Annals of Mathematics **183** (2016), 1057–1071
http://dx.doi.org/10.4007/annals.2016.183.3.7

Errata of "Isoparametric hypersurfaces with $(g, m) = (6, 2)$"

By Reiko Miyaoka

Abstract

We give a correction of the proof of the homogeneity of isoparametric hypersurfaces with $(g, m) = (6, 2)$.

1. Introduction

In [2], [3], and [4], we discuss the homogeneity of isoparametric hypersurfaces M with six principal curvatures by investigating the kernel of the shape operators of the focal submanifolds. In fact, M is homogeneous if and only if the kernel is independent of the normal direction [1], [3, §15]. Using this, we reprove the homogeneity for multiplicity $m = 1$ in [2], [4] and try to prove it for $m = 2$ in [3]. However, in Sections 8 and 13.3 of [3], there are some inappropriate arguments.

The purpose of this paper is to correct Section 8 and Proposition 13.6 in [3], where the argument to exclude the case $\dim K = 1, 2$ or $\dim E(c) = 4$ fails. The correction is now achieved. In Section 3 we rewrite the entire Section 8 [3]. We exclude the case $\dim E = 4$ in Section 5 and the case $\dim E = 5$ in Section 6. Then in Sections 7 and 8, we settle the case $\dim E(c)$ or $\dim E = 6$.

Thus we obtain

Theorem 1.1. *Isoparametric hypersurfaces with* $(g, m) = (6, 2)$ *are homogeneous.*

Remark 1.2. In addition to the revision of Sections 8 and 13.3 of [3], we need some minor changes as follows: There are typos: In (i) on page 81, Y_1^V and Y_2^V should be $Y_{\bar{1}}^V$ and $Y_{\bar{2}}^V$. In (94) on page 84, $\frac{1}{\sqrt{3}}$ in $\hat{e}_2$ and $\hat{e}_4$ should be $\sqrt{3}$. The notation $\boldsymbol{v}_i$ in the fourth to ninth lines of page 95 might be confusing, and we had better replace it by, say, $\boldsymbol{w}_i$. All other parts of [3] are correct as they are.

Partly supported by Grants-in-Aid for Scientific Research, 2334012, The Ministry of Education, Japan.

Acknowledgements. The author is grateful to Uwe Abresch and Anna Siffert for discovering the error and bringing it to her attention. She really would like to thank the referees for patient checking and suitable comments on this Errata. She also would like to express her hearty thanks to Annals of Mathematics for permission to submit the correction.

2. A brief summary of Sections 1–7 in [3]

Let M be an isoparametric hypersurface in S^{13} with $(g, m) = (6, 2)$, where g is the number of distinct principal curvatures and m is the multiplicity which is common among different principal curvatures when $g = 6$. The ambient sphere S^{13} is singularly foliated by parallel hypersurfaces of M and two focal submanifolds $M_\pm$. Choosing a unit normal vector field ξ of M, we denote the principal curvatures by $\lambda_1 > \cdots > \lambda_6$ and their curvature distributions by D_i, $i = 1, \ldots, 6$. We take an orthonormal frame $e_i, e_{\bar{i}}$ of each D_i. We write $\underline{i}$ for i and $\bar{i}$. Consider the focal submanifold M_+ at which each leaf $L_6(p)$ of D_6 collapses into a point $\bar{p} = \cos\theta\, p + \sin\theta\, \xi_p$ where $p \in M$ and $\theta = \cot^{-1}\lambda_6$. Then $T_{\bar{p}}M_+ = \oplus_{i=1}^5 D_i(q)$ and $T_{\bar{p}}^\perp M_+ = \mathbb{R}\eta_q \oplus D_6(q)$ hold for all $q \in D_6(p)$ and $\eta_q = -\sin\theta q + \cos\theta\xi_q$. Another focal submanifold M_- is obtained by replacing D_6 by D_1 and θ by $\bar{\theta} = \cot^{-1}\lambda_1$. Note that $T_{\bar{p}}M_- = \oplus_{i=2}^6 D_i(q)$ and $T_{\bar{p}}^\perp M_- = \mathbb{R}\bar{\eta}_q \oplus D_1(q)$ for all $q \in D_1(p)$ and $\bar{\eta}_q = -\sin\bar{\theta}q + \cos\bar{\theta}\xi_q$. By the argument in Sections 1–7 and 15 of [3], we know the following:

FACT. (1) *The shape operators $B_{\boldsymbol{n}}$ of M_+ with respect to a unit normal $\boldsymbol{n} \in T^\perp M_+$ are isospectral with eigenvalues $\mu_1 = \sqrt{3} = -\mu_5$, $\mu_2 = 1/\sqrt{3} = -\mu_4$ and $\mu_3 = 0$. The eigenspace of μ_i of B_{η_p} is given by $D_i(p)$.*

(2) *At the focal point $\bar{p}$, the unit sphere S^2 in $T_{\bar{p}}^\perp M_+$ is identified with the leaf $L_6(p)$ of D_6. Take $\zeta = e_6(p)$ in $T_{\bar{p}}^\perp M_+$. The geodesic $c = \{p(t)\}$ of $S^2 = L_6(p)$ through p in the direction ζ corresponds to a one parameter family of normal vectors $\cos t\,\eta + \sin t\,\zeta$ of M_+. Then the shape operator $L(t) = \cos tB_\eta + \sin tB_\zeta$ of M_+ has $\ker L(t) = D_3(p(t))$.*

(3) *M is homogeneous if and only if $\ker L(t)$ is independent of t and ζ, namely, if and only if D_3 is invariant on each L_6.*

All these hold if we replace M_+ by M_- and index i by $i + 1$ modulo 6.

Now, for a geodesic c of $L_6(p)$, put

$$E(c) = \mathrm{span}_t \ker L(t). \tag{1}$$

Then Theorem 1.1 is proved if we show $\dim E(c) = 2$ for any c of any L_6 (see [3, §15]). Recall [3, (42)]

$$\begin{aligned} E(c) &= \mathrm{span}\{e_{\underline{3}}(q), \nabla_{e_6}^k e_{\underline{3}}(q),\, k = 1, 2, \ldots\} \\ W(c) &= \mathrm{span}\{\bar{\nabla}_{e_{\underline{3}}} e_6(q), \nabla_{e_6}^k \bar{\nabla}_{e_{\underline{3}}} e_6(q),\, k = 1, 2, \ldots\}, \end{aligned} \tag{2}$$

which do not depend on the choice of $q \in c$. Note that

Lemma 2.1 ([3, Lemmas 5.3, 5.4, and (46) of Lemma 6.1]). *$W(c) \subset E(c)^{\perp}$. Moreover, $L(t)$ maps $E(c)$ onto $W(c)$ for any t, and $\dim W(c) = \dim E(c) - 2$ holds.*

For a fixed $L_6(p)$, we put

$$E = \operatorname{span}\{E(c) \mid c: \text{ a geodesic of } L_6(p)\}. \tag{3}$$

3. **Dimension of $E(c)$**

To investigate the dimension of $E(c)$ or E under the supposition $\dim E(c) > 2$, we need a special frame of $D_3(t)$ along a geodesic $c = \{p(t)\}$ of $L_6(p)$, parametrized by t so that $p(0) = p(2\pi)$. For a vector field $v(t)$ along c, we call $v(t)$ *even* when $v(t+\pi) = v(t)$, and *odd* when $v(t+\pi) = -v(t)$. We sometimes denote $p(t) = c(t)$.

Lemma 3.1. *If $e_3(t)$ is an even (odd, resp.) vector along c, then $e_3(t)$, $\nabla_{e_6} e_3(t)$, $\nabla^2_{e_6} e_3(t), \dots$ are all even (odd, resp.) vectors. On the other hand, $\bar{\nabla}_{e_3} e_6(t)$, $\nabla_{e_6} \bar{\nabla}_{e_3} e_6(t)$, $\nabla^2_{e_6} \bar{\nabla}_{e_3} e_6(t)$, $\dots$ are all odd (even, resp.) vectors.*

Proof. The former is clear from $\nabla^k_{e_6} e_3(t+\pi) = \nabla^k_{e_6} e_3(t)$. The latter follows from $L(t+\pi) = -L(t)$ and $L(t)(\nabla_{e_6} e_3(t)) = c_1 \bar{\nabla}_{e_3} e_6(t)$ (see [3, Lemma 5.1, (36)]). Then its derivatives in the direction $e_6(t)$ are all odd. The case when $e_3(t)$ is odd is similar. □

Lemma 3.2. *$\dim E(c)$ must be even.*

Proof. There are no odd dimensional subspace of TM_+ parallel along c and consisting of odd vectors, because of the continuity of the determinant of a moving frame. By [3, Lemma 7.7], we can choose $e_3(t), e_{\bar{3}}(t)$ so that $E(c)$ consists of all even or all odd vectors. By Lemma 3.1, evenness and oddness of the vectors in $E(c)$ and in $W(c)$ are opposite. Since both $E(c)$ and $W(c)$ are parallel and $\dim W(c) = \dim E(c) - 2$ (Lemma 2.1), $\dim E(c)$ must be even. □

Lemma 3.3. *If a differentiable field $e_3(t)$ spans a 2-dimensional space $K = \operatorname{span}\{e_3(t)\}$, then $e_3(t)$ is an odd vector.*

Remark 3.4. A typical case is when $e_3(t) = \cos t\, \boldsymbol{u} + \sin t\, \boldsymbol{v}$ for orthonormal vectors $\boldsymbol{u}$ and $\boldsymbol{v}$. Usually, the coefficient functions are general odd functions and $\boldsymbol{u}$ and $\boldsymbol{v}$ are not necessarily orthonormal.

Proof. Assume $\dim K = 2$; then it follows $\nabla_{e_6} e_3(p) \not\equiv 0$ modulo $D_3(p)$ ([3, Rem. 5.2]). Using $q = p(\pi/2)$, we can express $K = \operatorname{span}\{e_3(p), e_3(q)\}$. Thus we have

$$e_3(t) = a(t) e_3(p) + b(t) e_3(q) \in K. \tag{4}$$

Recall [3, (37)]

$$B_\zeta(e_3(p)) = -\bar{\nabla}_{e_3} e_6(p).$$

Because $e_3(q) \in \mathrm{Ker} L(\pi/2) = \ker B_\zeta$, exchanging p and q, we have

$$B_\eta(e_3(q)) = \bar{\nabla}_{e_3} e_6(q),$$

since $B_\eta = -L(\pi/2 + \pi/2)$ and $B_\zeta = L(\pi/2)$. Therefore, denoting $c(t) = \cos t$ and $s(t) = \sin t$, by (4) we have

$$\begin{aligned} 0 = L(t)e_3(t) &= (c(t)B_\eta + s(t)B_\zeta)(a(t)e_3(p) + b(t)e_3(q)) \\ &= b(t)c(t)B_\eta(e_3(q)) + a(t)s(t)B_\zeta(e_3(p)) \\ &= b(t)c(t)\bar{\nabla}_{e_3} e_6(q) - a(t)s(t)\bar{\nabla}_{e_3} e_6(p) \end{aligned}$$

for all t. From this it follows

$$\bar{\nabla}_{e_3} e_6(q) = u\bar{\nabla}_{e_3} e_6(p) \tag{5}$$

for some nonzero u. Thus $W = L(t)K$ is a 1-dimensional space consisting of $\bar{\nabla}_{e_3} e_6(t)$ which is a nonzero and hence a positive scalar multiple of $\bar{\nabla}_{e_3} e_6(p)$ (see [3, Rem. 5.2]). Then $\bar{\nabla}_{e_3} e_6(t)$ is an even vector, and so $e_3(t)$ is an odd vector. □

Lemma 3.4. *If there exists a constant e_3 along two geodesics c and $\bar{c}$ of $L_6(p)$, then e_3 is constant all over $L_6(p)$.*

Proof. Recall that if e_3 coincides at two nonantipodal points on a geodesic c, then e_3 is constant along c ([3, Lemma 7.1]). Thus if e_3 is constant along $c \cup \bar{c}$, e_3 is constant along any geodesic joining a point on c and a point on c', and hence by the continuity, constant all over L_6. □

Let $e_3(t), e_{\bar{3}}(t)$ be an orthonormal frame of $D_3(t)$ along a geodesic $c(t)$. For each t, put $W(t) = \mathrm{span}\,\{\bar{\nabla}_{e_3} e_6(t), \bar{\nabla}_{e_{\bar{3}}} e_6(t)\} \subset W(c)$.

Lemma 3.5. $\dim W(t)$ *is independent of t and takes values* 0, 1 *or* 2.

Proof. If $\bar{\nabla}_{e_3} e_6(t_0)$ and $\bar{\nabla}_{e_{\bar{3}}} e_6(t_0)$ are dependent at some t_0, then there exists $e_3'(t_0) = ae_3(t_0) + be_{\bar{3}}(t_0)$ such that $\nabla_{e_3'} e_6 = 0$, and hence e_3' is constant along c (see [3, Lemma 7.1]). Thus $\dim W(t) = 1$ unless $e_{\bar{3}}'(t)$, which is orthogonal to $e_3'(t_0)$, is also constant, in which case $\dim W(t) = 0$. Therefore, we have $\dim W(t) = 0$, 1 or 2 independent of t. □

Let Γ be the space of oriented geodesics of $L_6(p)$ for each p, which is diffeomorphic to S^2. Then $d : \Gamma \ni c \mapsto d(c) = \dim W(t) \in \{0, 1, 2\}$ is well defined by this lemma and is lower-semicontinuous. Thus $\mathcal{U} = \{c \in \Gamma \mid d(c) = \max_\Gamma d\}$ is an open subset of Γ. When $\max_\Gamma d = 0$, $D_3 = D_3(p)$ is constant along $L_3(p)$. Consider the following cases:

(i) $\max_\Gamma d = 1$,
(ii) $\max_\Gamma d = 2$.

LEMMA 3.6. *When* (i) *is the case, there exists* e_3 *which is constant all over* $L_6(p)$.

Proof. Since $\mathcal{U}$ is open, we may assume that a family of geodesics c^s through p in the direction $e_6^s(p) = \cos s e_6(p) + \sin s e_{\bar{6}}(p)$ belongs to $\mathcal{U}$. Then for each s, some $e_3^s(p) \in D_3(p)$ is constant along c^s. If $e_3^0(p) = e_3^s(p)$ holds for some $0 < s < \pi$, then $e_3 = e_3^0(p)$ is constant all over $L_6(p)$ by Lemma 3.4.

When $e_3^0(p)$ and $e_3^s(p)$ are independent in $D_3(p)$ for all $s \not\equiv 0$ modulo π, $e_3^s(p)$ lies in $D_3(p) \cap D_3(p_s)$ for each $p_s \in c^s \cap \gamma$, where γ is any fixed geodesic transversal to c^s. Hence $e_3^s(p) \in E(\gamma)$ spans the 2-dimensional space $K = D_3(p)$ along γ, where K is as in Lemma 3.3. Also, without loss of generality, we may consider that there exists a constant $e_{\bar{3}}$ along γ, and so $E(\gamma) \subset D_3(p) + \{e_{\bar{3}}\}$. However since $\dim E(\gamma)$ is even (Lemma 3.2), this implies $E(\gamma) = D_3(p)$. Because γ is any geodesic transversal to c^s, $E = D_3(p)$ follows from [3, Lemma 7.3], which is not the case. □

PROPOSITION 3.7. *If there exists some geodesic* c *of* $L_6(p)$ *such that* $\dim E(c) > 2$, *then* (i) *never occurs on* $M_\pm$.

Proof. Note that $\dim F(\gamma) > 2$ also holds by [3, Lemma 7.6]. We may consider $d(\gamma)$ defined for a geodesic γ of $L_1(p)$, where (i) or (ii) occurs similarly. Assume (i) is the case for M_-. Choose any $p_1 \in L_6(p)$, and let p_3 be as in [3, Fig. 1]. Then on $L_1(p_3)$, there exists $e_4(p_3)$ which is constant all over $L_1(p_3)$ by the previous lemma, and so is $e_6(p_1)$ all over $L_3(p_1)$. This means $0 = \nabla_{e_{\underline{1}}} e_4(p_3) = \nabla_{e_{\underline{3}}} e_6(p_1)$, and hence along the geodesic c of $L_6(p_1)$ in the direction e_6, D_3 is constant ([3, Rem. 5.2]). Since $p_1 \in L_6(p)$ is arbitrarily, this means that at each point of $L_6(p)$, there exists a geodesic along which D_3 is constant. Thus by [3, Lemma 7.3], $\dim E = 2$ follows, a contradiction. Thus (i) cannot occur on M_-, and neither on M_+. □

LEMMA 3.8. *When* (ii) *is the case, the subset* $\mathcal{U}_1 = \{c \in \Gamma \mid d(c) \leq 1\}$ *has no interior points.*

Proof. Lemma 3.6 and the proof of Proposition 3.7 are valid on $\mathcal{U}_1$ if it has interior points. □

We call $c \in \mathcal{U}$ "generic." Up to here, we do not assume a specific value of $\dim E(c)$.

4. $\dim E(c) = 4$

When $\dim E(c) > 2$ for some geodesic c of $L_6(p)$, we only need to consider the case (ii) by Proposition 3.7.

LEMMA 4.1. *When* $\dim E(c) = 4$ *for* $c \in \mathcal{U}$, *we can take* $e_3(t)$ *so that* $\bar{\nabla}_{e_3} e_6(t)$ *is parallel to* $\bar{\nabla}_{e_3} e_6(p)$, *and* $K = \mathrm{span}_t\{e_3(t)\}$ *is of dimension* 2. *We* *can express* $e_3(t) = a(t)e_3(p) + b(t)\nabla_{e_6}e_3(p)$, *or* $\tilde{a}(t)e_3(p) + \tilde{b}(t)e_3(q)$, *where* $a(t), b(t), \tilde{a}(t), \tilde{b}(t)$ *are odd functions, and* $q \in c$ *is not antipodal to* p.

Proof. Since (ii) is the case, $\dim W(t) = 2$ for each t. Since $W(t)$ and $\bar{\nabla}_{e_3}e_6(p)$ are contained in $W(c)$ which is of dimension 2 (Lemma 2.1), we can find $\tilde{e}_3(t)$ so that $\bar{\nabla}_{\tilde{e}_3}e_6(t)$ is parallel to $\bar{\nabla}_{e_3}e_6(p)$. We rewrite $\tilde{e}_3(t)$ by $e_3(t)$, and put $K = \mathrm{span}_t\{e_3(t)\}$. From $\dim L(t)K = 1$, $\dim K = 2$ or 3 follows. If $\dim K = 3$, $\ker L(t) \subset K$ for any t, which contradicts that $e_{\bar{3}}(p)$ is not contained in K, since $\bar{\nabla}_{e_{\bar{3}}}e_6(p)$ is independent of $\bar{\nabla}_{e_3}e_6(p)$ (see Lemma 7.1 [3]). The remaining part is as in the proof of Lemma 3.3. □

Remark 4.2. Replacing $e_3(t)$ by $e_{\bar{3}}(t)$, we may consider that $e_{\bar{3}}(t)$ also spans a 2-dimensional subspace $K_2(c)$ of $E(c)$. Thus we have $E(c) = K_1(c) + K_2(c)$, which is not necessarily an orthogonal decomposition, where

$$K_1(c) = \mathrm{span}\{e_3(p), \nabla_{e_6}e_3(p)\}, \quad K_2(c) = \mathrm{span}\{e_{\bar{3}}(p), \nabla_{e_6}e_{\bar{3}}(p)\}.$$

5. $\dim E = 4$

In this section, we exclude the case $\dim E = 4$ where $E = \mathrm{span}_c\, E(c)$.

Suppose $\dim E = 4$, and let S^3_E be the unit sphere of $E \cong \mathbb{R}^4$. For each $x \in L_6(p)$, consider the unit circle $S^1_x \subset D_3(x) \subset E$, where $D_3(x) = \ker B_{\eta_x}$.

When there is no constant e_3 along *any* geodesic of $L_6(p)$, S^1_x does not intersect S^1_y for x, y belonging to an open hemisphere U of $L_6(p)$, since $e_3(x) = e_3(y)$ implies that e_3 is constant along the geodesic joining x and y; see [3, Lemma 7.1]. Thus if y moves in an open neighborhood $U' \subset U$ of x, namely, in 2-parameters (s,t), S^1_y moves in 2-parameters in S^3_E without intersection continuously and hence generates an open neighborhood $\Omega \cong U' \times S^1$ of $e_3(x)$ in S^3_E.

Lemma 5.1. *When* $\dim E = 4$, *let* $S = \cup_{x \in L_6(p)} S^1_x \subset S^3_E$. *If along* any *geodesic of* $L_6(p)$ *there is no constant* e_3, *then* $S = S^3_E$.

Proof. Obviously, S is a nonempty closed subset of S^3_E. On the other hand, for $e_3(x) \in S$ at $x \in L_6(p)$, the above Ω is an open neighborhood of $e_3(x)$ contained in S. Hence S is open. Since S^3_E is connected, the lemma follows. □

Lemma 5.2. *When* $\dim E = 4$, *there exists a constant* e_3 *along some geodesic* c.

Proof. We have a rank 2 vector bundle over $L_6(p)$ with fiber $D_3(x)$ at $x \in L_6(p)$. Suppose that along any geodesic of $D_6(p)$, there is no constant e_3. Then for any $v \in S^3_E$, there exists $x \in L_6(p)$ such that $e_3(x) = v$ by Lemma 5.1. Here, for any antipodal pair $x, -x$ of $L_6(p)$, $D_3(x) = D_3(-x)$ and so $S^1_x = S^1_{-x}$ holds. On the other hand, under our assumption, $D_3(y) \cap D_3(x) = \{0\}$ if $y \neq -x$ and so $S^1_x \cap S^1_y = \emptyset$.

Thus we can define $\pi : S^3_E \to L_6(p)/\mathbb{Z}_2$ with the local triviality $\pi^{-1}(U') \cong U' \times S^1$ where U' is as above, and obtain an S^1 fibration $\pi : S^3_E \to L_6(p)/\mathbb{Z}_2 \cong S^2/\mathbb{Z}_2 = \mathbb{R}P^2$. However, this is impossible by the Thom-Gysin sequence. Namely, if there exists an S^1 bundle $(S^3_E, \mathbb{R}P^2, S^1)$, in the exact sequence for the $\mathbb{Z}_2$ homology of this bundle,

$$\to H_q(S^3_E) \to H_q(\mathbb{R}P^2) \to H_{q-2}(\mathbb{R}P^2) \to H_{q-1}(S^3_E) \to,$$

putting $q = 3$, we have a contradiction. □

Let c be a geodesic appearing in the lemma on which $e_3(t)$ is constant, or equally, $\nabla_{e_6} e_3(t) = 0$ holds. Let $p \in c$ and c be in the direction e_6. Along a *generic* geodesic c^s $(s \neq 0, \pi)$ in the direction $e^s_6 = \cos s e_6 + \sin s\, e_{\bar{6}}$ at p, take $e^s_3(t)$ spanning the 2-dimensional space $K^s_1 = \{e_3(p), \nabla_{e^s_6} e_3(p)\}$, which is possible by Proposition 3.7. Here, K^s_1 is independent of $s(\neq 0, \pi)$, because

$$\nabla_{e^s_6} e_3(p) = \cos s \nabla_{e_6} e_3(p) + \sin s \nabla_{e_{\bar{6}}} e_3(p) = \sin s \nabla_{e_{\bar{6}}} e_3(p).$$

Thus for any $s, s'(\neq 0, \pi)$ and $q \in c^s$, there exists $x \in c^{s'}$ such that $e^s_3(q) = e^{s'}_3(x)$ (see Lemma 4.1).

Now, take $q \in L_6(p) \setminus c$ first, and let c^s be the geodesic through p, q. Then above argument implies that for any $s'(\neq 0, \pi, s)$, there exists $x \in c^{s'}$ such that $e_3(q) = e_3(x)$. Hence e_3 is constant along the geodesic γ joining q and x by [3, Lemma 7.1]. As q is arbitrary, this implies the case (i), which contradicts Proposition 3.7. Thus we obtain

Proposition 5.3. *Neither* $\dim E = 4$ *nor* $\dim F = 4$ *can occur.*

6. $\dim E(c) = 4$ and $\dim E > 4$

Next, when $\dim E(c) = 4$, we show $\dim E = 6$. Along generic geodesics c and $\bar{c}$ through p, put

$$(6)\quad \hat{E} = E(c) + E(\bar{c}) = D_3(p) + \operatorname{span}\{\nabla_{e_6} e_3(p), \nabla_{e_6} e_{\bar{3}}(p), \nabla_{e_{\bar{6}}} e_3(p), \nabla_{e_{\bar{6}}} e_{\bar{3}}(p)\}.$$

Lemma 6.1. $\hat{E} = E$ *and* $\dim E = 6$.

Proof. Let c^s be the geodesic through p in the direction $e^s_6 = \cos s\, e_6 + \sin s\, e_{\bar{6}}$. By Proposition 3.7 and Lemma 4.1, it is easy to see $E(c^s) \subset \hat{E}$. For any geodesic γ transversal to c^s, take $p^s \in c^s \cap \gamma$. Then from $D_3(p^s) \subset E(c^s) \subset \hat{E}$ for every s, we know $E(\gamma) \subset \hat{E}$. Since γ is arbitrary, we conclude $\hat{E} = E = \operatorname{span}_\gamma E(\gamma)$, which is parallel along $L_6(p)$. By Lemma 4.1 again, vectors spanning $\hat{E}$ in (6) are odd. Thus we obtain $\dim E = 6$ by Proposition 5.3. □

Now, put

$$W = \operatorname{span}_{s,t}\{\bar{\nabla}_{e_{\underline{3}}} e^s_6(t)\} = \operatorname{span}\{\bar{\nabla}_{e_3} e_6(p), \bar{\nabla}_{e_{\bar{3}}} e_6(p), \bar{\nabla}_{e_3} e_{\bar{6}}(p), \bar{\nabla}_{e_{\bar{3}}} e_{\bar{6}}(p)\}.$$

Proposition 6.2. *When* $\dim E(c) = 4$, W *is orthogonal to* E, *and all the shape operators* $L(s,t) = \cos s \cos t B_\eta + \cos s \sin t B_\zeta + \sin s B_{\bar\zeta}$ *map* E *onto* W, *where* $\zeta = e_6$ *and* $\bar\zeta = e_{\bar 6}$.

Proof. From [3, (43)], at any point of L_6,

$$\langle \nabla_{e_{\underline{6}}} e_{\underline{3}}, \bar\nabla_{e_{\underline{3}}} e_{\underline{6}} \rangle = 0 \tag{7}$$

holds if two $e_{\underline{6}}$ are both e_6, or both $e_{\bar 6}$, or by the global symmetry (at p_3 for M_-), if two $e_{\underline{3}}$ are both e_3, or both $e_{\bar 3}$. Hence we need to show

$$\langle \nabla_{e_6} e_3, \bar\nabla_{e_{\bar 3}} e_{\bar 6} \rangle = 0, \tag{8}$$

$$\langle \nabla_{e_6} e_{\bar 3}, \bar\nabla_{e_3} e_{\bar 6} \rangle = 0. \tag{9}$$

Since $0 = \langle \nabla_{e_6 + e_{\bar 6}} e_3, \bar\nabla_{e_{\bar 3}} (e_6 + e_{\bar 6}) \rangle = \langle \nabla_{e_6} e_3, \bar\nabla_{e_{\bar 3}} e_{\bar 6} \rangle + \langle \nabla_{e_{\bar 6}} e_3, \bar\nabla_{e_{\bar 3}} e_6 \rangle$, it is sufficient to show either one of (8) or (9). Recall that $e_3(t)$ is chosen as in Lemma 4.1 along c, and we extend $e_{\bar 3}(t)$, which is orthogonal to $e_3(t)$, to $e_{\bar 3}(s,t)$ as in Lemma 4.1 along each geodesic $\bar c^t(s)$ through $c(t)$ in the direction $e_{\bar 6}(t)$. Then at $p^t_\pm \in c \cap \bar c^t$, we have

$$\langle \nabla_{e_6} e_3(p^t_+), \bar\nabla_{e_{\bar 3}} e_{\bar 6}(p^t_+) \rangle = -\langle \nabla_{e_6} e_3(p^t_-), \bar\nabla_{e_{\bar 3}} e_{\bar 6}(p^t_-) \rangle$$

since $\nabla_{e_6} e_3(t)$ is odd and $\bar\nabla_{e_{\bar 3}} e_{\bar 6}$ is even. Thus we have $p_0 \in c$ at which $\langle \nabla_{e_6} e_3(p_0), \bar\nabla_{e_{\bar 3}} e_{\bar 6}(p_0) \rangle = 0$, namely, (8), and hence (9) hold. Thus W is orthogonal to E (by (2) and the statement after it). Since E is parallel and of dimension 6, $W = E^\perp$ is parallel, and $B_\eta(E) = W$.

We know already that $L(s,0) = \cos s B_\eta + \sin s B_{\bar\zeta}$ maps $E(\bar c)$ onto $W(\bar c) \subset W$ ([3, Lemma 5.4]). Thus we need to show that $B_{\bar\zeta}$ maps $\nabla_{e_6} e_{\underline{3}}$ into W. Using [3, (36)], this follows from

$$\begin{aligned} B_{\bar\zeta}(\nabla_{e_6} e_{\underline{3}}) &= c_0 \nabla_{e_{\bar 6}} \Big(B_\eta(\nabla_{e_6} e_{\underline{3}}) \Big) - c_0 B_\eta(\nabla_{e_{\bar 6}} \nabla_{e_6} e_{\underline{3}}) \\ &= c_0 c_1 \nabla_{e_{\bar 6}} \bar\nabla_{e_{\underline{3}}} e_6 - c_0 B_\eta(\nabla_{e_{\bar 6}} \nabla_{e_6} e_{\underline{3}}). \end{aligned}$$

In fact, all the second derivatives such as $\nabla_{e_{\bar 6}} \nabla_{e_6} e_{\underline{3}}$ are contained in E since E is parallel, and $\nabla_{e_{\bar 6}} \bar\nabla_{e_{\underline{3}}} e_6 \in W$ since $W = E^\perp$ is parallel. Hence $B_{\bar\zeta}$ maps E onto W. Similarly, B_ζ maps E onto W. □

By this proposition, even when $\dim E(c) = 4$, we can express

$$L(t) = \cos t\, B_\eta + \sin t\, B_\zeta = \begin{pmatrix} 0 & R \\ {}^tR & S \end{pmatrix}, \quad T = {}^tRR, \tag{10}$$

with respect to the decomposition $E^6 \oplus W^4$ for any $\zeta \in D_6(p)$. In particular, we can apply the argument [3, §§9–13.2] to this case replacing $E(c)$ by E, and putting $Y = 0$ in [3, (106)]. All the results hold as in the case $\dim E(c) = 6$. Among the most important are Proposition 12.2 and Corollary 12.3, where under the assumption $ab \not\equiv 0$, σ and τ become constant along c. The arguments in [3, §13] are true except for the proof of Proposition 13.6 and Lemma 13.9.

7. Eigenvalues of T

Recall [3, Props. 10.1 and 10.3]. Then in both cases (A) $\dim E(c) = 6$, and (B) $\dim E(c) = 4$ with $\dim E = 6$, B_η is given by one of the following with respect to $E(c) \oplus W(c)$, and $E \oplus W$, respectively:

(0) $ab \neq 0$, $T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$, $T_1 = \begin{pmatrix} \sigma & 0 \\ 0 & 1/\sigma \end{pmatrix}$, $T_2 = \begin{pmatrix} \tau & 0 \\ 0 & 1/\tau \end{pmatrix}$, $S = \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix}$, $S_1 = \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}$, $S_2 = \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix}$, $\sigma + \dfrac{1}{\sigma} + a^2 = \dfrac{10}{3}$, $\tau + \dfrac{1}{\tau} + b^2 = \dfrac{10}{3}$.

(I) $a = b = 0$ and $T = \begin{pmatrix} \bar{T} & 0 \\ 0 & \bar{T} \end{pmatrix}$, $\bar{T} = \begin{pmatrix} 3 & 0 \\ 0 & 1/3 \end{pmatrix}$, $S = 0$.

(II) $a \neq 0$, $b = 0$ and $T = \begin{pmatrix} T_1 & 0 \\ 0 & \bar{T} \end{pmatrix}$, $S = \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix}$.

In fact, if $ab \equiv 0$ holds in an open neighborhood of the space of geodesics of $L_6(p)$, either (I) or (II) occurs since $\sigma + 1/\sigma + a^2 = 10/3$ and a similar formula holds for τ, b [3, (72)]. Note that $a \neq 0$ is equivalent with $\alpha\beta \neq 0$, as the latter implies $\sigma \neq 1/3, 3$. Similarly, $b \neq 0$ corresponds to $\gamma\delta \neq 0$ (the last line of [3, Prop. 11.1]). Therefore, Case (0) occurs only when $ab \not\equiv 0$ which is the case $\alpha\beta, \gamma\delta \not\equiv 0$.

The argument in [3, §§12, 13.1, 13.2], treating the case $ab \not\equiv 0$ are quite important, and Corollary 12.3 is most notable. Based on these results, we show

Proposition 7.1. *When $ab \not\equiv 0$, $\sigma = \tau \in (1/3, 3)$ holds.*

Proof. In the following, we use the notation in [3, §12] and the orthonormal basis $X_{\underline{i}}$, $Z_{\underline{i}}$ given by [3, (91), (92)].

Because σ, τ are constant along the geodesic c by [3, Cor. 12.3], differentiating $L(t)X_{\underline{i}}(t) = \nu_{\underline{i}} Z_{\underline{i}}(t)$ by t where $\nu_1 = \sqrt{\sigma}, \nu_2 = 1/\sqrt{\sigma}, \nu_{\bar{1}} = \sqrt{\tau}, \nu_{\bar{2}} = 1/\sqrt{\tau}$, we obtain

$$L_t(t)X_{\underline{i}}(t) + L(t)\dot{X}_{\underline{i}}(t) = \nu_{\underline{i}} \dot{Z}_{\underline{i}}(t).$$

Note that $\dot{X}_{\underline{i}}(t) = H(t)X_{\underline{i}}(t), \dot{Z}_{\underline{i}}(t) = H(t)Z_{\underline{i}}(t)$ by [3, (27)], where we use again that ν_i's are constant. Hence putting $t = 0$, and denoting $X_{\underline{i}}(0) = X_{\underline{i}}$ etc., we have

$$B_\zeta X_{\underline{i}} = -B_\eta H(0) X_{\underline{i}} + \nu_{\underline{i}} H(0) Z_{\underline{i}}. \tag{11}$$

Since $ab \neq 0$, using [3, (116)], we may put $H(0) = \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix}$, where

$$J_1 = \begin{pmatrix} H_0 & X & Y \\ -{}^tX & H_1 & Z \\ -{}^tY & -{}^tZ & H_2 \end{pmatrix}, \begin{pmatrix} H_1 & Z \\ -{}^tZ & H_2 \end{pmatrix} = \begin{pmatrix} 0 & x & y & z \\ -x & 0 & u & v \\ -y & -u & 0 & w \\ -z & -v & -w & 0 \end{pmatrix}.$$

Then (11) is expressed as

$$\begin{pmatrix} 0 & M \\ {}^tM & N \end{pmatrix} \begin{pmatrix} X_{\underline{i}} \\ 0 \end{pmatrix} = -\begin{pmatrix} 0 & A \\ {}^tA & D \end{pmatrix} \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix} \begin{pmatrix} X_{\underline{i}} \\ 0 \end{pmatrix} + \nu_i \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix} \begin{pmatrix} 0 \\ Z_{\underline{i}} \end{pmatrix},$$

and hence we obtain

$$B_\zeta X_{\underline{i}} = {}^tMX_{\underline{i}} = -{}^tAJ_1X_{\underline{i}} + \nu_i J_2 Z_{\underline{i}}. \tag{12}$$

Here and there, we abuse $\left(\begin{smallmatrix} 0 \\ V \end{smallmatrix}\right) = V$ or $\left(\begin{smallmatrix} V \\ 0 \end{smallmatrix}\right) = V$, if $V \in E$ or $V \in W$ is clear. Since we can express

$$A = \begin{pmatrix} 0_{2,4} \\ \bar{A} \end{pmatrix}, \quad \bar{A} = \operatorname{diag}\begin{pmatrix} \sqrt{\sigma} & 1/\sqrt{\sigma} & \sqrt{\tau} & 1/\sqrt{\tau} \end{pmatrix}, \tag{13}$$

where $0_{i,j}$ denote the $i \times j$ zero matrix, from $X_{\underline{i}} \perp D_3$ we have

$$\begin{aligned} {}^tAJ_1X_{\underline{i}} &= \begin{pmatrix} 0_{4,2} & {}^t\bar{A} \end{pmatrix} \begin{pmatrix} H_0 & X & Y \\ -{}^tX & H_1 & Z \\ -{}^tY & -{}^tZ & H_2 \end{pmatrix} \begin{pmatrix} 0_{2,1} \\ X_{\underline{i}} \end{pmatrix} \\ &= {}^t\bar{A} \begin{pmatrix} H_1 & Z \\ -{}^tZ & H_2 \end{pmatrix} X_{\underline{i}} = \begin{pmatrix} 0 & x\sqrt{\sigma} & y\sqrt{\sigma} & z\sqrt{\sigma} \\ -x/\sqrt{\sigma} & 0 & u/\sqrt{\sigma} & v/\sqrt{\sigma} \\ -y\sqrt{\tau} & -u\sqrt{\tau} & 0 & w\sqrt{\tau} \\ -z/\sqrt{\tau} & -v/\sqrt{\tau} & -w/\sqrt{\tau} & 0 \end{pmatrix} X_{\underline{i}}. \end{aligned}$$

Now, suppose $\sigma \neq \tau$, namely, $a^2 \neq b^2$. Then by [3, Prop. 13.3], [3, (138)] follows, and hence differentiating U_2 at $t = 0$, we have

$$J_2 Z_{\underline{i}} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} Z_{\underline{i}}.$$

Substituting these into (12), we obtain

$$\begin{aligned} {}^tMX_1 &= -\sqrt{\sigma}Z_{\bar{1}} + x/\sqrt{\sigma}Z_2 + y\sqrt{\tau}Z_{\bar{1}} + z/\sqrt{\tau}Z_{\bar{2}}, \\ {}^tMX_2 &= -x\sqrt{\sigma}Z_1 + u\sqrt{\tau}Z_{\bar{1}} + v/\sqrt{\tau}Z_{\bar{2}}, \\ {}^tMX_{\bar{1}} &= \sqrt{\tau}Z_1 - y\sqrt{\sigma}Z_1 - u/\sqrt{\sigma}Z_2 + w/\sqrt{\tau}Z_{\bar{2}} \\ {}^tMX_{\bar{2}} &= -z\sqrt{\sigma}Z_1 - v/\sqrt{\sigma}Z_2 - w\sqrt{\tau}Z_{\bar{1}}. \end{aligned}$$

Therefore, putting ${}^tM = \begin{pmatrix} l_1 & l_2 & l_3 & l_4 & l_5 & l_6 \end{pmatrix}$, by (12) we have

$$\begin{pmatrix} l_3 & l_4 & l_5 & l_6 \end{pmatrix} = \begin{pmatrix} 0 & -x\sqrt{\sigma} & \sqrt{\tau} - y\sqrt{\sigma} & -z\sqrt{\sigma} \\ x/\sqrt{\sigma} & 0 & -u/\sqrt{\sigma} & -v/\sqrt{\sigma} \\ -\sqrt{\sigma} + y\sqrt{\tau} & u\sqrt{\tau} & 0 & -w\sqrt{\tau} \\ z/\sqrt{\tau} & v/\sqrt{\tau} & w/\sqrt{\tau} & 0 \end{pmatrix}. \tag{14}$$

From this and (13), it follows

$$
{}^tMA = \begin{pmatrix} 0 & -x & \tau - y\sqrt{\sigma\tau} & -z\sqrt{\sigma/\tau} \\ x & 0 & -u\sqrt{\tau/\sigma} & -v/\sqrt{\sigma\tau} \\ -\sigma + y\sqrt{\sigma\tau} & u\sqrt{\tau/\sigma} & 0 & -w \\ z\sqrt{\sigma/\tau} & v/\sqrt{\sigma\tau} & w & 0 \end{pmatrix}. \tag{15}
$$

Therefore, we obtain

$$
{}^tAM + {}^tMA = \begin{pmatrix} 0 & 0 & \tau-\sigma & 0 \\ 0 & 0 & 0 & 0 \\ \tau-\sigma & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{16}
$$

On the other hand, we know

$$
{}^tAA = \mathrm{diag}\begin{pmatrix} \sigma & 1/\sigma & \tau & 1/\tau \end{pmatrix}, \tag{17}
$$

and so

$$
{}^tMM = U_2{}^tAA{}^tU_2 = \begin{pmatrix} (\sigma+\tau)/2 & 0 & (\sigma-\tau)/2 & 0 \\ 0 & 1/\sigma & 0 & 0 \\ (\sigma-\tau)/2 & 0 & (\sigma+\tau)/2 & 0 \\ 0 & 0 & 0 & 1/\tau \end{pmatrix} \tag{18}
$$

follows, where U_2 is given by [3, (138)]. Thus in ${}^t(cA+sM)(cA+sM) = c^2({}^tAA) + s^2({}^tMM) + cs({}^tAM + {}^tMA)$, where $c = \cos t, s = \sin t$, the second and the fourth columns and rows make $\begin{pmatrix} 1/\sigma & 0 \\ 0 & 1/\tau \end{pmatrix}$. On the other hand, the first and the third columns and rows yield

$$
\begin{pmatrix} c^2\sigma + s^2(\sigma+\tau)/2 & s^2(\sigma-\tau)/2 + cs(\tau-\sigma) \\ s^2(\sigma-\tau)/2 + cs(\tau-\sigma) & c^2\tau + s^2(\sigma+\tau)/2 \end{pmatrix},
$$

which has eigenvalues σ and τ for all c, s. Then as its determinant

$$
(c^2\sigma + s^2(\sigma+\tau)/2)(c^2\tau + s^2(\sigma+\tau)/2) - \{s^2(\sigma-\tau)/2 + cs(\tau-\sigma)\}^2
$$

should be identically $\sigma\tau$, noting the coefficient of cs^3, we obtain $\sigma = \tau$, a contradiction. Thus when $ab \neq 0$, $\sigma = \tau \neq 3, 1/3$, occurs. □

8. **Proof of Proposition 13.6 of** [3]

In the proof of Proposition 13.6 in [3], the exclusion of $\dim K = 4$ or $\dim K = 2$ fails in Lemma 13.9, where we use an incorrect result in [3, §8]. In both cases (A) $\dim E(c) = 6$ and (B) $\dim E(c) = 4$ and $\dim E = 6$, we give a correct proof here.

First, we remark that Case (II) is excluded in [3, Prop. 14.1] independent of the other argument, and the proof is also applicable to E when (B) occurs. Therefore, we may consider only the cases (0) and (I).

We emphasize $\alpha\beta \not\equiv 0$ in Case (0). In this case, $W(c)$ (Case (A)), or W (Case (B)) is contained in the space spanned by vectors given by [3, (92)], where $\sigma = \tau, \alpha = \gamma, \beta = \delta$ by Proposition 7.1:
(19)

$$\begin{aligned} Z_1 &= \tfrac{1}{\sqrt{\sigma}}\Big(\sqrt{3}\alpha(e_1 - e_5) + \tfrac{\beta}{\sqrt{3}}(e_2 - e_4)\Big),\ Z_2 = \beta(e_1 + e_5) - \alpha(e_2 + e_4), \\ Z_{\bar{1}} &= \tfrac{1}{\sqrt{\sigma}}\Big(\sqrt{3}\alpha(e_{\bar{1}} - e_{\bar{5}}) + \tfrac{\beta}{\sqrt{3}}(e_{\bar{2}} - e_{\bar{4}})\Big),\ Z_{\bar{2}} = \beta(e_{\bar{1}} + e_{\bar{5}}) - \alpha(e_{\bar{2}} + e_{\bar{4}}). \end{aligned}$$

Here $Z_2, Z_{\bar{2}}$ are parallel along c ([3, Prop. 13.4]).

8.1. *Case* (A).

PROPOSITION 8.1. *When Case* (0) *occurs*, *Case* (A) *is impossible.*

Proof. Suppose Case (0) and Case (A) occur. We restate the argument in the beginning of §13.3 [3]. Since $\dim W(c) = 4$, denoting by $Z_{\bar{2}}^{\perp}$ the orthogonal complement of $Z_{\bar{2}}$ in $W(c)$, we know $\dim\Big(Z_{\bar{2}}^{\perp} \cap W(t)\Big) = 3 + 2 - 4 \geq 1$. Thus we can choose $e_3(t)$ so that $\nabla_{e_3} e_6(t) \in Z_{\bar{2}}^{\perp}$ for all t. Then $K = \mathrm{span}\{e_3(t)\}$ is mapped into $Z_{\bar{2}}^{\perp}$ by $L(t)$, and so $\dim K \leq 5$. As we know $\dim K \neq 3, 5$ by the first part of Lemma 13.9, and by Lemma 13.10 of [3], which are correct, we may consider the case $\dim K = 4$ or 2.

When $\dim K = 4$, $L(t)K = \mathrm{span}\{Z_1(t), Z_{\bar{1}}(t), Z_2\}$ for each t. Thus K contains $e_3(t), X_1(t), X_{\bar{1}}(t), X_2(t)$, which implies that

$$K = \mathrm{span}\{e_3(t), X_1(t), X_{\bar{1}}(t), X_2(t)\}$$

for each t. Then the orthogonal complement of K in $E(c)$ is given by $K^{\perp} = \mathrm{span}\{e_{\bar{3}}(t), X_{\bar{2}}(t)\}$ for each t, which is parallel along c. Thus using a frame at p, we may express $K = \mathrm{span}\{e_3, X_1, X_2, X_{\bar{1}}\}$ and $K^{\perp} = \mathrm{span}\{e_{\bar{3}}(t)\} = \mathrm{span}\{e_{\bar{3}}, X_{\bar{2}}\}$.

Since Z_2 and $Z_{\bar{2}}$ are constant along c, $Z_{\bar{2}}^s = \cos s\, Z_2 + \sin s\, Z_{\bar{2}}$ is constant along c for each s. Apply the above argument to $Z_{\bar{2}}^s$ for $s \not\equiv \pi/2$ modulo π. Namely, if we take $e_3^s(t)$ along c so that $\nabla_{e_3^s} e_6(t)$ is orthogonal to $Z_{\bar{2}}^s$, the space $K^s = \mathrm{span}\{e_3^s(t)\}$ is of dimension 4 or 2. If $\dim K^s = 4$, then $e_{\bar{3}}^s(t)$ which is orthogonal to $e_3^s(t)$ spans the 2-dimensional space $(K^s)^{\perp} = \{e_{\bar{3}}^s, X_{\bar{2}}^s\}$, where $X_{\bar{2}}^s = \cos s\, X_2 + \sin s\, X_{\bar{2}}$. Since $e_{\bar{3}}(t)$ and $e_{\bar{3}}^s(t)$ are independent because so are $\nabla_{e_{\bar{3}}} e_6(t)$ and $\nabla_{e_{\bar{3}}^s} e_6(t)$, we obtain

$$D_3(t) = \mathrm{span}\{e_{\bar{3}}(t), e_{\bar{3}}^s(t)\} \subset \{e_{\bar{3}}, e_{\bar{3}}^s, X_{\bar{2}}, X_{\bar{2}}^s\},$$

which implies $\dim E(c) = 4$ because of (1), a contradiction. Thus $\dim K^s = 2$, but again in this case, $e_{\bar{3}}(t)$ and $e_{\bar{3}}^s(t)$ are independent, and we have

$$D_3(t) = \mathrm{span}\{e_{\bar{3}}(t), e_{\bar{3}}^s(t)\} \subset \{e_{\bar{3}}, e_{\bar{3}}^s, X_{\bar{2}}, X_{\bar{2}}^s\},$$

where $X_{\bar{2}}^s = -\sin s\, X_2 + \cos s\, X_{\bar{2}}$, which contradicts $\dim E(c) = 6$. The case $\dim K = 2$ is similarly excluded. $\square$

8.2. *Case* (B).

PROPOSITION 8.2. *When* (B) *occurs*, *Case* (0) *is impossible. Hence Case* (0) *never occurs.*

Proof. When (B) is the case, Lemma 6.1 implies that $E = E(c) + E(\bar{c})$ is of dimension 6 and $W = W(c) + W(\bar{c})$ is of dimension 4, where $\bar{c}$ is a geodesic orthogonal to c at p. In fact, this is true for generic $\bar{c}$ transversal to c.

By [3, Prop. 13.4] applied to W, $Z_2, Z_{\bar{2}}$ are constant. Also by Lemma 4.1, we may consider that $K = \mathrm{span}\{e_3(t)\}$ and $\bar{K} = \mathrm{span}\{e_{\bar{3}}(t)\}$ are 2-dimensional, and $Z_2 = \nabla_{e_3} e_6(t)/|\nabla_{e_3} e_6(t)|$, $Z_{\bar{2}} = \nabla_{e_{\bar{3}}} e_6(t)/|\nabla_{e_{\bar{3}}} e_6(t)|$ hold. Thus we obtain

$$W(c) = \mathrm{span}\{Z_2, Z_{\bar{2}}\}. \tag{20}$$

As we assume Case (0) for generic geodesic c^s in the direction $e_6^s = \cos s\, e_6 + \sin s\, e_{\bar{6}}$, there exist $Z_2^s, Z_{\bar{2}}^s$ constant along c^s and $W(c^s) = \mathrm{span}\{Z_2^s, Z_{\bar{2}}^s\}$. Note that these $Z_2^s, Z_{\bar{2}}^s$ are *different* from those in the last subsection (which was along c). Since $W(c^s) \subset W = \{Z_1, Z_2, Z_{\bar{1}}, Z_{\bar{2}}\}$, we may express

$$\begin{aligned} Z_2^s &= \beta^s(e_1^s + e_5^s) - \alpha^s(e_2^s + e_4^s) = x^s Z_1 + y^s Z_2 + z^s Z_{\bar{1}} + w^s Z_{\bar{2}}, \\ Z_{\bar{2}}^s &= \beta^s(e_{\bar{1}}^s + e_{\bar{5}}^s) - \alpha^s(e_{\bar{2}}^s + e_{\bar{4}}^s) = \bar{x}^s Z_1 + \bar{y}^s Z_2 + \bar{z}^s Z_{\bar{1}} + \bar{w}^s Z_{\bar{2}} \end{aligned} \tag{21}$$

for some $e_{\underline{i}}^s \in D_i(p)$ and α^s, β^s. As their D_1 component and D_5 component have the same length, we obtain

$$\begin{aligned} &\left(x^s \frac{\sqrt{3}\alpha}{\sqrt{\sigma}} + y^s \beta\right)^2 + \left(z^s \frac{\sqrt{3}\alpha}{\sqrt{\sigma}} + w^s \beta\right)^2 \\ &\qquad = \left(-x^s \frac{\sqrt{3}\alpha}{\sqrt{\sigma}} + y^s \beta\right)^2 + \left(-z^s \frac{\sqrt{3}\alpha}{\sqrt{\sigma}} + w^s \beta\right)^2 \end{aligned}$$

for each s, and a similar formula holds for $\bar{x}^s$ etc. Here, $\sigma = 2\Big(3\alpha^2 + \beta^2/3\Big)$ as in [3, (99)]. From this and $\alpha\beta \neq 0$, it follows

$$x^s y^s + z^s w^s = 0, \quad \bar{x}^s \bar{y}^s + \bar{z}^s \bar{w}^s = 0.$$

Rotating $Z_2^s, Z_{\bar{2}}^s$ in $W(c^s)$, we may assume $\bar{y}^s \equiv 0$ for each s. Moreover, since $e_6^s = \cos s\, e_6 + \sin s\, e_{\bar{6}}$ is odd in s, $y^s = \langle \nabla_{e_3} e_6^s, Z_2 \rangle$ is *odd in s*. Hence there exists some s_0 such that $y^{s_0} = 0$, and we have

$$z^{s_0} w^{s_0} = 0 \quad \text{and} \quad \bar{z}^{s_0} \bar{w}^{s_0} = 0. \tag{22}$$

LEMMA 8.3. *Under the above assumption*, $W(c^{s_0}) = \mathrm{span}\{Z_1, Z_{\bar{1}}\}$ *holds.*

Proof. For the moment, we omit s_0 in (22). We have four cases. The case $z = \bar{z} = 0$ causes $W(c^{s_0}) = \mathrm{span}\{Z_1, Z_{\bar{2}}\}$, which is impossible in view of (21) (see also (19)). Next, when $w = \bar{w} = 0$ holds, the conclusion follows. When

$w = \bar{z} = 0$, we have

$$Z_2^{s_0} = xZ_1 + zZ_{\bar{1}}, \quad Z_{\bar{2}}^{s_0} = \bar{x}Z_1 + \bar{w}Z_{\bar{2}}.$$

Since Z_2^s and$Z_{\bar{2}}^s$ are orthogonal, we have $x\bar{x} = 0$. If $\bar{x} = 0$, $Z_{\bar{2}}^{s_0} = Z_{\bar{2}}$, then by (21), $Z_2^{s_0} = xZ_1 + zZ_{\bar{1}}$ is impossible. Thus $x = 0$ holds, and from (21), we obtain $\bar{w} = 0$, and the conclusion follows. The case $z = \bar{w} = 0$ is similar. □

Proof of Proposition 8.2. As we can apply the above argument at any point $q \in c$, there exists s_q such that along the geodesic $c_q = c^{s_q}$ through q, $W(c_q) = \text{span}\{Z_2^{s_q}, Z_{\bar{2}}^{s_q}\} = \text{span}\{Z_1(q), Z_{\bar{1}}(q)\} = \text{span}\{Z_2(q), Z_{\bar{2}}(q)\}^\perp = \text{span}\{Z_2, Z_{\bar{2}}\}^\perp$, since $\{Z_2, Z_{\bar{2}}\}$ is parallel along c. Thus putting $H = \{Z_2, Z_{\bar{2}}\}^\perp$, we obtain $W(c_q) = H$ for any $q \in c$.

Now, let $c_1 = c^{s_p}$ and $c_2 = c^{s_q}$ for any $q \in c$, $p \neq \pm q$. Note that $W(c_1) = H = W(c_2)$. For $x \in c_1 \cap c_2$, we can express $E(c_i) = D_3(x) \oplus J_i$ for some 2-dimensional J_i perpendicular to $D_3(x)$, $i = 1, 2$, which are mapped by B_{η_x} onto H. Hence, $J_1 = J_2$, and so $E(c_1) = E(c_2)$ holds. Next, for any geodesic γ transversal to c_1 and c_2, take $x_i \in \gamma \cap c_i$. Then $\dim E(\gamma) = 4$ implies $E(\gamma) = D_3(x_1) + D_3(x_2) \subset E(c_1) + E(c_2) = E(c_1)$. Thus we obtain $E(\gamma) = E(c_1)$. Since any point $y \in L_6(p)$ lies on some geodesic transversal to c_1 and c_2 unless y lies on c_1 or c_2, $D_3(y) \subset E(c_1)$ always holds. Hence $E = E(c_1)$ and $\dim E = 4$ follows, which contradicts Proposition 5.3. □

By this proposition and by the remark in the beginning of this section, only Case (I) is possible on both $M_\pm$, which is excluded in [3, Prop. 14.4]. Note that the argument is available to both cases (A) and (B). Thus we obtain

THEOREM 8.4. *The focal submanifolds of an isoparametric hypersurface with* $(g, m) = (6, 2)$ *have the shape operators* $B_{\boldsymbol{n}}$ *whose kernel does not depend on* $\boldsymbol{n}$.

This proves Theorem 1.1 by the argument in Section 15 of [3].

References

[1] R. MIYAOKA, The linear isotropy group of $G_2/\mathrm{SO}(4)$, the Hopf fibering and isoparametric hypersurfaces, *Osaka J. Math.* **30** (1993), 179–202. MR 1233508. Zbl 0815.53070. Available at http://projecteuclid.org/euclid.ojm/1200784357.

[2] R. MIYAOKA, The Dorfmeister-Neher theorem on isoparametric hypersurfaces, *Osaka J. Math.* **46** (2009), 695–715. MR 2583325. Zbl 05644233. Available at http://projecteuclid.org/euclid.ojm/1256564202.

[3] R. MIYAOKA, Isoparametric hypersurfaces with $(g, m) = (6, 2)$, *Ann. of Math.* **177** (2013), 53–110. MR 2999038. Zbl 06146417. http://dx.doi.org/10.4007/annals.2013.177.1.2.

[4] R. Miyaoka, Remarks on "The Dorfmeister-Neher theorem on isoparametric hypersurfaces" [MR 2583325], *Osaka J. Math.* **52** (2015), 373–376. MR 3326615. Zbl 1318.53056. Available at http://projecteuclid.org/euclid.ojm/1427202891.

(Received: August 11, 2013)
(Revised: September 13, 2015)

Mathematical Institute, Tohoku University, Sendai, Japan
E-mail: r-miyaok@m.tohoku.ac.jp