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On the stability threshold for the 3D Couette flow in Sobolev regularity

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Abstract

We study Sobolev regularity disturbances to the periodic, plane Couette flow in the 3D incompressible Navier-Stokes equations at high Reynolds number $\mathbf{Re}$. Our goal is to estimate how the stability threshold scales in $\mathbf{Re}$: the largest the initial perturbation can be while still resulting in a solution that does not transition away from Couette flow. In this work we prove that initial data that satisfies $\|u_{\text{in}}\|_{H^\sigma} \leq \delta \mathbf{Re}^{-3/2}$ for any $\sigma > 9/2$ and some $\delta = \delta(\sigma) > 0$ depending only on σ is global in time, remains within $O(\mathbf{Re}^{-1/2})$ of the Couette flow in L^2 for all time, and converges to the class of "2.5-dimensional" streamwise-independent solutions referred to as *streaks* for times $t \gtrsim \mathbf{Re}^{1/3}$. Numerical experiments performed by Reddy et. al. with "rough" initial data estimated a threshold of $\sim \mathbf{Re}^{-31/20}$, which shows very close agreement with our estimate.

Contents

1. **Introduction**

1.1. *Presentation of the problem.* We consider the 3-dimensional Navier-Stokes equation with inverse Reynolds number $\nu = \mathbf{Re}^{-1} > 0$

$$\begin{cases} \partial_t v - \nu\Delta v + v \cdot \nabla v = -\nabla p, \\ \nabla \cdot v = 0 \end{cases}$$

set on $\mathbb{T} \times \mathbb{R} \times \mathbb{T}$; in other words, $v(t,x,y,z) \in \mathbb{R}^3$ and $p(t,x,y,z) \in \mathbb{R}$ are functions of $(t,x,y,z) \in \mathbb{R}_+ \times \mathbb{T} \times \mathbb{R} \times \mathbb{T}$. (The torus $\mathbb{T}$ is the periodized interval $[0,1]$.) The simplest nontrivial stationary solution is the *Couette flow* $(y,0,0)^t$. Despite the apparent simplicity, understanding the stability of this flow at high Reynolds number ($\nu \to 0$) is of enduring interest as a canonical, but subtle, problem in hydrodynamic stability and has been studied regularly throughout the history of fluid mechanics (along with several variants); see, e.g., [Kel87], [Rom73], [OK80], [TA92], [TTRD93], [RSBH98], [Cha02], [LK02] for a small representative subset or the texts [DR81], [SH01], [Yag12] and the references therein.

Denoting u for the perturbation of the Couette flow (that is, we set $v = (y,0,0)^t + u$), then it satisfies

(1.1)
$$\begin{cases} \partial_t u - \nu\Delta u + y\partial_x u + \begin{pmatrix} u^2 \\ 0 \\ 0 \end{pmatrix} - \nabla\Delta^{-1} 2\partial_x u^2 = -u \cdot \nabla u + \nabla\Delta^{-1}(\partial_i u^j \partial_j u^i), \\ \nabla \cdot u = 0, \\ u(t=0) = u_{\mathrm{in}}. \end{cases}$$

In this work, we want to answer the following question in the inviscid limit $\nu \to 0$:

> *Given σ, what is the smallest $\gamma > 0$ such that if the initial perturbation is such that $\|u_{\mathrm{in}}\|_{H^\sigma} = \varepsilon < \nu^\gamma$, then u remains close to the Couette flow (in a suitable sense) and converges back to the Couette flow as $t \to \infty$?*

Hence, the goal is not just to prove that the 3D Couette flow is nonlinearly stable in a suitable sense (this is straightforward for (1.1)) but to estimate the *stability threshold* — the size of the *largest* ball around zero in H^σ such that all solutions remain close to Couette. It is also of interest to determine the dynamics of solutions near the threshold [SH01].

1.2. *Background and previous work.* Understanding the stability and instability of laminar shear flows at high Reynolds number has been a classical question in applied fluid mechanics since the early experiments of Reynolds [Rey83]; see, e.g., the texts [DR81], [SH01], [Yag12]. In 3D hydrodynamics, one of the most ubiquitous phenomena is that of *subcritical transition*: when a laminar flow becomes unstable and transitions to turbulence in experiments or computer simulations at sufficiently high Reynolds number despite perhaps being spectrally stable. In fact, the flows in question can be nonlinearly asymptotically stable at all Reynolds numbers, despite being unstable for all practical purposes [Rom73]; see also [KLH94], [LK02]. It was suggested by Lord Kelvin [Kel87] that indeed the flow may be stable, but the stability threshold is decreasing as $\nu \to 0$, resulting in transition at a finite Reynolds number in any real system. Hence, the goal is, *given a norm* $\|\cdot\|_X$, to determine a $\gamma = \gamma(X)$ such that

$$\begin{aligned} \|u_{\text{in}}\|_X \lesssim \nu^\gamma &\implies \text{stability}, \\ \|u_{\text{in}}\|_X \gg \nu^\gamma &\implies \text{possible instability}. \end{aligned}$$

Of course we do not know *a priori* that the stability threshold is a power law. In the applied literature, γ is often referred to as the *transition threshold*. The γ is expected to depend nontrivially on the norm X (as observed in, for example, the numerical experiments of [RSBH98]).

Many works in applied mathematics and physics have been devoted to estimating γ; see, e.g., [BGM15a], [SH01], [Yag12] and the references therein. The linearized problem is nonnormal and permits several kinds of transient growth mechanisms:

(A) a transient un-mixing effect known as the Orr mechanism, noticed by Orr in 1907 in the context of 2D Couette flow [Orr07];
(B) the 3D lift-up effect, which rearranges mean streamwise momentum to deform the shear flow away from Couette, noticed first by Ellingsen and Palm [EP75] (see also [Lan80]);
(C) the transient growth of higher derivatives due to mixing; and
(D) a transient vorticity stretching.

Trefethen et. al. [TTRD93] considered the implications that nonnormal effects could have in the weakly nonlinear regime, in particular, forwarding the idea that the nonlinearity could repeatedly re-excite the transient growth, producing a "nonlinear bootstrap" scenario. The authors of [TTRD93] conjecture that $\gamma > 1$ for (1.1); a number of works have taken these, and related, ideas further to make conjectures generally giving $1 \leq \gamma \leq 7/4$; see, e.g., [GG94], [BDT95], [Wal95], [BT97], [LHR94], [Cha02]. Unfortunately, many of these authors do not carefully consider how the regularity of the initial data may affect the answer, despite the fact that the strength of the transient growth mechanisms is deeply tied to the regularity since the Couette flow can move information from small scales to large scales. (See Section 1.4 or [BM13], [BGM15a]; in fact, the sensitivity was noted by Reynolds [Rey83].) However, a few take the regularity into account, in particular, Reddy et. al. [RSBH98], where numerical experiments estimated $\gamma \approx 5/4$ for smooth initial data and

$\gamma \approx 31/20$ for "noisy" data. More recent numerical experiments have since suggested $\gamma \approx 1$ for smooth data [DBL10].

In this paper we consider Sobolev regularity data and prove that if the initial perturbation satisfies $\|u_{\mathrm{in}}\|_{H^\sigma} \le \delta\nu^{3/2}$ for $\sigma > 9/2$ and δ depending only on σ, then the solution stays within $O(\nu^{1/2})$ of the Couette flow, is attracted back to the class of x-independent solutions (referred to here as *streaks*) for $t \gtrsim \nu^{-1/3}$, and finally converges back to equilibrium as $t \to \infty$. Note that this result is very closely matched by the numerical estimate $\gamma \approx 31/20$ of [RSBH98]; see Remark 1.2 below for more discussions on regularity and the over-estimations in numerical experiments. The main result is stated in Theorem 1.1 below, the main bootstrap argument is set up in Section 2, and the requisite estimates constitute the remainder of the paper.

The main stabilizing effect is the *mixing-enhanced dissipation* wherein the mixing due to the Couette flow results in anomalously fast dissipation timescales (first derived by Lord Kelvin [Kel87]); see Section 1.4 for more discussion or previous works such as [RY83], [DN94], [LB01], [BL94], [BBG01], [BW13], [CKRZ08], [BMV16], [BCZ15]. ([DN94] are the first to the authors' knowledge to observe that this is important for understanding (1.1).) *Inviscid damping*, first derived by Orr [Orr07] in 2D and later noticed to be a hydrodynamic analogue of Landau damping (see, e.g., [BS03], [Ryu99], [MV11]), also plays a role in suppressing certain nonlinear effects.

Nonlinear stability of the Couette flow in Sobolev topology has been considered previously in the case of the bounded, infinite channel, that is, $y \in [-1, 1]$ and $x \in \mathbb{R}$ (which can of course lead to further complications, due to the presence of boundary layers), first by Romanov [Rom73], with later improvements by [KLH94] and [LK02]. This last paper seems to give the best mathematically rigorous result to date for this geometry, namely $\gamma \le 4$. In [BGM15a], [BGM15b], we study the stability threshold in Gevrey-α for $\alpha \in (1, 2)$ for (1.1). (Gevrey class was first introduced in [Gev18].) Roughly speaking, in [BGM15a] we prove that $\gamma = 1$ in these topologies (consistent with the numerical results of [DBL10]) and in [BGM15b] we study the dynamics of solutions that are as large as $\nu^{2/3-\delta}$. Note that the numerical over-estimation of [RSBH98], 5/4 vs. 1, is more pronounced in Gevrey than in Sobolev; see Remark 1.2.

All previous work in fluid mechanics and kinetic theory that depend on mixing as the stabilizing mechanism in models with strong nonlinear resonances are in infinite regularity. (Indeed, the resonances in (1.1) are far more problematic than those in 2D Navier-Stokes/Euler [BM13] or Vlasov-Poisson [MV11].) In this work we are looking for the boundary (in terms of γ) between when finite regularity results are possible and when infinite regularity seems to be required; see Section 1.6 for a more indepth discussion of the relationship between this work and previous related infinite regularity results in [MV11], [BMM16], [BM13], [BMV16], [You16], [BGM15a], [BGM15b]. We remark that there exists some finite regularity results in certain kinetic theory models [FR16], [FG16], [Die16], however, this is possible only because the nonlinearities being studied satisfy stringent nonresonance conditions.

1.3. *Streak solutions.* The first basic property to notice about (1.1) is that it admits a wide class of so-called "2.5-dimensional" solutions, which are often referred to as *streaks*, due to the streak-like appearance of the relatively fast fluid in experiments and computations [TTRD93], [SH01], [TE05], [BDDM98]. We will see that all solutions below the threshold converge to these streak solutions for $t \gtrsim \nu^{-1/3}$, and hence these solutions describe the fully 3D nonlinear dynamics for long times.

PROPOSITION 1.1 (Streak solutions). *Let* $\nu \in [0,\infty)$, $u_{\mathrm{in}} \in H^{5/2+}$ *be divergence-free and independent of* x, *that is,* $u_{\mathrm{in}}(x,y,z) = u_{\mathrm{in}}(y,z)$, *and denote by* $u(t)$ *the corresponding unique strong solution to* (1.1) *with initial data* u_{in}. *Then* $u(t)$ *is global in time, and for all* $T>0$, $u(t) \in L^\infty((0,T);H^{5/2+}(\mathbb{R}^3))$. *Moreover, the pair* $(u^2(t),u^3(t))$ *solves the* 2D *Navier-Stokes/Euler equations in* $(y,z)\in\mathbb{R}\times\mathbb{T}$:

$$\partial_t u^i + (u^2,u^3)\cdot\nabla u^i = -\partial_i p + \nu\Delta u^i, \qquad i\in\{2,3\}, \tag{1.2a}$$

$$\partial_y u^2 + \partial_z u^3 = 0, \tag{1.2b}$$

and u^1 *solves the* (*linear*) *forced advection-diffusion equation*

$$\partial_t u^1 + (u^2,u^3)\cdot\nabla u^1 = -u^2 + \nu\Delta u^1. \tag{1.3}$$

1.4. *Linear effects.* Four linear effects will play a key role in the analysis to come: lift up, inviscid damping, enhanced dissipation, and vortex stretching. We present quickly the linearized problem and how these four effects arise.

1.4.1. *The linearized problem.* The linearized problem reads

$$\begin{cases}\partial_t u - \nu\Delta u + y\partial_x u + \begin{pmatrix} u^2 \\ 0 \\ 0\end{pmatrix} - \nabla\Delta^{-1}2\partial_x u^2 = 0,\\ u(t=0)=u_{\mathrm{in}}.\end{cases}$$

Switch to the independent variables $(\overline{x},y,z)=(x-ty,y,z)$ by setting $\overline{u}(t,\overline{x},y,z) = u(t,x,y,z)$; it solves

$$\begin{cases}\partial_t \overline{u} - \nu\Delta_L\overline{u} + \begin{pmatrix} \overline{u}^2 \\ 0 \\ 0\end{pmatrix} - \nabla_L\Delta_L^{-1}2\partial_{\overline{x}}\overline{u}^2 = 0,\\ \overline{u}(t=0)=\overline{u}_{\mathrm{in}},\end{cases} \tag{1.4}$$

where $\nabla_L = (\partial_{\overline{x}}, \partial_y - t\partial_{\overline{x}}, \partial_z)$ and $\Delta_L = \nabla_L\cdot\nabla_L$.

1.4.2. *Lift up*. Consider first the projection onto zero frequencies in $\bar{x}$ (equivalently x) of the above equation. (For a function $f(t, x, y, z)$, we denote $f_0(y, z) = \int f(x, y, z)\, dx$.) Note that $\bar{u}_0 = u_0$, and hence it reads

$$\begin{cases} \partial_t u_0 - \nu \Delta u_0 = -\begin{pmatrix} u_0^2 \\ 0 \\ 0 \end{pmatrix}, \\ u_0(t=0) = (u_{\mathrm{in}})_0. \end{cases}$$

The solution of this linear problem is given by

$$u = \begin{pmatrix} e^{\nu t \Delta}\left[(u_{\mathrm{in}}^1)_0 - t(u_{\mathrm{in}}^2)_0\right] \\ e^{\nu t \Delta}(u_{\mathrm{in}}^2)_0 \\ e^{\nu t \Delta}(u_{\mathrm{in}}^3)_0 \end{pmatrix}$$

The linear growth predicted by this formula for times $t \lesssim \frac{1}{\nu}$ is known as the *lift up effect*, and was first noticed by Ellingsen and Palm [EP75] (see also [Lan80]). This nonnormal transient growth turns out to be a primary source of instability in (1.1) for small data; note also that this effect is not present in 2D due to the vanishing of u_0^2 by incompressibility in that case. For smooth data of size ε, we can expect at best the bounds,

$$\|u_0^1\|_{L^\infty H^s}^2 + \nu \|u_0^1\|_{L^2 H^s}^2 \lesssim \left(\frac{\varepsilon}{\nu}\right)^2.$$

1.4.3. *Inviscid damping*. Turning now to nonzero frequencies in $\bar{x}$, denoted for a function $f(\bar{x}, y, z)$ by $f_{\neq} = f - f_0$, observe that the linearized problem satisfied by $\overline{q}^2_{\neq} = \Delta_L \overline{u}^2_{\neq}$ reads

$$\begin{cases} \partial_t \overline{q}^2 - \nu \Delta_L \overline{q}^2 = 0, \\ \overline{q}^2(t=0) = \overline{q}^2_{\mathrm{in}}. \end{cases} \tag{1.5}$$

For smooth data of size ε, this gives a global bound on q^2 of order ε. This unknown was first introduced by Kelvin [Kel87] and is often used when studying the stability of parallel shear flows; see, e.g., [Cha02], [SH01]. The velocity field can be recovered by the formula $\bar{u}^2_{\neq} = \Delta_L^{-1} \overline{q}^2_{\neq}$, or, in Fourier (denoting k, η, l for the dual variables of $\bar{x}$, y, z respectively)

$$\widehat{\overline{u}^2_{\neq}} = \frac{1}{k^2 + (\eta - kt)^2 + l^2} \widehat{\overline{q}^2_{\neq}}. \tag{1.6}$$

Due to the bound $\frac{1}{k^2 + (\eta - kt)^2 + l^2} \lesssim \frac{\langle \eta \rangle^2}{\langle kt \rangle^2}$, this leads to a decay estimate of the type

$$\|\overline{u}^2_{\neq}\|_{H^s} \lesssim \frac{1}{t^2} \left\| \overline{q}^2_{\neq} \right\|_{H^{s+2}}. \tag{1.7}$$

This decay mechanism is known as *inviscid damping*; indeed, notice that the decay rate is independent of ν and is true also for the linearized 3D Euler equations. For the nonlinear problem, we will mostly depend on $L^2_t H^s_x$ estimates, in which case we can expect estimates such as

$$\|t\bar{u}^2_{\neq}\|_{L^2H^s} \lesssim \varepsilon \tag{1.8}$$

(ε standing for the size of the data). The regularity loss in (1.7) is required to control the transient growth in (1.6) for $\eta k > 0$; modes that are tilted against the shear and are subsequently unmixed to large scales before being mixed to small scales. This nonnormal effect was pointed out by Orr [Orr07] in 2D, however, it will remain important in 3D. Orr referred to the time $t = \eta/k$ as the *critical time*, a terminology we also use below.

1.4.4. *Enhanced dissipation.* In order to understand enhanced dissipation better, consider the model scalar problem, such as that solved by $\bar{q}^2$ above in (1.5),

$$\begin{cases} \partial_t w_{\neq} - \nu\Delta_L w_{\neq} = 0, \\ w_{\neq}(t=0) = (w_{\mathrm{in}})_{\neq}. \end{cases}$$

Taking the Fourier transform, the problem can be recast as

$$\begin{cases} \partial_t \widehat{w_{\neq}} - \nu(k^2 + (\eta - kt)^2 + l^2)\widehat{w_{\neq}} = 0, \\ \widehat{w_{\neq}}(t=0) = \widehat{(w_{\mathrm{in}})_{\neq}}. \end{cases}$$

Thus $\widehat{w_{\neq}}(t,k,\eta,l) = e^{-\nu\int_0^t (k^2+(\eta-k\tau)^2+l^2)d\tau}\widehat{(w_{\mathrm{in}})_{\neq}}$. Due to the inequality

$$\int_0^t (k^2 + (\eta - k\tau)^2 + l^2)d\tau \gtrsim t^3,$$

for the linear problem we get the decay

$$\|w_{\neq}\|_{H^s} \lesssim \varepsilon e^{-c\nu t^3}.$$

This decay is much faster than the standard viscous dissipation; indeed, the characteristic time scale for dissipation in nonzero-in-x modes is order $\sim \nu^{-1/3}$ instead of ν^{-1}. We refer to this phenomenon as *enhanced dissipation*; as mentioned above, it has been studied in several contexts previously; see, e.g., [RY83], [DN94], [LB01], [CKRZ08], [BW13], [BMV16], [BCZ15]. In this work, we will use L^2 time-integrated estimates of the type

$$\|w_{\neq}\|_{L^2H^s} \lesssim \frac{\varepsilon}{\nu^{1/6}} \quad \text{and} \quad \|tw_{\neq}\|_{L^2H^s} \lesssim \frac{\varepsilon}{\sqrt{\nu}}.$$

1.4.5. *Vorticity stretching and kinetic energy cascade.* The control of $\bar{q}^2$ provides the rapid decay of $\bar{u}^2$ via inviscid damping, which can then be integrated to understand the evolution of $\bar{u}^1$ and $\bar{u}^3$ in (1.4). In particular, we see that for times $1 \ll t \ll \nu^{-1/3}$, $\bar{u}^{1,3}_{\neq}$ are essentially time-independent, and hence over these times $u^{1,3}_{\neq}$ are being mixed like a passive scalar by the Couette flow. Hence, over these time scales we see a forward cascade of *kinetic energy.* (This persists on the nonlinear level as well [BGM15a].) Due to the negative order of the Biot-Savart law, it is easy to see that a forward cascade of kinetic energy is only possible if there is an accompanying vorticity stretching; this can also be confirmed by studying (1.4) in vorticity form.

Finally, we summarize the linear behavior here.

PROPOSITION 1.2 (Linearized Navier-Stokes). *Let $u_{\rm in}$ be a divergence-free, smooth vector field. The solution to the linearized Navier-Stokes $u(t)$ with initial data $u_{\rm in}$ satisfies the following for some $c \in (0,1/3)$:*

$$\left\| \bar{u}^2_{\neq}(t) \right\|_{H^\sigma} \lesssim \langle t \rangle^{-2} e^{-c\nu t^3} \left\| u^2_{\rm in} \right\|_{H^{\sigma+2}}, \tag{1.9a}$$

$$\left\| \bar{u}^{1,3}_{\neq}(t) \right\|_{H^\sigma} \lesssim e^{-c\nu t^3} \left\| u_{\rm in} \right\|_{H^{\sigma+7}} \tag{1.9b}$$

and the formulas

$$u^1_0(t,y,z) = e^{\nu t \Delta} \left(u^1_{in\ 0} - t u^2_{in\ 0} \right), \tag{1.10a}$$

$$u^2_0(t,y,z) = e^{\nu t \Delta} u^2_{in\ 0}, \tag{1.10b}$$

$$u^3_0(t,y,z) = e^{\nu t \Delta} u^3_{in\ 0}. \tag{1.10c}$$

1.5. *Statement of results.* We now state our main results.

THEOREM 1.1. *For all $\sigma > 9/2$, there exists $\delta = \delta(\sigma)$ such that if $\nu \in (0,1)$ and $u_{\rm in}$ is divergence-free with*

$$\varepsilon = \| u_{\rm in} \|_{H^\sigma} < \delta \nu^{3/2}, \tag{1.11}$$

then the resulting strong solution to (1.1) is global in time and there exists a function $\psi(t,y,z)$ satisfying

$$\|\psi\|^2_{L^\infty H^\sigma} + \nu \|\nabla \psi\|^2_{L^2 H^\sigma} \lesssim \frac{\varepsilon^2}{\nu^2},$$

such that, denoting by U^i, $i \in \{1,2,3\}$ the velocity field u^i in the new coordinates

$$U^i(t, x - ty - t\psi(t,y,z), y + \psi(t,y,z), z) = u^i(t,x,y,z),$$

the solution $u(t)$ to (1.1) with initial data $u_{\rm in}$ is global in time and satisfies the following estimates:

$$\left\| u^1_0 \right\|_{L^\infty H^\sigma} + \sqrt{\nu} \left\| \nabla u^1_0 \right\|_{L^2 H^\sigma} \lesssim \frac{\varepsilon}{\nu}, \tag{1.12a}$$

$$\left\| u^{2,3}_0 \right\|_{L^\infty H^\sigma} + \sqrt{\nu} \left\| \nabla u^{2,3}_0 \right\|_{L^2 H^\sigma} \lesssim \varepsilon, \tag{1.12b}$$

$$\left\| U^2_{\neq} \right\|_{L^\infty H^{\sigma-2}} + \left\| \nabla_L U^2_{\neq} \right\|_{L^2 H^{\sigma-3}} + \left\| t U^2_{\neq} \right\|_{L^2 H^{\sigma-4}} \lesssim \varepsilon, \tag{1.12c}$$

$$\left\| U^1_{\neq} \right\|_{L^\infty H^{\sigma-3}} + \sqrt{\nu} \left\| t U^1_{\neq} \right\|_{L^2 H^{\sigma-4}} \lesssim \varepsilon, \tag{1.12d}$$

$$\left\| U^3_{\neq} \right\|_{L^\infty H^{\sigma-2}} + \sqrt{\nu} \left\| t U^3_{\neq} \right\|_{L^2 H^{\sigma-3}} \lesssim \varepsilon. \tag{1.12e}$$

Remark 1.1. The latter terms in (1.12d) and (1.12e) emphasize the effect of enhanced dissipation, discussed above in Section 1.4. In particular, the scaling of the $L^2H^{\sigma-4}$ norm of $tU^i_{\neq}$ is far better at small ν than what would be true of the heat equation. The second two estimates in (1.12c) emphasize the effect of inviscid damping: notice indeed that the decay does not depend on ν.

Remark 1.2. How optimal are the assumptions of the theorem?

- As mentioned previously, numerics in [RSBH98] estimated a threshold for "noisy data" at $\varepsilon \sim \nu^{31/20}$; Theorem 1.1 shows that the stability threshold is slightly better. In light of the numerical evidence, it is reasonable to conjecture that Theorem 1.1 is sharp in terms of γ over some range of Sobolev spaces.
- By parabolic smoothing, it should be possible to slightly weaken (1.11) to something like $u_{\text{in}} = u_S + u_R$ with $\|u_S\|_{H^{9/2+}} + C\nu^{\frac{9}{4}-\frac{\alpha}{2}}\|u_R\|_{H^\alpha} < \delta\nu^{3/2}$ for a universal C at least over some range of $\alpha \in (5/2, 9/2)$. This is a local-in-time effect that is totally independent of Theorem 1.1 (though it may be a nontrivial refinement of the local theory for (1.1)). This is qualitatively consistent with the numerical over-estimation observed in [RSBH98] and others: numerical algorithms will inevitably introduce noise at the smallest scales of the simulation and hence possibly over-estimate γ; indeed, more recent computations carried out in [DBL10] are closer to the $\gamma \approx 1$ in the case of smooth data. This also suggests that the Sobolev regularity γ is more robust to low-regularity noise than the infinite regularity γ (which requires exponentially small noise [BGM15a]), which is consistent with the mentioned numerical observations.

1.6. *Brief discussion of the results and new ideas.* Our work shows that it may now be feasible to build a mathematical theory of subcritical instabilities in fluid mechanics and possibly also in related fields, such as magnetohydrodynamics. This seems especially possible in finite regularity, as the methods here are significantly more tractable than those in infinite regularity [BGM15a], [BGM15b]. Indeed, in the proof of Theorem 1.1, we need to use methods that differ significantly from those used in the infinite regularity works [MV11], [BMM16], [BM13], [BMV16], [You16], [BGM15a], [BGM15b]. In all of these previous works, the infinite regularity class is used to absorb the potential frequency cascade due to weakly nonlinear effects in a process related to classical Cauchy-Kovalevskaya-type arguments in, e.g., [Nir72], [Nis77], [FT89], [LO97] (see Section 2.3 for more precise discussions) or, in the case of [MV11], via a Nash-Moser-type iteration. Here this is clearly not an option, and hence we need to rule out any such cascade with the *least possible* amount of dissipation; something that will require a different kind of understanding of the weakly nonlinear effects in the pressure and a more precise understanding of the interplay between the enhanced dissipation and vortex stretching. The starting point for this is the linear analysis of Section 1.4, and based on this, Fourier multipliers that precisely encode the interplay between the dissipation and possible growth are designed. These multipliers are then used to make energy estimates that lose the minimal amount of information from the linear terms; see Section 2.3 for specifics and context with existing ideas in, e.g., [FT89], [Ali01], [BM13] and others. (In particular, we need multipliers that more precisely capture the effect of dissipation than in [BGM15a], [BGM15b].)

Once we have understood and quantified the linear terms, one needs to understand how this linear behavior interacts with the nonlinearity. For this, of critical importance in the proof is the precise structure of the nonlinearity, which contains a number of null structures. Similar to null forms for quasilinear wave equations, introduced in [Kla82], the null structures encountered in the present paper cancel possible interactions between large modes or derivatives of the solution. The simplest is that the nonlinearity in (1.1) does not allow u_0^1 to directly interact with itself in a nonlinear way (this is essentially how Proposition 1.1 works); however a similar structure also limits the way $u_{\neq}^1$ and u_0^1 interact. Another slightly more subtle structure is that, since the nonlinearity is comprised of forms of the type $u^j \partial_j u^i$, the large growth of y derivatives is crucially counter-balanced by the inviscid damping of u^2 in nonlinear terms. Indeed, this is why quantifying the inviscid damping of u^2 is important for the proof to work. Similarly, the $u^1\partial_x$ and $u^3\partial_z$ structure pairs less problematic derivatives with the more problematic $u^{1,3}$. Since the inviscid damping is important, a key physical mechanism to understand is how the streak and the kinetic energy cascade interact nonlinearly in the y derivative of the pressure, that is, the nonlinear term: $-\partial_y \Delta^{-1}\left(\partial_z u_0^1 \partial_x u_{\neq}^3\right)$. Controlling this term is one of the main challenges, which is done in Section 3.1.2, and in it, *all* of the linear effects outlined in Section 1.4 are playing a role (which is why it is very important that these are treated precisely). See Section 2 below for more details on the proof and techniques.

2. **Preliminaries and outline of the proof**

2.1. *Notation.*

2.1.1. *Miscellaneous.* Given two quantities A and B, we denote $A \lesssim B$ if there exists a constant C such that $A \le CB$. This constant might depend on σ, but not on δ, ν, C_0 or C_1 (the two latter quantities remain to be defined), provided that δ is chosen sufficiently small. That is, implicit constants such as, e.g., $C_0\delta$ are omitted for simplicity. We similarly denote $A \ll B$ if $A \le \delta_0 B$ for a small constant $\delta_0 \in (0,1)$ to emphasize the small size of the implicit constant. Finally, we write $\langle x\rangle = \sqrt{1+x^2}$.

2.1.2. *Fourier Analysis.* The Fourier transform of a function $f(X,Y,Z)$, denoted $\widehat{f}(k,\eta,\ell)$ or $\mathcal{F}f$, is such that

$$f(X,Y,Z) = \sum_k \int_{\eta\in\mathbb{R}} \sum_\ell \widehat{f}(k,\eta,\ell) e^{2\pi i(kX+\eta Y+\ell Z)}\, d\eta,$$
$$\widehat{f}(k,\eta,\ell) = \int_{X\in\mathbb{T}} \int_{Y\in\mathbb{R}} \int_{Z\in\mathbb{T}} f(X,Y,Z) e^{-i2\pi(kX+\eta Y+\ell Z)}\, dX\, dY\, dZ.$$

The Fourier multiplier with symbol $m(k,\eta,\ell)$ is such that

$$m(D)f = \mathcal{F}^{-1} m(k,\eta,\ell)\mathcal{F}f.$$

The projections on the zero frequency in X of a function $f(X,Y,Z)$ are denoted by

$$P_0 f = f_0 = \int f(X,Y,Z)\, dX,$$

while

$$P_{\neq} f = f_{\neq} = f - P_0 f.$$

2.1.3. *Functional spaces.* The Sobolev space H^N is given by the norm

$$\|f\|_{H^N} = \|\langle D\rangle^N f\|_{L^2}.$$

Recall that, for $s > \frac{3}{2}$, H^s is an algebra: $\|fg\|_{H^s} \lesssim \|f\|_{H^s}\|g\|_{H^s}$.

We will sometimes use the notation H^{s+} for $H^{s+\kappa}$, where κ can be taken arbitrarily small, with (implicit) constants depending on κ.

For a function of space and time $f = f(t,x)$, and times $a < b$, the Banach space $L^p(a,b;H^N)$ is given by the norm

$$\|f\|_{L^p(a,b;H^N)} = \left\| \|f\|_{H^N} \right\|_{L^p(a,b)}.$$

For simplicity of notation, we usually simply write $\|f\|_{L^pH^N}$ since the time-interval of integration in this work will be the same basically everywhere.

2.1.4. *Littlewood-Paley decomposition and paraproduct.* Start with θ a smooth, nonnegative function supported in the annulus $B(0,5)\setminus B(0,1)$ of $\mathbb{R}^3$, and such that $\sum_{j=-\infty}^{+\infty} \theta\left(\frac{\xi}{2^j}\right) = 1$ for $\xi \neq 0$, and define the Fourier multipliers

$$P_j = \theta\left(\frac{D}{2^j}\right), \qquad P_{\leq J} = \sum_{j=-\infty}^{J} \theta\left(\frac{D}{2^j}\right), \qquad P_{>J} = 1 - P_{\leq J}.$$

These Fourier multipliers enable us to split the product into two pieces such that each corresponds to the interaction of high frequencies of one function with low frequencies of the other:

$$fg = f_{Hi}g_{Lo} + f_{Lo}g_{Hi},$$

with

$$f_{Hi}g_{Lo} = \sum_j P_j f P_{\leq j} g, \qquad f_{Lo}g_{Hi} = \sum_j P_{\leq j-1} f P_j g.$$

(The lack of symmetry in this formula is irrelevant.) We record the estimate

$$\|f_{Hi}g_{Lo}\|_{H^s} \lesssim \|f\|_{H^s}\|g\|_{H^\sigma} \qquad \text{for } s>0,\ \sigma > \tfrac{3}{2}. \tag{2.1}$$

We further note that if g depends only on two variables, say y and z, then we have

$$\|f_{Hi}g_{Lo}\|_{H^s} \lesssim \|f\|_{H^s}\|g\|_{H^\sigma} \qquad \text{for } s>0,\ \sigma > 1. \tag{2.2}$$

2.2. *Re-formulation of the equations.* First, we reformulate the equations to make them more amenable to long-time, nonlinear analysis.

2.2.1. *Change of dependent variables.* In order to understand the linearized equation in Section 1.4, it is important to use the unknown $q^2 = \Delta u^2$. In linear or formal weakly nonlinear analyses (see, e.g., [Cha02, SH01] and the references therein) it is natural to couple q^2 with the vertical component of the vorticity, however, we will also need to change independent variables to adapt to the mixing caused by u_0^1, which makes this approach very problematic. Therefore, it is more convenient to work with the set of unknowns $q^i = \Delta u^i$ (as observed in [BGM15a]). These unknowns satisfy the system

$$\begin{cases} \partial_t q^1 + y\partial_x q^1 - \nu\Delta q^1 + 2\partial_{xy}u^1 + q^2 - 2\partial_{xx}u^2 \\ \qquad\qquad = -u\cdot\nabla q^1 - q^j\partial_j u^1 - 2\partial_i u^j \partial_{ij} u^1 + \partial_x\left(\partial_i u^j \partial_j u^i\right), \\ \partial_t q^2 + y\partial_x q^2 - \nu\Delta q^2 = -u\cdot\nabla q^2 - q^j\partial_j u^2 - 2\partial_i u^j \partial_{ij} u^2 + \partial_y\left(\partial_i u^j \partial_j u^i\right), \\ \partial_t q^3 + y\partial_x q^3 - \nu\Delta q^3 + 2\partial_{xy}u^3 - 2\partial_{xz}u^2 \\ \qquad\qquad = -u\cdot\nabla q^3 - q^j\partial_j u^3 - 2\partial_i u^j \partial_{ij} u^3 + \partial_z\left(\partial_i u^j \partial_j u^i\right), \\ q(t=0) = q_{\mathrm{in}}. \end{cases} \tag{2.3}$$

2.2.2. *Change of independent variables.* The x-component of the streak, u_0^1, is expected to be as large as $O(\varepsilon\nu^{-1})$ (again from Section 1.4), which is far too large to be balanced directly by the dissipation. (It is not hard to check this would require $\varepsilon \ll \nu^2$.) Hence, we remove the fast mixing action due to the streak itself, an approach also used in [BGM15a] for the same reason. There is essentially no choice in the change of coordinates we can employ — it is dictated uniquely by the desired properties and the structure of the equation. Although the coordinate transform is described in detail in [BGM15a], because it is still a central tool for our analysis here, we will describe briefly the motivation for its design. Define the coordinate transform with the ansatz as in [BGM15a],

$$X = x - ty - t\psi(t,y,z), \tag{2.4a}$$

$$Y = y + \psi(t,y,z), \tag{2.4b}$$

$$Z = z. \tag{2.4c}$$

We denote

$$\psi_y(t,Y,Z) = \partial_y\psi(t,y,z), \quad \psi_z(t,Y,Z) = \partial_z\psi(t,y,z).$$

To distinguish between old and new coordinates, we capitalize u^i and q^i, while ψ itself becomes C:

$$U^i(t,X,Y,Z) = u^i(t,x,y,z), \quad Q^i(t,X,Y,Z) = q^i(t,x,y,z),$$
$$C(t,Y,Z) = \psi(t,y,z),$$

where we are using the shorthand $X = X(t,x,y,z)$, $Y = Y(t,x,y,z)$, and $Z = Z(t,x,y,z)$. Notice that ψ_y, ψ_z, and C are related as follows:

$$\psi_y = \frac{\partial_Y C}{1-\partial_Y C} = \partial_Y C\sum_{j=0}^{\infty}(\partial_Y C)^j, \tag{2.5a}$$

$$\psi_z = \frac{\partial_Z C}{1-\partial_Y C} = \partial_Z C\sum_{j=0}^{\infty}(\partial_Y C)^j. \tag{2.5b}$$

In the new coordinates, differential operators are modified as follows: denoting $f(t,x,y,z) = F(t,X,Y,Z)$,

$$\begin{aligned}\nabla f(t,x,y,z) &= \begin{pmatrix}\partial_x f\\ \partial_y f\\ \partial_z f\end{pmatrix} = \begin{pmatrix}\partial_X F\\ (1+\psi_y)(\partial_Y - t\partial_X)F\\ (\partial_Z + \psi_z(\partial_Y - t\partial_X))F\end{pmatrix}\\ &= \begin{pmatrix}\partial_X^t F\\ \partial_Y^t F\\ \partial_Z^t F\end{pmatrix} = \nabla_t F(t,X,Y,Z).\end{aligned}$$

It will be useful to isolate the “linear part” of ∇_t (that is, the contribution associated with the linearized problem), which we denote ∇_L:

$$\nabla_L = \begin{pmatrix}\partial_X\\ \partial_Y - t\partial_X\\ \partial_Z\end{pmatrix} = \begin{pmatrix}\partial_X\\ \partial_Y^L\\ \partial_Z\end{pmatrix}.$$

Using the notation

$$\Delta_L = \nabla_L\cdot\nabla_L = \partial_X^2 + (\partial_Y^L)^2 + \partial_Z^2 \tag{2.6a}$$

$$G = (1+\psi_y)^2 + \psi_z^2 - 1, \tag{2.6b}$$

the Laplacian transforms as

$$\begin{aligned}\Delta f = \Delta_t F &= \left((\partial_X)^2 + (\partial_Y^t)^2 + (\partial_Z^t)^2\right) F \\ &= \Delta_L F + G\partial_{YY}^L F + 2\psi_z \partial_{ZY}^L F + \Delta_t C \partial_Y^L F.\end{aligned}$$

One of the motivations for the ansatz (2.4) is so that the symbol of Δ_t, $\sigma(\Delta_t)(t, y, k, \eta)$ (as a pseudo-differential operator) fails to be elliptic precisely when $\eta = kt$ — the same as Δ_L. (By convention, η is the wave number associated with Y and k is the wave number associated with X.) This property makes it possible to effectively approximate Δ_t^{-1} with Δ_L^{-1}, provided that ψ remains sufficiently small. (Lemmas of this type are outlined in Appendix A.2.) We will also need the modified Laplacian

$$\widetilde{\Delta}_t F = \Delta_t F - \Delta_t C \partial_Y^L F = \Delta_L F + G\partial_{YY}^L F + 2\psi_z \partial_{YZ}^L F. \tag{2.7}$$

Next, we describe how to choose ψ effectively. Suppose that f satisfies the passive scalar equation:

$$\partial_t f + y\partial_x f + u \cdot \nabla f = \nu \Delta f.$$

Then from the above considerations, we have

$$\partial_t F + \begin{pmatrix} u^1 - t(1+\partial_y\psi)u^2 - t\partial_z\psi u^3 - \partial_t(t\psi) + \nu t\Delta\psi \\ (1+\partial_y\psi)u^2 + \partial_z\psi u^3 + \partial_t\psi - \nu\Delta\psi \\ u^3 \end{pmatrix} \cdot \nabla_{X,Y,Z} F = \nu\widetilde{\Delta}_t F. \tag{2.8}$$

The primary contribution from the background streak in the velocity field is given by the $u_0^1 - tu_0^2$ in the first component, so it is natural to choose the contributions involving ψ to balance this by making the definition

$$\frac{d}{dt}(t\psi) + u_0 \cdot \nabla (t\psi) = u_0^1 - tu_0^2 + \nu t\Delta\psi, \tag{2.9a}$$

$$\lim_{t\to 0} t\psi(t) = 0. \tag{2.9b}$$

In fact, making a slightly different choice, e.g., by attempting to drop the higher order $u_0 \cdot \nabla(t\psi)$ term in (2.9a), does not seem to work, in the sense that ψ remains too large to get reasonable estimates over long times. Hence, (2.9a) appears to be the only feasible choice, given the ansatz (2.4). The mild coordinate singularity at $t = 0$ will be irrelevant, as this coordinate transform will only be applied for $t \geq 1$.

Let us now apply the choices (2.4) and (2.9a) to (2.3). Define g and $\widetilde{U}_0$ (which will be the X-independent part of the velocity in the new coordinates) by

$$g = \frac{1}{t}(U_0^1 - C), \quad \widetilde{U}_0 = \begin{pmatrix} 0 \\ g \\ U_0^3 \end{pmatrix}.$$

Computing from (2.9a), (2.3), and (1.1) gives the following system (we refer to [BGM15a] for more details):

(2.10)
$$\begin{cases} Q_t^1 - \nu\widetilde{\Delta}_t Q^1 + Q^2 + 2\partial_{XY}^t U^1 - 2\partial_{XX} U^2 \\ \qquad = -\widetilde{U}_0 \cdot \nabla Q^1 - U_{\neq} \cdot \nabla_t Q^1 - Q^j \partial_j^t U^1 - 2\partial_i^t U^j \partial_{ij}^t U^1 + \partial_X(\partial_i^t U^j \partial_j^t U^i), \\ Q_t^2 - \nu\widetilde{\Delta}_t Q^2 \\ \qquad = -\widetilde{U}_0 \cdot \nabla Q^2 - U_{\neq} \cdot \nabla_t Q^2 - Q^j \partial_j^t U^2 - 2\partial_i^t U^j \partial_{ij}^t U^2 + \partial_Y^t(\partial_i^t U^j \partial_j^t U^i), \\ Q_t^3 - \nu\widetilde{\Delta}_t Q^3 + 2\partial_{XY}^t U^3 - 2\partial_{XZ}^t U^2 \\ \qquad = -\widetilde{U}_0 \cdot \nabla Q^3 - U_{\neq} \cdot \nabla_t Q^3 - Q^j \partial_j^t U^3 - 2\partial_i^t U^j \partial_{ij}^t U^3 + \partial_Z^t(\partial_i^t U^j \partial_j^t U^i), \end{cases}$$

coupled with the equations that must be solved to find the coordinate system itself:

$$\begin{cases} \partial_t C + \widetilde{U}_0 \cdot \nabla C = g - U_0^2 + \nu\widetilde{\Delta}_t C, \\ \partial_t g + \widetilde{U}_0 \cdot \nabla g = -\frac{2}{t} g - \frac{1}{t}(U_{\neq} \cdot \nabla_t U_{\neq}^1)_0 + \nu\widetilde{\Delta}_t g. \end{cases} \tag{2.11}$$

Although most work is done directly on the system (2.10), (2.11), for certain steps it will be useful to use the momentum form of the equations

(2.12)
$$\partial_t U - \nu\widetilde{\Delta}_t U + \widetilde{U}_0 \cdot \nabla U + U_{\neq} \cdot \nabla_t U = \begin{pmatrix} -U^2 \\ 0 \\ 0 \end{pmatrix} + \nabla_t \Delta_t^{-1} 2\partial_X U^2 + \nabla_t \Delta_t^{-1}(\partial_i^t U^j \partial_j^t U^i).$$

2.2.3. *Shorthand*. It will be quite convenient to use shorthand for the various terms appearing in the above equations and to be able to distinguish whether interacting modes have zero or nonzero X frequency. Let us start with linear terms, appearing in the equations for Q^k, $k = 1, 3$:

$$\begin{aligned} \mathrm{LU} &= Q^2 && \text{(lift up term)}, \\ \mathrm{LS} &= 2\partial_{XY}^t U^k && \text{(linear stretching term)}, \\ \mathrm{LP} &= -2\partial_{Xk}^t U^2 && \text{(linear pressure term)}. \end{aligned}$$

Next, consider the nonlinear terms in (2.10). In the following, i, j run in $\{1, 2, 3\}$, while ε_1 and ε_2 may be 0 or $\neq$):

$$\begin{aligned} \mathcal{T}_{0,\varepsilon_1} &= \widetilde{U}_0 \cdot \nabla Q_{\varepsilon_1}^k && \text{(transport term)}, \\ \mathcal{T}_{\neq,\varepsilon_1} &= U_{\neq} \cdot \nabla_t Q_{\varepsilon_1}^k && \text{(transport term)}, \\ \mathrm{NLS1}(j, \varepsilon_1, \varepsilon_2) &= Q_{\varepsilon_1}^j \partial_j^t U_{\varepsilon_2}^k && \text{(nonlinear stretching term)}, \\ \mathrm{NLS2}(i, j, \varepsilon_1, \varepsilon_2) &= 2\partial_i^t U_{\varepsilon_1}^j \partial_{ij}^t U_{\varepsilon_2}^k && \text{(nonlinear stretching term)}, \\ \mathrm{NLP}(i, j, \varepsilon_1, \varepsilon_2) &= \partial_k^t(\partial_j^t U_{\varepsilon_1}^i \partial_i^t U_{\varepsilon_2}^j) && \text{(nonlinear pressure term)}. \end{aligned}$$

We will often abuse notation and, for instance, denote indifferently NLS1 for this term and its contribution to an energy estimate. The origin of these terms is more or less clear except perhaps the "stretching" terminology, which is due to the similarity these terms have with the vortex stretching term in the 3D Navier-Stokes equations in vorticity form.

2.3. *The Fourier multipliers.* At this point, our work here will depart from the infinite regularity case [BGM15a].

The multiplier m: stretching versus dissipation. Our focus here is the following linear equation:

$$\partial_t f + 2\partial_{XY}^L \Delta_L^{-1} f - \nu \Delta_L f = 0,$$

which occurs as some of the main linear terms governing Q^1 and Q^3 in (2.10). This equation can be seen as a competition between the linear stretching term $2\partial_{XY}^L \Delta_L^{-1} f$ and the dissipation term $\nu \Delta_L f$. Taking the Fourier transform, it becomes

$$\partial_t \widehat{f} + 2\frac{k(\eta - kt)}{k^2 + (\eta - kt)^2 + \ell^2} \widehat{f} + \nu \left(k^2 + (\eta - kt)^2 + \ell^2\right) \widehat{f} = 0.$$

If $k \neq 0$, the factor $2\frac{k(\eta-kt)}{k^2+(\eta-kt)^2+\ell^2}$ is positive for $t < \frac{\eta}{k}$, in which case it amounts to damping on $\widehat{f}$; and it is negative for $t > \frac{\eta}{k}$, in which case it corresponds to an amplification of $\widehat{f}$. As for the factor $\nu \left(k^2 + (\eta - kt)^2 + \ell^2\right)$, this gives enhanced dissipation for $k \neq 0$. We start with the following inequality, which compares the sizes of these two factors: uniformly in (k, η, ℓ), if $k \neq 0$, then

$$\nu \left(k^2 + (\eta - kt)^2 + \ell^2\right) \gg \frac{|k(\eta - kt)|}{k^2 + (\eta - kt)^2 + \ell^2} \quad \text{if} \quad |t - \frac{\eta}{k}| \gg \nu^{-1/3}.$$

Indeed, $\frac{|k(\eta-kt)|}{\nu(k^2+(\eta-kt)^2+\ell^2)^2} \leq \frac{|t-\frac{\eta}{k}|}{\nu(1+|t-\frac{\eta}{k}|^2)^2}$, and it is easy to check that $\frac{x}{\nu(1+x^2)^2} \ll 1$ for $|x| \gg \nu^{-1/3}$.

To summarize, stretching overcomes dissipation if $0 < t - \frac{\eta}{k} \lesssim \nu^{-1/3}$. To deal with this range of t, we introduce the multiplier m. Define $m(t, k, \eta, \ell)$ by $m(t = 0, k, \eta, \ell) = 1$ and the following ordinary differential equation:

$$\frac{\dot{m}}{m} = \begin{cases} 0 & \text{if } t \notin \left[\frac{\eta}{k}, \frac{\eta}{k} + 1000\nu^{-1/3}\right], \\ \frac{2k(\eta-kt)}{k^2+(\eta-kt)^2+\ell^2} & \text{if } t \in \left[\frac{\eta}{k}, \frac{\eta}{k} + 1000\nu^{-1/3}\right]. \end{cases}$$

This multiplier is such that if f solves the above equation and $0 < t - \frac{\eta}{k} < 1000\nu^{-1/3}$, then mf solves

$$\partial_t (mf) - \nu \Delta_L (mf) = 0,$$

and this equation is perfectly well behaved! That is, the growth that f undergoes is balanced by the decay of the multiplier m; this is especially useful since the growth is highly anisotropic in frequency. Conveniently, it turns out that m is given by a closed formula:

(1) if $k = 0$: $m(t, 0, \eta, \ell) = 1$;
(2) if $k \neq 0$, $\frac{\eta}{k} < -1000\nu^{-1/3}$: $m(t, k, \eta, \ell) = 1$;
(3) if $k \neq 0$, $-1000\nu^{-1/3} < \frac{\eta}{k} < 0$:
   - $m(t, k, \eta, \ell) = \frac{k^2+\eta^2+\ell^2}{k^2+(\eta-kt)^2+\ell^2}$ if $0 < t < \frac{\eta}{k} + 1000\nu^{-1/3}$,
   - $m(t, k, \eta, \ell) = \frac{k^2+\eta^2+\ell^2}{k^2+(1000k\nu^{-1/3})^2+\ell^2}$ if $t > \frac{\eta}{k} + 1000\nu^{-1/3}$;

(4) if $k \neq 0$, $\frac{\eta}{k} > 0$:
   - $m(t, k, \eta, \ell) = 1$ if $t < \frac{\eta}{k}$,
   - $m(t, k, \eta, \ell) = \frac{k^2+\ell^2}{k^2+(\eta-kt)^2+\ell^2}$ if $\frac{\eta}{k} < t < \frac{\eta}{k} + 1000\nu^{-1/3}$,
   - $m(t, k, \eta, \ell) = \frac{k^2+\ell^2}{k^2+(1000k\nu^{-1/3})^2+\ell^2}$ if $t > \frac{\eta}{k} + 1000\nu^{-1/3}$.

Notice, in particular, that

$$\nu^{2/3} \lesssim m(t, k, \eta, \ell) \leq 1. \tag{2.13}$$

Further, we point out the following key inequality, which shows that the growth is exactly balanced by Δ_L:

$$m(t, k, \eta, l) \gtrsim \frac{k^2 + l^2}{k^2 + l^2 + |\eta - kt|^2}. \tag{2.14}$$

Additional multipliers bounded from below by a positive constant. We will use several additional multipliers, which unlike m, are bounded above and below uniformly in ν and frequency. Multipliers M^0 and M^1 are used to balance the growth due to the linear pressure terms as well as some of the leading order nonlinear terms. The multiplier M^2 plays an especially crucial role by compensating for the transient slow-down of the enhanced dissipation near the critical times, and hence this multiplier will be ultimately how we quantify accelerated dissipation *without* regularity loss — of crucial importance to our methods and not possible with the techniques employed in the infinite regularity works [BMV16], [BGM15a].

Define M^i, $i = 0, 1, 2$ as follows: $M^i(t = 0, k, \eta, \ell) = 1$ and

- if $k = 0$, $M^i(t, k, \eta, \ell) = 1$ for all t;
- if $k \neq 0$, $\frac{\dot{M}^0}{M^0} = \frac{-k^2}{k^2+\ell^2+(\eta-kt)^2}$;
- if $k \neq 0$, $\frac{\dot{M}^1}{M^1} = \frac{-2\langle k\ell\rangle}{k^2+\ell^2+(\eta-kt)^2}$;
- if $k \neq 0$, $\frac{\dot{M}^2}{M^2} = -\frac{\nu^{1/3}}{\left[\nu^{1/3}|t-\frac{\eta}{k}|\right]^{1+\kappa}+1}$,

where $\kappa \in (0, 1/2)$ is a small, fixed constant. It is easy to check that these multipliers satisfy

$$0 < c < M^i(t, k, \eta, l) \leq 1$$

for a universal constant c. Define then

$$M = M^0 M^1 M^2.$$

To see the usefulness of M, consider the weighted energy estimate

$$\frac{1}{2}\frac{d}{dt}\left\|MQ^3_{\neq}\right\|^2_{H^N} = -\left\|\sqrt{-\dot{M}M}Q^3_{\neq}\right\|^2_{H^N} + \langle MU^3, M\partial_t Q^3_{\neq}\rangle_{H^N}.$$

In order to bound the latter term, we may firstly use some of the negative term coming from $\dot{M}$, and secondly, if we can control the term by something like

$$\langle MQ^3, M\partial_t Q^3_{\neq}\rangle_{H^N} \leq \frac{1}{2}\left\|\sqrt{-\dot{M}M}Q^3_{\neq}\right\|^2_{H^N} - \frac{\nu}{2}\left\|\nabla_L MQ^3\right\|^2_{H^N} + \varepsilon^3\mathcal{E}(t),$$

where $\mathcal{E}$ is uniformly bounded in L^1_t, then *both* $\sqrt{\nu}\left\|\nabla_L MQ^3\right\|_{L^2H^N}$ and $\left\|\sqrt{-\dot{M}M}Q^3_{\neq}\right\|_{L^2H^N}$ are bounded! The usefulness of this estimate is emphasized by the following very important lemma (the proof of which is immediate from the definition of M^2), which shows how to deduce L^2 in time enhanced dissipation without losing any regularity.

Lemma 2.1. *For $k \neq 0$, there holds*

$$1 \lesssim \nu^{-1/6}\sqrt{-\dot{M}^2M^2}(k,\eta,l) + \nu^{1/3}\left|k,\eta - kt, l\right|.$$

As a corollary, the following holds for any f and $\alpha \geq 0$:

$$\|f_{\neq}\|_{L^2H^\alpha} \lesssim \nu^{-1/6}\left(\left\|\sqrt{-\dot{M}^2M^2}f_{\neq}\right\|_{L^2H^\alpha} + \nu^{1/2}\left\|\nabla_L f_{\neq}\right\|_{L^2H^\alpha}\right).$$

Note that Lemma 2.1 also holds with M^2 replaced by the full M, as will be used frequently below.

The use of norms with time-decaying norms is quite classical when working in infinite regularity; see, e.g., the Cauchy-Kowalevskaya theorems of [Nir72], [Nis77]. The use of dissipation-like terms that appear in L^2-based infinite regularity estimates goes back to [FT89]; see also related ideas in, e.g., [LO97], [KV09], [CGP11], [MV11] and the references therein. The ghost energy of Alinhac [Ali01] for quasilinear wave equations uses $O(1)$ time-dependent weights in the norms, and the $O(1)$ multiplier M is a Fourier-side analogue; this general idea has been used several times [Zil14], [BGM15a], [BGM15b]. Combining ideas like the ghost energy with the Cauchy-Kowalevskaya-type ideas are multipliers such as $m(t,\nabla)$, which are *not* $O(1)$ (m^{-1} is bounded only by $O(\nu^{-2/3})$); this is significantly more complicated, as will be clear from the proof. In the context of nonlinear mixing, this general idea was introduced for infinite regularity in [BM13] and extended further in [BMV16], [BGM15a], [BGM15b]. However, m is very different from ideas appearing in these infinite regularity works, as we must use very differently the interplay between the dissipation and destabilizing effects.

2.4. *Bootstrap.* In the following, it will be convenient (for bookkeeping purposes) to introduce

$$N = \sigma - 2 > \frac{5}{2}.$$

First, we have the following standard lemma. One can, for example, apply the energy methods in [MB02] and [LO97] (for the analyticity). We omit the proof for brevity.

LEMMA 2.2 (Local existence, continuation, and propagation of analyticity). *Let u_{in} be divergence-free and satisfy* (1.11). *Then there exists a $T^\star > 0$ independent of ν such that there is a unique strong solution to* (1.1) *$u(t) \in C([0, T^\star]; H^{N+2})$ that satisfies the initial data and is real analytic for $t \in (0, T^\star]$. Moreover, there exists a maximal time of existence T_0 with $T^\star < T_0 \leq \infty$ such that the solution $u(t)$ remains unique and real analytic on $(0, T_0)$ and, if for some $\tau \leq T_0$ we have $\limsup_{t \nearrow \tau} \|u(t)\|_{H^N} < \infty$, then $\tau < T_0$.*

By similar considerations (see Section 2.7), for ε sufficiently small, there are no issues getting estimates on q^i, u^i, and ψ until $t = 2$.

LEMMA 2.3. *For $\varepsilon\nu^{-3/2}$ sufficiently small and constants C_0, C_1 sufficiently large* (*chosen below*), *the following estimates hold for $t \in [0, 2]$:*

$$\left\| q^1(t) \right\|_{H^N} + \left\| q^3(t) \right\|_{H^N} \leq 2C_0\varepsilon, \tag{2.15a}$$

$$\left\| q^2(t) \right\|_{H^N} \leq 2\varepsilon, \tag{2.15b}$$

$$\left\| u^1(t) \right\|_{H^{N+2}} + \left\| u^3(t) \right\|_{H^{N+2}} \leq 2C_0\varepsilon, \tag{2.15c}$$

$$\left\| u^2(t) \right\|_{H^{N+2}} \leq 2\varepsilon, \tag{2.15d}$$

$$\left\| t\psi(t) \right\|_{H^N} \leq 2C_1\varepsilon. \tag{2.15e}$$

Lemma 2.3 shows that we only need to worry about times $t > 1$, for which we now move to the coordinate system defined in Section 2.2.2; for details on converting the estimates to and from these coordinates, see Section 2.7 below. From now on, all time norms are taken over the interval $[1, T]$ unless otherwise stated; that is, all norms are defined via

$$\|f\|_{L^p H^s} := \left\| \|f(t)\|_{H^s} \right\|_{L^p([1,T])}.$$

Fix C_0, C_1, and C_2 large constants determined by the proof below, and let T be the largest time $T \geq 1$ such that the following estimates hold on $[1, T]$ (see Lemma 2.7 below for a proof that $T \geq 2$): the bounds on Q,

$$\left\| \langle t \rangle^{-1} Q_0^1(t) \right\|_{L^\infty H^N} \leq 8C_0\varepsilon, \tag{2.16a}$$

$$\left\| Q_0^1 \right\|_{L^\infty H^N} + \nu^{1/2} \left\| \nabla Q_0^1 \right\|_{L^2 H^N} \leq 8C_0\varepsilon\nu^{-1}, \tag{2.16b}$$

$$\left\| mMQ^1_{\neq} \right\|_{L^\infty H^N} + \nu^{1/2} \left\| mM\nabla_L Q^1_{\neq} \right\|_{L^2 H^N} + \left\| \sqrt{-\dot{M}M} mQ^1_{\neq} \right\|_{L^2 H^N} \leq 8C_0 \varepsilon \nu^{-1/3}, \tag{2.16c}$$

$$\left\| m^{1/2} MQ^2 \right\|_{L^\infty H^N} + \nu^{1/2} \left\| m^{1/2} M\nabla_L Q^2 \right\|_{L^2 H^N} + \left\| \sqrt{-\dot{M}M} m^{1/2} Q^2 \right\|_{L^2 H^N} \leq 8\varepsilon, \tag{2.16d}$$

$$\left\| mMQ^3 \right\|_{L^\infty H^N} + \nu^{1/2} \left\| mM\nabla_L Q^3 \right\|_{L^2 H^N} + \left\| \sqrt{-\dot{M}M} mQ^3 \right\|_{L^2 H^N} \leq 8C_0 \varepsilon, \tag{2.16e}$$

$$\left\| MQ^2_{\neq} \right\|_{L^\infty H^{N-1}} + \nu^{1/2} \left\| M\nabla_L Q^2_{\neq} \right\|_{L^2 H^{N-1}} + \left\| \sqrt{-\dot{M}M} Q^2_{\neq} \right\|_{L^2 H^{N-1}} \leq 8\varepsilon, \tag{2.16f}$$

the bounds on U,

$$\left\| \langle t \rangle^{-1} U^1_0 \right\|_{L^\infty H^{N-1}} \leq 8C_0 \varepsilon, \tag{2.17a}$$

$$\left\| U^1_0 \right\|_{L^\infty H^{N-1}} + \nu^{1/2} \left\| \nabla U^1_0 \right\|_{L^2 H^{N-1}} \leq 8C_0 \varepsilon \nu^{-1}, \tag{2.17b}$$

$$\| U^2_0 \|_{L^\infty H^{N-1}} + \nu^{1/2} \| U^2_0 \|_{L^2 H^{N-1}} + \nu^{1/2} \| \nabla U^2_0 \|_{L^2 H^{N-1}} \leq 8\varepsilon, \tag{2.17c}$$

$$\| U^3_0 \|_{L^\infty H^{N-1}} + \nu^{1/2} \| \nabla U^3_0 \|_{L^2 H^{N-1}} \leq 8C_0 \varepsilon, \tag{2.17d}$$

$$\left\| MU^1_{\neq} \right\|_{L^\infty H^{N-1}} + \nu^{1/2} \left\| \nabla_L MU^1_{\neq} \right\|_{L^2 H^{N-1}} + \left\| \sqrt{-\dot{M}M} U^1_{\neq} \right\|_{L^2 H^{N-1}} \leq 8C_0 \varepsilon, \tag{2.17e}$$

$$\left\| U^2_0 \right\|_{L^1 L^2} \leq 8C_2 \varepsilon \nu^{-1}, \tag{2.17f}$$

and the bounds on the coordinate system

$$\| g \|_{L^\infty H^{N+2}} + \nu^{1/2} \| \nabla g \|_{L^2 H^{N+2}} \leq 8C_0 \varepsilon, \tag{2.18a}$$

$$\left\| t^2 g \right\|_{L^\infty H^{N-1}} + \nu^{1/2} \left\| t^2 \nabla g \right\|_{L^2 H^{N-1}} \leq 8C_0 \varepsilon, \tag{2.18b}$$

$$\| C \|_{L^\infty H^{N+2}} + \nu^{1/2} \| \nabla C \|_{L^2 H^{N+2}} \leq 8C_1 \varepsilon \nu^{-1}. \tag{2.18c}$$

Note that Lemma 2.3 implies $T > 2$; see Section 2.7 and the continuity of the constant in Lemma 2.5. The goal is then to prove that $T = +\infty$, which follows immediately from the following (and that all of these norms are continuous in time).

PROPOSITION 2.1. *Assume that* $\|u_{\rm in}\|_{H^{N+2}} \leq \varepsilon \leq \delta \nu^{3/2}$, $\nu \in (0,1)$, *and that, for some* $T > 1$, *the estimates* (2.17), (2.16), (2.18) *hold on* $[1,T]$. *Then for* δ *sufficiently small depending only on* σ, C_0, C_1, *and* C_2 (*in particular,* *independent of* T), *these same estimates hold with all the occurrences of* 8 *on the right-hand side replaced by* 4.

That Proposition 2.1 implies Theorem 1.1 is proved in Lemma 2.8 below. The proof of Proposition 2.1 comprises the remainder of the paper.

Before going any further, let us comment on the choice of the *a priori* estimates (2.16), (2.17), and (2.18).

- Most of these estimates are the natural ones for the linearized problems, given the multipliers chosen above: consider, for instance, (2.16e), which is typical. It comprises a global bound in H^N weighted by m, $\|mMQ^3\|_{L^\infty H^N}$ (natural due to the linear stretching, as discussed in Section 2.3), a bound accounting for the viscous dissipation, $\nu^{1/2}\|mM\nabla_L Q^3\|_{L^\infty H^N}$, and finally a bound corresponding to the dissipation-like structure arising from the multipliers as explained in Section 2.3, $\|\sqrt{-\dot{M}M}mQ^3\|_{L^2H^N}$.
- For the modes that grow linearly in the absence of viscosity, we add estimates incorporating a weight $\langle t\rangle^{-1}$: this gives (2.16a) and (2.17a).
- The estimate (2.16c) loses $\nu^{-1/3}$ on the right-hand side compared to the linearized estimate. (This loss occurs when estimating the lift up term in Section 5.1.1.) It might not be optimal, but it is sufficient to close the bootstrap when coupled with (2.17e).
- Finally, one of the main subtleties are the two estimates on Q^2 in (2.16d) and (2.16f). The nonlinear effect of high frequencies can be quite dramatic near the critical times, and the leading order nonlinear term in the Q^2 equation, which turns out is $\mathrm{NLP}(1,3,0,\neq)$ (see the treatment in Section 3.1.2 below), cannot be bounded in H^N uniformly in t and ν if we only assume $\varepsilon \lesssim \nu^{3/2}$. This term is a very 3D nonlinear interaction involving the Orr mechanism, the stretching of Q^3, and the lift-up effect of U_0^1 all at once. By allowing Q^2 to grow near the critical time until the dissipation can balance the growth, quantified by the inclusion of the decaying $m^{1/2}$ in the norm, one can complete an estimate — hence (2.16d). However, any growth of Q^2 in turn limits the inviscid damping of U^2, and the decay provided by the inviscid damping provides a kind of null structure that diminishes the effect of certain nonlinear terms that would otherwise be uncontrollable. The solution to this issue is to pay regularity and get a better uniform estimate at lower frequencies, as expressed in (2.16f) — the gap of one derivative is roughly analogous to the fact that paying one derivative will give one power of t^{-1} decay in an estimate such as (1.7).

2.5. *Choice of constants.* Four constants have not been specified yet: $\delta \geq \varepsilon\nu^{-3/2}$, which appears in the statement of Theorem 1.1, and C_0, C_1, and C_2, which appear in the above bootstrap estimates. In the course of the proof, we choose them small such that

$$\frac{1}{C_0} + \frac{C_0}{C_1} + C_1\delta + C_0\delta < \frac{1}{A}$$

for a universal constant $A = A(\sigma)$ that depends only on σ. The constant C_2 is a fixed, universal constant. Specifically, this means that one first fixes C_0, then C_1 dependent on C_0, and then finally δ small relative to both.

2.6. *Estimates following immediately from the bootstrap hypotheses.* This section outlines some of the consequences of the bootstrap hypotheses.

The first lemma is a simple result of the Sobolev product law, the geometric series representation (2.5), and the bootstrap hypotheses. (Also recall the shorthand (2.6b).)

LEMMA 2.4. *Under the bootstrap hypotheses, for $\varepsilon\nu^{-1}$ sufficiently small and $1 < s \leq N+2$,*

$$\|\psi_y\|_{H^s} + \|\psi_z\|_{H^s} + \|G\|_{H^s} \lesssim \|\nabla C\|_{H^s}.$$

As a consequence, there holds for all $i, j \in \{X, Y, Z\}$,

$$\left\|\partial_Y^t f\right\|_{H^s} \lesssim \left\|\partial_Y^L f\right\|_{H^s} \qquad s \leq N+1, \tag{2.19a}$$

$$\left\|\partial_Z^t f\right\|_{H^s} \lesssim \|\partial_Z f\|_{H^s} + \varepsilon\nu^{-1}\left\|\partial_Y^L f\right\|_{H^s} \lesssim \|\nabla_L f\|_{H^s} \qquad s \leq N+1, \tag{2.19b}$$

$$\|\Delta_t f_{\neq}\|_{H^s} + \left\|\partial_i^t\partial_j^t f_{\neq}\right\|_{H^s} \lesssim \|\Delta_L f_{\neq}\|_{H^s} \qquad s \leq N, \tag{2.19c}$$

$$\|\Delta_t f_0\|_{H^s} + \left\|\partial_i^t\partial_j^t f_0\right\|_{H^s} \lesssim \|\Delta f_0\|_{H^s} + \varepsilon\nu^{-1}\|\nabla f_0\|_{H^s} \quad s \leq N. \tag{2.19d}$$

Similarly, by using also Lemma A.1, *we have for all $\alpha \in [0,1]$ and $3/2 < s \leq N$,*

$$\left\|m^\alpha \partial_j^t f_{\neq}\right\|_{H^s} \lesssim (1 + \|\nabla C\|_{H^{s+2\alpha}})\|\nabla_L m^\alpha f\|_{H^s}. \tag{2.20}$$

Remark 2.1. Note that for $s + 2\alpha \leq N+1$, the leading factor in (2.20) can be ignored by the $L^\infty H^{N+2}$ control on C for ν sufficiently small depending on C_1.

Remark 2.2. Lemma 2.4, particularly (2.19), is used so frequently throughout the proof that, for the sake of brevity, we do not always make explicit mention of it.

An important consequence of (2.19) is that in many places, the difference between ∂_i^t and ∂_i^L is irrelevant, however, the difference cannot be neglected everywhere. For example, for $s > 3/2$, there holds (note that ψ_y is independent of X),

$$\langle f, \partial_Y^t g\rangle_{H^s} \lesssim \|\nabla_L f\|_{H^s}\|g\|_{H^s} + \left\|\nabla^2 C\right\|_{H^s}\|f\|_{H^s}\|g\|_{H^s}; \tag{2.21}$$

indeed this is proved by integrating by parts and Cauchy-Schwarz. If $s \leq N$ and the frequency of f is nonzero, then the second term in (2.21) is controlled by the first:

$$\langle f_{\neq}, \partial_Y^t g \rangle_{H^s} \lesssim \|\nabla_L f_{\neq}\|_{H^s} \|g\|_{H^s} . \tag{2.22}$$

Note that for ν sufficiently small, the implicit constant does not depend on C_1. However, at the zero frequency, we need both terms in (2.21). Similar inequalities hold also for ∂_Z^t. One also has the following variant for $\alpha \in [0,1]$, which is useful in many places: for $3/2 < s \leq N$,

$$\langle m^\alpha f_{\neq}, m^\alpha \left(\partial_Y^t g\right) \rangle_{H^s} \lesssim \|\nabla_L m^\alpha f_{\neq}\|_{H^s} \left\| m^{\min(\alpha,1/2)} g \right\|_{H^s} . \tag{2.23}$$

As above, for ν sufficiently small, the implicit constant does not depend on C_1.

The next proposition consists of those estimates that follow directly from the estimates on Q^i and the elliptic lemmas detailed in Section A.2. These elliptic lemmas provide the technical tools for understanding Δ_t^{-1}, important for recovering U^i from $Q^i = \Delta_t U^i$.

Proposition 2.2 (Basic *a priori* estimates on the velocity in H^N). *Under the bootstrap hypotheses, for $\varepsilon\nu^{-3/2}$ sufficiently small, the following additional estimates hold*:

$$\left\| \langle t \rangle^{-1} U_0^1(t) \right\|_{L^\infty H^{N+2}} \lesssim C_0 \varepsilon, \tag{2.24a}$$

$$\left\| U_0^1 \right\|_{L^\infty H^{N+2}} + \nu^{1/2} \left\| \nabla U_0^1 \right\|_{L^2 H^{N+2}} \lesssim C_0 \varepsilon \nu^{-1}, \tag{2.24b}$$

$$\left\| U_0^2 \right\|_{L^\infty H^{N+2}} + \nu^{1/2} \left\| \nabla U_0^2 \right\|_{L^2 H^{N+2}} \lesssim \varepsilon, \tag{2.24c}$$

$$\left\| U_0^3 \right\|_{L^\infty H^{N+2}} + \nu^{1/2} \left\| \nabla U_0^3 \right\|_{L^2 H^{N+2}} \lesssim C_0 \varepsilon, \tag{2.24d}$$

$$\left\| U_{\neq}^1 \right\|_{L^\infty H^N} + \nu^{1/2} \left\| \nabla_L U_{\neq}^1 \right\|_{L^2 H^N} + \left\| \sqrt{-\dot{M}M} U_{\neq}^1 \right\|_{L^2 H^N} \lesssim C_0 \varepsilon \nu^{-1/3}, \tag{2.24e}$$

$$\left\| U_{\neq}^2 \right\|_{L^\infty H^N} + \nu^{1/2} \left\| \nabla_L U_{\neq}^2 \right\|_{L^2 H^N} + \left\| \sqrt{-\dot{M}M} U_{\neq}^2 \right\|_{L^2 H^N} \lesssim \varepsilon, \tag{2.24f}$$

$$\left\| U_{\neq}^3 \right\|_{L^\infty H^N} + \nu^{1/2} \left\| \nabla_L U_{\neq}^3 \right\|_{L^2 H^N} + \left\| \sqrt{-\dot{M}M} U_{\neq}^3 \right\|_{L^2 H^N} \lesssim C_0 \varepsilon, \tag{2.24g}$$

$$\begin{aligned} &\left\| m \Delta_L U_{\neq}^1 \right\|_{L^\infty H^N} + \nu^{1/2} \left\| m \nabla_L \Delta_L U_{\neq}^1 \right\|_{L^2 H^N} \nu^{-1/3} \\ &\qquad + \left\| \sqrt{-\dot{M}M} m \Delta_L U_{\neq}^1 \right\|_{L^2 H^N} \lesssim C_0 \varepsilon, \end{aligned} \tag{2.24h}$$

$$\begin{aligned} &\left\| m^{1/2} \Delta_L U_{\neq}^2 \right\|_{L^\infty H^N} + \nu^{1/2} \left\| m^{1/2} \nabla_L \Delta_L U_{\neq}^2 \right\|_{L^2 H^N} \\ &\qquad + \left\| \sqrt{-\dot{M}M} m^{1/2} \Delta_L U_{\neq}^2 \right\|_{L^2 H^N} \lesssim \varepsilon, \end{aligned} \tag{2.24i}$$

$$\left\| m\Delta_L U^3_{\neq} \right\|_{L^\infty H^N} + \nu^{1/2} \left\| m\nabla_L \Delta_L U^3_{\neq} \right\|_{L^2 H^N} + \left\| \sqrt{-\dot{M}M} m\Delta_L U^3_{\neq} \right\|_{L^2 H^N} \lesssim C_0 \varepsilon. \tag{2.24j}$$

Proof. The estimates on the zero frequencies follow from Lemma A.3 and the bootstrap hypotheses.

By (2.14), the estimates (2.24h), (2.24i), and (2.24j) imply (2.24e), (2.24f), and (2.24g). The estimates (2.24h), (2.24i), and (2.24j) follow from applying Lemmas A.4, A.6, and A.7 and using the bootstrap hypotheses on Q and C. □

The next proposition details the inviscid damping of U^2 and the enhanced dissipation.

PROPOSITION 2.3. *Under the bootstrap hypotheses, the following additional estimates hold:*

- *the enhanced dissipation of* Q^i:

$$\left\| mQ^1_{\neq} \right\|_{L^2 H^N} \lesssim C_0 \varepsilon \nu^{-1/2}, \tag{2.25a}$$

$$\left\| m^{1/2} Q^2_{\neq} \right\|_{L^2 H^N} + \left\| Q^2_{\neq} \right\|_{L^2 H^{N-1}} \lesssim \varepsilon \nu^{-1/6}, \tag{2.25b}$$

$$\left\| mQ^3_{\neq} \right\|_{L^2 H^N} \lesssim C_0 \varepsilon \nu^{-1/6}; \tag{2.25c}$$

- *the enhanced dissipation and inviscid damping of* U^i:

$$\left\| \nabla_L U^2_{\neq} \right\|_{L^2 H^N} \lesssim \varepsilon \nu^{-1/6}, \tag{2.26a}$$

$$\left\| \nabla_L U^2_{\neq} \right\|_{L^2 H^{N-1}} \lesssim \varepsilon, \tag{2.26b}$$

$$\left\| \Delta_{X,Z} U^3_{\neq} \right\|_{L^2 H^N} \lesssim C_0 \varepsilon \nu^{-1/6}, \tag{2.26c}$$

$$\left\| \Delta_{X,Z} U^1_{\neq} \right\|_{L^2 H^N} \lesssim C_0 \varepsilon \nu^{-1/2}, \tag{2.26d}$$

$$\left\| U^1_{\neq} \right\|_{L^2 H^{N-1}} \lesssim C_0 \varepsilon \nu^{-1/6}; \tag{2.26e}$$

- *the enhanced dissipation of* tU^i:

$$\left\| t\partial_X U^1_{\neq} \right\|_{L^2 H^{N-1}} \lesssim C_0 \varepsilon \nu^{-5/6}, \tag{2.27a}$$

$$\left\| t\partial_X U^1_{\neq} \right\|_{L^2 H^{N-2}} \lesssim C_0 \varepsilon \nu^{-1/2}, \tag{2.27b}$$

$$\left\| t\partial_X U^3_{\neq} \right\|_{L^2 H^{N-1}} \lesssim C_0 \varepsilon \nu^{-1/2}, \tag{2.27c}$$

$$\left\| t\partial_X U^2_{\neq} \right\|_{L^2 H^{N-2}} \lesssim \varepsilon, \tag{2.27d}$$

$$\left\| t\partial_X U^2_{\neq} \right\|_{L^2 H^{N-1}} \lesssim \varepsilon \nu^{-1/6}. \tag{2.27e}$$

Proof. The enhanced dissipation estimates on Q^i follow from Lemma 2.1 and the bootstrap estimates on Q. Turning to (2.26a), by (2.14) and Lemma 2.1,

$$\begin{aligned}\left\|\nabla_L U^2_{\neq}\right\|_{H^N} &\lesssim \left\|m^{1/2}\Delta_L U^2\right\|_{H^N}\\ &\lesssim \nu^{-1/6}\left(\left\|m^{1/2}\sqrt{-\dot{M}M}\Delta_L U^2\right\|_{H^N} + \nu^{1/2}\left\|m^{1/2}\nabla_L\Delta_L U^2\right\|_{H^N}\right),\end{aligned}$$

from which the estimate follows from Proposition 2.2. For (2.26b), first note that by the definition of M, for $k \neq 0$ there holds

$$\begin{aligned}|k,\eta-kt,\ell| &\le \frac{|k|}{|k|+|\eta-kt|+|\ell|}\,|k,\eta-kt,\ell|^2\\ &\lesssim \sqrt{-\dot{M}^0M^0}\,|k,\eta-kt,\ell|^2 \le \sqrt{-\dot{M}M}\,|k,\eta-kt,\ell|^2.\end{aligned}$$

It hence follows that

$$\left\|\nabla_L U^2_{\neq}\right\|_{H^{N-1}} \lesssim \left\|\sqrt{-\dot{M}M}\Delta_L U^2\right\|_{H^{N-1}},$$

from which the estimate follows from Lemma A.7 and the *a priori* estimate (2.16f).

To deduce (2.26c), we use (2.14) followed by Lemma 2.1 to derive

$$\begin{aligned}\left\|\Delta_{X,Z} U^3_{\neq}\right\|_{H^N} &\lesssim \left\|m\Delta_L U^3_{\neq}\right\|_{H^N}\\ &\lesssim \nu^{-1/6}\left(\left\|m\sqrt{-\dot{M}M}\Delta_L U^3_{\neq}\right\|_{H^N} + \nu^{1/2}\left\|m\nabla_L\Delta_L U^3_{\neq}\right\|_{H^N}\right),\end{aligned}$$

after which the estimates follow from Proposition 2.2. The estimates on U^1 in (2.26d) and (2.26e) follow similarly.

Turn next to the enhanced dissipation estimates involving powers of t in (2.27). For example, we have by (2.14), $|kt| \lesssim \langle \eta - kt\rangle\langle\eta\rangle$, and Proposition 2.2,

$$\left\|t\partial_X U^1_{\neq}\right\|_{H^{N-1}} \lesssim \left\|\nabla_L m\Delta_L U^1_{\neq}\right\|_{H^N},$$

and similarly for $t\partial_X U^3$. For $t\partial_X U^2$, we again use $|kt| \lesssim \langle \eta - kt\rangle\langle\eta\rangle$:

$$\left\|t\partial_X U^2_{\neq}\right\|_{H^{N-2}} \lesssim \left\|\nabla_L U^2_{\neq}\right\|_{H^{N-1}},$$

which is then controlled by (2.26b), and similarly for the analogous inequality in H^{N-1}. □

In what follows we will use the shorthand

(2.28a) $$A = m^{1/2}M\langle D\rangle^N,$$

(2.28b) $$B = mM\langle D\rangle^N.$$

2.7. *Equivalence of coordinate systems*. Coordinate systems of the general type (2.4) have been used in [BM13], [BMV16], [BGM15a], [BGM15b], and here we may follow a similar scheme for how to transfer information from one coordinate system to the other; we will give a sketch for completeness. In Sobolev regularity the technical details are significantly simpler as compositions behave well in finite regularity classes. In particular, we have the following composition lemma; if $s' \in \mathbb{N}$, this is immediate from the Faá di Bruno formula and Sobolev embedding; for fractional s, see, e.g., [IKT13] for a proof.

LEMMA 2.5 (Sobolev composition). *Let $s > 5/2$, $s \geq s' \geq 0$, $g \in H^s$ be such that $\|\nabla g\|_\infty < 1$. Then, there exists a constant*

$$C_S = C_S(s, s', \|g\|_{H^s}, \|\nabla g\|_{L^\infty}) > 1$$

such that for all $f \in H^{s'}$, there holds

$$\|f \circ (Id + g)\|_{H^{s'}} \leq C_S \|f\|_{H^{s'}}.$$

Moreover, if $\|g\|_{H^s} \searrow 0$, then $C_S \searrow 1$.

We also need a Sobolev inverse function theorem, which follows by straightforward arguments using Lemma 2.5.

LEMMA 2.6 (Sobolev inverse function theorem). *Let $s > 5/2$. Then, there exists an $\varepsilon_0 = \varepsilon_0(s)$ such that if $\|\alpha\|_{H^s} \leq \varepsilon_0$, then there exists a unique solution β to*

$$\beta(y) = \alpha(y + \beta(y)),$$

which satisfies $\|\beta\|_{H^s} \lesssim \|\alpha\|_{H^s}$.

The next step is to prove Lemma 2.3 and also deduce that we may take $T > 1$, the T such that the bootstrap hypotheses (2.16), (2.17), and (2.18) hold. Hence, we do not need to worry about the coordinate singularity at $t = 0$.

LEMMA 2.7. *For $\varepsilon\nu^{-3/2}$ sufficiently small, Lemma* 2.3 *holds, we may take $2 \leq T$ (defined in Section* 2.4 *above), and for $t \leq 2$, the inequalities* (2.16), (2.17), *and* (2.18) *all hold with constant* 2 *instead of* 8.

Proof. As in analogous lemmas in [BM13], [BGM15a], the proof is done by using the linearized coordinate transform. Indeed, define

$$\bar{x} = x - ty, \tag{2.29a}$$

$$h^i(t, \bar{x}, y, z) = q^i(t, \bar{x} + ty, y, z), \tag{2.29b}$$

$$v^i(t, \bar{x}, y, z) = u^i(t, \bar{x} + ty, y, z); \tag{2.29c}$$

note that $v^i = \Delta_L^{-1} h^i$. These satisfy natural analogues of (2.10) and (2.12). Using standard (inviscid) energy methods, it is easy to propagate H^N regularity on these unknowns to $t = 2$ (or any other fixed, finite time) by choosing ε sufficiently small. Next, we need to solve for $(\bar{x}, y, z)$ in terms of (X, Y, Z) and then apply Lemma 2.5. From (2.9a) it is straightforward via classical energy methods to derive $\|t\psi\|_{H^{N+2}} \lesssim \varepsilon$ for $t \in [0, 2]$. For $t \in [1/2, 2]$, this yields good estimates on $\psi(t, y, z) = Y(t, y, z) - y$ and $X(t, x, y, z) = \bar{x}(t, x, y) - t\psi(t, y, z)$. We then write

$$\begin{aligned} \bar{x}(t, X, Y, Z) &= X + t\psi(t, y(t, Y, Z), z(t, Y, Z)), \\ y(t, Y, Z) &= Y + \psi(t, y(t, Y, Z), z(t, Y, Z)), \\ z(t, Y, Z) &= Z. \end{aligned}$$

To solve for $y(t, Y, Z)$, we rewrite this equation as

$$y(t, Y, Z) - Y = \psi(t, Y - (y(t, Y, Z) - Y), Z)$$

and apply Lemma 2.6 by choosing ε sufficiently small. Using y and z we also derive $\bar{x}$ in terms of X, Y, Z. Hence, Lemma 2.5 and (2.29) complete the proof of the lemma for $\varepsilon\nu^{-1}$ sufficiently small. (In particular, one can ensure that the constant lost due to changing coordinate systems is arbitrarily close to 1 due to the continuity of C_S in Lemma 2.5.) □

In order to move information back to the original variables, as in [BM13], [BMV16], [BGM15a], we first move to the coordinate system (X, y, z). Hence, write

$$\bar{q}^i(t, X, y, z) = Q^i(t, X, Y(t, y, z), Z)$$

and

$$\bar{u}^i(t, X, y, z) = U^i(t, X, Y(t, y, z), Z).$$

(Recall that $Z = z$.) This lemma also proves that Proposition 2.1 implies Theorem 1.1.

Lemma 2.8. *For $\varepsilon < \delta\nu^{3/2}$ with δ sufficiently small, the bootstrap hypotheses imply that all the estimates in Propositions* 2.2 *and* 2.3 *hold also for $\bar{q}^i$ and $\bar{u}^i$ (with different implicit constants).*

In particular, for $\varepsilon\nu^{-3/2}$ sufficiently small, Proposition 2.1 *implies Theorem* 1.1.

Proof. Notice that $Z(y, z) = z$ and $Y(y, z) - y = \psi(y, z)$, and hence we need estimates on ψ, however, from (2.18), we only have estimates on C, ψ_y and ψ_z in (Y, Z) coordinates. Hence, we need to solve for $y = y(t, Y, z)$. To this end, write $Y - y = C(t, Y, Z) = C(t, y + (Y - y), Z)$ and then apply Lemma 2.6 to solve for $Y - y(t, Y, z) = \beta(t, Y, z)$. Lemma 2.6 moreover provides the uniform estimate $\|y(t, Y, z) - Y\|_{H^{N+2}} \lesssim \varepsilon\nu^{-1}$. With the bootstrap hypotheses, (2.4), and Lemma 2.5, this completes the lemma. Indeed, by the definition of X in (2.4), Theorem 1.1 follows immediately. □

3. Energy estimates on Q^2

In this section, we prove that, under the assumptions of Proposition 2.1 (in particular, the bootstrap assumptions (2.17), (2.16), and (2.18)), the inequalities (2.16d) and (2.16f) hold, with 8 replaced by 4 on the right-hand side.

3.1. *H^N estimate on Q^2.* An energy estimate gives (recall the shorthand (2.28))

$$\begin{aligned}
&\frac{1}{2}\|Mm^{1/2}Q^2(T)\|_{H^N}^2 + \nu\|\nabla_L Mm^{1/2}Q^2\|_{L^2H^N}^2 + \|\sqrt{-\dot{M}M}m^{1/2}Q^2\|_{L^2H^N}^2 \\
&\quad\le \frac{1}{2}\|Mm^{1/2}Q^2(1)\|_{H^N}^2 + \int_1^T\int AQ^2A\left[-(\widetilde{U}_0\cdot\nabla + U_{\neq}\cdot\nabla_t)Q^2 - Q^j\partial_j^t U^2\right. \\
&\qquad \left.- \partial_i^t U^j\partial_{ij}^t U^2 + \partial_Y^t(\partial_i^t U^j\partial_j^t U^i) + \nu(\widetilde{\Delta}_t - \Delta_L)Q^2\right] dV\,dt \\
&\quad= \frac{1}{2}\|Mm^{1/2}Q^2(1)\|_{H^N}^2 + \mathcal{T} + \mathrm{NLS1} + \mathrm{NLS2} + \mathrm{NLP} + \mathrm{DE}.
\end{aligned}$$

3.1.1. *Transport nonlinearity.* Decompose the transport nonlinearity by frequency:

$$\begin{aligned}
\mathcal{T} &= \int_1^T\int AQ^2A\left(\tilde{U}_0\cdot\nabla Q_0^2 + \tilde{U}_0\cdot\nabla Q_{\neq}^2\right) dV\,dt \\
&\quad + \int AQ^2A\left(U_{\neq}\cdot\nabla_t Q_0^2 + U_{\neq}\cdot\nabla_t Q_{\neq}^2\right) dV\,dt \\
&= \mathcal{T}_{00} + \mathcal{T}_{0\neq} + \mathcal{T}_{\neq 0} + \mathcal{T}_{\neq\neq}.
\end{aligned}$$

Further decompose $\mathcal{T}_{00}$ into

$$\begin{aligned}
\mathcal{T}_{00} &= \int_1^T\int \langle D\rangle^N Q_0^2\langle D\rangle^N\left(g\partial_Y Q_0^2\right)\,dV\,dt \\
&\quad + \int \langle D\rangle^N Q_0^2\langle D\rangle^N\left(U_0^3\partial_Z Q_0^2\right)\,dV\,dt = \mathcal{T}_{00}^2 + \mathcal{T}_{00}^3.
\end{aligned}$$

To bound $\mathcal{T}_{00}^2$, split g into low and high frequencies:

$$\begin{aligned}
\mathcal{T}_{00}^2 &= \int_1^T\int \langle D\rangle^N Q_0^2\langle D\rangle^N\left(P_{\le 1}g\partial_Y Q_0^2\right)\,dV\,dt \\
&\quad + \int_1^T\int \langle D\rangle^N Q_0^2\langle D\rangle^N\left(P_{>1}g\partial_Y Q_0^2\right)\,dV\,dt \\
&\lesssim \|Q_0^2\|_{L^\infty H^N}\|g\|_{L^2L^2}\|\nabla Q_0^2\|_{L^2H^N} + \|Q_0^2\|_{L^\infty H^N}\|\nabla g\|_{L^2H^N}\|\nabla Q_0^2\|_{L^2H^N} \\
&\lesssim \varepsilon^3\nu^{-1/2-1/2} = \varepsilon^3\nu^{-1},
\end{aligned}$$

where the last line followed from the bootstrap hypotheses. (Note that (2.18b) is used to deduce $\|g\|_{L^2L^2}\lesssim\varepsilon$.) To bound $\mathcal{T}_{00}^3$, observe that either the first Q_0^3 factor, or the U_0^3 factor, must have nonzero Z frequency — or the contribution is zero. Therefore (using also Proposition 2.2),

$$\begin{aligned}\mathcal{T}_{00}^3 &\lesssim \|Q_0^2\|_{L^\infty H^N}\|\nabla U_0^3\|_{L^2H^N}\|\nabla Q_0^2\|_{L^2H^N}\\ &\quad + \|U_0^3\|_{L^\infty H^N}\|\nabla Q_0^2\|_{L^2H^N}^2 \lesssim \varepsilon^3\nu^{-1/2-1/2} = \varepsilon^3\nu^{-1}.\end{aligned}$$

For the $\mathcal{T}_{0\neq}$ term, we apply the paraproduct decomposition defined above in Section 2.1.4:

$$\begin{aligned}\mathcal{T}_{0\neq} &= \int_1^T\int AQ_{\neq}^2 A\left(\left(\tilde{U}_0\right)_{Hi}\cdot\left(\nabla Q_{\neq}^2\right)_{Lo}\right)dV\,dt\\ &\quad + \int_1^T\int AQ_{\neq}^2 A\left(\left(\tilde{U}_0\right)_{Lo}\cdot\left(\nabla Q_{\neq}^2\right)_{Hi}\right)dV\,dt\\ &= \mathcal{T}_{0\neq;\mathrm{HL}} + \mathcal{T}_{0\neq;LH}.\end{aligned}$$

Consider the LH term first, which we write out as follows:

$$\begin{aligned}\mathcal{T}_{0\neq;LH} &= \int_1^T\int AQ^2 A\left(g_{Lo}(\partial_Y - t\partial_X)(Q_{\neq}^2)_{Hi} + (U_0^3)_{Lo}(\partial_Z Q_{\neq}^2)_{Hi}\right)dV\,dt\\ &\quad + \int_1^T\int AQ^2 A\left(g_{Lo}t\partial_X(Q_{\neq}^2)_{Hi}\right)dV\,dt.\end{aligned}$$

By (2.1), (2.13), and the bootstrap hypotheses,

$$\begin{aligned}&\mathcal{T}_{0\neq;LH}\\ &\lesssim \nu^{-1/3}\left\|AQ_{\neq}^2\right\|_{L^2L^2}\left(\|\langle t\rangle g\|_{L^\infty H^{3/2+}} + \left\|U_0^3\right\|_{L^\infty H^{3/2+}}\right)\left\|\nabla_L AQ^2\right\|_{L^2L^2} \lesssim \varepsilon^3\nu^{-1},\end{aligned}$$

where note that we applied (2.25). Similarly, for the HL term we have by (2.13) and (2.25b) from Proposition 2.3 (using $N > 5/2$),

$$\begin{aligned}\mathcal{T}_{0\neq;\mathrm{HL}} &\lesssim \left\|AQ_{\neq}^2\right\|_{L^2L^2}\left(\|g\|_{L^\infty H^N} + \left\|U_0^3\right\|_{L^\infty H^N}\right)\left\|\nabla Q_{\neq}^2\right\|_{L^2H^{3/2+}}\\ &\lesssim \nu^{-1/3}\left\|AQ_{\neq}^2\right\|_{L^2L^2}^2\left(\|g\|_{L^\infty H^N} + \left\|U_0^3\right\|_{L^\infty H^N}\right)\\ &\lesssim \varepsilon^3\nu^{-2/3}.\end{aligned}$$

Consider next $\mathcal{T}_{\neq 0}$. By the product rule, the bootstrap hypotheses, and Proposition 2.3 (specifically (2.26a) and (2.26c)), we have

$$\mathcal{T}_{\neq 0} \lesssim \left\|AQ_{\neq}^2\right\|_{L^\infty L^2}\left\|U_{\neq}^{2,3}\right\|_{L^2H^N}\left\|\nabla Q_0^2\right\|_{L^2H^N} \lesssim \varepsilon^3\nu^{-2/3}.$$

Consider finally $\mathcal{T}_{\neq\neq}$, by (2.13),

$$\mathcal{T}_{\neq\neq} \lesssim \nu^{-1/3}\left\|AQ^2\right\|_{L^\infty L^2}\|U_{\neq}\|_{L^2H^N}\left\|\nabla_L AQ^2\right\|_{L^2L^2} \lesssim \varepsilon^3\nu^{-4/3}.$$

3.1.2. *Nonlinear pressure terms.* Recall the shorthands defined in Section 2.2.3. Consider first the $\mathrm{NLP}(0,0)$ terms, which are straightforward. We first apply (2.21), which results in an error term when the derivative lands on the coefficients, and then we apply Lemma 2.4:

$$\begin{aligned}
\mathrm{NLP}(i,j,0,0) &\lesssim \left\|\nabla Q_0^2\right\|_{L^2H^N} \|\nabla U_0^{2,3}\|_{L^2H^N}\|\nabla U_0^{2,3}\|_{L^\infty H^{3/2+}} \\
&\quad + \|\Delta C\|_{L^2H^N} \left\|Q_0^2\right\|_{L^\infty H^N} \|\nabla U_0^{2,3}\|_{L^2H^N}\|\nabla U_0^{2,3}\|_{L^\infty H^{3/2+}} \\
&\lesssim \varepsilon^3\nu^{-1}.
\end{aligned}$$

Next turn to the $\mathrm{NLP}(0,\neq,i,j)$ terms, which include one of the leading order nonlinear terms, $\mathrm{NLP}(1,3,0,\neq)$. Consider this problematic term first, and expand with a paraproduct as described in Section 2.1.4,

$$\begin{aligned}
\mathrm{NLP}(1,3,0,\neq) &= \int_1^T \int AQ^2 A\partial_Y^t \left(\left(\partial_Z^t U_0^1\right)_{Hi}\left(\partial_X U_{\neq}^3\right)_{Lo}\right) dV dt \\
&\quad + \int_1^T \int AQ^2 A\partial_Y^t \left(\left(\partial_Z^t U_0^1\right)_{Lo}\left(\partial_X U_{\neq}^3\right)_{Hi}\right) dV dt \\
&= P_{\mathrm{HL}} + P_{\mathrm{LH}}.
\end{aligned}$$

For the LH term we have, using (2.23), Lemma A.1, and the inequality $|m^{1/2}\partial_X| \lesssim m\sqrt{-\dot{M}M}(-\Delta_L)$ that follows from (2.14),

$$\begin{aligned}
P_{\mathrm{LH}} &\lesssim \|\nabla_L AQ^2\|_{L^2L^2}\|\partial_Z^t U_0^1\|_{L^\infty H^{5/2+}}\|m^{1/2}\partial_X U_{\neq}^3\|_{L^2H^N} \\
&\lesssim \|\nabla_L AQ^2\|_{L^2L^2}\|\partial_Z^t U_0^1\|_{L^\infty H^{5/2+}} \left\|m\sqrt{-\dot{M}M}\Delta_L U_{\neq}^3\right\|_{L^2H^N} \\
&\lesssim \varepsilon^3\nu^{-1/2-1} = \varepsilon^3\nu^{-3/2},
\end{aligned}$$

which suffices for $\varepsilon\nu^{-3/2} \ll 1$; hence this term uses sharply the smallness requirement. For the HL term we can apply (2.22) and deduce using (2.27),

$$P_{\mathrm{HL}} \lesssim \left\|\nabla_L m^{1/2} MQ^2\right\|_{L^2H^N} \left\|\langle t\rangle^{-1}\nabla U_0^1\right\|_{L^\infty H^N} \left\|\langle t\rangle\partial_X U_{\neq}^3\right\|_{L^2H^{3/2+}} \lesssim \nu^{-1}\varepsilon^3.$$

This completes $\mathrm{NLP}(1,3,0,\neq)$ term; $\mathrm{NLP}(1,2,0,\neq)$ is similar.

Consider next $\mathrm{NLP}(i,j,0,\neq)$ with $i,j \neq 1$. For these terms we do not need a sophisticated argument; using (2.22) and Proposition 2.2 there holds,

$$\begin{aligned}
&\mathrm{NLP}(i,j,0,\neq)\mathbf{1}_{i,j\neq 1} \\
&\qquad \lesssim \left\|\nabla_L AQ^2\right\|_{L^2L^2} \left\|\nabla U_0^j\right\|_{L^\infty H^N} \left\|\nabla_L U_{\neq}^i\right\|_{L^2H^N} \mathbf{1}_{i,j\neq 1} \lesssim \varepsilon^3\nu^{-1}.
\end{aligned}$$

Turn next to $\mathrm{NLP}(i,j,\neq,\neq)$. We expand with a paraproduct and by symmetry, we only have to consider the case when i is in "high frequency." By

(2.21) (note that the leading factor could have zero X frequency),

$$\begin{aligned}
\mathrm{NLP}(i,j,\neq,\neq) &= \int_1^T \int AQ^2 A\partial_Y^t \left(\left(\partial_j^t U_{\neq}^i\right)_{Hi} \left(\partial_i^t U_{\neq}^j\right)_{Lo} \right) dV\,dt \\
&\lesssim \left(\left\| \nabla_L AQ^2 \right\|_{L^2L^2} + \left\| AQ^2 \right\|_{L^\infty L^2} \left\| \nabla C \right\|_{L^2H^{N+1}} \right) \\
&\quad \times \left\| \nabla_L U_{\neq}^i \right\|_{L^\infty H^N} \left\| \partial_i^t U_{\neq}^j \right\|_{L^2H^{3/2+}} \\
&\lesssim \nu^{-1/3} \left(\left\| \nabla_L AQ^2 \right\|_{L^2L^2} + \left\| AQ^2 \right\|_{L^\infty L^2} \left\| \nabla C \right\|_{L^2H^{N+1}} \right) \\
&\quad \times \left\| \Delta_L m U_{\neq}^i \right\|_{L^\infty H^N} \left\| \partial_i^t U_{\neq}^j \right\|_{L^2H^{N-1}}
\end{aligned}$$

At this point, we distinguish two cases: $i = 1$ and $i \neq 1$. First, notice that by the divergence-free condition, $N > 3/2$, $\varepsilon \ll \nu^{3/2}$, (2.14), Proposition 2.3, and the bootstrap hypotheses, there follows

$$\begin{aligned}
\left\| \partial_X U^1 \right\|_{L^2H^{N-1}} &\leq \left\| \partial_Y^t U_{\neq}^2 \right\|_{L^2H^{N-1}} + \left\| \partial_Z^t U_{\neq}^3 \right\|_{L^2H^{N-1}} \\
&\lesssim \left(1 + \left\| \nabla C \right\|_{L^\infty H^{N-1}}\right) \left\| \nabla_L U_{\neq}^2 \right\|_{L^2H^{N-1}} \\
&\quad + \left\| U_{\neq}^3 \right\|_{L^2H^N} + \left\| \nabla C \right\|_{L^\infty H^{N-1}} \left\| \nabla_L U_{\neq}^3 \right\|_{L^2H^{N-1}} \\
&\lesssim \left(1 + \varepsilon\nu^{-1}\right) \left\| \nabla_L U_{\neq}^2 \right\|_{L^2H^{N-1}} \\
&\quad + \left\| U_{\neq}^3 \right\|_{L^2H^N} + \varepsilon\nu^{-1} \left\| \nabla_L m \Delta_L U_{\neq}^3 \right\|_{L^2H^{N-1}} \\
&\lesssim \varepsilon + \varepsilon\nu^{-1/6} + \varepsilon.
\end{aligned} \tag{3.1}$$

Hence, if $i = 1$, we have the following by applying by Proposition 2.3 (and (3.1) in the case $j = 1$):

$$\begin{aligned}
\mathrm{NLP}(1,j,\neq,\neq) &\lesssim \nu^{-1/3} \left(\left\| \nabla_L AQ^2 \right\|_{L^2L^2} + \left\| AQ^2 \right\|_{L^\infty L^2} \left\| \nabla C \right\|_{L^2H^{N+1}} \right) \\
&\quad \times \| \Delta_L m U_{\neq}^1 \|_{L^\infty H^N} \| \partial_X U^j \|_{L^2H^{N-1}} \\
&\lesssim \varepsilon^3 \nu^{-1/3-1/2-1/3-1/6} = \varepsilon^3 \nu^{-4/3}.
\end{aligned}$$

On the other hand, if $i \neq 1$, by Proposition 2.2, (2.16d), and (in the case $j = 1$) (2.17e), the following holds:

$$\begin{aligned}
\mathrm{NLP}(i,j,\neq,\neq) &\lesssim \nu^{-1/3} \left(\left\| \nabla_L AQ^2 \right\|_{L^2L^2} + \left\| AQ^2 \right\|_{L^\infty L^2} \left\| \nabla C \right\|_{L^2H^{N+1}} \right) \\
&\quad \times \| \Delta_L m U_{\neq}^i \|_{L^\infty H^N} \| \nabla_L U_{\neq}^j \|_{L^2H^{N-1}} \\
&\lesssim \varepsilon^3 \nu^{-1/3-1/2-1/2} = \varepsilon^3 \nu^{-4/3}.
\end{aligned}$$

This completes the nonlinear pressure terms.

3.1.3. *Nonlinear stretching terms.* Starting with NLS1$(0,0,j)$ (note that $j \neq 1$) and using (2.19),

$$\begin{aligned}\mathrm{NLS1}(0,0,j) &\lesssim \left\| Q_0^2 \right\|_{L^\infty H^N} \left\| \Delta_t U_0^j \right\|_{L^2 H^N} \left\| \nabla U_0^2 \right\|_{L^2 H^N} \\ &\lesssim \left\| Q_0^2 \right\|_{L^\infty H^N} \left\| \nabla U_0^j \right\|_{L^2 H^{N+1}} \left\| \nabla U_0^2 \right\|_{L^2 H^N} \\ &\lesssim \varepsilon^3 \nu^{-1}.\end{aligned}$$

By Propositions 2.2 and 2.3 (note that $j = 1$ is permitted),

$$\mathrm{NLS1}(0,\neq,j) \lesssim \left\| A Q_{\neq}^2 \right\|_{L^2 L^2} \left\| Q_0^j \right\|_{L^\infty H^N} \left\| \nabla_L U_{\neq}^2 \right\|_{L^2 H^N} \lesssim \varepsilon^3 \nu^{-4/3}$$

and, using (2.13) and Proposition 2.3 (note that here $j \neq 1$),

$$\mathrm{NLS1}(\neq,0,j) \lesssim \left\| A Q_{\neq}^2 \right\|_{L^2 L^2} \left\| Q_{\neq}^j \right\|_{L^2 H^N} \left\| \nabla U_0^2 \right\|_{L^\infty H^N} \lesssim \varepsilon^3 \nu^{-1}.$$

Similarly (note that $j = 1$ is permitted),

$$\mathrm{NLS1}(\neq,\neq,j) \lesssim \left\| A Q^2 \right\|_{L^\infty L^2} \left\| Q_{\neq}^j \right\|_{L^2 H^N} \left\| \nabla_L U_{\neq}^2 \right\|_{L^2 H^N} \lesssim \varepsilon^3 \nu^{-4/3}.$$

Recall the second stretching term, NLS2, is written $\partial_i^t U^j \partial_{ij}^t U^2$. The contributions from the NLS2$(0,0)$ terms are treated in the same manner as NLS1$(0,0)$ above and are hence omitted for brevity. Turning to the nonzero frequencies, we have by Lemma 2.4, (2.14), (2.13), Proposition 2.3, and Proposition 2.2,

$$\begin{aligned}\mathrm{NLS2}(0,\neq,j) &\lesssim \left\| A Q_{\neq}^2 \right\|_{L^2 L^2} \left\| \partial_i^t U_0^j \right\|_{L^\infty H^N} \left\| \partial_{ij}^t U_{\neq}^2 \right\|_{L^2 H^N} \mathbf{1}_{j\neq 1} \\ &\quad + \left\| A Q_{\neq}^2 \right\|_{L^2 L^2} \left\| \partial_i^t U_0^1 \right\|_{L^\infty H^N} \left\| \partial_X \partial_i^t U_{\neq}^2 \right\|_{L^2 H^N} \\ &\lesssim \nu^{-1/3} \left\| A Q_{\neq}^2 \right\|_{L^2 L^2} \left\| \nabla U_0^j \right\|_{L^\infty H^N} \left\| m^{1/2} \Delta_L U_{\neq}^2 \right\|_{L^2 H^N} \mathbf{1}_{j\neq 1} \\ &\quad + \left\| A Q_{\neq}^2 \right\|_{L^2 L^2} \left\| \nabla U_0^1 \right\|_{L^\infty H^N} \left\| m^{1/2} \Delta_L U_{\neq}^2 \right\|_{L^2 H^N} \\ &\lesssim \varepsilon^3 \nu^{-1/3-1/6-1/6} + \varepsilon^3 \nu^{-1/6-1-1/6} \lesssim \varepsilon^3 \nu^{-4/3}.\end{aligned}$$

Similarly, we have (note that in this case $j \neq 1$)

$$\begin{aligned}\mathrm{NLS2}(\neq,0,j) &\lesssim \left\| A Q_{\neq}^2 \right\|_{L^2 L^2} \left\| \partial_i^t U_{\neq}^j \right\|_{L^2 H^N} \left\| \nabla U_0^2 \right\|_{L^\infty H^{N+1}} \\ &\lesssim \varepsilon^3 \nu^{-1/6-1/2} = \varepsilon^3 \nu^{-2/3}\end{aligned}$$

and, using Lemma 2.4, (2.13), Lemma 2.1, and Propositions 2.2 and 2.3, we have

$$\begin{aligned}\mathrm{NLS2}(\neq,\neq,j) &\lesssim \left\| A Q^2 \right\|_{L^\infty L^2} \left\| \partial_i^t U_{\neq}^j \right\|_{L^2 H^N} \left\| \Delta_L U_{\neq}^2 \right\|_{L^2 H^N} \\ &\lesssim \varepsilon^3 \nu^{-1/2-1/3-1/3-1/6} = \varepsilon^3 \nu^{-4/3}.\end{aligned}$$

3.1.4. *Dissipation error terms.* These terms are given by (recall the shorthand (2.6b))

$$\mathrm{DE} = \nu \int AQ^2 A \left(G \partial_{YY}^L Q^2 + 2\psi_z \partial_{YZ}^L Q^2 \right) dV = \mathcal{E}_1 + \mathcal{E}_2.$$

Both $\mathcal{E}_1$ and $\mathcal{E}_2$ are treated similarly, hence only consider $\mathcal{E}_1$. Arguing as in (2.21) and (2.13) (recall G is independent of X), by Lemma 2.4, we have

$$\begin{aligned}
\mathcal{E}_1 &\lesssim \nu^{2/3} \left\| G \right\|_{L^\infty H^N} \left\| \nabla_L AQ^2 \right\|_{L^2 L^2}^2 \\
&\quad + \nu^{2/3} \left\| \nabla G \right\|_{L^2 H^N} \left\| AQ^2 \right\|_{L^\infty L^2} \left\| \nabla_L AQ^2 \right\|_{L^2 L^2} \\
&\lesssim \varepsilon^3 \nu^{2/3-1-1/2-1/2} = \varepsilon^3 \nu^{-4/3},
\end{aligned}$$

which suffices.

3.2. H^{N-1} *estimate on* Q^2. Recall that a crucial strategy of the current approach is to confirm that the extra $m^{1/2}$ on Q^2 can be removed in H^{N-1}. As in Section 3.1, an energy estimate gives

$$\begin{aligned}
\frac{1}{2} &\| MQ^2(T) \|_{H^{N-1}}^2 + \nu \| \nabla_L MQ^2 \|_{L^2 H^{N-1}}^2 + \| \sqrt{-\dot{M}M} Q^2 \|_{L^2 H^{N-1}}^2 \\
&\leq \frac{1}{2} \| MQ^2(1) \|_{H^{N-1}}^2 + \int_1^T \int \langle D \rangle^{N-1} MQ^2 \langle D \rangle^{N-1} M \\
&\qquad \times \Big[-(\widetilde{U_0} \cdot \nabla + U_{\neq} \cdot \nabla_t) Q^2 - Q^j \partial_j^t U^2 - \partial_i^t U^j \partial_{ij}^t U^2 \\
&\qquad\quad + \partial_Y^t (\partial_i^t U^j \partial_j^t U^i) + \nu (\widetilde{\Delta}_t - \Delta_L) Q^2 \Big] \, dV \, dt \\
&= \frac{1}{2} \| MQ^2(1) \|_{H^{N-1}}^2 + \mathcal{T} + \mathrm{NLS1} + \mathrm{NLS2} + \mathrm{NLP} + \mathrm{DE}.
\end{aligned}$$

Nearly every step in this estimate is similar to those done in Section 3.1, indeed, the presence of $m^{1/2}$ in Section 3.1 is used only to control the $\mathrm{NLP}(1, 3, 0, \neq)$ term in Section 3.1.2. The $\mathcal{T}$ is bounded as in Section 3.1.1 and is hence omitted. (However, notice that this requires the H^N estimate (2.16d) here; this detail is due to our only assuming $N - 1 > 3/2$, where normally $H^{5/2+}$ is natural for closing energy estimates on a system such as (2.10).) Similarly, the dissipation error terms DE are controlled as in Section 3.1.4.

The $\mathrm{NLP}(0, 0)$ terms are treated as in Section 3.1.2 Now let us see how the reduction of one derivative allows to eliminate the use of $m^{1/2}$ in the treatment of $\mathrm{NLP}(1, 3, 0, \neq)$. By (2.22), (2.27), and $N - 1 > 3/2$,

$$\begin{aligned}
\mathrm{NLP}(1, 3, 0, \neq) &\lesssim \left\| \nabla_L MQ^2 \right\|_{L^2 H^{N-1}} \left\| \langle t \rangle^{-1} \nabla U_0^1 \right\|_{L^\infty H^{N-1}} \left\| \langle t \rangle \partial_X U_{\neq}^3 \right\|_{L^2 H^{N-1}} \\
&\lesssim \varepsilon^3 \nu^{-1/2-1/2} = \varepsilon^3 \nu^{-1}.
\end{aligned}$$

This suffices for the $\mathrm{NLP}(1, 3, 0, \neq)$ term; the $\mathrm{NLP}(1, 2, 0, \neq)$ is similar. The other NLP terms can be treated as in Section 3.1.2, and the NLS1 and NLS2 terms can be treated as in Section 3.1.3 above, and hence these contributions are also omitted. This completes the H^{N-1} estimate (2.16f).

4. **Energy estimate on Q^3**

In this section, we prove that, under the assumptions of Proposition 2.1 (in particular, the bootstrap assumptions (2.17), (2.16), (2.18)), the inequality (2.16e) holds, with 8 replaced by 4 on the right-hand side.

An energy estimate gives (recall the shorthand (2.28))

$$
\begin{aligned}
&\frac{1}{2}\|BQ^3(T)\|_{L^2}^2 + \nu\|\nabla_L BQ^3\|_{L^2L^2}^2 \|m\sqrt{-\dot{M}M}Q^3\|_{L^2H^N}^2 + \|M\sqrt{-\dot{m}m}Q^3\|_{L^2H^N}^2 \\
&\qquad = \frac{1}{2}\|BQ^3(1)\|_{L^2}^2 + \int_1^T \int BQ^3 B\left[-2\partial_{XY}^t U^3 + 2\partial_{XZ}^t U^2\right. \\
&\qquad - (\widetilde{U}_0 \cdot \nabla + U_{\neq} \cdot \nabla_t)Q^3 - Q^j \partial_j^t U^3 - 2\partial_i^t U^j \partial_{ij}^t U^3 \\
&\qquad \left. + \partial_Z^t \left(\partial_i^t U^j \partial_j^t U^i\right) + \nu(\widetilde{\Delta}_t - \Delta_L)Q^3\right] dV\, dt \\
&\qquad = \frac{1}{2}\|BQ^3(1)\|_{L^2}^2 + \mathrm{LS} + \mathrm{LP} + \mathcal{T} + \mathrm{NLS1} + \mathrm{NLS2} + \mathrm{NLP} + \mathrm{DE}.
\end{aligned}
\tag{4.1}
$$

4.1. *The linear stretching term* LS. The linear stretching term can be split into (recall the shorthands (2.6b) and (2.28))

$$
\begin{aligned}
\mathrm{LS} &= \int_1^T \int BQ^3 B\left[-2\partial_{XY}^L \Delta_L^{-1}\right. \\
&\qquad\qquad \left. \times d[Q^3 - G\partial_{YY}^L U^3 - 2\psi_z \partial_{YZ}^L U^3 - \Delta_t C \partial_Y^L U^3]\right] dV dt \\
&\quad + \int_1^T \int BQ^3 B\left(\psi_y \partial_{XY}^t U^3\right) dV dt \\
&= \mathrm{LS}_1 + \mathrm{LS}_2 + \mathrm{LS}_3 + \mathrm{LS}_4 + \mathrm{LS}_5.
\end{aligned}
$$

The leading order term, LS_1, is absorbed by the left-hand side of (4.1): indeed, by construction of m in Section 2.3, we have

$$
\mathrm{LS}_1 \leq \|M\sqrt{-\dot{m}m}Q^3\|_{L^2H^N}^2 + \frac{\nu}{2}\|\nabla_L BQ^3\|_{L^2L^2}^2.
$$

We turn next to the error terms. For LS_2, we apply Lemma A.1, use that $|k| \lesssim |k, \eta - kt, l| \sqrt{-\dot{M}^0 M^0}$, and (2.14) together with Lemma 2.1 and Proposition 2.2 to deduce

$$
\begin{aligned}
\mathrm{LS}_2 &= -2\int_1^T \int B\partial_{XY}^L \Delta_L^{-1} Q^3 B\left(G\partial_{YY}^L U^3\right) dV\, dt \\
&\lesssim \left\|\sqrt{-\dot{M}^0 M^0} m Q^3\right\|_{L^2H^N} \|G\|_{L^\infty H^{N+1}} \left\|m^{1/2}\Delta_L U_{\neq}^3\right\|_{L^2H^N} \\
&\lesssim \varepsilon^3 \nu^{-1-1/3-1/6} = \varepsilon^3 \nu^{-3/2},
\end{aligned}
$$

which suffices for $\varepsilon\nu^{-3/2} \ll 1$; notice that this is a sharp use of the smallness conditions. The term LS_3 can be estimated in the same way as LS_2. The LS_4 term is estimated with a slight variation, using 2.14 and $|k| \lesssim |k, \eta - kt, l| \sqrt{-\dot{M}^0 M^0}$:

$$\mathrm{LS}_4 \lesssim \left\| \sqrt{-\dot{M}^0 M^0} m Q^3_{\neq} \right\|_{L^2 H^N} \|C\|_{L^\infty H^{N+2}} \left\| \nabla_L U^3_{\neq} \right\|_{L^2 H^N} \lesssim \varepsilon^3 \nu^{-3/2}.$$

Turn to LS_5. Here we use Lemma A.1, $|k| \lesssim |k, \eta - kt, l| \sqrt{-\dot{M}^0 M^0}$, and (2.13) to deduce

$$\begin{aligned}
\mathrm{LS}_5 &\lesssim \left\| B Q^3_{\neq} \right\|_{L^2 L^2} \|\nabla C\|_{L^\infty H^{N+1}} \left\| \sqrt{-\dot{M}^0 M^0} m^{1/2} \Delta_L U^3_{\neq} \right\|_{L^2 H^N} \\
&\lesssim \varepsilon^3 \nu^{-1/6-1-1/3} = \varepsilon^3 \nu^{-3/2},
\end{aligned}$$

which suffices for $\varepsilon\nu^{-3/2} \ll 1$.

4.2. *The linear pressure term* LP. The linear pressure term is split into two contributions:

$$\begin{aligned}
\mathrm{LP} &= \int_1^T \int B Q^3 B \partial^L_{XZ} U^2_{\neq} dV dt + \int_1^T \int B Q^3 B \left(\psi_z \partial^L_{XY} U^2_{\neq} \right) dV dt \\
&= \mathrm{LP}_1 + \mathrm{LP}_2.
\end{aligned}$$

By definition of M^1,

$$\mathrm{LP}_1 \lesssim \left\| \sqrt{-\dot{M}^1 M^1} m Q^3 \right\|_{L^2 H^N} \left\| \sqrt{-\dot{M}^1 M^1} m^{1/2} \Delta_L U^2_{\neq} \right\|_{L^2 H^N} \lesssim C_0^{-1} \left(C_0 \varepsilon \right)^2,$$

which suffices by choosing C_0 sufficiently large. For LP_1, we have, similar to LS_5 above, by the definition of M and Lemma A.7 along with Lemma A.1,

$$\mathrm{LP}_2 \lesssim \|\nabla C\|_{L^\infty H^{N+1}} \left\| M m Q^3_{\neq} \right\|_{L^2 H^N} \left\| \sqrt{-\dot{M} M} m^{1/2} \Delta_L U^2_{\neq} \right\|_{L^2 H^N} \lesssim \varepsilon^3 \nu^{-7/6},$$

which is sufficient.

4.3. *Transport nonlinearity.* The interaction of nonzero frequencies will require more precision here than in Section 3.1.1. As in Section 3.1.1, we subdivide based on frequency:

$$\begin{aligned}
\mathcal{T} &= -\int_1^T \int B Q^3 B \left(\tilde{U}_0 \cdot \nabla Q^3_0 + \tilde{U}_0 \cdot \nabla Q^3_{\neq} \right) dV dt \\
&\quad - \int_1^T \int B Q^3 B \left(U_{\neq} \cdot \nabla_t Q^3_0 + U_{\neq} \cdot \nabla_t Q^3_{\neq} \right) dV dt \\
&= \mathcal{T}_{00} + \mathcal{T}_{0\neq} + \mathcal{T}_{\neq 0} + \mathcal{T}_{\neq\neq}.
\end{aligned}$$

The $\mathcal{T}_{00}$ term is treated as in Section 3.1.1 and is hence omitted for brevity. The $\mathcal{T}_{0\neq}$ term is treated analogously to the corresponding term in Section 3.1.1

via a paraproduct decomposition, yielding (applying Propositions 2.2 and 2.3 as above)

$$\begin{aligned}
\mathcal{T}_{0\neq} &\lesssim \left\| BQ^3_{\neq} \right\|_{L^2L^2} \left(\|g\|_{L^\infty H^N} + \left\| U^3_0 \right\|_{L^\infty H^N} \right) \left\| \nabla Q^3_{\neq} \right\|_{L^2 H^{3/2+}} \\
* &\quad + \left\| BQ^3_{\neq} \right\|_{L^2L^2} \left(\|\langle t\rangle g\|_{L^\infty H^{3/2+}} + \left\| U^3_0 \right\|_{L^\infty H^{3/2+}} \right) \left\| \nabla_L M Q^3_{\neq} \right\|_{L^2 H^N} \\
&\lesssim \nu^{-2/3} \left\| BQ^3_{\neq} \right\|^2_{L^2L^2} \left(\|g\|_{L^\infty H^N} + \left\| U^3_0 \right\|_{L^\infty H^N} \right) \\
&\quad + \nu^{-2/3} \left\| BQ^3_{\neq} \right\|_{L^2L^2} \left(\|\langle t\rangle g\|_{L^\infty H^{3/2+}} + \left\| U^3_0 \right\|_{L^\infty H^{3/2+}} \right) \left\| \nabla_L BQ^3_{\neq} \right\|_{L^2L^2} \\
&\lesssim \varepsilon^3 \nu^{-1} + \varepsilon^3 \nu^{-4/3}.
\end{aligned}$$

Consider next $\mathcal{T}_{\neq 0}$, which follows from Proposition 2.3 and the bootstrap hypotheses

$$\mathcal{T}_{\neq 0} \lesssim \left\| BQ^3_{\neq} \right\|_{L^\infty L^2} \left\| U^{2,3}_{\neq} \right\|_{L^2 H^N} \left\| \nabla Q^3_0 \right\|_{L^2 H^N} \lesssim \varepsilon^3 \nu^{-2/3}.$$

Finally consider $\mathcal{T}_{\neq\neq}$. First, divide up based on the presence of $U^1_{\neq}$:

$$\mathcal{T}_{\neq\neq} = \int BQ^3 B \left(U^j_{\neq} \partial^t_j Q^3_{\neq} \right) dV dt = \mathcal{T}^1_{\neq\neq} + \mathcal{T}^2_{\neq\neq} + \mathcal{T}^3_{\neq\neq}.$$

The latter two terms can be treated in a straightforward manner using (2.13) and Propositions 2.2 and 2.3,

$$\mathcal{T}^{2,3}_{\neq\neq} \lesssim \nu^{-2/3} \left\| BQ^3 \right\|_{L^\infty L^2} \left\| U^{2,3}_{\neq} \right\|_{L^2 H^N} \left\| \nabla_L BQ^3 \right\|_{L^2L^2} \lesssim \varepsilon^3 \nu^{-4/3}.$$

Next, decompose $\mathcal{T}^1_{\neq\neq}$ via a paraproduct

$$\begin{aligned}
\mathcal{T}^1_{\neq\neq} &= \int BQ^3 B \left((U^1_{\neq})_{Hi} (\partial_X Q^3_{\neq})_{Lo} \right) dV dt \\
&\quad + \int BQ^3 B \left((U^1_{\neq})_{Lo} (\partial_X Q^3_{\neq})_{Hi} \right) dV dt \\
&= \mathcal{T}^1_{\neq\neq;\mathrm{HL}} + \mathcal{T}^1_{\neq\neq;LH}.
\end{aligned}$$

For the HL term, we have the following by (2.13) and Propositions 2.2 and 2.3 (specifically, (2.25c) and (2.26d)):

$$\begin{aligned}
\mathcal{T}^1_{\neq\neq;\mathrm{HL}} &\lesssim \left\| BQ^3 \right\|_{L^\infty L^2} \left\| U^1_{\neq} \right\|_{L^2 H^N} \left\| Q^3_{\neq} \right\|_{L^2 H^{5/2+}} \\
&\lesssim \nu^{-2/3} \left\| BQ^3 \right\|_{L^\infty L^2} \left\| U^1_{\neq} \right\|_{L^2 H^N} \left\| BQ^3_{\neq} \right\|_{L^2L^2} \lesssim \varepsilon^3 \nu^{-4/3}.
\end{aligned}$$

For the LH term, we use the better estimate on $\left\| U^1_{\neq} \right\|_{H^{N-1}}$ from Proposition 2.3 (and (2.13)):

$$\mathcal{T}^1_{\neq\neq;LH} \lesssim \nu^{-2/3} \left\| BQ^3 \right\|_{L^\infty L^2} \left\| U^1_{\neq} \right\|_{L^2 H^{3/2+}} \left\| \partial_X BQ^3 \right\|_{L^2L^2} \lesssim \varepsilon^3 \nu^{-4/3}.$$

This completes the transport nonlinearity.

4.4. *Nonlinear pressure and stretching terms.*

4.4.1. *The stretching terms.* Recall the shorthands defined in Section 2.2.3. The NLS$i(0,0)$ terms are treated as in Section 3.1.3 and are hence omitted here .

For NLS1, we get from Proposition 2.3 (note $j \neq 1$)

$$\begin{aligned} \mathrm{NLS1}(j,\neq,0) &\lesssim \left\| BQ^3_{\neq} \right\|_{L^2L^2} \left\| Q^j_{\neq} \right\|_{L^2H^N} \left\| \nabla U^3_0 \right\|_{L^\infty H^N} \\ &\lesssim \varepsilon^3 \nu^{-1/6-2/3-1/6} = \varepsilon^3\nu^{-1}. \end{aligned}$$

Similarly, by Propositions 2.3 and 2.2,

$$\begin{aligned} \mathrm{NLS1}(j,0,\neq) &\lesssim \left\| BQ^3_{\neq} \right\|_{L^2L^2} \left\| Q^j_0 \right\|_{L^\infty H^N} \left\| \partial^t_j U^3_{\neq} \right\|_{L^2H^N} \\ &\lesssim \varepsilon^3 \left(\nu^{-1/6-1-1/6}\mathbf{1}_{j=1} + \nu^{-1/6-1/2}\mathbf{1}_{j\neq 1} \right) \\ &\lesssim \varepsilon^3\nu^{-4/3}. \end{aligned}$$

For the interaction of nonzero frequencies, we use the slight variant

$$\begin{aligned} \mathrm{NLS1}(j,\neq,\neq) &\lesssim \left\| BQ^3 \right\|_{L^\infty L^2} \left\| Q^j_{\neq} \right\|_{L^2H^N} \left\| \partial^t_j U^3_{\neq} \right\|_{L^2H^N} \\ &\lesssim \varepsilon^3 \left(\nu^{-2/3-1/6-1/2}\mathbf{1}_{j\neq 1} + \nu^{-1-1/6-1/6}\mathbf{1}_{j=1} \right) \\ &\lesssim \varepsilon^3\nu^{-4/3}, \end{aligned}$$

which is sufficient for $\varepsilon\nu^{-4/3} \ll 1$.

Turn next to NLS2; first by Proposition 2.3 (since $j \neq 1$),

$$\begin{aligned} \mathrm{NLS2}(i,j,\neq,0) &\lesssim \left\| BQ^3_{\neq} \right\|_{L^2L^2} \left\| \nabla_L U^j_{\neq} \right\|_{L^2H^N} \left\| \nabla U^3_0 \right\|_{L^\infty H^{N+1}} \\ &\lesssim \varepsilon^3 \nu^{-1/6-1/2} = \varepsilon^3\nu^{-2/3}, \end{aligned}$$

which suffices. Consider next NLS2$(0,\neq,1)$, which requires a slightly more precise treatment. Via a paraproduct decomposition, Lemma 2.4, and Lemma A.1, there holds

$$\begin{aligned} \mathrm{NLS2}(i,1,0,\neq) \lesssim &\left\| BQ^3_{\neq} \right\|_{L^2L^2} \left\| \nabla U^1_0 \right\|_{L^\infty H^{5/2+}} \left\| m^{1/2} \partial^t_i \partial_X U^3_{\neq} \right\|_{L^2H^N} \\ &+ \left\| BQ^3_{\neq} \right\|_{L^2L^2} \left\| \langle t \rangle^{-1} \partial^t_i U^1_0 \right\|_{L^\infty H^{N+1}} \left\| m^{1/2} \langle t \rangle \partial^t_i \partial_X U^3_{\neq} \right\|_{L^2H^{3/2+}}. \end{aligned}$$

Using (2.20), (2.14), and $t\,|k| \lesssim \langle \eta \rangle \langle \eta - kt \rangle$, followed by Proposition 2.2 and (2.25), we have

$$\begin{aligned} \mathrm{NLS2}(i,1,0,\neq) \lesssim &\left\| BQ^3_{\neq} \right\|_{L^2L^2} \left\| U^1_0 \right\|_{L^\infty H^{N+2}} \left\| m\Delta_L U^3_{\neq} \right\|_{L^2H^N} \\ &+ \left\| BQ^3_{\neq} \right\|_{L^2L^2} \left\| \langle t \rangle^{-1} U^1_0 \right\|_{L^\infty H^{N+2}} \left\| m\nabla_L \Delta_L U^3_{\neq} \right\|_{L^2H^{5/2+}} \\ \lesssim &\,\varepsilon^3 \nu^{-1/6-1-1/6} + \varepsilon^3\nu^{-1/6-1/2} = \varepsilon^3\nu^{-4/3}, \end{aligned}$$

which is sufficient. For $j \neq 1$ contributions, we have

$$\mathrm{NLS2}(i,j,0,\neq) \lesssim \left\| BQ^3_{\neq} \right\|_{L^2L^2} \left\| \nabla U^j_0 \right\|_{L^\infty H^{N+1}} \left\| m^{1/2} \Delta_L U^3_{\neq} \right\|_{L^2H^N} \lesssim \varepsilon^3 \nu^{-2/3}.$$

Turn finally to the interaction of nonzero frequencies; using (2.14), (2.13), and Proposition 2.2,

$$\begin{aligned}
\mathrm{NLS2}(i,j,\neq,\neq) &\lesssim \left\| BQ^3 \right\|_{L^\infty L^2} \left\| \nabla_L U^{2,3}_{\neq} \right\|_{L^2H^N} \left\| \Delta_L U^3_{\neq} \right\|_{L^2H^N} \\
&\quad + \left\| BQ^3 \right\|_{L^\infty L^2} \left\| \nabla_L U^1_{\neq} \right\|_{L^2H^N} \left\| \partial_X \nabla_L U^3_{\neq} \right\|_{L^2H^N} \\
&\lesssim \nu^{-1} \left\| BQ^3 \right\|_{L^\infty L^2} \left\| m\Delta_L U^{2,3}_{\neq} \right\|_{L^2H^N} \left\| m\Delta_L U^3_{\neq} \right\|_{L^2H^N} \\
&\quad + \nu^{-2/3} \left\| BQ^3 \right\|_{L^\infty L^2} \left\| m\Delta_L U^1_{\neq} \right\|_{L^2H^N} \left\| m\Delta_L U^3_{\neq} \right\|_{L^2H^N} \\
&\lesssim \varepsilon^3 \nu^{-1-1/6-1/6} = \varepsilon^3 \nu^{-4/3},
\end{aligned}$$

which completes the treatment of the stretching terms.

4.4.2. *The pressure term* NLP. The treatment of the $\mathrm{NLP}(i,j,0,0)$ term is the same as that in Section 3.1.2 and is hence omitted here. For the leading order term involving $i = 1$, we begin by subdividing via a paraproduct decomposition:

$$\begin{aligned}
\mathrm{NLP}(1,j,0,\neq) &= \int_1^T \int BQ^3 B\partial^t_Z \left((\partial^t_j U^1_0)_{Hi} (\partial_X U^j_{\neq})_{Lo} \right) dV\, dt \\
&\quad + \int_1^T \int BQ^3 B\partial^t_Z \left((\partial^t_j U^1_0)_{Lo} (\partial_X U^j_{\neq})_{Hi} \right) dV\, dt \\
&= P_{\mathrm{HL}} + P_{\mathrm{LH}}.
\end{aligned}$$

By (2.23) and (2.27), we have, since $j \neq 1$,

$$P_{\mathrm{HL}} \lesssim \left\| \nabla_L BQ^3_{\neq} \right\|_{L^2L^2} \left\| \langle t \rangle^{-1} \partial^t_j U^1_0 \right\|_{L^\infty H^N} \left\| \langle t \rangle \partial_X U^j_{\neq} \right\|_{L^2H^{3/2+}} \lesssim \varepsilon^3 \nu^{-1}.$$

By (2.23), Lemma A.1, (2.13), Lemma 2.4, and the inequality $1 \lesssim (|k| + |l| + |\eta - kt|)\sqrt{-\dot{M}^0 M^0}$ we deduce

$$\begin{aligned}
P_{\mathrm{LH}} &\lesssim \left\| \nabla_L BQ^3_{\neq} \right\|_{L^2L^2} \left\| \partial^t_j U^1_0 \right\|_{L^\infty H^{5/2+}} \left\| \sqrt{-\dot{M}M} m^{1/2} \partial_X \nabla_L U^j_{\neq} \right\|_{L^2H^N} \\
&\lesssim \left\| \nabla_L BQ^3_{\neq} \right\|_{L^2L^2} \left\| U^1_0 \right\|_{L^\infty H^{N+2}} \left\| \sqrt{-\dot{M}M} m\Delta_L U^j_{\neq} \right\|_{L^2H^N} \\
&\lesssim \varepsilon^3 \nu^{-3/2},
\end{aligned}$$

which completes the $\mathrm{NLP}(1,j,0,\neq)$ terms for $\varepsilon \nu^{-3/2} \ll 1$. For the $\mathrm{NLP}(i \neq 1, j, 0, \neq)$ terms, a much simpler argument is possible; indeed, by (2.22) and Proposition 2.3,

$$\mathrm{NLP}(i \neq 1, j, 0, \neq) \lesssim \left\| \nabla_L BQ^3_{\neq} \right\|_{L^2L^2} \left\| \partial^t_j U^i_0 \right\|_{L^\infty H^N} \left\| \nabla_L U^j_{\neq} \right\|_{L^2H^N} \lesssim \varepsilon^3 \nu^{-1}.$$

Turn next to the $\mathrm{NLP}(1,j,\neq,\neq)$ terms. By the paraproduct decomposition we deduce

$$\begin{aligned}\mathrm{NLP}(1,j,\neq,\neq) \lesssim \left\|\nabla_L BQ^3\right\|_{L^\infty L^2} \Big(& \left\|\partial_t^j U^1_{\neq}\right\|_{L^\infty H^N} \left\|\partial_X U^j_{\neq}\right\|_{L^2 H^{3/2+}} \\ & + \left\|\partial_j^t U^1_{\neq}\right\|_{L^\infty H^{3/2+}} \left\|\partial_X U^j_{\neq}\right\|_{L^2 H^N}\Big).\end{aligned}$$

For $j=1$, we use (2.14), (3.1), Proposition 2.3, and Lemma 2.2 to deduce,

$$\mathrm{NLP}(1,1,\neq,\neq) \lesssim \left\|\nabla_L BQ^3\right\|_{L^\infty L^2} \left\|m\Delta_L U^1_{\neq}\right\|_{L^\infty H^N} \left\|\partial_X U^1_{\neq}\right\|_{L^2 H^{3/2+}} \lesssim \varepsilon^3\nu^{-1}.$$

For $j \neq 1$, we use instead (using (2.14), Lemma 2.1, and Lemma 2.2),

$$\begin{aligned}&\mathrm{NLP}(1,j,\neq,\neq)\mathbf{1}_{j\neq 1} \\ &\quad\lesssim \left\|\nabla_L BQ^3\right\|_{L^\infty L^2} \Big(\left\|\nabla_L U^1_{\neq}\right\|_{L^\infty H^N} \left\|\partial_X U^j_{\neq}\right\|_{L^2 H^{3/2+}} \\ &\qquad + \left\|\nabla_L U^1_{\neq}\right\|_{L^\infty H^{3/2+}} \left\|\partial_X U^j_{\neq}\right\|_{L^2 H^N}\Big)\mathbf{1}_{j\neq 1} \\ &\quad\lesssim \nu^{-1/3}\left\|\nabla_L BQ^3\right\|_{L^\infty L^2} \left\|m\Delta_L U^1_{\neq}\right\|_{L^\infty H^N} \left\|m\Delta_L U^j_{\neq}\right\|_{L^2 H^N}\mathbf{1}_{j\neq 1} \\ &\quad\lesssim \varepsilon^3\nu^{-1/3-1/2-1/3-1/6} = \varepsilon^3\nu^{-4/3}.\end{aligned}$$

For the other $\neq,\neq$ terms, we use the following via (2.21), (2.14), Lemma 2.1, and Proposition 2.2:

$$\begin{aligned}\mathrm{NLP}(i,j,\neq,\neq)\mathbf{1}_{i,j\neq 1} &\lesssim \left\|\nabla_L BQ^3\right\|_{L^2 L^2} \left\|\nabla_L U^i_{\neq}\right\|_{L^\infty H^N} \left\|\nabla_L U^j_{\neq}\right\|_{L^2 H^N}\mathbf{1}_{i,j\neq 1} \\ &\lesssim \nu^{-2/3}\left\|\nabla_L BQ^3\right\|_{L^2 L^2} \left\|m\Delta_L U^i_{\neq}\right\|_{L^\infty H^N} \left\|m\Delta_L U^j_{\neq}\right\|_{L^2 H^N}\mathbf{1}_{i,j\neq 1} \\ &\lesssim \varepsilon^3\nu^{-4/3}.\end{aligned}$$

This completes all of the nonlinear pressure terms.

4.5. *Dissipation error terms.* Next turn to the dissipation error terms,

$$\mathrm{DE} = \nu\int\int BQ^3 B\left(G\partial^L_{YY}Q^3 + 2\psi_z\partial^L_{ZY}Q^3\right)dV\,dt = \mathcal{E}_1 + \mathcal{E}_2.$$

We will need a slightly more refined treatment here than was used in Section 3.1.4. As $\mathcal{E}_1$ is slightly harder, we will just treat this term and omit the treatment of $\mathcal{E}_2$ for brevity. At the zero X frequency we have, via integration by parts and the product rule,

$$\begin{aligned}\mathcal{E}_{1;0} &= \nu\int\int BQ^3_0 B\left(G\partial_{YY}Q^3_0\right)dV\,dt \\ &\lesssim \nu\left\|Q^3_0\right\|_{L^\infty H^N}\left\|\nabla G\right\|_{L^2 H^N}\left\|\nabla Q^3_0\right\|_{L^2 H^N} + \nu\left\|G\right\|_{L^\infty H^N}\left\|\nabla Q^3_0\right\|^2_{L^2 H^N} \\ &\lesssim \varepsilon^3\nu^{-1}.\end{aligned}$$

Turn next to the nonzero frequencies. Via integration by parts, Lemma A.1, and (2.13),

$$
\begin{aligned}
\mathcal{E}_{1;\neq} &= \nu \int BQ^3_{\neq} B\left(G\partial^L_{YY} Q^3_{\neq}\right) dV\,dt \\
&\lesssim \nu \left\|\nabla_L BQ^3_{\neq}\right\|_{L^2L^2} \|G\|_{L^\infty H^{N+1}} \left\|m^{1/2}\nabla_L Q^3_{\neq}\right\|_{L^2H^N} \\
&\quad + \nu \left\|BQ^3_{\neq}\right\|_{L^2L^2} \|G\|_{L^\infty H^{N+1}} \left\|\nabla_L Q^3_{\neq}\right\|_{L^2H^N} \\
&\lesssim \varepsilon^3 \nu^{-4/3}.
\end{aligned}
$$

This completes the treatment of the dissipation error terms.

5. Energy estimates on Q^1

The energy estimates on Q^1 are generally much simpler than estimates on Q^3 as the bounds (2.16c), (2.16a), and (2.16b) are so much weaker than (2.16e). (The lift-up effect growth is generally much larger than what the nonlinear terms could do in this regime.)

5.1. *Energy estimate on $Q^1_{\neq}$ in H^N.* An energy estimate gives (recall the shorthand (2.28))

$$
\begin{aligned}
&\frac{1}{2}\|BQ^1_{\neq}(T)\|^2_{L^2} + \nu\|\nabla_L BQ^1_{\neq}\|^2_{L^2L^2} \\
&\qquad + \|m\sqrt{-\dot{M}M}Q^1_{\neq}\|^2_{L^2H^N} + \|M\sqrt{-\dot{m}m}Q^1_{\neq}\|^2_{L^2H^N} \\
&\quad = \frac{1}{2}\|BQ^1_{\neq}(1)\|^2_{L^2} + \int_1^T \int BQ^1_{\neq} B\left[-Q^2_{\neq} - 2\partial^t_{XY}U^1_{\neq} + 2\partial_{XX}U^2_{\neq}\right. \\
&\qquad - ((\widetilde{U}_0\cdot\nabla + U_{\neq}\cdot\nabla_t)Q^1)_{\neq} - (Q^j\partial^t_i U^1)_{\neq} - 2(\partial^t_i U^j \partial^t_i\partial^t_j U^1)_{\neq} \\
&\qquad \left. + \partial_X\left(\partial^t_i U^j \partial^t_j U^i\right) + \nu(\widetilde{\Delta}_t - \Delta_L)Q^1_{\neq}\right] dV\,dt \\
&\quad = \frac{1}{2}\|mMQ^1_{\neq}(1)\|^2_{H^N} + \mathrm{LU} + \mathrm{LS} + \mathrm{LP} \\
&\qquad + \mathcal{T} + \mathrm{NLS1} + \mathrm{NLS2} + \mathrm{NLP} + \mathrm{DE}.
\end{aligned}
$$

Several terms above can be estimated exactly like the corresponding terms for Q^3, namely, LS, LP, and DE. Therefore, we omit the estimates of these terms for brevity and only treat the others.

5.1.1. *The lift up term* LU. The lift-up effect term is treated via Proposition 2.3, which implies

$$
\mathrm{LU} = -\int_1^T \int BQ^1_{\neq} BQ^2_{\neq} dV\,dt \lesssim \left\|BQ^1_{\neq}\right\|_{L^2L^2} \left\|AQ^2_{\neq}\right\|_{L^2L^2} \lesssim C_0\varepsilon^2\nu^{-2/3},
$$

which is consistent with the estimate as stated for C_0 chosen sufficiently large.

5.1.2. *The stretching and pressure terms* NLS1, NLS2, *and* NLP. We will focus on NLS1 and NLS2; the NLP terms can be treated analogously to the latter.

Consider first the NLS1$(0, \neq, 1)$ terms. Using a paraproduct decomposition as has been done several times above and applying Lemma A.1, we get

$$\begin{aligned}
\mathrm{NLS1}(0, \neq, 1) &\lesssim \left\| BQ^1_{\neq} \right\|_{L^2L^2} \left\| Q^1_0 \right\|_{L^\infty H^{5/2+}} \left\| \partial_X m^{1/2} U^1_{\neq} \right\|_{L^2H^N} \\
&\quad + \left\| BQ^1_{\neq} \right\|_{L^2L^2} \left\| Q^1_0 \right\|_{L^\infty H^N} \left\| \partial_X U^1_{\neq} \right\|_{L^2H^{3/2+}} \\
&\lesssim \nu^{-2/3} \varepsilon^2 \left(\varepsilon \nu^{-4/3} \right).
\end{aligned}$$

For corresponding terms with $j \neq 1$, an easier treatment is available:

$$\begin{aligned}
\mathrm{NLS1}(0, \neq, j \neq 1) &\lesssim \left\| BQ^1_{\neq} \right\|_{L^2L^2} \left\| Q^j_0 \right\|_{L^\infty H^N} \left\| \nabla_L U^1_{\neq} \right\|_{L^2H^N} \\
&\lesssim \varepsilon^{-1/3-1/6-1/2-1/3} = \left(\varepsilon \nu^{-2/3} \right) \nu^{-2/3} \varepsilon^2.
\end{aligned}$$

Similarly (noting that $j \neq 1$ by the nonlinear structure),

$$\mathrm{NLS1}(j, \neq, 0) \lesssim \left\| BQ^1_{\neq} \right\|_{L^2L^2} \left\| m^{1/2} Q^j_{\neq} \right\|_{L^2H^N} \left\| U^1_0 \right\|_{L^\infty H^{N+2}} \lesssim \left(\varepsilon \nu^{-4/3} \right) \varepsilon^2 \nu^{-2/3}.$$

Finally, for the NLS1$(i, \neq, \neq)$ terms, we may use another straightforward argument. By (2.14),

$$\begin{aligned}
\mathrm{NLS1}(j, \neq, \neq) &\lesssim \left\| BQ^1_{\neq} \right\|_{L^2L^2} \left\| Q^1_{\neq} \right\|_{L^\infty H^N} \left\| \partial_X U^1_{\neq} \right\|_{L^2H^N} \\
&\quad + \left\| BQ^1_{\neq} \right\|_{L^2L^2} \left\| Q^{2,3}_{\neq} \right\|_{L^\infty H^N} \left\| \nabla_L U^1_{\neq} \right\|_{L^2H^N} \\
&\lesssim \left(\varepsilon \nu^{-4/3} \right) \nu^{-2/3} \varepsilon^2.
\end{aligned}$$

This completes the NLS1 terms.

Turning to NLS2, we have first, since $j \neq 1$,

$$\begin{aligned}
\mathrm{NLS2}(i, j, \neq, 0) &\lesssim \| BQ^1 \|_{L^2L^2} \| \nabla_L U^{2,3}_{\neq} \|_{L^2H^N} \| \nabla U^1_0 \|_{L^\infty H^{N+1}} \\
&\lesssim \left(\varepsilon \nu^{-4/3} \right) \varepsilon^2 \nu^{-2/3}.
\end{aligned}$$

Next, we rely on Lemma A.1 and (2.20) (for the $j = 1$ case), (2.14), and (2.13),

$$\begin{aligned}
\mathrm{NLS2}(i, j, 0, \neq) &\lesssim \| BQ^1_{\neq} \|_{L^2L^2} \| \nabla_t U^1_0 \|_{L^\infty H^{N+1}} \| m^{1/2} \partial_X \nabla_L U^1_{\neq} \|_{L^2H^N} \mathbf{1}_{j=1} \\
&\quad + \| BQ^1_{\neq} \|_{L^2L^2} \| \nabla_t U^{2,3}_0 \|_{L^\infty H^N} \| \Delta_L U^1_{\neq} \|_{L^2H^N} \mathbf{1}_{j\neq 1} \\
&\lesssim \| BQ^1_{\neq} \|_{L^2L^2} \| U^1_0 \|_{L^\infty H^{N+2}} \| m \Delta_L U^1_{\neq} \|_{L^2H^N} \mathbf{1}_{j=1} \\
&\quad + \nu^{-2/3} \| BQ^1_{\neq} \|_{L^2L^2} \| U^{2,3}_0 \|_{L^\infty H^{N+1}} \| m \Delta_L U^1_{\neq} \|_{L^2H^N} \mathbf{1}_{j\neq 1} \\
&\lesssim \left(\varepsilon \nu^{-4/3} \right) \varepsilon^2 \nu^{-2/3}.
\end{aligned}$$

Finally, (2.22), a paraproduct decomposition, (2.14), and Propositions 2.2 and 2.3 imply

$$\begin{aligned}\mathrm{NLS2}(i,j,\neq,\neq) \lesssim \|BQ^1_{\neq}\|_{L^2L^2} &\Big(\|\nabla_L U^i_{\neq}\|_{L^\infty H^{3/2+}}\|\Delta_L U^1_{\neq}\|_{L^2H^N} \\ &\quad + \|\nabla_L U^i_{\neq}\|_{L^\infty H^N}\|\Delta_L U^1_{\neq}\|_{L^2H^{3/2+}}\Big)\mathbf{1}_{i\neq 1} \\ &+ \|BQ^1_{\neq}\|_{L^2L^2}\Big(\|\nabla_L U^1_{\neq}\|_{L^\infty H^{3/2+}}\|\partial_X\nabla_L U^1_{\neq}\|_{L^2H^N} \\ &\quad + \|\nabla_L U^1_{\neq}\|_{L^\infty H^N}\|\partial_X\nabla_L U^1_{\neq}\|_{L^2H^{3/2+}}\Big) \\ &\lesssim \mathbf{1}_{i\neq 1}\left(\varepsilon\nu^{-4/3}\right)\varepsilon^2\nu^{-2/3} + \mathbf{1}_{i=1}\left(\varepsilon\nu^{-1}\right)\varepsilon^2\nu^{-2/3},\end{aligned}$$

which suffices for $\varepsilon\nu^{-4/3} \ll 1$.

5.1.3. *Transport nonlinearity.* These terms can mostly be treated as in Section 4.3, however, one must check the contributions from Q^1_0. As in Section 3.1.1 and Section 4.3, we subdivide based on frequency (note that the slight difference since we are only focusing on nonzero frequencies here):

$$\begin{aligned}\mathcal{T} &= \int_1^T\int BQ^1_{\neq}B\left(\tilde{U}_0\cdot\nabla Q^1_{\neq}\right)dV\,dt \\ &\quad + \int_1^T\int BQ^1_{\neq}B\left(U_{\neq}\cdot\nabla_t Q^1_0 + U_{\neq}\cdot\nabla_t Q^1_{\neq}\right)dV\,dt \\ &= \mathcal{T}_{0\neq} + \mathcal{T}_{\neq 0} + \mathcal{T}_{\neq\neq}.\end{aligned}$$

The terms $\mathcal{T}_{0\neq}$ and $\mathcal{T}_{\neq\neq}$ can be treated as in Section 4.3 and are hence omitted for the sake of brevity. Hence, turn to the remaining $\mathcal{T}_{\neq 0}$. Here we have (note the nonlinear structure that eliminates $U^1_{\neq}$)

$$\begin{aligned}\mathcal{T}_{\neq 0} &\lesssim \left\|BQ^1_{\neq}\right\|_{L^2L^2}\left\|U^{2,3}_{\neq}\right\|_{L^\infty H^N}\left\|\nabla Q^1_0\right\|_{L^2H^N} \lesssim \varepsilon^3\nu^{-1/3-1/6-3/2} \\ &\lesssim \nu^{-2/3}\varepsilon^2\left(\varepsilon\nu^{-4/3}\right).\end{aligned}$$

5.2. *Long-time energy estimate on* Q^1_0. In this section we improve the estimate (2.16b). First, Q^1_0 solves the equation

$$\begin{aligned}\partial_t Q^1_0 - \nu\widetilde{\Delta}_t Q^1_0 + Q^2_0 = &-((\widetilde{U}_0\cdot\nabla + U_{\neq}\cdot\nabla_t)Q^1)_0 \\ &- (Q^j\partial^t_j U^1)_0 - 2(\partial^t_i U^j\partial^t_j\partial^t_i U^1)_0.\end{aligned}$$

An energy estimate gives

$$\begin{aligned}&\frac{1}{2}\|Q^1_0(T)\|^2_{H^N} + \nu\|\nabla_L Q^1_0\|^2_{L^2H^N} = \frac{1}{2}\|(Q^1(1))_0\|^2_{H^N} \\ &\quad + \int_1^T\int\langle D\rangle^N Q^1_0\langle D\rangle^N\Big[-Q^2_0 - ((\widetilde{U}_0\cdot\nabla + U_{\neq}\cdot\nabla_t)Q^1)_0 - (Q^j\partial^t_j U^1)_0 \\ &\qquad -2(\partial^t_i U^j\partial^t_j\partial^t_i U^1)_0 + \nu(\widetilde{\Delta}_t - \Delta_L)Q^1_0\Big]\,dV\,dt \\ &= \frac{1}{2}\|(Q^1(1))_0\|^2_{H^N} + \mathrm{LU} + \mathcal{T} + \mathrm{NLS1} + \mathrm{NLS2} + \mathrm{DE}.\end{aligned}$$

5.2.1. *The lift up term* LU. Using Lemma 2.4 and Proposition 2.2,

$$\begin{aligned}
\mathrm{LU} = &-\int_1^T \int \langle D\rangle^N Q_0^1 \langle D\rangle^N \left[\Delta U_0^2 + G\partial_{YY} U_0^2 + 2\psi_z \partial_{YZ} U_0^1 \right.\\
&\qquad\qquad\qquad\qquad \left. + \Delta_t C \partial_Y U_0^2\right] dV dt \\
&\lesssim (1 + \|\nabla C\|_{L^\infty H^N}) \left\|\nabla Q_0^1\right\|_{L^2 H^N} \left\|\nabla U_0^2\right\|_{L^2 H^N} \\
&\quad + \|\nabla C\|_{L^2 H^{N+1}} \left\|Q_0^1\right\|_{L^\infty H^N} \left\|\nabla U_0^2\right\|_{L^2 H^N} \\
&\lesssim C_0 \varepsilon^2 \nu^{-2},
\end{aligned}$$

which is consistent with the bootstrap argument provided C_0 is chosen sufficiently large.

5.2.2. *Transport nonlinearity.* Similar to Section 5.1.3, we subdivide based on frequency:

$$\begin{aligned}
\mathcal{T} &= \int_1^T \int \langle D\rangle^N Q_0^1 \langle D\rangle^N \left(\tilde{U}_0 \cdot \nabla Q_0^1\right) dV dt \\
&\quad + \int_1^T \int \langle D\rangle^N Q_0^1 \langle D\rangle^N \left(U_{\neq} \cdot \nabla_t Q_{\neq}^1\right) dV dt \\
&= \mathcal{T}_0 + \mathcal{T}_{\neq}.
\end{aligned}$$

The zero frequencies $\mathcal{T}_0$ can be treated as in Section 3.1.1 and are hence omitted for brevity. For the nonzero frequencies, first apply the divergence-free condition:

$$\mathcal{T}_{\neq} = \int_1^T \int \langle D\rangle^N Q_0^1 \langle D\rangle^N \nabla_t \cdot \left(U_{\neq} Q_{\neq}^1\right)_0 dV dt.$$

Due to the X average, the contribution from $U_{\neq}^1$ is crucially eliminated as well as the term involving $-t\partial_X$ in ∂_Y^L. Hence, by (2.21), (2.13) and Propositions 2.2 and 2.3,

$$\begin{aligned}
\mathcal{T}_{\neq} &\lesssim \left\|\nabla Q_0^1\right\|_{L^2 H^N} \left\|U_{\neq}^{2,3}\right\|_{L^2 H^N} \left\|Q_{\neq}^1\right\|_{L^\infty H^N} \\
&\quad + \left\|Q_0^1\right\|_{L^\infty H^N} \|C\|_{L^\infty H^{N+2}} \left\|U_{\neq}^{2,3}\right\|_{L^2 H^N} \left\|Q_{\neq}^1\right\|_{L^2 H^N} \\
&\lesssim \nu^{-2} \varepsilon^2 \left(\varepsilon \nu^{-2/3} + \varepsilon^2 \nu^{-4/3}\right),
\end{aligned}$$

which suffices.

5.2.3. *Nonlinear stretching terms.* Consider first $\mathrm{NLS1}(0,0)$, which are treated similar to $\mathrm{NLS}(0,0)$ and $\mathrm{NLP}(0,0)$ terms above: by Proposition 2.2 (and the fact that $j \neq 1$),

$$\mathrm{NLS1}(0,0) \lesssim \left\|Q_0^1\right\|_{L^\infty H^N} \left\|\nabla U_0^j\right\|_{L^2 H^{N+1}} \left\|\partial_j^t U_0^1\right\|_{L^2 H^N} \lesssim \frac{\varepsilon}{\nu}\left(\frac{\varepsilon^2}{\nu^2}\right),$$

which is sufficient. $\mathrm{NLS2}(0,0)$ is treated similarly and is hence omitted.

Turn next to NLS1($\neq,\neq$):

$$\mathrm{NLS1}(\neq,\neq,j\neq 1)\lesssim \left\|Q_0^1\right\|_{L^\infty H^N}\left\|Q_{\neq}^j\right\|_{L^2H^N}\left\|\partial_j^t U_{\neq}^1\right\|_{L^2H^N}\lesssim \frac{\varepsilon}{\nu^{2/3}}\left(\frac{\varepsilon^2}{\nu^2}\right)$$
$$\mathrm{NLS1}(\neq,\neq,1)\lesssim \left\|Q_0^1\right\|_{L^\infty H^N}\left\|Q_{\neq}^1\right\|_{L^2H^N}\left\|\partial_X U_{\neq}^1\right\|_{L^2H^N}\lesssim \frac{\varepsilon}{\nu^{2/3}}\left(\frac{\varepsilon^2}{\nu^2}\right),$$

which is sufficient. The NLS2($\neq,\neq$) term is treated analogously and is hence omitted for brevity.

5.2.4. *The dissipation error terms* DE. These are controlled as in Section 3.1.4 and are hence omitted for brevity.

5.3. *Short-time energy estimate on Q_0^1 in H^N.* Here we deduce (2.16a), which we refer to as a "short-time" estimate since it provides a superior estimate on $\|Q_0^1(t)\|_{H^N}$ for $t\ll\nu^{-1}$ versus the "long-time" estimate $\|Q_0^1(t)\|_{H^N}\lesssim \varepsilon\nu^{-1}$.

For this estimate (and the similar (2.17a)), we use a slightly different method from that which we have applied for most estimates in the paper. Consider the differential equality

$$\frac{1}{2}\frac{d}{dt}\left(\langle t\rangle^{-2}\left\|Q_0^1(t)\right\|_{H^N}^2\right)=-\frac{t}{\langle t\rangle^4}\left\|Q_0^1(t)\right\|_{H^N}^2-\langle t\rangle^{-2}\int\langle D\rangle^N Q_0^1\langle D\rangle^N Q_0^2 dV$$
$$-\nu\langle t\rangle^{-2}\left\|\nabla Q_0^1\right\|_{H^N}^2+\mathcal{NL}, \tag{5.1}$$

where using the shorthand from Section 2.2.3 analogous to that used in Section 5.2,

$$\mathcal{NL}=\mathcal{T}+\mathrm{NLS1}+\mathrm{NLS2}+\mathrm{DE}$$

denotes the contributions from all of the nonlinear terms. For the lift-up effect term, by (2.16d),

$$-\langle t\rangle^{-2}\int\langle D\rangle^N Q_0^1\langle D\rangle^N Q_0^2 dV\le\langle t\rangle^{-2}\left\|Q_0^1\right\|_{H^N}\left\|Q_0^2\right\|_{H^N}\le\langle t\rangle^{-2}8\varepsilon\|Q_0^1\|_{H^N},$$

and hence (5.1) becomes

$$\frac{1}{2}\frac{d}{dt}\left(\langle t\rangle^{-2}\left\|Q_0^1(t)\right\|_{H^N}^2\right)+\nu\langle t\rangle^{-2}\left\|\nabla Q_0^1\right\|_{H^N}^2$$
$$\le\frac{1}{\langle t\rangle^2}\left(8\varepsilon-\frac{t}{\langle t\rangle^2}\left\|Q_0^1(t)\right\|_{H^N}\right)\left\|Q_0^1(t)\right\|_{H^N}+\mathcal{NL}. \tag{5.2}$$

It follows from this differential inequality (and continuity) that if $\mathcal{NL}\le\frac{1}{2}\nu\langle t\rangle^{-2}\|\nabla Q_0^1(t)\|_{H^N}^2+f(t)$, with $\|f\|_{L^1}\le C_0\varepsilon^2$, then (2.16a) holds for C_0 sufficiently large. Indeed, let $t\in(a,b)\subset[1,T]$, where $a<b$ is such that

$\langle s\rangle^{-1}\left\|Q^1_0(s)\right\|_{H^N} \geq 8\langle 1\rangle\varepsilon$ for all $s \in (a,b)$ and $\langle a\rangle^{-1}\left\|Q^1_0(a)\right\|_{H^N} \leq 8\langle 1\rangle\varepsilon$. Then, (5.2) implies

$$\frac{1}{2}\frac{d}{dt}\left(\langle t\rangle^{-2}\left\|Q^1_0(t)\right\|^2_{H^N}\right) + \frac{1}{2}\nu\langle t\rangle^{-2}\left\|\nabla Q^1_0(t)\right\|^2_{H^N} \leq f(t),$$

and hence, by integrating from a to t,

$$\langle t\rangle^{-2}\left\|Q^1_0(t)\right\|^2_{H^N} \leq 64\langle 1\rangle^2\varepsilon^2 + C_0\varepsilon^2.$$

By continuity, for C_0 sufficiently large, this implies $\langle t\rangle^{-1}\left\|Q^1_0(t)\right\|_{H^N} \leq 4C_0\varepsilon$.

5.3.1. *Transport nonlinearity.* As in Section 5.2.2, we divide the transport nonlinearity into two pieces:

$$\mathcal{T} = \langle t\rangle^{-2}\int \langle D\rangle^N Q^1_0 \langle D\rangle^N\left(\tilde{U}_0\cdot\nabla Q^1_0 + \left(U_{\neq}\cdot\nabla_t Q^1_{\neq}\right)_0\right)dV = \mathcal{T}_0 + \mathcal{T}_{\neq}.$$

The first term is treated analogously to the treatment in Section 3.1.1:

$$\begin{aligned}\mathcal{T}_0 &\lesssim \left\|U^3_0\right\|_{H^N}\left\|\langle t\rangle^{-1}\nabla Q^1_0\right\|^2_{H^N}\\ &\quad + \left(\|g\|_{L^2} + \|\nabla g\|_{H^N} + \left\|\nabla U^3_0\right\|_{H^N}\right)\left\|\langle t\rangle^{-1}\nabla Q^1_0\right\|_{H^N}\left\|\langle t\rangle^{-1}Q^1_0\right\|_{H^N}\\ &\lesssim \varepsilon\left\|\langle t\rangle^{-1}\nabla Q^1_0\right\|^2_{L^2H^N} + \varepsilon\left(\|g\|^2_{L^2} + \left\|\nabla U^3_0\right\|^2_{H^N} + \|\nabla g\|^2_{H^N}\right);\end{aligned}$$

the first term is absorbed by the dissipation in (5.2), and the latter term integrates to $O(\varepsilon^3\nu^{-1})$.

For $\mathcal{T}_{\neq}$, we first use the divergence-free condition as in Section 5.2.2:

$$\mathcal{T}_{\neq} = \langle t\rangle^{-2}\int \langle D\rangle^N Q^1_0\langle D\rangle^N \nabla_t\cdot\left(U_{\neq}Q^1_{\neq}\right)_0 dV,$$

which eliminates the contribution from U^1 and $-t\partial_X$. By (2.21), (2.13), and Proposition 2.2, and for any constant K,

$$\begin{aligned}\mathcal{T}_{\neq} &\lesssim \langle t\rangle^{-2}\left\|\nabla Q^1_0\right\|_{H^N}\left\|U^{2,3}_{\neq}\right\|_{H^N}\left\|Q^1_{\neq}\right\|_{H^N}\\ &\quad + \langle t\rangle^{-2}\left\|Q^1_0\right\|_{H^N}\|C\|_{H^{N+2}}\left\|U^{2,3}_{\neq}\right\|_{H^N}\left\|Q^1_{\neq}\right\|_{H^N}\\ &\lesssim \frac{\nu}{K}\langle t\rangle^{-2}\left\|\nabla Q^1_0\right\|^2_{H^N} + \frac{K}{\nu}\langle t\rangle^{-2}\left\|U^{2,3}_{\neq}\right\|^2_{H^N}\left\|Q^1_{\neq}\right\|^2_{H^N}\\ &\quad + \langle t\rangle^{-2}\left\|Q^1_0\right\|_{H^N}\|C\|_{H^{N+2}}\left\|U^{2,3}_{\neq}\right\|_{H^N}\left\|Q^1_{\neq}\right\|_{H^N}.\end{aligned}$$

The first term is absorbed by the dissipation in (5.2) for K sufficiently large; the remaining terms integrate to $\varepsilon^4\nu^{-3}$ and $\varepsilon^4\nu^{-2-1/6}$ using the L^∞ controls on U, Q, and C, which is sufficient. (Note that $\varepsilon^4\nu^{-3} \ll \varepsilon^2$ is borderline as it requires $\varepsilon\nu^{-3/2} \ll 1$.)

5.3.2. *Nonlinear stretching.* Turn first to the interaction of zero frequencies. Consider NLS1 (noting that $j \neq 1$):

$$\begin{aligned}\mathrm{NLS1}(0,0) &= \langle t\rangle^{-2}\int \langle D\rangle^N Q_0^1 \langle D\rangle^N \left(\Delta_t U_0^j \partial_j^t U_0^1\right) dV\\ &\lesssim \langle t\rangle^{-2}\left\|Q_0^1\right\|_{H^N}\left\|\nabla U_0^j\right\|_{H^{N+1}}\left\|\nabla U_0^1\right\|_{H^N}.\end{aligned}$$

Hence, by Proposition 2.2 and Cauchy-Schwarz in time,

$$\int_1^T \mathrm{NLS1}(0,0)dt \lesssim \left\|\langle t\rangle^{-1}Q_0^1\right\|_{L^\infty H^N}\left\|U_0^j\right\|_{L^\infty H^{N+2}}\left\|\nabla U_0^1\right\|_{L^2 H^N} \lesssim \varepsilon^3\nu^{-3/2},$$

which is sufficient for $\varepsilon\nu^{-3/2}$ sufficiently small. The $\mathrm{NLS2}(0,0)$ terms can be treated similarly and are hence omitted for the sake of brevity.

Consider next $\mathrm{NLS1}(\neq,\neq)$. Using that $\nabla_t \cdot Q = 0$ due to the divergence-free condition and (2.21), we have for any K,

$$\begin{aligned}\mathrm{NLS1}(\neq,\neq) &= \langle t\rangle^{-2}\int \langle D\rangle^N Q_0^1 \langle D\rangle^N \left(Q_{\neq}^j \partial_j^t U_{\neq}^1\right)_0 dV\\ &= \langle t\rangle^{-2}\int \langle D\rangle^N Q_0^1 \langle D\rangle^N \partial_j^t \left(Q_{\neq}^j U_{\neq}^1\right)_0 dV\\ &\lesssim \langle t\rangle^{-2}\left(\left\|\nabla Q_0^1\right\|_{H^N} + \left\|\nabla C\right\|_{H^{N+1}}\left\|Q_0^1\right\|_{H^N}\right)\left\|Q_{\neq}^{2,3}\right\|_{H^N}\left\|U_{\neq}^1\right\|_{H^N}\\ &\lesssim \frac{\nu}{K}\langle t\rangle^{-2}\left\|\nabla Q_0^1\right\|_{H^N}^2 + \frac{K}{\nu}\langle t\rangle^{-2}\left\|Q_{\neq}^{2,3}\right\|_{H^N}^2\left\|U_{\neq}^1\right\|_{H^N}^2\\ &\quad + \langle t\rangle^{-2}\left\|\nabla C\right\|_{H^{N+1}}\left\|Q_0^1\right\|_{H^N}\left\|Q_{\neq}^{2,3}\right\|_{H^N}\left\|U_{\neq}^1\right\|_{H^N}.\end{aligned}$$

For K large, the first term is absorbed by the dissipation. By the L^∞ controls from Proposition 2.2, the second factor integrates to $\varepsilon^2\left(\varepsilon^2\nu^{-3}\right)$ and the third factor integrates to $\varepsilon^2\left(\varepsilon^4\nu^{-4}\right)$, both of which are sufficient. The $\mathrm{NLS2}(\neq,\neq)$ term is treated similarly and is hence omitted.

5.3.3. *Dissipation error estimates.* Write

$$\mathrm{DE} = \langle t\rangle^{-2}\nu\int \langle D\rangle^N Q_0^1 \langle D\rangle^N \left(G\partial_{YY}Q_0^1 + 2\psi_z\partial_{YZ}Q_0^1\right) dV = \mathcal{E}_1 + \mathcal{E}_2.$$

We only bound $\mathcal{E}_1$; $\mathcal{E}_2$ is bounded in the same manner. Via integration by parts and the Sobolev product rule,

$$\begin{aligned}\mathcal{E}_1 &\lesssim \langle t\rangle^{-2}\nu\left\|G\right\|_{H^N}\left\|\nabla Q_0^1\right\|_{H^N}^2 + \langle t\rangle^{-2}\nu\left\|\nabla G\right\|_{H^N}\left\|Q_0^1\right\|_{H^N}\left\|\nabla Q_0^1\right\|_{H^N}\\ &\lesssim \langle t\rangle^{-2}\varepsilon\left\|\nabla Q_0^1\right\|_{H^N}^2 + \varepsilon\nu^2\left\|\nabla C\right\|_{H^{N+1}}^2.\end{aligned}$$

The first term is absorbed by the leading order dissipation in (5.2), and the other term integrates to $O(\varepsilon^3\nu^{-1})$, which suffices. This completes the short-time energy estimate on Q_0^1.

6. Energy estimate on $U^1_{\neq}$

In this section we deduce the control (2.17e). This is relatively easy due to the lower regularity, however, there are some differences here from previous arguments due to the fact that we are working in velocity form. The entire point of this estimate is that by working directly on the velocity, it is easier to take advantage of the inviscid damping from (2.26b) in the lift-up effect term, which is the reason for the large growth of $Q^1_{\neq}$.

From the momentum equations, the nonzero frequencies of U^1 solve

$$\begin{aligned}\partial_t U^1_{\neq} - \nu\widetilde{\Delta}_t U^1_{\neq} &= -U^2_{\neq} + 2\partial_{XX}\Delta_t^{-1}U^2_{\neq}\\ &\quad - \left(\left[\widetilde{U}_0\cdot\nabla + U_{\neq}\cdot\nabla_t\right]U^1\right)_{\neq} + \partial_X\Delta_t^{-1}(\partial_i^t U^j\partial_j^t U^i)_{\neq}.\end{aligned}$$

An energy estimate gives

$$\begin{aligned}&\frac{1}{2}\|MU^1_{\neq}(T)\|^2_{H^{N-1}} + \nu\|M\nabla_L U^1_{\neq}\|^2_{L^2H^{N-1}} + \|\sqrt{-\dot{M}M}U^1_{\neq}\|^2_{L^2H^{N-1}}\\ &= \frac{1}{2}\|MU^1_{\neq}(1)\|^2_{H^{N-1}}\\ &+ \int_1^T\int\langle D\rangle^{N-1}MU^1_{\neq}\langle D\rangle^{N-1}M\left[-U^2_{\neq} + 2\partial_X^2\Delta_t^{-1}U^2_{\neq}\right.\\ &\quad\left. -(\left[\widetilde{U}_0\cdot\nabla + U_{\neq}\cdot\nabla_t\right]U^1)_{\neq} + (\partial_X\Delta_t^{-1}\partial_i^t U^j\partial_j^t U^i)_{\neq} + \nu(\widetilde{\Delta}_t - \Delta_L)U^1_{\neq}\right]dV\,dt\\ &= \frac{1}{2}\|MU^1_{\neq}(1)\|^2_{H^{N-1}} + \mathrm{LU} + \mathrm{LP} + \mathcal{T} + \mathrm{NLP} + \mathrm{DE}.\end{aligned}$$

6.1. *Lift-up effect.* Start with the lift-up effect term, which can be bounded through the inviscid damping estimate we have on $U^2_{\neq}$ in H^{N-1} in (2.26b). In particular, since $1 \lesssim \sqrt{-\dot{M}^0M^0}|\nabla_L|$ (see Section 2.3),

$$\mathrm{LU} \lesssim \left\|\sqrt{\dot{M}M}U^1_{\neq}\right\|_{L^2H^{N-1}}\left\|\nabla_L U^2_{\neq}\right\|_{L^2H^{N-1}} \lesssim C_0\varepsilon^2,$$

which is sufficient for C_0 chosen sufficiently big. We remark that the simplicity and effectiveness of this estimate is the reason we are working with $U^1_{\neq}$.

6.2. *Linear pressure.* We now turn to the linear pressure term, LP, which we bound by relying first on the inequality $1 \lesssim \sqrt{-\dot{M}^0M^0}|\nabla_L|$, and then on Lemmas A.6 and Proposition 2.3,

$$\begin{aligned}\mathrm{LP} &\lesssim \|\sqrt{\dot{M}M}U^1_{\neq}\|_{L^2H^{N-1}}\|\nabla_L\Delta_L\Delta_t^{-1}U^2_{\neq}\|_{L^2H^{N-1}}\\ &\lesssim \|\sqrt{\dot{M}M}U^1_{\neq}\|_{L^2H^{N-1}}\left[\|\nabla_L U^2_{\neq}\|_{L^2H^{N-1}} + \|\nabla C\|_{L^\infty H^{N+1}}\|U^2_{\neq}\|_{L^2H^{N-1}}\right]\\ &\lesssim C_0\varepsilon^2,\end{aligned}$$

which is sufficient for C_0 sufficiently large. Note that the inviscid damping of U^2 is also very important here.

6.3. *Transport nonlinearity.* Turning to the transport term, we subdivide analogous to what has been applied in, e.g., Section 3.1.1:

$$\begin{aligned}\mathcal{T} &= \int_1^T \int \langle D\rangle^{N-1} M U^1_{\neq} \langle D\rangle^{N-1} M \left[\widetilde{U}_0 \cdot \nabla U^1_{\neq} + U_{\neq} \cdot \nabla_t U^1_{\neq} + U_{\neq} \cdot \nabla_t U^1_0\right] dV\, dt \\ &= \mathcal{T}_{0\neq} + \mathcal{T}_{\neq\neq} + \mathcal{T}_{\neq 0}.\end{aligned}$$

The term $\mathcal{T}_{0\neq}$ is treated like the analogous term in Section 3.1.1 and is hence omitted. The term $\mathcal{T}_{\neq 0}$ is treated via the following, using Proposition 2.2 and (2.27) (also $N-1 > 3/2+$):

$$\begin{aligned}\mathcal{T}_{\neq 0} &\lesssim \left\| M U^1_{\neq} \right\|_{L^2 H^{N-1}} \left\| \langle t\rangle U^{2,3}_{\neq} \right\|_{L^2 H^{N-1}} \left\| \langle t\rangle^{-1} \nabla U^1_0 \right\|_{L^\infty H^{N-1}} \\ &\lesssim \varepsilon^3 \nu^{-1/6-1/2} = \varepsilon^3 \nu^{-2/3}.\end{aligned}$$

For $\mathcal{T}_{\neq\neq}$, we may also apply a straightforward argument:

$$\begin{aligned}\mathcal{T}_{\neq\neq} &\lesssim \left\| M U^1_{\neq} \right\|_{L^\infty H^{N-1}} \left\| U^{1,2,3}_{\neq} \right\|_{L^2 H^{N-1}} \left\| \nabla_L U^1_{\neq} \right\|_{L^2 H^{N-1}} \\ &\lesssim \varepsilon^3 \nu^{-1/6-1/2} = \varepsilon^3 \nu^{-2/3}.\end{aligned}$$

This completes the transport terms.

6.4. *Nonlinear pressure.* The nonlinear pressure term can be split into one piece, for which both velocity fields have nonzero X frequency, and its complement:

$$\begin{aligned}\mathrm{NLP} &= \int_1^T \int \langle D\rangle^{N-1} M U^1_{\neq} \langle D\rangle^{N-1} M \partial_X \Delta_t^{-1} (\partial_i^t U^j_{\neq} \partial_j^t U^i_{\neq} + 2\partial_i^t U^j_0 \partial_j^t U^i_{\neq})\, dV \\ &= \mathrm{NLP}_{\neq} + \mathrm{NLP}_0.\end{aligned}$$

Treating $\mathrm{NLP}_{\neq}$ is straightforward: using the divergence-free condition and Lemma A.5, we have

$$\begin{aligned}\mathrm{NLP}_{\neq} &= \int_1^T \int \langle D\rangle^{N-1} M U^1_{\neq} \langle D\rangle^{N-1} M \Delta_t^{-1} \partial_X \partial_i^t (U^j_{\neq} \partial_j^t U^i_{\neq})\, dV \\ &\lesssim \| U^1_{\neq} \|_{L^2 H^{N-1}} \| U_{\neq} \|_{L^\infty H^{N-1}} \| \nabla_L U_{\neq} \|_{L^2 H^{N-1}} \lesssim \varepsilon^3 \nu^{-1/6-1/2} = \varepsilon^3 \nu^{-2/3}.\end{aligned}$$

The NLP_0 terms are bounded similarly, except for the ones involving U^1_0 — to these we now turn. Using the divergence-free condition,

$$\begin{aligned}&\int_1^T \int \langle D\rangle^{N-1} M U^1_{\neq} \langle D\rangle^{N-1} M \partial_X \Delta_t^{-1} (2\partial_i^t U^1_0 \partial_X U^i_{\neq})\, dV \\ &\quad = \int_0^T \int \langle D\rangle^{N-1} M U^1_{\neq} \langle D\rangle^{N-1} M \Delta_t^{-1} \partial_X \partial_i^t (2U^1_0 \partial_X U^i_{\neq})\, dV \\ &\quad \lesssim \left\| M U^1_{\neq} \right\|_{L^2 H^{N-1}} \left\| U^1_0 \right\|_{L^\infty H^{N-1}} \left(\| \partial_X U^2 \|_{L^2 H^{N-1}} + \| \partial_X U^3 \|_{L^2 H^{N-1}} \right) \\ &\quad \lesssim \varepsilon^3 \nu^{-1/6-1-1/6} = \varepsilon^3 \nu^{-4/3}.\end{aligned}$$

This completes the nonlinear pressure terms.

6.5. *Dissipation error.* Finally, the dissipation error is easily dealt with via the same method we have used several times previously: integrating by parts in the second equality,

$$\begin{aligned}
\mathrm{DE} &= \nu \int_1^T \int \langle D\rangle^{N-1} M U^1_{\neq} \langle D\rangle^{N-1} M (G\partial^L_{YY} + 2\psi_z \partial^L_{YZ}) U^1_{\neq})\, dV\, dt \\
&= -\nu \int_1^T \int \langle D\rangle^{N-1} M \partial^L_Y U^1_{\neq} \langle D\rangle^{N-1} M (G\partial^L_{Y} + 2\psi_z \partial_{Z}) U^1_{\neq})\, dV\, dt \\
&\quad - \nu \int_1^T \int \langle D\rangle^{N-1} M U^1_{\neq} \langle D\rangle^{N-1} M (\partial_Y G\partial^L_{Y} + 2\partial_Y \psi_y \partial_{Z}) U^1_{\neq})\, dV\, dt \\
&\lesssim \nu \left[\|C\|_{L^\infty H^N} \|\nabla_L U^1_{\neq}\|^2_{L^2 H^{N-1}} + \|\nabla C\|_{L^2 H^N} \|U^1_{\neq}\|_{L^\infty H^{N-1}} \|\nabla_L U^1_{\neq}\|_{L^2 H^{N-1}} \right] \\
&\lesssim \varepsilon^3 \nu^{-1}.
\end{aligned}$$

This completes the estimate on $U^1_{\neq}$.

7. **Estimates on C and g**

7.1. *Energy estimate on C.* In this section, we prove that, under the assumptions of Proposition 2.1 (in particular, the bootstrap assumptions (2.17), (2.16), (2.18)), the inequality (2.18c) holds, with 8 replaced by 4 on the right-hand side. Recall (2.11). An energy estimate gives

$$\begin{aligned}
&\frac{1}{2}\|C(T)\|^2_{H^{N+2}} + \nu \|\nabla_L C\|^2_{L^2 H^{N+2}} \\
&= \frac{1}{2}\|C(1)\|^2_{H^{N+2}} \\
&\quad + \int_1^T \int \langle D\rangle^{N+2} C \langle D\rangle^{N+2} \left[-\tilde{U}_0 \cdot \nabla C + g - U_0^2 + \nu(\widetilde{\Delta}_t - \Delta_L) C \right] dV\, dt \\
&= \frac{1}{2}\|C(1)\|^2_{H^{N+2}} + \mathcal{T} + L1 + \mathrm{L2} + \mathrm{DE}.
\end{aligned}$$

The transport nonlinearity $\mathcal{T}$ can be treated in the same manner as in the Q^i_0 energy estimates above and are hence omitted for brevity.

7.1.1. *The linear term* L1. Distinguish first between high and low frequencies:

$$\mathrm{L1} = \int_1^T \int \langle D\rangle^{N+2} C \langle D\rangle^{N+2} \left[P_{\leq 1} g + P_{>1} g \right] dV\, dt = \mathrm{L1}_L + \mathrm{L1}_H.$$

Low frequencies are estimated by taking advantage of the decay of g:

$$\mathrm{L1}_L \lesssim \|C\|_{L^\infty L^2} \|g\|_{L^1 L^2} \lesssim \frac{C_1 \varepsilon}{\nu}(C_0 \varepsilon) = \frac{C_0 \nu}{C_1} \left(\frac{C_1 \varepsilon}{\nu} \right)^2,$$

while high frequencies are estimated with the help of the viscous dissipation:

$$\mathrm{L1}_H \lesssim \|\nabla C\|_{L^2 H^N} \|\nabla g\|_{L^2 H^N} \lesssim \frac{C_1 \varepsilon}{\nu^{3/2}} \frac{C_0 \varepsilon}{\sqrt{\nu}} = \frac{C_0}{C_1} \left(\frac{C_1 \varepsilon}{\nu} \right)^2.$$

Both are consistent with the Proposition 2.1 for $C_1 \gg C_0$.

7.1.2. *The linear term* L2. The approach is analogous to the above term. Separating first high and low frequencies,

$$\mathrm{L2} = \int_1^T \int \langle D\rangle^{N+2} C \langle D\rangle^{N+2} \left[P_{\leq 1} U_0^2 + P_{>1} U_0^2\right] \, dV \, dt = \mathrm{L2}_L + \mathrm{L2}_H,$$

we estimate low frequencies with the help of (2.17f),

$$\mathrm{L2}_L \lesssim \|C\|_{L^\infty L^2} \|U_0^2\|_{L^1 L^2} \lesssim \frac{C_1\varepsilon}{\nu} \frac{C_0\varepsilon}{\nu} = \frac{C_0}{C_1}\left(\frac{C_1\varepsilon}{\nu}\right)^2,$$

and high frequencies using viscous dissipation,

$$\mathrm{L2}_H \lesssim \|\nabla_L C\|_{L^2 H^N} \|\nabla_L U_0^2\|_{L^2 H^N} \lesssim \frac{C_1\varepsilon}{\nu^{3/2}} \frac{C_0\varepsilon}{\sqrt{\nu}} = \frac{C_0}{C_1}\left(\frac{C_1\varepsilon}{\nu}\right)^2.$$

This completes the treatment of the linear terms.

7.1.3. *Dissipation error terms.* Write

$$\mathrm{DE} = \nu \int_1^T \int \langle D\rangle^{N+2} C \langle D\rangle^{N+2} \left(G\partial_{YY} C + 2\psi_z \partial_{YZ} C\right) dV dt = \mathcal{E}_1 + \mathcal{E}_2.$$

The two error terms are treated exactly the same, so consider only $\mathcal{E}_1$. Using a paraproduct decomposition,

$$\begin{aligned}
\mathcal{E}_1 &= \nu \int_1^T \int \langle D\rangle^{N+2} C \langle D\rangle^{N+2} \left(G_{Hi}\partial_{YY} C_{Lo} + G_{Lo}\partial_{YY} C_{Hi}\right) dV dt \\
&\lesssim \nu \left\|C\right\|_{L^\infty H^{N+2}} \left\|G\right\|_{L^2 H^{N+2}} \left\|\nabla C\right\|_{L^2 H^{5/2+}} \\
&\quad + \nu \left(\left\|\nabla C\right\|_{L^2 H^{N+2}} \left\|G\right\|_{L^\infty H^{3/2+}} + \left\|C\right\|_{L^\infty H^{N+2}} \left\|\nabla G\right\|_{L^2 H^{3/2+}}\right) \left\|\nabla C\right\|_{L^2 H^{N+2}} \\
&\lesssim \varepsilon^3 \nu^{-3},
\end{aligned}$$

which is sufficient for $\varepsilon\nu^{-1} \ll 1$.

7.2. *Estimates on g.* In this section, we prove that, under the assumptions of Proposition 2.1 (in particular, the bootstrap assumptions (2.17), (2.16), and (2.18)), the inequalities (2.18b) and (2.18a) hold, with 8 replaced by 4 on the right-hand side.

7.2.1. *Decay estimate on g in H^{N-1}.* In this section we improve (2.18b). Recall (2.11). Therefore, an energy estimate gives

$$\begin{aligned}
&\frac{1}{2}\|T^2 g(T)\|_{H^{N-1}}^2 + \nu \|t^2 \nabla_L g\|_{L^2 H^{N-1}} \\
&= \frac{1}{2}\|g(1)\|_{H^{N-1}}^2 + \int_1^T \int t^4 \langle D\rangle^{N-1} g \langle D\rangle^{N-1} \Big[-\tilde{U}_0 \cdot \nabla g - \frac{1}{t}(U_{\neq} \cdot \nabla_t U_{\neq}^1)_0 \\
&\qquad\qquad + \nu(\widetilde{\Delta}_t - \Delta_L) g\Big] \, dV \, dt \\
&= \frac{1}{2}\|g(1)\|_{H^{N-1}}^2 + \mathcal{T}_0 + \mathcal{T}_{\neq} + \mathrm{DE};
\end{aligned}$$

notice the cancellation between the derivative of the time weight and the damping term. The estimates of $\mathcal{T}_0$ and DE are obtained similarly to the treatment in Section 7.1 and are hence omitted for brevity. However, a new element appears in the estimate of $\mathcal{T}_{\neq}$. First, notice that

$$\mathcal{T}_{\neq} = -\int_1^T \int \langle D\rangle^{N-1} t^2 g \langle D\rangle^{N-1} t \Big[\underbrace{(U^1_{\neq}\partial_X U^1_{\neq})_0}_{0} + (U^2_{\neq}\partial^t_Y U^1_{\neq})_0 \\ + (U^3_{\neq}\partial^t_Z U^1_{\neq})_0 \Big] \, dV \, dt.$$

Therefore, by (2.27) and Proposition 2.2, it follows that

$$\mathcal{T}_{\neq} \lesssim \|t^2 g\|_{L^\infty H^{N-1}} \left(\|\langle t\rangle U^2_{\neq}\|_{L^2 H^{N-1}} + \|\langle t\rangle U^3_{\neq}\|_{L^2 H^{N-1}} \right) \|\nabla_L U^1_{\neq}\|_{L^2 H^{N-1}} \\ \lesssim \varepsilon^3 \nu^{-1}.$$

This completes the improvement of the estimate (2.18b).

7.2.2. *Energy estimate on g in H^{N+2}.* From (2.11), an energy estimate on g gives

$$\begin{aligned} &\frac{1}{2}\|g(T)\|^2_{H^{N+2}} + \nu \|\nabla_L g\|^2_{L^2 H^{N+2}} \\ &= \frac{1}{2}\|g(1)\|^2_{H^{N+2}} + \int_1^T \int \langle D\rangle^{N+2} g \langle D\rangle^{N+2} \Big[-\tilde{U}_0 \cdot \nabla g - \frac{2g}{t} \\ &\qquad - \frac{1}{t}(U_{\neq} \cdot \nabla_t U^1_{\neq})_0 + \nu(\widetilde{\Delta}_t - \Delta_L) g \Big] \, dV \, dt \\ &= \frac{1}{2}\|g(1)\|^2_{H^{N+2}} + \mathcal{T}_0 + L + \mathcal{T}_{\neq} + \mathrm{DE}. \end{aligned}$$

Observe that L does not need to be estimated, since it has a favorable sign. All other terms appearing in the right-hand side can be estimated following the same pattern as in many other instances in this paper (hence these are omitted for the sake of brevity), except for $\mathcal{T}_{\neq}$, to which we now turn. Observe that

$$\mathcal{T}_{\neq} \leq \|g\|_{L^\infty H^{N+2}} \left\| \frac{1}{t}(U_{\neq} \cdot \nabla_t U^1_{\neq})_0 \right\|_{L^1 H^{N+2}} \lesssim C_0 \varepsilon \left\| \frac{1}{t}(U_{\neq} \cdot \nabla_t U^1_{\neq})_0 \right\|_{L^1 H^{N+2}}.$$

This last factor can, in turn, be estimated by

$$\left\| \frac{1}{t}(U_{\neq} \cdot \nabla_t U^1_{\neq})_0 \right\|_{L^1 H^{N+2}} \lesssim \left\| \frac{1}{t}(U_{\neq} \cdot \nabla_t U^1_{\neq})_0 \right\|_{L^1 L^2} + \left\| \Delta \frac{1}{t}(U_{\neq} \cdot \nabla_t U^1_{\neq})_0 \right\|_{L^1 H^N}.$$

The first term on the right-hand side is easily estimated (using that $N-1 > 3/2$ for Sobolev embedding):

$$\left\| \frac{1}{t}(U_{\neq} \cdot \nabla_t U^1_{\neq})_0 \right\|_{L^1 L^2} \lesssim \|U_{\neq}\|_{L^\infty H^{N-1}} \left\| \nabla_L U^1_{\neq} \right\|_{L^2 H^{N-1}} \lesssim \varepsilon^2 \nu^{-1/2}.$$

For the second term, we use that for any function f, $\Delta f_0 = (\Delta_L f)_0$ as well as the identity $(U^1_{\neq}\partial_X U^1_{\neq})_0 = 0$ (which was used in Section 7.2.1 above as well), we obtain

$$\begin{aligned}
&\left\| \Delta \frac{1}{t}(U_{\neq} \cdot \nabla_t U^1_{\neq})_0 \right\|_{L^1 H^N} = \left\| \frac{1}{t}(\Delta_L [U^{2,3}_{\neq} \cdot \nabla_t U^1_{\neq}])_0 \right\|_{L^1 H^N} \\
&\lesssim \left\| \frac{1}{t}((\Delta_L U^{2,3}_{\neq}) \cdot \nabla_t U^1_{\neq})_0 \right\|_{L^1 H^N} \\
&\quad + \left\| \frac{1}{t}(U^{2,3}_{\neq} \cdot \Delta_L \nabla_t U^1_{\neq})_0 \right\|_{L^1 H^N} + \left\| \frac{1}{t}((\nabla_L U^{2,3}_{\neq}) \cdot \nabla_L \nabla_t U^1_{\neq})_0 \right\|_{L^1 H^N} \\
&\lesssim \|\Delta_L U^{2,3}_{\neq}\|_{L^\infty H^N} \|\nabla_L U^1_{\neq}\|_{L^2 H^N} \\
&\quad + \|U^{2,3}_{\neq}\|_{L^\infty H^N} \|\nabla_L \Delta_L U^1_{\neq}\|_{L^2 H^N} + \|\nabla_L U^{2,3}_{\neq}\|_{L^2 H^N} \|\Delta_L U^1_{\neq}\|_{L^\infty H^N} \\
&\lesssim \varepsilon^2 \nu^{-1-1/2} = \varepsilon^2 \nu^{-3/2},
\end{aligned}$$

where in the last line we used (2.13) and Lemma 2.2. This completes the improvement of (2.18a) for $\varepsilon\nu^{-3/2} \ll 1$. (Note the sharp use of the hypotheses.)

8. **Zero frequency velocity estimates**

The purpose of these estimates are to deduce low frequency controls on the velocity. First, observe that by the discussion in Section 2.7, it suffices to prove these estimates on u^i_0, rather than U^i_0. Indeed, due to Lemma 2.5 and the estimate $\|C\|_{L^\infty H^{N+2}} \lesssim \varepsilon\nu^{-1}$, for $\varepsilon\nu^{-1} \ll 1$, we may move from one coordinate system to another, in particular,

$$\|U^i_0\|_{H^s} \approx \|u^i_0\|_{H^s}, \tag{8.1a}$$

$$\left\|U^i_{\neq}\right\|_{H^s} \approx \left\|\bar{u}^i_{\neq}\right\|_{H^s}; \tag{8.1b}$$

recall the definition of $\bar{u}^i$ from Section 2.7.

8.1. *Decay of* U^2_0. In this section, we improve the estimate (2.17f). First, due to the divergence-free condition, $\widehat{u}^2_0(k = 0, \eta, l = 0) = 0$, thus $Qu^2_0 = u^2_0$, where Q projects on the Fourier modes for which k or $l \neq 0$. Therefore, u^2_0 solves

$$\begin{aligned}
\partial_t u^2_0 - \nu\Delta u^2_0 &= -Q(u \cdot \nabla u^2)_0 + Q\partial_y \Delta^{-1}(\partial_i u^j \partial_j u^i)_0 \\
&= -Q(u_0 \cdot \nabla u^2_0) + Q(\partial_y \Delta^{-1}(\partial_i u^j_0 \partial_j u^i_0)) \\
&\quad - Q(u_{\neq} \cdot \nabla u^2_{\neq})_0 + Q\partial_y \Delta^{-1}(\partial_i u^j_{\neq} \partial_j u^i_{\neq})_0 \\
&= QT_0 + QP_0 + QT_{\neq} + QP_{\neq},
\end{aligned}$$

with data $(u^2_{\rm in})_0$. Duhamel's formulation then reads

$$\begin{aligned}
u^2_0 &= e^{\nu t\Delta}(u^2_{\rm in})_0 + \int_0^t e^{\nu(t-s)\Delta}(QT_0(s) + QP_0(s) + QT_{\neq}(s) + QP_{\neq}(s))\, ds \\
&= \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV} + \mathrm{V}.
\end{aligned}$$

Due to the spectral gap made possible via Q, there holds

$$\|e^{\nu t\Delta}Qf\|_{L^2} \lesssim e^{-\nu t}\|f\|_{L^2} \quad \text{and} \quad \|e^{\nu t\Delta}\nabla Qf\|_{L^2} \lesssim \frac{1}{\sqrt{\nu t}}e^{-\nu t}\|f\|_{L^2}, \tag{8.2}$$

so that

$$\left\|\int_0^t e^{\nu(t-s)\Delta}QF(s)\,ds\right\|_{L^1L^2} \lesssim \frac{1}{\nu}\|F\|_{L^1L^2},$$
$$\left\|\int_0^t e^{\nu(t-s)\Delta}\nabla QF(s)\,ds\right\|_{L^1L^2} \lesssim \frac{1}{\nu}\|F\|_{L^1L^2}.$$

Therefore, one obtains immediately

$$\|\mathrm{I}\|_{L^1L^2} \lesssim \frac{1}{\nu}\|u_{\mathrm{in}}\|_{L^2} \lesssim \frac{\varepsilon}{\nu}.$$

Next, by the divergence-free condition on u and Sobolev embedding,

$$\begin{aligned}\|\mathrm{II}\|_{L^1L^2} &= \left\|\int_0^t e^{\nu(t-s)\Delta}Q[\partial_y(u_0^2)^2 + \partial_z(u_0^2u_0^3)]\,ds\right\|_{L^1L^2} \\ &\lesssim \frac{1}{\nu}\left[\|(u_0^2)^2\|_{L^1L^2} + \|u_0^2u_0^3\|_{L^1L^2}\right] \\ &\lesssim \frac{1}{\nu}\|u_0^2\|_{L^1L^2}\left[\|u_0^2\|_{L^\infty H^{N-1}} + \|u_0^3\|_{L^\infty H^{N-1}}\right] \lesssim \frac{\varepsilon}{\nu}\|u_0^2\|_{L^1L^2},\end{aligned}$$

which is sufficient for $\varepsilon\nu^{-1} \ll 1$ as it can be absorbed into the left-hand side of the estimate. Similarly, we claim that the same bound holds for III:

$$\|\mathrm{III}\|_{L^1L^2} \lesssim \frac{\varepsilon}{\nu}\|u_0^2\|_{L^1L^2}.$$

Indeed, let us look at QP_0 which, since u is divergence-free, can be written $Q\partial_{yij}\Delta^{-1}(u_0^ju_0^i)$. If i or j is equal to 2, then the same proof as for II applies. If both i and j are equal to 3, use the divergence-free condition on u, namely, $\partial_z u_0^3 = -\partial_y u_0^2$, to reduce matters to the previous case.

Next turn to estimates IV and V. Due to the zero mode projection and the divergence-free constraint, first note

$$(u_{\neq}\cdot\nabla u_{\neq}^2)_0 = \left(\nabla\cdot(u_{\neq}u_{\neq}^2)\right)_0 = \left(\partial_y(\bar{u}_{\neq}^2\bar{u}_{\neq}^2)\right)_0 + \left(\partial_z(\bar{u}_{\neq}^3\bar{u}_{\neq}^2)\right)_0.$$

Therefore, by (8.2) we have

$$\begin{aligned}\|\mathrm{IV}\|_{L^1L^2} &\lesssim \nu^{-1}\left(\left\|\bar{u}_{\neq}^2\right\|_{L^1L^4}^2 + \left\|\bar{u}^3\bar{u}_{\neq}^2\right\|_{L^1L^2}\right) \\ &\lesssim \nu^{-1}\left\|\bar{u}_{\neq}^2\right\|_{L^1L^2}\left(\left\|\bar{u}_{\neq}^2\right\|_{L^\infty H^{N-1}} + \left\|\bar{u}_{\neq}^3\right\|_{L^\infty H^{N-1}}\right) \\ &\lesssim \nu^{-1}\varepsilon^2,\end{aligned}$$

where the last line followed from (8.1) and (2.27); note the use of the inviscid damping on $\bar{u}^2_{\neq}$. We may apply a similar treatment for V; indeed, by the zero mode projection and the divergence-free constraint,

$$\begin{aligned}\partial_y \Delta^{-1}(\partial_i u^j_{\neq} \partial_j u^i_{\neq})_0 &= \partial_y \Delta^{-1}(\partial_i \partial_j \left(u^j_{\neq} u^i_{\neq}\right))_0 \\ &= \partial_y \Delta^{-1}\left(\partial_{yy}\left(\bar{u}^2_{\neq}\bar{u}^2_{\neq}\right) + 2\partial_{yz}\left(\bar{u}^2_{\neq}\bar{u}^3_{\neq}\right) + \partial_{zz}\left(\bar{u}^3_{\neq}\bar{u}^3_{\neq}\right)\right)_0.\end{aligned}$$

By (8.2) we have

$$\begin{aligned}\|\mathrm{V}\|_{L^1L^2} &\lesssim \nu^{-1}\left\|\bar{u}^i_{\neq}\bar{u}^j_{\neq}\right\|_{L^1H^1}\mathbf{1}_{i\neq 1}\mathbf{1}_{j\neq 1} \\ &\lesssim \nu^{-1}\left\|\bar{u}^i_{\neq}\right\|_{L^2H^{N-1}}\left\|\bar{u}^j_{\neq}\right\|_{L^2H^{N-1}}\mathbf{1}_{i\neq 1}\mathbf{1}_{j\neq 1} \lesssim \nu^{-4/3}\varepsilon^2\end{aligned}$$

which is sufficient for $\varepsilon\nu^{-4/3} \ll 1$.

Gathering all the above estimates, we obtain that, for a constant K,

$$\|u^2_0\|_{L^1L^2} \leq K\frac{\varepsilon}{\nu} + K\frac{\varepsilon^2}{\nu^{4/3}} + K\frac{\varepsilon}{\nu}\|u^2_0\|_{L^1L^2},$$

which, by (8.1), improves (2.17f) for $\varepsilon\nu^{-3/2} = \delta$ sufficiently small.

8.2. *Uniform bound on* U^1_0. As discussed above, it suffices to consider the velocity in the original coordinates, u^1_0, which solves

$$\partial_t u^1_0 - \nu\Delta u^1_0 = -u^2_0 - (u\cdot\nabla u^1)_0.$$

An energy estimate gives

$$\begin{aligned}&\frac{1}{2}\|u^1_0(t)\|^2_{H^{N-1}} + \nu\|\nabla u^1_0\|^2_{L^2H^{N-1}} \\ &\quad= \frac{1}{2}\|(u^1_{\mathrm{in}})_0\|^2_{H^{N-1}} - \int_0^T\int \langle D\rangle^{N-1}u^1_0\langle D\rangle^{N-1}\left[u^2_0 - (u\cdot\nabla u^1)_0\right]\,dV\,dt \\ &\quad= \frac{1}{2}\|(u^1_{\mathrm{in}})_0\|^2_{H^{N-1}} + \mathrm{LU} + \mathcal{T}.\end{aligned}$$

To estimate the lift up term, use that u^2_0 always has a nonzero z frequency by incompressibility together with the algebra property of H^{N-1} to obtain

$$\mathrm{LU} \leq \|\nabla u^1_0\|_{L^2H^{N-1}}\|\nabla u^2_0\|_{L^2H^{N-1}} \lesssim \frac{C_0\varepsilon}{\nu^{3/2}}\frac{\varepsilon}{\sqrt{\nu}} = \frac{1}{C_0}\left(\frac{C_0\varepsilon}{\nu}\right)^2,$$

which suffices for C_0 sufficiently large. Split the transport term into the contribution of zero and nonzero frequencies (in X):

$$\begin{aligned}\mathcal{T} &= \int_0^T\int \langle D\rangle^{N-1}u^1_0\langle D\rangle^{N-1}\left[u^2_0\partial_y u^1_0 + u^3_0\partial_z u^1_0 + (u_{\neq}\cdot\nabla u^1_{\neq})_0\right]\,dV\,ds \\ &= \mathcal{T}_0 + \mathcal{T}_{\neq}.\end{aligned}$$

To estimate $\mathcal{T}_0$, consider first the term involving (roughly speaking) $u^1_0 u^2_0 \partial_y u^1_0$; to bound it, we will again us that the z frequency of u^2_0 cannot be zero. Consider next the term involving $u^1_0 u^3_0 \partial_z u^1_0$; to bound it, we will use that at least two of the factors u_0^1, u_0^3, and $\partial_z u_0^1$ must have nonzero z frequency. This leads to the estimate

$$\begin{aligned}\mathcal{T}_0 &\lesssim \|u_0^1\|_{L^\infty H^{N-1}} \|\nabla u_0^2\|_{L^2 H^{N-1}} \|\nabla u_0^1\|_{L^2 H^{N-1}} \\ &\quad + \|u_0^1\|_{L^\infty H^{N-1}} \|\nabla u_0^3\|_{L^2 H^{N-1}} \|\nabla u_0^1\|_{L^2 H^{N-1}} \\ &\quad + \|\nabla u_0^1\|_{L^\infty H^{N-1}} \|u_0^3\|_{L^2 H^{N-1}} \|\nabla u_0^1\|_{L^2 H^{N-1}} \\ &\lesssim \frac{C_0\varepsilon}{\nu} \frac{C_0\varepsilon}{\sqrt{\nu}} \frac{C_0\varepsilon}{\nu^{3/2}} \leq \frac{C_0\varepsilon}{\nu} \left(\frac{C_0\varepsilon}{\nu}\right)^2,\end{aligned}$$

which suffices for $\varepsilon\nu^{-1}$ sufficiently small.

To estimate $\mathcal{T}_{\neq}$ we use the projection to zero frequency to note

$$(u_{\neq} \cdot \nabla u_{\neq}^1)_0 = (\bar{u}_{\neq}^2 \cdot (\partial_y - t\partial_y u_0^1 \partial_x)\bar{u}_{\neq}^1)_0 + (\bar{u}_{\neq}^3 \cdot \partial_z \bar{u}_{\neq}^1)_0$$

(note that the $\bar{u}^1 \partial_X \bar{u}^1$ is eliminated), which implies (using also $N - 1 > 3/2$)

$$\begin{aligned}\mathcal{T}_{\neq} &\lesssim \|u_0^1\|_{L^\infty H^{N-1}} \|\bar{u}_{\neq}^{2,3}\|_{L^2 H^{N-1}} \|\nabla_L \bar{u}_{\neq}^1\|_{L^2 H^{N-1}} \\ &\lesssim \frac{C_0\varepsilon}{\nu} \frac{C_0\varepsilon}{\nu^{1/6}} \frac{C_0\varepsilon}{\sqrt{\nu}} \leq C_0\varepsilon\nu^{1/3} \left(\frac{C_0\varepsilon}{\nu}\right)^2,\end{aligned}$$

which suffices for ε sufficiently small. This completes the energy estimate on u_0^1.

8.3. *Short time estimate on* U_0^1. We also need to deduce (2.17a). For this, we combine the techniques of Section 5.3 combined with the methods applied in Section 8.2. We omit the treatment for brevity as the details follow analogously. (Note that the main change from Section 8.2 is the way the lift-up effect is treated.)

8.4. *Uniform bound on* U_0^2. In this section we improve the bound (2.17c). As discussed above, we may perform estimates on u_0^2 rather than U_0^2. In the original coordinates, u_0^2 solves the equation

$$\partial_t u_0^2 - \nu\Delta u_0^2 = -(u \cdot \nabla u^2)_0 + \partial_y \Delta^{-1}(\partial_i u^j \partial_j u^i)_0.$$

An energy estimate gives

$$\begin{aligned}&\frac{1}{2}\|u_0^2(T)\|_{H^{N-1}}^2 + \nu\|\nabla_L u_0^2\|_{L^2 H^{N-1}}^2 \\ &\quad = \frac{1}{2}\|(u_{\mathrm{in}}^2)_0\|_{H^{N-1}}^2 \\ &\qquad + \int_0^T \int \langle D\rangle^{N-1} u_0^2 \langle D\rangle^{N-1} \left[-(u\cdot\nabla u^2)_0 + \partial_y\Delta^{-1}(\partial_i u^j \partial_j u^i)_0\right] dV\, dt \\ &\quad = \frac{1}{2}\|(u_{\mathrm{in}}^2)_0\|_{H^{N-1}}^2 + \mathcal{T} + \mathrm{NLP}.\end{aligned}$$

The transport term $\mathcal{T}$ can be treated as for u_0^1 in Section 8.2; we omit the details. Turning to the nonlinear pressure term, it can be written, using that

u is divergence-free, as

$$\begin{aligned}\mathrm{NLP} &= \int_0^T \int \langle D\rangle^{N-1} u_0^2 \langle D\rangle^{N-1} \partial_y \Delta^{-1} \left[\partial_i(u_0^j \partial_j u_0^i) + \partial_i(u_{\neq}^j \partial_j u_{\neq}^i)_0\right] \\ &= \mathrm{NLP}_0 + \mathrm{NLP}_{\neq}.\end{aligned}$$

In order to bound $\mathrm{NLP}_{\neq}$, once again we use the remark that, due to the X average,

$$(\partial_i u_{\neq}^j \partial_j u_{\neq}^i)_0 = \partial_{ij}(\bar{u}^i \bar{u}^j)_0 \mathbf{1}_{i\neq 1}\mathbf{1}_{j\neq 1}.$$

Therefore,

$$\mathrm{NLP}_{\neq} \lesssim \|u_0^2\|_{L^\infty H^{N-1}} \|\bar{u}_{\neq}^{2,3}\|_{L^2 H^{N-1}} \|\nabla_L \bar{u}_{\neq}^{2,3}\|_{L^2 H^{N-1}} \lesssim \varepsilon^3 \nu^{-2/3}.$$

Since i and j can only be equal to 2 or 3, NLP_0 can be estimated by

$$\begin{aligned}\mathrm{NLP}_0 &\lesssim \|u_0^2\|_{L^2 H^{N-1}} \left(\|u_0^2\|_{L^\infty H^{N-1}} + \|u_0^3\|_{L^\infty H^{N-1}}\right) \\ &\quad \times \left(\|\nabla u_0^2\|_{L^2 H^{N-1}} + \|\nabla u_0^3\|_{L^2 H^{N-1}}\right) \\ &\lesssim \varepsilon^3 \nu^{-1}.\end{aligned}$$

This gives the desired bound on $\|u_0^2\|^2_{L^\infty H^{N-1}} + \nu\|\nabla_L u_0^2\|^2_{L^2 H^{N-1}}$ for $\varepsilon\nu^{-1}$ sufficiently small.

8.5. *Uniform bound on U_0^3.* As already explained above, we perform estimates on u_0^3, which solves

$$\partial_t u_0^3 - \nu \Delta u_0^3 = -(u\cdot\nabla u^3)_0 + \partial_y \Delta^{-1}(\partial_i u^j \partial_j u^i)_0.$$

An energy estimate gives

$$\begin{aligned}\frac{1}{2}\|u_0^3(T)\|^2_{H^{N-1}} + \nu\|\nabla u_0^3\|^2_{L^2 H^{N-1}} &= \frac{1}{2}\|(u_{\mathrm{in}}^3)_0\|^2_{H^{N-1}} \\ &\quad + \int_0^T \int \langle D\rangle^{N-1} u_0^2 \langle D\rangle^{N-1}\left[-(u\cdot\nabla u^3)_0 + \partial_z \Delta^{-1}(\partial_i u^j \partial_j u^i)_0\right] dV\, dt \\ &= \frac{1}{2}\|(u_{\mathrm{in}}^3)_0\|^2_{H^{N-1}} + \mathcal{T} + \mathrm{NLP}.\end{aligned}$$

The estimate on $\mathcal{T}$ is similar to that done on u_0^1 and hence is omitted for brevity. The estimate on NLP requires a slight variant of what is done for u_0^2. First,

$$\begin{aligned}\mathrm{NLP} &= \int_0^T \int \langle D\rangle^{N-1} u_0^3 \langle D\rangle^{N-1} \partial_z \partial_i \Delta^{-1}\left[(u_0^j \partial_j u_0^i) + (u_{\neq}^j \partial_j u_{\neq}^i)_0\right] \\ &= \mathrm{NLP}_0 + \mathrm{NLP}_{\neq}.\end{aligned}$$

The treatment of $\mathrm{NLP}_{\neq}$ is the same as for u_0^2 and is hence omitted. Turn next to NLP_0. If $i = j = 3$, then at least two of the three factors must have a nonzero z frequency, and hence we have

$$\begin{aligned}\mathrm{NLP}_0 &\lesssim \left\| u_0^3 \right\|_{L^\infty H^{N-1}} \left\| \nabla u_0^3 \right\|_{L^2 H^{N-1}}^2 + \left\| u_0^3 \right\|_{L^\infty H^{N-1}} \left\| u_0^2 \right\|_{L^2 L^{N-1}} \left\| \nabla u_0^3 \right\|_{L^2 L^{N-1}} \\ &\quad + \left\| u_0^3 \right\|_{L^\infty H^{N-1}} \left\| u_0^2 \right\|_{L^2 H^{N-1}} \left\| \nabla u_0^2 \right\|_{L^2 L^{N-1}} \\ &\lesssim \frac{\varepsilon}{\nu}\varepsilon^2,\end{aligned}$$

which suffices for $\varepsilon\nu^{-1}$ sufficiently small. Notice that we used $\|u_0^2\|_{L^2 H^{N-1}} \lesssim \varepsilon\nu^{-1/2}$; one way to deduce this is via incompressibility,

$$\left\| u_0^2 \right\|_{L^2 H^{N-1}} \leq \left\| \partial_z u_0^2 \right\|_{L^2 H^{N-1}}.$$

This completes all of the zero frequency velocity estimates.

Appendix A. Commutation and elliptic estimates

A.1. *Commutator-like estimates.* In this section we outline some technical pointwise estimates on the Fourier multipliers we are employing; these essentially become product-rule type estimates in practice.

Lemma A.1 (Commutator-type estimate on m). *For all t, k, l, η, ξ, there holds*

$$m(t,k,\eta,l) \lesssim \langle \eta - \xi, l - l' \rangle^2 m(t,k,\xi,l').$$

Proof. Clearly it suffices to show that

$$\frac{m(t,k,\eta,l)}{m(t,k,\eta',l')} \lesssim \langle l - l' \rangle^2 + \langle \eta - \eta' \rangle^2.$$

Due to the definition of m, this estimate is proved by distinguishing several cases, depending on how t, $\frac{\eta}{k}$, and $\frac{\eta'}{k}$ compare. Since all these cases are fairly similar, we will only consider three of them for brevity:

- If $\frac{\eta}{k} > 0$, $\frac{\eta'}{k} > 0$, $t > \frac{\eta}{k} + 1000\nu^{-1/3}$ and $t > \frac{\eta'}{k} + 1000\nu^{-1/3}$, then

$$\begin{aligned}\frac{m(t,k,\eta,l)}{m(t,k,\eta',l')} &= \frac{k^2 + l^2}{k^2 + (l')^2} \frac{k^2 + (1000\nu^{-1/3}k)^2 + (l')^2}{k^2 + (1000\nu^{-1/3}k)^2 + l^2} \\ &\lesssim \frac{1 + L^2}{1 + (L')^2} \frac{\nu^{-2/3} + (L')^2}{\nu^{-2/3} + L^2},\end{aligned}$$

  where we set $L = \frac{l}{k}$ and $L' = \frac{l'}{k}$. Since

$$\frac{1 + L^2}{1 + (L')^2} \frac{\nu^{-2/3} + (L')^2}{\nu^{-2/3} + L^2} - 1 \lesssim \frac{\nu^{-2/3}(L^2 - (L')^2)}{\langle L' \rangle^2 (\nu^{-2/3} + L^2)},$$

  we deduce the desired bound.

- If $\frac{\eta}{k} > 0$, $\frac{\eta'}{k} > 0$, and $t < \frac{\eta}{k}$ and $t > \frac{\eta'}{k} + 1000\nu^{-1/3}$, then

$$\frac{m(t,k,\eta,l)}{m(t,k,\eta',l')} \lesssim \frac{\nu^{-2/3} + (L')^2}{1 + (L')^2} \lesssim 1 + \nu^{-2/3} \lesssim \langle \eta - \eta' \rangle^2.$$

- If $0 < \frac{\eta}{k} < t < \frac{\eta}{k} + 1000\nu^{-1/3}$ and $0 < \frac{\eta'}{k} < t < \frac{\eta'}{k} + 1000\nu^{-1/3}$, then

$$\frac{m(t,k,\eta,l)}{m(t,k,\eta',l')} = \frac{(1+L^2)(1+(t-H')^2+(L')^2)}{(1+(t-H)^2+L^2)(1+(L')^2)},$$

where we set $L = \frac{l}{k}$, $L' = \frac{l'}{k}$, $H = \frac{\eta}{k}$, and $H' = \frac{\eta'}{k}$. Since

$$\begin{aligned}
&\frac{(1+L^2)(1+(t-H')^2+(L')^2)}{(1+(t-H)^2+L^2)(1+(L')^2)} - 1 \\
&\qquad = \frac{(t-H)^2(L^2-(L')^2) + L^2(2t-H-H')(H-H')}{(1+(t-H)^2+L^2)(1+(L')^2)} \\
&\qquad \lesssim \frac{|(L')^2 - L^2|}{1+(L')^2} + \frac{L^2|t-H||H-H'|}{(1+(t-H)^2+L^2)(1+(L')^2)} \\
&\qquad\quad + \frac{L^2|H-H'|^2}{(1+(t-H)^2+L^2)(1+(L')^2)} \\
&\qquad \lesssim \langle L - L' \rangle^2 + \langle H - H' \rangle^2,
\end{aligned}$$

the desired bound follows. ☐

Lemma A.2 (Commutator-type estimate on $\sqrt{-\dot{M}M}$). *For all t, k, l, l', η, and η', there hold the following estimates*:

$$\sqrt{-\dot{M}^0 M^0}(t,k,\eta,l) \lesssim \langle \eta-\eta', l-l' \rangle \sqrt{-\dot{M}^0 M^0}(t,k,\eta',l'), \tag{A.1a}$$

$$\sqrt{-\dot{M}^1 M^1}(t,k,\eta,l) \lesssim \langle \eta-\eta', l-l' \rangle^{3/2} \sqrt{-\dot{M}^1 M^1}(t,k,\eta',l'), \tag{A.1b}$$

$$\sqrt{-\dot{M}^2 M^2}(t,k,\eta,l) \lesssim \langle \nu^{1/3}\,|\eta-\eta'| \rangle^{(1+\kappa)/2} \sqrt{-\dot{M}^2 . M^2}(t,k,\eta',l'). \tag{A.1c}$$

Proof. All of these estimates follow immediately from the definition of M^i in Section 2.3. ☐

A.2. *Elliptic lemmas.* This section concerns estimates on Δ_t^{-1} involving the Fourier multipliers m, $\dot{M}$, and ∇_L. All of these estimates are based on comparing Δ_t^{-1} to Δ_L^{-1}. The estimates here differ from the analogous estimates employed previously in [BM13], [BMV16], [BGM15a], [BGM15b] due to the much lower regularity and the fact that the coefficients are a little smaller here (relative to the primary unknowns).

The first estimate concerns inverting Δ_t at zero X frequencies.

LEMMA A.3 (Zero mode elliptic regularity). *Under the bootstrap hypotheses, for $\varepsilon\nu^{-1}$ sufficiently small, there hold for any $1 < s \leq N$,*

$$\left\|\Delta_t^{-1}\phi_0\right\|_{H^{s+2}} \lesssim \|\phi_0\|_{H^s} + \left\|\Delta_t^{-1}\phi_0\right\|_{L^2}, \tag{A.2a}$$

$$\left\|\nabla\Delta_t^{-1}\phi_0\right\|_{H^{s+1}} \lesssim \|\nabla\phi_0\|_{H^{s-1}} + \left\|\nabla\Delta_t^{-1}\phi_0\right\|_{L^2}, \tag{A.2b}$$

$$\left\|\Delta\Delta_t^{-1}\phi_0\right\|_{H^s} \lesssim \|\phi_0\|_{H^s} + \varepsilon\nu^{-1}\left\|\nabla\Delta_t^{-1}\phi_0\right\|_{L^2}. \tag{A.2c}$$

Proof. Consider (A.2a). First,

$$\left\|\Delta_t^{-1}\phi_0\right\|_{H^{s+2}} \leq \left\|\Delta\Delta_t^{-1}\phi_0\right\|_{H^s} + \left\|\Delta_t^{-1}\phi_0\right\|_{L^2}.$$

From the definition of Δ_t, we have

$$\begin{aligned}\left\|\Delta\Delta_t^{-1}\phi_0\right\|_{H^s} \leq \|\phi_0\|_{H^s} &+ \left\|\left(G\partial_{YY}\Delta_t^{-1}\phi_0\right)\right\|_{H^s} \\ &+ 2\left\|\left(\psi_z\partial_{YZ}\Delta_t^{-1}\phi_0\right)\right\|_{H^s} + \left\|\left(\Delta_t C\partial_Y\Delta_t^{-1}\phi_0\right)\right\|_{H^s}.\end{aligned} \tag{A.3}$$

For the first two error terms, we simply have

$$\begin{aligned}\left\|\left(G\partial_{YY}\Delta_t^{-1}\phi_0\right)\right\|_{H^s} + 2\left\|\left(\psi_z\partial_{YZ}\Delta_t^{-1}\phi_0\right)\right\|_{H^s} &\lesssim \|\nabla C\|_{H^s}\left\|\Delta\Delta_t^{-1}\phi_0\right\|_{H^s} \\ &\lesssim \varepsilon\nu^{-1}\left\|\Delta\Delta_t^{-1}\phi_0\right\|_{H^s},\end{aligned}$$

which is then absorbed on the left-hand side of (A.3) for $\varepsilon\nu^{-1} \ll 1$. For the last error term, we use the product rule and a frequency decomposition:

$$\begin{aligned}\left\|\left(\Delta_t C\partial_Y\Delta_t^{-1}\phi_0\right)\right\|_{H^s} &\leq \left\|\left(\Delta_t C\partial_Y P_{\leq 1}\Delta_t^{-1}\phi_0\right)\right\|_{H^s} + \left\|\left(\Delta_t C\partial_Y P_{>1}\Delta_t^{-1}\phi_0\right)\right\|_{H^s} \\ &\lesssim \|C\|_{H^{s+2}}\left\|\Delta_t^{-1}\phi_0\right\|_{L^2} + \|C\|_{H^{s+2}}\left\|\Delta\Delta_t^{-1}\phi_0\right\|_{H^s}.\end{aligned}$$

The latter term is again absorbed on the left-hand side of (A.3) for $\varepsilon\nu^{-1} \ll 1$ (since $s \leq N$), and the former is consistent with the right-hand side of (A.2a). Estimate (A.2b) follows by similar considerations.

Estimate (A.2c) follows from

$$\begin{aligned}\left\|\Delta\Delta_t^{-1}\phi_0\right\|_{H^s} &\leq \|\phi_0\|_{H^s} + \left\|G\partial_{YY}\Delta_t^{-1}\phi_0\right\|_{H^s} \\ &\quad + 2\left\|\psi_z\partial_{YZ}\Delta_t^{-1}\phi_0\right\|_{H^s} + \left\|\Delta_t C\partial_Y\Delta_t^{-1}\phi_0\right\|_{H^s}. \\ &\lesssim \|\phi_0\|_{H^s} + \varepsilon\nu^{-1}\left\|\Delta\Delta_t^{-1}\phi_0\right\|_{H^s} + \varepsilon\nu^{-1}\left\|\nabla\Delta_t^{-1}\phi_0\right\|_{H^s} \\ &\lesssim \|\phi_0\|_{H^s} + \varepsilon\nu^{-1}\left\|\Delta\Delta_t^{-1}\phi_0\right\|_{H^s} + \varepsilon\nu^{-1}\left\|P_{<1}\left(\nabla\Delta_t^{-1}\phi_0\right)\right\|_{H^s} \\ &\lesssim \|\phi_0\|_{H^s} + \varepsilon\nu^{-1}\left\|\Delta\Delta_t^{-1}\phi_0\right\|_{H^s} + \varepsilon\nu^{-1}\left\|\nabla\Delta_t^{-1}\phi_0\right\|_{L^2}.\end{aligned}$$

The second term is then absorbed on the left-hand side. □

LEMMA A.4. *Under the bootstrap hypotheses, for $\varepsilon\nu^{-4/3}$ sufficiently small, there holds for any $\alpha \in [0,1]$, $3/2 < s \leq N$,*

$$\left\| m^\alpha \Delta_L \Delta_t^{-1} \phi_{\neq} \right\|_{H^s} \lesssim \left\| m^\alpha \phi_{\neq} \right\|_{H^s}.$$

Proof. Writing $P = \Delta_t^{-1}\phi$ gives

$$\Delta_L P = \phi - G\partial_Y^L \partial_Y^L P - 2\psi_z \partial_Y^L \partial_Z P - \Delta_t C \partial_Y^L P.$$

Applying $\langle D \rangle^s m^\alpha$ to both sides gives

$$\left\| m^\alpha \Delta_L \Delta_t^{-1} \phi_{\neq} \right\|_{H^s} \lesssim \left\| m^\alpha \phi_{\neq} \right\|_{H^s} + \sum_{j=1}^{3} \mathcal{E}_j, \tag{A.4}$$

where

$$\mathcal{E}_1 = \| m^\alpha G \partial_Y^L \partial_Y^L P \|_{H^s}, \quad \mathcal{E}_2 = \| m^\alpha \psi_z \partial_Y^L \partial_Z P \|_{H^s}, \quad \mathcal{E}_3 = \| m^\alpha \Delta_t C \partial_Y^L P \|_{H^s}.$$

By Lemma A.1 we can deduce

$$\begin{aligned} \mathcal{E}_1 + \mathcal{E}_2 &\lesssim \| \nabla C \|_{H^{s+\min(2\alpha,1)}} \left\| m^{\min(1/2,\alpha)} \Delta_L P \right\|_{H^s} \\ &\lesssim \nu^{-\max(0,2\alpha-1)/3} \| \nabla C \|_{H^{s+\min(2\alpha,1)}} \| m^\alpha \Delta_L P \|_{H^s}. \end{aligned}$$

However, since $s \leq N$, by the bootstrap hypotheses,

$$\nu^{-\max(0,2\alpha-1)/3} \| \nabla C \|_{H^{s+\min(2\alpha,1)}} \lesssim \varepsilon \nu^{-4/3} \ll 1,$$

and this error can be absorbed by the left-hand side of the estimate in (A.4). For $\mathcal{E}_3$, we apply (2.14):

$$\mathcal{E}_3 \lesssim \left\| \nabla^2 C \right\|_{H^s} \| \nabla_L P \|_{H^s} \lesssim \| \nabla C \|_{H^{s+1}} \left\| m^{\min(1/2,\alpha)} \Delta_L P \right\|_{H^s},$$

and from here we may proceed as in $\mathcal{E}_{1,2}$ above. □

LEMMA A.5. *Under the bootstrap hypotheses, for $\varepsilon\nu^{-4/3}$ sufficiently small, there holds for any $3/2 < s \leq N$,*

$$\left\| \Delta_t^{-1} \partial_i^t \partial_j^t \phi_{\neq} \right\|_{H^s} \lesssim \| \phi_{\neq} \|_{H^s}.$$

Proof. The first observation is that Δ_t^{-1} and $\partial_i^t \partial_j^t$ *commute*; indeed one need only undo the coordinate transform, commute them as Fourier multipliers, and then redo the coordinate transform. Therefore, the estimate is the same as

$$\left\| \partial_i^t \partial_j^t \Delta_t^{-1} \phi_{\neq} \right\|_{H^s} \lesssim \| \phi_{\neq} \|_{H^s}.$$

By the $L^\infty H^{N+2}$ control on C and the projection to nonzero frequencies, we have

$$\left\| \partial_i^t \partial_j^t \Delta_t^{-1} \phi_{\neq} \right\|_{H^s} \lesssim \left\| \Delta_L \Delta_t^{-1} \phi_{\neq} \right\|_{H^s}.$$

Hence, the desired result now follows from Lemma A.4. □

LEMMA A.6. *Under the bootstrap hypotheses, for $\varepsilon\nu^{-4/3}$ sufficiently small, there holds for any $\alpha \in [0,1]$, $3/2 < s \leq N$,*

$$\left\|\nabla_L m^\alpha \Delta_L \Delta_t^{-1}\phi_{\neq}\right\|_{H^s} \lesssim \|\nabla_L m^\alpha \phi_{\neq}\|_{H^s} + \|\nabla C\|_{H^{s+2}} \nu^{-\max(0,2\alpha-1)/3} \|m^\alpha \phi_{\neq}\|_{H^{3/2+}} . \tag{A.5}$$

Remark A.1. For $s \leq N-1$, the second term in (A.5) can be removed.

Proof. Define $P = \Delta_t^{-1}\phi_{\neq}$. As in the proof of Lemma A.4 above,

$$\left\|\nabla_L m^\alpha \Delta_L \Delta_t^{-1}\phi_{\neq}\right\|_{H^s} \lesssim \|\nabla_L m^\alpha \phi_{\neq}\|_{H^s} + \sum_{j=1}^{3} \mathcal{E}_j, \tag{A.6}$$

where

$$\mathcal{E}_1 = \|\nabla_L m^\alpha G\partial_Y^L\partial_Y^L P\|_{H^s}, \quad \mathcal{E}_2 = \|\nabla_L m^\alpha \psi_z \partial_Y^L \partial_Z P\|_{H^s}, \\ \mathcal{E}_3 = \|\nabla_L m^\alpha \Delta_t C\partial_Y^L P\|_{H^s}.$$

By the product rule and that G does not depend on X,

$$\mathcal{E}_1 \lesssim \left\|m^\alpha G\partial_Y^L\partial_Y^L\nabla_L P\right\|_{H^s} + \left\|m^\alpha \nabla G\partial_Y^L\partial_Y^L P\right\|_{H^s}.$$

Expanding each term with a paraproduct decomposition (see Section 2.1.4) and Lemma A.1, we have

$$\begin{aligned}\mathcal{E}_1 &\lesssim \|G\|_{H^{1+2\alpha+}} \|\nabla_L m^\alpha \Delta_L P\|_{H^s} + \|G\|_{H^{s+\min(2\alpha,1)}} \left\|\nabla_L m^{\min(1/2,\alpha)}\Delta_L P\right\|_{H^{3/2+}} \\ &\quad + \|\nabla G\|_{H^{1+\min(2\alpha,1)}} \left\|m^{\min(1/2,\alpha)}\Delta_L P\right\|_{H^s} \\ &\quad + \|\nabla G\|_{H^{s+\min(2\alpha,1)}} \left\|m^{\min(1/2,\alpha)}\Delta_L P\right\|_{H^{3/2+}}.\end{aligned}$$

Using Lemma 2.4, we have

$$\begin{aligned}\mathcal{E}_1 &\lesssim \|\nabla C\|_{H^{1+2\alpha+}} \|\nabla_L m^\alpha \Delta_L P\|_{H^s} \\ &\quad + \|\nabla C\|_{H^{s+\min(2\alpha,1)}} \left\|\nabla_L m^{\min(1/2,\alpha)}\Delta_L P\right\|_{H^{3/2+}} \\ &\quad + \|\nabla C\|_{H^{2+\min(2\alpha,1)}} \left\|m^{\min(1/2,\alpha)}\Delta_L P\right\|_{H^s} \\ &\quad + \|\nabla C\|_{H^{s+1+\min(2\alpha,1)}} \left\|m^{\min(1/2,\alpha)}\Delta_L P\right\|_{H^{3/2+}}.\end{aligned}$$

By (2.13), (2.14), and $N > 2$, we have

$$\begin{aligned}\mathcal{E}_1 &\lesssim \left(\|C\|_{H^{N+2}} + \nu^{-\max(0,2\alpha-1)/3}\|\nabla C\|_{H^{s+\min(2\alpha,1)}}\right) \|\nabla_L m^\alpha \Delta_L P\|_{H^s} \\ &\quad + \nu^{-\max(0,2\alpha-1)/3} \|\nabla C\|_{H^{3+}} \|m^\alpha \Delta_L P\|_{H^s} \\ &\quad + \nu^{-\max(0,2\alpha-1)/3} \|\nabla C\|_{H^{s+2}} \|m^\alpha \Delta_L P\|_{H^{3/2+}} \\ &\lesssim \varepsilon\nu^{-4/3} \|\nabla_L m^\alpha \Delta_L P\|_{H^s} + \nu^{-\max(0,2\alpha-1)/3} \|\nabla C\|_{H^{s+2}} \|m^\alpha \Delta_L P\|_{H^s}.\end{aligned}$$

By $\varepsilon\nu^{-4/3}$ sufficiently small, the first term can be absorbed on the right-hand side of (A.6), and the second term is consistent with the stated (A.5) by Lemma A.4. By Lemma 2.4, the treatment of $\mathcal{E}_2$ is exactly the same and is hence omitted for brevity.

For $\mathcal{E}_3$, we apply again Leibniz's rule and (2.14):

$$\begin{aligned}
\mathcal{E}_3 &\lesssim \|\Delta C\|_{H^{1+2\alpha+}} \|m^\alpha \Delta_L P\|_{H^s} + \|\Delta C\|_{H^{2+}} \|\nabla_L P\|_{H^s} \\
&\quad + \|\Delta C\|_{H^s} \|\Delta_L P\|_{H^{3/2+}} + \|\Delta C\|_{H^{s+1}} \|\nabla_L P\|_{H^{3/2+}} \\
&\lesssim \|C\|_{H^{N+2}} \|m^\alpha \nabla_L \Delta_L P\|_{H^s} + \|\Delta C\|_{H^s} \left\|m^{\min(1/2,\alpha)} \nabla_L \Delta_L P\right\|_{H^{3/2+}} \\
&\quad + \|\Delta C\|_{H^{s+1}} \left\|m^{\min(1/2,\alpha)} \Delta_L P\right\|_{H^{3/2+}} \\
&\lesssim \nu^{-\max(0,2\alpha-1)/3} \|C\|_{H^{N+2}} \|m^\alpha \nabla_L \Delta_L P\|_{H^s} \\
&\quad + \|\Delta C\|_{H^{s+1}} \left\|m^{\min(1/2,\alpha)} \Delta_L P\right\|_{H^{3/2+}} \\
&\lesssim \varepsilon\nu^{-4/3} \|m^\alpha \nabla_L \Delta_L P\|_{H^s} + \nu^{-\max(0,2\alpha-1)/3} \|\nabla C\|_{H^{s+2}} \|m^\alpha \Delta_L P\|_{H^{3/2+}}.
\end{aligned}$$

As above, for $\varepsilon\nu^{-4/3}$ sufficiently small, the first term can be absorbed on the right-hand side of (A.6), and the second term is consistent with the stated (A.5) by Lemma A.4. Also note that if $s + 3 \leq N + 2$, then the latter term can be absorbed on the right-hand side of (A.6) for $\varepsilon\nu^{-4/3} \ll 1$, as claimed in Remark A.1. □

Lemma A.7. *Suppose $i \in \{0, 1, 2\}$. Under the bootstrap hypotheses, for $\varepsilon\nu^{-3/2}$ sufficiently small, there holds for any $\alpha \in [0, 1]$, $0 \leq s \leq N$,*

$$\begin{aligned}
\left\|\sqrt{-\dot{M}^i M^i} m^\alpha \Delta_L \Delta_t^{-1} \phi_{\neq}\right\|_{H^s} &\lesssim \left\|\sqrt{-\dot{M} M} m^\alpha \phi_{\neq}\right\|_{H^s} \\
&\quad + (\varepsilon\nu^{-3/2})\nu^{1/2} \left\|\nabla_L m^\alpha \Delta_L \Delta_t^{-1} \phi_{\neq}\right\|_{H^s}.
\end{aligned} \tag{A.7}$$

Proof. Writing $P = \Delta_t^{-1}\phi$, applying the multiplier $\sqrt{-\dot{M}^i M^i}\langle D\rangle^s m^\alpha$ to both sides of the equation, and taking L^2 norms gives

$$\left\|\sqrt{-\dot{M}^i M^i} m^\alpha \Delta_L \Delta_t^{-1} \phi_{\neq}\right\|_{H^s} \lesssim \left\|\sqrt{-\dot{M} M} m^\alpha \phi_{\neq}\right\|_{H^s} + \sum_{j=1}^{3} \mathcal{E}_j, \tag{A.8}$$

where

$$\mathcal{E}_1 = \|\sqrt{-\dot{M} M} m^\alpha G \partial_Y^L \partial_Y^L P\|_{H^s}, \quad \mathcal{E}_2 = \|\sqrt{-\dot{M} M} m^\alpha \psi_z \partial_Y^L \partial_Z P\|_{H^s},$$
$$\mathcal{E}_3 = \|\sqrt{-\dot{M} M} m^\alpha \Delta_t C \partial_Y^L P\|_{H^s}.$$

Similar to the arguments employed in the other elliptic lemmas, via a paraproduct decomposition, Lemma A.1, and Lemma A.2, we get

$$\begin{aligned}\mathcal{E}_1 + \mathcal{E}_2 &\lesssim \nu^{-\max(0,2\alpha-1)/3} \left\|\nabla C\right\|_{H^{5/2+\min(1,2\alpha)+}} \left\|\sqrt{-\dot{M}M} m^\alpha \Delta_L P\right\|_{H^s} \\ &\quad + \nu^{-\max(0,2\alpha-1)/3} \left\|\nabla C\right\|_{H^{s+\min(1,2\alpha)}} \left\|m^\alpha \Delta_L P\right\|_{H^{3/2+}}.\end{aligned}$$

However, by Lemma 2.1 we have

$$\left\|m^\alpha \Delta_L P\right\|_{H^{3/2+}} \lesssim \nu^{-1/6} \left(\left\|\sqrt{-\dot{M}M} m^\alpha \Delta_L P\right\|_{H^s} + \nu^{1/2} \left\|\nabla_L m^\alpha \Delta_L P\right\|_{H^s} \right),$$

which implies (along with $N \geq \max(s, 5/2+)$)

$$\mathcal{E}_1 + \mathcal{E}_2 \lesssim (\varepsilon\nu^{-3/2}) \left\|\sqrt{-\dot{M}M} m^\alpha \Delta_L P\right\|_{H^s} + (\varepsilon\nu^{-3/2})\nu^{1/2} \left\|\nabla_L m^\alpha \Delta_L P\right\|_{H^s}.$$

For $\varepsilon\nu^{-3/2}$ sufficiently small, the first term is absorbed in the left-hand side of (A.8) whereas the latter term is consistent with (A.7).

Consider next the error term $\mathcal{E}_3$, which by a paraproduct decomposition, Lemma 2.1, (2.14), and the lower bound on m, is

$$\begin{aligned}\mathcal{E}_3 &\lesssim \left\|\Delta C\right\|_{H^{5/2+}} \left\|\sqrt{-\dot{M}M} \nabla_L P\right\|_{H^s} + \left\|\nabla^2 C\right\|_{H^s} \left\|\nabla_L P\right\|_{3/2+} \\ &\lesssim \nu^{-\max(0,2\alpha-1)/3} \left\|C\right\|_{H^{N+2}} \left\|\sqrt{-\dot{M}M} m^\alpha \Delta_L P\right\|_{H^s} \\ &\quad + \nu^{-\max(0,2\alpha-1)/3} \left\|\nabla C\right\|_{H^{s+1}} \left\|m^\alpha \Delta_L P\right\|_{3/2+},\end{aligned}$$

from which the result follows in the same way as for $\mathcal{E}_{1,2}$. □

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