θ and not on the choices of θ, $\mathfrak{a}$, ψ_F or β. The same applies to

$$k_0(\Theta) = k_0(\beta, \mathfrak{a})/e_\mathfrak{a}, \tag{4.2.2}$$

where $k_0(\beta, \mathfrak{a})$ is the "critical exponent" of [11, (1.4.5)]. Recall that $k_0(\Theta) = -\infty$ when $\deg\Theta = 1$. Otherwise, $k_0(\Theta)$ is a negative rational number satisfying $-k_0(\Theta) \leqslant m_\Theta$.

If $\mathfrak{a}$ is a hereditary order in $A = \mathrm{End}_F(V)$, the realization of the trivial element $\mathbf{0}$ of $\mathcal{E}(F)$ on $\mathfrak{a}$ is the trivial character of the group $U^1_\mathfrak{a} = 1{+}\mathfrak{p}$, where $\mathfrak{p} = \mathrm{rad}\,\mathfrak{a}$. We set

$$\begin{gathered} \deg\mathbf{0} = e_\mathbf{0} = f_\mathbf{0} = 1, \\ m_\mathbf{0} = 0. \end{gathered} \tag{4.2.3}$$

The following observation will be useful later.

Proposition. *Let* x *be a positive rational number, say* $x = a/b$, *for relatively prime positive integers* a, b. *There exists* $\Theta \in \mathcal{E}(F)$ *such that* $m_\Theta = x$ *and* $e_\Theta = \deg\Theta = b$.

Proof. Let E/F be a totally ramified field extension of degree b, and choose $\alpha \in E$ of valuation $-a$. The element α is then minimal over F. If $\mathfrak{a}$ is the unique E-pure hereditary $\mathfrak{o}_F$-order in $\mathrm{End}_F(E)$, the quadruple $[\mathfrak{a}, a, 0, \alpha]$ is a simple stratum. The endo-class Θ of any $\theta \in \mathcal{C}(\mathfrak{a}, \alpha)$ has the required properties. □

4.3. Let $\Theta \in \mathcal{E}(F)$, $\Theta \neq \mathbf{0}$. We attach to Θ a finite set $\mathcal{S}_\Theta$ of positive rational numbers, to be called the *normalized jumps* of Θ. We choose a realization $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$ of Θ, as in 4.2. We first attach to $[\mathfrak{a}, m, 0, \beta]$ a finite set $\mathrm{S}_{[\mathfrak{a},\beta]}$ of positive integers t, such that $-k_0(\beta, \mathfrak{a}) \leqslant t \leqslant m$.

We proceed by induction on the degree $[F[\beta]{:}F]$. If β is minimal over F, in particular if $\beta \in F^\times$, we put $\mathrm{S}_{[\mathfrak{a},\beta]} = \{m\}$. If β is not minimal over F, we set $r = -k_0(\beta, \mathfrak{a})$. Thus $0 < r < m$. We choose a simple stratum $[\mathfrak{a}, m, r, \gamma]$ equivalent to $[\mathfrak{a}, m, r, \beta]$ [11, (2.4.1)]. Thus $[\mathfrak{a}, m, 0, \gamma]$ is simple and $[F[\gamma]{:}F] < [F[\beta]{:}F]$. The set $\mathrm{S}_{[\mathfrak{a},\gamma]}$ has been defined inductively, and its least element is either m or $-k_0(\gamma, \mathfrak{a})$. In either case, the least element is strictly greater than r. We define

$$\mathrm{S}_{[\mathfrak{a},\beta]} = \mathrm{S}_{[\mathfrak{a},\gamma]} \cup \{r\}.$$

Remark. If we have another simple stratum $[\mathfrak{a}', m', 0, \beta]$ in $\mathrm{End}_F(V')$ for some V', then $\mathrm{S}_{[\mathfrak{a}',\beta]} = \{xe_{\mathfrak{a}}/e_{\mathfrak{a}'} : x \in \mathrm{S}_{[\mathfrak{a},\beta]}\}$, as follows from [11, (1.4.13)].

We define

$$\mathcal{S}_\Theta = \{s/e_{\mathfrak{a}} : s \in \mathrm{S}_{[\mathfrak{a},\beta]}\}. \tag{4.3.1}$$

The least element of $\mathcal{S}_\Theta$ is thus either m_Θ or $-k_0(\Theta)$.

Lemma. *The set $\mathcal{S}_\Theta$ depends only on Θ, and not on any of the choices θ, ψ_F, $[\mathfrak{a}, m, 0, \beta]$.*

Proof. This follows from [11, (3.5.4)]. □

Definition. Let $x \in \mathbb{R}$, $x \geqslant 0$, $x \notin \mathcal{S}_\Theta$.

(1) If $x < \min \mathcal{S}_\Theta$, set $\gamma_x = \beta$.
(2) If $x > m_\Theta = \max \mathcal{S}_\Theta$, set $\gamma_x = 0$.
(3) Otherwise, let $y = t/e_{\mathfrak{a}}$ be the least element of $\mathcal{S}_\Theta$ such that $y > x$, and let $[\mathfrak{a}, m, t{-}1, \gamma_x]$ be a simple stratum equivalent to $[\mathfrak{a}, m, t{-}1, \beta]$.

Set $E_x = F[\gamma_x]$, and define

$$\begin{aligned} d_\Theta(x) &= [E_x{:}F], \\ e_\Theta(x) &= e(E_x|F), \\ \mathfrak{c}_\Theta(x) &= \mathfrak{c}(\gamma_x). \end{aligned} \tag{4.3.2}$$

Proposition. *The quantities* (4.3.2) *depend only on x and Θ. If $y_1 < y_2$ are successive elements of the set $\{0, \infty\} \cup \mathcal{S}_\Theta$, the functions* (4.3.2) *are constant in the region $y_1 < x < y_2$.*

Proof. This follows, via an inductive argument, from the properties recalled in 4.2. □

Observation. The proposition notwithstanding, all the invariants (4.3.2) of Θ are defined purely in terms of an element β giving rise to a realization of Θ.

4.4. Let $\Theta \in \mathcal{E}(F)$, $\Theta \neq \mathbf{0}$, be as 4.3. We define a function $\Phi_\Theta(x)$, $x \geqslant 0$. To start with, assume $x \notin \mathcal{S}_\Theta$ and set

$$\Phi_\Theta(x) = \begin{cases} x, & x > m_\Theta, \\ \frac{\mathfrak{c}_\Theta(x)}{d_\Theta(x)^2} + \frac{x}{d_\Theta(x)}, & 0 < x < m_\Theta. \end{cases} \tag{4.4.1}$$

Proposition.

(1) *The function $\Phi_\Theta(x)$ of (4.4.1) extends uniquely to a continuous function on the half-line $x \geqslant 0$, that is,*

$$\lim_{x \to y-} \Phi_\Theta(x) = \lim_{x \to y+} \Phi_\Theta(x), \quad y \in \mathcal{S}_\Theta.$$

(2) *The function Φ_Θ is piecewise linear, convex and strictly increasing.*
(3) *If $x \notin \mathcal{S}_\Theta$, then $\Phi'_\Theta(y) = d_\Theta(x)^{-1}$ for y ranging over some open neighborhood of x.*
(4) *The discontinuities of the derivative Φ'_Θ are the elements x of $\mathcal{S}_\Theta$* except *when $E_{m_\Theta - \epsilon} = F$ for some $\epsilon > 0$. In that case, Φ'_Θ is continuous at m_Θ.*

Proof. Assertion (1) is given by 4.1 Proposition above together with 3.1 Proposition of [5]. Part (2) then follows from (4.4.1) and 4.3 Proposition. Part (3) follows from the definition and 4.3 Proposition, part (4) from the definition. □

The trivial element $\mathbf{0}$ of $\mathcal{E}(F)$ is dealt with via the explicit formula

$$\Phi_{\mathbf{0}}(x) = x, \quad x \geqslant 0. \tag{4.4.2}$$

In all cases, we call Φ_Θ the *structure function* of $\Theta \in \mathcal{E}(F)$.

Complements. *Let $[\mathfrak{a}, m, 0, \beta]$ be a simple stratum.*

(1) *For $i = 1, 2$, let $\theta_i \in \mathcal{C}(\mathfrak{a}, \beta)$. If Θ_i is the endo-class of θ_i, then $\Phi_{\Theta_1} = \Phi_{\Theta_2}$.*
(2) *Let $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$ have endo-class Θ. The character $\check{\theta} = \theta^{-1}$ of $H^1(\beta, \mathfrak{a})$ lies in $\mathcal{C}(\mathfrak{a}, -\beta)$ and its endo-class $\Theta^\vee$ satisfies $\Phi_{\Theta^\vee} = \Phi_\Theta$.*

Proof. Both assertions follow directly from the observation concluding 4.3. □

4.5. The functions Φ_Θ reflect the approximation properties intrinsic to the concept of endo-class.

Proposition. *For $i = 1, 2$, let $[\mathfrak{a}, m_i, 0, \beta_i]$ be a simple stratum in* $\mathrm{End}_F(V)$ *for a finite-dimensional F-vector space V. Let $\theta_i \in \mathcal{C}(\mathfrak{a}, \beta_i)$, and let Θ_i be the* *endo-class of* θ_i. *If* $t \geqslant 0$ *is an integer such that the restrictions* $\theta_i \mid H^{1+t}(\beta_i, \mathfrak{a})$ *intertwine in* $\mathrm{Aut}_F(V)$, *then*

$$\Phi_{\Theta_1}(x) = \Phi_{\Theta_2}(x), \quad x \geqslant t/e_{\mathfrak{a}}.$$

Proof. Choose a simple stratum $[\mathfrak{a}, m_i, t, \gamma_i]$ equivalent to $[\mathfrak{a}, m_i, t, \beta_i]$. In particular, $H^{1+t}(\beta_i, \mathfrak{a}) = H^{1+t}(\gamma_i, \mathfrak{a})$ and the character $\theta_i^t = \theta_i \mid H^{1+t}(\beta_i, \mathfrak{a})$ lies in $\mathcal{C}(\mathfrak{a}, t, \gamma_i)$. The truncated simple characters θ_i^t intertwine and and so are conjugate in $\mathrm{Aut}_F(V)$ [11, (3.5.11)]. From [11, (3.5.1)] we know that $[F[\gamma_1] : F] = F[\gamma_2] : F]$, $e(F[\gamma_1]|F) = e(F[\gamma_2]|F)$ and $k_0(\gamma_1, \mathfrak{a}) = k_0(\gamma_2, \mathfrak{a})$. The proposition now follows from the definition (4.4.1), the observation of 4.3 and induction along the stratum $[\mathfrak{a}, m_1, 0, \beta_1]$. □

4.6. Let $\Theta \in \mathcal{E}(F)$, and let K/F be a finite, tamely ramified field extension. Let $\Theta_i^K \in \mathcal{E}(K)$, $1 \leqslant i \leqslant r$, be the set of K/F-*lifts* of Θ [2]. If $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$ is a realization of Θ, the Θ_i^K are in bijection with the simple components of the semisimple K-algebra $K \otimes_F F[\beta]$. The relation between Φ_Θ and the functions $\Phi_{\Theta_i^K}$ is, in general, quite intricate but we shall only need a special case.

Say that $\Theta \in \mathcal{E}(F)$ is *totally wild* if $e_\Theta = \deg \Theta = p^r$ for an integer $r \geqslant 0$. If $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$ is a realization of Θ, then Θ is totally wild if and only if the field extension $F[\beta]/F$ is totally wildly ramified.

Proposition. *Let* $\Theta \in \mathcal{E}(F)$ *be totally wild. If* K/F *is a finite, tamely ramified field extension, then* Θ *has a unique* K/F-*lift* Θ^K. *If* $e = e(K|F)$, *then*

$$\Phi_{\Theta^K}(x) = e\,\Phi_\Theta(x/e), \quad x \geqslant 0.$$

Proof. Let $\deg \Theta = p^a$, $a \geqslant 0$. If V is an F-vector space of dimension p^a, there is a simple stratum $[\mathfrak{a}_0, m, 0, \beta]$ in $\mathrm{End}_F(V)$ such that $\mathcal{C}(\mathfrak{a}_0, \beta)$ contains a character θ_0 of endo-class Θ. If $E = F[\beta]$, then E/F is totally ramified of degree $p^a = \deg \Theta = e_\Theta$.

The algebra $K \otimes_F E$ is a field, which we denote KE. In particular, Θ admits a unique K/F-lift Θ^K. Let $A = \mathrm{End}_F(KE)$, and let $\mathfrak{a}$ be the unique KE-pure hereditary $\mathfrak{o}_F$-order in A. The quadruple $[\mathfrak{a}, em, 0, \beta]$ is a simple stratum in A, and there is a simple character $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$ of endo-class Θ. As $e_{\mathfrak{a}} = ep^a$, so

$$\mathcal{S}_\Theta = \{x/ep^a : x \in \mathrm{S}_{[\mathfrak{a},\beta]}\},$$

in the notation of 4.3.

Let $B = \mathrm{End}_K(KE)$ be the A-centralizer of K and $\mathfrak{b} = \mathfrak{a} \cap B$. Thus $\mathfrak{b}$ is the unique KE-pure hereditary $\mathfrak{o}_K$-order in B. The stratum $[\mathfrak{b}, em, 0, \beta]$ is simple [2, (2.4)]. Further, $H^1(\beta, \mathfrak{b}) = H^1(\beta, \mathfrak{a}) \cap B^\times$, and the character $\theta^K = \theta \mid H^1(\beta, \mathfrak{b})$ lies in $\mathcal{C}(\mathfrak{b}, \beta, \psi_K)$, where $\psi_K = \psi_F \circ \mathrm{Tr}_{K/F}$ [2, (7.7)]. The endo-class of θ^K over K is then Θ^K.

Lemma. *The sets* $\mathrm{S}_{[\mathfrak{b},\beta]}$, $\mathrm{S}_{[\mathfrak{a},\beta]}$ *are equal.*

Proof. We proceed by induction along β, in the manner of many proofs in [11]. Suppose first that β is minimal over F. It is then minimal over K [2, (2.4)], and the field extensions $F[\beta]/F$, $K[\beta]/K$ are totally ramified of the same degree p^a. The lemma holds in this case.

We therefore assume $r = -k_0(\beta,\mathfrak{a}) < em$. Let $s = -k_0(\beta,\mathfrak{b})$. According to [2] (2.4), we have $s \geqslant r$. We show that $s = r$ in this case: assume for a contradiction that $s > r$. We choose a simple stratum $[\mathfrak{b}, em, s{-}1, \gamma]$ in B, equivalent to $[\mathfrak{b}, em, s{-}1, \beta]$, such that $[\mathfrak{a}, em, s{-}1, \gamma]$ is simple: this we may do by [2, (3.8)]. Certainly $[\mathfrak{a}, em, s{-}1, \gamma]$ is equivalent to $[\mathfrak{a}, em, s{-}1, \beta]$, which is not simple. It follows from [11, (2.4.1)] that $F[\gamma]/F$ is totally wildly ramified and $[F[\gamma]{:}F] < [F[\beta]{:}F]$. Thus $[K[\gamma]{:}K] < [K[\beta]{:}K]$, implying that $[\mathfrak{b}, em, s{-}1, \beta]$ is not simple. This eliminates the possibility $s > r$.

We conclude that the sets $\mathrm{S}_{[\mathfrak{a},\beta]}$, $\mathrm{S}_{[\mathfrak{b},\beta]}$ have the same least element $r = -k_0(\beta,\mathfrak{a})$. We choose a simple stratum $[\mathfrak{b}, em, r, \gamma]$, equivalent to $[\mathfrak{b}, em, r, \beta]$, such that $[\mathfrak{a}, em, r, \gamma]$ is simple. By definition, $\mathrm{S}_{[\mathfrak{a},\beta]} = \{r\} \cup \mathrm{S}_{[\mathfrak{a},\gamma]}$ and $\mathrm{S}_{[\mathfrak{b},\beta]} = \{r\} \cup \mathrm{S}_{[\mathfrak{b},\gamma]}$. Inductively, $\mathrm{S}_{[\mathfrak{a},\gamma]} = \mathrm{S}_{[\mathfrak{b},\gamma]}$, and the lemma is proved. $\square$

We deduce that

$$\mathcal{S}_{\Theta^K} = \{ey : y \in \mathcal{S}_\Theta\}. \tag{4.6.1}$$

The proof of the lemma also shows that, if $x > 0$ and $x \notin \mathcal{S}_\Theta$, then

$$(KE)_{ex} = KE_x \quad \text{whence} \quad d_\Theta(x) = d_{\Theta^K}(ex), \tag{4.6.2}$$

in the notation of 4.3.

Set $\phi(x) = e\Phi_\Theta(x/e)$. The functions ϕ and Φ_{Θ^K} are continuous, and smooth outside of $\mathcal{S}_{\Theta^K}$. Also, by (4.6.2), $\phi'(x) = \Phi'_\Theta(x)$ for $x \notin \mathcal{S}_{\Theta^K}$. In other words, $\phi(x) = \Phi_{\Theta^K}(x){+}c$ for a constant c. However, for x sufficiently large, $\phi(x) = \Phi_{\Theta^K}(x) = x$, so $c = 0$ as required to prove the proposition. $\square$

5. **Ultrametric on simple characters**

We re-examine the conductor formula of [10], interpreting it in terms of the structure functions Φ_Θ of 4.4 and a canonical ultrametric on the set $\mathcal{E}(F)$ of endo-classes of simple characters over F.

5.1. Let Θ_1, $\Theta_2 \in \mathcal{E}(F)$, $\Theta_i \neq \mathbf{0}$. There is an F-vector space V of finite dimension, and a hereditary order $\mathfrak{a}$ in $\mathrm{End}_F(V)$, such that $\mathfrak{a}$ carries realizations of both Θ_i. That is, there are simple strata $[\mathfrak{a}, m_i, 0, \beta_i]$ in $\mathrm{End}_F(V)$ and simple characters $\theta_i \in \mathcal{C}(\mathfrak{a}, \beta_i)$ such that θ_i is of endo-class Θ_i.

Let $l \geqslant 0$ be the least integer such that the characters $\theta_i \mid H^{l+1}(\beta_i, \mathfrak{a})$ intertwine (and are therefore conjugate [11, (3.5.11)]) in $\mathrm{Aut}_F(V)$. We define

$$\mathbb{A}(\Theta_1, \Theta_2) = \mathbb{A}(\Theta_2, \Theta_1) = l/e_{\mathfrak{a}}. \tag{5.1.1}$$

The definition is independent of all choices; see the discussion in [10, 6.15]. One may treat the trivial class $\mathbf{0}$ on the same basis, but it is quicker to simply define

$$\mathbb{A}(\Theta, \mathbf{0}) = m_\Theta, \quad \Theta \in \mathcal{E}(F). \tag{5.1.2}$$

Proposition.

(1) *Let $\Theta, \Theta' \in \mathcal{E}(F)$. If $m_\Theta < m_{\Theta'}$, then $\mathbb{A}(\Theta, \Theta') = m_{\Theta'}$.*
(2) *If $\Theta_1, \Theta_2 \in \mathcal{E}(F)$, then $\mathbb{A}(\Theta_1, \Theta_2) = 0$ if and only if $\Theta_1 = \Theta_2$.*
(3) *If $\Theta_1, \Theta_2, \Theta_3 \in \mathcal{E}(F)$, then*

$$\mathbb{A}(\Theta_1, \Theta_3) \leqslant \max\{\mathbb{A}(\Theta_1, \Theta_2), \mathbb{A}(\Theta_2, \Theta_3)\}. \tag{5.1.3}$$

Proof. Part (1) follows from [11, (2.6.3)]. In (2), we find a hereditary order $\mathfrak{a}$ in some $A = \mathrm{End}_F(V)$, a simple stratum $[\mathfrak{a}, m_i, 0, \beta_i]$ and a simple character $\theta_i \in \mathcal{C}(\mathfrak{a}, 0, \beta_i)$ of endo-class Θ_i, $i = 1, 2$. The assertion $\mathbb{A}(\Theta_1, \Theta_2) = 0$ is equivalent to the characters θ_i of $H^1(\beta_i, \mathfrak{a})$ being conjugate in $\mathrm{Aut}_F(V)$. This, in turn, is equivalent to $\Theta_1 = \Theta_2$.

In (3), we may take simultaneous realizations $\theta_i \in \mathcal{C}(\mathfrak{a}, \beta_i)$ of Θ_i, $i = 1, 2, 3$, in some $G = \mathrm{Aut}_F(V)$. Let t_{ij} be the least nonnegative integer such that $\theta_i \mid H^{1+t_{ij}}(\beta_i, \mathfrak{a})$ is G-conjugate to $\theta_j \mid H^{1+t_{ij}}(\beta_i, \mathfrak{a})$. Thus $\mathbb{A}(\Theta_i, \Theta_j) = t_{ij}/e_\mathfrak{a}$. By symmetry, we may assume that $t_{12} \leqslant t_{23}$. Replacing the θ_i by conjugates, we may further assume that

$$\begin{aligned} H^{1+t_{12}}(\beta_1, \mathfrak{a}) &= H^{1+t_{12}}(\beta_2, \mathfrak{a}), \\ H^{1+t_{23}}(\beta_2, \mathfrak{a}) &= H^{1+t_{23}}(\beta_3, \mathfrak{a}) \end{aligned}$$

and that

$$\begin{aligned} \theta_1(g) &= \theta_2(g), \quad g \in H^{1+t_{12}}(\beta_1, \mathfrak{a}), \\ \theta_2(h) &= \theta_3(h), \quad h \in H^{1+t_{23}}(\beta_2, \mathfrak{a}) = H^{1+t_{23}}(\beta_1, \mathfrak{a}). \end{aligned}$$

Thus θ_1 agrees with θ_3 on $H^{1+t_{23}}(\beta_i, \mathfrak{a})$. It follows that $t_{13} \leqslant t_{23}$, as required to prove (5.1.3). □

In summary, the pairing $\mathbb{A}$ defines an *ultrametric* on the set $\mathcal{E}(F)$. It is natural to re-state 4.5 Proposition in terms of $\mathbb{A}$.

Corollary. *If $\Theta, \Theta' \in \mathcal{E}(F)$, then $\Phi_\Theta(a) = \Phi_{\Theta'}(a)$ for all $a \geqslant \mathbb{A}(\Theta, \Theta')$.*

5.2. In some circumstances, a different language is clearer. Let $[\mathfrak{a}, m, 0, \beta]$ be a simple stratum in some matrix algebra $A = \mathrm{End}_F(V)$, and let $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$. Let $\epsilon > 0$, and let t be the greatest integer such that $t/e_\mathfrak{a} < \epsilon$. In particular, $t \geqslant 0$. The *ϵ-truncation of θ*, denoted $\mathrm{tc}_\epsilon(\theta)$, is the character $\theta \mid H^{1+t}(\beta, \mathfrak{a})$.

Using the general machinery of [2, §8], we may form the endo-class of $\mathrm{tc}_\epsilon(\theta)$: if Θ is the endo-class of θ, we denote the endo-class of $\mathrm{tc}_\epsilon(\theta)$ by $\mathrm{tc}_\epsilon(\Theta)$. This depends only on Θ and ϵ. The definition of the ultrametric $\mathbb{A}$ then implies

Proposition. *Let $\epsilon > 0$. If $\Theta_1, \Theta_2 \in \mathcal{E}(F)$, then $\mathbb{A}(\Theta_1, \Theta_2) < \epsilon$ if and only if $\mathrm{tc}_\epsilon(\Theta_1) = \mathrm{tc}_\epsilon(\Theta_2)$.*

5.3. The following, more delicate, property is needed in certain situations.

Density Lemma. *Let $\Theta \in \mathcal{E}(F)$. The set $\{\mathbb{A}(\Theta, \Xi) : \Xi \in \mathcal{E}(F)\}$ is dense in the half line $\{x \geqslant 0 : x \in \mathbb{R}\}$.*

Proof. Let $x \in \mathbb{Q}$, $x > 0$. If $x > m_\Theta$, there exists $\Xi \in \mathcal{E}(F)$ such that $m_\Xi = x$, by 4.2 Proposition. This gives $\mathbb{A}(\Theta, \Xi) = x$, by 5.1 Proposition, so it is enough to treat the case $x < m_\Theta$.

Lemma. *Let $[\mathfrak{a}, m, 0, \beta]$ be a simple stratum in a matrix algebra $A = \mathrm{M}_n(F)$, and let $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$. Let k be an integer, $1 \leqslant k \leqslant m$. There exists a simple stratum $[\mathfrak{a}, m, 0, \beta']$ in A and $\theta' \in \mathcal{C}(\mathfrak{a}, \beta')$ such that*
(1) *$H^k(\beta', \mathfrak{a}) = H^k(\beta, \mathfrak{a})$;*
(2) *θ' agrees with θ on $H^{k+1}(\beta, \mathfrak{a})$; and*
(3) *the characters θ, θ' do not intertwine on $H^k(\beta, \mathfrak{a})$.*

Proof. We first reduce to the case in which the stratum $[\mathfrak{a}, m, k{-}1, \beta]$ is simple. Suppose it is not. We choose a simple stratum $[\mathfrak{a}, m, k{-}1, \gamma]$ equivalent to $[\mathfrak{a}, m, k{-}1, \beta]$. Directly from the definitions in [11, Ch. 3], we have

$$H^k(\gamma, \mathfrak{a}) = H^k(\beta, \mathfrak{a}), \quad \mathcal{C}(\mathfrak{a}, k{-}1, \gamma) = \mathcal{C}(\mathfrak{a}, k{-}1, \beta).$$

In particular, there exists $\xi \in \mathcal{C}(\mathfrak{a}, \gamma)$ agreeing with θ on $H^k(\beta, \mathfrak{a})$. We may now work with the pair (γ, ξ) in place of (β, θ).

We revert to our original notation, assuming that $[\mathfrak{a}, m, k{-}1, \beta]$ is simple. Let B denote the A-centralizer of β, and let $\mathfrak{b} = \mathfrak{a} \cap B$. We choose a simple stratum $[\mathfrak{b}, k, k{-}1, \alpha]$ in B. Writing $\mathfrak{p} = \mathrm{rad}\,\mathfrak{a}$, let $a \in \mathfrak{p}^{-k}$ satisfy $s_\beta(a) = \alpha$, where $s_\beta : A \to B$ is a tame corestriction relative to $F[\beta]/F$. The stratum $[\mathfrak{a}, m, k{-}1, \beta{+}a]$ is then equivalent to a simple stratum $[\mathfrak{a}, m, k{-}1, \beta']$ [11, (2.2.3)]. Let ψ_a denote the character

$$1{+}x \longmapsto \psi_F(\mathrm{tr}_A(ax)), \quad x \in \mathfrak{p}^k,$$

of $U^k_\mathfrak{a}$. The character $\theta' = \theta\psi_a$ of $H^k(\beta', \mathfrak{a}) = H^k(\beta, \mathfrak{a})$ then lies in $\mathcal{C}(\mathfrak{a}, k{-}1, \beta')$ and agrees with θ on $H^{1+k}(\beta, \mathfrak{a})$. However, 2.8 Proposition of [13] implies that the characters θ, θ' of $H^k(\beta, \mathfrak{a})$ do not intertwine. □

In the context of the lemma, let θ, θ' have endo-class Θ, Θ' respectively. Thus $\mathbb{A}(\Theta, \Theta') = k/e_\mathfrak{a}$. The only restrictions on the rational number $k/e_\mathfrak{a}$ are that $e_\mathfrak{a}$ be divisible by e_Θ and $k/e_\mathfrak{a} \leqslant m_\Theta$. Such values are dense in the region $0 < x < m_\Theta$. □

5.4. We recall the notation of the introduction: $\mathcal{A}_n^0(F)$ is the set of equivalence classes of irreducible cuspidal representations of $\mathrm{GL}_n(F)$. We set $\widehat{\mathrm{GL}}_F = \bigcup_{n\geqslant 1} \mathcal{A}_n^0(F)$ and, for $\pi \in \widehat{\mathrm{GL}}_F$, we write $\mathrm{gr}(\pi) = n$ to indicate $\pi \in \mathcal{A}_n^0(F)$. Let $\check{\pi}$ be the contragredient of π.

A representation $\pi \in \widehat{\mathrm{GL}}_F$ contains a simple character θ_π. The conjugacy class of θ_π in $\mathrm{GL}_n(F)$ is determined uniquely by π; see, for instance, Corollary 1 of [7]. In particular, π determines the endo-class $\Theta = \Theta_\pi$ of θ.

We recall the definition of the *Swan exponent* $\mathrm{sw}(\pi_1 \times \pi_2)$ of a pair of representations $\pi_1, \pi_2 \in \widehat{\mathrm{GL}}_F$. Set $n_i = \mathrm{gr}(\pi_i)$, let ψ be a nontrivial character of F, let s be a complex variable and q the cardinality of the residue class field of F. Let $\varepsilon(\pi_1 \times \pi_2, s, \psi)$ be the Rankin–Selberg local constant of [20] and [25]. This is a monomial in q^{-s} of degree $n_1 n_2 c(\psi) + \mathrm{Ar}(\pi_1 \times \pi_2)$, where $c(\psi)$ is an integer depending only on ψ, and the *Rankin–Selberg exponent* $\mathrm{Ar}(\pi_1 \times \pi_2)$ is an integer depending only on the π_i. Define an integer $d(\pi_1, \pi_2)$ as the number of unramified characters χ of $F^\times$ such that $\chi\pi_1 \cong \check{\pi}_2$. In particular, $d(\pi_1, \pi_2) = 0$ if $n_1 \neq n_2$. The Swan exponent is then

$$\mathrm{sw}(\pi_1 \times \pi_2) = \mathrm{Ar}(\pi_1 \times \pi_2) - n_1 n_2 + d(\pi_1, \pi_2).$$

Reformulating 6.5 Theorem of [10] in our present notation, we find

Conductor formula. *For $i = 1, 2$, let $\pi_i \in \widehat{\mathrm{GL}}_F$ and set $\Theta_i = \Theta_{\pi_i}$. If $a = \mathbb{A}(\Theta_1, \Theta_2)$, then*

$$\frac{\mathrm{sw}(\check{\pi}_1 \times \pi_2)}{\mathrm{gr}(\pi_1)\,\mathrm{gr}(\pi_2)} = \Phi_{\Theta_1}(a) = \Phi_{\Theta_2}(a). \tag{5.4.1}$$

If we take $\pi \in \mathcal{A}_n^0(F)$ and let ι be the trivial character of $\mathrm{GL}_1(F)$, we get the special case (cf. (5.1.2))

$$\mathrm{sw}(\pi \times \iota)/n = \mathrm{sw}(\pi)/n = m_{\Theta_\pi}. \tag{5.4.2}$$

Proposition. *Let $\Theta \in \mathcal{E}(F)$, and let $\pi \in \widehat{\mathrm{GL}}_F$ satisfy $\Theta_\pi = \Theta$. The function Φ_Θ is the unique continuous function on the positive real axis such that*

$$\frac{\mathrm{sw}(\check{\pi} \times \rho)}{\mathrm{gr}(\pi)\,\mathrm{gr}(\rho)} = \Phi_\Theta\big(\mathbb{A}(\Theta, \Theta_\rho)\big)$$

for all $\rho \in \widehat{\mathrm{GL}}_F$.

Proof. This follows from (5.4.1), the continuity of the function Φ_Θ (4.4 Proposition) and the Density Lemma of 5.3. □

The proposition has a consequence that is useful when making more general conductor estimates, although we do not need it here. For $i = 1, 2$, let $\Theta_i \in \mathcal{E}(F)$ and choose $\pi_i \in \widehat{\mathrm{GL}}_F$ such that $\Theta_i = \Theta_{\pi_i}$. Let $\mathrm{gr}(\pi_i) = n_i$. The quantity

$$\varsigma(\pi_1, \pi_2) = \mathrm{sw}(\check{\pi}_1 \times \pi_2)/n_1 n_2$$

depends only on the Θ_i, not on the choices of π_i: this is a consequence of the proposition. We therefore write $\varsigma(\Theta_1,\Theta_2)=\varsigma(\pi_1,\pi_2)$.

COROLLARY. *The pairing ς on the set $\mathcal{E}(F)$ satisfies the ultrametric inequality: if $\Theta_1,\Theta_2,\Theta_3\in\mathcal{E}(F)$, then*

$$\varsigma(\Theta_1,\Theta_2)\leqslant\max\{\varsigma(\Theta_1,\Theta_3),\varsigma(\Theta_3,\Theta_2)\}.$$

Proof. The pairing ς is symmetric: $\varsigma(\Theta_1,\Theta_2)=\varsigma(\Theta_2,\Theta_1)$. We may assume, by symmetry, that $\mathbb{A}(\Theta_1,\Theta_3)\leqslant\mathbb{A}(\Theta_3,\Theta_2)$. The function Φ_{Θ_3} is increasing, so

$$\varsigma(\Theta_1,\Theta_3)=\Phi_{\Theta_3}(\mathbb{A}(\Theta_1,\Theta_3))\leqslant\Phi_{\Theta_3}(\mathbb{A}(\Theta_2,\Theta_3))=\varsigma(\Theta_2,\Theta_3).$$

We are thus reduced to checking that $\varsigma(\Theta_1,\Theta_2)\leqslant\varsigma(\Theta_2,\Theta_3)$. However, the ultrametric inequality for $\mathbb{A}$ and our hypothesis give $\mathbb{A}(\Theta_1,\Theta_2)\leqslant\mathbb{A}(\Theta_3,\Theta_2)$ so

$$\varsigma(\Theta_1,\Theta_2)=\Phi_{\Theta_2}(\mathbb{A}(\Theta_1,\Theta_2))\leqslant\Phi_{\Theta_2}(\mathbb{A}(\Theta_3,\Theta_2))=\varsigma(\Theta_3,\Theta_2),$$

as required. □

5.5. We give a property of the ultrametric $\mathbb{A}$ relative to tame lifting, as in [2] (see also 4.6 above). For clarity, we temporarily write $\mathbb{A}_F$ for the canonical ultrametric on $\mathcal{E}(F)$ and $\mathbb{A}_K$ for that on $\mathcal{E}(K)$, where K/F is a finite tame extension.

PROPOSITION. *Let $\Theta,\Upsilon\in\mathcal{E}(F)$, and let K/F be a finite tame extension with $e(K|F)=e$. If Θ_i, $1\leqslant i\leqslant r$ are the K/F-lifts of Θ and Υ_j, $1\leqslant j\leqslant s$, those of Υ, then*

$$e\,\mathbb{A}_F(\Theta,\Upsilon)=\min_{i,j}\mathbb{A}_K(\Theta_i,\Upsilon_j)=\min_{j}\mathbb{A}_K(\Theta_1,\Upsilon_j).$$

Proof. From (9.8) Theorem of [2] we deduce that $e\,\mathbb{A}_F(\Theta,\Upsilon)\leqslant\mathbb{A}_K(\Theta_i,\Upsilon_j)$ for all i and j. On the other hand, [2, (9.12) Corollary] implies that, for any i, there exists j such that $e\,\mathbb{A}_F(\Theta,\Upsilon)\geqslant\mathbb{A}_K(\Theta_i,\Upsilon_j)$, whence the result follows. □

6. Comparison via the Langlands correspondence

We use the local Langlands correspondence to connect the preceding lines of thought.

6.1. We recall formally some matters mentioned in the introduction. Using the notation of 5.4, the Langlands correspondence is a canonical bijection

$$\widehat{\mathrm{GL}}_F\longrightarrow\widehat{\mathcal{W}}_F,$$
$$\pi\longmapsto{}^L\pi,$$

with, among others, the following properties:

$$\begin{aligned} \dim {}^L\pi &= \mathrm{gr}(\pi), \\ {}^L(\check\pi) &= \big({}^L\pi\big)^\vee, \qquad\qquad \pi,\rho \in \widehat{\mathrm{GL}}_F. \\ \varepsilon(\pi\times\rho, s,\psi) &= \varepsilon({}^L\pi\otimes {}^L\rho, s,\psi), \end{aligned} \tag{6.1.1}$$

Here, the second ε is the Langlands–Deligne local constant. The correspondence also respects twisting with characters. The definition of $\mathrm{sw}(\pi_1\times\pi_2)$ in 5.4 thus implies

$$\mathrm{sw}(\pi\times\rho) = \mathrm{sw}({}^L\pi\otimes {}^L\rho), \quad \pi,\rho\in\widehat{\mathrm{GL}}_F.$$

We prefer to write $\varsigma(\pi_1\times\pi_2) = \mathrm{sw}(\pi_1\times\pi_2)/\mathrm{gr}(\pi_1)\mathrm{gr}(\pi_2)$, so that

$$\varsigma(\check\pi\times\rho) = \varsigma({}^L\check\pi\otimes {}^L\rho), \quad \pi,\rho\in\widehat{\mathrm{GL}}_F. \tag{6.1.2}$$

A representation $\pi\in\widehat{\mathrm{GL}}_F$ determines an endo-class $\Theta_\pi\in\mathcal{E}(F)$, as recalled in 5.4. On the other hand, a representation $\sigma\in\widehat{\mathcal{W}}_F$ determines an orbit $[\sigma;0]^+\in\mathcal{W}_F\backslash\widehat{\mathcal{P}}_F$, as in 2.2.

First ramification theorem. *Let $\Theta\in\mathcal{E}(F)$, and choose $\pi\in\widehat{\mathrm{GL}}_F$ such that $\Theta_\pi=\Theta$. The conjugacy class ${}^L\Theta = [{}^L\pi;0]^+\in\mathcal{W}_F\backslash\widehat{\mathcal{P}}_F$ depends only on Θ and not on the choice of π. The map*

$$\begin{aligned} \mathcal{E}(F) &\longrightarrow \mathcal{W}_F\backslash\widehat{\mathcal{P}}_F, \\ \Theta &\longmapsto {}^L\Theta, \end{aligned} \tag{6.1.3}$$

is a canonical bijection.

Proof. See [5, 8.2 Theorem], [9, 6.1]. □

6.2. Let $\Theta\in\mathcal{E}(F)$. Choose $\pi\in\widehat{\mathrm{GL}}_F$ such that $\Theta_\pi=\Theta$, and write ${}^L\pi=\sigma$. The decomposition function Σ_σ depends only on $[\sigma;0]^+={}^L\Theta$, so we use the notation $\Sigma_\sigma=\Sigma_{{}^L\Theta}$. Combining (6.1.2) with (3.1.4) and 5.4 Proposition, we find

$$\begin{aligned} \Phi_\Theta(\mathbb{A}(\Theta,\Theta_\rho)) &= \varsigma(\check\pi\times\rho) = \varsigma({}^L\check\pi\otimes {}^L\rho) \\ &= \Sigma_{{}^L\Theta}(\Delta({}^L\Theta, {}^L\Theta_\rho)), \quad \rho\in\widehat{\mathrm{GL}}_F. \end{aligned}$$

In other words,

$$\Phi_\Theta(\mathbb{A}(\Theta,\Upsilon)) = \Sigma_{{}^L\Theta}(\Delta({}^L\Theta,{}^L\Upsilon)), \quad \Theta,\Upsilon\in\mathcal{E}(F). \tag{6.2.1}$$

We accordingly define the *Herbrand function* Ψ_Θ of Θ by

$$\Psi_\Theta = \Phi_\Theta^{-1}\circ\Sigma_{{}^L\Theta}, \qquad \Theta\in\mathcal{E}(F). \tag{6.2.2}$$

Proposition. *Let $\Theta\in\mathcal{E}(F)$.*

(1) *The function Ψ_Θ is continuous, strictly increasing and piecewise linear in the region $x\geqslant 0$. It is smooth except at a finite set of points.*

(2) *It satisfies* $\Psi_\Theta(0) = 0$ *and* $\Psi_\Theta(x) = x$ *for* $x \geqslant m_\Theta$.

Proof. Part (1) combines 4.4 Proposition with 3.1 Proposition. In part (2), we choose $\pi \in \widehat{\mathrm{GL}}_F$ such that $\Theta_\pi = \Theta$ and set $\sigma = {}^L\pi$. Thus

$$\varPhi_\Theta(0) = \varsigma(\check\pi \times \pi) = \varsigma(\check\sigma \otimes \sigma) = \varSigma_{^L\Theta}(0),$$

whence $\Psi_\Theta(0) = 0$. By (5.4.2), $m_\Theta = \varsigma(\pi) = \varsigma(\sigma) = \mathrm{sl}(\sigma)$, so the second assertion in (2) follows from (3.1.2) and (4.4.1). □

6.3. We pause to tie up some loose ends. Since $\mathrm{sw}(\pi) = \mathrm{sw}({}^L\pi)$ and $\mathrm{gr}(\pi) = \dim {}^L\pi$, 2.4 Proposition follows from 4.2 Proposition. The Density Lemma of 3.3 follows from that of 5.3 and the continuity of the strictly increasing function Ψ_Θ^{-1}. This proves 3.3 Proposition.

6.4. We prove our first main result.

Higher Ramification Theorem. *Let* $\Theta \in \mathcal{E}(F)$, *let* $\epsilon > 0$ *and* $\delta = \Psi_\Theta(\epsilon)$. *If* $\varUpsilon \in \mathcal{E}(F)$, *then*

$$\begin{aligned} \Delta({}^L\Theta, {}^L\varUpsilon) < \epsilon \quad &\Longleftrightarrow \quad \mathbb{A}(\Theta, \varUpsilon) < \delta, \\ \Delta({}^L\Theta, {}^L\varUpsilon) \leqslant \epsilon \quad &\Longleftrightarrow \quad \mathbb{A}(\Theta, \varUpsilon) \leqslant \delta. \end{aligned}$$

Proof. Let $\Theta \in \mathcal{E}(F)$ and $\delta > 0$. The endo-class Θ determines the function $\varPhi_\Theta$ and the orbit ${}^L\Theta \in \mathcal{W}_F\backslash\widehat{\mathcal{P}}_F$, whence it determines the function $\varSigma_{^L\Theta}$. For $\varUpsilon \in \mathcal{E}(F)$, the condition $\mathbb{A}(\Theta, \varUpsilon) < \delta$ implies

$$\Delta({}^L\Theta, {}^L\varUpsilon) = \varSigma_{^L\Theta}^{-1}\varPhi_\Theta(\mathbb{A}(\Theta, \varUpsilon)) = \Psi_\Theta^{-1}(\mathbb{A}(\Theta, \varUpsilon)) < \Psi_\Theta^{-1}(\delta),$$

since the function Ψ_Θ is strictly increasing (6.2 Proposition). Indeed, the converse holds for the same reason: if $\Delta({}^L\Theta, {}^L\varUpsilon) < \Psi_\Theta^{-1}(\delta)$, then $\mathbb{A}(\Theta, \varUpsilon) < \delta$. The same argument proves the second assertion. □

6.5. We give a more concrete variant of the main theorem. We first need a technical result.

Lemma. *If* $\Theta, \varUpsilon \in \mathcal{E}(F)$ *and* $x \geqslant \mathbb{A}(\Theta, \varUpsilon)$, *then* $\Psi_\Theta^{-1}(x) = \Psi_\varUpsilon^{-1}(x)$.

Proof. Let $\delta > \mathbb{A}(\Theta, \varUpsilon)$, and set $\epsilon = \Psi_\Theta^{-1}(\delta)$. An endo-class $\varXi \in \mathcal{E}(F)$ thus satisfies $\mathbb{A}(\varXi, \varUpsilon) < \delta$ if and only if $\mathbb{A}(\varXi, \Theta) < \delta$. The second condition is equivalent to $\Delta({}^L\varXi, {}^L\Theta) < \epsilon$ by the theorem, while the first is equivalent to $\Delta({}^L\varUpsilon, {}^L\varXi) < \Psi_\varUpsilon^{-1}(\delta)$. On the other hand,

$$\Delta({}^L\varUpsilon, {}^L\varXi) \leqslant \max\big\{\Delta({}^L\varUpsilon, {}^L\Theta), \Delta({}^L\Theta, {}^L\varXi)\big\} < \epsilon.$$

It follows that $\Psi_\varUpsilon^{-1}(\delta) \leqslant \epsilon = \Psi_\Theta^{-1}(\delta)$ for $\delta > \mathbb{A}(\Theta, \varUpsilon)$. By symmetry,

$$\Psi_\varUpsilon^{-1}(\delta) = \Psi_\Theta^{-1}(\delta), \quad \delta > \mathbb{A}(\Theta, \varUpsilon).$$

By continuity, the relation holds for $\delta \geqslant \mathbb{A}(\Theta, \varUpsilon)$. □

Remark. Under the hypotheses of the lemma, we may equally deduce that $\Psi_\Theta(y) = \Psi_\Upsilon(y)$ when $y \geqslant \Delta({}^L\Theta, {}^L\Upsilon)$.

We now use the notation of 5.2 for truncated endo-classes.

Corollary.

(1) *Let* $\Theta \in \mathcal{E}(F)$ *and* $\delta > 0$. *There is a unique pair* (ϵ, ξ), *where* $\epsilon > 0$ *and* $\xi \in \mathcal{W}_F\backslash\widehat{\mathcal{R}}_F(\epsilon)$, *with the following property*: *a representation* $\pi \in \widehat{\mathrm{GL}}_F$ *satisfies* $\mathrm{tc}_\delta(\Theta_\pi) = \mathrm{tc}_\delta(\Theta)$ *if and only if the representation* ξ *is equivalent to a component of* ${}^L\pi \mid \mathcal{R}_F(\epsilon)$.

(2) *Let* $\epsilon > 0$ *and* $\xi \in \mathcal{W}_F\backslash\widehat{\mathcal{R}}_F(\epsilon)$. *There exist* $\Theta \in \mathcal{E}(F)$ *and* $\delta > 0$ *with the following property*: *a representation* $\pi \in \widehat{\mathrm{GL}}_F$ *satisfies* $\mathrm{tc}_\delta(\Theta_\pi) = \mathrm{tc}_\delta(\Theta)$ *if and only if the representation* ξ *is equivalent to a component of* ${}^L\pi \mid \mathcal{R}_F(\epsilon)$. *The pair* (ϵ, ξ) *determines the truncated endo-class* $\mathrm{tc}_\delta(\Theta)$ *uniquely.*

Proof. The proofs of the two parts are virtually identical, so we treat only (1). Set $\epsilon = \Psi_\Theta^{-1}(\delta)$, and let ξ be the conjugacy class of an irreducible component of ${}^L\Theta$ on $\mathcal{R}_F(\epsilon)$. From 6.4 Theorem, a class $\Upsilon \in \mathcal{E}(F)$ satisfies $\mathbb{A}(\Upsilon, \Theta) < \delta$ if and only if $\Delta({}^L\Upsilon, {}^L\Theta) < \epsilon$. The first of these conditions is equivalent to $\mathrm{tc}_\delta(\Upsilon) = \mathrm{tc}_\delta(\Theta)$ (5.2 Proposition) while the second is equivalent to ${}^L\Upsilon$ containing ξ, by the definition of Δ. All assertions now follow. $\square$

7. The Herbrand function of an endo-class

We give a procedure for determining the Herbrand function Ψ_Θ of an endo-class $\Theta \in \mathcal{E}(F)$.

7.1. Fundamental to the method is the following lifting property.

Proposition. *Let* K/F *be a finite*, *tame extension*, *and set* $e(K|F) = e$. *If* $\Theta \in \mathcal{E}(F)$ *and if* $\Theta^K \in \mathcal{E}(K)$ *is a* K/F*-lift of* Θ, *then*

$$\Psi_{\Theta^K}(x) = e\,\Psi_\Theta(e^{-1}x), \quad x \geqslant 0. \tag{7.1.1}$$

Proof. Using transitivity of tame lifting, we reduce immediately to the case where the tame extension K/F is Galois. Write $\Gamma = \mathrm{Gal}(K/F)$, and let $\Upsilon \in \mathcal{E}(F)$. Let Υ^K be a K/F-lift of Υ. Write $\mathbb{A}_F$, $\mathbb{A}_K$ for the canonical ultrametrics on $\mathcal{E}(F)$, $\mathcal{E}(K)$ respectively. We choose the lift Υ^K so that

$$\mathbb{A}_K(\Theta^K, \Upsilon^K) \leqslant \mathbb{A}_K(\Theta^K, \gamma\Upsilon^K), \quad \gamma \in \Gamma.$$

The function Ψ_{Θ^K} is strictly increasing, so writing Δ_F, Δ_K for the canonical ultrametrics on $\mathcal{W}_F\backslash\widehat{\mathcal{P}}_F$, $\mathcal{W}_K\backslash\widehat{\mathcal{P}}_K$ respectively, we have

$$\Delta_K({}^L\Theta^K, {}^L\Upsilon^K) \leqslant \Delta_K({}^L\Theta^K, {}^L(\gamma\Upsilon^K)), \quad \gamma \in \Gamma.$$

The canonical bijection $\mathcal{E}(K) \to \mathcal{W}_K\backslash\widehat{\mathcal{P}}_K$ is $\varGamma$-equivariant, so this reads

$$\Delta_K({}^L\varTheta^K, {}^L\varUpsilon^K) \leqslant \Delta_K({}^L\varTheta^K, \gamma\, {}^L\varUpsilon^K), \quad \gamma \in \varGamma,$$

whence

$$\begin{aligned} \mathbb{A}_F(\varTheta, \varUpsilon) &= e^{-1}\mathbb{A}_K(\varTheta^K, \varUpsilon^K), \\ \Delta_F({}^L\varTheta, {}^L\varUpsilon) &= e^{-1}\Delta_K({}^L\varTheta^K, {}^L\varUpsilon^K), \end{aligned}$$

by 5.5 Proposition, 2.6 Proposition respectively. Therefore,

$$\begin{aligned} \mathbb{A}_F(\varTheta, \varUpsilon) &= \Psi_\varTheta\big(\Delta_F({}^L\varTheta, {}^L\varUpsilon)\big) = \Psi_\varTheta\big(e^{-1}\Delta_K({}^L\varTheta^K, {}^L\varUpsilon^K)\big) \\ &= \Psi_\varTheta(e^{-1}\Psi_{\varTheta^K}^{-1}\big(\mathbb{A}_K(\varTheta^K, \varUpsilon^K)\big). \end{aligned}$$

We write $y = \mathbb{A}_F(\varTheta, \varUpsilon)$ to get

$$y = \Psi_\varTheta(e^{-1}\Psi_{\varTheta^K}^{-1}(ey)). \tag{7.1.2}$$

The Density Lemma of 5.2 says that the set of values $y = \mathbb{A}_F(\varTheta, \varUpsilon)$, $\varUpsilon \in \mathcal{E}(F)$, is dense on the positive real axis, so (7.1.2) holds for all $y > 0$. Writing $z = \Psi_{\varTheta^K}^{-1}(ey)$, we get $e^{-1}\Psi_{\varTheta^K}(z) = \Psi_\varTheta(e^{-1}z)$, as required. □

Remark. Given $\varTheta \in \mathcal{E}(F)$, the definitions in [2] (or see [9, 6.3]) give a finite tame extension K/F for which $\varTheta$ has a totally wild K/F-lift. The proposition therefore reduces the problem of computing $\Psi_\varTheta$ to the case where $\varTheta$ is *totally wild*.

When $\varTheta$ is totally wild and K/F is tamely ramified, there is a simple relation (4.6) connecting $\varPhi_\varTheta$ and $\varPhi_{\varTheta^K}$. Likewise for $\varSigma_{^L\varTheta}$ and $\varSigma_{^L\varTheta^K}$ (3.2). However, for general $\varTheta$, the relations between $\varPhi_\varTheta$ and $\varPhi_{\varTheta^K}$, and between $\varSigma_{^L\varTheta}$ and $\varSigma_{^L\varTheta^K}$, are rather intricate. The symmetry indicated by the proposition can be viewed as a refinement of the Tame Parameter Theorem of [9, 6.3].

7.2. Recall that $\sigma \in \widehat{\mathcal{W}}_F$ is *totally wild* if $\sigma|_{\mathcal{P}_F}$ is irreducible. Equivalently, the orbit $[\sigma, 0]^+ \in \mathcal{W}_F\backslash\widehat{\mathcal{P}}_F$ has exactly one element. Write $\widehat{\mathcal{W}}_F^{\mathrm{wr}}$ for the set of totally wild classes in $\widehat{\mathcal{W}}_F$. In particular, any $\sigma \in \widehat{\mathcal{W}}_F^{\mathrm{wr}}$ has dimension p^r for some $r \geqslant 0$.

Lemma. *A representation $\sigma \in \widehat{\mathcal{W}}_F$ is totally wild if and only if $\sigma = {}^L\pi$ for $\pi \in \widehat{\mathrm{GL}}_F$ such that $\mathrm{gr}(\pi) = \deg \varTheta_\pi$ and $\varTheta_\pi$ is totally wild.*

Proof. This follows from [9, 6.3]. □

7.3. Totally wild representations of $\mathcal{W}_F$ exhibit simple ultrametric behavior with respect to twisting by characters.

Proposition. *Let $\sigma \in \widehat{\mathcal{W}}_F^{\mathrm{wr}}$ and let c be a positive integer. If χ is a character of $\mathcal{W}_F$ of conductor c, then $\Delta(\sigma, \chi \otimes \sigma) \leqslant c$. If $\varSigma'_\sigma$ is continuous at c, then $\Delta(\sigma, \chi \otimes \sigma) = c$.*

Proof. Suppose $c > \mathrm{sl}(\sigma)$. The definition of Σ_σ (3.1.2) shows that Σ'_σ is continuous at c. Also $\mathrm{sl}(\chi\otimes\sigma) = c > \mathrm{sl}(\sigma)$, whence $\Delta(\sigma,\chi\otimes\sigma) = c$. We assume, therefore, that $c \leqslant \mathrm{sl}(\sigma)$. The representations σ, $\chi\otimes\sigma$ are $\mathcal{R}_F^+(c)$-isomorphic, so $\Delta(\sigma,\chi\otimes\sigma)\leqslant c$. The distance $\Delta(\sigma,\chi\otimes\sigma)$ is strictly less than c if and only if $\chi|_{\mathcal{R}_F(c)}$ occurs in $\check\sigma\otimes\sigma|_{\mathcal{R}_F(c)}$. Suppose this condition holds. Since χ is trivial on $\mathcal{R}_F^+(c)$, the definition now shows that Σ'_σ is discontinuous at c. □

7.4. We recall how the set $\mathcal{E}(F)$ carries a canonical action of the group of characters of $F^\times$.

Let $\Theta\in\mathcal{E}(F)$, and let χ be a character of $F^\times$. If $\deg\Theta = 1$, then Θ is the endo-class of a character θ of U_F^1 and $\chi\Theta$ is the endo-class of the (possibly trivial) character $\theta\chi \mid U_F^1$. Assume that $\deg\Theta > 1$, and choose a realization $\theta\in\mathcal{C}(\mathfrak{a},\beta)$ of Θ, relative to a simple stratum $[\mathfrak{a},m,0,\beta]$ in a matrix algebra $\mathrm{End}_F(V)$. Define a character $\chi\theta$ of $H^1(\beta,\mathfrak{a})$ by

$$\chi\theta(h) = \chi(\det h)\,\theta(h),\quad h\in H^1(\beta,\mathfrak{a}).$$

Lemma. *Let $k = \mathrm{sw}(\chi)\geqslant 1$, and let $c\in F^\times$ satisfy $\chi(1{+}x) = \psi_F(cx)$ for $2v_F(x) > k$. If $m' = \max\{m, nk\}$, the quadruple $[\mathfrak{a},m',0,\beta{+}c]$ is a simple stratum and $\chi\theta\in\mathcal{C}(\mathfrak{a},\beta{+}c)$.*

Proof. See [12, appendix]. □

Denote by $\chi\Theta$ the endo-class of $\chi\theta$. If $\pi\in\mathcal{A}_n^0(F)$ and $\Theta=\Theta_\pi$, then $\chi\Theta$ is the endo-class $\Theta_{\chi\pi}$ of the representation $\chi\pi : g\mapsto\chi(\det g)\pi(g)$, $g\in\mathrm{GL}_n(F)$.

The lemma shows that if Θ is totally wild, then so is $\chi\Theta$ for any χ. In a more general setting, the following is a direct consequence of the definitions (4.4.1), (3.1.2), on noting that ${}^L(\chi\Theta) = \chi\otimes{}^L\Theta$ (in the obvious notation).

Proposition. *Let $\Theta\in\mathcal{E}(F)$. If χ is a character of $F^\times$, then $\Phi_{\chi\Theta} = \Phi_\Theta$ and $\Sigma_{{}^L(\chi\Theta)} = \Sigma_{{}^L\Theta}$. Consequently, $\Psi_{\chi\Theta} = \Psi_\Theta$.*

7.5. Our main result gives a procedure for calculating the Herbrand function Ψ_Θ of any $\Theta\in\mathcal{E}(F)$. As noted in 7.1, it is enough to treat the case where Θ is totally wild.

If $\Theta\in\mathcal{E}(F)$ is totally wild and if K/F is a finite tame extension, let $\Theta^K\in\mathcal{E}(K)$ be the unique K/F-lift of Θ. Denote by $\mathbb{A}_K$ the canonical ultrametric on $\mathcal{E}(K)$.

Interpolation Theorem. *Let $\Theta\in\mathcal{E}(F)$ be totally wild. The function Ψ_Θ has the following properties*:

(1) *It is continuous, strictly increasing and piecewise linear.*
(2) *The derivative Ψ'_Θ is continuous except at a finite set of points.*
(3) *There is a finite set D of positive real numbers such that*
  (a) *if K/F is a finite tame extension, with $e = e(K|F)$, and*

(b) *if χ is a character of $K^\times$ such that $e^{-1}\mathrm{sw}(\chi) \notin D$,*
then

$$\mathbb{A}_K(\Theta^K, \chi\Theta^K) = e\Psi_\Theta(e^{-1}\mathrm{sw}(\chi)). \tag{7.5.1}$$

These properties determine Ψ_Θ uniquely.

Proof. The function Ψ_Θ certainly satisfies (1) by 6.2 Proposition, while (2) follows directly from the definitions of the functions $\varPhi_\Theta$, $\varSigma_\sigma$. Condition (3) determines $\Psi_\Theta(x)$ at a set of points x dense in the positive real axis. Since Ψ_Θ is continuous, it is thereby determined completely.

We have to show that Ψ_Θ has property (3). Let $\deg\Theta = p^r$ and let $\pi \in \mathcal{A}^0_{p^r}(F)$ satisfy $\Theta_\pi = \Theta$. Set $\sigma = {}^L\pi \in \widehat{\mathcal{W}}^{\mathrm{wr}}_F$. If K/F is a finite tame extension, set $\sigma^K = \sigma|_{\mathcal{W}_K}$ and define $\pi^K \in \mathcal{A}^0_{p^r}(K)$ by ${}^L\pi^K = \sigma^K$. In particular, $\Theta_{\pi^K} = \Theta^K$ [9, 6.2 Proposition].

Let χ be a character of $K^\times$, of conductor $k \geqslant 1$, such that $\varSigma'_\sigma$ is continuous at k/e, $e = e(K|F)$. By 3.2 Proposition, $\varSigma_{\sigma^K}(x) = e\varSigma_\sigma(x/e)$, so $\varSigma'_{\sigma^K}$ is continuous at k. By (5.4.1), $\varsigma(\check\pi^K \times \chi\pi^K) = \varPhi_{\Theta^K}(\mathbb{A}_K(\Theta^K, \chi\Theta^K))$. By 7.3 Proposition,

$$\varsigma(\check\sigma^K \otimes \chi \otimes \sigma^K) = \varSigma_{\sigma^K}(\Delta_K(\sigma^K, \chi\otimes\sigma^K)) = \varSigma_{\sigma^K}(k),$$

where Δ_K is the canonical pairing on $\widehat{\mathcal{W}}_K$. By 4.6 Proposition, $\varPhi_{\Theta^K}(x) = e\varPhi_\Theta(x/e)$. Altogether,

$$\begin{aligned}\varsigma(\check\pi^K \times \chi\pi^K) &= \varPhi_{\Theta^K}\big(\mathbb{A}_K(\chi\Theta^K, \Theta^K)\big) = e\,\varPhi_\Theta\big(e^{-1}\mathbb{A}_K(\chi\Theta^K, \Theta^K)\big)\\ &= \varsigma(\check\sigma^K \otimes \chi \otimes \sigma^K) = \varSigma_{\sigma^K}(k)\\ &= e\varSigma_\sigma(k/e),\end{aligned}$$

whence $\mathbb{A}_K(\chi\Theta^K, \Theta^K) = e\,\varPhi_\Theta^{-1} \circ \varSigma_\sigma(k/e) = e\Psi_\Theta(k/e)$. Thus (3) holds relative to any set D containing the discontinuities of $\varSigma'_\sigma$. These are finite in number by 3.1 Proposition. □

Remark. In this proof, it was necessary to exclude only the discontinuities of $\varSigma'_\sigma$. The result does not assert that the function Ψ_Θ is smooth elsewhere. Indeed, Example 1 of 8.5 below gives a case in which $\varSigma'_\sigma$ has one discontinuity, while Ψ'_Θ has two, in the relevant range $0 < x < m_\Theta$.

7.6. We describe the function Ψ_Θ, for $\Theta \in \mathcal{E}(F)$ totally wild, on part of its range. We have already noted in 6.2 that $\Psi_\Theta(0) = 0$ and that $\Psi_\Theta(x) = x$ when $x > m_\Theta$.

It is convenient to first dispose of a special case. If $\deg\Theta = 1$, the definitions of the various functions give

$$\Psi_\Theta(x) = \varPhi_\Theta(x) = \varSigma_{{}^L\Theta}(x) = x, \quad x \geqslant 0, \tag{7.6.1}$$

so we henceforward exclude this case.

Proposition. *Let $\Theta \in \mathcal{E}(F)$ be totally wild of degree p^r, $r \geqslant 1$, and suppose that $m_\Theta = ap^{t-r}$ for integers a and t satisfying $a \not\equiv 0 \pmod p$ and $0 \leqslant t < r$.*

(1) *There exists $\epsilon > 0$ such that $\Psi'_\Theta(x) = p^{-r}$ for $0 < x < \epsilon$.*
(2) *There exists $\delta > 0$ such that*

$$\Psi'_\Theta(x) = p^{r-t}, \quad m_\Theta - \delta < x < m_\Theta.$$

Proof. Part (1) follows from the definitions (4.4.1) and (3.1.2) on noting that, if $\sigma \in \widehat{\mathcal{W}}_F^{\mathrm{wr}}$, there exists $\epsilon > 0$ such that σ is irreducible on $\mathcal{R}_F(\epsilon)$.

In part (2), write $m_\Theta = p^{-r}m = p^{t-r}a$. The class Θ has a realization $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$ for a simple stratum $[\mathfrak{a}, m, 0, \beta]$ in $\mathrm{M}_{p^r}(F)$. We choose a simple stratum $[\mathfrak{a}, m, m-1, \alpha]$ equivalent to $[\mathfrak{a}, m, m-1, \beta]$, and we set $E = F[\alpha]$. The extension E/F is totally ramified of degree p^s, where $0 \leqslant s \leqslant r$. The integer m satisfies $m/e_\mathfrak{a} = -\upsilon_F(\alpha)/p^s$, or $m = -p^{r-s}\upsilon_F(\alpha)$. If $s = 0$ or, equivalently, if $\alpha \in F$, then m is divisible by p^r, contrary to hypothesis. Thus $s \geqslant 1$ and $\alpha \notin F$. Since α is minimal over F, the valuation $\upsilon_E(\alpha)$ is not divisible by p. In other words, $t = r-s$ and $a = -\upsilon_E(\alpha)$. The definition (4.4.1) gives $\Phi'_\Theta(x) = [E{:}F]^{-1} = p^{t-r}$ in a region $m_\Theta - \delta_1 < x < m_\Theta$, $\delta_1 > 0$.

Let $\sigma \in \widehat{\mathcal{W}}_F^{\mathrm{wr}}$ satisfy $[\sigma; 0]^+ = {}^L\Theta$. Since $\mathrm{sl}(\sigma) = \varsigma(\sigma) = m_\Theta$, the restriction of σ to $\mathcal{R}_F(m_\Theta)$ is a sum of characters of $\mathcal{R}_F(m_\Theta)$ trivial on $\mathcal{R}_F^+(m_\Theta)$. These are all conjugate under $\mathcal{P}_F$ and so, by 1.2 Lemma 2, they are all the same. Therefore, every irreducible component of $\check\sigma \otimes \sigma$ contains the trivial character of $\mathcal{R}_F(m_\Theta)$. By 2.1 Proposition 1, $\check\sigma \otimes \sigma$ is trivial on $\mathcal{R}_F(m_\Theta - \delta_2)$, for some $\delta_2 > 0$. In that region, Σ'_σ has value 1, whence the result follows with $\delta = \min\{\delta_1, \delta_2\}$. □

Remark. In the context of the proposition, consider the case where $m_\Theta = ap^{t-r}$, with $a \not\equiv 0 \pmod p$, but $t \geqslant r$. Following the argument through, the element α of the proof lies in F, so there is a character χ of $F^\times$ such that $m_{\chi\Theta} < m_\Theta$. In light of 7.4 Proposition, nothing is lost by excluding this case.

7.7. *Example*. Say that $\Theta \in \mathcal{E}(F)$ is *essentially tame* if, for some finite, tamely ramified extension K/F, Θ has a K/F-lift of degree 1: equivalently, e_Θ is relatively prime to p.

Corollary. *For $\Theta \in \mathcal{E}(F)$, the following conditions are equivalent*:

(1) $\Psi_\Theta(x) = x$, $x \geqslant 0$;
(2) $\Psi'_\Theta(x)$ *is continuous in the region* $0 < x < m_\Theta$;
(3) Θ *is essentially tame.*

Proof. Let $\Theta \in \mathcal{E}(F)$, and let K/F be a finite tame extension such that Θ has a totally wild K/F-lift Θ^K. Thus $\Psi_\Theta(x) = \Psi_{\Theta^K}(ex)/e$, $e = e(K|F)$. If

Θ is essentially tame, then $\deg \Theta^K = 1$. Both (1) and (2) hold for Ψ_{Θ^K}, hence also for Ψ_Θ by 7.1 Proposition.

Conversely, suppose that Θ is not essentially tame. The totally wild endo-class Θ^K then has degree p^r for some $r \geqslant 1$, and there is a character χ of $K^\times$ so that $\chi\Theta^K$ satisfies the hypotheses of 7.6 Proposition. Therefore, both (1) and (2) fail for the function $\Psi_{\chi\Theta^K} = \Psi_{\Theta^K}$, hence also for Ψ_Θ. $\square$

8. **The decomposition function**

We analyze some features of the decomposition function Σ_σ, $\sigma \in \widehat{\mathcal{W}}_F$, taking the view that Σ_σ has been given somehow, without prior knowledge of σ.

8.1. We examine the discontinuities of the derivative Σ'_σ, using only group-theoretic methods.

Proposition. *Let $\sigma \in \widehat{\mathcal{W}}_F$, let $\epsilon > 0$, and let σ_ϵ be an irreducible component of $\sigma \mid \mathcal{R}_F(\epsilon)$. Let Γ_ϵ be the group of characters of $\mathcal{R}_F(\epsilon)/\mathcal{R}_F^+(\epsilon)$. The following are equivalent*:

(1) *The function Σ'_σ is continuous at ϵ.*
(2) *The representation $\chi\otimes\sigma_\epsilon$ is not $\mathcal{W}_F$-conjugate to σ_ϵ for any $\chi \in \Gamma_\epsilon$, $\chi \neq 1$.*

Proof. An exercise in elementary representation theory yields

Lemma. *Suppose that the representation $\sigma_\epsilon \mid \mathcal{R}_F^+(\epsilon) = \sigma_\epsilon^+$ is irreducible. The map $\chi \mapsto \chi\otimes\sigma_\epsilon$ is a bijection between the group Γ_ϵ and the set of isomorphism classes of irreducible smooth representations of $\mathcal{R}_F(\epsilon)$ that contain σ_ϵ^+.*

We prove the proposition. For $\epsilon > 0$, define

$$d_\epsilon = \dim \operatorname{Hom}_{\mathcal{R}_F(\epsilon)}(1, \check\sigma\otimes\sigma),$$
$$d_\epsilon^+ = \dim \operatorname{Hom}_{\mathcal{R}_F^+(\epsilon)}(1, \check\sigma\otimes\sigma).$$

The step function Σ'_σ is continuous at a point $\epsilon > 0$ if and only if it is constant on a neighborhood of ϵ. This is equivalent to the condition $d_\epsilon = d_\epsilon^+$.

Let m_ϵ be the multiplicity of σ_ϵ in $\sigma \mid \mathcal{R}_F(\epsilon)$ and l_ϵ the number of $\mathcal{W}_F$-conjugates of σ_ϵ. Define m_ϵ^+ and l_ϵ^+ analogously, relative to an irreducible component σ_ϵ^+ of $\sigma_\epsilon \mid \mathcal{R}_F^+(\epsilon)$. Thus $d_\epsilon = l_\epsilon m_\epsilon^2$ and $d_\epsilon^+ = l_\epsilon^+ m_\epsilon^{+2}$. Moreover, $l_\epsilon m_\epsilon$ and $l_\epsilon^+ m_\epsilon^+$ are the Jordan–Hölder lengths of the restrictions $\sigma \mid \mathcal{R}_F(\epsilon)$ and $\sigma \mid \mathcal{R}_F^+(\epsilon)$ respectively.

Suppose that condition (2) holds. In particular, $\sigma_\epsilon \not\cong \sigma_\epsilon\otimes\chi$ for any character $\chi \in \Gamma_\epsilon$, $\chi \neq 1$. This implies that $\sigma_\epsilon \mid \mathcal{R}_F^+(\epsilon)$ is irreducible, so set $\sigma_\epsilon^+ = \sigma_\epsilon \mid \mathcal{R}_F^+(\epsilon)$. The lemma implies that σ_ϵ is the unique irreducible component of $\sigma \mid \mathcal{R}_F(\epsilon)$ containing σ_ϵ^+. Thus $m_\epsilon = m_\epsilon^+$ and the representations

σ_ϵ, σ_ϵ^+ have the same $\mathcal{W}_F$-isotropy, whence $l_\epsilon^+ = l_\epsilon$. Therefore, $d_\epsilon = d_\epsilon^+$ and Σ'_σ is continuous at ϵ.

Suppose now that (2) fails. If $\sigma_\epsilon \mid \mathcal{R}_F^+(\epsilon)$ is reducible, certainly Σ'_σ cannot be continuous at ϵ. We therefore assume the contrary and let c be the number of $\chi \in \Gamma_\epsilon$ such that $\chi \otimes \sigma_\epsilon$ is $\mathcal{W}_F$-conjugate to σ_ϵ. Thus $c > 1$ by hypothesis and, by the lemma, $l_\epsilon = c\, l_\epsilon^+$. Correspondingly, $m_\epsilon^+ = cm_\epsilon$, so $d_\epsilon^+ = cd_\epsilon > d_\epsilon$ and Σ'_σ is not continuous at ϵ. □

Remark. We draw attention to one step in the preceding proof: if the conditions of the proposition are satisfied, then $\sigma_\epsilon \mid \mathcal{R}_F^+(\epsilon)$ is *irreducible*.

8.2. To prepare for the main result, we develop some ideas from Galois theory.

Let $\sigma \in \widehat{\mathcal{W}}_F$, and assume $\dim \sigma > 1$. Define $\bar\sigma$ to be the *projective* representation defined by σ: that is, if $\dim \sigma = n$, then $\bar\sigma$ is the composition of σ with the canonical map $\mathrm{GL}_n(\mathbb{C}) \to \mathrm{PGL}_n(\mathbb{C})$. The image of $\bar\sigma$ is finite and $\mathrm{Ker}\, \bar\sigma$ is of the form $\mathcal{W}_E$ for a finite Galois extension E/F. We call E/F the *centric field* of σ. Let T/F be the maximal tamely ramified sub-extension of E/F: we call T/F the *tame centric field* of σ.

Definition. Let $\sigma \in \widehat{\mathcal{W}}_F^{\mathrm{wr}}$.

(1) Define $D(\sigma)$ as the group of characters χ of $\mathcal{W}_F$ such that $\chi \otimes \sigma \cong \sigma$.
(2) Write $\sigma_0^+ = \sigma \mid \mathcal{P}_F \in \widehat{\mathcal{P}}_F$. Define $D_0(\sigma)$ as the group of characters ϕ of $\mathcal{P}_F$ such that $\phi \otimes \sigma_0^+ \cong \sigma_0^+$.

Restriction of characters gives a canonical homomorphism $D(\sigma) \to D_0(\sigma)$. A character ϕ of $\mathcal{P}_F$ lies in $D_0(\sigma)$ if and only if it is a component of $\check\sigma_0^+ \otimes \sigma_0^+$, whence $|D_0(\sigma)| \leqslant (\dim \sigma)^2$.

On the other hand, provided $\dim \sigma > 1$, the group $D_0(\sigma)$ is not trivial. For, σ_0^+ factors through an irreducible representation, call it ρ, of a finite quotient G of $\mathcal{P}_F$. Since G is a finite p-group and $\dim \rho$ has dimension strictly greater than 1, ρ is induced from a representation of a subgroup H of G of index p. It follows that $\rho \cong \phi \otimes \rho$ for any character ϕ of G/H. Viewed as a character of $\mathcal{P}_F$, $\phi \in D_0(\sigma)$.

The representation σ_0^+ is stable under conjugation by $\mathcal{W}_F$, so $\mathcal{W}_F$ acts on $D_0(\sigma)$, with $\mathcal{P}_F$ acting trivially. The $\mathcal{W}_F$-stabilizer of a character $\phi \in D_0(\sigma)$ is thus of the form $\mathcal{W}_{T_\phi}$ for a finite tame extension T_ϕ/F. The kernel of the canonical map $\mathcal{W}_F \to \mathrm{Aut}\, D_0(\sigma)$ is therefore

$$\mathcal{W}_{T_I} = \bigcap_{\phi \in D_0(\sigma)} \mathcal{W}_{T_\phi},$$

where T_I/F is a finite, tamely ramified, Galois extension. We call T_I/F the *imprimitivity field* of σ.

If K/F is a finite tame extension, the representation $\sigma^K = \sigma \mid \mathcal{W}_K$ is irreducible and lies in $\widehat{\mathcal{W}}_K^{\mathrm{wr}}$. It agrees with σ on $\mathcal{P}_K = \mathcal{P}_F$, so $D_0(\sigma^K) = D_0(\sigma)$.

PROPOSITION. *If $\sigma \in \widehat{\mathcal{W}}_F^{\mathrm{wr}}$ has tame centric field T/F and imprimitivity field T_I/F, then $T_I \subset T$. The canonical map $D(\sigma^{T_I}) \to D_0(\sigma)$ is an isomorphism.*

Proof. We first note

LEMMA 1. *If $\zeta \in D(\sigma)$ is tamely ramified, then $\zeta = 1$.*

Proof. The kernel of ζ is $\mathcal{W}_K$, for a finite, cyclic, tame extension K/F. The relation $\zeta \otimes \sigma \cong \sigma$ implies that σ is reducible on $\mathcal{W}_K$. Since $\mathcal{P}_F \subset \mathcal{W}_K$, it is also reducible on $\mathcal{P}_F$, contrary to hypothesis. □

LEMMA 2. *Let K/F be a finite tame extension. The restriction map $D(\sigma^K) \to D_0(\sigma)$ is an isomorphism of $D(\sigma^K)$ with the group $D_0(\sigma)^{\mathcal{W}_K}$ of $\mathcal{W}_K$-fixed points in $D_0(\sigma)$.*

Proof. Lemma 8.1 implies that the map $D(\sigma^K) \to D_0(\sigma)$ is injective. Its image is clearly contained in $D_0(\sigma)^{\mathcal{W}_K}$. Let $\zeta \in D_0(\sigma)^{\mathcal{W}_K}$. Thus ζ admits extension to a character $\tilde{\zeta}$ of $\mathcal{W}_K$ [9, 1.3 Proposition]. The representations σ^K, $\tilde{\zeta} \otimes \sigma^K$ agree on $\mathcal{P}_K$ so (loc. cit.) there is a tame character χ of $\mathcal{W}_K$ such that $\chi\tilde{\zeta} \otimes \sigma^K \cong \sigma^K$. Therefore, $\chi\tilde{\zeta} \in D(\sigma^K)$, as required. □

By the definition of T, we have $\mathcal{W}_T = \mathcal{P}_F\mathcal{W}_E$. A character $\zeta \in D_0(\sigma)$ is effectively a character of $\mathcal{P}_F\mathcal{W}_E/\mathcal{W}_E$, and hence a character of $\mathcal{W}_T$. In particular, $\mathcal{W}_T$ fixes ζ, whence $\mathcal{W}_T \subset \mathcal{W}_{T_\zeta}$. Therefore, $\mathcal{W}_T \subset \mathcal{W}_{T_I}$, or $T \supset T_I$, as required to complete the proof of the proposition. □

8.3. Let $\sigma \in \widehat{\mathcal{W}}_F^{\mathrm{wr}}$. Say that σ is *absolutely wild* if its tame centric field is F. That is, if E is the centric field of σ, then E/F is totally wildly ramified.

THEOREM. *Let $\sigma \in \widehat{\mathcal{W}}_F^{\mathrm{wr}}$ be absolutely wild of dimension p^r, $r \geqslant 1$. If $a > 0$ is the least discontinuity of Σ'_σ, then a is an integer and*

$$a = \min\{\mathrm{sw}(\chi) : \chi \in D(\sigma), \chi \neq 1\}.$$

Proof. Nothing is changed by tensoring σ with a tame character of $\mathcal{W}_F$. We may therefore assume that σ is a representation of $\mathrm{Gal}(\widetilde{E}/F)$, where $\widetilde{E}/F$ is totally wildly ramified.

The group $D(\sigma) \cong D_0(\sigma)$ is nontrivial. We accordingly define

$$c = \min\{\mathrm{sw}(\chi) : \chi \in D(\sigma), \chi \neq 1\}.$$

Any character $\phi \in D(\sigma)$ occurs as an irreducible component of $\check{\sigma} \otimes \sigma$, so the definitions in 3.1 imply that Σ'_σ is discontinuous at c. Thus $c \geqslant a$.

If $0 < \epsilon < a$, condition (2) of 8.1 Proposition holds at ϵ, so $\sigma_\epsilon = \sigma \mid \mathcal{R}_F(\epsilon)$ is irreducible (8.1 Remark). It follows that $\sigma_a = \sigma \mid \mathcal{R}_F(a)$ is also irreducible. Since Σ'_σ is discontinuous at a, there is a nontrivial character χ of $\mathcal{R}_F(a)/\mathcal{R}_F^+(a)$ such that $\sigma_a \otimes \chi$ is $\mathcal{W}_F$-conjugate to σ_a, say $\sigma_a^g \cong \sigma_a \otimes \chi$ for some $g \in \mathcal{W}_F$. However, $\sigma_a = \sigma \mid \mathcal{R}_F(a)$ and surely $\sigma^g \cong \sigma$. Thus $\sigma_a \cong \sigma_a \otimes \chi$, whence σ_a is reducible on $\mathcal{R}_F^+(a)$. Consequently, σ is reducible on $\mathcal{R}_F^+(a)$. As σ is effectively a representation of a finite p-group, it is induced from a representation of an open normal subgroup of $\mathcal{W}_F$, of index p and containing $\mathcal{R}_F^+(a)$. That is, there is a nontrivial character ϕ of $\mathcal{W}_F$, trivial on $\mathcal{R}_F^+(a)$, such that $\sigma \otimes \phi \cong \sigma$. Therefore, $c \leqslant \mathrm{sw}(\phi) \leqslant a$, giving $c = a$, as required. $\square$

The proof of the theorem relies on σ being absolutely wild, but the result extends to the general case of $\sigma \in \widehat{\mathcal{W}}_F^{\mathrm{wr}}$.

Corollary. *Let $\sigma \in \widehat{\mathcal{W}}_F^{\mathrm{wr}}$ have dimension p^r, $r \geqslant 1$. Let T_I/F be the imprimitivity field of σ, and set $e = e(T_I|F)$. The least discontinuity a of Σ'_σ is given by*

$$a = \min\{\mathrm{sw}(\chi)/e : \chi \in D(\sigma^{T_I}), \chi \neq 1\}.$$

In particular, a is p-integral.

Proof. We apply the theorem to the absolutely wild representation $\sigma^T \in \widehat{\mathcal{W}}_T^{\mathrm{wr}}$, where T/F is the tame centric field of σ. If c is the least discontinuity of Σ'_σ, then $e(T|F)c$ is that of Σ'_{σ^T}. If $\phi \in D(\sigma^T)$, then $\phi = \chi|_{\mathcal{W}_T}$, for a unique $\chi \in D(\sigma^{T_I})$ (8.2 Proposition), and $\mathrm{sw}(\phi) = e(T|T_I)\,\mathrm{sw}(\chi)$. $\square$

8.4. Consider, as an example, the case where $\deg\Theta = p$, $\Theta \in \mathcal{E}(F)$. Thus Θ is either essentially tame or totally wild. The first case is covered by 7.7, so we assume Θ to be totally wild. Write $m_\Theta = m/p$. Twisting with a character of $F^\times$ changes nothing, so we further assume $m \not\equiv 0 \pmod p$. This case is analyzed in [22], to which we refer for details.

Directly from (4.4.1) and 4.1 Proposition, we have $\Phi_\Theta(0) = m(p-1)/p^2$ and $\Phi'_\Theta(x) = p^{-1}$, $0 < x < m/p$. On the other side, $\Sigma_{L\Theta}(0) = \Phi_\Theta(0)$. So, in this p-dimensional case, the Herbrand function Ψ_Θ may be read directly from the decomposition function Σ_σ. In particular, the derivatives Ψ'_Θ, Σ'_σ have the same discontinuities in the region $0 < x < m_\Theta = \varsigma(\sigma)$.

The only possible values for $\Sigma'_{L\Theta}$ are p^{-2}, p^{-1} and 1. The function Σ'_σ has either one or two discontinuities. If it has two discontinuities, at $a < b$, say, then $\sigma \mid \mathcal{R}_F(a)$ is irreducible, $\sigma \mid \mathcal{R}_F^+(b)$ is a multiple of a character, leaving only the possibility that $\sigma \mid \mathcal{R}_F^+(a)$ is a sum of p distinct characters. If σ is absolutely wild, it is induced from a character χ of $\mathcal{W}_K$, with K/F cyclic. The extension K/F is uniquely determined, and χ is determined up to conjugation by elements of $\mathrm{Gal}(K/F)$. Moreover, $\mathrm{sw}(\chi) = a$.

If Σ'_σ has only one discontinuity a, examination of the graph of Ψ_Θ reveals that $a = m/(p+1)$. In this case, σ is, loosely speaking, a "Heisenberg representation." More exactly, the finite p-group $\sigma(\mathcal{P}_F)$ is *extra special of class* 2.

8.5. We mention a couple of examples, with the aim of giving the reader a broader perspective.

In dimension p^r, $r \geqslant 2$, the Herbrand function can be quite complicated. As an example, take $p = 2$ and assume that F contains a primitive cube root of unity. Let $\tau \in \widehat{\mathcal{W}}_F^{\mathrm{wr}}$ have dimension 2 and Swan exponent 1. Thus τ belongs to a class of representation, sometimes called "epipelagic," analyzed completely in [8]. In the present case, there is a simple stratum $[\mathfrak{a}, 1, 0, \alpha]$ in $\mathrm{M}_2(F)$ such that $[\tau; 0]^+ = {}^L\Theta$, where Θ is the endo-class of the unique simple character $\theta \in \mathcal{C}(\mathfrak{a}, \alpha)$. There exists $\tau' \in \widehat{\mathcal{W}}_F^{\mathrm{wr}}$, satisfying the same conditions, attached to a simple stratum $[\mathfrak{a}, 1, 0, \alpha']$, such that $(\det \alpha')^3 \not\equiv (\det \alpha)^3 \pmod{U_F^1}$. The derivatives Σ'_τ, $\Sigma'_{\tau'}$ each have a single discontinuity, lying at $\frac{1}{3}$. A short argument, starting from 5.1 Theorem and 5.2 Corollary of [8], shows that $\sigma = \tau \otimes \tau'$ is irreducible and that Σ'_σ has a unique discontinuity, again at $\frac{1}{3}$. Also, $\varsigma(\sigma) = \varsigma(\tau) = \varsigma(\tau') = \frac{1}{2}$, so $\mathrm{sw}(\sigma) = 2$.

Define $\Xi \in \mathcal{E}(F)$ by ${}^L\Xi = [\sigma; 0]^+$. This is the endo-class of some $\xi \in \mathcal{C}(\mathfrak{A}, \beta)$, where $[\mathfrak{A}, 2, 0, \beta]$ is a simple stratum in $\mathrm{M}_4(F)$ in which $F[\beta]/F$ is totally ramified of degree 4. The only possibility is $k_0(\beta, \mathfrak{A}) = -1$, so we can write down Φ_Ξ directly from its definition. We find that Φ'_Ξ has a unique discontinuity at $\frac{1}{4}$ and $\Phi_\Xi(0) = \frac{5}{16}$. Summarizing,

Example 1. In the range $0 < x < m_\Xi = \frac{1}{2}$, the function Ψ'_Ξ has discontinuities at $\frac{1}{3}$ and $\frac{3}{8}$. Consequently,

$$\Psi'_\Xi(x) = \begin{cases} \frac{1}{4}, & 0 < x < \frac{1}{3}, \\ 4, & \frac{1}{3} < x < \frac{3}{8}, \\ 2, & \frac{3}{8} < x < \frac{1}{2}. \end{cases}$$

One can push this case just one step further, with surprising consequences. There is a third representation $\tau'' \in \widehat{\mathcal{W}}_F^{\mathrm{wr}}$ of the same sort, attached to a stratum $[\mathfrak{a}, 1, 0, \alpha'']$ in $\mathrm{M}_2(F)$, such that $(\det \alpha'')^3$ is not congruent to either $(\det \alpha)^3$ or $(\det \alpha')^3$ modulo U_F^1. Applying Theorem 1 of [4], one shows

Example 2. The representation $\rho = \tau \otimes \tau' \otimes \tau''$ is irreducible, totally wild, and $\mathrm{sw}(\rho) = 3$. Let $[\rho; 0]^+ = {}^L\Upsilon$, $\Upsilon \in \mathcal{E}(F)$. In the range $0 < x < m_\Upsilon = \frac{3}{8}$, each of the functions $\Psi'_\Upsilon(x)$, $\Sigma'_\rho(x)$ has a unique discontinuity, which occurs at $x = \frac{1}{3}$.

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