

A sharp counterexample to local existence of low regularity solutions to Einstein equations in wave coordinates

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Abstract

We give a sharp counterexample to local existence of low regularity solutions to Einstein equations in wave coordinates. We show that there are initial data in H^2 satisfying the wave coordinate condition such that there is no solution in H^2 to Einstein equations in wave coordinates for any positive time. This result is sharp since Klainerman-Rodnianski and Smith-Tataru proved existence for the same equations with slightly more regular initial data.

1. Introduction

The Einstein vacuum equations $R_{\mu\nu} = 0$ in wave coordinates become a system on nonlinear wave equations, called the reduced Einstein equations:

$$(1.0.1) \quad \widetilde{\square}_g g_{\mu\nu} = F_{\mu\nu}(g)[\partial g, \partial g].$$

The metric, in addition, is assumed to satisfy the wave coordinate condition

$$(1.0.2) \quad \partial_\alpha(\sqrt{|g|}g^{\alpha\beta}) = 0, \quad \text{where } |g| = |\det(\partial g/\partial x)|,$$

which is preserved by the reduced equations if it is satisfied initially and if the data satisfies the so-called constraint equations. Here $F_{\mu\nu}(g)[\partial g, \partial g]$ are quadratic forms in ∂g with coefficients depending on g , and the reduced wave operator is given by

$$(1.0.3) \quad \widetilde{\square}_g = g^{\alpha\beta}\partial_\alpha\partial_\beta.$$

We are considering the initial value problem with low regularity data. Given initial data in Sobolev spaces H^s ,

$$(1.0.4) \quad g|_{t=0} = g^0 \in H^s, \quad \partial_t g|_{t=0} = g^1 \in H^{s-1},$$

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we are asking for which s we can obtain a local solution in H^s , i.e.,

$$(1.0.5) \quad g(t, \cdot) \in H^s, \quad \partial_t g(t, \cdot) \in H^{s-1}, \quad 0 \leq t \leq T$$

for some $T > 0$, given that initial data satisfy the constraint equations and the wave coordinate condition. In 1952 Choquet-Bruhat proved that this is true for large s . More recently Klainerman-Rodnianski [KR05], respectively Smith-Tataru [ST05], proved local existence in H^s , for $s > 2$, for Einstein equations in wave coordinates. The result in [ST05] is in fact for more general quasilinear equations of the above form. (See also a recent work of Wang [Wan14].) Moreover, Klainerman-Rodnianski-Szeftel [KRS15] recently proved that one has local existence of bounded curvature solutions to Einstein equations if the curvature is bounded initially. However, that does not imply existence in wave coordinates.

We in fact show that one does not in general have local existence in H^2 for Einstein equations in wave coordinates:

THEOREM 1.1. *For any $\varepsilon > 0$, there is a domain of dependence D and there is a smooth solution to Einstein equations in wave coordinates in D such that*

$$(1.0.6) \quad \|g(0, \cdot) - m\|_{H^2(D_0)} + \|\partial_t g(0, \cdot)\|_{H^1(D_0)} \leq \varepsilon,$$

where m is the Minkowski metric, but for any $t > 0$,

$$(1.0.7) \quad \|g(t, \cdot)\|_{H^2(D_t)} + \|\partial_t g(t, \cdot)\|_{H^1(D_t)} = \infty,$$

where $D_t = \{x; (t, x) \in D\}$. Moreover, the curvature tensor satisfies

$$(1.0.8) \quad \|R(t, \cdot)\|_{L^2(D_t)} \leq C\varepsilon$$

for any t . (Here domain of dependence is an open subset of the upper half-space such that the backward light cone from any point in it is also contained in it.)

Remark. By a recent result Czimek [Czi16], data as above can be extended to data on \mathbf{R}^3 in H^2 satisfying the constraint equations and the wave coordinate condition.

To put the result in the theorem in context we recall that in Lindblad [Lin96], [Lin98], counterexamples to local existence in H^2 were given for the semilinear equation

$$(1.0.9) \quad \square\phi = (\underline{L}\phi)^2,$$

respectively, for the quasilinear equation

$$(1.0.10) \quad \square\phi = \phi \underline{L}^2\phi,$$

where $\underline{L} = \partial_t - \partial_{x_1}$. The counterexample for the semilinear equation is much stronger, and the quasilinear counterexample is just due to concentration of

characteristics. On the other hand, it was shown in Klainerman-Machedon [KM95] that there is local existence H^s , for any $s > 3/2$, for systems that satisfy the null condition, in particular, for

$$(1.0.11) \quad \square\phi = (\partial_t\phi)^2 - |\nabla_x\phi|^2.$$

Einstein equations in wave coordinates do not satisfy the null condition. However, as was shown in Lindblad-Rodnianski [LR05], they satisfy a weak null condition in a null frame, and the semilinear terms can be modeled by the system

$$(1.0.12) \quad \square\phi_2 = -(\underline{L}\phi_1)^2, \quad \square\phi_1 = 0$$

that satisfies the weak null condition. The same argument used to give a counterexample for the systems (1.0.9) and (1.0.10) in H^2 also gives a counterexample in H^2 for the model problem (1.0.12):

PROPOSITION 1.2. *For any $\varepsilon > 0$, there is a smooth solution $\phi = (\phi_1, \phi_2)$ to (1.0.12) in $D = \{(t, x); (x_1 - 1)^2 + x_2^2 + x_3^2 < (1 - t)^2\}$ such that*

$$(1.0.13) \quad \|\phi(0, \cdot)\|_{H^2(D_0)} + \|\partial_t\phi(0, \cdot)\|_{H^1(D_0)} \leq \varepsilon,$$

but for any $t > 0$,

$$(1.0.14) \quad \|\phi(t, \cdot)\|_{H^2(D_t)} + \|\partial_t\phi(t, \cdot)\|_{H^1(D_t)} = \infty,$$

where $D_t = \{x; (t, x) \in D\}$. Moreover, the data can be extended so that

$$(1.0.15) \quad \|\phi(0, \cdot)\|_{H^2(\mathbf{R}^3)} + \|\partial_t\phi(0, \cdot)\|_{H^1(\mathbf{R}^3)} \lesssim \varepsilon.$$

The proof of this is accomplished by finding explicit solutions of the system depending on (t, x_1) only inside the domain of dependence D that satisfy the conditions. Its easy to check that for any function χ_1 ,

$$\phi_1(t, x) = \chi_1(x_1 - t), \quad \phi_2(t, x) = -t\chi_2(x_1 - t)$$

solves the system if

$$\chi_2(x_1) = 2 \int_0^{x_1} \chi_1'(s)^2 ds.$$

Let

$$\chi_1(x_1) = \int_0^{x_1} \epsilon |\log |s/4||^\alpha ds, \quad 1/4 < \alpha < 1/2,$$

in which case

$$\chi_2(x_1) = 2 \int_0^{x_1} \epsilon^2 |\log |s/4||^{2\alpha} ds.$$

We have

$$(1.0.16) \quad \|\phi_1(t, \cdot)\|_{H^2(D_t)} \sim \|\chi_1''\|_{L^2(D_t)}, \quad \|\phi_2(t, \cdot)\|_{H^2(D_t)} \sim t\|\chi_2''\|_{L^2(D_t)},$$

and a calculation shows that

$$(1.0.17) \quad \int_{D_t} \chi_i''(t-x_1)^2 dx \sim \int_t^{2-t} |\chi_i''(x_1-t)|^2 (x_1-t) dx_1 \quad \begin{cases} < \infty & \text{if } i = 1, \\ = \infty & \text{if } i = 2, \end{cases}$$

from which the first part of the proposition follows. The second part of the proposition is obtained by multiplying with a cutoff $\chi((x_2^2 + x_3^2)/x_1)$.

Note that in the example, derivatives tangential to the characteristic surfaces $t - x_1 = c$ are better behaved than transversal derivatives.

Modulo terms that satisfy the null condition or cubic terms that are smaller because of the smallness in the construction above, we have

$$(1.0.18) \quad \widetilde{\square}_g g_{\mu\nu} \sim P(\partial_\mu g, \partial_\nu g), \quad \text{where} \quad P(h, k) = \frac{1}{4} h_\alpha^\alpha k_\beta^\beta - \frac{1}{2} h^{\alpha\beta} k_{\alpha\beta}.$$

Expressing this in a null frame $L = \partial_t + \partial_{x_1}$, $\underline{L} = \partial_t - \partial_{x_1}$, $A, B = \partial_{x_2}, \partial_{x_3}$,

$$(1.0.19) \quad \widetilde{\square}_g g_{TU} \sim 0, \quad T \in \{L, A, B\}, \quad U \in \{\underline{L}, L, A, B\},$$

$$(1.0.20) \quad \widetilde{\square}_g g_{\underline{L}\underline{L}} \sim P(\partial_{\underline{L}} g, \partial_{\underline{L}} g).$$

The linearized version of the wave coordinate condition reads

$$(1.0.21) \quad -m^{\mu\nu} \partial_\mu g_{\nu\gamma} + \frac{1}{2} m^{\mu\nu} \partial_\gamma g_{\mu\nu} \sim 0,$$

which expressed in a null frame becomes

$$(1.0.22) \quad -\frac{1}{2} \partial_{\underline{L}} g_{L\gamma} - \frac{1}{2} \partial_L g_{\underline{L}\gamma} + \partial_2 g_{2\gamma} + \partial_3 g_{3\gamma} - \frac{1}{2} \partial_\gamma (-g_{\underline{L}\underline{L}} + g_{22} + g_{33}) \sim 0.$$

Modulo tangential derivatives $\partial_L, \partial_2, \partial_3$ that we expect to be better, the wave coordinate condition reads

$$(1.0.23) \quad \partial_{\underline{L}} g_{LL} \sim 0, \quad \partial_{\underline{L}} g_{L2} \sim 0, \quad \partial_{\underline{L}} g_{L3} \sim 0, \quad \partial_{\underline{L}}(g_{22} + g_{33}) \sim 0,$$

which implies that

$$(1.0.24) \quad P(\partial_{\underline{L}} g, \partial_{\underline{L}} g) \sim -\frac{1}{2} ((\partial_{\underline{L}} g_{22})^2 + (\partial_{\underline{L}} g_{33})^2 + 2(\partial_{\underline{L}} g_{23})^2).$$

Consistent with this we choose

$$(1.0.25) \quad g_{22} = 1 + \chi_1(x_1 - t), \quad g_{33} = 1 - \chi_1(x_1 - t)$$

and

$$(1.0.26) \quad g_{23} = g_{L2} = g_{L3} = 0.$$

These components solve the homogeneous wave equations (1.0.19). In order to also solve the remaining wave equation (1.0.20) we must have

$$(1.0.27) \quad g_{\underline{L}\underline{L}} = -t\chi_2(x_1 - t).$$

In order to satisfy the remaining wave coordinate condition for g_{LL} we must have

$$(1.0.28) \quad \partial_L g_{LL} - 2\delta^{AB} \partial_A g_{B\bar{L}} = 0.$$

To satisfy this we finally define

$$(1.0.29) \quad g_{B\bar{L}} = -\frac{1}{4}x^B \chi_2(x_1 - t),$$

which also satisfies the wave equation (1.0.19).

Based on the above linearized approximation, we make the nonlinear ansatz in the table below, with $\tilde{\chi}_2$ a modification of χ_2 :

g_{YZ}	L	\bar{L}	2	3
L	0	-2	0	0
\bar{L}	-2	$-t\tilde{\chi}_2$	$-\frac{1}{4}x_2(1 + \chi_1)\tilde{\chi}_2$	$-\frac{1}{4}(1 + \chi_1)^{-1}x_3\tilde{\chi}_2$
2	0	$-\frac{1}{4}x_2(1 + \chi_1)\tilde{\chi}_2$	$1 + \chi_1$	0
3	0	$-\frac{1}{4}x_3(1 + \chi_1)^{-1}\tilde{\chi}_2$	0	$(1 + \chi_1)^{-1}$

This modification is obtained by trying to modify the metric above in order for it to satisfy the nonlinear wave coordinate condition. The reason this can be done is that we first choose the metric so that $\det g = 1$, in which the wave coordinate condition becomes a linear equation for the inverse of the metric

$$\partial_\mu g^{\mu\nu} = \partial_L g^{L\nu} + \partial_{\bar{L}} g^{\bar{L}\nu} + \partial_1 g^{1\nu} + \partial_2 g^{2\nu} = 0,$$

solved in the same way we solved the linearized equation.

As it turns out, with a metric of the form in the table, the only nonvanishing component of the curvature tensor is $R_{A\bar{L}B\bar{L}} \neq 0$. With $\tilde{\chi}_2$ satisfying

$$\tilde{\chi}'_2 - 2(\chi'_1)^2(1 + \chi_1)^{-2} - \tilde{\chi}_2^2/16 = 0,$$

we have that the Ricci curvature R_{LL} is equal to $g^{AB}R_{A\bar{L}B\bar{L}} = 0$. In the quasilinear case the domain has to be opened up slightly away from the characteristic $t = x_1$, $x_2 = x_3 = 0$, to make sure the boundary of the domain is non-timelike and hence a domain of dependence. Since the metric is a small perturbation of the Minkowski metric in L^∞ , the light cones are close to those of Minkowski and we only have to insure that the boundary is non-timelike. Let D be the domain

$$(1.0.30) \quad D = \{(t, x); (x_1 - 1)^2 H(x_1 - 1) + x_2^2/4 + x_3^2/4 < (1 - t)^2\},$$

where $H(x_1 - 1) = 1$ when $x_1 < 1$, and $H(x_1 - 1) = 1/4$ when $x_1 > 1$. The boundary consists of two parts $C = C_1 \cup C_2$, where

$$(1.0.31) \quad C_1 = \{(t, x); x_1 < 1, (x_1 - 1)^2 + x_2^2/4 + x_3^2/4 = (1 - t)^2\}$$

and

$$(1.0.32) \quad C_2 = \{(t, x); x_1 \geq 1, (x_1 - 1)^2/4 + x_2^2/4 + x_3^2/4 = (1 - t)^2\}.$$

C_2 is clearly non-timelike as is C_1 when $x_2^2 + x_3^2 \geq c > 0$ since this is true for the Minkowski metric with some room. In null coordinates $u = (t - x_1)/2$, $v = (t + x_1)/2$, C_1 is given by

$$4(1 - v)u + x_2^2/4 + x_3^2/4 = 0.$$

The conormal is given by

$$n = 2(1 - t)dt - 2(1 - x_1)dx_1 + x_2dx_2/2 + x_3dx_3/2.$$

Now it is easy to see that the inverse of the metric takes the following form:

g^{YZ}	L	\underline{L}	2	3
L	g^{LL}	$-\frac{1}{2}$	$-\frac{1}{8}x_2\tilde{\chi}_2$	$-\frac{1}{8}x_3\tilde{\chi}_2$
\underline{L}	$-\frac{1}{2}$	0	0	0
2	$-\frac{1}{8}x_2\tilde{\chi}_2$	0	$(1 + \chi_1)^{-1}$	0
3	$-\frac{1}{8}x_3\tilde{\chi}_2$	0	0	$1 + \chi_1$

It follows that

$$|g^{\alpha\beta}n_\alpha n_\beta - m^{\alpha\beta}n_\alpha n_\beta| \lesssim (|\chi_1| + |\tilde{\chi}_2|)(x_2^2 + x_3^2) + |g^{LL}|u^2,$$

where $|u| \lesssim x_2^2 + x_3^2$ on C_1 and

$$m^{\alpha\beta}n_\alpha n_\beta = -(x_2^2 + x_3^2).$$

Hence if N is the normal to C_1 , then

$$g_{\alpha\beta}N^\alpha N^\beta \leq 0,$$

so C_1 is non-timelike.

2. The heuristic argument and illposedness for the model system

2.1. *The reduced Einstein equations.* Let g be a solution of Einstein equations

$$(2.1.1) \quad R_{\mu\nu} = 0,$$

in harmonic coordinates

$$(2.1.2) \quad \partial_\alpha(\sqrt{|g|}g^{\alpha\beta}) = 0, \quad \beta = 0, \dots, 3.$$

Denote the reduced wave operator by

$$\tilde{\square} = g^{\alpha\beta}\partial_\alpha\partial_\beta.$$

Let $h_{\alpha\beta} = g_{\alpha\beta} - m_{\alpha\beta}$, where m is the Minkowski metric. Then by [LR05] we have

$$(2.1.3) \quad \tilde{\square}_g h_{\mu\nu} = F_{\mu\nu}(h)(\partial h, \partial h),$$

where F is a quadratic form in ∂h with coefficients depending on h :

$$F_{\mu\nu}(h)(\partial h, \partial h) = P(\partial_\mu h, \partial_\nu h) + Q_{\mu\nu}(\partial h, \partial h) + G_{\mu\nu}(h)(\partial h, \partial h).$$

Here

$$P(h, k) = \frac{1}{4}h_\alpha^\alpha k_\beta^\beta - \frac{1}{2}h^{\alpha\beta}k_{\alpha\beta},$$

where the indices are raised with respect to the Minkowski metric, $Q_{\mu\nu}$ are linear combinations of the standard null forms and $G_{\mu\nu}$ contains only cubic terms.

We want to construct a counterexample to local existence in H^2 . First, by [KM95], semilinear equations satisfying the classical null condition have local existence in H^2 , so we can neglect these terms in a heuristic argument. The counterexamples we construct below will be singular along a light ray in such a way that h vanishes exactly at the light cone, and therefore $|G_{\mu\nu}| \lesssim |h| |\partial h|^2$ will actually be more regular than $|\partial h|^2$ so also this term can be neglected in the heuristic argument. Inside a light cone, the counterexample we construct will be a function of (t, x_1) only with a singularity along $t - x_1 = 0$, but more regular in the $t + x_1$ direction. We therefore expect the derivatives in the $t - x_1$ direction to be worse than derivatives in other directions, so expanding the metric in a null frame $L = \partial_t + \partial_1$, $\underline{L} = \partial_t - \partial_1$, $A, B = \partial_2, \partial_3$, we see that ∂_μ is to leading order $\frac{1}{2}L_\mu \partial_{\underline{L}}$. We have $g^{\alpha\beta} = m^{\alpha\beta} - h^{\alpha\beta} + O(h^2)$, where $h^{\alpha\beta} = m^{\alpha\mu}m^{\beta\nu}h_{\mu\nu}$ and $h^{\alpha\beta}\partial_\alpha\partial_\beta$ is to leading order $h_{LL}\partial_{\underline{L}}^2$, where $h_{LL} = h_{\alpha\beta}L^\alpha L^\beta$. Similarly, $P(\partial_\mu h, \partial_\nu h)$ is to leading order given by $\underline{L}_\mu \underline{L}_\nu P(\partial_{\underline{L}} h, \partial_{\underline{L}} h)/4$. Hence expanding h in a null frame $h_{UV} = h_{\mu\nu}U^\mu U^\nu$, the reduced Einstein equations become to highest order

$$(2.1.4) \quad (\square - h_{LL}\partial_{\underline{L}}^2) h_{TU} \sim 0,$$

$$(2.1.5) \quad (\square - h_{LL}\partial_{\underline{L}}^2) h_{\underline{L}\underline{L}} \sim P(\partial_{\underline{L}} h, \partial_{\underline{L}} h),$$

where T is any tangential frame component $T \in \{L, A, B\}$ and U is any frame component $U \in \{\underline{L}, L, A, B\}$. By [LR05],

$$(2.1.6) \quad \begin{aligned} P(p, k) &= \frac{1}{4} \delta^{AB} (2p_{AL}k_{BL} + 2p_{AL}k_{BL} - p_{AB}k_{LL} - p_{LL}k_{AB}) \\ &\quad - \frac{1}{8} (p_{LL}k_{LL} + p_{LL}k_{LL}) \\ &\quad - \frac{1}{4} \delta^{AB} \delta^{A'B'} (2p_{AA'}k_{BB'} - p_{AB}k_{A'B'}). \end{aligned}$$

The system simplifies further because, as we shall see next, the wave coordinate condition implies that

$$(2.1.7) \quad \partial_{\underline{L}} h_{LL} \sim 0, \quad \partial_{\underline{L}} h_{L2} \sim 0, \quad \partial_{\underline{L}} h_{L3} \sim 0, \quad \partial_{\underline{L}} (h_{22} + h_{33}) \sim 0,$$

which implies that

$$(2.1.8) \quad P(\partial_{\underline{L}} h, \partial_{\underline{L}} h) \sim -\frac{1}{2} ((\partial_{\underline{L}} h_{22})^2 + (\partial_{\underline{L}} h_{33})^2 + 2(\partial_{\underline{L}} h_{23})^2)$$

and that after a possible change of variables, we can also neglect the term $h_{LL} \partial_{\underline{L}}^2$.

2.2. Illposedness for the model problem. Consider the following semilinear system:

$$(2.2.1) \quad \square \phi_2 = -(\underline{L}\phi_1)^2, \quad \square \phi_1 = 0, \quad \text{where } \underline{L} = \partial_t - \partial_{x^1}.$$

Our first result using the techniques from [Lin96] is illposedness for this system:

LEMMA 2.1. *Let $\epsilon > 0$, and set*

$$\chi_1(x_1) = \int_0^{x_1} \epsilon |\log |s/4||^\alpha ds, \quad 0 < \alpha < 1/2.$$

There is $\Psi_1 \in H^2(\mathbb{R}^3)$ such that

$$\Psi_1(x) = \chi_1(x_1) \quad \text{in } B_0 = \{x \in \mathbb{R}^3; (x_1 - 1)^2 + x_2^2 + x_3^2 < 1\}$$

and

$$\|\Psi_1\|_{\dot{H}^2} \leq C_\alpha \epsilon,$$

supp $\Psi_1 \subset \{x; |x| \leq 2\}$ and $\text{singsupp } \Psi_1 = \{0\}$.

Let

$$\chi_2(x_1) = 2 \int_0^{x_1} \chi_1'(s)^2 ds = 2 \int_0^{x_1} \epsilon^2 |\log |s/4||^{2\alpha} ds, \quad 1/4 < \alpha < 1/2.$$

There is $\Psi_{2t} \in \dot{H}^1(\mathbb{R}^3)$ such that

$$\Psi_{2t}(x) = \chi_2(x_1 - t) \quad \text{in } B_t = \{x \in \mathbb{R}^3; (x_1 - 1)^2 + x_2^2 + x_3^2 < (1 - t)^2\}.$$

For $0 \leq t < 1$, we have

$$\|\Psi_{2t}\|_{\dot{H}^2(B_t)} = \infty.$$

Proof. We have

$$\begin{aligned} \int_{B_0} |\chi_1''(x_1)|^2 dx &= \int_0^2 |\chi_1''(x_1)|^2 \left(\int_{x_2^2+x_3^2 \leq 2x_1-x_1^2} dx_2 dx_3 \right) dx_1 \\ &= \int_0^2 |\chi_1''(x_1)|^2 \pi(2x_1-x_1^2) dx_1 \\ &\leq \int_0^2 \frac{2\epsilon^2 \pi \alpha^2 x_1 dx_1}{x_1^2 |\log|x_1/4||^{2(1-\alpha)}} < \infty. \end{aligned}$$

Thus $\|\Psi_1\|_{\dot{H}^2(B_0)} \leq C_\alpha \epsilon$, and it follows from extension theorems in Stein [Ste70] (see page 181), that it can be extended to a function in $H^2(\mathbb{R}^3)$ with comparable norm. Moreover, the extension can be chosen to satisfy the above support and singular support properties.

Moreover, if $0 \leq t < 1$, then

$$\begin{aligned} \int_{B_t} |\chi_2''(x_1-t)|^2 dx &= \int_t^{2-t} |\chi_2''(x_1-t)|^2 \left(\int_{x_2^2+x_3^2 \leq (2-(x_1+t))(x_1-t)} dx_2 dx_3 \right) dx_1 \\ &= \int_t^{2-t} |\chi_2''(x_1-t)|^2 \pi(2-(x_1+t))(x_1-t) dx_1 \\ &\geq 2 \int_0^{1-t} \frac{\epsilon^4 (1-t) \pi (2\alpha)^2 x_1 dx_1}{x_1^2 |\log|x_1/4||^{2(1-2\alpha)}} = \infty. \quad \square \end{aligned}$$

The data we will choose for (2.2.1) are

$$\phi_1(0, x) = \Psi_1(x), \quad \partial_t \phi_1(0, x) = -\partial_{x_1} \Psi_1(x).$$

Note now that by a domain of dependency argument, the solution of (2.2.1) inside the cone $\Lambda = \{(t, x); |x - (1, 0, 0)| \leq 1 - t, t \geq 0\}$ only depends on the data inside the ball B_0 . Since data inside the ball B_0 only depends on x_1 , the solution ϕ_1 inside Λ satisfies

$$(\partial_{x_1} - \partial_t)(\partial_t + \partial_{x_1})\phi_1(t, x_1) = 0.$$

It follows that $(\partial_t + \partial_{x_1})\phi_1 = 0$ in Λ and hence

$$\phi_1(t, x_1) = \chi_1(x_1 - t), \quad (t, x) \in \Lambda.$$

Hence

$$(\partial_{x_1} - \partial_t)(\partial_t + \partial_{x_1})\phi_2(t, x_1) = -\left((\partial_{x_1} - \partial_t)\phi_1(t, x_1) \right)^2 = -4\chi_1'(x_1 - t)^2.$$

We now choose data

$$\phi_2(0, x) = 0, \quad \partial_t \phi_2(0, x) = -\Psi_{20}(x).$$

It then follows that in Λ ,

$$\phi_2(t, x) = -t\chi_2(x_1 - t).$$

Hence by the estimate in the lemma,

$$\|\phi_2(t, \cdot)\|_{H^2} = \infty \quad \text{if } 0 < t < 1.$$

On the other hand, it easily follows from standard Strichartz estimates that

$$\|\phi_2(t, \cdot)\|_{H^{2-\delta}} < \infty \quad \text{if } \delta > 0.$$

2.3. The wave coordinate condition. We prefer to work with lower indices since the nonlinearity is more transparent in this case. We collect two standard linear algebra results about the derivative of the determinant of a matrix and the inverse of a matrix:

LEMMA 2.2. *Let $|g| = |\det g|$. We have*

$$(2.3.1) \quad \partial_\alpha |g| = |g| g^{\mu\nu} \partial_\alpha g_{\mu\nu},$$

$$(2.3.2) \quad \partial_\alpha g^{\mu\nu} = -g^{\mu\mu_1} g^{\nu\nu_1} \partial_\alpha g_{\nu_1\mu_1}.$$

We convert the constraint equations

$$\partial_\alpha (\sqrt{|g|} g^{\alpha\beta}) = 0, \quad \beta = 0, \dots, 3,$$

using the lemma above. We get

$$-\sqrt{|g|} g^{\alpha\alpha_1} g^{\beta\beta_1} \partial_\alpha g_{\alpha_1\beta_1} + \frac{1}{2} g^{\alpha\beta} \sqrt{|g|} g^{\mu\nu} \partial_\alpha g_{\mu\nu} = 0.$$

Apply $g_{\beta\gamma}$, divide by $\sqrt{|g|}$ and relabel the indices ($\alpha \rightarrow \mu$, $\alpha_1 \rightarrow \nu$) to arrive at

$$(2.3.3) \quad -g^{\mu\nu} \partial_\mu g_{\nu\gamma} + \frac{1}{2} g^{\mu\nu} \partial_\gamma g_{\mu\nu} = 0,$$

which is the form that we will use.

Write down the linearization of the wave coordinate condition (2.3.3) that for small h is good approximation of the wave coordinate condition:

$$(2.3.4) \quad -m^{\mu\nu} \partial_\mu h_{\nu\gamma} + \frac{1}{2} m^{\mu\nu} \partial_\gamma h_{\mu\nu} = 0.$$

Define the basis (null frame) by

$$(2.3.5) \quad L = \partial_t + \partial_{x_1}, \quad \underline{L} = \partial_t - \partial_{x_1}, \quad \partial_A = \partial_{x_A}, \quad A = 2, 3.$$

We use the basis from (2.3.5) in (2.3.4). For $\gamma = L$, we have

$$\frac{1}{2} \partial_{\underline{L}} h_{LL} + \frac{1}{2} \partial_L h_{\underline{L}\underline{L}} - \delta^{AB} \partial_A h_{BL} + \frac{1}{2} \partial_L (-h_{\underline{L}\underline{L}} + \delta^{AB} h_{AB}) = 0.$$

For $\gamma = \underline{L}$,

$$\frac{1}{2} \partial_{\underline{L}} h_{\underline{L}\underline{L}} + \frac{1}{2} \partial_L h_{LL} - \delta^{AB} \partial_A h_{B\underline{L}} + \frac{1}{2} \partial_{\underline{L}} (-h_{\underline{L}\underline{L}} + \delta^{AB} h_{AB}) = 0.$$

For $\gamma = C \in \{2, 3\}$,

$$\frac{1}{2} \partial_{\underline{L}} h_{LC} + \frac{1}{2} \partial_L h_{\underline{L}C} - \delta^{AB} \partial_A h_{BC} + \frac{1}{2} \partial_C (-h_{\underline{L}\underline{L}} + \delta^{AB} h_{AB}) = 0.$$

In the first two equations, the $h_{\underline{L}\underline{L}}$ coefficient cancels and, therefore, we can write the linearized wave coordinate condition as follows:

$$(2.3.6a) \quad \partial_{\underline{L}} h_{\underline{L}\underline{L}} - 2\delta^{AB} \partial_A h_{B\underline{L}} + \partial_{\underline{L}} (\delta^{AB} h_{AB}) = 0,$$

$$(2.3.6b) \quad \partial_L h_{\underline{L}\underline{L}} - 2\delta^{AB} \partial_A h_{B\underline{L}} + \partial_{\underline{L}} (\delta^{AB} h_{AB}) = 0,$$

$$(2.3.6c) \quad \partial_{\underline{L}} h_{LC} + \partial_L h_{\underline{L}C} - 2\delta^{AB} \partial_A h_{BC} + \partial_C (-h_{\underline{L}\underline{L}} + \delta^{AB} h_{AB}) = 0.$$

Recall that our solution is

$$h_{AB} \sim \phi_1,$$

$$h_{\underline{L}\underline{L}} \sim \phi_2,$$

where $\phi_1 \sim \chi_1(x_1 - t)$ and $\phi_2 \sim -t\chi_2(x_1 - t)$ inside the cone $|x| \leq 1 - t$ with $\phi_1 \in H^2$ while $\phi_2 \in H^{2-\delta} \setminus H^2$.

2.3.1. *Eliminating truly bad parts.* We would like to eliminate the components that are differentiated by \underline{L} in (2.3.6a)–(2.3.6c) as they would not have the same regularity as derivatives of L, A . Therefore, identifying these terms in (2.3.6a)–(2.3.6c), respectively, we set

$$(2.3.7) \quad h_{LL} = 0,$$

$$(2.3.8) \quad \delta^{AB} h_{AB} = h_{22} + h_{33} = 0,$$

$$(2.3.9) \quad h_{LC} = 0.$$

We cannot set $h_{AB} = 0$, but it is enough to have

$$(2.3.10) \quad \begin{aligned} h_{22} &= -h_{33} = \phi_1, \\ h_{23} &= 0. \end{aligned}$$

2.3.2. *Satisfying the first linearized wave coordinate condition (2.3.6a).* With (2.3.7), (2.3.8), and (2.3.9), the first constraint is satisfied automatically.

2.3.3. *Satisfying the second linearized wave coordinate condition (2.3.6b).* With the choice h_{AB} as in (2.3.10), the constraint (2.3.6b) becomes

$$\partial_L h_{\underline{L}\underline{L}} - 2\delta^{AB} \partial_A h_{B\underline{L}} = 0.$$

Since $\partial_L h_{\underline{L}\underline{L}} = -t\chi_2$ inside the cone, this suggests we define

$$(2.3.11) \quad h_{B\underline{L}} = -\frac{1}{4} x^B \chi_2.$$

Observe that $x^B \phi_2 \in H^2$ since near the singular point $x_1 = t$ of ϕ_2 , inside the cone, we have $|x^B| \lesssim (t - x_1)^{\frac{1}{2}}$, which makes the appropriate expression integrable and prevents the singularity.

2.3.4. *Satisfying the third linearized wave coordinate condition (2.3.6c).*

We have

$$\partial_A h_{BC} = 0$$

by (2.3.10). Also,

$$\partial_L h_{B\bar{L}} = 0$$

by (2.3.11). Combining this with (2.3.7) and (2.3.9), we see that the last constraint (2.3.6c) is reduced to

$$\partial_C h_{L\bar{L}} = 0,$$

which suggests

$$h_{L\bar{L}} = 0.$$

To summarize, in the $L, \bar{L}, \partial_2, \partial_3$ basis and in that order, $h_{\alpha\beta}$ is

$$h_{YZ}|_{\{|x|<1-t\}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -t\chi_2 & -\frac{1}{4}x_2\chi_2 & -\frac{1}{4}x_3\chi_2 \\ 0 & -\frac{1}{4}x_2\chi_2 & \chi_1 & 0 \\ 0 & -\frac{1}{4}x_3\chi_2 & 0 & -\chi_1 \end{pmatrix}.$$

3. The solution inside the cone

The goal of this section is to build on the ideas of Section 2 to obtain a solution of the Einstein equations inside the cone and wave coordinates for it, such that the metric in these coordinates has a finite H^2 norm at time zero and is infinite at all other times. For this end, let D be the domain

$$(3.0.1) \quad D = \{(t, x); (x_1 - 1)^2 H(x_1 - 1) + x_2^2/4 + x_3^2/4 < (1 - t)^2\},$$

where $H(x_1 - 1) = 1$ when $x_1 < 1$, and $H(x_1 - 1) = 1/4$ when $x_1 > 1$. Set $D_t = \{x; (t, x) \in D\}$. Our goal is to prove the following statement:

THEOREM 3.1. *There exist a spacetime (D, g) and coordinates $x_\alpha : D \rightarrow \mathbb{R}$, $\alpha = 0, 1, 2, 3$ such that*

- *the metric g satisfies the Einstein vacuum equation*

$$\text{Ric}(g) = 0 \quad \text{on } D;$$

- *the coordinates x_α are wave coordinates*

$$\partial_\alpha(\sqrt{|g|}g^{\alpha\beta}) = 0 \quad \text{on } D \quad \beta = 0, 1, 2, 3;$$

- *the metric g has finite initial data in $H^2(D_0) \times H^1(D_0)$:*

$$\|g_{\alpha\beta}\|_{H^2(D_0)} + \|\partial_t g_{\alpha\beta}\|_{H^1(D_0)} < \infty, \quad \alpha, \beta \in \{0, 1, 2, 3\};$$

- *the $H^2(D_t)$ norm of g_{00} at any other time t is infinite:*

$$\|g_{00}\|_{H^2(D_t)} = \infty, \quad \forall t \in (-1, 1) \setminus \{0\}.$$

We prove the theorem by describing an explicit example for such a metric g and coordinates x_α . The coordinates x_α are the standard coordinates on \mathbb{R}^{1+3} :

$$x_\alpha((y_0, y_1, y_2, y_3)) = \delta_\alpha^\beta y_\beta.$$

We will also write $t = x_0$. We use the rest of this section to specify g and verify that the hypotheses of Theorem 3.1 are satisfied. We define the following vector fields:

$$(3.0.2) \quad \begin{aligned} L &= \partial_t + \partial_{x_1}, \\ \underline{L} &= \partial_t - \partial_{x_1}. \end{aligned}$$

We complete $\{L, \underline{L}\}$ to a basis by adding $\partial_A = \partial_{x_A}, A = 2, 3$. In what follows we will use A, B to denote an index from a set $\{2, 3\}$. Since L, \underline{L} are constant coefficient vector fields, we will abuse the notation and treat L, \underline{L} as fictitious indices as well. For example, $\partial_L f = \partial_t f + \partial_{x_1} f$ or $g_{L\underline{L}} = \langle L, \underline{L} \rangle_g$.

Remark. Since $\{L, \underline{L}, \partial_A | A = 2, 3\}$ form a basis and have constant coefficients, we use this basis instead of the standard one in all subsequent derivations.

We can now specify the metric.

Definition 3.2. The nonzero coefficients of the metric g in the basis above are as follows:

$$(3.0.3) \quad g_{L\underline{L}} = -2,$$

$$(3.0.4) \quad g_{\underline{L}\underline{L}} = -t\tilde{\chi}_2(x_1 - t),$$

$$(3.0.5) \quad g_{AB} = \delta_{AB}\chi_{1A}(x_1 - t),$$

and

$$(3.0.6) \quad g_{AL} = -\frac{1}{4}x_A\chi_{2A}(x_1 - t),$$

where

$$\chi_{12} = 1 + \chi_1 = \frac{1}{\chi_{13}}.$$

Here χ_1 was defined in Lemma 2.1, $\tilde{\chi}_2$ is a slight modification of χ_2 that will be defined in Lemma 3.3 below and

$$\chi_{2A} = \chi_{1A}\tilde{\chi}_2.$$

The rest of the coefficients are given by symmetry.

Remark. Unless we specify otherwise, the argument of the χ -functions will be $x_1 - t$.

The coefficients of g are summarized in Table 1.

g_{YZ}	L	\underline{L}	2	3
L	0	-2	0	0
\underline{L}	-2	$-t\tilde{\chi}_2$	$-\frac{1}{4}x_2(1+\chi_1)\tilde{\chi}_2$	$-\frac{1}{4}(1+\chi_1)^{-1}x_3\tilde{\chi}_2$
2	0	$-\frac{1}{4}x_2(1+\chi_1)\tilde{\chi}_2$	$1+\chi_1$	0
3	0	$-\frac{1}{4}x_3(1+\chi_1)^{-1}\tilde{\chi}_2$	0	$(1+\chi_1)^{-1}$

Table 1. The coefficients g_{YZ} of the metric.

Thus we reduce the proof of Theorem 3.1 to the following three lemmas:

LEMMA 3.3. *Let $\tilde{\chi}_2$ satisfy*

$$\tilde{\chi}'_2 - 2(\chi'_1)^2(1+\chi_1)^{-2} - \frac{1}{16}\tilde{\chi}_2^2 = 0.$$

Then the metric g defined in Definition 3.2 satisfies $\text{Ric}(g) = 0$.

LEMMA 3.4. *Let g be the metric defined in Definition 3.2. Then the standard coordinates satisfy the wave coordinate condition (2.1.2).*

LEMMA 3.5. *The metric g in Definition 3.2 satisfies*

$$\begin{aligned} \|g_{\alpha\beta}\|_{H^2(D_0)} + \|\partial_t g_{\alpha\beta}\|_{H^1(D_0)} &< \infty, \quad \alpha, \beta \in \{0, 1, 2, 3\}, \\ \|g_{00}\|_{H^2(D_{\delta'})} &= \infty, \quad \forall \delta' \in (-\delta, \delta) \setminus \{0\}. \end{aligned}$$

Lemmas 3.3 and 3.4 are given by direct computation for which we will provide some intermediate steps. The following statement is a straightforward observation:

CLAIM 3.6. *We have the following equalities:*

- (1) $\sqrt{|g|} = 2$;
- (2) *the nonzero coefficients of the inverse metric g^{YZ} are as follows:*

$$\begin{aligned} g^{LL} &= \frac{1}{4}t\tilde{\chi}_2 + \frac{1}{64}\left(x_2^2(1+\chi_1) + x_3^2(1+\chi_1)^{-1}\right)\tilde{\chi}_2^2, \\ g^{LA} &= -\frac{1}{8}x_A\chi_{2A}(\chi_{1A})^{-1}, \\ g^{L\underline{L}} &= -\frac{1}{2}, \\ g^{AB} &= \delta_{AB}(\chi_{1A})^{-1} \end{aligned}$$

and their symmetric counterparts.

g^{YZ}	L	\underline{L}	2	3
L	$\frac{1}{4}t\tilde{\chi}_2 + \frac{1}{64}(x_2^2(1+\chi_1) + x_3^2(1+\chi_1)^{-1})\tilde{\chi}_2^2$	$-\frac{1}{2}$	$-\frac{1}{8}x_2\tilde{\chi}_2$	$-\frac{1}{8}x_3\tilde{\chi}_2$
\underline{L}	$-\frac{1}{2}$	0	0	0
2	$-\frac{1}{8}x_2\tilde{\chi}_2$	0	$(1+\chi_1)^{-1}$	0
3	$-\frac{1}{8}x_3\tilde{\chi}_2$	0	0	$1+\chi_1$

 Table 2. The coefficients g^{YZ} of the metric.

We summarize g^{YZ} in Table 2.

Proof of Lemma 3.3. We use the slightly nonstandard definition of Christoffel symbols from [LR05, (3.1)]:

$$(3.0.7) \quad \begin{aligned} \Gamma_{\alpha}^{\beta}{}_{\gamma} &= \frac{1}{2}g^{\beta\delta}(\partial_{\alpha}g_{\delta\gamma} + \partial_{\gamma}g_{\delta\alpha} - \partial_{\delta}g_{\alpha\gamma}), \\ \Gamma_{\alpha\beta\gamma} &= g_{\beta\delta}\Gamma_{\alpha}^{\delta}{}_{\gamma} = \frac{1}{2}(\partial_{\alpha}g_{\beta\gamma} + \partial_{\gamma}g_{\beta\alpha} - \partial_{\beta}g_{\alpha\gamma}). \end{aligned}$$

The following Christoffel symbols are not zero:

$$\begin{aligned} \Gamma_{\underline{L}\underline{L}\underline{L}} &= \Gamma_{\underline{L}\underline{L}\underline{L}} = -\Gamma_{\underline{L}\underline{L}\underline{L}} = -\frac{\tilde{\chi}_2}{2}, \\ \Gamma_{\underline{L}\underline{L}\underline{L}} &= -\frac{\tilde{\chi}_2}{2} + t\tilde{\chi}'_2, \\ \Gamma_{\underline{L}\underline{A}\underline{L}} &= \frac{1}{2}x_a\chi'_{2A}, \\ \Gamma_{\underline{L}\underline{B}\underline{A}} &= \Gamma_{\underline{A}\underline{B}\underline{L}} = -\delta_{AB}\chi'_{1A}, \\ \Gamma_{\underline{A}\underline{L}\underline{B}} &= \delta_{AB}\left(-\frac{1}{4}\chi_{2A} + \chi'_{1A}\right), \end{aligned}$$

whereas $\Gamma_{\underline{L}\underline{L}\underline{L}} = \Gamma_{\underline{L}\underline{L}\underline{L}} = \Gamma_{\underline{L}\underline{L}\underline{L}} = 0$ and

$$\Gamma_{\underline{A}\underline{L}\underline{L}} = \Gamma_{\underline{A}\underline{L}\underline{L}} = \Gamma_{\underline{L}\underline{A}\underline{L}} = \Gamma_{\underline{A}\underline{L}\underline{L}} = \Gamma_{\underline{L}\underline{A}\underline{L}} = \Gamma_{\underline{A}\underline{L}\underline{B}} = \Gamma_{\underline{A}\underline{B}\underline{L}} = \Gamma_{\underline{A}\underline{B}\underline{C}} = 0.$$

With the convention (3.0.7), we have the following formula for the curvature (see also [LR05, 3.10]):

$$R_{\mu\alpha\nu\beta} = \partial_{\beta}\Gamma_{\mu\alpha\nu} - \partial_{\nu}\Gamma_{\mu\alpha\beta} + \Gamma_{\nu\lambda\alpha}\Gamma_{\mu}^{\lambda}{}_{\beta} - \Gamma_{\alpha\lambda\beta}\Gamma_{\mu}^{\lambda}{}_{\nu}.$$

We will split the curvatures into two nontensors, which represent the linear and the quadratic parts:

$$\begin{aligned} R_{\mu\alpha\nu\beta} &= R_{\mu\alpha\nu\beta}^{\text{lin}} + R_{\mu\alpha\nu\beta}^{\text{quad}}, \\ R_{\mu\alpha\nu\beta}^{\text{lin}} &= \partial_\beta \Gamma_{\mu\alpha\nu} - \partial_\nu \Gamma_{\mu\alpha\beta}, \\ R_{\mu\alpha\nu\beta}^{\text{quad}} &= g^{\lambda\gamma} \Gamma_{\nu\lambda\alpha} \Gamma_{\mu\gamma\beta} - g^{\lambda\gamma} \Gamma_{\alpha\lambda\beta} \Gamma_{\mu\gamma\nu}. \end{aligned}$$

We claim that the only nontrivial components of the nontensors $R^{\text{lin}}, R^{\text{quad}}$ are $R_{\underline{A}\underline{L}\underline{B}\underline{L}}^{\text{lin}}$ and $R_{\underline{A}\underline{L}\underline{B}\underline{L}}^{\text{quad}}$. In fact, this follows from the symmetries if we can show that they vanish if at least one index is L or at least three indices are A, B, C . For $R_{\underline{A}\underline{L}\underline{B}\underline{L}}^{\text{lin}}$, the first follows since the only components of $\partial_\beta \Gamma_{\mu\alpha\nu}$ with at least one L are $\partial_{\underline{L}} \Gamma_{\underline{L}\underline{L}\underline{L}}$, $\partial_{\underline{L}} \Gamma_{\underline{L}\underline{L}\underline{L}} = \partial_{\underline{L}} \Gamma_{\underline{L}\underline{L}\underline{L}}$ and $\partial_{\underline{L}} \Gamma_{\underline{L}\underline{L}\underline{L}}$, but they are seen to cancel each other when appearing in R^{lin} . Secondly, $\Gamma_{ABC} = 0$ and $\partial_A \Gamma_{B\underline{C}\underline{L}} = \partial_A \Gamma_{\underline{L}CB} = 0$ and $\partial_A \Gamma_{B\underline{L}C} = 0$, which concludes the proof of the statement for R^{lin} . For R^{quad} , the first follows since the only combination of $g^{\lambda\gamma} \Gamma_{\nu\lambda\alpha} \Gamma_{\mu\gamma\beta}$ with one index L , say $\nu = L$, is $g^{\underline{L}\underline{L}} \Gamma_{\underline{L}\underline{L}\underline{L}} \Gamma_{\underline{L}\underline{L}\underline{L}}$, which will cancel when appearing in R^{quad} . Secondly, if three of the indices of $g^{\lambda\gamma} \Gamma_{\nu\lambda\alpha} \Gamma_{\mu\gamma\beta}$ are A, B, C , say $\nu = A$ and $\alpha = B$ and $\beta = C$, then in fact $g^{\underline{L}\underline{L}} \Gamma_{\underline{A}\underline{L}\underline{B}} \Gamma_{\underline{\mu}LC} = 0$, which concludes the proof of the claim.

We have

$$\begin{aligned} R_{\underline{A}\underline{L}\underline{B}\underline{L}}^{\text{lin}} &= \delta_{AB} \left(\frac{1}{2} \chi'_{2A} - 2\chi''_{1A} \right), \\ R_{\underline{A}\underline{L}\underline{B}\underline{L}}^{\text{quad}} &= \delta_{AB} \left[\chi_{1A}^{-1} (\chi'_{1A})^2 + \frac{1}{4} \tilde{\chi}_2 \left(-\frac{1}{4} \chi_{2A} + \chi'_{1A} \right) \right]. \end{aligned}$$

The last follows since

$$\begin{aligned} (3.0.8) \quad R_{\underline{A}\underline{L}\underline{B}\underline{L}}^{\text{quad}} &= \Gamma_{B\underline{C}\underline{L}} \Gamma_{A\underline{L}}^C - \Gamma_{\underline{L}\underline{L}\underline{L}} \Gamma_{A\underline{B}}^L - \Gamma_{\underline{L}\underline{L}\underline{L}} \Gamma_{A\underline{B}}^L \\ &= g^{CD} \Gamma_{B\underline{C}\underline{L}} \Gamma_{A\underline{D}\underline{L}} - g^{\underline{L}\underline{L}} \Gamma_{\underline{L}\underline{L}\underline{L}} \Gamma_{A\underline{L}\underline{B}}. \end{aligned}$$

With this we compute¹

$$\begin{aligned} g^{AB} R_{\underline{A}\underline{L}\underline{B}\underline{L}}^{\text{lin}} &= \tilde{\chi}_2 - 4 \frac{(\chi'_1)^2}{(1 + \chi_1)^2}, \\ g^{AB} R_{\underline{A}\underline{L}\underline{B}\underline{L}}^{\text{quad}} &= (\chi'_1)^2 \left[\frac{2}{(1 + \chi_1)^2} \right] - \frac{1}{16} \tilde{\chi}_2^2. \end{aligned}$$

¹We assume the summation convention on A, B .

We use this to compute the only nonzero component of the Ricci curvature $\text{Ric}_{\underline{LL}}$:

$$\begin{aligned}\text{Ric}_{\underline{LL}} &= g^{AB} R_{\underline{AL}\underline{BL}}^{\text{lin}} + g^{AB} R_{\underline{AL}\underline{BL}}^{\text{quad}} \\ &= \tilde{\chi}'_2 - 2 \frac{(\chi'_1)^2}{(1 + \chi_1)^2} - \frac{1}{16} \tilde{\chi}_2^2. \quad \square\end{aligned}$$

Proof of Lemma 3.4. Since $\sqrt{|g|}$ is constant by item (1) of Claim 3.6, we use some elementary linear algebra to rewrite the wave coordinate condition (2.1.2) as

$$(3.0.9) \quad g^{\mu\nu} \partial_\mu g_{\nu\gamma} = 0, \quad \gamma = 0, \dots, 3.$$

Denote

$$d_\gamma = g^{\mu\nu} \partial_\mu g_{\nu\gamma}.$$

Our goal is to show $d_\gamma = 0$ for $\gamma = 0, \dots, 3$. Instead, we will show $d_L = d_{\underline{L}} = d_A = 0$ for $A = 2, 3$, which is equivalent since L, \underline{L}, A form a basis of constant coefficient vector fields. The fact that $d_L = 0$ is obvious, since the metric coefficients of the form g_{XL} are constant. For $d_{\underline{L}}$, we write

$$\begin{aligned}d_{\underline{L}} &= g^{LL} \partial_L g_{\underline{L}\underline{L}} + g^{L\underline{L}} \partial_L g_{\underline{L}\underline{L}} + g^{LA} \partial_L g_{A\underline{L}} \\ &\quad + g^{\underline{L}\underline{L}} \partial_{\underline{L}} g_{\underline{L}\underline{L}} + g^{A\underline{L}} \partial_A g_{\underline{L}\underline{L}} + g^{AB} \partial_A g_{B\underline{L}}.\end{aligned}$$

Since coefficients g_{XL} are constant, we drop their derivatives

$$d_{\underline{L}} = g^{L\underline{L}} \partial_L g_{\underline{L}\underline{L}} + g^{LA} \partial_L g_{A\underline{L}} + g^{AB} \partial_A g_{B\underline{L}}.$$

Observe that $\partial_L g_{A\underline{L}} = \frac{1}{4} \partial_L (x_A \chi_{2A} (x_1 - t)) = 0$. Therefore,

$$\begin{aligned}d_{\underline{L}} &= g^{L\underline{L}} \partial_L g_{\underline{L}\underline{L}} + g^{AB} \partial_A g_{B\underline{L}} \\ &= g^{L\underline{L}} \partial_L g_{\underline{L}\underline{L}} + g^{22} \partial_2 g_{2\underline{L}} + g^{33} \partial_3 g_{3\underline{L}} \\ &= -\frac{1}{2} \partial_L (-t \tilde{\chi}_2) - \frac{1}{4} (1 + \chi_1)^{-1} \partial_2 (x_2 (1 + \chi_1) \tilde{\chi}_2) \\ &\quad - \frac{1}{4} (1 + \chi_1) \partial_3 (x_3 (1 + \chi_1)^{-1} \tilde{\chi}_2) \\ &= 0,\end{aligned}$$

since $\partial_L (t \tilde{\chi}_2 (x_1 - t)) = \tilde{\chi}_2 (x_1 - t)$. Lastly,

$$\begin{aligned}d_A &= g^{LL} \partial_L g_{LA} + g^{L\underline{L}} \partial_L g_{\underline{L}A} + g^{LB} \partial_L g_{AB} \\ &\quad + g^{\underline{L}\underline{L}} \partial_{\underline{L}} g_{LA} + g^{BL} \partial_B g_{LA} + g^{BC} \partial_B g_{CA}.\end{aligned}$$

We drop derivatives of the constant coefficients g_{XL} :

$$d_A = g^{L\underline{L}} \partial_L g_{\underline{L}A} + g^{LB} \partial_L g_{AB} + g^{BC} \partial_B g_{CA}.$$

Next, observe that $g_{\underline{L}A}$ depends only on $x_1 - t$ and x_A , thus $\partial_L g_{\underline{L}A} = 0$. Similarly, g_{AB} depends only on $x_1 - t$, therefore $\partial_L g_{AB} = 0$. Similarly, g_{CA}

depends only on $t - x_1$, and therefore $\partial_B g_{CA} = 0$. Thus we arrive at the conclusion

$$d_A = 0,$$

which completes the proof. □

Proof of Lemma 3.5. The function χ_1 has been analyzed in Lemma 2.1. Thus, to prove the lemma, it is enough to establish the following:

$$\begin{aligned} \tilde{\chi}_2 &\in H^1(D_t) \setminus H^2(D_t), \quad t \in [0, 1], \\ x^A \tilde{\chi}_2 &\in H^2(D_0), \quad A = 2, 3. \end{aligned}$$

Recall that $\tilde{\chi}_2$ satisfies

$$(3.0.10) \quad \frac{d}{dx} \tilde{\chi}_2 = 2(\chi'_1(x))^2(1 + \chi_1(x))^{-2} + \frac{1}{16} \tilde{\chi}_2(x)^2.$$

We choose $\tilde{\chi}_2(0) = 0$. Then by integrating (3.0.10), we can show that $\tilde{\chi}_2$ is bounded by 2 for $|y| \leq 1$ if we adjust ϵ in the definition of χ_1 , so that $\int_0^1 (\chi'_1)^2 \leq 1$, and apply the bootstrap assumption $\tilde{\chi}_2 \leq 4$ for the integral $\int_0^1 \tilde{\chi}_2^2$. The same argument works to show that $\tilde{\chi}'_2 \in L^2(D_0)$. To show that $\tilde{\chi}''_2 \notin L^2$, we differentiate (3.0.10). We will have

$$(3.0.11) \quad \tilde{\chi}''_2 = 2(\chi'_1(y))^2(1 + \chi_1^2)^{-2} + F(\chi_1, \chi'_1, \tilde{\chi}_2, \tilde{\chi}'_2),$$

where F will have a smooth dependance on $\chi_1, \tilde{\chi}_2$ and a polynomial in $\chi'_1, \tilde{\chi}'_2$. Since $\chi'_1, \tilde{\chi}'_2 \in L^p$ for any $p < \infty$, we conclude that

$$\|F(\chi_1, \chi'_1, \tilde{\chi}_2, \tilde{\chi}'_2)\|_{L^2} \leq C < \infty.$$

Also since χ_1 is bounded, we can bound $(1 + \chi_1(y))^{-2} \geq c > 0$. Therefore, applying the same logic as in Lemma 2.1, we will arrive at

$$\int_{D_t} \tilde{\chi}''_2(x_1 - t)^2 \geq c \int_0^{1-t} \frac{\epsilon^4(1-t)\pi(2\alpha)^2 x_1 dx_1}{x_1^2 |\log |x_1/4||^{2(1-2\alpha)}} - \frac{1}{2} C^2 = \infty.$$

Thus it remains to show that $x_A \tilde{\chi}_2 \in H^2(B_0)$. Without loss of generality, put $A = 2$. The only estimate that is not addressed above is $x_2 \tilde{\chi}''_2 \in L^2$, since we have already shown $\frac{\partial}{\partial x_2}(x_2 \tilde{\chi}_2) \in H^1$. We use (3.0.11) to obtain the following estimate, which concludes the proof of the lemma:

$$\begin{aligned} &\int_{D_t} x_2^2 |\tilde{\chi}''_2(x_1 - t)|^2 dx \\ &\lesssim \int_t^{2-t} |\tilde{\chi}''_2(x_1 - t)|^2 \left(\int_{x_2^2 + x_3^2 \leq (2-(x_1+t))(x_1-t)} x_2^2 dx_2 dx_3 \right) dx_1 \\ &\lesssim \int_t^{2-t} |\tilde{\chi}''_2(x_1 - t)|^2 \pi(2 - (x_1 + t))^2 (x_1 - t)^2 dx_1 \\ &\lesssim 2 \int_0^{1-t} \frac{\epsilon^4(1-t)^2 \pi(2\alpha)^2 dx_1}{|\log |x_1/4||^{2(1-2\alpha)}} < \infty. \end{aligned} \quad \square$$

4. Taking into account the bending of the light cones

To take into account the bending of the light cones in the metric we need to open up our domain slightly to ensure it is spacelike or null. We will therefore replace our domain. Let D be the domain

$$(4.0.1) \quad D = \{(t, x); (x_1 - 1)^2 H(x_1 - 1) + x_2^2/4 + x_3^2/4 < (1 - t)^2\},$$

where $H(x_1 - 1) = 1$ when $x_1 < 1$, and $H(x_1 - 1) = 1/4$ when $x_1 > 1$. The boundary consist of two parts $C = C_1 \cup C_2$, where

$$(4.0.2) \quad C_1 = \{(t, x); x_1 < 1, (x_1 - 1)^2 + x_2^2/4 + x_3^2/4 = (1 - t)^2\}$$

and

$$(4.0.3) \quad C_2 = \{(t, x); x_1 \geq 1, (x_1 - 1)^2/4 + x_2^2/4 + x_3^2/4 = (1 - t)^2\}.$$

A conormal to C_1 is given by

$$n = 2(1 - t)dt - 2(1 - x_1)dx_1 + x_2 dx_2/2 + x_3 dx_3/2,$$

or expressed in the L, \underline{L}, A, B coordinates $u = (t - x_1)/2$ and $v = (t + x_1)/2$,

$$n = 4(1 - v)du - 4udv + x_2 dx_2/2 + x_3 dx_3/2.$$

Hence,

$$\begin{aligned} g_{\alpha\beta} N^\alpha N^\beta &= g^{\alpha\beta} n_\alpha n_\beta \\ &= g^{uu} n_u n_u + g^{vv} n_v n_v + 2g^{uv} n_u n_v + 2g^{uA} n_u n_A + g^{AB} n_A n_B \\ &= 16(1 - v)u + \left(\frac{1}{4}(u + v) + \frac{1}{64}(x_2^2(1 + \chi_1) + x_3^2(1 + \chi_1)^{-1})\tilde{\chi}_2\right)\tilde{\chi}_2 16u^2 \\ &\quad - (x_2^2 + x_3^2)(1 - v)\tilde{\chi}_2 + (1 + \chi_1)^{-1}x_2^2\frac{1}{4} + (1 + \chi_1)x_3^2\frac{1}{4} \\ &= 16(1 - v)u + \frac{1}{4}(x_2^2 + x_3^2) + \frac{1}{4}(x_3^2 - (1 + \chi_1)^{-1}x_2^2)\chi_1 - (x_2^2 + x_3^2)(1 - v)\tilde{\chi}_2 \\ &\quad + \left(\frac{1}{4}(u + v) + \frac{1}{64}(x_2^2(1 + \chi_1) + x_3^2(1 + \chi_1)^{-1})\tilde{\chi}_2\right)\tilde{\chi}_2 16u^2. \end{aligned}$$

The surface is in the uv coordinates given by

$$4(1 - v)u + x_2^2/4 + x_3^2/4 = 0.$$

Therefore it is clear that if N is the normal to C_1 , then

$$g_{\alpha\beta} N^\alpha N^\beta \leq 0,$$

with equality only if $u = 0$. This proves that C_1 is spacelike apart from when $u = 0$, where it is null.

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