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Embedded self-similar shrinkers of genus 0

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Dedicated to Professor Leon Simon on the occasion of his seventieth birthday

Abstract

We confirm a well-known conjecture that the round sphere is the only compact, embedded self-similar shrinking solution of mean curvature flow in $\mathbb{R}^3$ with genus 0. More generally, we show that the only properly embedded self-similar shrinkers in $\mathbb{R}^3$ with vanishing intersection form are the sphere, the cylinder, and the plane. This answers two questions posed by T. Ilmanen.

1. Introduction

This paper is concerned with self-similar shrinking solutions to the mean curvature flow in $\mathbb{R}^3$. A surface $M \subset \mathbb{R}^3$ is called a self-similar shrinker if it satisfies the equation $H = \frac{1}{2} \langle x, \nu \rangle$, where ν and H denote the unit normal vector and the mean curvature, respectively. This condition guarantees that the surface M moves by homotheties when evolved by the mean curvature flow.

The classification of self-similar solutions to geometric flows is a central problem with important implications for the analysis of singularities. Indeed, Huisken's montonicity formula [15] implies that any tangent flow to a compact solution of mean curvature flow is a self-similar shrinker (see also [10] and [12]). The simplest example of a compact self-similar shrinker in $\mathbb{R}^3$ is the round sphere of radius 2 centered at the origin. G. Drugan [11] has recently constructed an example of a self-similar shrinker of genus 0 which is immersed but fails to be embedded. Angenent [1] has constructed an example of an embedded self-similar shrinker of genus 1. Moreover, N. Kapouleas, S. Kleene, and N.M. Møller [18] have constructed new examples of noncompact self-similar shrinkers using gluing techniques. These examples are embedded and have high genus.

A well-known conjecture asserts that the round sphere of radius 2 should be the only embedded self-similar shrinker of genus 0. Our main result confirms this conjecture:

Theorem 1. *Let M be a compact, embedded self-similar shrinker in $\mathbb{R}^3$ of genus* 0. *Then M is a round sphere.*

In view of the examples constructed by Drugan and Angenent, the assumptions that M is embedded and has genus 0 are both necessary. In that respect, Theorem 1 shares some common features with Lawson's Conjecture on embedded minimal tori in S^3 (cf. [19]). This conjecture was recently confirmed in [3]; see [6] for a survey.

In the noncompact case, our arguments imply the following:

Theorem 2. *Suppose that M is a properly embedded self-similar shrinker in $\mathbb{R}^3$ with the property that any two loops in M have vanishing intersection number* mod 2. *Then M is a round sphere or a cylinder or a plane.*

Theorem 2 confirms two conjectures of T. Ilmanen, the Wiggly Plane Conjecture and the Planar Domain Conjecture (cf. [17]). We note that the topological assumption in Theorem 2 is equivalent to the condition that M is homeomorphic to an open subset of S^2; this follows, e.g., from the simple exhaustion theorem in Section 4 in [14].

We next discuss some related results. In 1990, G. Huisken [15] proved that the round sphere is the only compact self-similar shrinking solution with positive mean curvature. Using a similar argument, Huisken was able to show that a noncompact self-similar shrinker which has bounded curvature and positive mean curvature must be a cylinder (cf. [16]). Moreover, K. Ecker and G. Huisken proved that a self-similar shrinker which can be written as an entire graph must be a plane (cf. [13, p. 471]). In a remarkable recent work, T. Colding and W. Minicozzi [9] proved that a self-similar shrinker which is a stable critical point of a certain entropy functional must be a sphere or a cylinder or a plane. Furthermore, T. Colding, T. Ilmanen, W. Minicozzi, and B. White recently showed that the round sphere has smallest entropy among all compact self-similar shrinkers (see [8]). We note that L. Wang [23] has obtained a classification of self-similar shrinkers which are asymptotic to cones. X. Wang [24] has proved a uniqueness result for convex translating solutions to the mean curvature flow which can be expressed as graphs over $\mathbb{R}^3$. Furthermore, we recently obtained a classification of steady gradient Ricci solitons in dimension 3 and 4 under a noncollapsing assumption (cf. [4], [5]).

We now sketch the main ideas involved in the proof of Theorem 1. Suppose that M is a compact, embedded self-similar shrinker in $\mathbb{R}^3$ of genus 0. In the first step, we show that, for any plane $P \subset \mathbb{R}^3$ which passes through the origin, the intersection $M \cap P$ consists of a single Jordan curve which is piecewise C^1. This argument is inspired in part by the two-piece property for embedded minimal surfaces in S^3 (cf. Ros [21]). We next prove that M is star-shaped.

Indeed, if $\langle \bar{x}, \nu(\bar{x}) \rangle = 0$ for some point $\bar{x} \in M$, then we consider the tangent plane P to M at $\bar{x}$. Clearly, P passes through the origin, so the intersection $M \cap P$ consists of a single Jordan curve. On the other hand, $M \cap P$ contains at least two arcs which intersect transversally at $\bar{x}$. This gives a contradiction. Having established that M is star-shaped, it follows that the mean curvature of M does not change sign. Huisken's theorem then implies that M is a round sphere, thereby completing the proof of Theorem 1.

The proof of Theorem 2 uses similar techniques; this is discussed in Section 4.

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2. **The key estimate**

We begin by collecting some basic identities for self-similar shrinkers in $\mathbb{R}^3$.

Proposition 3. *Let Σ be a self-similar shrinker in $\mathbb{R}^3$. Moreover, suppose that Ξ is a smooth vector field on $\mathbb{R}^3$, and let ξ denote the projection of Ξ to the tangent plane of Σ. Then*

$$\mathrm{div}_\Sigma \xi - \frac{1}{2} \langle x, \xi \rangle = \sum_{i=1}^{2} \langle \bar{D}_{e_i} \Xi, e_i \rangle - \frac{1}{2} \langle x, \Xi \rangle.$$

Here, $\bar{D}$ denotes the Levi-Civita connection on the ambient space $\mathbb{R}^3$, and $\{e_1, e_2\}$ is a local orthonormal frame on Σ.

Proof. Since Σ is a self-similar shrinker, we have $H = \frac{1}{2} \langle x, \nu \rangle$. This implies

$$\begin{aligned} \mathrm{div}_\Sigma \xi - \frac{1}{2} \langle x, \xi \rangle &= \sum_{i=1}^{2} \langle \bar{D}_{e_i} \Xi, e_i \rangle - H \langle \Xi, \nu \rangle - \frac{1}{2} \langle x, \xi \rangle \\ &= \sum_{i=1}^{2} \langle \bar{D}_{e_i} \Xi, e_i \rangle - \frac{1}{2} \langle x, \nu \rangle \langle \Xi, \nu \rangle - \frac{1}{2} \sum_{i=1}^{2} \langle x, e_i \rangle \langle \Xi, e_i \rangle \\ &= \sum_{i=1}^{2} \langle \bar{D}_{e_i} \Xi, e_i \rangle - \frac{1}{2} \langle x, \Xi \rangle. \end{aligned}$$

This proves the assertion. □

Corollary 4. *Let Σ be a self-similar shrinker in $\mathbb{R}^3$. Suppose that $F : \mathbb{R}^3 \to \mathbb{R}$ is a smooth function, and let $f : \Sigma \to \mathbb{R}$ denote the restriction of F to Σ. Then*

$$\Delta_\Sigma f - \frac{1}{2} \langle x, \nabla^\Sigma f \rangle = \sum_{i=1}^{2} (\bar{D}^2 F)(e_i, e_i) - \frac{1}{2} \langle x, \bar{\nabla} F \rangle.$$

Here, $\bar{\nabla} F$ and $\bar{D}^2 F$ denote gradient and Hessian of F with respect to the Euclidean metric, and $\{e_1, e_2\}$ is a local orthonormal frame on Σ.

Proof. Apply Proposition 3 to the gradient vector field $\Xi = \bar{\nabla} F$. □

It is well known that self-similar shrinkers can be characterized as critical points of a functional. More precisely, Σ is a self-similar shrinker if and only if Σ is a critical point of the functional

$$\mathscr{F}(\Sigma) = \int_\Sigma e^{-\frac{|x|^2}{4}}.$$

Following Colding and Minicozzi, we define a differential operator L on Σ by

$$Lf = \Delta_\Sigma f + |A|^2 f + \frac{1}{2} f - \frac{1}{2} \langle x, \nabla^\Sigma f \rangle$$

(cf. [9, eq. (4.13)]). The second variation of $\mathscr{F}$ is given by

$$-\int_\Sigma e^{-\frac{|x|^2}{4}} f\, Lf = \int_\Sigma e^{-\frac{|x|^2}{4}} \Big(|\nabla^\Sigma f|^2 - |A|^2 f^2 - \frac{1}{2} f^2 \Big),$$

where $f : \bar{\Sigma} \to \mathbb{R}$ is a test function which has compact support and vanishes along the boundary of Σ (see [9, Th. 4.14]).

We next consider a self-similar shrinker whose boundary is contained in a plane. In this case, we can use the height function as a test function in the stability inequality. This leads to the following result:

Proposition 5. *Let Σ be a smooth surface in $\mathbb{R}^3$ with boundary $\partial\Sigma = \Gamma$, and let $k \geq 4$. Suppose that $H = \frac{1}{2} \langle x, \nu \rangle$ on $\Sigma \cap \{|x| \leq k\}$. Moreover, suppose that the stability inequality*

$$0 \leq -\int_\Sigma e^{-\frac{|x|^2}{4}} f\, Lf$$

holds for every smooth function $f : \bar{\Sigma} \to \mathbb{R}$ which vanishes on the set $\Gamma \cup (\bar{\Sigma} \cap \{|x| \geq k\})$. Finally, we assume that $\Gamma \cap \{|x| \leq k\} \subset \{x \in \mathbb{R}^3 : \langle a, x \rangle = 0\}$ for some unit vector $a \in \mathbb{R}^3$. Then

$$\int_{\Sigma \cap \{|x| \leq \sqrt{k}\}} |A|^2 e^{-\frac{|x|^2}{4}} \langle a, x \rangle^2 \leq \frac{C}{\log k} \int_{\Sigma \cap \{\sqrt{k} \leq |x| \leq k\}} e^{-\frac{|x|^2}{4}},$$

where C is a positive constant independent of k.

Proof. Let us fix a smooth cutoff function $\eta : (-\infty, \infty) \to [0,1]$ satisfying $\eta = 1$ on $(-\infty, \frac{1}{2}]$, $\eta = 0$ on $[1, \infty)$, and $\eta' \leq 0$ on $(-\infty, \infty)$. We define a smooth function $F : \mathbb{R}^3 \to \mathbb{R}$ by

$$F(x) = \langle a, x \rangle\, \eta\Big(\frac{\log |x|}{\log k}\Big).$$

Note that

$$\langle x, \bar{\nabla} F \rangle = F + \frac{1}{\log k} \langle a, x \rangle\, \eta'\Big(\frac{\log |x|}{\log k}\Big).$$

Moreover, we have

$$|\bar{D}^2 F| \leq \frac{C}{\log k}\,\frac{1}{|x|}\,1_{\{\sqrt{k} \leq |x| \leq k\}},$$

where C is a positive constant independent of k. This implies

$$|F|\,|\bar{D}^2 F| \leq \frac{C}{\log k}\,1_{\{\sqrt{k} \leq |x| \leq k\}}.$$

Let $f : \bar{\Sigma} \to \mathbb{R}$ denote the restriction of F to $\bar{\Sigma}$. Using Corollary 4, we obtain

$$\begin{aligned}
&- f\Big(\Delta_\Sigma f - \frac{1}{2}\,\langle x, \nabla^\Sigma f\rangle\Big) \\
&= -F\bigg(\sum_{i=1}^{2} (\bar{D}^2 F)(e_i, e_i) - \frac{1}{2}\,\langle x, \bar{\nabla} F\rangle\bigg) \\
&= -F\sum_{i=1}^{2} (\bar{D}^2 F)(e_i, e_i) + \frac{1}{2}\,F^2 + \frac{1}{2\log k}\,\langle a, x\rangle^2\,\eta\Big(\frac{\log |x|}{\log k}\Big)\,\eta'\Big(\frac{\log |x|}{\log k}\Big) \\
&\leq \frac{C}{\log k}\,1_{\{\sqrt{k} \leq |x| \leq k\}} + \frac{1}{2}\,f^2.
\end{aligned}$$

In the last step, we have used the inequality $\eta' \leq 0$. Consequently,

$$-f\,Lf \leq \frac{C}{\log k}\,1_{\{\sqrt{k} \leq |x| \leq k\}} - |A|^2\,f^2.$$

Note that f vanishes on the set $\Gamma \cup (\bar{\Sigma} \cap \{|x| \geq k\})$. Using f as a test function in the stability inequality gives

$$\begin{aligned}
0 &\leq -\int_\Sigma e^{-\frac{|x|^2}{4}}\,f\,Lf \\
&\leq \frac{C}{\log k}\int_{\Sigma \cap \{\sqrt{k} \leq |x| \leq k\}} e^{-\frac{|x|^2}{4}} - \int_\Sigma |A|^2\,e^{-\frac{|x|^2}{4}}\,f^2 \\
&\leq \frac{C}{\log k}\int_{\Sigma \cap \{\sqrt{k} \leq |x| \leq k\}} e^{-\frac{|x|^2}{4}} - \int_{\Sigma \cap \{|x| \leq \sqrt{k}\}} |A|^2\,e^{-\frac{|x|^2}{4}}\,\langle a, x\rangle^2.
\end{aligned}$$

This proves the assertion. ∎

3. Proof of Theorem 1

In this section, we describe the proof of Theorem 1. Let M be a compact, embedded self-similar shrinker in $\mathbb{R}^3$ of genus 0. Moreover, suppose that M is not a round sphere. By Theorem 4.1 in [15], the mean curvature H must change sign. In particular, we can find a point $\bar{x} \in M$ such that $H(\bar{x}) = 0$. Using the shrinker equation, we conclude that $\langle \bar{x}, \nu(\bar{x})\rangle = 0$. For abbreviation, let $a := \nu(\bar{x})$ and $Z := \{x \in M : \langle a, x\rangle = 0\}$. Clearly, $\bar{x} \in Z$. The structure of the set Z is described in the following lemma.

LEMMA 6. *The set $Z = \{x \in M : \langle a, x \rangle = 0\}$ is a union of finitely many C^1-arcs which meet at isolated points. More precisely, for each point $x_0 \in Z$, there exists an open neighborhood $U \subset M$ of x_0 such that $Z \cap U$ is a union of m C^1-arcs which intersect transversally at x_0. Here, m can be characterized as the order of vanishing of the function $x \mapsto \langle a, x \rangle$ at x_0.*

Proof. The set Z can be viewed as the nodal set of a solution of an elliptic equation. Indeed, it follows from Corollary 4 that the function $f(x) := \langle a, x \rangle$ satisfies the equation

$$\Delta_M f - \frac{1}{2} \langle x, \nabla^M f \rangle = -\frac{1}{2} f$$

(see also [9, Lemma 3.20]). This identity can be rewritten as

$$\Delta_M(e^{-\frac{|x|^2}{8}} f) = h\, e^{-\frac{|x|^2}{8}} f,$$

where $h := e^{\frac{|x|^2}{8}} \Delta_M(e^{-\frac{|x|^2}{8}}) - \frac{1}{2}$. If we apply Lemma 2.4 and Theorem 2.5 in [7] to the function $e^{-\frac{|x|^2}{8}} f$, the assertion follows. □

We now continue with the proof of Theorem 1. In view of our choice of $\bar{x}$ and a, the function $x \mapsto \langle a, x \rangle$ vanishes to order $m \geq 2$ at the point $\bar{x}$. Consequently, there exists an open neighborhood $U \subset M$ of $\bar{x}$ such that $Z \cap U$ is a union of at least two C^1-arcs which intersect transversally at x_0. In particular, Z cannot be a Jordan curve. Hence, we can find a closed Jordan curve Γ with the property that Γ is piecewise C^1 and $\Gamma \subsetneq Z$. Since M has genus 0, Γ bounds a disk in M.

The complement $\mathbb{R}^3 \setminus M$ has two connected components which we denote by Ω and $\tilde{\Omega}$. To fix notation, let us assume that Ω is unbounded and $\tilde{\Omega}$ is bounded.

PROPOSITION 7. *There exists a smooth surface $\Sigma \subset \Omega$ such that $\bar{\Sigma} \setminus \Sigma = \Gamma$ and $|A|^2 = \langle x, \nu \rangle = 0$ at each point on Σ.*

Proof. For k sufficiently large, we denote by $\mathscr{C}_k$ the set of all embedded disks $S \subset \bar{\Omega} \cap \{|x| \leq 2k\}$ with the property that $\partial S = \Gamma$. The fact that Γ bounds a disk in M implies that $\mathscr{C}_k$ is nonempty if k is sufficiently large. Moreover, we choose a smooth cutoff function $\psi_k : [0, \infty) \to [0, 1]$ satisfying $\psi_k = 0$ on $[0, k]$ and $\psi_k'(2k) > k$. We now consider the functional

$$\mathscr{F}_k(S) = \int_S e^{-\frac{|x|^2}{4} + \psi_k(|x|)}$$

for $S \in \mathscr{C}_k$. We can interpret $\mathscr{F}_k$ as the area functional for the conformal metric $e^{-\frac{|x|^2}{4} + \psi_k(|x|)}\, \delta_{ij}$. For k sufficiently large, the region $\bar{\Omega} \cap \{|x| \leq 2k\}$ is a mean convex domain with respect to this conformal metric. Therefore, general results from [20] guarantee that there exists a smooth embedded surface

$\Sigma_k \in \mathscr{C}_k$ which minimizes the functional $\mathscr{F}_k$. Since Σ_k is a global minimizer for the functional $\mathscr{F}_k$, it is easy to see that

$$\sup_k \mathscr{F}_k(\Sigma_k) < \infty.$$

This implies

$$\sup_k \int_{\Sigma_k} e^{-\frac{|x|^2}{4}} < \infty.$$

Using the first variation formula, we deduce that $H = \frac{1}{2}\langle x, \nu\rangle$ on $\Sigma_k \cap \{|x| \leq k\}$. Finally, the stability inequality implies that

$$0 \leq -\int_{\Sigma_k} e^{-\frac{|x|^2}{4}} f\, Lf$$

for every smooth function $f : \bar{\Sigma}_k \to \mathbb{R}$ which vanishes on the set

$$\Gamma \cup (\bar{\Sigma}_k \cap \{|x| \geq k\}).$$

Using Proposition 5, we obtain

$$\begin{aligned}(1)\qquad &\limsup_{k\to\infty} \int_{\Sigma_k \cap \{|x| \leq \sqrt{k}\}} |A|^2\, e^{-\frac{|x|^2}{4}}\, \langle a, x\rangle^2 \\ &\leq \limsup_{k\to\infty} \frac{C}{\log k} \int_{\Sigma_k \cap \{\sqrt{k} \leq |x| \leq k\}} e^{-\frac{|x|^2}{4}} = 0.\end{aligned}$$

Finally, it follows from Theorem 3 in [22] that

$$\limsup_{k\to\infty} \sup_{\Sigma_k \cap W} |A|^2 < \infty$$

for every compact set $W \subset \mathbb{R}^3 \setminus \Gamma$. Hence, after passing to a subsequence if necessary, the surfaces Σ_k converge in $C^\infty_{loc}(\mathbb{R}^3 \setminus \Gamma)$ to a smooth surface $\Sigma \subset \mathbb{R}^3 \setminus \Gamma$ which satisfies the shrinker equation $H = \frac{1}{2}\langle x, \nu\rangle$. Using (1), we conclude that Σ is totally geodesic. In particular, $\langle x, \nu\rangle = 0$ at each point on Σ. Moreover, it is easy to see that $\Sigma \subset \bar{\Omega}$. Since M is not totally geodesic, the strict maximum principle implies that Σ cannot touch M. Consequently, $\Sigma \subset \Omega$.

We next show that $\Gamma \subset \bar{\Sigma}$. If $\Gamma \setminus \bar{\Sigma} \neq \emptyset$, we can construct a one-form α on $\mathbb{R}^3$ such that α has compact support, $\alpha = 0$ in an open neighborhood of $\bar{\Sigma}$, $d\alpha = 0$ in an open neighborhood of Γ, and $\int_\Gamma \alpha \neq 0$. This implies $\int_{\Sigma_k} d\alpha = \int_\Gamma \alpha \neq 0$ for each k, and $\int_{\Sigma_k} d\alpha \to 0$ as $k \to \infty$. This is a contradiction. Thus, $\Gamma \subset \bar{\Sigma}$. Since $\bar{\Sigma} \setminus \Sigma \subset \Gamma$, we conclude that $\bar{\Sigma} \setminus \Sigma = \Gamma$. This completes the proof of Proposition 7. □

Proposition 8. *There exists a smooth surface $\tilde{\Sigma} \subset \tilde{\Omega}$ such that $\bar{\tilde{\Sigma}} \setminus \tilde{\Sigma} = \Gamma$ and $|A|^2 = \langle x, \nu\rangle = 0$ at each point on $\tilde{\Sigma}$.*

Proof. We consider the set $\tilde{\mathscr{C}}$ of all embedded disks $S \subset \bar{\tilde{\Omega}}$ with boundary $\partial S = \Gamma$. As above, $\tilde{\mathscr{C}}$ is nonempty since Γ bounds a disk in M. We now consider the functional $\mathscr{F}$ defined in Section 2. The functional $\mathscr{F}$ can be viewed as the area functional for the conformal metric $e^{-\frac{|x|^2}{4}}\, \delta_{ij}$. Clearly, $\bar{\tilde{\Omega}}$

is a mean convex domain with respect to this conformal metric. Using results in [20], we can find a smooth embedded surface $\tilde{\Sigma} \in \tilde{\mathscr{C}}$ which minimizes the functional $\mathscr{F}$. The first variation formula implies that the surface $\tilde{\Sigma}$ satisfies $H = \frac{1}{2} \langle x, \nu \rangle$. Moreover, the stability inequality gives

$$0 \leq -\int_{\tilde{\Sigma}} e^{-\frac{|x|^2}{4}} f \, Lf$$

for every smooth function $f : \bar{\tilde{\Sigma}} \to \mathbb{R}$ which vanishes on the boundary Γ. Using Proposition 5 with k sufficiently large, we obtain

$$\int_{\tilde{\Sigma}} |A|^2 \, e^{-\frac{|x|^2}{4}} \, \langle a, x \rangle^2 = 0.$$

Consequently, $\tilde{\Sigma}$ is totally geodesic. This implies $\langle x, \nu \rangle = 0$ at each point on $\tilde{\Sigma}$. Finally, we clearly have $\tilde{\Sigma} \subset \bar{\tilde{\Omega}}$. Since M is not totally geodesic, the strict maximum principle implies that $\tilde{\Sigma}$ cannot touch M. Therefore, $\tilde{\Sigma} \subset \tilde{\Omega}$, as claimed. □

Proposition 9. *The unit normal vectors to Σ and $\tilde{\Sigma}$ are parallel to a at all points.*

Proof. Suppose that there exists a point $x \in \Sigma$ such that $\nu(x) = b$, where a and b are linearly independent. Let us define

$$\Sigma' = \{x \in \Sigma : \nu(x) = b\} \neq \emptyset.$$

By Proposition 7, Σ' is a subset of $\{x \in \mathbb{R}^3 : \langle b, x \rangle = 0\} \setminus \Gamma$. Moreover, Proposition 7 implies that Σ', viewed as a subset of $\{x \in \mathbb{R}^3 : \langle b, x \rangle = 0\} \setminus \Gamma$, is open and closed. On the other hand, we have

$$\{x \in \mathbb{R}^3 : \langle b, x \rangle = 0\} \cap \Gamma \subset \{x \in \mathbb{R}^3 : \langle a, x \rangle = \langle b, x \rangle = 0\} =: L.$$

Hence, the closure of Σ' is either an entire plane or a halfplane with boundary L. In the latter case, we have $L \subset \Gamma \subset M$, but this is impossible since M is compact. Consequently, the closure of Σ' is the entire plane $\{x \in \mathbb{R}^3 : \langle b, x \rangle = 0\}$. Since $\Sigma' \subset \Omega$, it follows that the plane $\{x \in \mathbb{R}^3 : \langle b, x \rangle = 0\}$ is contained in $\bar{\Omega}$, and M lies on one side of this plane. This contradicts the fact that $\int_M e^{-\frac{|x|^2}{4}} \langle b, x \rangle = 0$. Consequently, the normal vector to Σ is parallel to a at each point on Σ. An analogous argument shows that the normal vector to $\tilde{\Sigma}$ is parallel to a at each point on $\tilde{\Sigma}$. This completes the proof of Proposition 9. □

Combining Propositions 7, 8, and 9, we conclude that the surfaces Σ and $\tilde{\Sigma}$ are contained in the plane $\{x \in \mathbb{R}^3 : \langle a, x \rangle = 0\}$. Moreover, we have $\Sigma \subset \Omega$ and $\tilde{\Sigma} \subset \tilde{\Omega}$; in particular, Σ and $\tilde{\Sigma}$ are disjoint. Finally, Σ and $\tilde{\Sigma}$ have the same boundary Γ. Therefore, the union $\Sigma \cup \tilde{\Sigma} \cup \Gamma$, viewed as a subset of

$\{x \in \mathbb{R}^3 : \langle a, x \rangle = 0\}$, is open and closed. This implies

$$\{x \in \mathbb{R}^3 : \langle a, x \rangle = 0\} = \Sigma \cup \tilde{\Sigma} \cup \Gamma \subset \Omega \cup \tilde{\Omega} \cup \Gamma = (\mathbb{R}^3 \setminus M) \cup \Gamma.$$

Consequently,

$$\{x \in M : \langle a, x \rangle = 0\} \subset \Gamma.$$

In other words, the set Z coincides with Γ. This contradicts our choice of Γ. This completes the proof of Theorem 1. □

4. **Proof of Theorem 2**

In this final section, we discuss the proof of Theorem 2. Throughout this section, we assume that M is a properly embedded self-similar shrinker in $\mathbb{R}^3$. We first recall a well-known result, which is an immediate consequence of Brakke's local area bound (see [2] or [12, Prop. 4.9]):

Proposition 10. *For k large, the area of $M \cap \{|x| \leq k\}$ is at most $O(k^2)$.*

Proof. We sketch the proof for the convenience of the reader. By assumption, the surfaces $M_t = \sqrt{-t}\, M$ form a solution of mean curvature flow. Applying Proposition 4.9 in [12] (with $t_0 = 0$ and $\rho = 4$) gives

$$\operatorname{area}(M_t \cap \{|x| \leq 2\}) \leq 8 \operatorname{area}(M_{-1} \cap \{|x| \leq 4\})$$

for all $t \in [-1, 0)$. This implies

$$\operatorname{area}(M \cap \{|x| \leq 2k\}) \leq 8k^2 \operatorname{area}(M \cap \{|x| \leq 4\})$$

for $k \geq 1$. From this, the assertion follows. □

We will also need the following result, which is a special case of a much more general theorem of Brian White (see [25] for an announcement):

Theorem 11 (B. White [25]). *Suppose that M contains the line $\{x \in \mathbb{R}^3 : x_1 = x_2 = 0\}$, and M is disjoint from the halfplane $\{x \in \mathbb{R}^3 : x_1 < 0,\ x_2 = 0\}$. Then M is a plane.*

Proof. We again sketch an argument for the convenience of the reader. We define a vector field Ξ on $\mathbb{R}^3$ by $\Xi(x_1, x_2, x_3) = (-x_2, x_1, 0)$, and let ξ denote the projection of Ξ to the tangent plane of M. By assumption, M is disjoint from the halfplane $\{x \in \mathbb{R}^3 : x_1 < 0,\ x_2 = 0\}$. Hence, every point $x \in M \setminus \{x \in \mathbb{R}^3 : x_1 = x_2 = 0\}$ can be uniquely written in the form $x = (\sqrt{x_1^2 + x_2^2} \cos\theta, \sqrt{x_1^2 + x_2^2} \sin\theta, x_3)$ for some $\theta \in (-\pi, \pi)$. This defines a smooth function $\theta : M \setminus \{x \in \mathbb{R}^3 : x_1 = x_2 = 0\} \to (-\pi, \pi)$ satisfying $(x_1^2 + x_2^2)\, \nabla^M \theta = \xi$.

Using Proposition 3, we obtain

$$\mathrm{div}_M \xi - \frac{1}{2} \langle x, \xi \rangle = \sum_{i=1}^2 \langle \bar{D}_{e_i} \Xi, e_i \rangle - \frac{1}{2} \langle x, \Xi \rangle = 0.$$

In the last step, we have used that Ξ is a Killing vector field in ambient space. This implies

$$\begin{aligned} \mathrm{div}_M(e^{-\frac{|x|^2}{4}} \theta \, \xi) &= e^{-\frac{|x|^2}{4}} \theta \Big(\mathrm{div}_M \xi - \frac{1}{2} \langle x, \xi \rangle \Big) + e^{-\frac{|x|^2}{4}} \langle \nabla^M \theta, \xi \rangle \\ &= e^{-\frac{|x|^2}{4}} (x_1^2 + x_2^2) \, |\nabla^M \theta|^2 \end{aligned}$$

on $M \setminus \{x \in \mathbb{R}^3 : x_1 = x_2 = 0\}$. Integrating over $M \setminus \{x \in \mathbb{R}^3 : x_1 = x_2 = 0\}$ gives

$$\begin{aligned} 0 &= \int_{M \setminus \{x \in \mathbb{R}^3 : x_1 = x_2 = 0\}} \mathrm{div}_M(e^{-\frac{|x|^2}{4}} \theta \, \xi) \\ &= \int_{M \setminus \{x \in \mathbb{R}^3 : x_1 = x_2 = 0\}} e^{-\frac{|x|^2}{4}} (x_1^2 + x_2^2) \, |\nabla^M \theta|^2. \end{aligned}$$

Consequently, $\nabla^M \theta = 0$ at each point $x \in M \setminus \{x \in \mathbb{R}^3 : x_1 = x_2 = 0\}$. Thus, M is a plane. This completes the proof of Theorem 11. □

We now continue with the proof of Theorem 2. Let M be a properly embedded self-similar shrinker in $\mathbb{R}^3$ with the property that any two loops in M have vanishing intersection number mod 2. Moreover, we assume that M is neither a round sphere, nor a cylinder, nor a plane. By Proposition 10, M has polynomial area growth. By a theorem of Colding and Minicozzi, the mean curvature must change sign (see [9, Th. 10.1]). In particular, we can find a point $\bar{x} \in M$ such that $H(\bar{x}) = 0$, and hence $\langle \bar{x}, \nu(\bar{x}) \rangle = 0$. As above, we put $a := \nu(\bar{x})$. We now consider two cases.

Case 1: Suppose that the sets $\{x \in M : \langle a, x \rangle > 0\}$ and $\{x \in M : \langle a, x \rangle < 0\}$ are both connected. In this case, we can construct two loops with the property that the first loop is contained in $\{x \in M : \langle a, x \rangle > 0\} \cup \{\bar{x}\}$, the second loop is contained in $\{x \in M : \langle a, x \rangle < 0\} \cup \{\bar{x}\}$, and the two loops intersect transversally at $\bar{x}$. This contradicts our assumption that any two loops in M have vanishing intersection number mod 2.

Case 2: For the remainder of this section, we will assume that one of the sets $\{x \in M : \langle a, x \rangle > 0\}$ and $\{x \in M : \langle a, x \rangle < 0\}$ is not connected. Without loss of generality, we may assume that $\{x \in M : \langle a, x \rangle > 0\}$ is disconnected. Let D be an arbitrary connected component of $\{x \in M : \langle a, x \rangle > 0\}$. Moreover, let Ω and $\tilde{\Omega}$ denote the connected components of $\mathbb{R}^3 \setminus M$.

PROPOSITION 12. *There exists a smooth surface $\Sigma \subset \Omega$ such that $\bar{\Sigma} \setminus \Sigma = \partial D$ and $|A|^2 = \langle x, \nu \rangle = 0$ at each point on Σ.*

Proof. By Sard's Lemma, we can find a sequence of numbers $r_k \in (2k, 3k)$ such that the sphere $\{|x| = r_k\}$ intersects M transversally. By smoothing out the domain $\Omega \cap \{|x| < r_k\}$, we can construct an open domain Ω_k with smooth boundary such that $\Omega_k \cap \{|x| \leq 2k\} = \Omega \cap \{|x| \leq 2k\}$ and $\Omega_k \subset \{|x| \leq 3k\}$. Moreover, we can find a smooth function $\chi_k : \bar{\Omega}_k \to [0, 1]$ such that $\chi_k = 0$ on the set $\bar{\Omega}_k \cap \{|x| \leq k\}$ and $\bar{\Omega}_k$ is a mean convex domain with respect to the conformal metric $e^{-\frac{|x|^2}{4} + \chi_k(x)}\, \delta_{ij}$.

By Sard's Lemma, we can find a real number $\rho_k \in (k, 2k)$ such that the sphere $\{|x| = \rho_k\}$ intersects M and ∂D transversally. Clearly, the curve $\Gamma_k = \partial(D \cap \{|x| < \rho_k\})$ satisfies $\Gamma_k \cap \{|x| \leq k\} \subset \partial D \subset \{x \in \mathbb{R}^3 : \langle a, x \rangle = 0\}$. Let Σ_k be a surface which minimizes the modified area functional

$$\int_S e^{-\frac{|x|^2}{4} + \chi_k(x)}$$

among all embedded, orientable surfaces $S \subset \bar{\Omega}_k$ with boundary $\partial S = \Gamma_k$. Clearly,

$$\sup_k \int_{\Sigma_k} e^{-\frac{|x|^2}{4}} < \infty.$$

Moreover, the first variation formula implies that $H = \frac{1}{2}\, \langle x, \nu \rangle$ at each point on $\Sigma_k \cap \{|x| \leq k\}$. Using the stability inequality together with Proposition 5, we conclude that

$$\begin{aligned} &\limsup_{k \to \infty} \int_{\Sigma_k \cap \{|x| \leq \sqrt{k}\}} |A|^2\, e^{-\frac{|x|^2}{4}}\, \langle a, x \rangle^2 \\ &\qquad \leq \limsup_{k \to \infty} \frac{C}{\log k} \int_{\Sigma_k \cap \{\sqrt{k} \leq |x| \leq k\}} e^{-\frac{|x|^2}{4}} = 0. \end{aligned} \tag{2}$$

Finally, it follows from results in [22] that

$$\limsup_{k \to \infty} \sup_{\Sigma_k \cap W} |A|^2 < \infty$$

for every compact set $W \subset \mathbb{R}^3 \setminus \partial D$. Hence, after passing to a subsequence, the surfaces Σ_k converge in $C^\infty_{\text{loc}}(\mathbb{R}^3 \setminus \partial D)$ to a smooth surface $\Sigma \subset \mathbb{R}^3 \setminus \partial D$ which satisfies the shrinker equation $H = \frac{1}{2}\, \langle x, \nu \rangle$. Using (2), we conclude that Σ is totally geodesic. In particular, $\langle x, \nu \rangle = 0$ at each point on Σ. Moreover, it is easy to see that $\Sigma \subset \bar{\Omega}$. Since M is not totally geodesic, the strict maximum principle implies that Σ cannot touch M. Consequently, $\Sigma \subset \Omega$. Arguing as above, we obtain $\bar{\Sigma} \setminus \Sigma = \partial D$. $\square$

PROPOSITION 13. *There exists a smooth surface $\tilde{\Sigma} \subset \Omega$ such that $\bar{\tilde{\Sigma}} \setminus \tilde{\Sigma} = \partial D$ and $|A|^2 = \langle x, \nu \rangle = 0$ at each point on $\tilde{\Sigma}$.*

Proof. Analogous to Proposition 12. □

PROPOSITION 14. *The unit normal vectors to* Σ *and* $\tilde{\Sigma}$ *are parallel to* a *at all points.*

Proof. Suppose that there exists a point on Σ or $\tilde{\Sigma}$ where the unit normal vector is not parallel to a. Without loss of generality, we may assume that there exists a point $x \in \Sigma$ such that $\nu(x) = b$, where a and b are linearly independent. Let

$$\Sigma' = \{x \in \Sigma : \nu(x) = b\} \neq \emptyset.$$

As above, Σ' is a subset of $\{x \in \mathbb{R}^3 : \langle b, x\rangle = 0\} \setminus \partial D$, which is both open and closed. Since

$$\{x \in \mathbb{R}^3 : \langle b, x\rangle = 0\} \cap \partial D \subset \{x \in \mathbb{R}^3 : \langle a, x\rangle = \langle b, x\rangle = 0\} =: L,$$

it follows that the closure of Σ' is either an entire plane or a halfplane with boundary L. If the closure of Σ' is a halfplane with boundary L, then we have $L \subset \partial D \subset M$ and $\Sigma' \subset \Omega \subset \mathbb{R}^3 \setminus M$. In other words, M contains the line L, and M is disjoint from the halfplane Σ'. Hence, it follows from Theorem 11 that M is a plane, contrary to our assumption. Consequently, the closure of Σ' is the entire plane $\{x \in \mathbb{R}^3 : \langle b, x\rangle = 0\}$. Since $\Sigma' \subset \Omega$, it follows that the plane $\{x \in \mathbb{R}^3 : \langle b, x\rangle = 0\}$ is contained in $\bar{\Omega}$, and M lies on one side of this plane. As above, this contradicts the fact that $\int_M e^{-\frac{|x|^2}{4}} \langle b, x\rangle = 0$. This completes the proof. □

It follows from Propositions 12, 13, and 14 that Σ and $\tilde{\Sigma}$ are contained in the plane $\{x \in \mathbb{R}^3 : \langle a, x\rangle = 0\}$. Moreover, Σ and $\tilde{\Sigma}$ are disjoint, and have the same boundary ∂D. Putting these facts together, we conclude that

$$\{x \in \mathbb{R}^3 : \langle a, x\rangle = 0\} = \Sigma \cup \tilde{\Sigma} \cup \partial D \subset \Omega \cup \tilde{\Omega} \cup \partial D = (\mathbb{R}^3 \setminus M) \cup \partial D.$$

Thus, $\{x \in M : \langle a, x\rangle = 0\} = \partial D$. This implies $\{x \in M : \langle a, x\rangle > 0\} = D$. In particular, the set $\{x \in M : \langle a, x\rangle > 0\}$ is connected, contrary to our assumption. This completes the proof of Theorem 2. □

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