

# On the structure of $\mathcal{A}$ -free measures and applications

By GUIDO DE PHILIPPIS and FILIP RINDLER

## Abstract

We establish a general structure theorem for the singular part of  $\mathcal{A}$ -free Radon measures, where  $\mathcal{A}$  is a linear PDE operator. By applying the theorem to suitably chosen differential operators  $\mathcal{A}$ , we obtain a simple proof of Alberti's rank-one theorem and, for the first time, its extensions to functions of bounded deformation (BD). We also prove a structure theorem for the singular part of a finite family of normal currents. The latter result implies that the Rademacher theorem on the differentiability of Lipschitz functions can hold only for absolutely continuous measures and that every top-dimensional Ambrosio–Kirchheim metric current in  $\mathbb{R}^d$  is a Federer–Fleming flat chain.

## 1. Introduction

Consider a finite Radon measure  $\mu$  on an open set  $\Omega \subset \mathbb{R}^d$  with values in  $\mathbb{R}^m$  that is  $\mathcal{A}$ -free for a  $k$ 'th-order linear constant-coefficient PDE operator  $\mathcal{A}$  ( $k \in \mathbb{N}$ ), i.e.,

$$(1.1) \quad \mathcal{A}\mu := \sum_{|\alpha| \leq k} A_\alpha \partial^\alpha \mu = 0 \quad \text{in } \mathcal{D}'(\Omega; \mathbb{R}^n).$$

Here,  $A_\alpha \in \mathbb{R}^{n \times m}$  and  $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d}$  for each multi-index  $\alpha = (\alpha_1, \dots, \alpha_d) \in (\mathbb{N} \cup \{0\})^d$ . A central question about (1.1) asks what can be said about the *singular part*  $\mu^s$  of solutions  $\mu = g\mathcal{L}^d + \mu^s$  ( $\mu^s \perp \mathcal{L}^d$ ). Besides Alberti's celebrated rank-one theorem [1] for  $\mathcal{A} = \text{curl}$ , not much is known at present.

In this respect we recall that the *wave cone*

$$\Lambda_{\mathcal{A}} := \bigcup_{|\xi|=1} \ker \mathbb{A}^k(\xi) \subset \mathbb{R}^m \quad \text{with} \quad \mathbb{A}^k(\xi) := (2\pi i)^k \sum_{|\alpha|=k} A_\alpha \xi^\alpha,$$

where  $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_d^{\alpha_d}$ , plays a crucial role in the compensated compactness theory for sequences of  $\mathcal{A}$ -free maps [29], [30], [42], [43], [16], [39]. Indeed,  $\Lambda_{\mathcal{A}}$

contains the values that an oscillating or concentrating sequence of functions is expected to attain. The corresponding *characteristic*  $\xi$ 's determine the allowed directions of oscillations and concentrations.

Since the singular part  $\mu^s$  of a measure contains “condensed” concentrations, it is natural to conjecture that for a measure  $\mu$  solving (1.1), the *polar*  $\frac{d\mu}{d|\mu|}$ , i.e., the Radon–Nikodým derivative of  $\mu$  with respect to its total variation measure  $|\mu|$ , must lie in the wave cone at almost all singular points. For  $\mathcal{A} = \text{curl}$ , this was conjectured by Ambrosio and De Giorgi in [14] and proved by Alberti in [1]. Our main result asserts the truth of this conjecture in full generality:

**THEOREM 1.1.** *Let  $\Omega \subset \mathbb{R}^d$  be an open set, let  $\mathcal{A}$  be a  $k$ 'th-order linear constant-coefficient differential operator as above, and let  $\mu \in \mathcal{M}(\Omega; \mathbb{R}^m)$  be an  $\mathcal{A}$ -free Radon measure on  $\Omega$  with values in  $\mathbb{R}^m$ . Then,*

$$\frac{d\mu}{d|\mu|}(x) \in \Lambda_{\mathcal{A}} \quad \text{for } |\mu|^s\text{-a.e. } x \in \Omega.$$

*Remark 1.2.* Note that (perhaps surprisingly) we do not need to require  $\mathcal{A}$  to satisfy Murat's constant-rank condition [31].

*Remark 1.3.* Let us point out that Theorem 1.1 is also valid in the situation

$$(1.2) \quad \mathcal{A}\mu = \sigma \quad \text{for some } \sigma \in \mathcal{M}(\Omega; \mathbb{R}^n).$$

This can be reduced to the setting of Theorem 1.1 by defining  $\tilde{\mu} = (\mu, \sigma) \in \mathcal{M}(\mathbb{R}^d; \mathbb{R}^{m+n})$  and  $\tilde{\mathcal{A}}$  (with an additional 0'th-order term) such that (1.2) is equivalent to  $\tilde{\mathcal{A}}\tilde{\mu} = 0$ . It is easy to check that, if  $k \geq 1$ , then  $\Lambda_{\tilde{\mathcal{A}}} = \Lambda_{\mathcal{A}} \times \mathbb{R}^n$  and that for  $|\mu|$ -a.e. point,  $\frac{d\mu}{d|\mu|}$  is proportional to  $\frac{d\tilde{\mu}}{d|\tilde{\mu}|}$ .

*Remark 1.4.* Using essentially the same proof, Theorem 1.1 can be further extended to the setting of *variable-coefficient* linear differential operators  $\mathcal{A} = \sum_{\alpha} A_{\alpha}(x)\partial^{\alpha}$  with the coefficients satisfying suitable regularity assumptions. In this setting, the conclusion reads

$$\frac{d\mu}{d|\mu|}(x) \in \Lambda_{\mathcal{A}}(x) := \bigcup_{|\xi|=1} \ker \mathbb{A}_x^k(\xi) \quad \text{for } |\mu|^s\text{-a.e. } x,$$

where

$$\mathbb{A}_x^k(\xi) := \sum_{|\alpha|=k} (2\pi i)^k A_{\alpha}(x)\xi^{\alpha}.$$

Similar statements can be obtained if  $\mu$  solves some pseudo-differential equations.

By applying Theorem 1.1 to suitably chosen differential operators, we easily obtain several remarkable consequences, which are outlined below. In

particular, we provide a simple proof of Alberti’s rank-one theorem and, for the first time, its extensions to functions of bounded deformation (BD). We also prove a structure theorem for the singular part of a finite family of normal currents in the spirit of the rank-one theorem. By relying on the results of Alberti and Marchese [4] and of Schioppa [40], the latter result immediately implies that the Rademacher theorem can hold only for absolutely continuous measures and that every top-dimensional Ambrosio–Kirchheim metric current in  $\mathbb{R}^d$  is a Federer–Fleming flat chain (a part of the so-called “flat chain conjecture”, see [11, §11]).

1.1. *Rank-one property of BV-derivatives.* As already mentioned above, in [1] Alberti solved a conjecture of Ambrosio and De Giorgi [14] by showing the rank-one property for the singular part of the gradients of BV-functions (also see [15], [2]). Besides its theoretical interest, the rank-one theorem has many applications in the theory of functions of bounded variation; we just mention the following: lower-semicontinuity and relaxation [9], [18], [26], integral representation theorems [12], Young measure theory [25], [38], [24], approximation theory [27], and the study of continuity equations with BV-vector fields [6]. (In the latter case the use of the rank-one theorems can however be avoided; see [6, Rem. 3.7] and [7].) We refer to [10, Ch. 5] for further history.

**THEOREM 1.5** (Alberti’s rank-one theorem). *Let  $\Omega \subset \mathbb{R}^d$  be an open set and let  $u \in \text{BV}(\Omega; \mathbb{R}^\ell)$ . Then, for  $|D^s u|$ -a.e.  $x \in \Omega$ , there exist  $a(x) \in \mathbb{R}^\ell \setminus \{0\}$ ,  $b(x) \in \mathbb{R}^d \setminus \{0\}$  such that*

$$\frac{dD^s u}{d|D^s u|}(x) = a(x) \otimes b(x).$$

Alberti’s rank-one theorem easily follows by choosing  $\mathcal{A} = \text{curl}$  in Theorem 1.1. Let us also mention that Massaccesi and Vittone have recently given a short and elegant proof of the rank-one property based on the theory of sets of finite perimeter [28].

As already observed by Alberti in [1, Th. 4.13], Theorem 1.5 implies the validity of a similar property for higher-order derivatives. A direct proof of this fact can also be obtained as a corollary of our Theorem 1.1:

**THEOREM 1.6** (Rank-one theorem for higher-order derivatives). *Let  $\Omega \subset \mathbb{R}^d$  be an open set and let  $u \in L^1(\Omega; \mathbb{R}^\ell)$  with  $D^r u \in \mathcal{M}(\Omega; \text{SLin}^r(\mathbb{R}^d; \mathbb{R}^\ell))$  for some  $r \in \mathbb{N}$ , where  $\text{SLin}^r(\mathbb{R}^d; \mathbb{R}^\ell)$  contains all symmetric  $r$ -linear maps from  $\mathbb{R}^d$  to  $\mathbb{R}^\ell$ . Then, for  $|(D^r u)^s|$ -a.e.  $x \in \Omega$ , there exist  $a(x) \in \mathbb{R}^\ell \setminus \{0\}$ ,  $b(x) \in \mathbb{R}^d \setminus \{0\}$  such that*

$$\frac{d(D^r u)^s}{d|(D^r u)^s|}(x) = a(x) \otimes \underbrace{b(x) \otimes \cdots \otimes b(x)}_{r \text{ times}}.$$

1.2. *Polar density theorem for BD-functions.* The proofs in [1] and in [28] of Alberti's rank-one theorem strongly rely on the structure of functions of bounded variation and on their link with the theory of sets of finite perimeter. In particular, so far it has remained open whether a similar statement is valid for the larger class of functions of *bounded deformation*, i.e., those functions  $u \in L^1(\Omega; \mathbb{R}^d)$  whose symmetric part of the (distributional) derivative is a measure,

$$Eu := \frac{Du + (Du)^T}{2} \in \mathcal{M}(\Omega; \mathbb{R}_{\text{sym}}^{d \times d}).$$

We collect all these functions into the set  $\text{BD}(\Omega)$ ; see [45], [44], [8] for a detailed account of the theory of this space.

The extension of Alberti's rank-one theorem to the space of functions of bounded deformation follows from our main Theorem 1.1 with the appropriate choice of the differential operator  $\mathcal{A}$ :

**THEOREM 1.7.** *Let  $\Omega \subset \mathbb{R}^d$  be an open set and let  $u \in \text{BD}(\Omega)$ . Then, for  $|E^s u|$ -a.e.  $x \in \Omega$ , there exist  $a(x), b(x) \in \mathbb{R}^d \setminus \{0\}$  such that*

$$\frac{dE^s u}{d|E^s u|}(x) = a(x) \odot b(x),$$

where we define the symmetrized tensor product as  $a \odot b := (a \otimes b + b \otimes a)/2$  for  $a, b \in \mathbb{R}^d$ .

This theorem has consequences for the structure theory of BD-functions and lower semicontinuity theory. (In the lower semicontinuity theory our structure theorem can, however, be avoided at the price of some mild restrictions on the functional; see [36] for BD and [37] for an analogous result in BV.) Some of these consequences will be explored in future work.

Further, in [44], [20], [13] it is motivated why the space

$$U(\Omega) := \{ u \in \text{BD}(\Omega) : \text{div } u \in L^2(\Omega) \}$$

is the appropriate space for elasto-plasticity theory in the geometrically linear setting. For this space we immediately get the following structure result:

**COROLLARY 1.8.** *Let  $\Omega \subset \mathbb{R}^d$  be an open set and let  $u \in U(\Omega)$ . Then, for  $|E^s u|$ -a.e.  $x \in \Omega$ , there exist  $a(x), b(x) \in \mathbb{R}^d \setminus \{0\}$  with*

$$a(x) \perp b(x)$$

such that

$$\frac{dE^s u}{d|E^s u|}(x) = a(x) \odot b(x).$$

1.3. *Normal currents, the Rademacher theorem, and metric currents.* Our next application of Theorem 1.1 deals with finite families of (Euclidean) normal currents, by which we obtain some consequences concerning the differentiability of Lipschitz functions and the theory of metric currents. We assume the reader to be familiar with the theory of currents and with basic multilinear algebra. We refer to [17, Chs. 1, 4] and Section 3 below for the relevant notations and definitions.

To motivate our result, recall that any  $(d - 1)$ -dimensional normal current  $T \in \mathbf{N}_{d-1}(\mathbb{R}^d)$  without boundary ( $\partial T = 0$ ) can be identified via Hodge duality with the derivative of a function  $u \in \text{BV}_{\text{loc}}(\mathbb{R}^d; \mathbb{R})$ , that is,  $T = \star Du$ . Using this identification and the fact that  $\dim \Lambda_{d-1}(V) = 1$  if and only if  $\dim(V) = d - 1$ , Theorem 1.5 can be rephrased as follows.

COROLLARY 1.9. *Let*

$$T_1 = \vec{T}_1 \|T_1\|, \dots, T_r = \vec{T}_r \|T_r\| \in \mathbf{N}_{d-1}(\mathbb{R}^d)$$

be  $(d - 1)$ -dimensional boundaryless normal currents, i.e.,  $\partial T_i = 0$  for  $i = 1, \dots, r$ . Let further  $\mu \in \mathcal{M}_+(\mathbb{R}^d)$  be a positive Radon measure such that

$$\mu \ll \|T_i\| \quad \text{for } i = 1, \dots, r.$$

Then, for  $\mu^s$ -a.e.  $x \in \mathbb{R}^d$ , there exists a  $(d - 1)$ -dimensional subspace  $V_x \subset \mathbb{R}^d$  such that  $\vec{T}_1(x), \dots, \vec{T}_r(x) \in \Lambda_{d-1}(V_x)$ .

As another simple application of Theorem 1.1 we can generalize the above statement to finite families of normal currents (not necessarily of the same dimension).

THEOREM 1.10. *Let  $\Omega \subset \mathbb{R}^d$  be an open set and let  $T_1 = \vec{T}_1 \|T_1\| \in \mathbf{N}_{k_1}(\Omega), \dots, T_r = \vec{T}_r \|T_r\| \in \mathbf{N}_{k_r}(\Omega)$  be normal currents, where  $k_1, \dots, k_r \in \{1, \dots, d\}$ ,  $r \in \mathbb{N}$ . Let further  $\mu \in \mathcal{M}_+(\Omega)$  be a positive Radon measure such that*

$$\mu \ll \|T_i\| \quad \text{for } i = 1, \dots, r.$$

Then, for  $\mu^s$ -a.e.  $x \in \Omega$ , there exists a 1-covector  $\omega_x \in \Lambda^1(\mathbb{R}^d) \setminus \{0\}$  such that

$$\vec{T}_1(x) \lrcorner \omega_x = \dots = \vec{T}_r(x) \lrcorner \omega_x = 0.$$

Equivalently, for  $\mu^s$ -a.e.  $x \in \Omega$ ,  $\vec{T}_1(x) \in \Lambda_{k_1}(\ker \omega_x), \dots, \vec{T}_r(x) \in \Lambda_{k_r}(\ker \omega_x)$ .

Remark 1.11. Let us note in passing the following curious consequence of the above result: It is well known that, apart from the trivial cases  $k \in \{1, d - 1, d\}$ , the orienting vector  $\vec{T}$  of a  $k$ -dimensional normal current  $T$  need not be simple, i.e., of the form  $\vec{T}(x) = v_1(x) \wedge \dots \wedge v_k(x)$ ,  $v_i(x) \in \mathbb{R}^d$ . However, if  $\dim V = (d - 1)$ , then every  $w \in \Lambda_{d-2}(V)$  is necessarily simple. Thus, we

have that for  $T \in \mathbf{N}_{d-2}^{\text{loc}}(\mathbb{R}^d)$ , the simplicity of  $\vec{T}$  holds  $\|T\|^s$ -almost everywhere. Note that the current

$$T = (e_1 \wedge e_2 + e_3 \wedge e_4) \mathcal{H}^4 \llcorner \{x_5 = 0\} \in \mathbf{N}_2^{\text{loc}}(\mathbb{R}^5)$$

shows that this statement is false for  $k$ -dimensional currents with  $1 < k < (d - 2)$ .

A particularly relevant instance of Theorem 1.10 is obtained when  $r = d$  and  $k_1 = \dots = k_d = 1$ . In view of the subsequent applications, let us state it in a slightly different (but equivalent) formulation:

**COROLLARY 1.12.** *Let  $T_1 = \vec{T}_1 \|T_1\|, \dots, T_d = \vec{T}_d \|T_d\| \in \mathbf{N}_1(\mathbb{R}^d)$  be one-dimensional normal currents such that there exists a positive Radon measure  $\mu \in \mathcal{M}_+(\mathbb{R}^d)$  with the following properties:*

- (i)  $\mu \ll \|T_i\|$  for  $i = 1, \dots, d$ ;
- (ii) for  $\mu$ -a.e.  $x$ ,  $\text{span}\{\vec{T}_1(x), \dots, \vec{T}_d(x)\} = \mathbb{R}^d$ .

*Then,  $\mu \ll \mathcal{L}^d$ .*

This answers the question about a higher-dimensional analogue of [2, Prop. 8.6]. By the trivial identification of one-dimensional normal currents with vector-valued measures, Corollary 1.12 can be stated in the following equivalent formulation, which in a sense is dual to Theorem 1.5. It can be also directly inferred from Theorem 1.1.

**COROLLARY 1.13.** *Let  $\mu \in \mathcal{M}(\Omega; \mathbb{R}^{d \times d})$  be a matrix-valued measure such that*

$$\text{div } \mu \in \mathcal{M}(\Omega; \mathbb{R}^d).$$

*Then,*

$$\text{rank} \left( \frac{d\mu}{d|\mu|}(x) \right) \leq d - 1 \quad \text{for } |\mu|^s\text{-a.e. } x \in \Omega.$$

It has been noted in several places that the validity of the rank-one theorem for maps  $u \in \text{BV}(\mathbb{R}^2; \mathbb{R}^2)$  has some direct implications concerning differentiability of Lipschitz functions and the structure of top-dimensional metric currents in the plane [34], [2], [3], [35], [4], [40]. Relying on [4], [40], we use Corollary 1.12 to extend these results to every dimension. In particular, Theorem 1.15 below provides a positive answer to the case  $k = d$  of the “flat chain conjecture” stated in [11, §11]; see [40, Th. 1.6] for the case  $k = 1$ .

**THEOREM 1.14.** *Let  $\mu \in \mathcal{M}_+(\mathbb{R}^d)$  be a positive Radon measure such that every Lipschitz map  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  is differentiable  $\mu$ -almost everywhere. Then,  $\mu \ll \mathcal{L}^d$ .*

**THEOREM 1.15.** *Let  $T \in \mathbf{M}_d^{\text{met}}(\mathbb{R}^d)$  be an Ambrosio–Kirchheim metric current of dimension  $d$ ; see [11]. Then,  $\|T\| \ll \mathcal{L}^d$ . In particular, the space*

of  $d$ -dimensional metric currents in  $\mathbb{R}^d$  coincides with the space of Federer–Fleming  $d$ -dimensional flat chains,  $\mathbf{M}_d^{\text{met}}(\mathbb{R}^d) = \mathbf{F}_d(\mathbb{R}^d)$ .

Let us mention that the last two theorems will also follow by a stronger result announced by Csörnyei and Jones in [23], namely, that for every Lebesgue null set  $E \subset \mathbb{R}^d$ , there exists a Lipschitz map  $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$  that is nowhere differentiable in  $E$ ; see the discussion in the introduction of [4] for a detailed account of these type of results.

1.4. *Sketch of the proof.* We conclude this introduction with an outline of the main ideas behind the proof of Theorem 1.1. Let us assume for simplicity that  $\mathcal{A}$  is a first-order homogeneous operator,  $\mathcal{A} = \sum_{\ell} A_{\ell} \partial_{\ell}$ . Assume by contradiction that there is a set  $E$  of positive  $|\mu|^s$ -measure such that the polar vector  $\frac{d\mu}{d|\mu|}(x)$  is not in the wave cone  $\Lambda_{\mathcal{A}}$  for every  $x \in E$ . One can then find a point  $x_0 \in E$  and a sequence  $r_j \downarrow 0$  such that

$$\text{w}^*\text{-}\lim_{j \rightarrow \infty} \frac{(T^{x_0, r_j})_{\#} \mu}{|\mu|(B_{r_j}(x_0))} = \text{w}^*\text{-}\lim_{j \rightarrow \infty} \frac{(T^{x_0, r_j})_{\#} \mu^s}{|\mu|^s(B_{r_j}(x_0))} = P_0 \nu,$$

where  $T^{x, r} : \mathbb{R}^d \rightarrow \mathbb{R}^d$  is the dilation map  $T^{x, r}(y) = (y - x)/r$ ,  $T_{\#}^{x, r}$  denotes the push-forward operator (that is, for any measure  $\sigma$  and Borel set  $B$ ,  $[(T^{x, r})_{\#} \sigma](B) := \sigma(x + rB)$ ),  $\nu \in \text{Tan}(x_0, |\mu|) = \text{Tan}(x_0, |\mu|^s)$  is a nonzero tangent measure in the sense of Preiss [33], and

$$P_0 := \frac{d\mu}{d|\mu|}(x_0) \notin \Lambda_{\mathcal{A}}.$$

Moreover, one easily checks that

$$\sum_{\ell=1}^d A_{\ell} P_0 \partial_{\ell} \nu = 0 \quad \text{in } \mathcal{D}'(\Omega; \mathbb{R}^n).$$

By taking the Fourier transform of the above equation, we get

$$\mathbb{A}(\xi) P_0 \widehat{\nu}(\xi) = 0, \quad \xi \in \mathbb{R}^d.$$

Having assumed that  $P_0 \notin \Lambda_{\mathcal{A}}$ , this implies  $\text{supp } \widehat{\nu} = \{0\}$  and thus  $\nu \ll \mathcal{L}^d$ . The latter fact, however, is not by itself a contradiction to  $\nu \in \text{Tan}(x_0, |\mu|^s)$ . Indeed, Preiss [33] provided an example of a purely singular measure that has only multiples of Lebesgue measure as tangents. (We also refer to [32] for a measure that has *every* measure as a tangent at almost every point.)

On the other hand,  $P_0 \notin \Lambda_{\mathcal{A}}$  implies that  $\mathbb{A}(\xi) P_0 \neq 0$ , so one can hope for some sort of “elliptic regularization” that forces not only  $\nu \ll \mathcal{L}^d$  but also  $|\mu|^s \ll \mathcal{L}^d$  in a neighborhood of  $x_0$ . In fact, this is (almost) the case: Inspired by Allard’s Strong Constancy Lemma in [5] and using some basic

pseudo-differential calculus, we can show that in the above situation, not only

$$\nu_j := \frac{(T^{x_0, r_j})\# \mu^s}{|\mu|^s(B_{r_j}(x_0))} \xrightarrow{*} \nu \ll \mathcal{L}^d$$

but that, crucially, this convergence also holds in the total variation norm,

$$|\nu_j - \nu|(B_1) \rightarrow 0.$$

Since  $\nu_j \perp \mathcal{L}^d$ , this latter fact easily gives a contradiction to  $\nu \ll \mathcal{L}^d$  and concludes the proof of the theorem.

*Acknowledgments.* The authors would like to thank A. Massaccesi and D. Vittone for useful discussions. G. D. P. is supported by the MIUR SIR-grant “Geometric Variational Problems” (RBSI14RVEZ) and F. R. acknowledges the support from an EPSRC Research Fellowship on “Singularities in Nonlinear PDEs” (EP/L018934/1).

## 2. Proof of the main theorem

2.1. *Notation.* We denote by  $\mathcal{M}(\Omega; \mathbb{R}^m)$  the space of all finite Radon measures on an open set  $\Omega \subset \mathbb{R}^d$  with values in  $\mathbb{R}^m$  and by  $\mathcal{M}_+(\Omega)$  the space of positive Radon measures on  $\Omega$ . We write  $\mu = \text{w}^*\text{-lim}_{j \rightarrow \infty} \mu_j$  or  $\mu_j \xrightarrow{*} \mu$  for the local weak\*-convergence of  $\mu_j$  to  $\mu$ , that is  $\int \varphi d\mu_j \rightarrow \int \varphi d\mu$  for all  $\varphi \in C_c^0(\Omega)$ , the set of all continuous functions with compact support in  $\Omega$ . The  $d$ -dimensional Lebesgue measure is  $\mathcal{L}^d$ . Given a Borel set  $B \subset \Omega$  and a measure  $\mu \in \mathcal{M}(\Omega; \mathbb{R}^m)$  (or  $\mu \in \mathcal{M}_+(\Omega)$ ), we denote by  $\mu \llcorner B$  the restriction of  $\mu$  to  $B$ .

The Lebesgue–Radon–Nikodým decomposition of a Radon measure  $\mu \in \mathcal{M}(\Omega; \mathbb{R}^m)$  is given as

$$\mu = \frac{d\mu}{d|\mu|} |\mu| = \mu^a + \mu^s = g \mathcal{L}^d + \frac{d\mu}{d|\mu|} |\mu|^s,$$

where  $\frac{d\mu}{d|\mu|} \in L(\Omega, |\mu|; \mathbb{R}^m)$  is the polar of  $\mu$ , i.e., the Radon–Nikodým derivative of  $\mu$  with respect to  $\mu$ 's total variation measure  $|\mu| \in \mathcal{M}_+(\Omega)$ ,  $\mu^a \ll \mathcal{L}^d$  is the absolutely continuous part of  $\mu$  with density  $g \in L^1(\Omega)$ , and  $\mu^s \perp \mathcal{L}^d$  is the singular part of  $\mu$ . Note that here and in the following the terms “singular” and “absolutely continuous” are always understood with respect to the Lebesgue measure if not otherwise specified.

We will generically denote by  $\mathcal{A}$  a  $k$ 'th-order linear partial differential operator with constant coefficients that acts on smooth functions  $u \in C^\infty(\mathbb{R}^d; \mathbb{R}^m)$  as

$$\mathcal{A}u := \sum_{|\alpha| \leq k} A_\alpha \partial^\alpha u \in C^\infty(\mathbb{R}^d; \mathbb{R}^n),$$

where  $\alpha = (\alpha_1, \dots, \alpha_d) \in (\mathbb{N} \cup \{0\})^d$  is a multi-index,  $\partial^\alpha = \partial_1^{\alpha_1} \dots \partial_d^{\alpha_d}$ , and  $A_\alpha \in \mathbb{R}^{n \times m}$  are matrices. A vector-valued Radon measure  $\mu \in \mathcal{M}(\Omega; \mathbb{R}^m)$  is said to be  $\mathcal{A}$ -free if

$$\mathcal{A}\mu = 0 \quad \text{in } \mathcal{D}'(\Omega; \mathbb{R}^n).$$

Here,  $\mathcal{D}(\Omega; \mathbb{R}^n) = C_c^\infty(\Omega; \mathbb{R}^n)$  is the set of  $\mathbb{R}^n$ -valued test functions in  $\Omega$  with the usual topology and  $\mathcal{D}'(\Omega; \mathbb{R}^n)$  is the set of  $\mathbb{R}^n$ -valued distributions on  $\Omega$ .

Given  $\mathcal{A}$  as above, its *symbol*  $\mathbb{A}: \mathbb{R}^d \rightarrow \mathbb{R}^{n \times m}$  is defined as

$$\mathbb{A}(\xi) := \sum_{|\alpha| \leq k} (2\pi i)^{|\alpha|} A_\alpha \xi^\alpha, \quad \xi \in \mathbb{R}^d,$$

where  $\xi^\alpha := \xi_1^{\alpha_1} \dots \xi_d^{\alpha_d}$ . Note that for  $u$  in the Schwartz class  $\mathcal{S}(\mathbb{R}^d; \mathbb{R}^m)$ ,

$$\widehat{\mathcal{A}u}(\xi) = \mathbb{A}(\xi)\widehat{u}(\xi),$$

where for  $v \in \mathcal{S}(\mathbb{R}^d; \mathbb{R}^m)$  we denote by  $\widehat{v}$  its Fourier transform,

$$\widehat{v}(\xi) = \mathcal{F}[v](\xi) := \int v(x)e^{-2\pi i x \cdot \xi} dx, \quad \xi \in \mathbb{R}^d.$$

We also recall the definition of the *wave cone* associated to  $\mathcal{A}$  [42], [31], [43], [16]:

$$\Lambda_{\mathcal{A}} := \bigcup_{|\xi|=1} \ker \mathbb{A}^k(\xi) \subset \mathbb{R}^m \quad \text{with} \quad \mathbb{A}^k(\xi) := (2\pi i)^k \sum_{|\alpha|=k} A_\alpha \xi^\alpha.$$

2.2. *First-order operators.* For the sake of illustration, we first treat the case when  $\mathcal{A}$  is a first-order homogeneous constant-coefficient differential operator, namely,

$$(2.1) \quad \mathcal{A}\mu = \sum_{\ell=1}^d A_\ell \partial_\ell \mu = 0 \quad \text{in } \mathcal{D}'(\Omega; \mathbb{R}^n).$$

*Proof of Theorem 1.1 assuming (2.1).* We have

$$\Lambda_{\mathcal{A}} = \bigcup_{|\xi|=1} \ker \mathbb{A}(\xi), \quad \mathbb{A}(\xi) = \mathbb{A}^1(\xi) = 2\pi i \sum_{\ell=1}^d A_\ell \xi_\ell.$$

Let

$$E := \left\{ x \in \Omega : \frac{d\mu}{d|\mu|}(x) \notin \Lambda_{\mathcal{A}} \right\},$$

where the existence of  $\frac{d\mu}{d|\mu|}(x)$  in the sense of the Besicovitch derivation theorem (see [10, Th. 2.22]) is part of the definition of  $E$ .

Assume by contradiction that  $|\mu|^s(E) > 0$ . We now choose a point  $x_0 \in E$  and a sequence  $r_j \downarrow 0$  such that

$$(i) \quad \lim_{j \rightarrow \infty} \frac{|\mu|^a(B_{r_j}(x_0))}{|\mu|^s(B_{r_j}(x_0))} = 0 \quad \text{and} \quad \lim_{j \rightarrow \infty} \int_{B_{r_j}(x_0)} \left| \frac{d\mu}{d|\mu|}(x) - \frac{d\mu}{d|\mu|}(x_0) \right| d|\mu|^s(x) = 0;$$

(ii) there exists a positive Radon measure  $\nu \in \mathcal{M}_+(\mathbb{R}^d)$  with  $\nu \llcorner B_{1/2} \neq 0$  and such that

$$\nu_j := \frac{(T^{x_0, r_j})_{\#} |\mu|^s}{|\mu|^s(B_{r_j}(x_0))} \ast \nu;$$

(iii) for the polar vector, it holds that

$$P_0 := \frac{d\mu}{d|\mu|}(x_0) \notin \Lambda_{\mathcal{A}}$$

and there is a positive constant  $c > 0$  such that  $|\mathbb{A}(\xi)P_0| \geq c|\xi|$  for  $\xi \in \mathbb{R}^d$ .

Indeed, (i) holds at  $|\mu|^s$ -a.e. point by classical measure theory, (ii) follows by the fact that for  $|\mu|^s$ -a.e.  $x \in \Omega$  the space of tangent measures  $\text{Tan}(|\mu|^s, x)$  to  $|\mu|^s$  at  $x$  is nontrivial (see, for instance, [33, Th. 2.5] or [36, Lemma A.1]), and finally, (iii) follows from the assumption  $|\mu|^s(E) > 0$ .

We now claim that (i)–(iii) above imply that

$$(2.2) \quad 0 \neq \nu \llcorner B_{1/2} \ll \mathcal{L}^d,$$

$$(2.3) \quad \lim_{j \rightarrow \infty} |\nu_j - \nu|(B_{1/2}) = 0.$$

Before proving (2.2) and (2.3), let us show how to use them to conclude the proof. Recall that  $\nu_j \perp \mathcal{L}^d$ , and take Borel sets  $E_j \subset B_{1/2}$  with  $\mathcal{L}^d(E_j) = 0 = \nu(E_j)$  and  $\nu_j(E_j) = \nu_j(B_{1/2})$ . Then,

$$\nu_j(B_{1/2}) = \nu_j(E_j) \leq |\nu_j - \nu|(B_{1/2}) + \nu(E_j) = |\nu_j - \nu|(B_{1/2}) \rightarrow 0,$$

thanks to (2.3). Hence, we infer  $\nu(B_{1/2}) = 0$ , in contradiction to (2.2). Thus,  $|\mu|^s(E) = 0$ , concluding the proof of the theorem.

We are thus left to prove (2.2) and (2.3). Let us assume that  $x_0 = 0$  and set  $T^r := T^{x_0, r}$ . Clearly,

$$\mathcal{A}(T^r_{\#} \mu) = 0 \quad \text{in } \mathcal{D}'(B_1; \mathbb{R}^n).$$

Therefore, with  $\nu_j$  defined as in (ii) above and  $c_j := |\mu|^s(B_{r_j})^{-1}$ ,

$$(2.4) \quad \mathcal{A}(P_0 \nu_j) = \mathcal{A}(P_0 \nu_j - c_j T^r_{\#} \mu).$$

Let now  $\{\varphi_\varepsilon\}_{\varepsilon > 0}$  be a compactly supported, smooth, and positive approximation of the identity. By the lower semicontinuity of the total variation,

$$|\nu_j - \nu|(B_{1/2}) \leq \liminf_{\varepsilon \rightarrow 0} |\nu_j \ast \varphi_\varepsilon - \nu|(B_{1/2}).$$

Thus, for every  $j$ , we can find  $\varepsilon_j \leq 1/j$  such that

$$(2.5) \quad |\nu_j - \nu|(B_{1/2}) \leq |\nu_j \ast \varphi_{\varepsilon_j} - \nu|(B_{1/2}) + \frac{1}{j}.$$

We now convolve (2.4) with  $\varphi_{\varepsilon_j}$  to get

$$(2.6) \quad \mathcal{A}(P_0 u_j) = \mathcal{A}(V_j),$$

where we have set

$$u_j := \nu_j * \varphi_{\varepsilon_j}, \quad V_j := [P_0 \nu_j - c_j T_{\#}^{r_j} \mu] * \varphi_{\varepsilon_j}.$$

Note that  $u_j, V_j$  are smooth,  $u_j \geq 0$ , and

$$(2.7) \quad u_j \xrightarrow{*} \nu.$$

Moreover, recalling that  $x_0 = 0$  and  $c_j = |\mu|^s(B_{r_j})^{-1}$ , by the definition of  $V_j$ ,  $\nu_j$ ,  $P_0$  and standard properties of convolutions (see [10, Th. 2.2]), for  $\varepsilon_j \leq 1/4$ , it holds that

$$\begin{aligned} \int_{B_{3/4}} |V_j| \, dx &\leq \frac{|P_0 T_{\#}^{r_j} |\mu|^s - T_{\#}^{r_j} \mu|(B_1)}{|\mu|^s(B_{r_j})} \\ &\leq \frac{|P_0 |\mu|^s - \mu^s|(B_{r_j})}{|\mu|^s(B_{r_j})} + \frac{|\mu|^a(B_{r_j})}{|\mu|^s(B_{r_j})} \\ &= \int_{B_{r_j}} \left| \frac{d\mu}{d|\mu|}(0) - \frac{d\mu}{d|\mu|}(x) \right| d|\mu|^s(x) + \frac{|\mu|^a(B_{r_j})}{|\mu|^s(B_{r_j})}. \end{aligned}$$

Hence, by (i) above,

$$(2.8) \quad \lim_{j \rightarrow \infty} \int_{B_{3/4}} |V_j| \, dx = 0.$$

Take a cut-off function  $\chi \in \mathcal{D}(B_{3/4})$  with  $0 \leq \chi \leq 1$  and  $\chi \equiv 1$  on  $B_{1/2}$ . Then, (2.6) implies that

$$(2.9) \quad \mathcal{A}(P_0 \chi u_j) = \chi \mathcal{A}(P_0 u_j) + \mathcal{A}(P_0 \chi) u_j = \mathcal{A}(\chi V_j) + R_j,$$

where the remainder terms  $R_j := \mathcal{A}(P_0 \chi) u_j - \sum_{\ell} A_{\ell} V_j \partial_{\ell} \chi$  are smooth, compactly supported in  $B_1$ , and satisfy

$$\sup_j \int_{B_1} |R_j| \, dx \leq C$$

for some constant  $C$  thanks to (2.7) and (2.8). Taking the Fourier transform of (2.9), we obtain

$$[\mathbb{A}(\xi) P_0] \widehat{\chi u_j}(\xi) = \mathbb{A}(\xi) \widehat{\chi V_j}(\xi) + \widehat{R_j}(\xi).$$

Now multiply by  $[\mathbb{A}(\xi) P_0]^* = \overline{[\mathbb{A}(\xi) P_0]^T}$  and add  $\widehat{\chi u_j}(\xi)$  to both sides of the above equation to obtain

$$(1 + |\mathbb{A}(\xi) P_0|^2) \widehat{\chi u_j}(\xi) = [\mathbb{A}(\xi) P_0]^* \mathbb{A}(\xi) \widehat{\chi V_j}(\xi) + \widehat{\chi u_j}(\xi) + [\mathbb{A}(\xi) P_0]^* \widehat{R_j}(\xi),$$

which can be rewritten as

$$\begin{aligned} \widehat{\chi u_j}(\xi) &= \frac{[\mathbb{A}(\xi) P_0]^* \mathbb{A}(\xi) \widehat{\chi V_j}(\xi)}{1 + |\mathbb{A}(\xi) P_0|^2} + \frac{1 + 4\pi^2 |\xi|^2}{1 + |\mathbb{A}(\xi) P_0|^2} \cdot \frac{\widehat{\chi u_j}(\xi)}{1 + 4\pi^2 |\xi|^2} \\ &\quad + \frac{(1 + 4\pi^2 |\xi|^2)^{1/2} [\mathbb{A}(\xi) P_0]^*}{1 + |\mathbb{A}(\xi) P_0|^2} \cdot \frac{\widehat{R_j}(\xi)}{(1 + 4\pi^2 |\xi|^2)^{1/2}}. \end{aligned}$$

Hence,

$$(2.10) \quad \chi u_j = T_0[\chi V_j] + T_1[\chi u_j] + T_2[R_j] =: f_j + g_j + h_j$$

with

$$\begin{aligned} T_0[V] &:= \mathcal{F}^{-1} \left[ (1 + |\mathbb{A}(\xi)P_0|^2)^{-1} [\mathbb{A}(\xi)P_0]^* \mathbb{A}(\xi) \widehat{V}(\xi) \right], \\ T_1[u] &:= \mathcal{F}^{-1} \left[ m_1(\xi) (1 + 4\pi^2|\xi|^2)^{-1} \widehat{u}(\xi) \right], \\ T_2[R] &:= \mathcal{F}^{-1} \left[ m_2(\xi) (1 + 4\pi^2|\xi|^2)^{-1/2} \widehat{R}(\xi) \right], \end{aligned}$$

where we have set

$$\begin{aligned} m_1(\xi) &= (1 + |\mathbb{A}(\xi)P_0|^2)^{-1} (1 + 4\pi^2|\xi|^2), \\ m_2(\xi) &= (1 + |\mathbb{A}(\xi)P_0|^2)^{-1} (1 + 4\pi^2|\xi|^2)^{1/2} [\mathbb{A}(\xi)P_0]^*. \end{aligned}$$

By (iii) above,  $T_0$  is an operator associated with an Hörmander–Mihlin multiplier (meaning that it has a smooth symbol  $m_0(\xi)$  such that  $|\partial^\beta m_0(\xi)| \leq K|\xi|^{-|\beta|}$  for every multi-index  $|\beta| \leq \lfloor d/2 \rfloor + 1$  and some  $K > 0$ ). The  $L^1$ – $L^{1,\infty}$  estimates [21, Th. 5.2.7] in conjunction with (2.8) give

$$(2.11) \quad \sup_{\lambda \geq 0} \lambda \mathcal{L}^d(\{|f_j| > \lambda\}) \leq C \|\chi V_j\|_{L^1} \rightarrow 0.$$

Moreover, the operators  $T_1$  and  $T_2$  are compact from  $L_c^1(B_1)$  to  $L_{\text{loc}}^1(\mathbb{R}^d)$ , where  $L_c^1(B_1)$  is the set of  $L^1$ -functions vanishing outside  $B_1$ . Indeed, by Lemma 2.1 below, for every  $s > 0$ , the operator

$$(\text{Id} - \Delta)^{-s/2} w = \mathcal{F}^{-1} \left[ (1 + 4\pi^2|\xi|^2)^{-s/2} \widehat{w}(\xi) \right]$$

is compact from  $L_c^1(B_1)$  to  $L^p(\mathbb{R}^d)$  for  $1 < p < p(d, s)$  and by [21, Th. 5.2.7] the symbols  $m_1$  and  $m_2$  are  $L^p$ -multipliers. We conclude, in particular, that

$$\{g_j + h_j\}_j \text{ is precompact in } L_{\text{loc}}^1(\mathbb{R}^d).$$

From (2.8) we further get

$$(2.12) \quad \langle f_j, \varphi \rangle = \langle T_0[\chi V_j], \varphi \rangle = \langle \chi V_j, T_0^*[\varphi] \rangle \rightarrow 0 \quad \text{for every } \varphi \in \mathcal{D}(\mathbb{R}^d; \mathbb{R}^n),$$

where  $T_0^* : \mathcal{S}(\mathbb{R}^d; \mathbb{R}^n) \mapsto \mathcal{S}(\mathbb{R}^d; \mathbb{R}^m)$  is the adjoint of  $T_0$ . Since  $\chi u_j \geq 0$ , (2.10) gives that

$$f_j^- := \max\{0, -f_j\} \leq |g_j + h_j|.$$

As shown above, the family  $\{g_j + h_j\}_j$  is precompact in  $L_{\text{loc}}^1(\mathbb{R}^d)$  and thus the previous inequality implies the local equi-integrability of  $\{f_j^-\}$ . Together with (2.11), (2.12) and Lemma 2.2 below this yields  $f_j \rightarrow 0$  in  $L_{\text{loc}}^1(\mathbb{R}^d)$  and thus that the sequence  $\{\chi u_j\}$  is precompact in  $L_{\text{loc}}^1(\mathbb{R}^d)$ . Since also  $\chi u_j \xrightarrow{*} \chi \nu$  by (2.7), we deduce that  $\chi \nu \in L^1(\mathbb{R}^d)$ , which implies (2.2). Moreover,

$$\chi u_j \rightarrow \chi \nu \quad \text{in } L^1(\mathbb{R}^d),$$

which, taking into account (2.5), implies (2.3).  $\square$

2.3. *General operators.* We now treat the general situation, namely, the case of a measure  $\mu \in \mathcal{M}(\Omega; \mathbb{R}^n)$  satisfying

$$(2.13) \quad \mathcal{A}\mu = \sum_{|\alpha| \leq k} A_\alpha \partial^\alpha \mu = 0 \quad \text{in } \mathcal{D}'(\Omega; \mathbb{R}^n).$$

*Proof of Theorem 1.1.* As before, let us set

$$E := \left\{ x \in \Omega : \frac{d\mu}{d|\mu|}(x) \notin \Lambda_{\mathcal{A}} \right\}$$

and assume that  $|\mu|^s(E) > 0$ . Arguing as in the proof for first-order operators, we may find a point  $x_0 \in E$  satisfying (i) and (ii) above and also

(iii') for the polar vector, it holds that

$$P_0 := \frac{d\mu}{d|\mu|}(x_0) \notin \Lambda_{\mathcal{A}},$$

and there is a positive constant  $c > 0$  such that  $|\mathbb{A}^k(\xi)P_0| \geq c|\xi|^k$  for  $\xi \in \mathbb{R}^d$ .

We will show that (i), (ii) and (iii') together imply (2.2) and (2.3), and thus yield the desired contradiction.

Assuming that  $x_0 = 0$ , we note that (2.13) and a simple scaling argument give

$$\mathcal{A}^k(T_{\#}^r \mu) + \sum_{h=0}^{k-1} \mathcal{A}^h(r^{k-h} T_{\#}^r \mu) = 0,$$

where  $\mathcal{A}^h := \sum_{|\alpha|=h} A_\alpha \partial^\alpha$  is the  $h$ -homogeneous part of the operator  $\mathcal{A}$ . Hence, with  $\nu_j$  defined as in (ii) and  $c_j = |\mu|^s(B_{r_j})^{-1}$ ,

$$\sum_{|\alpha|=k} A_\alpha \partial^\alpha (P_0 \nu_j) = \sum_{|\alpha|=k} A_\alpha \partial^\alpha (P_0 \nu_j - c_j T_{\#}^{r_j} \mu) - \sum_{h=1}^{k-1} \mathcal{A}^h(r_j^{k-h} c_j T_{\#}^{r_j} \mu).$$

Mollification and localization now yield

$$(2.14) \quad \sum_{|\alpha|=k} A_\alpha \partial^\alpha (P_0 \chi u_j) = \sum_{|\alpha|=k} A_\alpha \partial^\alpha (\chi V_j) + R_j.$$

Here, as before,

$$u_j := \nu_j * \varphi_{\varepsilon_j}, \quad V_j = [P_0 \nu_j - c_j T_{\#}^{r_j} \mu] * \varphi_{\varepsilon_j},$$

where  $\chi \in \mathcal{D}(B_{3/4})$  with  $0 \leq \chi \leq 1$ ,  $\chi \equiv 1$  on  $B_{1/2}$ , and  $\varphi_{\varepsilon_j}$  is a sequence of mollifier such that (2.5) is satisfied. In particular, by (i),  $\|\chi V_j\|_{L^1} \rightarrow 0$ . Moreover, the remainder term  $R_j$  can be written as a finite sum of smooth-coefficient partial differential operators of order *at most*  $k-1$  applied to smooth

functions with bounded  $L^1$ -norm and compact support:

$$R_j = \sum_{h=0}^{k-1} \sum_{|\alpha|=h} b_\alpha(x) \partial^\alpha z_j^\alpha,$$

where  $b_\alpha(x) \in \mathcal{D}(B_{3/4})$ , the functions  $z_j^\alpha$  are smooth and compactly supported, and  $\sup_j \|z_j^\alpha\|_{L^1} \leq C$  for some constant  $C$ . Namely,  $R_j = R_j^1 + R_j^2 + R_j^3$ , where

$$\begin{aligned} R_j^1 &= \sum_{|\alpha|=k} \sum_{\substack{\beta+\gamma=\alpha \\ |\gamma|\geq 1}} c_{\beta\gamma} \partial^\gamma \chi \partial^\beta (A_\alpha P_0 \tilde{\chi} u_j), \\ R_j^2 &= \sum_{|\alpha|=k} \sum_{\substack{\beta+\gamma=\alpha \\ |\gamma|\geq 1}} c_{\beta\gamma} \partial^\gamma \chi \partial^\beta (A_\alpha \tilde{\chi} V_j), \\ R_j^3 &= \sum_{|\alpha|\leq k-1} \sum_{|\alpha|=h} \chi \partial^\alpha \left( \tilde{\chi} A_\alpha (r_j^{k-h} c_j T_\#^{r_j} \mu) * \varphi_{\varepsilon_j} \right) \end{aligned}$$

with  $c_{\beta\gamma} \in \mathbb{R}$ , and  $\tilde{\chi} \in \mathcal{D}(B_1)$  is identically equal to 1 on the support of  $\chi$ .

By taking the Fourier transform of (2.14) and performing the same computations as in the first part, but now multiplying with  $[\mathbb{A}^k(\xi)P_0]^*$  instead of  $[\mathbb{A}(\xi)P_0]^*$ , we obtain

$$(2.15) \quad \chi u_j = S_0[\chi V_j] + S_1[\chi u_j] + \tilde{R}_j,$$

where  $S_0$  and  $S_1$  are given by

$$\begin{aligned} S_0[V] &= \mathcal{F}^{-1} \left[ \frac{[\mathbb{A}^k(\xi)P_0]^* \mathbb{A}^k(\xi) \widehat{V}(\xi)}{1 + |\mathbb{A}^k(\xi)P_0|^2} \right], \\ S_1[u] &= \mathcal{F}^{-1} \left[ \frac{(1 + 4\pi^2|\xi|^2)^k}{1 + |\mathbb{A}^k(\xi)P_0|^2} \cdot \frac{\widehat{u}(\xi)}{(1 + 4\pi^2|\xi|^2)^k} \right]. \end{aligned}$$

Applying the Hörmander–Mihlin multiplier theorem and arguing as for first-order operators, we deduce that

$$\sup_{\lambda \geq 0} \lambda \mathcal{L}^d(\{|S_0[\chi V_j]| > \lambda\}) \leq C \|\chi V_j\|_{L^1(B_1)} \rightarrow 0 \quad \text{and} \quad S_0[\chi V_j] \xrightarrow{*} 0.$$

Moreover, the family  $\{S_1[\chi u_j]\}$  is precompact in  $L^1_{\text{loc}}(\mathbb{R}^d)$ . To conclude the proof it is enough to show that  $\{\tilde{R}_j\}$  is precompact in  $L^1_{\text{loc}}(\mathbb{R}^d)$ , since then the application of Lemma 2.2 as in the first part will imply the validity of (2.2) and (2.3). The generic term of  $\tilde{R}_j$  can be written as

$$f_j^\alpha = Q \circ (\text{Id} - \Delta)^{-\frac{k}{2}} \circ P_\alpha \circ (\text{Id} - \Delta)^{\frac{|\alpha|-k}{2}} [z_j^\alpha],$$

where  $0 \leq |\alpha| \leq (k - 1)$ ,  $\sup_j \|z_j^\alpha\|_{L^1} \leq C$ ,

$$Q[z] = \mathcal{F}^{-1} \left[ (1 + |\mathbb{A}^k(\xi)P_0|^2)^{-1} (1 + 4\pi^2|\xi|^2)^{k/2} \mathbb{A}^k(\xi) \widehat{z}(\xi) \right],$$

and  $P_\alpha$  is the  $k$ 'th-order pseudo-differential operator given by

$$P_\alpha[z](x) = \int b_\alpha(x) \frac{(2\pi i)^{|\alpha|} \xi^\alpha}{(1 + 4\pi^2 |\xi|^2)^{\frac{|\alpha|-k}{2}}} \widehat{z}(\xi) e^{2\pi i x \cdot \xi} d\xi, \quad x \in \mathbb{R}^d.$$

The composition  $(\text{Id} - \Delta)^{-k/2} \circ P_\alpha$  is a pseudo-differential operator of order 0 (see [41, Th. 2, Ch. VI]), and thus bounded from  $L^p(\mathbb{R}^d)$  to itself; see [41, Prop. 4, Ch. VI]. By (iii') and the Hörmander–Mihlin multiplier theorem, also  $Q$  is a bounded operator from  $L^p(\mathbb{R}^d)$  to itself. Since  $|\alpha| \leq k - 1$ , Lemma 2.1 below implies that  $(\text{Id} - \Delta)^{(|\alpha|-k)/2}$  is compact from  $L^1_c(B_1)$  to  $L^p(\mathbb{R}^d)$  for  $1 < p < p(d, |\alpha| - k)$ . We conclude that  $\{f_j^\alpha\}$  is precompact in  $L^1_{\text{loc}}(\mathbb{R}^d)$ . The validity of (2.2) and (2.3) now follows from (2.15) by arguing as before.  $\square$

2.4. *Auxiliary results.* Finally, we prove the two simple technical lemmas that have been used in the proofs above. The first is an  $L^1$ -compactness result in the spirit of the Sobolev embedding theorems. Since we have not been able to find a reference, we provide its simple proof.

LEMMA 2.1. *For  $u \in \mathcal{S}(\mathbb{R}^d)$  and  $s > 0$ , define*

$$(\text{Id} - \Delta)^{-s/2} u := \mathcal{F}^{-1}[(1 + 4\pi^2 |\xi|^2)^{-s/2} \widehat{u}(\xi)].$$

*Then,  $(\text{Id} - \Delta)^{-s/2}$  extends to a compact map from  $L^1_c(B_1)$  to  $L^p(\mathbb{R}^d)$  for  $1 \leq p < p(d, s)$ , where*

$$p(d, s) := \begin{cases} \frac{d}{d-s} & \text{if } s < d, \\ \infty & \text{otherwise,} \end{cases}$$

*and  $L^1_c(B_1) \subset L^1(\mathbb{R}^d)$  is the set of  $L^1$ -functions supported in  $B_1$ .*

*Proof.* For  $u$  in the Schwartz class, we can write

$$(\text{Id} - \Delta)^{-s/2} u = b(s, d) * u$$

where  $b(s, d) = \mathcal{F}^{-1}[(1 + 4\pi^2 |\xi|^2)^{-s/2}]$  is the Bessel potential of order  $s$ ; see [22, §6.1.2]. By classical estimates [22, Prop. 6.1.5],  $b(s, d) \in L^p$  for  $1 \leq p < p(d, s)$  so that by Young's inequality for convolutions,

$$(\text{Id} - \Delta)^{-s/2} u \in L^p \quad \text{for} \quad 1 \leq p < p(d, s)$$

(actually also for  $p = p(d, s)$  if  $s \neq d$ ). For every  $\varepsilon > 0$ , we can write

$$b(s, d) = b_{1,\varepsilon} + b_{2,\varepsilon} \quad \text{with} \quad b_{1,\varepsilon} \in C^1_c(\mathbb{R}^d) \quad \text{and} \quad \|b_{2,\varepsilon}\|_{L^1} < \varepsilon;$$

see [22, Prop. 6.1.6]. Thus,

$$(\text{Id} - \Delta)^{-s/2} u = b_{1,\varepsilon} * u + b_{2,\varepsilon} * u =: T_{1,\varepsilon}[u] + T_{2,\varepsilon}[u].$$

Because  $b_{1,\varepsilon} \in C^1_c(\mathbb{R}^d)$ ,  $T_{1,\varepsilon}$  is compact from  $L^1_c(B_1)$  to  $L^1(\mathbb{R}^d)$ . Moreover,

$$\|(\text{Id} - \Delta)^{-s/2} - T_{1,\varepsilon}\|_{L^1 \rightarrow L^1} \leq \|T_{2,\varepsilon}\|_{L^1 \rightarrow L^1} \leq \varepsilon,$$

so that  $(\text{Id} - \Delta)^{-s/2}$  is the limit in the uniform topology of compact operators and thus compact as well. The conclusion of the lemma now follows by Hölder’s inequality.  $\square$

The second lemma is a consequence of the Vitali convergence theorem.

LEMMA 2.2. *Let  $\{f_j\} \subset L^1_c(B_1)$  be a family of functions such that*

- (a)  $f_j \xrightarrow{*} 0$  in  $\mathcal{D}'(B_1)$ ;
- (b) *the negative parts of the  $f_j$ ’s tend to zero in measure, i.e.,*

$$\lim_{j \rightarrow \infty} \mathcal{L}^d(\{f_j^- > \lambda\}) = 0 \quad \text{for every } \lambda > 0;$$

- (c) *the sequence of negative parts  $\{f_j^-\}$  is equi-integrable,*

$$\lim_{\mathcal{L}^d(E) \rightarrow 0} \sup_{j \in \mathbb{N}} \int_{B_1} f_j^- \, dx = 0.$$

Then,  $f_j \rightarrow 0$  in  $L^1_{\text{loc}}(B_1)$ .

*Proof.* Let  $\varphi \in \mathcal{D}(B_1)$ ,  $0 \leq \varphi \leq 1$ . It is enough to show that

$$(2.16) \quad \lim_{j \rightarrow \infty} \int \varphi |f_j| \, dx = 0.$$

We write

$$\int \varphi |f_j| \, dx = \int \varphi f_j \, dx + 2 \int \varphi f_j^- \, dx \leq \int \varphi f_j \, dx + 2 \int f_j^- \, dx.$$

The first term on the right-hand side goes to 0 as  $j \rightarrow \infty$  by assumption (a). Thanks to the Vitali convergence theorem, assumptions (b) and (c) further give that also the second term vanishes in the limit. Hence, (2.16) follows.  $\square$

### 3. Applications

Theorems 1.5, 1.6 and 1.7 follow from Theorem 1.1 simply by applying it to the differential constraints that gradients, higher gradients, or symmetrized gradients, respectively, have to satisfy.

*Proof of Theorem 1.5.* Let  $\mu = (\mu_j^k) \in \mathcal{M}(\Omega; \mathbb{R}^{\ell \times d})$  be the (distributional) gradient of a function  $u \in \text{BV}(\Omega; \mathbb{R}^\ell)$ ,  $\mu = Du$ . Then,

$$0 = \partial_i \mu_j^k - \partial_j \mu_i^k \quad \text{for } i, j = 1, \dots, d; k = 1, \dots, \ell.$$

Setting

$$\mathcal{A} \mu := (\partial_j \mu_i^k - \partial_i \mu_j^k)_{i,j=1,\dots,d; k=1,\dots,\ell},$$

it is a simple algebraic exercise, carried out for instance in [19, Rem. 3.5(iii)], to compute that

$$\Lambda_{\mathcal{A}} = \{a \otimes \xi \in \mathbb{R}^{\ell \times d} : a \in \mathbb{R}^\ell, \xi \in \mathbb{R}^d \setminus \{0\}\}.$$

Corollary 1.5 then follows directly from Theorem 1.1.  $\square$

*Proof of Theorem 1.6.* For the operator

$$\mathcal{A}\mu := \left( \partial_j \mu_{\alpha_1 \dots \alpha_h i \alpha_{h+2} \dots \alpha_r}^k - \partial_i \mu_{\alpha_1 \dots \alpha_h j \alpha_{h+2} \dots \alpha_r}^k \right)_{i,j,\alpha_1,\dots,\alpha_n=1,\dots,d; k=1,\dots,\ell; h=1,\dots,r},$$

one can see that  $\mathcal{A}\mu = 0$  if and only if  $\mu$  is an  $r$ 'th-order derivative, and furthermore one can compute that

$$\Lambda_{\mathcal{A}} = \left\{ a \otimes \xi \otimes \dots \otimes \xi \in \text{SLin}^r(\mathbb{R}^d; \mathbb{R}^\ell) : a \in \mathbb{R}^\ell, \xi \in \mathbb{R}^d \setminus \{0\} \right\};$$

see [19, Ex. 3.10(d)] for the details. □

*Proof of Theorem 1.7.* Let  $\mu = (\mu_j^k) \in \mathcal{M}(\Omega, \mathbb{R}_{\text{sym}}^{d \times d})$  be the (distributional) symmetrized gradient of  $u \in \text{BD}(\Omega)$ ,  $\mu = Eu$ . Then, by direct computation (see [19, Ex. 3.10(e)]),

$$0 = \mathcal{A}\mu := \left( \sum_{i=1}^d \partial_{ik} \mu_i^j + \partial_{ij} \mu_i^k - \partial_{jk} \mu_i^i - \partial_{ii} \mu_j^k \right)_{j,k=1,\dots,d}.$$

These equations are often called the *Saint-Venant compatibility conditions* in applications. Hence, for  $M \in \mathbb{R}_{\text{sym}}^{d \times d}$ ,

$$-(4\pi)^{-2} \mathbb{A}(\xi)M = (M\xi) \otimes \xi + \xi \otimes (M\xi) - (\text{tr} M) \xi \otimes \xi - |\xi|^2 M,$$

which gives

$$\ker \mathbb{A}^2(\xi) = \ker \mathbb{A}(\xi) = \left\{ a \otimes \xi + \xi \otimes a : a \in \mathbb{R}^d \right\}.$$

Theorem 1.1 now implies the conclusion. □

*Proof of Corollary 1.8.* The only fact to show in addition to the assertion of Corollary 1.7 is that  $a(x) \cdot b(x) = 0$ . For  $Eu$ , we have the Lebesgue–Radon–Nikodým decomposition  $Eu = \mathcal{E}u \mathcal{L}^d + E^s u$  and thus

$$\text{div } u = \text{tr}(\mathcal{E}u) \mathcal{L}^d + a(x) \cdot b(x) |E^s u|.$$

Since  $\text{div } u \in L^2(\Omega)$ , we must have  $a(x) \cdot b(x) = 0$  for  $|E^s u|$ -a.e.  $x \in \Omega$ . □

Before proving Theorem 1.10, let us recall some simple facts concerning (Euclidean) currents and multi-linear algebra. We refer to [17] for more details.

Given a finite dimensional vector space  $V$ , we let  $\Lambda_k(V)$  be the set of  $k$ -vectors and  $\Lambda^k(V) \simeq (\Lambda_k(V))^*$  be the set of  $k$ -covectors. If  $v \in \Lambda_k(V)$  and  $\eta \in \Lambda^1(V)$ , then the interior product of  $\eta$  with  $v$  is the  $(k - 1)$ -vector  $v \lrcorner \eta \in \Lambda_{k-1}(V)$  defined by duality as  $\langle v \lrcorner \eta, \omega \rangle := \langle v, \eta \wedge \omega \rangle$  for every  $\omega \in \Lambda^{k-1}(V)$ ; see [17, §1.5].

Following [17, §4.1.7], we let

$$\mathcal{D}^k(\Omega) := \mathcal{D}(\Omega, \Lambda^k(\mathbb{R}^d)) \quad \text{and} \quad \mathcal{D}_k(\Omega) := \mathcal{D}'(\Omega, \Lambda_k(\mathbb{R}^d))$$

be the sets of compactly supported  $k$ -differential forms with smooth coefficients and the set of  $k$ -dimensional currents, respectively. For  $T \in \mathcal{D}_k(\Omega)$ , the

boundary  $\partial T \in \mathcal{D}_{k-1}(\Omega)$  is defined by duality with the exterior differential via  $\langle \partial T, \omega \rangle := \langle T, d\omega \rangle$ , where  $\omega \in \mathcal{D}^{k-1}(\Omega)$ . One easily checks that

$$(3.1) \quad \partial T = - \sum_{i=1}^d \partial_i T \lrcorner dx^i;$$

see [17, p. 356]. Here, for  $T \in \mathcal{D}_k(\Omega)$  and  $\eta \in C^\infty(\Omega; \Lambda^1(\mathbb{R}^d))$ ,  $T \lrcorner \eta \in \mathcal{D}_{k-1}(\Omega)$  is defined as  $\langle T \lrcorner \eta, \omega \rangle := \langle T, \eta \wedge \omega \rangle$ ,  $\omega \in \mathcal{D}^{k-1}(\Omega)$  and  $\partial_i T \in \mathcal{D}_k(\Omega)$  is defined by duality via  $\langle \partial_i T, \phi dx^{j_1} \wedge \dots \wedge dx^{j_k} \rangle = -\langle T, \partial_i \phi dx^{j_1} \wedge \dots \wedge dx^{j_k} \rangle$ .

We endow  $\Lambda_k(\mathbb{R}^d)$  with the mass norm; see [17, §1.8]. A  $k$ -current is said to have finite mass if it can be extended to a  $\Lambda_k(\mathbb{R}^d)$ -valued (finite) Radon measure, and we let  $\|T\|$  be the total variation of  $T$  and

$$\vec{T} := \frac{dT}{d\|T\|};$$

see [17, §4.1.7]. In this context, the Radon–Nikodým theorem reads as  $T = \vec{T} \|T\|$ . We denote by  $\mathbf{N}_k(\Omega)$  the set of  $k$ -dimensional normal currents, i.e., those currents such that  $T$  and  $\partial T$  both have finite mass. Note that the boundary of a  $k$ -dimensional normal current  $T$  can be seen as a  $\Lambda_{k-1}(\mathbb{R}^d)$ -valued Radon measure,  $\partial T \in \mathcal{M}(\Omega; \Lambda_{k-1}(\mathbb{R}^d))$ .

*Proof of Theorem 1.10.* Let us set

$$\mathbf{T} := (T_1, \dots, T_r) \in \mathcal{M}(\Omega; \Lambda_{k_1}(\mathbb{R}^d) \times \dots \times \Lambda_{k_r}(\mathbb{R}^d))$$

and note that the assumption of Theorem 1.10 can be rewritten as

$$\mathcal{A}\mathbf{T} := (\partial T_1, \dots, \partial T_r) \in \mathcal{M}(\Omega; \Lambda_{k_1-1}(\mathbb{R}^d) \times \dots \times \Lambda_{k_r-1}(\mathbb{R}^d)).$$

By applying Theorem 1.1 in conjunction with Remark 1.3 we deduce that for  $|\mathbf{T}|^s$ -a.e.  $x \in \Omega$ , there exists  $\xi_x \neq 0$  such that

$$(3.2) \quad \frac{d\mathbf{T}}{d|\mathbf{T}|}(x) \in \ker \mathbb{A}(\xi_x).$$

Thanks to (3.1), one easily checks that for  $\mathbf{v} = (v_1, \dots, v_r) \in \Lambda_{k_1}(\mathbb{R}^d) \times \dots \times \Lambda_{k_r}(\mathbb{R}^d)$ , it holds that

$$(3.3) \quad \mathbb{A}(\xi)\mathbf{v} = -2\pi i (v_1 \lrcorner \omega_\xi, \dots, v_r \lrcorner \omega_\xi) \in \Lambda_{k_1-1}(\mathbb{R}^d) \times \dots \times \Lambda_{k_r-1}(\mathbb{R}^d),$$

where  $\omega_\xi \in \Lambda^1(\mathbb{R}^d)$  is defined as  $\omega_\xi(w) := w \cdot \xi$ ,  $w \in \mathbb{R}^d$ .

Let  $\mu \in \mathcal{M}_+(\Omega)$  be as in the statement of the theorem, and note that since  $\mu \ll \|T_i\|$  for every  $i = 1, \dots, r$ , the Radon–Nikodým derivatives  $\frac{d|\mathbf{T}|}{d\|T_i\|}$  and  $\frac{dT_i}{d|\mathbf{T}|}$  exist  $\mu$ -almost everywhere. Then,

$$(3.4) \quad \vec{T}_i = \frac{d|\mathbf{T}|}{d\|T_i\|} \frac{dT_i}{d|\mathbf{T}|}.$$

Since clearly  $\mu^s \ll |\mathbf{T}|^s$ , the first part of the conclusion with  $\omega_x = \omega_{\xi_x}$  follows from (3.2), (3.3) and (3.4). It is now a simple exercise in linear algebra to see that the second part of the statement is equivalent to the first one.  $\square$

*Proof of Corollary 1.12.* By Theorem 1.10, assumption (i) implies that for  $\mu^s$ -a.e.  $x \in \mathbb{R}^d$ , there exists a  $(d - 1)$ -dimensional subspace  $V_x$  such that

$$\vec{T}_1(x), \dots, \vec{T}_d(x) \in V_x.$$

Assumption (ii) hence gives that  $\mu^s = 0$ , which is the desired conclusion.  $\square$

*Proof of Corollary 1.13.* Let  $\mu = (\mu_j^k) \in \mathcal{M}(\Omega; \mathbb{R}^{d \times d})$  and let

$$\mathcal{A}\mu := \operatorname{div} \mu = \left( \sum_{j=1}^d \partial_j \mu_j^k \right)_{k=1, \dots, d}.$$

Then, for  $M \in \mathbb{R}^{d \times d}$ ,  $\mathbb{A}(\xi)M = (2\pi i)M\xi$ , so that

$$\Lambda_{\mathcal{A}} = \left\{ M \in \mathbb{R}^{d \times d} : \operatorname{rank} M \leq d - 1 \right\}.$$

The conclusion follows from Theorem 1.1 and Remark 1.3.  $\square$

We will now show how to obtain Theorems 1.14 and 1.15 from Corollary 1.12. In order to do so, we assume the reader to be familiar with the work of Alberti–Marchese [4] concerning differentiability of Lipschitz functions, with the definition of metric currents given in [11], as well as with the work of Schioppa in [40]. We refer to these papers also for notations and definitions.

Let us start with the following lemma, which is essentially [4, Cor. 6.5].

LEMMA 3.1. *Let  $\mu \in \mathcal{M}_+(\mathbb{R}^d)$  be a finite positive Radon measure. Then the following are equivalent:*

- (i) *the decomposability bundle of  $\mu$  (see [4, §2.6]) is of full dimension,  $V(\mu, x) = \mathbb{R}^d$  for  $\mu$ -a.e.  $x \in \mathbb{R}^d$ ;*
- (ii) *there are  $d$  normal one-dimensional currents  $T_1 = \vec{T}_1 \|T_1\|, \dots, T_d = \vec{T}_d \|T_d\| \in \mathbf{N}_1(\mathbb{R}^d)$  such that  $\mu \ll \|T_i\|$  for  $i = 1, \dots, d$ , and*

$$\operatorname{span} \{ \vec{T}_1(x), \dots, \vec{T}_d(x) \} = \mathbb{R}^d \quad \text{for } \mu\text{-a.e. } x \in \mathbb{R}^d.$$

*Proof.* The implication (i)  $\Rightarrow$  (ii) is obtained by choosing (in a measurable way) for  $\mu$ -a.e.  $x \in \mathbb{R}^d$  a basis  $\{e_1(x), \dots, e_d(x)\}$  of  $V(\mu, x)$  and by applying to each  $e_i$  the implication (i)  $\Rightarrow$  (ii) of [4, Cor. 6.5]. For the other implication, simply notice that, by the implication (ii)  $\Rightarrow$  (i) of [4, Cor. 6.5],  $\vec{T}_i(x) \in V(\mu, x)$  for  $\mu$ -a.e.  $x \in \mathbb{R}^d$ .  $\square$

*Proof of Theorem 1.14.* By [4, Th. 1.1] the assumptions in the statement of the theorem are equivalent to  $V(\mu, x) = \mathbb{R}^d$  for  $\mu$ -a.e.  $x \in \mathbb{R}^d$ . This implies that  $\mu \ll \mathcal{L}^d$  by Lemma 3.1 and Corollary 1.12.  $\square$

*Proof of Theorem 1.15.* By [40, Th. 1.3] the mass measure  $\|T\|$  associated with a  $d$ -dimensional metric current  $T \in \mathbf{M}_d^{\text{met}}(\mathbb{R}^d)$  admits  $d$  independent Alberti representations, which, by the very definition of decomposability bundle (see [4, §2.6]) implies that  $V(\|T\|, x) = \mathbb{R}^d$  for  $\|T\|$ -a.e.  $x \in \mathbb{R}^d$ . Theorem 1.15 hence follows from Theorem 1.14; see also the discussion after Theorem 1.3 in [40].  $\square$

### References

- [1] G. ALBERTI, Rank one property for derivatives of functions with bounded variation, *Proc. Roy. Soc. Edinburgh Sect. A* **123** (1993), 239–274. MR 1215412. Zbl 0791.26008. <http://dx.doi.org/10.1017/S030821050002566X>.
- [2] G. ALBERTI, M. CSÖRNYEI, and D. PREISS, Structure of null sets in the plane and applications, in *European Congress of Mathematics*, Eur. Math. Soc., Zürich, 2005, pp. 3–22. MR 2185733. Zbl 1088.28002.
- [3] G. ALBERTI, M. CSÖRNYEI, and D. PREISS, Differentiability of Lipschitz functions, structure of null sets, and other problems, in *Proceedings of the International Congress of Mathematicians. Volume III*, Hindustan Book Agency, New Delhi, 2010, pp. 1379–1394. MR 2827846. Zbl 1251.26010.
- [4] G. ALBERTI and A. MARCHESE, On the differentiability of Lipschitz functions with respect to measures in the Euclidean space, *Geom. Funct. Anal.* **26** (2016), 1–66. MR 3494485. Zbl 06593116. <http://dx.doi.org/10.1007/s00039-016-0354-y>.
- [5] W. K. ALLARD, An integrality theorem and a regularity theorem for surfaces whose first variation with respect to a parametric elliptic integrand is controlled, in *Geometric Measure Theory and the Calculus of Variations* (Arcata, Calif., 1984), *Proc. Sympos. Pure Math.* **44**, Amer. Math. Soc., Providence, RI, 1986, pp. 1–28. MR 0840267. Zbl 0609.49028. <http://dx.doi.org/10.1090/pspum/044/840267>.
- [6] L. AMBROSIO, Transport equation and Cauchy problem for  $BV$  vector fields, *Invent. Math.* **158** (2004), 227–260. MR 2096794. Zbl 1075.35087. <http://dx.doi.org/10.1007/s00222-004-0367-2>.
- [7] L. AMBROSIO, Transport equation and Cauchy problem for non-smooth vector fields, in *Calculus of Variations and Nonlinear Partial Differential Equations, Lecture Notes in Math.* **1927**, Springer-Verlag, New York, 2008, pp. 1–41. MR 2408257. Zbl 1159.35041. [http://dx.doi.org/10.1007/978-3-540-75914-0\\_1](http://dx.doi.org/10.1007/978-3-540-75914-0_1).
- [8] L. AMBROSIO, A. COSCIA, and G. DAL MASO, Fine properties of functions with bounded deformation, *Arch. Rational Mech. Anal.* **139** (1997), 201–238. MR 1480240. Zbl 0890.49019. <http://dx.doi.org/10.1007/s002050050051>.
- [9] L. AMBROSIO and G. DAL MASO, On the relaxation in  $BV(\Omega; \mathbf{R}^m)$  of quasi-convex integrals, *J. Funct. Anal.* **109** (1992), 76–97. MR 1183605. Zbl 0769.49009. [http://dx.doi.org/10.1016/0022-1236\(92\)90012-8](http://dx.doi.org/10.1016/0022-1236(92)90012-8).
- [10] L. AMBROSIO, N. FUSCO, and D. PALLARA, *Functions of Bounded Variation and Free Discontinuity Problems*, *Oxford Math. Monogr.*, The Clarendon Press, Oxford University Press, New York, 2000. MR 1857292. Zbl 0957.49001.

- [11] L. AMBROSIO and B. KIRCHHEIM, Currents in metric spaces, *Acta Math.* **185** (2000), 1–80. MR 1794185. Zbl 0984.49025. <http://dx.doi.org/10.1007/BF02392711>.
- [12] G. BOUCHITTÉ, I. FONSECA, and L. MASCARENHAS, A global method for relaxation, *Arch. Rational Mech. Anal.* **145** (1998), 51–98. MR 1656477. Zbl 0921.49004. <http://dx.doi.org/10.1007/s002050050124>.
- [13] S. CONTI and M. ORTIZ, Dislocation microstructures and the effective behavior of single crystals, *Arch. Ration. Mech. Anal.* **176** (2005), 103–147. MR 2185859. Zbl 1064.74144. <http://dx.doi.org/10.1007/s00205-004-0353-2>.
- [14] E. DE GIORGI and L. AMBROSIO, New functionals in the calculus of variations, *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.* **82** (1988), 199–210 (1989). MR 1152641. Zbl 0715.49014. <http://dx.doi.org/10.1007/978-1-4757-6019-4.4>.
- [15] C. DE LELLIS, A note on Alberti’s rank-one theorem, in *Transport Equations and Multi-D Hyperbolic Conservation Laws, Lect. Notes Unione Mat. Ital.* **5**, Springer-Verlag, New York, 2008, pp. 61–74. MR 2504174. Zbl 1221.46030. [http://dx.doi.org/10.1007/978-3-540-76781-7\\_2](http://dx.doi.org/10.1007/978-3-540-76781-7_2).
- [16] R. J. DIPERNA, Compensated compactness and general systems of conservation laws, *Trans. Amer. Math. Soc.* **292** (1985), 383–420. MR 0808729. Zbl 0606.35052. <http://dx.doi.org/10.2307/2000221>.
- [17] H. FEDERER, *Geometric Measure Theory, Die Grundlehren der mathematischen Wissenschaften* **153**, Springer-Verlag, New York, 1969. MR 0257325. Zbl 0176.00801.
- [18] I. FONSECA and S. MÜLLER, Relaxation of quasiconvex functionals in  $BV(\Omega, \mathbf{R}^p)$  for integrands  $f(x, u, \nabla u)$ , *Arch. Rational Mech. Anal.* **123** (1993), 1–49. MR 1218685. Zbl 0788.49039. <http://dx.doi.org/10.1007/BF00386367>.
- [19] I. FONSECA and S. MÜLLER,  $\mathcal{A}$ -quasiconvexity, lower semicontinuity, and Young measures, *SIAM J. Math. Anal.* **30** (1999), 1355–1390. MR 1718306. Zbl 0940.49014. <http://dx.doi.org/10.1137/S0036141098339885>.
- [20] M. FUCHS and G. SEREGIN, *Variational Methods for Problems from Pasticity Theory and for Generalized Newtonian Fluids, Lecture Notes in Math.* **1749**, Springer-Verlag, New York, 2000. MR 1810507. Zbl 0964.76003. <http://dx.doi.org/10.1007/BFb0103751>.
- [21] L. GRAFAKOS, *Classical Fourier Analysis*, third ed., *Grad. Texts in Math.* **249**, Springer-Verlag, New York, 2014. MR 3243734. Zbl 1304.42001. <http://dx.doi.org/10.1007/978-1-4939-1194-3>.
- [22] L. GRAFAKOS, *Modern Fourier Analysis*, third ed., *Grad. Texts in Math.* **250**, Springer-Verlag, New York, 2014. MR 3243741. Zbl 1304.42002. <http://dx.doi.org/10.1007/978-1-4939-1230-8>.
- [23] P. JONES, Product formulas for measures and applications to analysis and geometry, talk given at the conference “Geometric and algebraic structures in mathematics”, Stony Brook University, May 2011. Available at <http://www.math.sunysb.edu/Videos/dennisfest/>.

- [24] B. KIRCHHEIM and J. KRISTENSEN, On rank one convex functions that are homogeneous of degree one, *Arch. Ration. Mech. Anal.* **221** (2016), 527–558. MR 3483901. Zbl 06587975. <http://dx.doi.org/10.1007/s00205-016-0967-1>.
- [25] J. KRISTENSEN and F. RINDLER, Characterization of generalized gradient Young measures generated by sequences in  $W^{1,1}$  and BV, *Arch. Ration. Mech. Anal.* **197** (2010), 539–598. MR 2660519. Zbl 1245.49060. <http://dx.doi.org/10.1007/s00205-009-0287-9>.
- [26] J. KRISTENSEN and F. RINDLER, Relaxation of signed integral functionals in BV, *Calc. Var. Partial Differential Equations* **37** (2010), 29–62. MR 2564396. Zbl 1189.49018. <http://dx.doi.org/10.1007/s00526-009-0250-5>.
- [27] J. KRISTENSEN and F. RINDLER, Piecewise affine approximations for functions of bounded variation, *Numer. Math.* **132** (2016), 329–346. MR 3447134. Zbl 06541934. <http://dx.doi.org/10.1007/s00211-015-0721-x>.
- [28] A. MASSACCESI and D. VITTONI, An elementary proof of the rank one theorem for BV functions, 2016. arXiv 1601.02903.
- [29] F. MURAT, Compacité par compensation, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **5** (1978), 489–507. MR 0506997. Zbl 0399.46022. Available at [http://www.numdam.org/item?id=ASNSP\\_1978\\_4\\_5\\_3\\_489\\_0](http://www.numdam.org/item?id=ASNSP_1978_4_5_3_489_0).
- [30] F. MURAT, Compacité par compensation. II, in *Proceedings of the International Meeting on Recent Methods in Nonlinear Analysis* (Rome, 1978), Pitagora, Bologna, 1979, pp. 245–256. MR 0533170. Zbl 0427.35008.
- [31] F. MURAT, Compacité par compensation: condition nécessaire et suffisante de continuité faible sous une hypothèse de rang constant, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **8** (1981), 69–102. MR 0616901. Zbl 0464.46034. Available at [http://www.numdam.org/item?id=ASNSP\\_1981\\_4\\_8\\_1\\_69\\_0](http://www.numdam.org/item?id=ASNSP_1981_4_8_1_69_0).
- [32] T. O’NEIL, A measure with a large set of tangent measures, *Proc. Amer. Math. Soc.* **123** (1995), 2217–2220. MR 1264826. Zbl 0827.28002. <http://dx.doi.org/10.2307/2160960>.
- [33] D. PREISS, Geometry of measures in  $\mathbf{R}^n$ : distribution, rectifiability, and densities, *Ann. of Math.* **125** (1987), 537–643. MR 0890162. Zbl 0627.28008. <http://dx.doi.org/10.2307/1971410>.
- [34] D. PREISS, Differentiability of Lipschitz functions on Banach spaces, *J. Funct. Anal.* **91** (1990), 312–345. MR 1058975. Zbl 0711.46036. [http://dx.doi.org/10.1016/0022-1236\(90\)90147-D](http://dx.doi.org/10.1016/0022-1236(90)90147-D).
- [35] D. PREISS and G. SPEIGHT, Differentiability of Lipschitz functions in Lebesgue null sets, *Invent. Math.* **199** (2015), 517–559. MR 3302120. Zbl 1317.26011. <http://dx.doi.org/10.1007/s00222-014-0520-5>.
- [36] F. RINDLER, Lower semicontinuity for integral functionals in the space of functions of bounded deformation via rigidity and Young measures, *Arch. Ration. Mech. Anal.* **202** (2011), 63–113. MR 2835863. Zbl 1258.49014. <http://dx.doi.org/10.1007/s00205-011-0408-0>.
- [37] F. RINDLER, Lower semicontinuity and Young measures in BV without Alberti’s rank-one theorem, *Adv. Calc. Var.* **5** (2012), 127–159. MR 2912698. Zbl 1239.49018. <http://dx.doi.org/10.1515/acv.2011.008>.

- [38] F. RINDLER, A local proof for the characterization of Young measures generated by sequences in BV, *J. Funct. Anal.* **266** (2014), 6335–6371. MR 3192455. Zbl 1305.49019. <http://dx.doi.org/10.1016/j.jfa.2014.03.010>.
- [39] F. RINDLER, Directional oscillations, concentrations, and compensated compactness via microlocal compactness forms, *Arch. Ration. Mech. Anal.* **215** (2015), 1–63. MR 3296143. Zbl 1327.35004. <http://dx.doi.org/10.1007/s00205-014-0783-4>.
- [40] A. SCHIOPPA, Poincaré inequalities for mutually singular measures, *Anal. Geom. Metr. Spaces* **3** (2015), 40–45. MR 3312276. Zbl 1310.26017. <http://dx.doi.org/10.1515/agms-2015-0003>.
- [41] E. M. STEIN, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton Math. Ser. **43**, Princeton Univ. Press, Princeton, NJ, 1993. MR 1232192. Zbl 0821.42001.
- [42] L. TARTAR, Compensated compactness and applications to partial differential equations, in *Nonlinear Analysis and Mechanics: Heriot-Watt Symposium, Vol. IV, Res. Notes in Math.* **39**, Pitman, Boston, 1979, pp. 136–212. MR 0584398. Zbl 0437.35004.
- [43] L. TARTAR, The compensated compactness method applied to systems of conservation laws, in *Systems of Nonlinear Partial Differential Equations* (Oxford, 1982), *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.* **111**, Reidel, Dordrecht, 1983, pp. 263–285. MR 0725524. Zbl 0536.35003.
- [44] R. TEMAM, *Problèmes Mathématiques en Plasticité*, *Math. Methods of Info. Sci.* **12**, Gauthier-Villars, Montrouge, 1983. MR 0711964. Zbl 0547.73026.
- [45] R. TEMAM and G. STRANG, Functions of bounded deformation, *Arch. Rational Mech. Anal.* **75** (1980/81), 7–21. MR 0592100. Zbl 0472.73031. <http://dx.doi.org/10.1007/BF00284617>.

(Received: January 25, 2016)

SCUOLA INTERNAZIONALE SUPERIORE DI STUDI AVANZATI, TRIESTE, ITALY  
*E-mail*: guido.dephilippis@sissa.it

MATHEMATICS INSTITUTE, UNIVERSITY OF WARWICK, COVENTRY, UNITED KINGDOM  
*E-mail*: F.Rindler@warwick.ac.uk