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Gromov-Hausdorff limits of Kähler manifolds and the finite generation conjecture

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Abstract

We study the uniformization conjecture of Yau by using the Gromov-Hausdorff convergence. As a consequence, we confirm Yau's finite generation conjecture. More precisely, on a complete noncompact Kähler manifold with nonnegative bisectional curvature, the ring of polynomial growth holomorphic functions is finitely generated. During the course of the proof, we prove if M^n is a complete noncompact Kähler manifold with nonnegative bisectional curvature and maximal volume growth, then M is biholomorphic to an affine algebraic variety. We also confirm a conjecture of Ni on the existence of polynomial growth holomorphic functions on Kähler manifolds with nonnegative bisectional curvature.

1. Introduction

In [34], Yau proposed to study the uniformization of complete Kähler manifolds with nonnegative curvature. In particular, one wishes to determine whether or not a complete noncompact Kähler manifold with positive bisectional curvature is biholomorphic to a complex Euclidean space. For this sake, Yau further asked in [34] (see also page 117 in [33]) whether or not the ring of polynomial growth holomorphic functions is finitely generated and whether or not dimension of the spaces of holomorphic functions of polynomial growth is bounded from above by the dimension of the corresponding spaces of polynomials on $\mathbb{C}^n$. Let us summarize Yau's questions in the three conjectures below:

Conjecture 1. *Let M^n be a complete noncompact Kähler manifold with positive bisectional curvature. Then M is biholomorphic to $\mathbb{C}^n$.*

Conjecture 2. *Let M^n be a complete noncompact Kähler manifold with nonnegative bisectional curvature. Then the ring $\mathcal{O}_P(M)$ is finitely generated.*

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Conjecture 3. *Let M^n be a complete noncompact Kähler manifold with nonnegative bisectional curvature. Then given any $d > 0$, $\dim(\mathcal{O}_d(M)) \leq \dim(\mathcal{O}_d(\mathbb{C}^n))$.*

On a complete Kähler manifold M, we say a holomorphic function f has polynomial growth, denoted by $f \in \mathcal{O}_d(M)$ if there exists some $C > 0$ such that $|f(x)| \leq C(1 + d(x, x_0))^d$ for all $x \in M$. Here x_0 is a fixed point on M. Let $\mathcal{O}_P(M) = \cup_{d>0}\mathcal{O}_d(M)$.

Conjecture 1 is open so far. However, there has been much important progress due to various authors. In earlier works, Mok-Siu-Yau [24] and Mok [23] considered embedding by using holomorphic functions of polynomial growth. Later, with the Kähler-Ricci flow, results were improved significantly. See, for example, [30], [31], [9], [27], [1], [8], [17].

Conjecture 3 was confirmed by Ni [26] with the assumption that M has maximal volume growth. Later, by using Ni's method, Chen-Fu-Le-Zhu [7] removed the extra condition. See also [22] for a different proof. The key of Ni's method is a monotonicity formula for heat flow on Kähler manifold with nonnegative bisectional curvature.

Despite great progress of Conjectures 1 and 3, not much was known about Conjecture 2. In [23], Mok proved the following:

Theorem 1.1 (Mok). *Let M^n be a complete noncompact Kähler manifold with positive bisectional curvature such that for some fixed point $p \in M$,*

- *scalar curvature $\leq \frac{C_0}{d(p,x)^2}$ for some $C_0 > 0$;*
- $\mathrm{Vol}(B(p, r)) \geq C_1 r^{2n}$ *for some $C_1 > 0$.*

Then M^n is biholomorphic to an affine algebraic variety.

In Mok's proof, the biholomorphism was given by holomorphic functions of polynomial growth. Therefore, $\mathcal{O}_P(M)$ is finitely generated. In the general case, it was proved by Ni [26] that the transcendental dimension of $\mathcal{O}_P(M)$ over $\mathbb{C}$ is at most n. However, this does not imply the finite generation of $\mathcal{O}_P(M)$. The main result in this paper is the confirmation of Conjecture 2 in the general case:

Theorem 1.2. *Let M^n be a complete noncompact Kähler manifold with nonnegative bisectional curvature. Then the ring $\mathcal{O}_P(M)$ is finitely generated.*

During the course of the proof, we obtain a partial result for Conjecture 1:

Theorem 1.3. *Let M^n be a complete noncompact Kähler manifold with nonnegative bisectional curvature. Assume M is of maximal volume growth. Then M is biholomorphic to an affine algebraic variety.*

Here maximal volume growth means $\mathrm{Vol}(B(p, r)) \geq Cr^{2n}$ for some $C > 0$. This seems to be the first uniformization type result without assuming the curvature upper bound.

If one wishes to prove Conjecture 1 by considering $\mathcal{O}_P(M)$, it is important to know when $\mathcal{O}_P(M) \neq \mathbb{C}$. In [26], Ni proposed the following interesting conjecture:

CONJECTURE 4. *Let M^n be a complete noncompact Kähler manifold with nonnegative bisectional curvature. Assume M has positive bisectional curvature at one point p. Then the following three conditions are equivalent*:

(1) $\mathcal{O}_P(M) \neq \mathbb{C}$;
(2) *M has maximal volume growth*;
(3) *there exists a constant C independent of r so that $⨍_{B(p,r)} S \leq \frac{C}{r^2}$. Here S is the scalar curvature and $⨍$ means the average.*

In the complex one-dimensional case, the conjecture is known to hold, e.g., [19]. For higher dimensions, Ni proved (1) implies (3) in [26]. The proof used the heat flow method. Then in [29], Ni and Tam proved that (3) also implies (1). Their proof employs the Poincaré-Lelong equation and the heat flow method. Thus, it remains to prove (1) and (2) are equivalent. Under some extra conditions, Ni [27] and Ni-Tam [28] were able to prove the equivalence of (1) and (2). In [21], the author proved that (1) implies (2). In fact, the condition that M has positive bisectional curvature at one point could be relaxed to that the universal cover of M is not a product of two Kähler manifolds.

In this paper, we prove that (2) also implies (1). Thus Conjecture 4 is solved in full generality. More precisely, we prove

THEOREM 1.4. *Let (M^n, g) be a complete Kähler manifold with nonnegative bisectional curvature and maximal volume growth. Then there exists a nonconstant holomorphic function of polynomial growth on M.*

The strategy of the proofs in this paper is very different from earlier works. Here we make use of several different techniques:

- the Gromov-Hausdorff convergence theory developed by Cheeger-Colding [2], [3], [4], [5], Cheeger-Colding-Tian [6];
- the heat flow method by Ni [26] and Ni-Tam [28], [29];
- the Hörmander L^2-estimate of $\overline{\partial}$ [16], [11];
- the three circle theorem [22].

We point out that recently, the Gromov-Hausdorff convergence theory was shown to be a very powerful tool to study Kähler manifolds; see, e.g., [12], [32].

The first key point is to prove Theorem 1.4. By Hörmander's L^2-technique, to produce holomorphic functions of polynomial growth, it suffices to construct strictly plurisubharmonic function of logarithmic growth. However, it is not obvious how to construct such functions by only assuming the maximal volume growth condition. In [24], [23], Mok-Siu-Yau and Mok considered the Poincaré-Lelong equation $\sqrt{-1}\partial\overline{\partial}u = \mathrm{Ric}$. When the curvature has pointwise quadratic decay, they were able to prove the existence of a solution with logarithmic growth. Later, Ni and Tam [28], [29] were able to relax the condition to that the curvature has average quadratic decay. Then it suffices to prove that maximal volume growth implies the average curvature decay.

We prove Theorem 1.4 by a different strategy. We first blow down the manifold. Then by using the Cheeger-Colding theory, heat flow technique and Hörmander L^2-theory, we construct holomorphic functions with controlled growth in a sequence of exhaustion domains on M. Then the three circle theorem ensures that we can take subsequence to obtain a nonconstant holomorphic function with polynomial growth.

Once Theorem 1.4 is proved, Hörmander's L^2-technique produces a lot of holomorphic functions of polynomial growth. It turns out $\mathcal{O}_P(M)$ separates points and tangent spaces on M. However, since the manifold is not compact, it does not follow directly that M is affine algebraic. To overcome this difficulty, we prove in Theorem 6.1 that the map given by $\mathcal{O}_d(M)$ is proper. We will use induction on the dimension of the splitting factor of a tangent cone.

Once we have proved the properness of the holomorphic map, it is straightforward to prove M is affine algebraic by using techniques from complex analytic geometry. Here the argument resembles some part in [12]. Then we conclude Conjecture 2 when the manifold has maximal volume growth. For the general case, we apply the main result in [21]. It suffices to handle the case when the universal cover of the manifold splits. Then we need to consider group actions. The final result follows from an algebraic result of Nagata [25].

This paper is organized as follows. In Section 2, we collect some preliminary results necessary for this paper. In Section 3, we prove a result which controls the size of a holomorphic chart when the manifold is Gromov-Hausdorff close to a Euclidean ball. As the first application, in Section 4 we prove a gap theorem for the complex structure of $\mathbb{C}^n$. Section 5 deals with the proof of Theorem 1.4. The proof of Theorem 6.1 is contained in Section 6. Finally, the proof of Theorem 1.2 is given Section 7.

There are two appendices. In Appendix A, we present a result of Ni-Tam in [28] which was not stated explicitly. (Here we are not claiming any credit.) In Appendix B, we introduce some results of Nagata [25] to conclude the proof of the main theorem.

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2. **Preliminary results**

First recall some convergence results for manifolds with Ricci curvature lower bound. Let (M_i^n, y_i, ρ_i) be a sequence of pointed complete Riemannian manifolds, where $y_i \in M_i^n$ and ρ_i is the metric on M_i^n. By Gromov's compactness theorem if (M_i^n, y_i, ρ_i) have a uniform lower bound of the Ricci curvature, then a subsequence converges to some $(M_\infty, y_\infty, \rho_\infty)$ in the Gromov-Hausdorff topology. See [14] for the definition and basic properties of Gromov-Hausdorff convergence.

Definition 2.1. Let $K_i \subset M_i^n \to K_\infty \subset M_\infty$ in the Gromov-Hausdorff topology. Assume that $\{f_i\}_{i=1}^\infty$ are functions on M_i^n and f_∞ is a function on M_∞. Assume that Φ_i are ε_i-Gromov-Hausdorff approximations and $\lim_{i\to\infty} \varepsilon_i = 0$. If $f_i \circ \Phi_i$ converges to f_∞ uniformly, we say $f_i \to f_\infty$ uniformly over $K_i \to K_\infty$.

In many applications, f_i are equicontinuous. The Arzela-Ascoli theorem applies to the case when the spaces are different. When

$$(M_i^n, y_i, \rho_i) \to (M_\infty, y_\infty, \rho_\infty)$$

in the Gromov-Hausdorff topology, any bounded, equicontinuous sequence of functions f_i has a subsequence converging uniformly to some f_∞ on M_∞.

Let the complete pointed metric space (M_∞^m, y) be the Gromov-Hausdorff limit of a sequence of connected pointed Riemannian manifolds, $\{(M_i^n, p_i)\}$, with $\mathrm{Ric}(M_i) \geq 0$. Here M_∞^m has Hausdorff dimension m with $m \leq n$. A tangent cone at $y \in M_\infty^m$ is a complete pointed Gromov-Hausdorff limit $((M_\infty)_y, d_\infty, y_\infty)$ of $\{(M_\infty, r_i^{-1}d, y)\}$, where d, d_∞ are the metrics of $M_\infty, (M_\infty)_y$ respectively, and $\{r_i\}$ is a positive sequence converging to 0.

Definition 2.2. A point $y \in M_\infty$ is called regular, if there exists some k so that every tangent cone at y is isometric to $\mathbb{R}^k$. A point is called singular, if it is not regular.

In [3], the following theorem was proved:

THEOREM 2.1. *Regular points are dense in the Gromov-Hausdorff limits of manifolds with Ricci curvature lower bound.*

Below is a result of Ni-Tam [28] on the heat flow on Kähler manifolds:

THEOREM 2.2. *Let M^n be a complete noncompact Kähler manifold with nonnegative bisectional curvature. Let u be a smooth function on M with compact support. Let*

$$v(x,t) = \int_M H(x,y,t)u(y)dy.$$

Here $H(x, y, t)$ *is the heat kernel of* M. *Let* $\eta(x, t)_{\alpha\overline{\beta}} = v_{\alpha\overline{\beta}}$ *and* $\lambda(x)$ *be the minimum eigenvalue for* $\eta(x, 0)$. *Let*

$$\lambda(x, t) = \int_M H(x, y, t)\lambda(y)dy.$$

Then $\eta(x, t) - \lambda(x, t)g_{\alpha\overline{\beta}}$ *is a nonnegative* $(1, 1)$ *tensor for* $t \in [0, T]$ *for* $T > 0$.

A detailed proof of this theorem is presented in Appendix A.

Recall the Hörmander L^2-theory:

THEOREM 2.3. *Let* (X, ω) *be a connected but not necessarily complete Kähler manifold with* $\mathrm{Ric} \geq 0$. *Assume* X *is Stein. Let* φ *be a* C^∞ *function on* X *with* $\sqrt{-1}\partial\overline{\partial}\varphi \geq c\omega$ *for some positive function* c *on* X. *Let* g *be a smooth* $(0, 1)$ *form satisfying* $\overline{\partial}g = 0$ *and* $\int_X \frac{|g|^2}{c} e^{-\varphi}\omega^n < +\infty$. *Then there exists a smooth function* f *on* X *with* $\overline{\partial}f = g$ *and* $\int_X |f|^2 e^{-\varphi}\omega^n \leq \int_X \frac{|g|^2}{c} e^{-\varphi}\omega^n$.

The proof can be found in [11, pp. 38–39]. Also compare Lemma 4.4.1 in [16]. Note that the theorem also applies to singular metrics with positive curvature in the current sense.

Recall the three circle theorem in [22]:

THEOREM 2.4. *Let* M *be a complete noncompact Kähler manifold with nonnegative holomorphic sectional curvature,* $p \in M$. *Let* f *be a holomorphic function on* M. *Let* $M(r) = \sup\limits_{B(p,r)} |f(x)|$. *Then* $\log M(r)$ *is a convex function of* $\log r$. *Therefore, given any* $k > 1$, $\frac{M(kr)}{M(r)}$ *is monotonic increasing.*

Theorem 2.4 has the following consequences:

COROLLARY 2.1. *Given the same condition as in Theorem* 2.4, *if* $f \in \mathcal{O}_d(M)$, *then* $\frac{M(r)}{r^d}$ *is nonincreasing.*

COROLLARY 2.2. *Given the same condition as in Theorem* 2.4, *if* $f(p) = 0$, *then* $\frac{M(r)}{r}$ *is nondecreasing.*

Remark 2.1. The three circle theorem is still true for holomorphic sections on nonpositive bundles. See page 17 of [22] for a proof.

Finally, we need the multiplicity estimate by by Ni [26] (see also [7]):

THEOREM 2.5. *Let* M^n *be a complete noncompact Kähler manifold with nonnegative bisectional curvature. Then* $\dim(\mathcal{O}_d(M)) \leq \dim(\mathcal{O}_d(\mathbb{C}^n))$.

Note that this result also follows from Corollary 2.1.

In this paper, we will denote by $\Phi(u_1, \ldots, u_k|\ldots)$ any nonnegative functions depending on $u_1, \ldots, u_k$ and some additional parameters such that when these parameters are fixed,

$$\lim_{u_k \to 0} \cdots \lim_{u_1 \to 0} \Phi(u_1, \ldots, u_k|\ldots) = 0.$$

Let $C(n), C(n,v)$ be large positive constants depending only on n or n, v, and let $c(n), c(n,v)$ be small positive constants depending only on n or n, v. The values might change from line to line.

3. **Construct holomorphic charts with uniform size**

In this section, we introduce the following proposition, which is crucial for the construction of holomorphic functions.

PROPOSITION 3.1. *Let M^n be a complete Kähler manifold with nonnegative bisectional curvature, $x \in M$. There exist $\varepsilon(n) > 0, \delta = \delta(n) > 0$ so that the following holds: For $\varepsilon < \varepsilon(n)$, if $d_{GH}(B(x, \frac{1}{\varepsilon}r), B_{\mathbb{C}^n}(0, \frac{1}{\varepsilon}r)) < \varepsilon r$, there exists a holomorphic chart $(w_1, \ldots, w_n)$ containing $B(x, \delta r)$ so that*

- $w_s(x) = 0 (1 \leq s \leq n)$;
- $\left|\sum_{s=1}^{n} |w_s|^2(y) - r^2(y)\right| \leq \Phi(\varepsilon|n) r^2$ *in* $B(x, \delta r)$ *— here* $r(y) = d(x, y)$;
- $|dw_s(y)| \leq C(n)$ *in* $B(x, \delta r)$.

Proof. By scaling, we may assume $r \gg 1$, to be determined. Set $R = \frac{r}{100} \gg 1$. According to the assumptions and the Cheeger-Colding theory [2] (see also equation (1.23) in [4]), there exist real harmonic functions $b_1, \ldots, b_{2n}$ on $B(x, 4r)$ so that

$$\fint_{B(x,2r)} \sum_j |\nabla(\nabla b_j)|^2 + \sum_{j,l} |\langle \nabla b_j, \nabla b_l \rangle - \delta_{jl}|^2 \leq \Phi(\varepsilon|n, r) \tag{3.1}$$

and

$$b_j(x) = 0 (1 \leq j \leq 2n), \qquad |\nabla b_j| \leq C(n) \tag{3.2}$$

in $B(x, 2r)$. Furthermore, the map $F(y) = (b_1(y), \ldots, b_{2n}(y))$ is a $\Phi(\varepsilon|n)r$ Gromov-Hausdorff approximation from $B(x, 2r)$ to $B_{\mathbb{R}^{2n}}(0, 2r)$. According to the argument above Lemma 9.14 in [6] (see also (20) in [21]), after a suitable orthogonal transformation, we may assume

$$\fint_{B(x,r)} |J\nabla b_{2s-1} - \nabla b_{2s}|^2 \leq \Phi(\varepsilon|n, r) \tag{3.3}$$

for $1 \leq s \leq n$. Set $w'_s = b_{2s-1} + \sqrt{-1} b_{2s}$. Then

$$\fint_{B(x,r)} |\overline{\partial} w'_s|^2 \leq \Phi(\varepsilon|n, r). \tag{3.4}$$

The idea is to perturb w'_s so that they become a holomorphic chart. We would like to apply the Hörmander L^2-estimate. First, we construct the weight function. Consider the function

$$h(y)=\sum_{j=1}^{2n}b_j^2(y).$$

Then in $B(x,r)$,

$$|h(y)-r^2(y)|\leq\Phi(\varepsilon|n)r^2. \tag{3.5}$$

By (3.2),

$$|\nabla h(y)|\leq C(n)r(y) \tag{3.6}$$

in $B(x,r)$. The real Hessian of h satisfies

$$\int_{B(x,5R)}\sum_{u,v}|h_{uv}(y)-2g_{uv}|^2\leq\Phi(\varepsilon|n,R). \tag{3.7}$$

Now consider a smooth function φ: $\mathbb{R}^+\to\mathbb{R}^+$ with $\varphi(t)=t$ for $0\leq t\leq 1$, $\varphi(t)=0$ for $t\geq 2$, and $|\varphi|,|\varphi'|,|\varphi''|\leq C(n)$. Let $H(x,y,t)$ be the heat kernel on M, and set

$$u(y)=5R^2\varphi\left(\frac{h(y)}{5R^2}\right), u_t(z)=\int_M H(z,y,t)u(y)dy. \tag{3.8}$$

Claim 3.1. *$u_1(z)$ satisfies that $(u_1)_{\alpha\overline{\beta}}(z)\geq c(n)g_{\alpha\overline{\beta}}>0$ in $B\big(x,\frac{R}{10}\big)$.*

Proof. Let $\lambda(y)$ be the lowest eigenvalue of the complex Hessian $u_{\alpha\overline{\beta}}$. By (3.7),

$$⨍_{B(x,5R)}|h_{\alpha\overline{\beta}}-2g_{\alpha\overline{\beta}}|^2\leq\Phi(\varepsilon|n,R).$$

Then there exists $E\subset B(x,5R)$ with

$$\mathrm{vol}(B(x,5R)\backslash E)\leq\Phi(\varepsilon|n,R),\qquad h_{\alpha\overline{\beta}}\geq\frac{1}{2}g_{\alpha\overline{\beta}} \tag{3.9}$$

on E. By (3.5), we may assume $h(y)\leq 5R^2$ in $B(x,2R)$. Then $u=h$ in $B(x,2R)$. We have

$$\begin{aligned}\left(\int_{B(x,2R)\backslash E}|\lambda^2(y)|dy\right)^{\frac{1}{2}}&\leq\left(\int_{B(x,4R)\backslash E}\sum_{\alpha,\beta}|h_{\alpha\overline{\beta}}|^2\right)^{\frac{1}{2}}\\&\leq 4\left(\int_{B(x,4R)\backslash E}\sum_{\alpha,\beta}|h_{\alpha\overline{\beta}}-2g_{\alpha\overline{\beta}}|^2dy\right)^{\frac{1}{2}}\\&\quad+4\left(\int_{B(x,4R)\backslash E}\sum_{\alpha,\beta}|2g_{\alpha\overline{\beta}}|^2dy\right)^{\frac{1}{2}}\\&\leq\Phi(\varepsilon|n,R)\end{aligned} \tag{3.10}$$

and

$$
\begin{aligned}
|\lambda| &\le |u_{\alpha\overline{\beta}}| \\
&= \left|\varphi' h_{\alpha\overline{\beta}} + \frac{\varphi''}{5R^2} h_\alpha h_{\overline{\beta}}\right| \\
&\le |\varphi'(h_{\alpha\overline{\beta}} - 2g_{\alpha\overline{\beta}})| + \left|2\varphi' g_{\alpha\overline{\beta}} + \frac{\varphi''}{5R^2} h_\alpha h_{\overline{\beta}}\right| \\
&\le C(n)(|h_{\alpha\overline{\beta}} - 2g_{\alpha\overline{\beta}}| + 1).
\end{aligned} \tag{3.11}
$$

Therefore,

$$
\int_{B(x,5R)} |\lambda(y)| dy \le C(n) R^{2n}. \tag{3.12}
$$

Let $\lambda(z,1) = \int H(z,y,1)\lambda(y)dy$. Note that by definition (3.8), u is supported in $B(x,4R)$. By (3.9), $\lambda \ge \frac{1}{2}$ in E. For $z \in B\left(x, \frac{R}{10}\right)$,

$$
\begin{aligned}
\int H(z,y,1)\lambda(y)dy &= \int_{B(x,4R)} H(z,y,1)\lambda(y)dy \\
&\ge \int_{B(x,2R)\setminus E} H(z,y,1)\lambda(y)dy \\
&\quad + \int_{B(x,4R)\setminus B(x,2R)} H(z,y,1)\lambda(y)dy \\
&+ \int_{E\cap B(z,1)} H(z,y,1)\lambda(y)dy.
\end{aligned} \tag{3.13}
$$

By heat kernel estimates of Li-Yau [20], $H(z,y,1) \ge c(n) > 0$ for $y \in B(z,1)$. Also, with volume comparison, we find $H(z,y,1) \le C(n)$ for $y, z \in B(x,4R)$.As a consequence,

$$
\begin{aligned}
&\int_{B(x,2R)\setminus E} |H(z,y,1)\lambda(y)| dy \\
&\qquad \le C(n) \int_{B(x,2R)\setminus E} |\lambda(y)| dy \\
&\qquad \le C(n) \left(\int_{B(x,2R)\setminus E} |\lambda^2(y)| dy \right)^{\frac{1}{2}} (\mathrm{vol}(B(x,2R)\setminus E))^{\frac{1}{2}} \\
&\qquad \le \Phi(\varepsilon | n, R),
\end{aligned} \tag{3.14}
$$

$$
\int_{E\cap B(z,1)} H(z,y,1)\lambda(y)dy \ge \frac{1}{2} \int_{E\cap B(z,1)} H(z,y,1)dy \ge c(n) > 0. \tag{3.15}
$$

Note that $d(y,z) \ge R$ for $y \in B(x,4R)\setminus B(x,2R)$. The heat kernel estimate says $H(y,z,1) \le C(n)e^{-\frac{R^2}{5}}$. Therefore, by (3.12),

$$
\int_{B(x,4R)\setminus B(x,2R)} |H(z,y,1)\lambda(y)| dy \le C(n) e^{-\frac{R^2}{5}} R^{2n} < \Phi\left(\frac{1}{R}\right). \tag{3.16}
$$

Putting (3.14), (3.15), (3.16) in (3.13), we find

$$\lambda(z,1) = \int H(z,y,1)\lambda(y)dy \geq c(n) - \Phi\left(\frac{1}{R}|n) - \Phi(\varepsilon|n,R\right) \tag{3.17}$$

for $z \in B\big(x, \frac{R}{10}\big)$. We first let R be large and then let ε be very small. Then $\lambda(z,1) > c(n)$. We conclude the proof of the claim from Theorem 2.2. □

Recall that u_t is defined in (3.8). We claim that there exists $\varepsilon_0 = \varepsilon_0(n) > 0$ so that for large R,

$$\min_{y\in\partial B(x,\frac{R}{20})} u_1(y) > 4 \sup_{y\in B(x,\varepsilon_0\frac{R}{20})} u_1(y). \tag{3.18}$$

This is a simple exercise by using the heat kernel estimate. One can also apply Proposition A.1 to conclude the proof. From now on, we freeze the value of R. That is to say, $R = R(n) > 0$ satisfies Claim 3.1, (3.18) and $\frac{R}{20}\varepsilon_0 > 100$.

Let Ω be the connected component of

$$\left\{y \in B\left(x, \frac{R}{20}\right) \middle| u_1(y) < 2 \sup_{y\in B(x,\varepsilon_0\frac{R}{20})} u_1(y)\right\}$$

containing $B(x, \varepsilon_0\frac{R}{20})$. Then Ω is relatively compact in $B(x, \frac{R}{20})$ and Ω is a Stein manifold by Claim 3.1.

Now we apply Theorem 2.3 to Ω, with the Kähler metric induced from M. Take smooth $(0,1)$ forms $g_s = \overline{\partial} w_s'$ defined in (3.4), and take the weight function $\psi = u_1$. We find smooth functions f_s in Ω with $\overline{\partial} f_s = g_s$ and

$$\int_\Omega |f_s|^2 e^{-\psi}\omega^n \leq \int_\Omega \frac{|g_s|^2}{c} e^{-\psi}\omega^n \leq \frac{\int_\Omega |\overline{\partial} w_s'|^2\omega^n}{c(n)} \leq \Phi(\varepsilon|n). \tag{3.19}$$

Here we used the fact that $r = 100R = 100R(n)$. By Proposition A.1, we find $\psi = u_1 \leq C(R,n) = C(n)$ in $B(x,R)$. Therefore,

$$\int_{B(x,10)} |f_s|^2\omega^n \leq \int_\Omega |f_s|^2\omega^n \leq \Phi(\varepsilon|n). \tag{3.20}$$

Note that $w_s = w_s' - f_s$ is holomorphic, as $\overline{\partial} w_s = \overline{\partial} w_s' - g_s = 0$. Since w_s' is harmonic (complex), f_s is also harmonic. By the mean value inequality [18] and Cheng-Yau's gradient estimate [10], we find that in $B(x,5)$,

$$|f_s| \leq \Phi(\varepsilon|n), \qquad |\nabla f_s| \leq \Phi(\varepsilon|n). \tag{3.21}$$

Therefore, equation (3.1) implies

$$\int_{B(x,4)} |(w_s)_i\overline{(w_t)_j} g^{i\overline{j}} - 2\delta_{st}| \leq \Phi(\varepsilon|n). \tag{3.22}$$

Claim 3.2. *$w_s (s = 1, \ldots, n)$ is a holomorphic chart in $B(x,1)$.*

Proof. Recall that $(b_1, \dots, b_{2n})$ is a $\Phi(\varepsilon|n)$ Gromov-Hausdorff approximation to the image in $\mathbb{R}^{2n}$. According to (3.21), on $B(x, 1)$, $w = (w_1, \dots, w_n)$ is also a $\Phi(\varepsilon|n)$ Gromov Hausdorff approximation to $B_{\mathbb{C}^n}(0, 1)$. Therefore, $w^{-1}(\overline{B_{\mathbb{C}^n}(0,1)})$ is compact in $B(x, 1 + \Phi(\varepsilon|n))$. First we prove that the degree d of the map w is 1. By (3.22) and the fact that holomorphic maps preserves the orientation, $d \geq 1$. We also have

(3.23)
$$\begin{aligned} d \cdot \mathrm{vol}(B_{\mathbb{C}^n}(0,1)) &= \frac{1}{(-2\sqrt{-1})^n} \int_{w^{-1}(B(0,1))} dw_1 \wedge d\overline{w_1} \wedge \cdots \wedge dw_n \wedge d\overline{w_n} \\ &\leq (1 + \Phi(\varepsilon|n)) \, \mathrm{Vol}(B(x, (1 + \Phi(\varepsilon|n))) + \Phi(\varepsilon|n) \end{aligned}$$

by (3.21) and (3.22). This means that if ε is sufficiently small, then $d = 1$. That is to say, $(w_1, \dots, w_n)$ is generically one to one in $B(x, 1)$. Moreover, $(w_1, \dots, w_n)$ must be a finite map: the preimage of a point must be a subvariety which is compact in the Stein manifold Ω, thus finitely many points. According to Proposition 14.7 on page 87 of [13], this is an isomorphism. □

We can make a small perturbation so that $w_s(x) = 0$ for $1 \leq s \leq n$. This completes the proof of Proposition 3.1. □

4. A gap theorem for the complex structure of $\mathbb{C}^n$

As the first application of Proposition 3.1, we prove a gap theorem for the complex structure of $\mathbb{C}^n$. The conditions are rather restrictive. However, we shall expand some of the ideas in later sections.

THEOREM 4.1. *Let M^n be a complete noncompact Kähler manifold with nonnegative bisectional curvature and $p \in M$. There exists $\varepsilon(n) > 0$ so that if $\varepsilon < \varepsilon(n)$ and*

$$\frac{\mathrm{vol}(B(p, r))}{r^{2n}} \geq \omega_{2n} - \varepsilon \tag{4.1}$$

for all $r > 0$, then M is biholomorphic to $\mathbb{C}^n$. Here ω_{2n} is the volume of the unit ball in $\mathbb{C}^n$. Furthermore, the ring $\mathcal{O}_P(M)$ is finitely generated. In fact, it is generated by n functions which form a coordinate in $\mathbb{C}^n$.

Proof. Consider the blow-down sequence $(M_i, p_i, g_i) = \left(M, p, \frac{1}{s_i^2} g\right)$ for $s_i \to \infty$. According to Proposition 3.1 and the Cheeger-Colding theory [2], if ε is sufficiently small, there exists a holomorphic chart $(w_1^i, \dots, w_n^i)$ on $B(p_i, 1)$. Moreover, the map $(w_1^i, \dots, w_n^i)$ is a $\Phi(\varepsilon|n)$ Gromov-Hausdorff approximation to $B_{\mathbb{C}^n}(0, 1)$. We may assume

$$w_s^i(p_i) = 0 \tag{4.2}$$

for $s = 1, \dots, n$. We can also regard w_s^i as holomorphic functions on $B(p, s_i) \subset M$. For each i, we can find a new basis v_s^i for $\operatorname{span}\{w_s^i\}$ so that

$$\int_{B(p,1)} v_s^i \overline{v_t^i} = \delta_{st}. \tag{4.3}$$

Set

$$M_s^i(r) = \sup_{x \in B(p,r)} |v_s^i(x)|. \tag{4.4}$$

Claim 4.1. $\frac{M_s^i(s_i)}{M_s^i(\frac{1}{2}s_i)} \leq 2 + \Phi(\varepsilon|n)$ *for* $1 \leq s \leq n$.

Proof. It suffices to prove this for $s = 1$. Let $v_1^i = \sum_{j=1}^n c_j^i w_j^i$. Without loss of generality, assume $|c_1^i| = \max_{1 \leq j \leq n} |c_j^i| > 0$. Then $\frac{v_1^i}{c_1^i} = w_1^i + \sum_{j=2}^n \alpha_{ij} w_j^i$ and $|\alpha_{ij}| \leq 1$. Since on M_i, $(w_1^i, \dots, w_n^i)$ is a $\Phi(\varepsilon|n)$ Gromov-Hausdorff approximation to $B_{\mathbb{C}^n}(0,1)$, we find

$$\begin{aligned} \frac{M_s^i(s_i)}{M_s^i(\frac{1}{2}s_i)} &= \frac{\sup_{x \in B(p,s_i)} \left| w_1^i(x) + \sum_{j=2}^n \alpha_{ij} w_j^i(x) \right|}{\sup_{x \in B(p,\frac{s_i}{2})} \left| w_1^i(x) + \sum_{j=2}^n \alpha_{ij} w_j^i(x) \right|} \\ &= \frac{\sup_{x \in B(p_i,1)} \left| w_1^i(x) + \sum_{j=2}^n \alpha_{ij} w_j^i(x) \right|}{\sup_{x \in B(p_i,\frac{1}{2})} \left| w_1^i(x) + \sum_{j=2}^n \alpha_{ij} w_j^i(x) \right|} \\ &\leq 2 + \Phi(\varepsilon|n). \end{aligned} \tag{4.5}$$

This concludes the proof. □

According to the three circle Theorem 2.4, $\frac{M_s^i(2r)}{M_s^i(r)}$ is monotonic increasing for $0 < r < \frac{1}{2}s_i$. Then Claim 4.1 implies

$$\frac{M_s^i(2r)}{M_s^i(r)} \leq 2 + \Phi(\varepsilon|n) \tag{4.6}$$

for $0 < r < \frac{1}{2}s_i$. From (4.3), we find $M_s^i(\frac{1}{2}) \leq C(n)$. Equation (4.6) implies

$$M_s^i(r) \leq C(n)(r^\alpha + 1) \tag{4.7}$$

for $\alpha = 1 + \Phi(\varepsilon|n)$. As $s_i \to \infty$, by taking subsequence, we can assume $v_s^i \to v_s$ uniformly on each compact set of M. Set

$$M_s(r) = \sup_{x \in B(p,r)} |v_s(x)|. \tag{4.8}$$

Then

$$M_s(r) \leq C(n)(r^\alpha + 1) \tag{4.9}$$

for $\alpha = 1 + \Phi(\varepsilon|n)$ and $r \geq 0$. We may assume $v_s \in \mathcal{O}_{\frac{3}{2}}(M)$. Note that v_s also satisfies

$$v_s(p) = 0 (1 \leq s \leq n), \quad \int_{B(p,1)} v_s \overline{v_t} = \delta_{st}. \tag{4.10}$$

Our goal is to prove that $(v_1, \dots, v_n)$ is a biholomorphism from M to $\mathbb{C}^n$.

Claim 4.2. *Let ε in* (4.1) *be sufficiently small* (*depending only on n*). *If we rescale each v_s so that $\sup_{B(p,1)} |v_s| = 1$, then in $B(p,1)$, $(v_1, \dots, v_n)$ is a $\frac{1}{100n}$-Gromov-Hausdorff approximation to $B_{\mathbb{C}^n}(0,1)$.*

Proof. We argue by contradiction. Assume there exists a positive sequence $\varepsilon_a \to 0 (a \in \mathbb{N})$, and let (M'_a, q_a) be a sequence of n-dimensional complete noncompact Kähler manifolds with nonnegative bisectional curvature and

$$\frac{\mathrm{vol}(B(q_a, r))}{r^{2n}} \geq \omega_{2n} - \varepsilon_a \tag{4.11}$$

for all $r > 0$. Assume there exist holomorphic functions $u_s^a (s = 1, \dots, n)$ on M'_a so that

$$u_s^a(q_a) = 0, \quad u_s^a \in \mathcal{O}_{\frac{3}{2}}(M'_a), \quad \int_{B(q_a,1)} u_s^a \overline{u_t^a} = c_{st}^a \delta_{st}, \quad \sup_{B(q_a,1)} |u_s^a| = 1. \tag{4.12}$$

Here c_{st}^a are constants. Assume in $B(q_a, 1)$ that $(u_1^a, \dots, u_n^a)$ is *not* a $\frac{1}{100n}$-Gromov-Hausdorff approximation to $B_{\mathbb{C}^n}(0,1)$. According to Cheeger-Colding theory [2] and (4.11), (M'_a, q_a) converges to $(\mathbb{R}^{2n}, 0)$ in the pointed Gromov-Hausdorff sense. By the three circle theorem and (4.12), we have a uniform bound for u_s^a in $B(q_a, r)$ for any $r > 0$. Let $a \to \infty$. Then there is a subsequence so that $u_s^a \to u_s$ uniformly on each compact set. Moreover, by Remark 9.3 of [6] (see also (21) in [21]), there is a natural linear complex structure on $\mathbb{R}^{2n}$. Thus we can identify the limit space with $\mathbb{C}^n$. By Lemma 4 in [21], the limits of holomorphic functions are still holomorphic. Moreover, $\{u_s\}$ satisfy (4.12), according to the three circle theorem. Thus u_s are all linear functions which form a standard complex coordinate in $\mathbb{C}^n$. Therefore, $(u_1, \dots, u_n)$ is an isometry from $B_{\mathbb{C}^n}(0,1)$ to $B_{\mathbb{C}^n}(0,1)$. This contradicts the assumption that $(u_1^a, \dots, u_n^a)$ is not a $\frac{1}{100n}$-Gromov-Hausdorff approximation to $B_{\mathbb{C}^n}(0,1)$. $\square$

According to Claim 4.2, $(v_1, \dots, v_n)(\partial B(p,1)) \cap B_{\mathbb{C}^n}(0, \frac{1}{2}) = \emptyset$. Let U be the connected component of $(v_1, \dots, v_n)^{-1}(B_{\mathbb{C}^n}(0, \frac{1}{2}))$ containing p. Then U is relatively compact in $B(p,1)$. We claim that $(v_1, \dots, v_n)(U)$ has complex dimension n. Otherwise, for a generic point q in $(v_1, \dots, v_n)(U)$, let the preimage in $B(p,1)$ be Σ_q. Then Σ_q has complex dimension at least one. But Σ_q

is a compact analytic set in a Stein manifold. Thus it contains only finitely many points. This is a contradiction.

Therefore, $dv_1 \wedge \cdots \wedge dv_n$ is not identically zero. By (4.9) and Cheng-Yau's gradient estimate, $|dv_i| \leq C(n)(r^{\Phi(\varepsilon|n)} + 1)$. Thus $|dv_1 \wedge \cdots \wedge dv_n| \leq C(n)(r^{\Phi(\varepsilon|n)} + 1)$. The canonical line bundle K_M has nonpositive curvature. Note that by the remark following Corollary 2.2, the three circle theorem also holds for holomorphic sections of nonpositive bundles. Therefore, if the holomorphic n-form $dv_1 \wedge \cdots \wedge dv_n$ vanishes at some point in M, then $|dv_1 \wedge \cdots \wedge dv_n|$ must be of at least linear growth, by Corollary 2.2. Therefore, $dv_1 \wedge \cdots \wedge dv_n$ is vanishing identically on M. This is a contradiction.

Next we prove that the map $(v_1, \ldots, v_n)$: $M \to \mathbb{C}^n$ is proper. Given any $R > 1$, we can define a norm $|\cdot|_R$ for the span of $v_1, \ldots, v_n$ induced by $\int_{B(p,R)} v_s \overline{v_t}$. There exists a basis $v_1^R, \ldots, v_n^R$ for the span of $v_1, \ldots, v_n$ so that

$$\int_{B(p,1)} v_s^R \overline{v_t^R} = \delta_{st}, \qquad \int_{B(p,R)} v_s^R \overline{v_t^R} = c(R)_{st}\delta_{st}. \tag{4.13}$$

That is, we diagonalize the two norms $|\cdot|_1$ and $|\cdot|_R$ simultaneously. Obviously we have

$$\sum_{s=1}^{n} |v_s(x)|^2 = \sum_{s=1}^{n} |v_s^R(x)|^2 \tag{4.14}$$

for any $x \in M$. To prove $(v_1, \ldots, v_n)$ is proper, it suffices to prove $\sum_{s=1}^{n} |v_s^R(x)|^2$ is large for $x \in \partial B(p,R)$ and large R. Define

$$w_s^R(x) = \frac{v_s^R(x)}{c_s^R}, \tag{4.15}$$

where c_s^R are positive constants so that

$$\sup_{x \in B(p,R)} w_s^R(x) = 1 \tag{4.16}$$

for $s = 1, \ldots, n$. Note that $\int_{B(p,1)} |v_s^R|^2 = 1$ and $v_s^R(p) = 0$. According to Corollary 2.2 and (4.13),

$$c_s^R \geq cR, \tag{4.17}$$

where $c = c(n) > 0$, $R > 1$. We can apply Claim 4.2 to v_s^R in $B(p,R)$. Here we have to rescale the radius to 1. Then we obtain that $(Rw_1^R, \ldots, Rw_n^R)$ is a $\frac{R}{100n}$-Gromov-Hausdorff approximation from $B(p,R)$ to $B_{\mathbb{C}^n}(0,R)$. In particular, for any $x \in \partial B(p,R)$, there exists some s_0 with $|w_{s_0}^R(x)| \geq \frac{1}{2n}$. Then

$$\begin{aligned} |v_{s_0}^R(x)| &= c_{s_0}^R |w_{s_0}^R(x)| \geq \frac{1}{2n} cR, \\ \sum_{s=1}^{n} |v_s(x)|^2 &= \sum_{s=1}^{n} |v_s^R(x)|^2 \geq c(n)R^2. \end{aligned} \tag{4.18}$$

The properness is proved.

As $dv_1 \wedge \cdots \wedge dv_n$ is not vanishing at any point on M and $(v_1, \ldots, v_n)$ is a proper map to $\mathbb{C}^n$, we conclude that $(v_1, \ldots, v_n)$ is a biholomorphism from M to $\mathbb{C}^n$.

Next we prove that $\mathcal{O}_P(M)$ is generated by $(v_1, \ldots, v_n)$. We can regard $(v_1, \ldots, v_n)$ as a global holomorphic coordinate system on M. If $f \in \mathcal{O}_d(M)$, we can think $f = f(v_1, \ldots, v_n)$. It suffices to prove the right-hand side is a polynomial. Indeed, $|f(x)| \leq C(1+d(x,p)^d)$. Note that by (4.18), $|f(v_1, \ldots, v_n)| \leq C\Big(\Big(\sum_{s=1}^{n} |v_s|^2\Big)^{\frac{d}{2}} + 1\Big)$. This proves f is a polynomial of $v_1, \ldots, v_n$. $\square$

5. **Proof of Theorem 1.4**

Proof. We only consider the case for $n \geq 2$. Otherwise, the result is known. Pick $p \in M$. Let

$$\alpha = \lim_{r\to\infty} \frac{\operatorname{vol}(B(p,r))}{r^{2n}} > 0. \tag{5.1}$$

Consider the blow-down sequence $(M_i, p_i, g_i) = \Big(M, p, \frac{1}{s_i^2} g\Big)$ for $s_i \to \infty$. By the Cheeger-Colding theory [2], a subsequence converges to a metric cone (X, p_∞, d_∞). Define

$$r(x) = d_\infty(x, p_\infty), \;\; x \in X, \qquad r_i(x) = d_{g_i}(x, p_i), \;\; x \in M_i. \tag{5.2}$$

Now pick two regular points $y_0, z_0 \in X$ with

$$r(y_0) = r(z_0) = 1, \qquad d_\infty(y_0, z_0) \geq c(n, \alpha) > 0. \tag{5.3}$$

Note that the latter inequality is guaranteed by Theorem 2.1. There exists $\delta_0 > 0$ satisfying

$$B(y_0, 2\delta_0) \cap B(z_0, 2\delta_0) = \emptyset, \;\; \delta_0 < \frac{1}{10} \tag{5.4}$$

and

$$\begin{aligned} d_{GH}\left(B\left(y_0, \frac{1}{\varepsilon}\delta_0\right), B_{\mathbb{R}^{2n}}\left(0, \frac{1}{\varepsilon}\delta_0\right)\right) &\leq \frac{1}{2}\varepsilon\delta_0, \\ d_{GH}\left(B\left(z_0, \frac{1}{\varepsilon}\delta_0\right), B_{\mathbb{R}^{2n}}\left(0, \frac{1}{\varepsilon}\delta_0\right)\right) &\leq \frac{1}{2}\varepsilon\delta_0. \end{aligned} \tag{5.5}$$

Here $\varepsilon = \frac{1}{2}\varepsilon(n)$, where $\varepsilon(n)$ is given by Proposition 3.1. Therefore, if i is sufficiently large, we can find points $y_i, z_i \in M_i$ with $r_i(y_i) = r_i(z_i) = 1$ and

$$\begin{aligned} d_{GH}\left(B\left(y_i, \frac{1}{\varepsilon}\delta_0\right), B_{\mathbb{R}^{2n}}\left(0, \frac{1}{\varepsilon}\delta_0\right)\right) &\leq \varepsilon\delta_0, \\ d_{GH}\left(B\left(z_i, \frac{1}{\varepsilon}\delta_0\right), B_{\mathbb{R}^{2n}}\left(0, \frac{1}{\varepsilon}\delta_0\right)\right) &\leq \varepsilon\delta_0. \end{aligned} \tag{5.6}$$

Let w_s^i and v_s^i be the local holomorphic charts around y_i and z_i constructed in Proposition 3.1. Note that they have uniform size (independent of i). By changing the value of δ_0, we may assume w^i_s, v^i_s are holomorphic charts in $B(y_i,\delta_0)$ and $B(z_i,\delta_0)$. Moreover,

$$|dw^i_s|, |dv^i_s| \le C(n), \qquad w^i_s(y_i)=0, \quad v^i_s(z_i)=0, \tag{5.7}$$

$$\left|\sum_{s=1}^n |w^i_s(y)|^2 - d_{g_i}(y,y_i)^2\right| \le \Phi(\varepsilon|n)\delta_0^2, \tag{5.8}$$

$$\left|\sum_{s=1}^n |v^i_s(z)|^2 - d_{g_i}(z,z_i)^2\right| \le \Phi(\varepsilon|n)\delta_0^2 \tag{5.9}$$

for $y \in B(y_i,\delta_0), z\in B(z_i,\delta_0)$. We need to construct a weight function on $B(p_i,R)$ for some large R to be determined later. The construction is similar to Proposition 3.1. Set

$$A_i = B(p_i,5R)\backslash B\left(p_i,\frac{1}{5R}\right). \tag{5.10}$$

By the Cheeger-Colding theory [2, (4.43) and (4.82)], there exists a smooth function ρ_i on M_i so that

$$\int_{A_i} |\nabla\rho_i - \nabla\frac{1}{2}r_i^2|^2 + |\nabla^2\rho_i - g_i|^2 < \Phi\left(\frac{1}{i}|R\right), \tag{5.11}$$

$$|\rho_i - \frac{r_i^2}{2}| < \Phi\left(\frac{1}{i}|R\right) \tag{5.12}$$

in A_i. According to (4.20)–(4.23) in [2],

$$\rho_i = \frac{1}{2}(\mathcal{G}_i)^{\frac{2}{2-2n}}, \qquad \Delta\mathcal{G}_i(x)=0, \;\; x\in B(p_i,10R)\backslash B\left(p_i,\frac{1}{10R}\right), \tag{5.13}$$

and

$$\mathcal{G}_i = r_i^{2-2n} \tag{5.14}$$

on $\partial(B(p_i,10R)\backslash B(p_i,\frac{1}{10R}))$. Now

$$|\nabla\rho_i(y)| = C(n)|\mathcal{G}_i|^{\frac{n}{1-n}}|\nabla\mathcal{G}_i(y)|. \tag{5.15}$$

By (5.12)–(5.14) and the Cheng-Yau's gradient estimate,

$$|\nabla\rho_i(y)| \le C(n)r_i(y) \tag{5.16}$$

for $y\in A_i$ and sufficiently large i. Now consider a smooth function $\overline{\varphi}$: $\mathbb{R}^+\to\mathbb{R}^+$ given by $\overline{\varphi}(t)=t$ for $t\ge 2$, $\overline{\varphi}(t)=0$ for $0\le t\le 1$, and $|\overline{\varphi}|, |\overline{\varphi}'|, |\overline{\varphi}''| \le C(n)$. Let

$$u_i(x) = \frac{1}{R^2}\overline{\varphi}(R^2\rho_i(x)). \tag{5.17}$$

We set $u_i(x)=0$ for $x\in B(p_i,\frac{1}{5R})$. Then u_i is smooth in $B(p_i,4R)$.

Claim 5.1. *For sufficiently large i, $\int_{B(p_i,4R)}|\nabla u_i - \nabla\frac{1}{2}r_i^2|^2 + |\nabla^2 u_i - g_i|^2 < \Phi(\frac{1}{R})$, $\left|u_i - \frac{r_i^2}{2}\right| < \Phi\left(\frac{1}{R}\right)$, and $|\nabla u_i| \le C(n)r_i$ in $B(p_i,4R)$.*

Proof. We have

$$\nabla u_i(x) = \overline{\varphi}'(R^2\rho_i(x))\nabla\rho_i(x), \tag{5.18}$$

$$\nabla^2 u_i(x) = R^2\overline{\varphi}''(R^2\rho_i(x))\nabla\rho_i \otimes \nabla\rho_i + \overline{\varphi}'(R^2\rho_i(x))\nabla^2\rho_i. \tag{5.19}$$

The proof follows from a routine calculation, by (5.12), (5.13), and (5.16). □

As in Proposition 3.1, consider a smooth function φ: $\mathbb{R}^+ \to \mathbb{R}^+$ with $\varphi(t) = t$ for $0 \le t \le 1$, $\varphi(t) = 0$ for $t \ge 2$, and $|\varphi|, |\varphi'|, |\varphi''| \le C(n)$. Set

$$v_i(z) = 3R^2\varphi\left(\frac{u_i(z)}{3R^2}\right), \quad v_{i,t}(z) = \int_M H_i(z,y,t)v_i(y)dy. \tag{5.20}$$

Here $H_i(x, y, t)$ is the heat kernel on M_i. Then v_i is supported in $B(p_i, 4R)$. By similar arguments as in Claim 3.1, we arrive at the following:

PROPOSITION 5.1. *$v_{i,1}(z)$ satisfies that $(v_1)_{\alpha\overline{\beta}}(z) \ge c(n,\alpha)g_{\alpha\overline{\beta}} > 0$ for $z \in B(p_i, \frac{R}{10})$. Here $\alpha > 0$ is given by* (5.1).

Now define

$$q_i(x) = 4n\left(\log\left(\sum_{s=1}^n |w_s^i|^2\right)\lambda\left(4\frac{\sum_{s=1}^n |w_s^i|^2}{\delta_0^2}\right) + \log\left(\sum_{s=1}^n |v_s^i|^2\right)\lambda\left(4\frac{\sum_{s=1}^n |v_s^i|^2}{\delta_0^2}\right)\right). \tag{5.21}$$

Here λ is a standard cut-off function $\mathbb{R}^+ \to \mathbb{R}^+$ with $\lambda(t) = 1$ for $0 \le t \le 1$ and $\lambda(t) = 0$ for $t \ge 2$. Note that by (5.8) and (5.9), $q_i(x)$ has compact support in $B(y_i, \delta_0) \cup B(z_i, \delta_0) \subset B(p_i, 2)$.

LEMMA 5.1. *$\sqrt{-1}\partial\overline{\partial}q_i \ge -C(n,\delta_0)\omega_i$. Moreover, $e^{-q_i(x)}$ is not locally integrable at y_i and z_i.*

Proof. We have

$$|\sqrt{-1}\partial\overline{\partial}|w_s^i|^2| = |\partial w_s^i \wedge \overline{\partial w_s^i}| \le |dw_s^i|^2 \le C(n) \tag{5.22}$$

in $B(y_i, \delta_0)$. When $\lambda'\left(4\frac{\sum_{s=1}^n |w_s^i|^2}{\delta_0^2}\right) \neq 0$,

$$\delta_0^2 \ge \sum_{s=1}^n |w_s^i|^2 \ge \frac{1}{4}\delta_0^2. \tag{5.23}$$

Also note that

$$\sqrt{-1}\partial\overline{\partial}\log\left(\sum_{s=1}^n |w_s^i|^2\right) \ge 0 \tag{5.24}$$

in the current sense. Then the proof of the first part follows from routine calculation.

For the second part, when $x \in B(y_i, \frac{\delta_0}{10})$,

$$e^{-q_i(x)} = \frac{1}{\left(\sum_{s=1}^{n} |w_s^i|^2\right)^{4n}}.$$

As $w_s^i(y_i) = 0$ for all s, a simple calculation shows $e^{-q_i(x)}$ is not locally integrable at y_i. The same argument works for z_i. $\square$

Putting Proposition 5.1 and Lemma 5.1 together, we find $C(n, \alpha, \delta_0) > 0$ so that

$$\sqrt{-1}\partial\overline{\partial}(q_i(x) + C(n, \alpha, \delta_0)v_{i,1}(x)) \geq \omega_i \tag{5.25}$$

in $B\left(p_i, \frac{R}{15}\right)$. Set

$$\psi_i(x) = q_i(x) + C(n, \alpha, \delta_0)v_{i,1}(x). \tag{5.26}$$

By the same argument as in Proposition 3.1, we find $\varepsilon_0 = \varepsilon_0(\alpha, n) > 0$ so that for sufficiently large R,

$$\min_{y \in \partial B(p_i, \frac{R}{20})} v_{i,1}(y) > 4 \sup_{y \in B\left(p_i, \varepsilon_0 \frac{R}{20}\right)} v_{i,1}(y). \tag{5.27}$$

Of course, we can assume

$$\frac{\varepsilon_0 R}{20} > 4. \tag{5.28}$$

From now on, we freeze the value of R. That is,

$$R = R(n, \alpha) > 0 \tag{5.29}$$

satisfies the all the conditions above. Let Ω_i be the connected component of

$$\left\{ y \in B\left(p_i, \frac{R}{20}\right) \middle| v_{i,1}(y) < 2 \sup_{y \in B\left(p_i, \varepsilon_0 \frac{R}{20}\right)} v_{i,1}(y) \right\}$$

containing $B\left(p_i, \varepsilon_0 \frac{R}{20}\right)$. Then Ω_i is relatively compact in $B\left(p_i, \frac{R}{20}\right)$ and Ω_i is a Stein manifold, by Proposition 5.1. Also $B(p_i, 3) \subset \Omega_i$.

Now consider a function $f_i(x) = 1$ for $x \in B\left(y_i, \frac{\delta_0}{4}\right)$; f_i has compact support in $B(y_i, \delta_0) \subset B(p_i, 2)$ and $|\nabla f_i| \leq C(n, \alpha, \delta_0)$. Then $f_i(z_i) = 0$. We solve the equation $\overline{\partial} h_i = \overline{\partial} f_i$ in Ω_i with

$$\int_{\Omega_i} |h_i|^2 e^{-\psi_i} \leq \int_{\Omega_i} |\overline{\partial} f_i|^2 e^{-\psi_i} \leq C(n, \alpha, \delta_0). \tag{5.30}$$

By Lemma 5.1, we have that e^{-q_i} is not locally integrable at y_i and z_i, $h_i(y_i) = h_i(z_i) = 0$. Define the holomorphic function $\mu_i = f_i - h_i$. Recall that by the construction, $f_i(y_i) = 1, f_i(z_i) = 0$. Then

$$\mu_i(y_i) = f_i(y_i) - h_i(y_i) = 1, \qquad \mu_i(z_i) = f_i(z_i) - h_i(z_i) = 0. \tag{5.31}$$

Therefore, μ_i is not constant on Ω_i. It is easy to see that $\psi_i(x) \leq C(n,\alpha,\delta_0)$ in $B(p_i,3)$. Then

$$\frac{1}{C(n,\alpha,\delta_0)}\int_{B(p_i,3)}|h_i|^2 \leq \int_{\Omega_i}|h_i|^2 e^{-\psi_i} \leq C(n,\alpha,\delta_0). \tag{5.32}$$

Thus

$$\int_{B(p_i,3)}|\mu_i|^2 \leq 2\int_{B(p_i,3)}(|h_i|^2+|f_i|^2) \leq C(n,\alpha,\delta_0). \tag{5.33}$$

Mean value inequality implies that

$$|\mu_i(x)| \leq C(n,\alpha,\delta_0) \tag{5.34}$$

for $x \in B(p_i,2)$. Therefore, the holomorphic function

$$\nu_i^*(x) = \mu_i(x) - \mu_i(p_i) \tag{5.35}$$

is uniformly bounded in $B(p_i,2)$. Set

$$M_i'(r) = \sup_{x\in B(p_i,r)}|\nu_i^*(x)|. \tag{5.36}$$

Then

$$M_i'(2) \leq C(n,\alpha,\delta_0). \tag{5.37}$$

On the other hand, by (5.31), we find

$$M_i'(1) \geq \frac{1}{2}. \tag{5.38}$$

Therefore,

$$\frac{M_i'(2)}{M_i'(1)} \leq C(n,\alpha,\delta_0). \tag{5.39}$$

Now we are ready to apply the three circle theorem. More precisely, we consider the rescale functions $\overline{\nu}_i^* = \beta_i\nu_i^*$ in $B(p,2s_i) \subset M$. Here β_i are constants so that

$$\int_{B(p,2)}|\overline{\nu}_i^*|^2 = 1. \tag{5.40}$$

This implies

$$|\overline{\nu}_i^*| \leq C(n,\alpha) \tag{5.41}$$

in $B(p,1)$. Set $M_i(r) = \sup\limits_{x\in B(p,r)}|\overline{\nu}_i^*|$. The three circle theorem says $\frac{M_i(2r)}{M_i(r)}$ is monotonic increasing for $0 < r \leq s_i$. By (5.39) and similar arguments as in (4.9), we obtain that

$$M_i(r) \leq C(n,\alpha,\delta_0)\left(r^{C(n,\alpha,\delta_0)}+1\right) \tag{5.42}$$

for all i and $s_i \geq r$. Let $i \to \infty$. A subsequence of $\overline{\nu}_i^*$ converges uniformly on each compact set to a holomorphic function v of polynomial growth. v cannot be constant, as v satisfies $v(p) = 0$ and $\int_{B(p,2)} |v|^2 = 1$. Moreover, the degree at infinity is bounded by $C(n, \alpha, \delta_0)$. □

Remark 5.1. By the Gromov compactness theorem, we can find $\delta_0 = \delta_0(n, \alpha), y_0, z_0$ satisfying (5.3), (5.4) and (5.5). Therefore, the degree of the holomorphic function at infinity is bounded by $C(n, \alpha)$. The dependence on α is obvious necessary if we look at the complex one-dimensional case.

COROLLARY 5.1. *Let M^n be a complete Kähler manifold with nonnegative bisectional curvature and maximal volume growth. Then the transcendental dimension of polynomial growth holomorphic functions is n. Moreover, $\mathcal{O}_P(M)$ separates points and tangents on M.*

Proof. From Theorem 1.4, there exists a nonconstant holomorphic function f of polynomial growth. First we assume that the universal cover of M does not split as products. Then by Theorem 3.1 in [28], if we run the heat flow for $\log(|f|^2 + 1)$, the function becomes strictly plurisubharmonic of logarithmic growth. Then we can apply Hörmander's L^2-estimate (for example, Theorem 5.2 in [26]) to conclude that $\mathcal{O}_P(M)$ separates points and tangents on M. Together with the multiplicity estimate Theorem 2.5, we proved that the transcendental dimension of holomorphic functions of polynomial growth over $\mathbb{C}$ is n. If the universal covering splits, we work on the universal covering space. Each factor must be of maximal volume growth. Then we can find nonconstant holomorphic functions of polynomial growth. Then we run the heat flow for each factor to obtain strictly plurisubharmonic functions of logarithmic growth. Then we add these function together, which is still strictly plurisubharmonic. Finally, to put these functions back to M, just observe that $\pi_1(M)$ is finite. Then we can symmetrize the function. Then it projects to M, still with logarithmic growth. Then the argument is the same for the nonsplitting case. □

Remark 5.2. In this case, one can actually prove M^n is biholomorphic to a quasi-affine variety. This follows from Mok's deep work in [23]. However, this is not enough to prove that $\mathcal{O}_P(M)$ is finitely generated. By using Theorem 6.1 below, we shall prove M is affine algebraic.

6. **A properness theorem**

PROPOSITION 6.1. *There exists $\varepsilon(n, v) > 0$ so that the following holds: Let (Y^n, q) be a complete Kähler manifold with nonnegative bisectional curvature and $\frac{\mathrm{vol}(B(q,r))}{r^{2n}} \geq v > 0$ for all $r > 0$. Assume that for some $0 < \varepsilon < \varepsilon(n, v)$,*

$$d_{GH}\left(B\left(q, \frac{1}{\varepsilon}R\right), B_X\left(o, \frac{1}{\varepsilon}R\right)\right) \leq \varepsilon R \tag{6.1}$$

for some metric cone (X,o) (o is the vertex) and $R>0$. Then there exist $N=N(v,n)\in\mathbb{N}, 1>\delta_1>5\delta_2>\delta=\delta(v,n)>0$ and holomorphic functions $g^1,\dots,g^N$ on $B(q,\delta_1 R)$ with $g^j(q)=0$ and

$$\min_{x\in\partial B(q,\frac{\delta_1}{3}R)}\sum_{j=1}^N|g^j(x)|^2>2\sup_{x\in B(q,\delta_2 R)}\sum_{j=1}^N|g^j(x)|^2. \tag{6.2}$$

Furthermore, for all j,

$$\frac{\sup_{x\in B(q,\frac{1}{2}\delta_1 R)}|g^j(x)|^2}{\sup_{x\in B(q,\frac{1}{3}\delta_1 R)}|g^j(x)|^2}\le C=C(n,v). \tag{6.3}$$

Remark 6.1. This proposition is a generalization of Proposition 3.1. Essentially it deals with the separation of points.

Proof. By rescaling, we see that if the proposition holds for some $R>0$, then it holds for all $R>0$ with the same parameter constants. Therefore, without loss of generality, we may assume R is sufficiently large, to be determined. Assume

$$X=\mathbb{R}^k\times Z. \tag{6.4}$$

We will do induction on k. For the case $k=2n$, the proposition reduces to Proposition 3.1. Assume the proposition holds for $k=2s$, but fails for $k=2s-2$. Then there exist complete Kähler manifolds $(Y_i^n,q_i)(i\in\mathbb{N})$ with nonnegative bisectional curvature and $\frac{\mathrm{vol}(B(q_i,r))}{r^{2n}}\ge v>0$ for all $r>0$, metric cones (X_i,o_i), and a sequence $R_i>0$ with

$$(X_i,o_i)=(\mathbb{R}^{2s-2},0)\times(Z_i,z_i^*),\quad d_{GH}(B(q_i,iR_i),B_{X_i}(o_i,iR_i))\le\frac{1}{i}R_i. \tag{6.5}$$

Furthermore, Proposition 6.1 does not hold uniformly for any subsequence (Y_{i_k},q_{i_k}). That is to say, there do not exist positive constants δ,C,N, $1>\delta_1^{i_k}>5\delta_2^{i_k}>\delta$ and holomorphic functions $g^j_{i_k}$ on $B(q_{i_k},\delta_1^{i_k}R_i)$ ($g^j_{i_k}(q_{i_k})=0$), satisfying (6.2) and (6.3) (replace R by R_i) for all k.

By rescaling (Y_i,q_i), we may assume that $R_i=R$ for all i. For notational simplicity, we still denote the rescaled manifolds by (Y_i,q_i). By Gromov compactness, after passing to a subsequence, we may assume (Y_i,q_i) converges in the pointed Gromov-Hausdorff sense to a metric space (X_0,o_0). By (6.5), (X_0,o_0) is a metric cone. Also there exists a sequence $s_i\to\infty$ with

$$(X_0,o_0)=(\mathbb{R}^{2s-2},0)\times(Z_0,z_0^*),\quad d_{GH}(B(q_i,s_iR),B_{X_0}(o_0,s_iR))<\Phi\left(\frac{1}{i}\right)R. \tag{6.6}$$

Observe that Z_0 does not split off the $\mathbb{R}^2$ factor, by the induction hypothesis. Our goal is to show that Proposition 6.1 holds uniformly for (Y_i, q_i) when i is sufficiently large. Of course, this would complete the induction.

The idea is this. For some $\delta_3 > 0$, we shall consider the set $\partial B(o_0, \frac{\delta_3 R}{3}) \cap (X_0 \backslash (\mathbb{R}^{2s-2} \times z_0^*))$. Then each tangent cone splits off $\mathbb{R}^{2s}$. Thus a small neighborhood is Gromov-Hausdorff close to a ball in $\mathbb{R}^{2s}$ times a metric cone. This is still true for Y_i, when i is sufficiently large. Then we can apply induction around these points. The Hörmander L^2-estimate of $\overline{\partial}$ can be applied to separate these points from q_i. For points near the slice $\mathbb{R}^{2s-2} \times z_0^*$, we can construct holomorphic coordinate functions as in Proposition 3.1 to separate from q_i.

For the reader's convenience, we break the proof of Proposition 6.1 into three parts.

Part I: *Basic setup.* As in (5.20), we have a nonnegative function $v_{i,1}$ so that in $B(q_i, 10R)$,

$$\sqrt{-1}\partial\overline{\partial} v_{i,1} \geq c(n,v)\omega_i > 0, \tag{6.7}$$

$$\inf_{y \in B(q_i, \frac{1}{2}R) \backslash B(q_i, \frac{1}{4}R)} v_{i,1}(y) > 4 \sup_{y \in B(q_i, \delta_3 R)} v_{i,1}(y), \tag{6.8}$$

$$\inf_{y \in B(q_i, \frac{\delta_3}{2}R) \backslash B(q_i, \frac{\delta_3}{4}R)} v_{i,1}(y) > 4 \sup_{y \in B(q_i, \delta_4 R)} v_{i,1}(y). \tag{6.9}$$

Here $\delta_s = \delta_s(n, v) > 0 (s = 3, 4)$. By Proposition A.1, we may also assume

$$4 \sup_{y \in B(q_i, \delta_4 R)} v_{i,1}(y) > \frac{1}{2}, \qquad \delta_3 R > 100. \tag{6.10}$$

Now we freeze the value of R. That is to say,

$$R = R(n, v) > 0. \tag{6.11}$$

Then

$$|v_{i,1}(y)| \leq C(R, n, v) = C(n, v), \ \ y \in B(q_i, R). \tag{6.12}$$

Let Ω_i be the connected component of $\left\{z | v_{i,1}(z) < \sup_{B(q_i, \delta_3 R)} v_{i,1}\right\}$ containing $B(q_i, \delta_3 R)$. As before, we see Ω_i is Stein.

According to (6.6) and (2.4)–(2.11) in [6], there exist harmonic functions $b_l^i (1 \leq l \leq 2s-2)$ in $B(q_i, 2R)$ with

$$\int_{B(q_i, R)} \sum_{1 \leq l_1, l_2 \leq 2s-2} |\langle \nabla b_{l_1}^i, \nabla b_{l_2}^i \rangle - \delta_{l_1 l_2}|^2 + \sum_l |\nabla^2 b_l^i|^2 < \Phi\left(\frac{1}{i} | n\right) \tag{6.13}$$

and

$$b_l^i(q_i) = 0, \qquad |\nabla b_l^i| \leq C(n) \tag{6.14}$$

in $B(q_i, R)$. Moreover, in $B(q_i, R)$, $(b_1^i, \dots, b_{2s-2}^i)$ approximates $(y_1, \dots, y_{2s-2})$ with error $\Phi(\frac{1}{i}|n)$. Here $(y_1, \dots, y_{2s-2})$ is the Euclidean coordinate of (X_0, o_0) $= (\mathbb{R}^{2s-2}, 0) \times (Z_0, z_0^*)$. By similar arguments as before, we may assume that

$$\int_{B(q_i,R)} |J\nabla b_{2m-1}^i - \nabla b_{2m}^i|^2 \leq \Phi\left(\frac{1}{i}|n\right) \tag{6.15}$$

for $1 \leq m \leq s-1$. Set $\tilde{w}_m^i = b_{2m-1}^i + \sqrt{-1}b_{2m}^i$. Then

$$\int_{B(q_i,R)} |\overline{\partial}\tilde{w}_m^i|^2 \leq \Phi\left(\frac{1}{i}|n\right). \tag{6.16}$$

By solving the $\overline{\partial}$ problem as before, we find holomorphic functions $w_m^i (1 \leq m \leq s-1)$ with

$$w_m^i(q_i) = 0, \qquad |w_m^i - \tilde{w}_m^i| \leq \Phi\left(\frac{1}{i}|n\right) \tag{6.17}$$

in $B(q_i, \frac{R}{2})$. Recall that δ_3 in (6.9). For sufficiently large i, define

$$E_i = \left\{x | x \in \partial B\left(q_i, \frac{\delta_3 R}{3}\right), \sum_{m=1}^{s-1} |w_m^i|^2 \leq \frac{(\delta_3 R)^2}{27}\right\}, \tag{6.18}$$

$$E = \left\{x | x \in \partial B_{\mathbb{R}^{2s-2}\times Z_0}\left((0, z_0^*), \frac{\delta_3 R}{3}\right), \sum_{k=1}^{2s-2} |y_k|^2 \leq \frac{(\delta_3 R)^2}{18}\right\}. \tag{6.19}$$

Then the limit of E_i is contained in E under the Gromov-Hausdorff approximation. Observe from the definition and (6.10), if $x \in \partial B(q_i, \frac{\delta_3 R}{3}) \backslash E_i$, then

$$\sum_{m=1}^{s-1} |w_m^i(x)|^2 > \frac{(\delta_3 R)^2}{27} > 1. \tag{6.20}$$

Part II: *The induction step.* For $x \in E$, let C_x be a tangent cone. Then C_x must split off a factor $\mathbb{R}^{2s-1}$, by Cheeger-Colding [2]. Since C_x is the Gromov-Hausdorff limit of Kähler manifolds with noncollapsed volume and nonnegative Ricci curvature, C_x splits off a factor $\mathbb{R}^{2s}$, by [6]. According to the induction hypothesis, let $\varepsilon(n, v, s) > 0$ be the constant $\varepsilon(n, v)$ in Proposition 6.1 when X splits off $\mathbb{R}^{2s}$. There exists

$$\frac{\delta_3 R}{20} > r_x > 0 \tag{6.21}$$

with

$$d_{GH}\left(B\left(x, \frac{1}{\varepsilon}r_x\right), B_W\left(w, \frac{1}{\varepsilon}r_x\right)\right) < \varepsilon r_x, \quad (W, w) = (\mathbb{R}^{2s}, 0)\times(H, h^*). \tag{6.22}$$

Here $\varepsilon = \frac{1}{2}\varepsilon(n, v, s)$ and (H, h^*) is a metric cone with vertex at h^*. Note that when $y \in E$ is sufficiently close to x,

$$d_{GH}\left(B\left(y, \frac{1}{\varepsilon}r_x\right), B_W\left(w, \frac{1}{\varepsilon}r_x\right)\right) < \varepsilon r_x. \tag{6.23}$$

By compactness, we can find a uniform positive lower bound of r_x, say

$$r_x \geq R_0(X_0) > 0. \tag{6.24}$$

By Gromov compactness, we actually have

$$r_x > R_0 = R_0(n, v) > 0. \tag{6.25}$$

Note that W is not necessarily equal or close to a tangent cone of x.

Then for sufficiently large i and any point $x_i \in E_i$,

$$d_{GH}\left(B\left(x_i, \frac{1}{\varepsilon} r_{x_i}\right), B_{W_i}\left(w_i, \frac{1}{\varepsilon} r_{x_i}\right)\right) < \varepsilon r_{x_i}, \tag{6.26}$$

$$\frac{\delta_3 R}{20} > r_{x_i} \geq R_0 > 0, \qquad (W_i, w_i) = (\mathbb{R}^{2s}, 0) \times (H_i, h_i^*). \tag{6.27}$$

Here ε is the same as in (6.22) and (H_i, h_i^*) is a metric cone with vertex at h_i^*. We can apply the induction to $B(x_i, \frac{1}{\varepsilon} r_{x_i})$ and the metric cone (W_i, w_i). By the induction hypothesis, there exist

$$1 > \delta_1 > 5\delta_2 > \delta(n, v), \qquad N = N(v, n) \in \mathbb{N} \tag{6.28}$$

and holomorphic functions $g_i^j (1 \leq j \leq N)$ in $B(x_i, \delta_1 r_{x_i})$ with

$$g_i^j(x_i) = 0, \qquad \min_{x \in \partial B\left(x_i, \frac{\delta_1 r_{x_i}}{3}\right)} \sum_{i=1}^{N} |g_i^j(x)|^2 > 2 \sup_{x \in B(x_i, \delta_2 r_{x_i})} \sum_{i=1}^{N} |g_i^j(x)|^2, \tag{6.29}$$

$$\frac{\sup_{x \in B(x_i, \frac{1}{2}\delta_1 r_{x_i})} |g_i^j(x)|^2}{\sup_{x \in B(x_i, \frac{1}{3}\delta_1 r_{x_i})} |g_i^j(x)|^2} \leq C(n, v). \tag{6.30}$$

By normalization, we can also assume

$$\sup_j \sup_{y \in B(x_i, \delta_2 r_{x_i})} |g_i^j(y)| = 2. \tag{6.31}$$

Note that by the three circle theorem,

$$\sup_{y \in B(x_i, \frac{r_{x_i}\delta_1}{2})} |g_i^j(y)| \leq C(n, v). \tag{6.32}$$

Set

$$F_i(x) = \sum_{j=1}^{N} |g_i^j|^2. \tag{6.33}$$

Let λ be a standard cut-off function $\mathbb{R}^+ \to \mathbb{R}^+$ given by $\lambda(t) = 1$ for $0 \leq t \leq 1$, $\lambda(t) = 0$ for $t \geq 2$, and $|\lambda'|, |\lambda''| \leq C(n)$. Consider

$$h_i(x) = 4n \log F_i(x) \lambda(F_i(x)). \tag{6.34}$$

By (6.29) and (6.31), $h_i(x)$ is supported in $B\left(x_i, \frac{\delta_1 r_{x_i}}{3}\right)$. Similar to Lemma 5.1, it is easy to check that

$$\sqrt{-1}\partial\bar{\partial} h_i(x) \geq -C(n, v)\omega_i. \tag{6.35}$$

By (6.7), there exists $\xi = \xi(n, v) > 0$ with

$$\sqrt{-1}\partial\overline{\partial}(\xi v_{i,1} + h_i) \geq \omega_i \tag{6.36}$$

in Ω_i. We will assume such ξ is large, to be determined later. Set

$$\phi(x) = \xi v_{i,1}(x) + h_i(x). \tag{6.37}$$

Now consider a function

$$\mu_i(x) = \varphi\left(\frac{d(x, x_i)}{\delta_1 r_{x_i}}\right). \tag{6.38}$$

Here $\varphi(t) = 1$ for $t \leq \frac{1}{3}$, $\varphi(t) = 0$ for $t \geq 1$, and $|\varphi'| \leq C(n)$. Then it is clear μ_i is supported in $B(x_i, \delta_1 r_{x_i})$. Also, by (6.27),

$$|\nabla \mu_i| \leq C(n, v). \tag{6.39}$$

We solve the $\overline{\partial}$ problem $\overline{\partial} s_i = \overline{\partial}\mu_i$ on Ω_i satisfying

$$\begin{aligned} \int_{\Omega_i} e^{-\phi}|s_i|^2 &\leq \int_{\Omega_i} e^{-\phi}|\overline{\partial}\mu_i|^2 \\ &= \int_{B(x_i,\delta_1 r_{x_i})\setminus B\left(x_i, \frac{\delta_1 r_{x_i}}{3}\right)} e^{-\phi}|\overline{\partial}\mu_i|^2 \\ &\leq \exp\left(-\xi \inf_{y \in B(q_i, \frac{\delta_3 R}{2})\setminus B(q_i, \frac{\delta_3 R}{4})} v_{i,1}(y)\right) C(n, v). \end{aligned} \tag{6.40}$$

Here we used that h_i is supported in $B\left(x_i, \frac{\delta_1 r_{x_i}}{3}\right)$. We also used that

$$B(x_i, \delta_1 r_{x_i}) \subset B\left(q_i, \frac{\delta_3 R}{2}\right) \Big\backslash B\left(q_i, \frac{\delta_3 R}{4}\right), \tag{6.41}$$

by (6.27). Observe that by (6.9), μ_i vanishes on $B\left(q_i, \frac{1}{2}\delta_4 R\right)$. Hence s_i is holomorphic in $B\left(q_i, \frac{1}{2}\delta_4 R\right)$. The mean value inequality implies that for $x \in B\left(q_i, \frac{\delta_4 R}{5}\right)$,

(6.42)

$$\begin{aligned} &|s_i(x)| \leq \frac{\int_{B(q_i,\delta_4 R)} |s_i|^2}{c(n,v)(\delta_4 R)^{2n}} \\ &\leq \exp\left(\xi \sup_{y \in B(q_i,\delta_4 R)} v_{i,1}(y)\right) \frac{\int_{\Omega_i} e^{-\phi}|s_i|^2}{c(n,v)(\delta_4 R)^{2n}} \\ &\leq \exp\left(-\xi\Bigg(\inf_{y \in B\left(q_i,\frac{\delta_3 R}{2}\right)\setminus B\left(q_i,\frac{\delta_3 R}{4}\right)} v_{i,1}(y) - \sup_{y \in B(q_i,\delta_4 R)} v_{i,1}(y)\Bigg)\right) \frac{1}{c(n,v)(\delta_4 R)^{2n}} \\ &\leq \frac{e^{-\frac{\xi}{4}}}{c(n,v)(\delta_4 R)^{2n}}. \end{aligned}$$

Here we used (6.9) and (6.10). If ξ is large (depending only on n, v), then we can make

$$|s_i(x)| \leq \frac{1}{10} \tag{6.43}$$

for $x \in B\left(q_i, \frac{\delta_4 R}{5}\right)$. Now we freeze the value of $\xi = \xi(n, v)$. Note that the local integrability of s_i forces $s_i(x_i) = 0$. Set

$$w_i^1(x) = \mu_i(x) - s_i(x). \tag{6.44}$$

Then w_1^i is holomorphic in Ω_i and

$$w_i^1(x_i) = 1, \qquad |w_i^1| \leq \frac{1}{10} \tag{6.45}$$

in $B\left(q_i, \frac{\delta_4 R}{5}\right)$. Set

$$f_i^1(x) = w_i^1(x) - w_i^1(q_i). \tag{6.46}$$

Then

$$f_i^1(q_i) = 0, \qquad |f_i^1(x_i)| \geq \frac{9}{10}. \tag{6.47}$$

By (6.40) and mean value inequality, we find

$$|f_i^1(x)| \leq C(n, v), \qquad |\nabla f_i^1(x)| \leq C(n, v), x \in B\left(q_i, \frac{2\delta_3 R}{3}\right). \tag{6.48}$$

Therefore, there exists $\delta_5(n, v) > 0$ so that

$$|f_i^1(x)| \geq \frac{1}{2} \tag{6.49}$$

in $B(x_i, \delta_5 R)$.

Part III: *Completion of the proof.* By a standard covering argument, we can take $x^j \in E$ ($j = 1, 2, \ldots, K$, $K = K(v, n)$) with

$$\cup_j B\left(x^j, \frac{\delta_5 R}{3}\right) \supset E, \quad \frac{\delta_3 R}{20} > r_{x^j} \geq R_0(n, v) > 0, \tag{6.50}$$

and

(6.51)
$$d_{GH}\left(B\left(x^j, \frac{1}{\varepsilon} r_{x^j}\right), B_{W^j}\left(w^j, \frac{1}{\varepsilon} r_{x^j}\right)\right) < \varepsilon r_{x^j}, \ (W^j, w^j) = (\mathbb{R}^{2s}, 0) \times (H^j, (h^j)^*).$$

Here $(H^j, (h^j)^*)$ is a metric cone with vertex at $(h^j)^*$ and ε is the same as in (6.21). Then for sufficiently large i, we can find $x_i^j \in E_i$, $j = 1, \ldots, K$ with

$$d_{GH}\left(\left(B(x_i^j, \frac{1}{\varepsilon} r_{x^j}\right), B_{W^j}\left(w_j, \frac{1}{\varepsilon} r_{x^j}\right)\right) < \varepsilon r_{x^j} \tag{6.52}$$

and

$$\cup_j B\left(x_i^j, \frac{\delta_5 R}{2}\right) \supset E_i. \tag{6.53}$$

Now we can apply the induction argument as in Part II for each geodesic ball $B(x_i^j, \frac{1}{\varepsilon} r_{x^j})$. We obtain holomorphic functions f_i^j on $B(q_i, \delta_3 R)$ satisfying

$$|f_i^j(x)| \geq \frac{1}{2} \tag{6.54}$$

for $x \in B(x_i^j, \delta_5 R)$ and

$$|f_i^j(x)| \leq C(n,v), \;\; x \in B\left(q_i, \frac{2\delta_3 R}{3}\right), \quad f_i^j(q_i) = 0. \tag{6.55}$$

Put $G_i(x) = \sum_{j=1}^{K} |f_i^j|^2 + \sum_{m=1}^{s-1} |w_m^i|^2$. Then by (6.17), (6.20), (6.53) and (6.55),

$$|\nabla G_i(x)| \leq C(n,v), \;\; x \in B\left(q_i, \frac{\delta_3 R}{2}\right), \;\; G_i(q_i) = 0, \tag{6.56}$$

and

$$|G_i(x)| \geq \frac{1}{4}, \;\; x \in \partial B\left(q_i, \frac{\delta_3 R}{3}\right). \tag{6.57}$$

Therefore, there exists $\delta_6 = \delta_6(n,v) > 0$ with

$$\sup_{x \in B(q_i, \delta_6 R)} |G_i(x)| \leq \frac{1}{10}. \tag{6.58}$$

Take $g_i^k = f_i^k$ for $k = 1, \ldots, K$ and $g_i^k = w_{k-K}^i$ for $K+1 \leq k \leq K+s-1$. Then we can find parameters δ, C, N so that (6.2) and (6.3) hold for (Y_i, q_i) when i is sufficiently large. This contradicts the assumption in the paragraph right below (6.4). The proof of Proposition 6.1 is complete. □

The following is the main theorem in this section:

THEOREM 6.1. *Let M^n be a complete noncompact Kähler manifold with nonnegative bisectional curvature and maximal volume growth. Then there exist finitely many polynomial growth holomorphic functions $f_1, \ldots, f_k$ so that $(f_1, \ldots, f_k)$ is a proper holomorphic map from M to $\mathbb{C}^k$.*

Proof. Pick a point $p \in M$. Put

$$v = \lim_{r \to \infty} \frac{\mathrm{vol}(B(p,r))}{r^{2n}} > 0. \tag{6.59}$$

For any sequence $r_i \to \infty$, set $(M_i, p_i) = (M, p, r_i^{-2} g)$. (We shall make r_i explicit in Proposition 6.2 below.) Then there exist $R_i'' \to \infty$ and metric cones (X_i, x_i^*) with

$$d_{GH}(B(p_i, R_i''), B_{X_i}(x_i^*, R_i'')) < \frac{1}{R_i''}. \tag{6.60}$$

Let $d_i(x) = d_i(x, p_i)$ for $x \in M_i$. Following the construction in (5.11) and Claim 5.1, we find a sequence $R_i' \to \infty$ and functions ρ_i on M_i satisfying

$$\int_{B(p_i,4R_i')} \left|\nabla\rho_i - \nabla\frac{1}{2}d_i^2\right|^2 + |\nabla^2\rho_i - g_i|^2 < \Phi\left(\frac{1}{i}\right). \tag{6.61}$$

Also, in $B(p_i, 4R_i')$,

$$\left|\rho_i - \frac{d_i^2}{2}\right| < \Phi\left(\frac{1}{i}\right), \quad |\nabla\rho_i| \le C(n)d_i. \tag{6.62}$$

As before, consider a smooth function φ: $\mathbb{R}^+ \to \mathbb{R}^+$ with $\varphi(t) = t$ for $0 \le t \le 1$, $\varphi(t) = 0$ for $t \ge 2$, and $|\varphi|, |\varphi'|, |\varphi''| \le C(n)$. Set

$$v_i(z) = 3(R_i')^2\varphi\left(\frac{\rho_i(z)}{3(R_i')^2}\right). \tag{6.63}$$

Then v_i is supported in $B(p_i, 4R_i')$. Let $H_i(x, y, t)$ be the heat kernel of M_i. Consider the function $\tau_i(x) = \log(1 + v_i(x))$, and define

$$\tau_{i,t}(z) = \int_{M_i} H_i(z, y, t)\tau_i(y)dy. \tag{6.64}$$

By (6.62), we have

$$\inf_{B(p_i,\frac{3}{2})\setminus B(p_i,\frac{30}{24})} \tau_i - \sup_{B(p_i,\frac{27}{24})} \tau_i \ge 2c(n, v) > 0. \tag{6.65}$$

Here τ_i is of logarithmic growth uniform for all i. By heat kernel estimates, there exists $t_0 = t_0(n, v) > 0$ so that

$$\inf_{B(p_i,\frac{3}{2})\setminus B(p_i,\frac{30}{24})} \tau_{i,t_0} - \sup_{B(p_i,\frac{27}{24})} \tau_{i,t_0} \ge c(n, v) > 0. \tag{6.66}$$

On a smooth Kähler metric cone, let r be the distance function to the vertex. Then $\sqrt{-1}\partial\overline{\partial}\log(1+\frac{1}{2}r^2)$ is a positive $(1, 1)$ form away from the vertex. Since τ_i resembles $\log(1 + \frac{1}{2}r^2)$, by similar arguments as in Proposition 5.1, we find that on $B(p_i, 5)$,

$$\sqrt{-1}\partial\overline{\partial}\tau_{i,t_0} \ge c(n, v) > 0. \tag{6.67}$$

By Proposition A.1, for any fixed R and sufficiently large i, on $B(p_i, R)$,

$$c(n, v)\log(d_i(x) + 2) - C(n, v) \le \tau_{i,t_0}(x) \le C(n, v)\log(d_i(x) + 2), \tag{6.68}$$

$$\sqrt{-1}\partial\overline{\partial}\tau_{i,t_0}(x) > 0, \ \ x \in B(p_i, R). \tag{6.69}$$

Therefore, there exist sequences $\widetilde{R}_i \to \infty$, $R_i \to \infty$, and $c_i \to \infty$ so that $\tau_{i,t_0}^{-1}(\{c|c \le c_i\}) \cap B(p_i, \widetilde{R}_i)$ is relatively compact in $B(p_i, \widetilde{R}_i)$. Also

$$\tau_{i,t_0}^{-1}(\{c|c \le c_i\}) \cap B(p_i, \widetilde{R}_i) \supset B(p_i, R_i), \tag{6.70}$$

$$\sqrt{-1}\partial\overline{\partial}\tau_{i,t_0} > 0 \tag{6.71}$$

in $\tau_{i,t_0}^{-1}(\{c|c \le c_i\}) \cap B(p_i, \widetilde{R}_i)$. Let Ω_i be the connected component of the open set $\tau_{i,t_0}^{-1}(\{c|c < c_i\})$ containing $B(p_i, R_i)$. Then Ω_i is a Stein manifold.

Let $\delta_1 = \delta_1(n, v)$ be given by Proposition 6.1. According to Proposition 6.1, there exist holomorphic functions $w_j^i (1 \le j \le K = K(n, v))$ on $B(p_i, 3)$ (here we take $R = \frac{3}{\delta_1}$ in Proposition 6.1) so that

$$w_j^i(p_i) = 0, \quad \max_j \sup_{B(p_i,1)} |w_j^i| = 1, \tag{6.72}$$

$$\min_{x\in\partial B(p_i,1)} \sum_{j=1}^{K} |w_j^i(x)|^2 > 2 \sup_{x\in B(p_i,\frac{3\delta_2}{\delta_1})} \sum_{i=1}^{K} |w_j^i(x)|^2, \tag{6.73}$$

$$\frac{\sup_{x\in B(p_i,\frac{3}{2})} |w_j^i(x)|^2}{\sup_{x\in B(p_i,1)} |w_j^i(x)|^2} \le C(n, v). \tag{6.74}$$

Then of course, on $B\left(p_i, \frac{3}{2}\right)$,

$$|w_j^i(x)| \le C(n, v). \tag{6.75}$$

Also, by the three circle theorem, we have

$$\max_j \sup_{B(p_i,\frac{3\delta_2}{\delta_1})} |w_j^i| \ge c(n, v) > 0. \tag{6.76}$$

Thus

$$\min_{x\in\partial B(p_i,1)} \sum_{j=1}^{K} |w_j^i(x)|^2 \ge c(n, v) > 0. \tag{6.77}$$

Now consider a cut off function $\lambda_i(x) = \lambda(d_i(x))$ with $\lambda_i = 1$ in $B(p_i, \frac{30}{24})$, where λ_i has compact support in $B(p_i, \frac{33}{24})$ and $|\nabla\lambda_i| \le C(n, v)$. Let $\tilde{w}_j^i = \lambda_i w_j^i$. Then $\overline{\partial}\tilde{w}_j^i$ is supported in $B(p_i, \frac{33}{24})\backslash B(p_i, \frac{30}{24})$. We solve the $\overline{\partial}$-problem $\overline{\partial}\tilde{f}_j^i = \overline{\partial}\tilde{w}_j^i$ in Ω_i with the weight function $\psi_i = \eta\tau_{i,t_0}$ where $\eta = \eta(n, v)$ is a very large number, to be determined. Then by (6.67),

$$\int_{\Omega_i} |\tilde{f}_j^i|^2 e^{-\psi_i} \le \frac{\int_{\Omega_i} |\overline{\partial}\tilde{w}_j^i|^2 e^{-\psi_i}}{c(n, v)}. \tag{6.78}$$

This implies that

$$\int_{B(p_i,\frac{27}{24})} |\tilde{f}_j^i|^2 e^{-\psi_i} \le \frac{\int_{B(p_i,\frac{3}{2})\backslash B(p_i,\frac{30}{24})} |\overline{\partial}\tilde{w}_j^i|^2 e^{-\psi_i}}{c(n, v)}. \tag{6.79}$$

Let

$$f_j^i(x) = \tilde{w}_j^i(x) - \tilde{f}_j^i(x) - (\tilde{w}_j^i(p_i) - \tilde{f}_j^i(p_i)). \tag{6.80}$$

By (6.66), (6.72), (6.73), (6.75), (6.77) and similar arguments as in (6.42), if $\eta = \eta(n, v)$ is large enough, we can make $|\tilde{f}^i_j|$ so small in $B(p_i, 1)$ that

$$C(n,v) \geq \min_{x \in \partial B(p_i,1)} \sum_{j=1}^{K} |f^i_j(x)|^2 > \frac{3}{2} \sup_{x \in B(p_i, \frac{3\delta_2}{\delta_1})} \sum_{i=1}^{K} |f^i_j(x)|^2 \geq c(n,v). \tag{6.81}$$

Now we freeze the value $\eta = \eta(n, v)$. (6.68) says ψ_i is of logarithmic growth uniform for all i. By (6.78) and the mean value inequality, we find $C = C(n, v) > 0$ so that for any $R > 0$, if i is sufficiently large,

$$|f^i_j(x)| \leq C(d_i(x)^C + 1) \tag{6.82}$$

for $x \in B(p_i, R)$. By passing to a subsequence, we can assume $(M_i, p_i) \to (M_\infty, p_\infty)$ in the pointed Gromov-Hausdorff sense. Also, f^i_j converges to f^∞_j, which is of polynomial growth of order C on M_∞.

For C in (6.82), let $V = \mathrm{span}\{g \in \mathcal{O}_{2C}(M) | g(p) = 0\}$ and let $k = \dim(V)$. Take a basis g_s of V satisfying

$$\int_{B(p,1)} g_s \overline{g_t} = \delta_{st}. \tag{6.83}$$

To prove Theorem 6.1, it suffices to prove the following:

Proposition 6.2. *There exist constants $R > 0$ and $c > 0$ with $\sum_s |g_s(x)|^2 \geq cd(x, p)^2$ for $d(x, p) \geq R$.*

Proof. Assume the proposition is not true. There exist $r_i \to \infty$ and points x_i with

$$d(p, x_i) = r_i, \sum_s |g_s(x_i)|^2 \leq \frac{r_i^2}{i}. \tag{6.84}$$

We follow the notation from (6.60) to (6.82). For each i, there exists a basis g^i_s of V with

$$\int_{B(p,1)} g^i_s \overline{g^i_t} = \delta_{st}, \qquad \int_{B(p_i,1)} g^i_s \overline{g^i_t} = \lambda^i_{st} \delta_{st}. \tag{6.85}$$

Then (6.83) and (6.85) imply

$$\sum_s |g_s|^2 = \sum_s |g^i_s|^2. \tag{6.86}$$

Note that by the three circle theorem and the mean value inequality,

$$\lambda^i_{ss} \geq c(n, v) r_i^2. \tag{6.87}$$

Then $h^i_s = \frac{g^i_s}{\sqrt{\lambda^i_{ss}}}$ satisfies

$$\int_{B(p_i,1)} h^i_s \overline{h^i_t} = \delta_{st}. \tag{6.88}$$

The three circle theorem and the mean value inequality imply

$$0 < c(n,v) \le \sup_{B(p_i,1)} |h_s^i(x)| \le C(n,v). \tag{6.89}$$

After passing to subsequence, we may assume that $M_i \to M_\infty$ and h_s^i, f_j^i all converge. Say $h_s^i \to h_s^\infty$ and $f_j^i \to f_j^\infty$ uniformly on each compact set. Clearly $h_s^\infty (s = 1, \dots, k)$ are linearly independent on M_∞.

Claim 6.1. *$\mathrm{span}\{f_j^\infty\} \subset \mathrm{span}\{h_s^\infty\}$ on M_∞.*

Proof. Assume the claim is not true. Set $V' = \mathrm{span}\{f_j^\infty, h_s^\infty\}$. Then $\dim(V') > k$. By the three circle theorem, f_j^∞, h_s^∞ are of polynomial growth of order $2C$. Take a basis $u_1, \dots, u_m$ of V', $m \ge k+1$. Therefore, $u_l (1 \le l \le m)$ are of polynomial growth of order $2C$. For any $f \in V'$, f satisfies the three circle theorem. That is, if $M(f,r) = \sup_{B(p_\infty, r)} |f(x)|$, then $\log M(f,r)$ is convex in terms of $\log r$. The reason is that f is a limit of holomorphic functions of polynomial growth on M_i. Write $u_l = \sum_{s=1}^{k} a_l^s h_s^\infty + \sum_{j=1}^{K} b_l^j f_j^\infty$. Here a_l^s, b_l^j are constants. Define $u_l^i = \sum_{s=1}^{k} a_l^s h_s^i + \sum_{j=1}^{K} b_l^j f_j^i$. Then $u_l^i \to u_l$ uniformly on each compact set. As u_l is a basis for V', for sufficiently large i, u_l^i are linearly independent on $B(p_i, 1)$. We can also regard u_l^i as holomorphic functions on $B(p, 3r_i)$ on M. Let v_l^i be a basis of $\mathrm{span}\{u_l^i\}$ with $\int_{B(p,1)} v_l^i \overline{v_t^i} = \delta_{lt}$. Let us write $v_l^i = \sum_{t=1}^{m} C_{lt}^i u_t^i$. Here C_{lt}^i are constants. We are interested in

$$F_{i,l} = \frac{\sup_{B(p_i,2)} |v_l^i|}{\sup_{B(p_i,1)} |v_l^i|} = \frac{\sup_{B(p_i,2)} \left| \sum_{t=1}^{m} C_{lt}^i u_t^i \right|}{\sup_{B(p_i,1)} \left| \sum_{t=1}^{m} C_{lt}^i u_t^i \right|}. \tag{6.90}$$

In the quotient, we can normalize the coefficients C_{lt}^i so that $\sup_{1 \le t \le m} |C_{lt}^i| = 1$. As u_l are linearly independent on M_∞, by a simple compactness argument and the three circle theorem for V' on M_∞, we see that for i sufficiently large, $1 \le l \le m$,

$$F_{i,l} \le (2+\varepsilon)^{2C} \tag{6.91}$$

for any given $\varepsilon > 0$. As before, we can apply the three circle theorem to find a subsequence of v_l^i converging to linearly independent holomorphic functions v_l on M, satisfying $v_l(p) = 0$ and $\deg(v_l) \le 2C$. As l is from 1 to m and $m > k$, this contradicts that $\dim(V) = k$. $\square$

Given Claim 6.1, we find f^i_j is almost in the span$\{h^i_s\}$. More precisely,

$$\lim_{i\to\infty}\sup_{B(p_i,1)}|f^i_j(x)-\sum_s c^i_{js}h^i_s|=0 \tag{6.92}$$

for $c^i_{js}=\int_{B(p_i,1)}f^i_j\overline{h^i_s}$. In particular, $|c^i_{js}|\leq C(n,v)$. By (6.81),

$$\min_{\partial B(p_i,1)}\sum_{j=1}^{K}|f^i_j(x)|^2>\frac{3}{2}\sup_{B\left(p_i,\frac{3\delta_2}{\delta_1}\right)}\sum_{j=1}^{K}|f^i_j(x)|^2\geq c(n,v)>0. \tag{6.93}$$

Hence

$$C(n,v)\min_{\partial B(p_i,1)}\sum_s|h^i_s|^2\geq c(n,v)>0. \tag{6.94}$$

By (6.87),

$$|h^i_s|^2=\frac{|g^i_s|^2}{\lambda^i_{ss}}\leq\frac{|g^i_s|^2}{cr_i^2}. \tag{6.95}$$

Then from (6.86),

$$\min_{\partial B(p,r_i)}\sum_s|g_s|^2=\min_{\partial B(p_i,1)}\sum_s|g_s|^2=\min_{\partial B(p_i,1)}\sum_s|g^i_s|^2\geq c(n,v)r_i^2>0. \tag{6.96}$$

This contradicts (6.84). □

The proof of Theorem 6.1 is complete. □

7. Completion of the proof of Theorem 1.2

First, we prove Theorem 1.2 under the assumption that the manifold has maximal volume growth.

THEOREM 7.1. *Let M be a complete noncompact Kähler manifold with nonnegative bisectional curvature and maximal volume growth. Then M is biholomorphic to an affine algebraic variety. Also the ring of holomorphic functions of polynomial growth is finitely generated.*

Proof. Given any $k\in\mathbb{N}$, let $n_k=\dim_{\mathbb{C}}(\mathcal{O}_k(M))$. Define a holomorphic map from M to $\mathbb{C}^{n_k}$ by $F_k(x)=(g_1(x),\ldots,g_{n_k}(x))$. Here $g_1,\ldots,g_{n_k}$ is a basis for $\mathcal{O}_k(M)$. When k is getting larger, we only add new functions to the basis. That is, we do not change the previous functions. Our goal is to prove that for sufficiently large k, F_k is a biholomorphism to an affine algebraic variety.

Below, the value of k might change from line to line; basically we shall increase its value in finite steps. First assume k is so large that the holomorphic functions constructed in Theorem 6.1 are in $\mathcal{O}_k(M)$ and they separate the tangent space at a point $p\in M$. Let Σ_k be the affine algebraic variety defined the integral ring generated by $g_1,\ldots,g_{n_k}$. Then $\dim(\Sigma_k)=n$, as the transcendental dimension of $(g_1,\ldots,g_{n_k})$ over $\mathbb{C}$ is n. Moreover, $\dim(F_k(M))=n$, as the tangent space at p is separated. By Theorem 6.1, F_k is a proper holomorphic map from M to $\mathbb{C}^{n_k}$. Hence the image of F_k is closed. By the proper mapping theorem, the image of F_k is an analytic subvariety of dimension n. As Σ_k is irreducible, $F_k(M) = \Sigma_k$.

Our argument below is very similar to some parts of [12]. Given any point in Σ_k, the preimage of F_k is a compact subvariety of M, as F_k is proper. As M is exhausted by Stein manifolds Ω_i, the preimages contain only finitely many points. Given a generic point $y \in \Sigma_k$, we can find polynomial growth holomorphic functions separating $F_k^{-1}(y)$. Therefore, by increasing k, we may assume F_k is generically one to one. Note that if $x \in \Sigma_k$ and the preimages of x contain more than one point, then x is in the singular set of Σ_k, say $S(\Sigma_k)$. Write $S(\Sigma_k)$ as a finite union of irreducible algebraic subvarieties $\Sigma'_s (1 \leq s \leq t_k)$. Set $h = \dim(S(\Sigma_k))$. Let us assume $\dim(\Sigma'_s) = \dim(S(\Sigma_k))$ for $1 \leq s \leq r_k \leq t_k$. For a generic point $x \in \Sigma'_s$, the preimages under F_k contain finitely many points. Therefore, we can increase the value of k so that the preimages of x and their tangent spaces are separated. In this way, the dimension of $S(\Sigma_k)$ is decreased. After finitely many steps, F_k becomes a biholomorphism from M to Σ_k which is affine algebraic.

Claim 7.1. *We can identify polynomial growth holomorphic functions on M with regular functions on Σ_k via F_k. Thus $\mathcal{O}_P(M)$ is finitely generated.*

Proof. First, by Theorem 3.2 in [15], regular functions on Σ_k are identified with the affine coordinate ring of Σ_k. Thus, any regular function is of polynomial growth. Recall that the transcendental dimension of $\mathcal{O}_P(M)$ is n over $\mathbb{C}$. By increasing k if necessary, we may assume the affine coordinate functions generate the field of $\mathcal{O}_P(M)$. Then every polynomial growth holomorphic function is rational on Σ_k and hence a regular function on Σ_k. □

The proof of Theorem 7.1 is complete. □

Next, we come to the finite generation in the general case. Let us rewrite Theorem 1.2 as follows:

Corollary 7.1. *Let M be a complete noncompact Kähler manifold with nonnegative bisectional curvature. Then the ring of holomorphic functions of polynomial growth is finitely generated.*

Proof. We first consider the case when the universal cover does not split. By Theorem 2 in [21], if there exists a nonconstant holomorphic function of polynomial growth on M, then M is of maximal volume growth. Then then the result follows from the theorem above.

In the general case, let $\widetilde{M}$ be the universal cover. Let G be the fundamental group of M. Let E be the set of G-invariant holomorphic functions of polynomial growth on $\widetilde{M}$. We can identify E with $\mathcal{O}_P(M)$. Given any $f \in E$, consider

$$u_t^f(x) = \int_{\widetilde{M}} H_{\widetilde{M}}(x, y, t) \log(|f(y)|^2 + 1)dy, \tag{7.1}$$

where $H_{\widetilde{M}}(x, y, t)$ is the heat kernel of $\widetilde{M}$. By Theorem 3.1 in [28], $\sqrt{-1}\partial\overline{\partial}u_t^f \geq 0$ for $t > 0$. Let D_f^t be the null space of $\sqrt{-1}\partial\overline{\partial}u_t^f$. Theorem 3.1 in [28] says D_f^t is a parallel distribution.

Claim 7.2. *D_f^t is invariant for $t > 0$. Then we define $D_f = D_f^t$, $t > 0$.*

Proof. By Theorem 2.1, part (ii) in [28] (see also the second sentence in the proof of Corollary 2.1 in [28]), if $t_1 > t_2 > 0$, then

$$\dim(D_f^{t_1}) \leq \dim(D_f^{t_2}). \tag{7.2}$$

The de Rham theorem says we can write $\widetilde{M} = N_1 \times N_2$ where $D_f^{t_2}$ is the tangent space of N_2. $u_{t_2}^f$ is of logarithmic growth by Proposition A.1. Moreover, $u_{t_2}^f$ is pluriharmonic on each slice $N_2$and hence harmonic on N_2. As N_2 has nonnegative bisectional curvature, the Ricci curvature of N_2 is nonnegative. By a theorem of Cheng-Yau [10], $u_{t_2}^f$ is constant on each slice N_2. That is to say, $u_{t_2}^f$ is a function on N_1. By uniqueness of the heat flow, $u_{t_1}^f$ is also constant on each slice of N_2. Combining this with (7.2), we obtain that $D_f^{t_1} = D_f^{t_2}$. $\square$

Hence, u_t^f is constant on N_2 for $t \geq 0$. This implies f is constant on the factor N_2. Now define the parallel distribution

$$D = \cap_{f \in E} D_f. \tag{7.3}$$

By the de Rham theorem, we can assume $\widetilde{M} = M_1 \times M_2$ where D is the tangent space of M_2. Then, for any $f \in E$, f is constant on the factor M_2. Note that D is invariant under G-action. Fix an inclusion i of a slice: $M_1 \hookrightarrow M_1 \times M_2$. Now for any $g \in G$, $g(i(M_1))$ must be another slice of M_1. Let π be the projection from $M_1 \times M_2$ to M_1. For $x \in M_1$ and $g \in G$, define a holomorphic isometry u_g of M_1 by $u_g(x) = \pi(g(i(x)))$. Of course, u_g is a subgroup of the holomorphic isometry group of M_1. Let G' be the closure of u_g. Then we can identify E with polynomial growth holomorphic functions on M_1 invariant under G'.

Claim 7.3. *G' is a compact group.*

Proof. It suffices to prove that for $x \in M_1$, $u_g(x)$ is bounded for $g \in G$. Assume this is not true. Then there exists a sequence $g_i \in G'$ with $x_i = g_i(x) \to \infty$ on M_1. Let $(U, z_1, \ldots, z_m)$ be a holomorphic chart on M_1 around x with $z(x) = 0$. Let $(U_i = g_i(U), z_s^i = z_s \circ g_i^{-1})$ be the holomorphic chart on U_i. By taking a subsequence if necessary, we may assume U_i are mutually disjoint.

We will use some construction in [26]. First, pick finitely many $f_j \in E$ so that $\sqrt{-1}\partial\overline{\partial}\sum_j u_1^{f_j} > 0$ on M_1. Let $u = \sum_j u_1^{f_j}$. Then u is a strictly plurisubharmonic function on M_1 with logarithmic growth. Moreover, u is invariant under G' action. Let $U^2 \subset\subset U^1 \subset\subset U$ be open sets containing x. Consider a smooth cut-off function φ with $\varphi = 1$ in U^2 and $\varphi = 0$ in $M_1 \backslash U^1$. Define $\varphi_i = \varphi \circ g_i^{-1}$. Then φ_i is supported in U_i. Let

$$\psi(x) = 4m \sum_i \varphi_i(x) \log\left(\sum_{s=1}^m |z_s^i(x)|^2\right) + Cu(x). \tag{7.4}$$

Here C is a positive constant so that $\sqrt{-1}\partial\overline{\partial}\psi \geq \omega$ on U_i. ω is the Kähler form on M_1. Then $\sqrt{-1}\partial\overline{\partial}\psi > 0$ on M_1. Now we solve the $\overline{\partial}$-problem $\overline{\partial}h_i = \overline{\partial}\varphi_i$ with

$$\int_{M_1} |h_i|^2 e^{-\psi} \leq \int_{M_1} |\overline{\partial}\varphi_i|^2 e^{-\psi}. \tag{7.5}$$

One sees that $\lambda_i = h_i - \varphi_i$ are holomorphic functions of polynomial growth. The growth orders are uniformly bounded. Moreover, $h_i(x_k) = 0$ for all $k \in \mathbb{N}$. Thus λ_i are linearly independent, as $\lambda_i(x_j) = h_i(x_j) - \varphi_i(x_j) = -\delta_{ij}$. This contradicts Theorem 2.5. □

Claim 7.4. *M_1 is of maximal volume growth.*

Proof. As M_1 is simply connected, write M_1 as a product of irreducible Kähler manifolds. For each factor, there exists a polynomial growth holomorphic function on M_1 which is not constant. Then it must be of maximal volume growth by Theorem 2 in [21]. □

By Claim 7.4 and Theorem 7.1, $\mathcal{O}_P(M_1)$ is finitely generated. $\mathcal{O}_P(M)$ is just the subring of $\mathcal{O}_P(M_1)$ invariant under G'. Since G' is compact, the finite generation of $\mathcal{O}_P(M)$ follows from a theorem of Nagata [25]. The detailed argument is in Appendix B.

□

Appendix A. Proof of Theorem 2.2

Proof. This part basically follows from [28]. For any $a > 0$, $\eta(x,t)$ satisfies

$$\left(\frac{\partial}{\partial t} - \Delta\right)\eta_{\gamma\overline{\delta}} = R_{\beta\overline{\alpha}\gamma\overline{\delta}}\eta_{\alpha\overline{\beta}} - \frac{1}{2}(R_{\gamma\overline{p}}\eta_{p\overline{\delta}} + R_{p\overline{\delta}}\eta_{\gamma\overline{p}}) \tag{A.1}$$

and

$$\int_M \|\eta(x,0)\| \exp(-ar^2(x))dx < \infty. \tag{A.2}$$

For the moment, we will assume

$$\lim_{r\to\infty}\inf\int_0^T\int_{B(p,r)}\|\eta\|^2(x,t)\exp(-ar^2(x))dxdt<\infty. \tag{A.3}$$

The proof is given at the end of this section.

Recall Corollary 1.1 in [28] with simplified the assumptions:

Proposition A.1. *Let (M^n,p) be a complete noncompact Kähler manifold with nonnegative bisectional curvature. $r(x)=d(x,p)$. Let u be a nonnegative function on M satisfying*

$$u(x)\le\exp(a+br(x)) \tag{A.4}$$

for some constants $a,b>0$. Let

$$v(x,t)=\int_M H(x,y,t)u(y)dy. \tag{A.5}$$

Here H is the heat kernel on M. Then given any $\varepsilon>0$, $T>0$, there exists $C(n,\varepsilon,a,b)>0$ such that for any x satisfying $r=r(x)\ge\sqrt{T}$,

$$C_1(n,\varepsilon)\inf_{B(x,\varepsilon r)}u\le v(x,t)\le C(n,\varepsilon,a,b)+\sup_{B(x,\varepsilon r)}u \tag{A.6}$$

for $0\le t\le T$. Here $C_1(n,\varepsilon)>0$.

Fix a point $p\in M$. Let $r(x)=d(x,p)$. Let $\phi(x)=\exp(r(x))$. Define

$$\phi(x,t)=e^t\int_M H(x,y,t)\phi(y)dy. \tag{A.7}$$

Then

$$\left(\frac{\partial}{\partial t}-\Delta\right)\phi=\phi \tag{A.8}$$

and

$$\phi(x,t)\ge ce^{c_1r} \tag{A.9}$$

for $0\le t\le T$, by Proposition A.1. Here c,c_1 are positive constants. Let

$$h(x,t)=\int_M H(x,y,t)\|\eta\|(y,0)dy. \tag{A.10}$$

The proposition below is just Lemma 2.2 in [28]. Note that we used that u has compact support on M.

Proposition A.2. *There exists a positive function $\tau(R)$ so that for $0\le t\le T$, $h(x,t)\le\tau(R)$ for $x\in B(p,2R)\backslash B\left(p,\frac{R}{2}\right)$. Moreover, $\lim_{R\to\infty}\tau(R)=0$.*

The next proposition is Lemma 2.1 in [28]. Note that (A.1), (A.2) and (A.3) are used.

Proposition A.3. *$\|\eta\|(x,t)$ is a subsolution of the heat equation. Moreover, $\|\eta\|(x,t) \le h(x,t)$.*

Given $\varepsilon > 0$, define

$$(\tilde{\eta})_{\alpha\overline{\beta}} = \eta_{\alpha\overline{\beta}} + (\varepsilon\phi - \lambda(x,t))g_{\alpha\overline{\beta}}. \tag{A.11}$$

At $t = 0$, $\tilde{\eta} > 0$. Also, for $0 \le t \le T$, if R is sufficiently large, by Proposition A.2, we have $\tilde{\eta} > 0$ on $\partial B(p,R)$. Suppose that at some $t_0 \in [0,T]$, $\tilde{\eta}(x_0,t_0) < 0$ for $x_0 \in \overline{B(p,R)}$. Then there exists $0 \le t_1 < T$ with $\tilde{\eta}(x,t) \ge 0$ for $x \in B(p,R)$ and $0 \le t \le t_1$. Moreover, the minimum eigenvalue of $\tilde{\eta}(x_1,t_1)$ is zero for some $x_1 \in B(p,R)$. (Note that x_1 cannot be on the boundary.) Now we apply the maximal principle. Let us assume

$$\tilde{\eta}(x_1,t_1)_{\gamma\overline{\gamma}} = 0 \tag{A.12}$$

for $\gamma \in T^{1,0}_{x_1}M, |\gamma| = 1$. We may diagonalize $\tilde{\eta}$ at (x_1,t_1). Of course, we can assume γ is one of the basis of the holomorphic tangent space. Then at (x_1,t_1),

$$\left(\frac{\partial}{\partial t} - \Delta\right)\tilde{\eta}_{\gamma\overline{\gamma}} \le 0. \tag{A.13}$$

On the other hand, by (A.1),

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta\right)\eta_{\gamma\overline{\gamma}} &= \sum_\alpha R_{\gamma\overline{\gamma}\alpha\overline{\alpha}}\eta_{\alpha\overline{\alpha}} - \sum_\alpha R_{\gamma\overline{\gamma}\alpha\overline{\alpha}}\eta_{\gamma\overline{\gamma}} \\ &= \sum_\alpha R_{\gamma\overline{\gamma}\alpha\overline{\alpha}}(\tilde{\eta}_{\alpha\overline{\alpha}} - \tilde{\eta}_{\gamma\overline{\gamma}}) \\ &\ge 0.\end{aligned} \tag{A.14}$$

Note that

$$\left(\frac{\partial}{\partial t} - \Delta\right)((\varepsilon\phi - \lambda(x,t))g_{\gamma\overline{\gamma}}) = \varepsilon\phi g_{\gamma\overline{\gamma}} > 0. \tag{A.15}$$

Hence at (x_1,t_1),

$$\left(\frac{\partial}{\partial t} - \Delta\right)\tilde{\eta}_{\gamma\overline{\gamma}} > 0. \tag{A.16}$$

This is a contradiction. Now let $R \to \infty$ and then $\varepsilon \to 0$. We proved that $\eta - \lambda(x,t)g_{\alpha\overline{\beta}} \ge 0$ for $0 \le t \le T$.

Next we verify (A.3). Basically we follow pages 487–488 in [28]. Note that our condition is more special. First, we have that $|v(x,t)| \le C$ for all x,t, as u has compact support. Note that

$$\left(\Delta - \frac{\partial}{\partial t}\right)v^2 = 2|\nabla v|^2. \tag{A.17}$$

Multiplying (A.17) by suitable cut-off functions, using integration by parts, we find

$$\int_0^T \fint_{B(p,r)} |\nabla v|^2 \leq C_1 \left(r^{-2} \int_0^{2T} \fint_{B(p,2r)} v^2 + \fint_{B(p,r)} u^2 \right) \leq C_2(T+1) \tag{A.18}$$

for $r \geq 1$. The Bochner formula gives

$$\left(\Delta - \frac{\partial}{\partial t} \right) |\nabla v|^2 \geq 2|\nabla^2 v|^2. \tag{A.19}$$

Multiplying (A.19) by suitable cut-off functions, using integration by parts, we find

(A.20)

$$\int_0^T \fint_{B(p,r)} |\nabla^2 v|^2 \leq C_3 \left(r^{-2} \int_0^{2T} \fint_{B(p,2r)} |\nabla v|^2 + \fint_{B(p,r)} |\nabla u|^2 \right) \leq C_2(T+1)$$

for $r \geq 1$. From this, (A.3) follows easily. $\square$

Appendix B. Some algebraic results of Nagata

We continue the proof of Theorem 1.2. The ring $R = \mathcal{O}_P(M_1)$ is finitely generated. We may assume the generators are in $F = \mathcal{O}_d(M_1)$ for some $d > 0$. Let $g_1, \ldots, g_l$ be a basis for F. Obviously F is an invariant space of G'. Then we may think $\mathcal{O}_P(M_1)$ is $\mathbb{C}[g_1, \ldots, g_l]/\alpha$. Here α is an ideal. Then the G' action on R is induced by the representation $G' \to \mathrm{GL}(l, \mathbb{C})$. Let $I_{G'}(R)$ be the subring of R fixed by G'. In [25, p. 370], the following definition was made:

Definition B.1. A group G is reductive if every rational representation is completely reducible.

It was pointed out on page 370 of [25] that all rational representations of G in [25] are given by some specific finite-dimensional representations of G. In our case, as G' is compact, every finite-dimensional representation (complex) is completely reducible. Therefore, according to the definition above, G' is reductive. In [25], the following was proved:

THEOREM B.1 (Nagata). *$I_G(R)$ is finitely generated if G is semi-reductive.*

It was pointed out in the first sentence of part 5, page 373 of [25] that a reductive group is obviously semi-reductive. Putting all these things together, we proved the finite generation of $I_{G'}(R) = \mathcal{O}_P(M)$.

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