Annals of Mathematics **183** (2016), 377–498
http://dx.doi.org/10.4007/annals.2016.183.2.1

Global solutions of the Euler–Maxwell two-fluid system in 3D

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Abstract

The fundamental "two-fluid" model for describing plasma dynamics is given by the Euler–Maxwell system, in which compressible ion and electron fluids interact with their own self-consistent electromagnetic field. We prove global stability of a constant neutral background, in the sense that irrotational, smooth and localized perturbations of a constant background with small amplitude lead to global smooth solutions in three space dimensions for the Euler–Maxwell system. Our construction is robust in dimension 3 and applies equally well to other plasma models such as the Euler–Poisson system for two-fluids and a relativistic Euler–Maxwell system for two fluids. Our solutions appear to be the first nontrivial global smooth solutions in all of these models.

Contents

The first author is supported in part by NSF grant #1209437 and Chinese NSF grant #10828103. The second author is supported in part by a Packard Fellowship and NSF grant DMS-1265818. The third author is supported in part by NSF grant DMS-1 142293. Both the first and the third authors thank the support of Beijing International Mathematical Research Center.

1. Introduction

A plasma is a collection of fast-moving charged particles. It is believed that more than 90% of the matter in the universe is in the form of plasma, from sparse intergalactic plasma, to the interior of stars to neon signs. In addition, understanding of the instability formation in plasma is one of the main challenges for nuclear fusion, in which charged particles are accelerated at high speed to create energy. We refer to [2], [9] for physics references in book form.

At high temperature and velocity, ions and electrons in a plasma tend to become two separate fluids due to their different physical properties (inertia, charge). One of the basic fluid models for describing plasma dynamics is the so-called "two-fluid" model, in which two compressible ion and electron fluids interact with their own self-consistent electromagnetic field. Such a Euler–Maxwell system describes the dynamical evolution of the functions $n_e, n_i : \mathbb{R}^3 \to \mathbb{R}$, $v_e, v_i, E, B : \mathbb{R}^3 \to \mathbb{R}^3$, which evolve according to the quasi-linear coupled system,

$$
\begin{aligned}
&\partial_t n_e + \operatorname{div}(n_e v_e) = 0,\\
&n_e m_e \left[\partial_t v_e + v_e \cdot \nabla v_e\right] + \nabla p_e = -n_e e \left[E + \frac{v_e}{c} \times B\right],\\
&\partial_t n_i + \operatorname{div}(n_i v_i) = 0,\\
&n_i M_i \left[\partial_t v_i + v_i \cdot \nabla v_i\right] + \nabla p_i = Z n_i e \left[E + \frac{v_i}{c} \times B\right],\\
&\partial_t B + c\nabla \times E = 0,\\
&\partial_t E - c\nabla \times B = 4\pi e \left[n_e v_e - Z n_i v_i\right],
\end{aligned}
\tag{1.1}
$$

together with the elliptic equations

$$
\operatorname{div}(B) = 0, \quad \operatorname{div}(E) = 4\pi e(Z n_i - n_e) \tag{1.2}
$$

and two equations of state expressing p_e and p_i in terms of n_e and n_i. These equations describe a plasma composed of electrons and one species of ions. The electrons have charge $-e$, density n_e, mass m_e, velocity v_e, and pressure p_e, and the ions have charge Ze, density n_i, mass M_i, velocity v_i, and pressure p_i.

In addition, c denotes the speed of light and E and B denote the electric and magnetic field. The two equations (1.2) are propagated by the dynamic flow, provided we assume that they are satisfied at the initial time.

The full Euler–Maxwell system (1.1) with constraint (1.2) forms the foundation of the "two-fluid" model in the plasma theory, which captures the complex dynamics of a plasma due to electromagnetic interactions present in the model. Even at the linear level, there are new ion-acoustic waves, Langmuir waves, as well as light waves etc. At the nonlinear level, the Euler–Maxwell system is the origin of many well-known dispersive PDE, such as KdV [22], KP [37], [42], Zakharov [46], Zakharov-Kuznetsov [37], [42] and NLS, which can be derived from (1.1) and (1.2) via different scaling and asymptotic expansions. We also refer to [7], [8], [12] for derivation of the cold-ion and quasi-neutral equations and to [3] for a study of a similar model for semiconductors.

In this paper we consider perturbations of the flat neutral equilibrium, namely $(n_e^0, v_e^0, n_i^0, v_i^0, E^0, B^0) = (Zn_0, 0, n_0, 0, 0, 0)$, for constant $n_0 > 0$ to the Euler–Maxwell system (1.1) and (1.2). From a PDE viewpoint, the full Euler–Maxwell system (1.1) with constraint (1.2) can be classified as a system of nonlinear hyperbolic conservation laws with *no dissipation and no relaxation effects.*[1]

In some cases, mostly under suitable irrotationality assumptions, systems of hyperbolic conservation laws can be reduced to systems of nonlinear wave equations. In the case of massless wave equations, the global theory for small data is reasonably well understood in three dimensions. Some key developments include the work of John [30] showing that blow-up in finite time can happen even for small smooth localized initial data of a semilinear wave equation, the construction of "almost global" solutions by John and Klainerman [31], the introduction of the vector field method by Klainerman [35], and the understanding of the role of "null structures," starting with the works of Klainerman [33], [36] and Christodoulou [4]. These results eventually led to the spectacular proof of Christodoulou and Klainerman [6] of the stability of the Minkowski space-time among solutions of the Einstein vacuum equations. An alternative, shorter proof of this stability result was given recently by Lindblad and Rodnianski [40], using the concept of "weak null structures" [39].

On the other hand, a classical result of Sideris [44] demonstrates that, for the compressible Euler equation for a neutral gas, shock waves will develop even for smooth irrotational initial data with small amplitude. This shock formation was recently further described in [5] (see also [1]).

[1] When dissipation or relaxation is present, one expects stronger decay, even at the level of the L^2-norm; see, e.g., [3], [41] and the references therein. In our case however, the evolution is time-reversible and we need a different mechanism of decay based on dispersion.

In our situation, the Euler–Maxwell system (1.1)–(1.2) cannot be reduced to systems of wave equations, decoupled at the linear level, even under irrotationality assumptions. However, in a (highly simplified) approximation, one can think that the system can be reduced to a coupled system of two Klein–Gordon equations with different speeds and no null structure and a wave-like equation with certain null structure at the origin (see (1.3)–(1.5) below).

While global results are classical in the case of scalar Klein–Gordon equations, starting with the application of the vector field method by Klainerman [34] and the introduction of the normal form transformation by Shatah [43] (see also [45], [10], [11]), it was pointed out by Germain [13] that there are key new difficulties in the case of two Klein–Gordon equations with different speeds. In this case, the vector field method does not seem to work well, due to the absence of a suitable "scaling" vector field, and there are large sets of space-time resonances (see (1.6)–(1.8)) that contribute in the analysis.

In [13] and [14], the authors study semilinear and quasilinear systems of two Klein–Gordon equations with different speeds in dimension three, using the "space-time resonance method," and prove global existence and scattering (with weak decay like $t^{-1/2}$), in certain cases that cover most parameters. A robust result in this direction, which gives time-integrability of the solution in L^∞ and works for all speeds, was obtained by two of the authors in [26]. The analysis in [26] can be regarded as a highly simplified model for the analysis in this paper.

The goal of this paper is to develop a flexible method that can be used to deal systematically with complicated physical coupled systems, such as the Euler–Maxwell system, at least in dimension 3. The strategy described here, initiated in the previous works [25], [26], shares some similarities with the space-time resonance method of Germain–Masmoudi–Shatah [16] (see also the recent work of Gustafson–Nakanishi–Tsai [24]). We introduce, however, a new analytic framework that involves function spaces localized in both space and frequency, which are naturally compatible with the introduction of fractional powers of the weights. This framework, which can also be combined naturally with partial vector field methods and modified scattering (see, for example, the recent paper [27]), allows us to analyze efficiently bilinear operators such as those in (1.6), with complicated oscillatory phases and large sets of resonances. In particular, we are able to describe precisely the geometric structure of sublevel sets of the space-time gradients of the relevant phase functions (see (1.7)) and use analytic techniques (such as localization in the Fourier space and L^2 orthogonality arguments) and the intrinsic curvature of these sets to control our bilinear operators. We find this approach more precise and flexible analytically, which is crucial to analyze the complicated phase functions (1.7) arising in the Euler–Maxwell system.

Our approach seems flexible and robust and can be extended to other quasilinear problems in 3D, such as the Euler–Poisson system for two-fluids and the relativistic Euler–Maxwell system for two fluids, which enjoy natural (Galilean or Lorentz) symmetry. Such models will be discussed in separate papers [20]. In all of these models, including the Euler–Maxwell model we consider in this paper, the solutions we construct appear to be the first smooth, nontrivial global solutions.

1.1. *Description of the method.* To analyze the global dynamics of solutions of the system (1.1) we use a combination of dispersive analysis and energy estimates, relying heavily on the Fourier transform. (See [4], [16], [17], [24], [34], [33], [43] for previous seminal works.) To overcome the quasilinear nature of the nonlinearity and ensure global existence, we use classical high-order energy estimates to make up for the loss of derivatives in the nonlinearity. Global existence follows if a lower regularity L^∞ norm decays faster than $1/t$.

This crucial decay property is established by semilinear analysis of systems of dispersive equations. Assuming also a suitable form of irrotationality, after normalizations the system (1.1) can be reduced to a system of quasilinear coupled equations of the form

$$(1.3)\qquad (\partial_t + i\Lambda_i)U_i = \mathcal{N}_i, \quad (\partial_t + i\Lambda_e)U_e = \mathcal{N}_e, \quad (\partial_t + i\Lambda_b)U_b = \mathcal{N}_b,$$

where U_i, U_e, U_b are complex-valued functions (corresponding roughly to the ion variables, the electron variables, and the Maxwell field respectively), and $\mathcal{N}_i, \mathcal{N}_e, \mathcal{N}_b$ are quadratic nonlinearities. The operators $\Lambda_i, \Lambda_e, \Lambda_b$ are pseudo-differential operators obtained by diagonalizing the system at the linear level, and their symbols are quite complicated. (See (3.4) for the precise formulas.) In a first approximation, one can think that the operators $\Lambda_i, \Lambda_e, \Lambda_b$ are defined by the symbols[2]

$$(1.4)\quad \Lambda_i(\xi) = |\xi|\sqrt{\frac{2+|\xi|^2}{1+|\xi|^2}}, \quad \Lambda_e(\xi) = C\sqrt{1+A|\xi|^2}, \quad \Lambda_b(\xi) = C\sqrt{1+B|\xi|^2},$$

where C is a sufficiently large constant, $A, B \in [1,\infty)$, and $B \geq 2A$. In other words, the system can be thought of as a coupled system of Klein–Gordon equations with different speeds for the variables U_e and U_b and a wave-like equation for the variable U_i. The nonlinearities are quasilinear; in a first approximation one can think of them as semilinear quadratic nonlinearities, of the form

$$(1.5)\qquad \mathcal{N}_i = |\nabla|(1-\Delta)^{-1/2}F_i(U,\overline{U}), \quad \mathcal{N}_e = F_e(U,\overline{U}), \quad \mathcal{N}_b = F_b(U,\overline{U}),$$

[2]We remark that Λ_i is related to the ion-acoustic waves, Λ_e is related to the Langmuir waves, and Λ_b is related to the light (electromagnetic) waves.

where $U = (U_i, U_e, U_b)$. The ion nonlinearity $\mathcal{N}_i$ has "null structure" at the origin in the frequency space, described by the operator $|\nabla|(1-\Delta)^{-1/2}$ in front of the nonlinearity, but no other relevant null structures appear to be present in the problem.

Expecting some form of scattering, we express the solution as free evolutions from profiles that vary more slowly in time, $U_\sigma(t) = e^{-it\Lambda_\sigma} V_\sigma(t)$, $\sigma \in \{e, i, b\}$. After suitable algebraic manipulations, and appropriate use of the Fourier transform, we need to study bilinear operators T of the form

$$\widehat{T[f,g]}(\xi) = \int_{\mathbb{R}} \int_{\mathbb{R}^3} e^{it\Phi(\xi,\eta)} m(\xi,\eta) \widehat{f}(\xi-\eta,t)\widehat{g}(\eta,t)\, d\eta dt, \tag{1.6}$$

with a phase Φ that is specific to each interaction and that is of the form

$$\Phi(\xi,\eta) = \Lambda_0(\xi) \pm \Lambda_1(\xi-\eta) \pm \Lambda_2(\eta), \quad \Lambda_j \in \{\Lambda_i, \Lambda_e, \Lambda_b\}. \tag{1.7}$$

As a first approximation, one may think of f, g as being smooth bump functions and m being essentially a smooth cutoff, and the main challenge is to estimate efficiently the infinite time integral. It then becomes clear that a key role is played by the properties of the function Φ and, in particular, by the points where it is stationary,

$$\nabla_{(t,\eta)}[t\Phi(\xi,\eta)] = 0, \quad \text{i.e.,} \quad \Phi(\xi,\eta) = 0 \text{ and } \nabla_\eta \Phi(\xi,\eta) = 0. \tag{1.8}$$

The collection of such points forms the *space-time resonant set*. This was already highlighted in [16] and forms the basis of the "space-time resonance method," as developed in several problems in [13], [14], [16], [17], [18]. In some situations, one has no or few fully stationary points and the task is mainly to propagate enough smoothness of $\widehat{f}$, $\widehat{g}$ to exploit (non)stationary-phase arguments.

However, in our case the space-time resonance set is very rich. It was already pointed out by Germain [13] that a key new difficulty arises even in the case of a system of two Klein–Gordon equations with equal masses and different speeds. More precisely, one should expect the existence of a finite number of 2-dimensional sets of space-time resonances of the form $\{(\xi,\eta) = (R_j\omega, r_j\omega) : \omega \in \mathbb{S}^2\}$ for certain values r_j, R_j that depend on the parameters.

In our case, the space-time resonance set is substantially more complicated. After a careful analysis of the interactions done in Appendix B, we isolate three different problematic space-time resonant sets $\mathcal{S}$.

- Case A: we have the case of smooth 2-dimensional spheres

  $$\mathcal{S}_A = \{(\xi,\eta) = (R_j\omega, r_j\omega),\ \omega \in \mathbb{S}^2\}, \quad R_j \neq 0,\ r_j \neq 0,\ j = 1, \dots, N.$$

  This case already appears in the analysis of Klein–Gordon equations. As in [26], we can perform an efficient stationary phase analysis and use additional refined orthogonality arguments to prove global existence with robust $(1+t)^{-1-\varepsilon}$ decay. The analysis in [26] in the case of Klein–Gordon equations with different speeds can be thought of as a (highly simplified) version of the analysis needed to cover this case. Our analysis relies on the fact that the space-time resonances are *nondegenerate*, in the sense that $\det[\nabla_\eta^2\Phi(\xi,\eta)] \neq 0$ on the space-time resonant set. This is used implicitly in Lemma 6.2 to give a precise description of the set of points where $|\nabla_\eta\Phi(\xi,\eta)| \leq \delta$. See the discussion in the introduction of [26] on this nondegeneracy condition.

- Case B: we have a first degenerate sphere

$$\mathcal{S}_B = \{(\xi,\eta) = (R'\omega, 0),\ \omega \in \mathbb{S}^2\}, \quad R' \neq 0$$

where, in addition, the phase is not smooth in η. In this case, we use the fact that the essential speed of propagation of the singular perturbation is slower than expected (qualitatively, $|\nabla_\xi\Phi(\xi,\eta)| \lesssim |\eta| \ll 1$ on the space-time resonant set), and a careful adaptation of the orthogonality analysis of Case A, keeping track of how the bounds deteriorate as $\eta \to 0$.

- Case C: the presence of an "ion-like" dispersion relation brings in a strong degenerate set at 0

$$\mathcal{S}_C = \{(\xi,\eta) = (0, r'\omega),\ \omega \in \mathbb{S}^2\}, \quad r' \neq 0 \text{ or } r' = 0.$$

Here the problem comes from the strong degeneracy of the phase. Similar problems already appeared for the Euler–Poisson equation for the ions (see (1.24)), but for (1.1), we need more refined multiplier estimate and orthogonality arguments, combined with additional finite speed of propagation estimates and use of the null-form structure of the nonlinearity $\mathcal{N}_i$. In the case of pure ion interactions, we also need to exploit the fact that the phase Λ_i is of the form $\Lambda_i(v) \approx A|v| - B|v|^3$ for $|v| \ll 1$, with $A, B > 0$ (compare with the simplified formula in (1.4)). This leads to the weak ellipticity bound (8.27), which plays an important role in the proofs in Section 8.

1.1.1. *Choice of the norms.* We employ and extend the method developed in [25], [26]. We seek an appropriate space B satisfying two requirements: (1) the bilinear operator T in (1.6) needs to be bounded

$$T : B \cap H^{N_0} \times B \cap H^{N_0} \to B \tag{1.9}$$

when applied on solutions of the system, and (2) the free flow of the linearized Euler–Maxwell system with initial data in the space B should belong to a space-like $L^1_t L^\infty_x$, which has sufficiently strong time decay to close the energy estimate.

In order to define such a space B, we measure localization both in space and in frequency. We quantify these "coordinates" all the way to the uncertainty principle and decompose an arbitrary function as a sum of "atoms":

$$f = \sum_{X\cdot N\geq 1} Q_X P_N f, \quad (Q_X f)(x) \simeq \mathbf{1}_{X\leq |x|\leq 2X}(x) f(x),$$
$$\widehat{(P_N f)}(\xi) \simeq \mathbf{1}_{N\leq|\xi|\leq 2N}(\xi)\hat{f}(\xi).$$

We can then define the norms for the space B on each atom. The simplest norm giving the appropriate decay would be a weighted space $x^{-1-\varepsilon}L^2$ and this is the main motivation for our "strong" norm B^1. Unfortunately, some interactions produce outputs that are not bounded in this norm around a 2D resonant sphere. To account for this, we also introduce another kind of atoms, the "weak" atoms, bounded only in B^2 that barely fail to be in $x^{-1}L^2$, but are essentially concentrated on the 2-dimensional resonant spheres. Finally, each atom is allowed to be a combination of the two above types:

$$\|f\| = \sup_{X\cdot N\geq 1} \|Q_X P_N f\|_{B_{X,N}},$$
$$\|g\|_{B_{X,N}} = \|g\|_{B^1_{X,N}+B^2_{X,N}} = \inf_{g=g_1+g_2}\{\|g_1\|_{B^1_{X,N}} + \|g_2\|_{B^2_{X,N}}\}.$$

We refer to Definition 4.1 for the precise definition of the Z norm and to Lemma A.5 in Appendix A for the proof that these norms yield the desired integrability upon application of the linear flow.

1.2. *Statement of the main result.* In order to state our main result, we normalize the Euler–Maxwell system in the following way. Assume the pressures are given by the formulas[3]

$$p_e = P_e\frac{n_e^2}{2}, \quad p_i = P_i Z^2\frac{n_i^2}{2}, \tag{1.10}$$

with constants P_e and P_i. The physical parameters are then the effective ion and electron temperatures

$$k_B T_e = n_0 P_e, \quad k_B T_i = n_0 Z P_i,$$

where k_B denotes the Boltzmann constant, with corresponding electron and ion thermal speeds[4]

$$V_e = \sqrt{\frac{n_0 P_e}{m_e}} = \sqrt{\frac{k_B T_e}{m_e}}, \quad V_i = \sqrt{\frac{n_0 P_i Z}{M_i}} = \sqrt{\frac{k_B T_i}{M_i}}.$$

[3]In fact, our approach allows us to treat any sufficiently smooth *barotropic* pressure law, in particular, the typical power law $p_e \sim n_e^{\gamma_e}$ for some $\gamma_e > 0$ and similarly for p_i. We use the particular quadratic laws for the pressure here only for the sake of concreteness and since it minimizes the nonlinear terms we have to consider.

[4]These correspond to the speed of inertial (linearized) waves if one neglects the electromagnetic field.

We also have the Debye length

$$\frac{1}{\lambda_D^2} = 4\pi e^2 \left[\frac{n_0}{k_B T_e} + \frac{Z n_0}{k_B T_i}\right] = 4\pi e^2 \left[\frac{1}{P_e} + \frac{1}{P_i}\right].$$

The Euler–Maxwell system can be adimensionalized to depend only on three parameters: the ratio of the electron to ion masses (per charge)

$$\varepsilon := Z m_e / M_i, \tag{1.11}$$

the ratio of the temperatures

$$T := P_e/P_i = Z T_e/T_i, \tag{1.12}$$

and the (normalized) ratio of the speed of light to the ion velocity

$$C_b := \varepsilon \frac{c^2}{V_i^2} = \frac{c^2}{V_e V_i}\sqrt{T\varepsilon} = \frac{c^2 m_e}{n_0 P_i}. \tag{1.13}$$

More precisely, let

$$\lambda := \sqrt{\frac{4\pi e^2}{P_i}}, \qquad \beta := \sqrt{\frac{4\pi n_0 Z e^2}{M_i}},$$

and

$$\begin{aligned} n_e(x,t) &= n_0\Big[n(\lambda x, \beta t) + 1\Big], & n_i(x,t) &= (n_0/Z)\Big[\rho(\lambda x, \beta t) + 1\Big],\\ v_e(x,t) &= (\beta/\lambda) v(\lambda x, \beta t), & v_i(x,t) &= (\beta/\lambda) u(\lambda x, \beta t),\\ E(x,t) &= (4\pi e n_0/\lambda)\tilde{E}(\lambda x, \beta t), & B(x,t) &= (c M_i \beta/(Ze))\tilde{B}(\lambda x, \beta t). \end{aligned} \tag{1.14}$$

The parameter β is the *ion plasma frequency*, and $\beta/\lambda = V_i$ is the ion thermal velocity. In terms of $n, v, \rho, u, \tilde{E}, \tilde{B}$, the system (1.1)–(1.2) becomes

$$\begin{aligned} &\partial_t n + \operatorname{div}((n+1)v) = 0,\\ &\varepsilon\left(\partial_t v + v\cdot\nabla v\right) + T\nabla n + \tilde{E} + v \times \tilde{B} = 0,\\ &\partial_t \rho + \operatorname{div}((\rho+1)u) = 0,\\ &\left(\partial_t u + u\cdot\nabla u\right) + \nabla\rho - \tilde{E} - u\times\tilde{B} = 0,\\ &\partial_t \tilde{B} + \nabla\times\tilde{E} = 0,\\ &\partial_t\tilde{E} - \frac{C_b}{\varepsilon}\nabla\times\tilde{B} = \left[(n+1)v - (\rho+1)u\right],\\ &\operatorname{div}(\tilde{B}) = 0, \quad \operatorname{div}(\tilde{E}) = \rho - n, \end{aligned} \tag{1.15}$$

where ε, T and C_b have been defined above. We will assume throughout the paper that

$$\varepsilon \leq 10^{-3}, \qquad T \in [1, 100], \qquad C_b \geq 6T. \tag{1.16}$$

We will make two additional simplifications. Using the system (1.15) it is easy to see that

$$\partial_t\big[\tilde{B}-\varepsilon\nabla\times v\big]=\nabla\times\big[v\times(\tilde{B}-\varepsilon\nabla\times v)\big],$$
$$\partial_t\big[\tilde{B}+\nabla\times u\big]=\nabla\times\big[u\times(\tilde{B}+\nabla\times u)\big].$$

Therefore, "*generalized irrotational flows*" with the property that

$$\tilde{B}=\varepsilon\nabla\times v=-\nabla\times u \tag{1.17}$$

are naturally preserved for all time. See Proposition 2.1(iii) below for precise details.

Our main theorem is as follows:

THEOREM 1.1. *Assume* (1.16). *Let* $N_0 = 10^4$ *and assume that*

$$\begin{aligned}&\|(n^0,v^0,\rho^0,u^0,\tilde{E}^0,\tilde{B}^0)\|_{H^{N_0}}+\|(n^0,v^0,\rho^0,u^0,\tilde{E}^0,\tilde{B}^0)\|_Z=\delta_0\le\overline{\delta},\\&\mathrm{div}(\tilde{E}^0)+n^0-\rho^0=0,\qquad \tilde{B}^0=\varepsilon\nabla\times v^0=-\nabla\times u^0,\end{aligned} \tag{1.18}$$

where $\overline{\delta}=\overline{\delta}(C_b,T,\varepsilon)>0$ *is sufficiently small and the* Z *norm is defined in Definition* 4.1. *Then there exists a unique global solution* $(n,v,\rho,u,\tilde{E},\tilde{B})\in C([0,\infty):H^{N_0})$ *of the system* (1.15) *with initial data*

$$(n(0),v(0),\rho(0),u(0),\tilde{E}(0),\tilde{B}(0))=(n^0,v^0,\rho^0,u^0,\tilde{E}^0,\tilde{B}^0).$$

Moreover, for any $t\in[0,\infty)$,

$$\begin{aligned}&\mathrm{div}(\tilde{E})(t)+n(t)-\rho(t)=0,\\&\tilde{B}(t)=\varepsilon\nabla\times v(t)=-\nabla\times u(t)\quad(\textit{generalized irrotationality})\end{aligned} \tag{1.19}$$

and, with $\beta:=1/100$,

$$\begin{aligned}&\|(n(t),v(t),\rho(t),u(t),\tilde{E}(t),\tilde{B}(t))\|_{H^{N_0}}\\&+\sup_{|\alpha|\le 4}(1+t)^{1+\beta/2}\|(D_x^\alpha n(t),D_x^\alpha v(t),D_x^\alpha\rho(t),D_x^\alpha u(t),D_x^\alpha\tilde{E}(t),D_x^\alpha\tilde{B}(t)\|_{L^\infty}\lesssim\delta_0.\end{aligned} \tag{1.20}$$

Our main result demonstrates that even though the Euler–Maxwell system (1.1) and (1.2) is much more complicated than the pure Euler system for a neutral gas, it is in fact *more stable* in the sense that global smooth solutions can persist globally without any shock formations. This is a stark and surprising contrast to Sideris's result for the pure Euler equations [44]. Our method is also valid for general pressure laws, the Euler–Poisson system as well as a relativistic Euler–Maxwell system.

Remark 1.2. We make a few remarks about the assumptions in Theorem 1.1.

- Condition (1.16) is needed for our careful analysis of the dispersion relations that appear in the study of the linearized system (see Lemma A.4 in Appendix A). It is consistent with the relevant physical ranges of the parameters.
- Our hypotheses imply, in particular, that the perturbation is *electrically neutral*, i.e.,
$$\int_{\mathbb{R}^3} \left[Zn_0(1+\rho^0(x)) - n_0(1+n^0(x)) \right] dx = 0.$$
This is, however, forced by (1.2) if we assume that the electric perturbation is integrable.
- The smallness assumption is needed: large deviations from an equilibrium do create shocks [23].

1.3. *Simplified models.* The blow-up result of Sideris for the pure compressible Euler equations [44] can be explained from the fact that small irrotational perturbations of a constant background for the pure compressible Euler equations satisfy a quasilinear wave equation without null-structure of the form

$$(\partial_{tt} - \Delta)\,\alpha = \mathcal{Q}(\alpha, \nabla\alpha, \nabla^2\alpha), \tag{1.21}$$

where α is related to the unknown and the right-hand side denotes a quadratic nonlinearity in up to two derivatives of α. This type of equation has slow decay of linear waves (decay like $1/t$) and strong resonances, and therefore blow-up or formation of shocks is expected.

The Euler–Maxwell system (1.15) contains a nonlinearity $\mathcal{Q}$ similar to the pure compressible Euler case. However, due to self-consistent electromagnetic interaction, the linearized Euler–Maxwell system exhibits much more complex and subtle linear and bilinear dispersive effects than the wave equation. The main task in the present work is to systematically track down and exploit such dispersive effects mathematically to preserve smoothness globally in time and prevent shock formation.

In order to put our result in the right context as well as to understand the wealth of dynamics involved in small perturbations of (1.1)–(1.2), we need to introduce some intermediate models. The Euler–Maxwell system (1.1) and (1.2) is such a “master equation” describing very rich and complex plasma dynamics that it contains several well-known simplified models in plasma physics. For instance, in all physical situations,[5] $m_e \ll M_i$. It is then natural to formally set $\varepsilon = 0$ in (1.15), which leads to simplified *one fluid* models for either

[5]Indeed, the ratio m_e/M_i is no bigger than the ratio of the electron mass to the proton mass that equals 1/1836.

ions ($M_i = 1$, $m_e = 0$) or electrons ($M_i = \infty$, $m_e = 1$). Moreover, if all the velocities are much smaller than the speed of light, then $C_b \gg 1$. Formally setting[6] $C_b = \infty$ and $B \equiv 0$ replaces the Maxwell equations by the simpler *Poisson equation*. We refer to [7], [8] for other examples.

In the following, we will consider the simplified models in a form that is consistent with the reformulation (1.15) given appropriate approximations. This might look somewhat different from the classical form of these models. However, after an appropriate rescaling the equations should be the same up to cubic and higher-order terms, which can be treated easily.

1.3.1. *Single-fluid models.* The simplest model is the *Euler–Poisson model for the electrons*

$$
\begin{aligned}
\partial_t n + \operatorname{div}((1+n)v) &= 0,\\
\partial_t v + v \cdot \nabla v + \nabla n &= \nabla \phi,\\
\Delta \phi &= n.
\end{aligned}
\tag{1.22}
$$

Here the magnetic field vanishes $B \equiv 0$, and the ions are treated as motionless with a constant density and only form a fixed charged background. Such a simplified system is used for describing Langmuir waves in the two-fluid theory. After a suitable change of unknown, (1.22) can be reformulated as

$$
(\partial_{tt} - \Delta + 1)\,\alpha = \mathcal{Q}(\alpha, \nabla\alpha, \nabla^2\alpha). \tag{1.23}
$$

The linearized Euler–Poisson system for irrotational flows is no longer the acoustic (wave) equation as in the pure Euler system (1.21), but the Klein–Gordon system with "mass term" created by the plasma frequency due to to the electrostatic interaction. Taking advantage of the much better properties of Klein–Gordon equations (faster time decay of linear waves like $t^{-3/2}$, absence of quadratic resonances), global smooth irrotational flows were constructed in [19] via the normal form method of Shatah [43]:

THEOREM 1.3 (Stability of a neutral equilibrium solution [19]). *Solutions of equation* (1.22) *with initial data* (n^0, v^0) *that are small, smooth, neutral and irrotational in the sense that*

$$
\int_{\mathbb{R}^3} n^0(x)dx = 0, \qquad \nabla \times v^0 \equiv 0
$$

remain globally smooth and decay to 0 *in* L^∞ *as* $t \to +\infty$.

The neutral assumption was later removed in [15], and this result was extended to two spatial dimensions independently in [25], [38] (see also [28],

[6]This is called the *electrostatic approximation.*

[29]). Theorem 1.3 was the first positive result indicating that the dispersive effect alone in the two-fluid theory may prevent shock formation,[7] and it started an investigation to understand to which extent the introduction of electromagnetic forces could stabilize the full Euler–Maxwell system.

Recently, further progress was made in this direction in the study of another simplified model: the *Euler–Poisson equation for the ions*:[8]

$$\begin{aligned} \partial_t\rho + \mathrm{div}((1+\rho)u) &= 0,\\ \partial_t u + u\cdot\nabla u + \nabla\rho &= -\nabla\phi,\\ -\Delta\phi &= \rho - \phi. \end{aligned} \tag{1.24}$$

Here the electron dynamics with constant temperature is decoupled from the ion dynamics via the Boltzmann relation. The model equation then becomes

$$\left(\partial_{tt} - \Delta + (-\Delta)(1-\Delta)^{-1}\right)\alpha = |\nabla|\mathcal{Q}(\alpha,\nabla\alpha). \tag{1.25}$$

This system has intermediate behavior between (1.21) and (1.23). The linearized solutions decay slowly (like $t^{-4/3}$) and create many strong degeneracies near the zero frequency, where the dispersion relation is similar to the wave dispersion up to third order (see λ_i in Lemma A.4). Nevertheless, the first and third authors were able to obtain an analogue of Theorem 1.3 for perturbations of a neutral equilibrium by using a variation on the normal form method, controlling bilinear multipliers with rough coefficients using arguments inspired by [24]. Here, a crucial property is the fact that the nonlinearity is an exact derivative, which helps compensate for the degeneracy at the 0 frequency.

1.3.2. *One-fluid models with magnetic fields.* Both systems (1.22) and (1.24) can be reduced (under the irrotational assumption) to a scalar quasilinear equation. This is no longer the case for one-fluid models with nontrivial magnetic fields, which yield quasilinear systems with different speeds. Bilinear interactions in quasilinear systems generically create resonant sets of 2-dimensional spheres in the phase space, which are very challenging to control. This was first studied in [13] for the case of semilinear systems of Klein–Gordon equations with different speeds (see also [11] for a study of a system with different masses) and led in [14] to the first construction of global smooth solutions

[7]Another way to prevent shock formation is to introduce exponential damping of the perturbation via dissipation or relaxation (see, e.g., [41]). We will not discuss this at all in this paper.

[8]In many works (including [21] and [12], [22], [37]), the Poisson relation in (1.24) is replaced by

$$-\Delta\phi = 1 + \rho - e^{\phi}$$

but, for small perturbations, this agrees with (1.24) up to nonlinear corrections that can be easily handled.

for the Euler–Maxwell equation for electrons,

$$
\begin{aligned}
\partial_t n + \operatorname{div}((1+n)v) &= 0,\\
\partial_t v + v\cdot\nabla v + \nabla n &= -\left[E + v\times B\right],\\
\partial_t B + \nabla\times E &= 0,\\
\partial_t E - C\nabla\times B &= (1+n)v,
\end{aligned}
\tag{1.26}
$$

with constraints $\operatorname{div}(B)=0$ and $\operatorname{div}(E)=n$.

THEOREM 1.4 (Stability in the Euler–Maxwell system for electrons [14], [26]). *A solution of* (1.26) *with initial data* (n^0, v^0, E^0, B^0) *small, smooth, compactly supported, neutral and irrotational in the sense that*

$$
\int_{\mathbb{R}^3} n^0(x)dx = 0,\quad \int_{\mathbb{R}^3} B^0(x)dx = 0,\quad \nabla\times v^0 + CB^0 \equiv 0
$$

remains global and smooth and decays to 0 *in* L^∞.

This was first shown in [14] under additional generic conditions on the parameters. Later in [26], the generic condition was removed and a stronger (integrable) decay was obtained, providing a robust approach even in the quasilinear case. The model system is

$$
\begin{aligned}
\left(\partial_{tt} - \Delta + 1\right)\alpha &= \mathcal{Q}_1(\alpha,\beta,\nabla\alpha,\nabla\beta,\nabla^2\alpha,\nabla^2\beta),\\
\left(\partial_{tt} - C\Delta + 1\right)\beta &= \mathcal{Q}_2(\alpha,\beta,\nabla\alpha,\nabla\beta,\nabla^2\alpha,\nabla^2\beta).
\end{aligned}
\tag{1.27}
$$

It is important to note that the speed of the electron fluids is different from the speed of the magnetic field, so that new analytical tools are needed to estimate the 2-dimensional resonant sphere in the phase space. The main result of [26] is the natural analogue of Theorem 1.3, and it is the foundation of the approach we use in this work. Note that in this case, we also need to introduce a decay condition on the initial data in order to be able to perform a more refined analysis of the solutions.

1.4. *Organization of the paper.* In Section 2, we obtain a classical local well-posedness result in the energy space. In Section 3, we reduce the Euler–Maxwell system (1.15) into a quasilinear dispersive system and identify the linearized system, together with the main structure of the nonlinearity. In Section 4, we introduce the function space Z (see 4.5) and prove the main Theorem 1.1 assuming boundedness of the relevant bilinear integral operators as in (1.6)–(1.9). In Section 5, we study the case of nonresonant interactions for localized atoms. Sections 6, 7, and 8 are then devoted to the study of the resonant interactions.

In Section 6, we study Case A resonant interactions. We first make use of an efficient parametrization $p^{\sigma;\mu,\nu}$ in (6.5), (6.9)–(6.11), then control precisely the output of interactions of "atoms" by carefully designed $B^2_{k,j}$ norm defined in (4.5) as well as additional L^2 orthogonality argument in the spirit of [26].

In Section 7, we study Case B resonant interactions. We make use of a precise analytic characterization of Case B (Lemma 7.2), decay estimates Lemma A.5, as well as an orthogonality argument to control the L^2 norm to complete the analysis.

Section 8 is devoted to the study of Case C. We take advantage of the geometry of angles between η, ξ and $\xi-\eta$ to obtain extra regularity to overcome the singularity near zero frequency.

Finally, in Appendix A, we isolate relevant information on the structure of the dispersion relations λ_i, λ_e and λ_b and provide various stationary-phase estimates that are needed throughout the proof, and in Appendix B we classify the quadratic resonances that may appear.

Acknowledgments. The third author expresses his thanks to B. Texier and A. Cerfon for interesting discussions and helpful references. We would also like to thank Yu Deng for valuable discussions that clarified several arguments.

2. **Energy estimates and the local existence theory**

The local existence theory for (1.15) is based on energy estimates. These in turn are obtained from the physical energy. The (local) energy identity reads

$$\partial_t \mathfrak{e} + \mathrm{div}\,[\mathfrak{J}_e + \mathfrak{J}_i + \mathfrak{J}_b] = 0,$$
$$\mathfrak{e} := T\frac{n^2}{2} + \varepsilon(n+1)\frac{|v|^2}{2} + \frac{\rho^2}{2} + (\rho+1)\frac{|u|^2}{2} + \frac{|\tilde{E}|^2}{2} + \frac{C_b}{\varepsilon}\frac{|\tilde{B}|^2}{2},$$
$$\mathfrak{J}_e := \left\{Tn + \varepsilon\frac{|v|^2}{2}\right\}(n+1)v, \quad \mathfrak{J}_i := \left\{\rho + \frac{|u|^2}{2}\right\}(\rho+1)u, \quad \mathfrak{J}_b := \frac{C_b}{\varepsilon}\tilde{E}\times\tilde{B}.$$

From this, we obtain our higher-order energies. For any $(n, v, \rho, u, \tilde{E}, \tilde{B}) \in H^N$, we define

$$\begin{aligned}\mathcal{E}_N := \sum_{|\gamma|\le N}\int_{\mathbb{R}^3} \Big[T|D^\gamma_x n|^2 + \varepsilon(1+n)|D^\gamma_x v|^2 + |D^\gamma_x \rho|^2 \\ + (\rho+1)|D^\gamma_x u|^2 + |D^\gamma_x \tilde{E}|^2 + \frac{C_b}{\varepsilon}|D^\gamma_x \tilde{B}|^2\Big]\,dx.\end{aligned} \tag{2.1}$$

The following proposition is our local regularity result:

PROPOSITION 2.1. (i) *There is* $\delta_1 \in (0,1]$ *such that if*

$$\|(n^0, v^0, \rho^0, u^0, \tilde{E}^0, \tilde{B}^0)\|_{H^4} \le \delta_1, \tag{2.2}$$

then there is a unique solution $(n, v, \rho, u, \tilde{E}, \tilde{B}) \in C([0,1] : H^4)$ *of the system* (1.15) *with*

$$(n(0), v(0), \rho(0), u(0), \tilde{E}(0), \tilde{B}(0)) = (n^0, v^0, \rho^0, u^0, \tilde{E}^0, \tilde{B}^0).$$

Moreover,

$$\sup_{t\in[0,1]} \|(n(t), v(t), \rho(t), u(t), \tilde{E}(t), \tilde{B}(t))\|_{H^4} \lesssim \|(n^0, v^0, \rho^0, u^0, \tilde{E}^0, \tilde{B}^0)\|_{H^4}.$$

(ii) *If* $N \geq 4$ *and* $(n^0, v^0, \rho^0, u^0, \tilde{E}^0, \tilde{B}^0) \in H^N$ *satisfies* (2.2), *then*

$$(n, v, \rho, u, \tilde{E}, \tilde{B}) \in C([0,1] : H^N),$$

and

$$\mathcal{E}_N(t') - \mathcal{E}_N(t) \lesssim \int_t^{t'} A(s)\mathcal{E}_N(s)\, ds \tag{2.3}$$

for any $t \leq t' \in [0,1]$, *where*

(2.4)

$$\begin{aligned} A(s) := {} & \|\nabla n(s)\|_{L^\infty} + \|v(s)\|_{L^\infty} + \|\nabla v(s)\|_{L^\infty} + \|\nabla\rho(s)\|_{L^\infty} \\ & + \|u(s)\|_{L^\infty} + \|\nabla u(s)\|_{L^\infty} + \|\nabla\tilde{E}(s)\|_{L^\infty} + \|\tilde{B}(s)\|_{L^\infty} + \|\nabla\tilde{B}(s)\|_{L^\infty}. \end{aligned}$$

(iii) *If* $(n^0, v^0, \rho^0, u^0, \tilde{E}^0, \tilde{B}^0) \in H^4$ *satisfies* (2.2) *and, in addition,*

$$\operatorname{div}(\tilde{E}^0) + n^0 - \rho^0 = 0, \qquad \tilde{B}^0 = \varepsilon\nabla \times v^0 = -\nabla \times u^0,$$

then, for any $t \in [0,1]$,

$$\operatorname{div}(\tilde{E})(t) + n(t) - \rho(t) = 0, \qquad \tilde{B}(t) = \varepsilon\nabla \times v(t) = -\nabla \times u(t). \tag{2.5}$$

Proof of Proposition 2.1. We multiply each equation by a suitable factor and rewrite the system (1.15) as a symmetric hyperbolic system,

$$\begin{aligned} & T\partial_t n + T\sum_{k=1}^{3} v_k\partial_k n + T(1+n)\sum_{k=1}^{3}\partial_k v_k = 0, \\ & \varepsilon(1+n)\partial_t v_j + T(1+n)\partial_j n + \varepsilon(1+n)\sum_{k=1}^{3} v_k\partial_k v_j \\ & \qquad = -(1+n)\tilde{E}_j - (1+n)\sum_{k,m=1}^{3} \in_{jmk} v_m\tilde{B}_k, \\ & \partial_t\rho + \sum_{k=1}^{3} u_k\partial_k\rho + (1+\rho)\sum_{k=1}^{3}\partial_k u_k = 0, \\ & (1+\rho)\partial_t u_j + (1+\rho)\partial_j\rho + (1+\rho)\sum_{k=1}^{3} u_k\partial_k u_j \\ & \qquad = (1+\rho)\tilde{E}_j + (1+\rho)\sum_{k,m=1}^{3} \in_{jmk} u_m\tilde{B}_k, \\ & \frac{C_b}{\varepsilon}\partial_t\tilde{B}_j + \frac{C_b}{\varepsilon}\sum_{k,m=1}^{3} \in_{jmk} \partial_m\tilde{E}_k = 0, \\ & \partial_t\tilde{E}_j - \frac{C_b}{\varepsilon}\sum_{k,m=1}^{3} \in_{jmk} \partial_m\tilde{B}_k = (1+n)v_j - (1+\rho)u_j. \end{aligned}$$

Then we apply Theorems II and III in [32] to prove the local existence claim in part (i) and the propagation of regularity claim in part (ii).

To verify the energy inequality (2.3) we let, for $P = D_x^\gamma$, $|\gamma| \leq N$,

$$\mathcal{E}'_P := \int_{\mathbb{R}^3} \big[T|Pn|^2 + \varepsilon(1+n)|Pv|^2 + |P\rho|^2 + (1+\rho)|Pu|^2 + |P\tilde{E}|^2 + \frac{C_b}{\varepsilon}|P\tilde{B}|^2\big]\,dx.$$

Then we calculate

$$\frac{d}{dt}\mathcal{E}'_P = \mathrm{I}_P + \mathrm{II}_P + \mathrm{III}_P + \mathrm{I}'_P + \mathrm{II}'_P + \mathrm{III}'_P + \mathrm{IV}_P,$$

where

$$\mathrm{I}_P := \int_{\mathbb{R}^3} 2TPn \cdot P\partial_t n\,dx, \qquad \mathrm{I}'_P := \int_{\mathbb{R}^3} 2P\rho \cdot P\partial_t \rho\,dx,$$

$$\mathrm{II}_P := \sum_{j=1}^3 \varepsilon \int_{\mathbb{R}^3} \partial_t n \cdot Pv_j \cdot Pv_j\,dx, \qquad \mathrm{II}'_P := \sum_{j=1}^3 \int_{\mathbb{R}^3} \partial_t \rho \cdot Pu_j \cdot Pu_j\,dx,$$

$$\mathrm{III}_P := \sum_{j=1}^3 \varepsilon \int_{\mathbb{R}^3} 2(1+n) \cdot Pv_j \cdot P\partial_t v_j\,dx,$$

$$\mathrm{III}'_P := \sum_{j=1}^3 \int_{\mathbb{R}^3} 2(1+\rho) \cdot Pu_j \cdot P\partial_t u_j\,dx,$$

$$\mathrm{IV}_P := \sum_{j=1}^3 \int_{\mathbb{R}^3} 2P\tilde{E}_j \cdot P\partial_t \tilde{E}_j\,dx + \sum_{j=1}^3 \int_{\mathbb{R}^3} 2\frac{C_b}{\varepsilon} P\tilde{B}_j \cdot P\partial_t \tilde{B}_j\,dx.$$

We use the general bound

$$\|D_x^\rho f \cdot D_x^{\rho'} g\|_{L^2} \lesssim \|\nabla_x f\|_{L^\infty} \|g\|_{H^M} + \|\nabla_x g\|_{L^\infty} \|f\|_{H^M}, \tag{2.6}$$

provided that $|\rho| + |\rho'| \leq M+1$, $M \geq 1$, and $|\rho|, |\rho'| \geq 1$. Using also the equations, we estimate

$$\Big|\mathrm{I}_P + \sum_{k=1}^3 \int_{\mathbb{R}^3} 2TPn \cdot (1+n) \cdot P\partial_k v_k\,dx\Big| \lesssim A(t)\|(n,v,\rho,u,\tilde{E},\tilde{B})\|_{H^N}^2,$$

$$\Big|\mathrm{II}_P\Big| \lesssim A(t)\|(n,v,\rho,u,\tilde{E},\tilde{B})\|_{H^N}^2,$$

$$\Big|\mathrm{III}_P + \sum_{j=1}^3 \int_{\mathbb{R}^3} \big[2TP\partial_j n \cdot (1+n) \cdot Pv_j + 2P\tilde{E}_j \cdot Pv_j \cdot (1+n)\big]\,dx\Big| \lesssim A(t)\|(n,v,\rho,u,\tilde{E},\tilde{B})\|_{H^N}^2$$

and, similarly,

$$\Big|\mathrm{I}'_P + \sum_{k=1}^{3}\int_{\mathbb{R}^3} 2P\rho\cdot(1+\rho)\cdot P\partial_k u_k\,dx\Big| \lesssim A(t)\|(n,v,\rho,u,\tilde{E},\tilde{B})\|_{H^N}^2,$$

$$\Big|\mathrm{II}'_P\Big| \lesssim A(t)\|(n,v,\rho,u,\tilde{E},\tilde{B})\|_{H^N}^2,$$

$$\Big|\mathrm{III}'_P + \sum_{j=1}^{3}\int_{\mathbb{R}^3}\big[2P\partial_j\rho\cdot(1+\rho)\cdot Pu_j - 2P\tilde{E}_j\cdot Pu_j\cdot(1+\rho)\big]\,dx\Big| \lesssim A(t)\|(n,v,\rho,u,\tilde{E},\tilde{B})\|_{H^N}^2.$$

In addition,

$$\Big|\mathrm{IV}_P - \sum_{j=1}^{3}\int_{\mathbb{R}^3} 2P\tilde{E}_j\cdot[Pv_j\cdot(1+n) - Pu_j\cdot(1+\rho)]\ dx\Big| \lesssim A(t)\|(n,v,\rho,u,\tilde{E},\tilde{B})\|_{H^N}^2.$$

Therefore,

$$\Big|\frac{d}{dt}\mathcal{E}'_P\Big| \lesssim A(t)\|(n,v,\rho,u,\tilde{E},\tilde{B})\|_{H^N}^2,$$

and the bound (2.3) follows since

$$\mathcal{E}_N = \sum_{P=D_x^\gamma,\,|\gamma|\le N}\mathcal{E}'_P \approx \|(n,v,\rho,u,\tilde{E},\tilde{B})\|_{H^N}^2.$$

Finally, to verify that the identities (2.5) are propagated by the flow, we let

$$X := n-\rho+\mathrm{div}(\tilde{E}),\qquad Y := \tilde{B}-\varepsilon\nabla\times v,\qquad Z := \tilde{B}+\nabla\times u.$$

Using the equations in (1.15) we calculate

$$\begin{aligned}\partial_t X &= \partial_t n - \partial_t\rho + \sum_{j=1}^{3}\partial_j\partial_t\tilde{E}_j\\ &= -\sum_{j=1}^{3}\partial_j[(1+n)v_j-(1+\rho)u_j] + \sum_{j=1}^{3}\partial_j[(1+n)v_j-(1+\rho)u_j] = 0,\end{aligned}$$

therefore $X\equiv 0$. Moreover

$$\partial_t\Big(\sum_{k=1}^{3}\partial_k\tilde{B}_k\Big) = 0,$$

therefore

$$\sum_{k=1}^{3}\partial_k\tilde{B}_k\equiv 0,\qquad \sum_{k=1}^{3}\partial_k Y_k\equiv 0,\qquad \sum_{k=1}^{3}\partial_k Z_k\equiv 0.$$

Finally we notice that

$$\partial_t Y = \nabla \times (v \times Y), \qquad \partial_t Z = \nabla \times (u \times Z).$$

Using energy estimates it follows easily that $Y \equiv 0, Z \equiv 0$, as desired. □

3. Derivation of the main dispersive system

The main part of this paper is devoted to obtain global time integrability of the function A defined in (2.4), so as to be able to propagate energy control using (2.3). In order to do this, one needs to turn the system (1.15)–(1.17) into a quasilinear system of dispersive equations. This is the purpose of this section. The main results are summarized in Proposition 3.2.

In the rest of the paper, we use the standard convention that repeated indices are summed. For $\xi \in \mathbb{R}^3$ and $\alpha = 1, 2, 3$, we define

$$\begin{aligned} &|\nabla|(\xi) := |\xi|, \qquad R_\alpha(\xi) := i\xi_\alpha/|\xi|, \qquad Q_{\alpha\beta}(\xi) := i \in_{\alpha\gamma\beta} \xi_\gamma/|\xi|, \\ &H_1(\xi) := \sqrt{1+|\xi|^2}, \qquad H_\varepsilon(\xi) := \varepsilon^{-1/2}\sqrt{1+T|\xi|^2}, \\ &\Lambda_b(\xi) := \varepsilon^{-1/2}\sqrt{1+\varepsilon+C_b|\xi|^2}. \end{aligned} \tag{3.1}$$

By a slight abuse of notation, we also let $|\nabla|, R_\alpha, Q, H_1, H_\varepsilon, \Lambda_b$ denote the operators on $\mathbb{R}^3$ defined by the corresponding Fourier multipliers. Notice that

$$Q^3 = Q \qquad \text{and} \qquad QA = |\nabla|^{-1}(\nabla \times A) \text{ for any vector-field } A.$$

Closer inspection of the system (1.15)–(1.17) shows a decoupling at the linear level of the magnetic unknowns curl(E), B and the electrostatic (Euler–Poisson) unknowns n, ρ, div(v) and div(u). More precisely, we may define

$$2U_b := \Lambda_b|\nabla|^{-1}Q\tilde{B} - iQ^2\tilde{E}, \qquad h := -|\nabla|^{-1}\mathrm{div}(v), \qquad g := -|\nabla|^{-1}\mathrm{div}(u).$$

Recalling that $\tilde{B} = \varepsilon\nabla\times v = -\nabla\times u$ and $\mathrm{div}(\tilde{E}) = \rho - n$, the functions U_b, h, g together with n, ρ allow us to recover all the physical unknowns, i.e.,

$$\begin{aligned} \tilde{B} &= 2\Lambda_b^{-1}|\nabla|Q\mathrm{Re}(U_b), \\ v &= \nabla|\nabla|^{-1}h + \frac{2}{\varepsilon}\Lambda_b^{-1}\mathrm{Re}(U_b), \\ u &= \nabla|\nabla|^{-1}g - 2\Lambda_b^{-1}\mathrm{Re}(U_b), \\ \tilde{E} &= -\nabla|\nabla|^{-2}[\rho - n] - 2\mathrm{Im}(U_b). \end{aligned} \tag{3.2}$$

Let

$$A_\alpha = 2\Lambda_b^{-1}\mathrm{Re}(U_{b,\alpha}).$$

In terms of n,h,ρ,g,U_b the system (1.15)–(1.17) becomes

(3.3)
$$\begin{aligned}
\partial_t n-|\nabla|h&=-\partial_\alpha\left[nR_\alpha h\right]-(1/\varepsilon)\partial_\alpha\left[nA_\alpha\right],\\
\partial_t\rho-|\nabla|g&=-\partial_\alpha\left[\rho R_\alpha g\right]+\partial_\alpha\left[\rho A_\alpha\right],\\
\partial_t h+|\nabla|^{-1}H_\varepsilon^2 n-\varepsilon^{-1}|\nabla|^{-1}\rho&=-(1/2)|\nabla|\left[R_\alpha hR_\alpha h\right]\\
&\quad-\varepsilon^{-1}|\nabla|\left[R_\alpha hA_\alpha\right]-(\varepsilon^{-2}/2)|\nabla|\left[A_\alpha A_\alpha\right],\\
\partial_t g-|\nabla|^{-1}n+|\nabla|^{-1}H_1^2\rho&=-(1/2)|\nabla|\left[R_\alpha gR_\alpha g\right]\\
&\quad+|\nabla|\left[R_\alpha gA_\alpha\right]-(1/2)|\nabla|\left[A_\alpha A_\alpha\right],\\
\partial_t U_{b,\alpha}+i\Lambda_bU_{b,\alpha}&=-(i/2)Q^2_{\alpha\beta}[nR_\beta h-\rho R_\beta g+\varepsilon^{-1}nA_\beta+\rho A_\beta],
\end{aligned}$$

where the left-hand sides of the equations above are linear in the variables n,h,ρ,g,U_b and the right-hand sides are quadratic.

We make linear changes of variables to diagonalize this system. Let

(3.4)
$$\begin{aligned}
\Lambda_e&:=\varepsilon^{-1/2}\sqrt{\frac{(1+\varepsilon)-(T+\varepsilon)\Delta+\sqrt{((1-\varepsilon)-(T-\varepsilon)\Delta)^2+4\varepsilon}}{2}},\\
\Lambda_i&:=\varepsilon^{-1/2}\sqrt{\frac{(1+\varepsilon)-(T+\varepsilon)\Delta-\sqrt{((1-\varepsilon)-(T-\varepsilon)\Delta)^2+4\varepsilon}}{2}},
\end{aligned}$$

such that

(3.5)
$$(\Lambda_e^2-H_\varepsilon^2)(H_\varepsilon^2-\Lambda_i^2)=\varepsilon^{-1},\qquad \Lambda_e^2-H_1^2=H_\varepsilon^2-\Lambda_i^2.$$

Let

(3.6)
$$R:=\sqrt{\frac{\Lambda_e^2-H_\varepsilon^2}{H_\varepsilon^2-\Lambda_i^2}},$$

and notice that

(3.7)
$$\Lambda_e^2-H_\varepsilon^2=\varepsilon^{-1/2}R,\qquad H_\varepsilon^2-\Lambda_i^2=\varepsilon^{-1/2}R^{-1}.$$

Let

(3.8)
$$\begin{aligned}
U_e&:=\frac{1}{2\sqrt{1+R^2}}\big[-\varepsilon^{1/2}|\nabla|^{-1}\Lambda_e n+R|\nabla|^{-1}\Lambda_e\rho-i\varepsilon^{1/2}h+iRg\big],\\
U_i&:=\frac{1}{2\sqrt{1+R^2}}\big[\varepsilon^{1/2}R|\nabla|^{-1}\Lambda_i n+|\nabla|^{-1}\Lambda_i\rho+i\varepsilon^{1/2}Rh+ig\big].
\end{aligned}$$

Note that, since $R(0)=\sqrt{\varepsilon}$ and $\rho-n=\mathrm{div}(\tilde{E})$, U_e is not singular at the 0 frequency, and since $\Lambda_i(0)=0$, neither is U_i. Using the system (3.3) it is easy to check that the complex variables U_e, U_i and U_b satisfy the identities

(3.9)
$$\begin{aligned}
(\partial_t+i\Lambda_e)U_e&=\mathcal{N}_e,\\
(\partial_t+i\Lambda_i)U_i&=\mathcal{N}_i,\\
(\partial_t+i\Lambda_b)U_{b,\alpha}&=\mathcal{N}_{b,\alpha},
\end{aligned}$$

where

(3.10)

$$\begin{aligned}
\Re(\mathcal{N}_e) &= \frac{\Lambda_e R_\alpha}{2\sqrt{1+R^2}}\big[\varepsilon^{1/2}(nR_\alpha h) - R(\rho R_\alpha g) + \varepsilon^{-1/2}(nA_\alpha) + R(\rho A_\alpha)\big],\\
\Im(\mathcal{N}_e) &= \frac{|\nabla|}{4\sqrt{1+R^2}}\big[\varepsilon^{-3/2}(\varepsilon R_\alpha h + A_\alpha)(\varepsilon R_\alpha h + A_\alpha)\\
&\qquad\qquad - R[(R_\alpha g - A_\alpha)(R_\alpha g - A_\alpha)]\big],\\
\Re(\mathcal{N}_i) &= \frac{-\Lambda_i R_\alpha}{2\sqrt{1+R^2}}\big[\varepsilon^{1/2}R(nR_\alpha h) + (\rho R_\alpha g) + \varepsilon^{-1/2}R(nA_\alpha) - (\rho A_\alpha)\big],\\
\Im(\mathcal{N}_i) &= \frac{-|\nabla|}{4\sqrt{1+R^2}}\big[\varepsilon^{-3/2}R[(\varepsilon R_\alpha h + A_\alpha)(\varepsilon R_\alpha h + A_\alpha)]\\
&\qquad\qquad + (R_\alpha g - A_\alpha)(R_\alpha g - A_\alpha)\big],\\
\Re(\mathcal{N}_{b,\alpha}) &= 0,\\
\Im(\mathcal{N}_{b,\alpha}) &= -(1/2)Q^2_{\alpha\beta}\left[nR_\beta h - \rho R_\beta g + \varepsilon^{-1}nA_\beta + \rho A_\beta\right].
\end{aligned}$$

The system (3.9) is our main dispersive system, which is diagonalized at the linear level. To analyze it we have to express the nonlinearities $\mathcal{N}_e$, $\mathcal{N}_i$, and $\mathcal{N}_{b,\alpha}$ in terms of the complex variables U_e, U_i, and U_b. Indeed, it follows from (3.8) that

$$\begin{aligned}
n &= \frac{-|\nabla|\varepsilon^{-1/2}}{\sqrt{1+R^2}\Lambda_e}(U_e + \overline{U_e}) + \frac{|\nabla|\varepsilon^{-1/2}R}{\sqrt{1+R^2}\Lambda_i}(U_i + \overline{U_i}),\\
\rho &= \frac{|\nabla|R}{\sqrt{1+R^2}\Lambda_e}(U_e + \overline{U_e}) + \frac{|\nabla|}{\sqrt{1+R^2}\Lambda_i}(U_i + \overline{U_i}),\\
h &= \frac{i\varepsilon^{-1/2}}{\sqrt{1+R^2}}(U_e - \overline{U_e}) + \frac{-i\varepsilon^{-1/2}R}{\sqrt{1+R^2}}(U_i - \overline{U_i}),\\
g &= \frac{-iR}{\sqrt{1+R^2}}(U_e - \overline{U_e}) + \frac{-i}{\sqrt{1+R^2}}(U_i - \overline{U_i}),\\
A_\alpha &= \Lambda_b^{-1}(U_{b,\alpha} + \overline{U_{b,\alpha}}).
\end{aligned} \tag{3.11}$$

We summarize now the main results we proved in this section. Recall first the definitions of the main multipliers

(3.12)

$$\begin{aligned}
\Lambda_e(\xi) &:= \varepsilon^{-1/2}\sqrt{\frac{(1+\varepsilon) + (T+\varepsilon)|\xi|^2 + \sqrt{((1-\varepsilon)+(T-\varepsilon)|\xi|^2)^2 + 4\varepsilon}}{2}},\\
\Lambda_i(\xi) &:= \varepsilon^{-1/2}\sqrt{\frac{(1+\varepsilon) + (T+\varepsilon)|\xi|^2 - \sqrt{((1-\varepsilon)+(T-\varepsilon)|\xi|^2)^2 + 4\varepsilon}}{2}},\\
\Lambda_b(\xi) &:= \varepsilon^{-1/2}\sqrt{1+\varepsilon+C_b|\xi|^2},
\end{aligned}$$

and

$$
\begin{aligned}
|\nabla|(\xi) &:= |\xi|, \qquad R_\alpha(\xi) := i\xi_\alpha/|\xi|,\\
Q_{\alpha\beta}(\xi) &:= i \in_{\alpha\gamma\beta} \xi_\gamma/|\xi|, \qquad H_1(\xi) := \sqrt{1+|\xi|^2},\\
H_\varepsilon(\xi) &:= \varepsilon^{-1/2}\sqrt{1+T|\xi|^2},\\
R(\xi) &:= [\Lambda_e(\xi)^2 - H_\varepsilon(\xi)^2]^{1/2}[H_\varepsilon(\xi)^2-\Lambda_i(\xi)^2]^{-1/2}.
\end{aligned}
\tag{3.13}
$$

The lemma below describes symbol-type properties of some of these multipliers.

Lemma 3.1. *In* $\mathbb{R}^3$, *we have*

$$
\Lambda_e^2 \ge H_\varepsilon^2 \ge H_1^2 \ge \Lambda_i^2 \ge |\nabla|^2, \qquad \Lambda_i^2 \lesssim |\nabla|^2,
\tag{3.14}
$$

and

$$
\begin{aligned}
\Lambda_e^2 - H_\varepsilon^2 = \varepsilon^{-1/2}R, &\qquad H_\varepsilon^2-\Lambda_i^2 = \varepsilon^{-1/2}R^{-1},\\
\Lambda_e(\xi)^2 - H_\varepsilon(\xi)^2 &= H_1(\xi)^2 - \Lambda_i(\xi)^2\\
&= \frac{2}{(1-\varepsilon)+(T-\varepsilon)|\xi|^2+\sqrt{\big((1-\varepsilon)+(T-\varepsilon)|\xi|^2\big)^2+4\varepsilon}}.
\end{aligned}
\tag{3.15}
$$

In addition, for $\alpha=(\alpha_1,\alpha_2,\alpha_3)$, *we have the symbol-type estimates*

$$
\begin{aligned}
&|D_\xi^\alpha \Lambda_e(\xi)| + |D_\xi^\alpha H_\varepsilon(\xi)| + |D_\xi^\alpha H_1(\xi)| \lesssim_{|\alpha|} (1+|\xi|)^{1-|\alpha|},\\
&|D_\xi^\alpha \Lambda_i(\xi)| + |D_\xi^\alpha |\nabla|(\xi)| \lesssim_{|\alpha|} |\xi|^{1-|\alpha|},\\
&|D_\xi^\alpha R(\xi)| \lesssim_{|\alpha|} (1+|\xi|)^{-2-|\alpha|}.
\end{aligned}
\tag{3.16}
$$

Proof of Lemma 3.1. The inequalities in (3.14) and the identities in (3.15) follow directly from definitions. The symbol-type estimates in (3.16) also follow from definitions and the additional formula

$$
R(\xi) = \frac{2\varepsilon^{1/2}}{(1-\varepsilon)+(T-\varepsilon)|\xi|^2+\sqrt{\big((1-\varepsilon)+(T-\varepsilon)|\xi|^2\big)^2+4\varepsilon}}. \qquad \square
$$

The following proposition is the main result in this section.

Proposition 3.2. *With* $N_0 = 10^4$ *as in Theorem* 1.1, *assume that*

$$
(n, v, \rho, u, \tilde{E}, \tilde{B}) \in C(I : H^{N_0})
$$

is a solution of the system (1.15)–(1.17)*, where* $I \subseteq \mathbb{R}$ *is an interval. Let* $\Lambda_e, \Lambda_i, \Lambda_b, |\nabla|, R_\alpha, Q, H_1, H_\varepsilon, R$ *denote the operators defined by the corresponding multipliers in* (3.12)–(3.13)*. Let*

$$\begin{aligned}
&h := -|\nabla|^{-1}\mathrm{div}(v), \qquad g := -|\nabla|^{-1}\mathrm{div}(u), \qquad A_\alpha := |\nabla|^{-1}Q_{\alpha\beta}\tilde{B}_\beta,\\
&U_e := \frac{1}{2\sqrt{1+R^2}}\big[-\varepsilon^{1/2}|\nabla|^{-1}\Lambda_e n + R|\nabla|^{-1}\Lambda_e\rho - i\varepsilon^{1/2}h + iRg\big],\\
&U_i := \frac{1}{2\sqrt{1+R^2}}\big[\varepsilon^{1/2}R|\nabla|^{-1}\Lambda_i n + |\nabla|^{-1}\Lambda_i\rho + i\varepsilon^{1/2}Rh + ig\big],\\
&U_b := [\Lambda_b|\nabla|^{-1}Q\tilde{B} - iQ^2\tilde{E}]/2
\end{aligned} \tag{3.17}$$

and, for $\alpha \in \{1,2,3\}$,

$$\begin{aligned}
&U_{e+} := U_e, \quad U_{e-} := \overline{U_e}, \quad U_{i+} := U_i,\\
&U_{i-} := \overline{U_i}, \quad U_{b+\alpha} := U_{b,\alpha}, \quad U_{b-\alpha} := \overline{U_{b,\alpha}}.
\end{aligned}$$

(i) *Then* $U_e, U_i, U_b \in C(I : H^{N_0})$ *and, for any* $t \in I$,

$$\|U_e(t)\|_{H^{N_0}} + \|U_i(t)\|_{H^{N_0}} + \|U_b(t)\|_{H^{N_0}} \lesssim \|(n(t), v(t), \rho(t), u(t), \tilde{E}(t), \tilde{B}(t))\|_{H^{N_0}}. \tag{3.18}$$

Moreover, the functions $U_e : \mathbb{R}^3 \times I \to \mathbb{C}$, $U_i : \mathbb{R}^3 \times I \to \mathbb{C}$, $U_b : \mathbb{R}^3 \times I \to \mathbb{C}^3$ *satisfy the dispersive system*

$$(\partial_t + i\Lambda_e)U_e = \mathcal{N}_e, \qquad (\partial_t + i\Lambda_i)U_i = \mathcal{N}_i, \qquad (\partial_t + i\Lambda_b)U_b = \mathcal{N}_b, \tag{3.19}$$

where the quadratic nonlinearities $\mathcal{N}_e, \mathcal{N}_i, \mathcal{N}_b$ *are given by*

$$\begin{aligned}
\mathcal{F}(\mathcal{N}_e)(\xi,t) &= c\sum_{\mu,\nu\in\mathcal{I}_0}\int_{\mathbb{R}^3} m_{e;\mu,\nu}(\xi,\eta)\widehat{U_\mu}(\xi-\eta,t)\widehat{U_\nu}(\eta,t)\,d\eta,\\
\mathcal{F}(\mathcal{N}_i)(\xi,t) &= c\sum_{\mu,\nu\in\mathcal{I}_0}\int_{\mathbb{R}^3} m_{i;\mu,\nu}(\xi,\eta)\widehat{U_\mu}(\xi-\eta,t)\widehat{U_\nu}(\eta,t)\,d\eta,\\
\mathcal{F}(\mathcal{N}_b)(\xi,t) &= c\sum_{\mu,\nu\in\mathcal{I}_0}\int_{\mathbb{R}^3} m_{b;\mu,\nu}(\xi,\eta)\widehat{U_\mu}(\xi-\eta,t)\widehat{U_\nu}(\eta,t)\,d\eta.
\end{aligned} \tag{3.20}$$

The set $\mathcal{I}_0$ *is given by*

$$\mathcal{I}_0 := \{e+, e-, i+, i-, b+1, b+2, b+3, b-1, b-2, b-3\}, \tag{3.21}$$

and the multipliers $m_{e;\mu,\nu} : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{C}$, $m_{i;\mu,\nu} : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{C}$, $m_{b;\mu,\nu} : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{C}^3$ *are as in Lemma* 3.3 *below.*

(ii) *The physical variables* $(n, \rho, v, u, \tilde{E}, \tilde{B})$ *can be expressed in terms of the complex variables* U_e, U_i, U_b *according to the formulas*

$$
\begin{aligned}
n &= \frac{-|\nabla|\varepsilon^{-1/2}}{\sqrt{1+R^2}\Lambda_e}(U_e + \overline{U_e}) + \frac{|\nabla|\varepsilon^{-1/2}R}{\sqrt{1+R^2}\Lambda_i}(U_i + \overline{U_i}),\\
\rho &= \frac{|\nabla|R}{\sqrt{1+R^2}\Lambda_e}(U_e + \overline{U_e}) + \frac{|\nabla|}{\sqrt{1+R^2}\Lambda_i}(U_i + \overline{U_i}),\\
v &= \nabla|\nabla|^{-1}h + \frac{2}{\varepsilon}\Lambda_b^{-1}Re(U_b),\\
h &= \frac{i\varepsilon^{-1/2}}{\sqrt{1+R^2}}(U_e - \overline{U_e}) + \frac{-i\varepsilon^{-1/2}R}{\sqrt{1+R^2}}(U_i - \overline{U_i}),\\
u &= \nabla|\nabla|^{-1}g - 2\Lambda_b^{-1}Re(U_b),\\
g &= \frac{-iR}{\sqrt{1+R^2}}(U_e - \overline{U_e}) + \frac{-i}{\sqrt{1+R^2}}(U_i - \overline{U_i}),\\
\tilde{E} &= -\nabla|\nabla|^{-2}\left[\rho - n\right] - 2Im(U_b),\\
\tilde{B} &= 2\Lambda_b^{-1}|\nabla|Q Re(U_b).
\end{aligned}
\tag{3.22}
$$

Proof of Proposition 3.2. The claim (3.18) is a consequence of (3.16) and the observation that $R(0) = \varepsilon^{1/2}$. The diagonalized dispersive system (3.19) and the identities (3.22) were derived earlier; see (3.9)–(3.10), (3.2), and (3.11). It remains only to prove the formulas (3.20), showing that the nonlinearities $\mathcal{N}_e, \mathcal{N}_i, \mathcal{N}_b$ can be expressed as bilinear forms in terms of the complex variables U_e, U_i, U_b. This is easy to see by inspecting the formulas (3.10) and (3.11). □

The precise formulas of the multipliers $m_{e;\mu,\nu}$, $m_{i;\mu,\nu}$, and $m_{b;\mu,\nu}$ are complicated. However, we do not use these formulas in the rest of the paper. We will only use the simple observation that these multipliers can be expressed as suitable products of multipliers satisfying inequalities of the Hörmander–Michlin type. More precisely, for any integer $n \geq 1$, let

$$
\mathcal{S}^n := \{q : \mathbb{R}^3 \to \mathbb{C} : \ \|q\|_{\mathcal{S}^n} := \sup_{\xi\in\mathbb{R}^3\setminus\{0\}} \sup_{|\alpha|\leq n} |\xi|^{|\alpha|}|D^\alpha_\xi q(\xi)| < \infty\}
\tag{3.23}
$$

and

$$
\mathcal{M} := \{m : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{C} : \ m(\xi,\eta) = q_1(\xi)\cdot q_2(\xi-\eta)\cdot q_3(\eta), \ \sup_{n\in\{1,2,3\}} \|q_n\|_{\mathcal{S}^{100}} \leq 1\}.
\tag{3.24}
$$

Lemma 3.3. *The multipliers* $m_{e;\mu,\nu}(\xi,\eta)$ *and* $m_{b,\alpha;\mu,\nu}(\xi,\eta)$, $\alpha \in \{1,2,3\}$, *can be written as finite sums of functions of the form*

$$
(1+|\xi|^2)^{1/2}\cdot m(\xi,\eta), \qquad m \in \mathcal{M}.
\tag{3.25}
$$

Similarly, the multipliers $m_{i;\mu,\nu}(\xi,\eta)$ *can be written as finite sums of functions of the form*

$$|\xi|\cdot m(\xi,\eta),\qquad m\in\mathcal{M}. \tag{3.26}$$

Remark 3.4. We notice that the multipliers $m_{i;\mu,\nu}$ satisfy better estimates at $\xi=0$ than the multipliers $m_{e;\mu,\nu}$ and $m_{b,\alpha;\mu,\nu}$; in particular, these multipliers vanish at the origin. This is an indication of a certain *null structure* of the system and is important in the analysis in Sections 7 and 8.

Proof of Lemma 3.3. The formulas (3.25) and (3.26) follow from the identities (3.10)–(3.11) and Lemma 3.1. Indeed, using (3.11) and Lemma 3.1, we notice first that the functions n,ρ,h,g,A_α can all be written as finite sums of Calderón–Zygmund operators applied to the complex variables $U_{e\pm},U_{i\pm},U_{b\pm\alpha}$, i.e., finite sums of expressions of the form

$$TU_{e\pm},\quad TU_{i\pm},\quad TU_{b\pm\alpha},\quad \text{where}\quad \widehat{Tf}(\xi)=q(\xi)\widehat{f}(\xi)\quad\text{for some}\quad q\in\mathcal{S}^{100}.$$

Then we again use Lemma 3.1 and the identities in (3.10) to complete the proof of the lemma. □

4. Main definitions and propositions

In this section we define our main function spaces and state two key propositions that concern properties of solutions of the dispersive system (3.19). Then we show how to use these propositions, together with the local regularity theory in Section 2 and linear dispersive bounds, to complete the proof of the main theorem.

We fix $\varphi:\mathbb{R}\to[0,1]$ as an even smooth function supported in $[-8/5,8/5]$ and equal to 1 in $[-5/4,5/4]$. For simplicity of notation, we also let $\varphi:\mathbb{R}^d\to[0,1]$ denote the corresponding radial function on $\mathbb{R}^d$, $d=2,3$. For $d\in\{1,2,3\}$, let

$$\varphi_k(x)=\varphi_{k,(d)}(x):=\varphi(|x|/2^k)-\varphi(|x|/2^{k-1})\quad\text{for any } k\in\mathbb{Z},\ x\in\mathbb{R}^d,$$
$$\varphi_I:=\sum_{m\in I\cap\mathbb{Z}}\varphi_m\quad\text{for any } I\subseteq\mathbb{R}.$$

Let

$$\mathcal{J}:=\{(k,j)\in\mathbb{Z}\times\mathbb{Z}_+:\ k+j\geq 0\}.$$

The restriction $j+k\geq 0$ is consistent with the uncertainty principle. In addition, we only control $j\geq 0$ since we are primarily interested in large spatial scales. For any $(k,j)\in\mathcal{J}$, let

$$\widetilde{\varphi}^{(k)}_j(x):=\begin{cases}\varphi_{(-\infty,-k]}(x) & \text{if } k+j=0 \text{ and } k\leq 0,\\ \varphi_{(-\infty,0]}(x) & \text{if } j=0 \text{ and } k\geq 0,\\ \varphi_j(x) & \text{if } k+j\geq 1 \text{ and } j\geq 1.\end{cases}$$

Notice that, for any $k \in \mathbb{Z}$ fixed,

$$\sum_{j \geq -\min(k,0)} \widetilde{\varphi}_j^{(k)} = 1.$$

For any interval $I \subseteq \mathbb{R}$, let

$$\widetilde{\varphi}_I^{(k)}(x) := \sum_{j \in I,\, (k,j) \in \mathcal{J}} \widetilde{\varphi}_j^{(k)}(x).$$

Let P_k, $k \in \mathbb{Z}$, denote the operator on $\mathbb{R}^3$ defined by the Fourier multiplier $\xi \to \varphi_k(\xi)$. Similarly, for any $I \subseteq \mathbb{R}$, let P_I denote the operator on $\mathbb{R}^3$ defined by the Fourier multiplier $\xi \to \varphi_I(\xi)$.

Definition 4.1. Let

$$\beta := 1/100, \qquad \alpha := \beta/2, \qquad \gamma := 3/2 - 4\beta. \tag{4.1}$$

We define

$$Z := \{ f \in L^2(\mathbb{R}^3) : \|f\|_Z := \sup_{(k,j) \in \mathcal{J}} \|\widetilde{\varphi}_j^{(k)}(x) \cdot P_k f(x)\|_{B_{k,j}} < \infty \} \tag{4.2}$$

where, with $\widetilde{k} := \min(k,0)$ and $k_+ := \max(k,0)$,

$$\|g\|_{B_{k,j}} := \inf_{g = g_1 + g_2} \big[\|g_1\|_{B^1_{k,j}} + \|g_2\|_{B^2_{k,j}} \big], \tag{4.3}$$

$$\|h\|_{B^1_{k,j}} := (2^{\alpha k} + 2^{10k}) \big[2^{(1+\beta)j} \|h\|_{L^2} + 2^{(1/2-\beta)\widetilde{k}} \|\widehat{h}\|_{L^\infty} \big], \tag{4.4}$$

and

$$\begin{aligned} \|h\|_{B^2_{k,j}} := 2^{10|k|}(2^{\alpha k} + 2^{10k}) \big[& 2^{(1-\beta)j} \|h\|_{L^2} + \|\widehat{h}\|_{L^\infty} \\ & + 2^{\gamma j} \sup_{R \in [2^{-j}, 2^k],\, \xi_0 \in \mathbb{R}^3} R^{-2} \|\widehat{h}\|_{L^1(B(\xi_0, R))} \big]. \end{aligned} \tag{4.5}$$

The Z norm is our main tool to capture the dispersive character of solutions. It has been introduced by two of the authors in [26], in the context of Klein–Gordon system with different speeds. It has two basic properties: (1) it gives integrable decay of the solution (see Lemma A.5), and (2) it can be propagated by the nonlinear flow (see Proposition 4.3). It is also invariant under the action of Calderón–Zygmund operators, which is a useful feature given the structure of the nonlinearities described in Proposition 3.2.

To understand the Z norm, one can think that the $B^1_{k,j}$ is the easiest norm that one would want to use; in particular, its x-integrability of the L^2-norm is sufficient to obtain the needed $1/t$ decay after we apply the linear flow. However, the $B^2_{k,j}$ norm is forced upon us by the presence of space-time resonances. It has slightly too weak decay, but this is compensated for by the last term that captures the 2-dimensional property of the support.

The component $B^2_{k,j}$ is important only at middle frequencies $|k| \lesssim 1$, when j is large; the factor $2^{10|k|}$ in front of the norm guarantees that the $B^2_{k,j}$ norm becomes less and less relevant when $|k|$ increases. One should think that this norm is used to measure functions that have thin, essentially 2-dimensional Fourier support contained in a neighborhood of the set of space-time resonances.

Finally, the weights in k in (4.4) are chosen such that at the uncertainty principle $k + j = 0$, all norms should be comparable for a normalized bump function supported essentially at frequency $\approx 2^k$ and distance $\lesssim 2^j$ from the origin.

The definition above shows that if $\|f\|_Z \leq 1$ then, for any $(k, j) \in \mathcal{J}$, one can decompose

$$\widetilde{\varphi}^{(k)}_j \cdot P_k f = (2^{\alpha k} + 2^{10k})^{-1}(g + h), \tag{4.6}$$

where[9]

$$g = g \cdot \widetilde{\varphi}^{(k)}_{[j-2,j+2]}, \qquad h = h \cdot \widetilde{\varphi}^{(k)}_{[j-2,j+2]}, \tag{4.7}$$

and

$$\begin{aligned} &2^{(1+\beta)j}\|g\|_{L^2} + 2^{(1/2-\beta)\widetilde{k}}\|\widehat{g}\|_{L^\infty} \lesssim 1,\\ &2^{(1-\beta)j}\|h\|_{L^2} + \|\widehat{h}\|_{L^\infty} + 2^{\gamma j} \sup_{R\in[2^{-j},2^k],\, \xi_0\in\mathbb{R}^3} R^{-2}\|\widehat{h}\|_{L^1(B(\xi_0,R))} \lesssim 2^{-10|k|}. \end{aligned} \tag{4.8}$$

In some of the easier estimates we will often use the weaker bound, obtained by setting $R = 2^k$,

$$\begin{aligned} &2^{(1+\beta)j}\|g\|_{L^2} + 2^{(1/2-\beta)\widetilde{k}}\|\widehat{g}\|_{L^\infty} \lesssim 1,\\ &2^{(1-\beta)j}\|h\|_{L^2} + \|\widehat{h}\|_{L^\infty} + 2^{\gamma j}\|\widehat{h}\|_{L^1} \lesssim 2^{-8|k|}. \end{aligned} \tag{4.9}$$

We are now ready to state our main propositions which concern solutions $U = (U_e, U_i, U_b)$ of the system (3.19)–(3.20) derived in Proposition 3.2. We claim first that smooth solutions that start with data in the space Z remain in the space Z, in a continuous way. More precisely:

Proposition 4.2. *Assume* $N_0 = 10^4$, $T_0 \geq 1$, *and* $U = (U_e, U_i, U_b) \in C([0, T_0] : H^{N_0})$ *is a solution of the system of equations* (3.19)–(3.20). *Assume that, for some* $t_0 \in [0, T_0]$,

$$e^{it_0\Lambda_\sigma} U_\sigma(t_0) \in Z \qquad \text{for } \sigma \in \{e, i, b\}. \tag{4.10}$$

[9]The support condition (4.7) can easily be achieved by starting with a decomposition $\widetilde{\varphi}^{(k)}_j \cdot P_k f = (2^{\alpha k} + 2^{10k})^{-1}(g' + h')$ that minimizes the $B_{k,j}$ norm up to a constant and then redefining $g := g' \cdot \widetilde{\varphi}^{(k)}_{[j-1,j+1]}$ and $h := h' \cdot \widetilde{\varphi}^{(k)}_{[j-1,j+1]}$.

Then there is

$$\tau = \tau\Big(T_0, \sup_{\sigma\in\{e,i,b\}} \|e^{it_0\Lambda_\sigma}U_\sigma(t_0)\|_Z, \sup_{\sigma\in\{e,i,b\}}\sup_{t\in[0,T_0]} \|U_\sigma(t)\|_{H^{N_0}}\Big) > 0$$

such that

$$\sup_{t\in[0,T_0]\cap[t_0,t_0+\tau]}\ \sup_{\sigma\in\{e,i,b\}} \|e^{it\Lambda_\sigma}U_\sigma(t)\|_Z \le 2 \sup_{\sigma\in\{e,i,b\}} \|e^{it_0\Lambda_\sigma}U_\sigma(t_0)\|_Z, \tag{4.11}$$

and the mapping $t \to e^{it\Lambda_\sigma}U_\sigma(t)$ *is continuous from* $[0,T_0]\cap[t_0,t_0+\tau]$ *to* Z, *for any* $\sigma \in \{e,i,b\}$.

The proof of Proposition 4.2 is very similar to the proof of Proposition 2.4 in [26]. For any integer $J \ge 0$ and $f \in H^{N_0}$, we define

$$\|f\|_{Z_J} := \sup_{(k,j)\in\mathcal{J}} 2^{\min(0,2J-2j)}\|\widetilde{\varphi}^{(k)}_j(x)\cdot P_kf(x)\|_{B_{k,j}},$$

compare with Definition 4.1, and notice that

$$\|f\|_{Z_J} \le \|f\|_Z, \qquad \|f\|_{Z_J} \lesssim_J \|f\|_{H^{N_0}}.$$

The main point is show that if $t \le t' \in [0,T_0]\cap[t_0,t_0+1]$ and $J \in \mathbb{Z}_+$ then

$$\begin{aligned}\sup_{\sigma\in\{e,i,b\}} &\|e^{it'\Lambda_\sigma}U_\sigma(t') - e^{it\Lambda_\sigma}U_\sigma(t)\|_{Z_J}\\ &\le \widetilde{C}|t'-t|(1+\sup_{s\in[t,t']}\sup_{\sigma\in\{e,i,b\}}\|e^{is\Lambda_\sigma}U_\sigma(s)\|_{Z_J})^2,\end{aligned}$$

with a suitable constant $\widetilde{C}$ that may depend only on

$$T_0, \qquad \sup_{\sigma\in\{e,i,b\}}\sup_{t\in[0,T_0]}\|U_\sigma(t)\|_{H^{N_0}}, \qquad \sup_{\sigma\in\{e,i,b\}}\|e^{it_0\Lambda_\sigma}U_\sigma(t_0)\|_Z.$$

This is very similar to the proof of the corresponding estimate (3.2) in [26], and we refer the reader there for the details.

The key proposition in the paper is the following bootstrap estimate:

Proposition 4.3. *Assume* $N_0 = 10^4$, $T_0 \ge 0$, *and* $U = (U_e,U_i,U_b) \in C([0,T_0] : H^{N_0})$ *is a solution of the system of equations* (3.19)–(3.20). *Assume that*

$$\sup_{t\in[0,T_0]}\sup_{\sigma\in\{e,i,b\}}\|e^{it\Lambda_\sigma}U_\sigma(t)\|_{H^{N_0}\cap Z} \le \delta_1 \le 1. \tag{4.12}$$

Then

$$\sup_{t\in[0,T_0]}\sup_{\sigma\in\{e,i,b\}}\|e^{it\Lambda_\sigma}U_\sigma(t) - U_\sigma(0)\|_Z \lesssim \delta_1^2, \tag{4.13}$$

where the implicit constant in (4.13) *may depend only on the constants* T, ε, C.

We prove Proposition 4.3 in Sections 5 and 6. In the rest of this section we show how to use these propositions and the local theory to complete the proof of Theorem 1.1.

4.1. *Proof of Theorem* 1.1. Theorem 1.1 is a consequence of Propositions 2.1, 3.2, 4.2, 4.3, and a linear dispersive estimate. Indeed, assume that we start with data $(n^0, v^0, \rho^0, u^0, \tilde{E}^0, \tilde{B}_0)$ as in (1.18), where $\overline{\delta}$ is taken sufficiently small. Using first Proposition 2.1, there is $T_1 \geq 1$ and a unique solution $(n, v, \rho, u, \tilde{E}, \tilde{B}) \in C([0, T_1] : H^{N_0})$ of the system (1.15), such that

$$(n(0), v(0), \rho(0), u(0), \tilde{E}(0), \tilde{B}(0)) = (n^0, v^0, \rho^0, u^0, \tilde{E}^0, \tilde{B}_0),$$

$$\text{div}(E)(t) + n(t) - \rho(t) = 0, \quad \tilde{B}(t) = \varepsilon\nabla \times v(t) = -\nabla \times u(t), \quad t \in [0, T_1], \tag{4.14}$$

and

$$\sup_{t\in[0,T_1]} \|(n(t), v(t), \rho(t), u(t), \tilde{E}(t), \tilde{B}(t))\|_{H^{N_0}} \leq \delta_0^{3/4}. \tag{4.15}$$

We can now apply Proposition 3.2 and construct the complex variables $U_e, U_i, U_b \in C([0, T_1] : H^{N_0})$ as in (3.17), which satisfy the dispersive system (3.19)–(3.20), and the uniform bound

$$\sup_{t\in[0,T_1]} \Big(\|U_e(t)\|_{H^{N_0}} + \|U_i(t)\|_{H^{N_0}} + \|U_b(t)\|_{H^{N_0}}\Big) \lesssim \delta_0^{3/4}. \tag{4.16}$$

Moreover, using the definition (3.17), the assumption (1.18), and Lemmas 3.1 and A.1, we have

$$\|U_e(0)\|_Z + \|U_i(0)\|_Z + \|U_b(0)\|_Z \lesssim \delta_0. \tag{4.17}$$

We are now ready to apply Proposition 4.2. Let T_2 denote the largest number in $(0, T_1]$ with the property that

$$\sup_{t\in[0,T_2)} \Big[\|e^{it\Lambda_e}U_e(t)\|_Z + \|e^{it\Lambda_i}U_i(t)\|_Z + \|e^{it\Lambda_b}U_b(t)\|_Z\Big] \leq \delta_0^{3/4}.$$

Such a $T_2 \in (0, T_1]$ exists, in view of (4.17) and Proposition 4.2. We apply now Proposition 4.3 on the intervals $[0, T_2(1 - 1/n)]$, $n = 2, 3, \ldots$, with $\delta_1 \approx \delta_0^{3/4}$. It follows that

$$\sup_{t\in[0,T_2)} \Big[\|e^{it\Lambda_e}U_e(t)\|_Z + \|e^{it\Lambda_i}U_i(t)\|_Z + \|e^{it\Lambda_b}U_b(t)\|_Z\Big] \lesssim \delta_0.$$

Using again Proposition 4.2, it follows that $T_2 = T_1$ and

$$\sup_{t\in[0,T_1]} \Big[\|e^{it\Lambda_e}U_e(t)\|_Z + \|e^{it\Lambda_i}U_i(t)\|_Z + \|e^{it\Lambda_b}U_b(t)\|_Z\Big] \lesssim \delta_0. \tag{4.18}$$

We can now return to the physical variables $(n, v, \rho, u, \tilde{E}, \tilde{B})$. Using the formulas in (3.22), the bounds (4.18), and the dispersive bounds (A.27) it follows that, for any $t \in [0, T_1]$ and $|\alpha| \leq 4$,

$$\begin{aligned}(1 + t)^{1+\beta/2}\big[&\|D_x^\alpha n(t)\|_{L^\infty} + \|D_x^\alpha \rho(t)\|_{L^\infty} + \|D_x^\alpha v(t)\|_{L^\infty} \\ &+ \|D_x^\alpha u(t)\|_{L^\infty} + \|D_x^\alpha \tilde{E}(t)\|_{L^\infty} + \|D_x^\alpha \tilde{B}(t)\|_{L^\infty}\big] \lesssim \delta_0.\end{aligned} \tag{4.19}$$

Recalling the definition (2.4) and the energy estimate (2.3), it follows that

$$\sup_{t\in[0,T_1]} \mathcal{E}_{N_0}(t) \lesssim \delta_0^2.$$

As a consequence, if the solution $(n, v, \rho, u, \tilde{E}, \tilde{B})$ satisfies the bound (4.15) on some interval $[0, T_1]$, then it has to satisfy the stronger bound

$$\sup_{t\in[0,T_1]} \|(n(t), v(t), \rho(t), u(t), \tilde{E}(t), \tilde{B}(t))\|_{H^{N_0}} \lesssim \delta_0.$$

Therefore, the solution can be extended globally, and the desired bound (1.20) follows using also (4.19). This completes the proof of Theorem 1.1.

5. Proof of Proposition 4.3, I: nonresonant interactions

In this section we start the proof of Proposition 4.3. We derive first several new formulas describing the solutions U_σ.

5.1. *Renormalizations.* Equations (3.19)–(3.20) give

$$[\partial_t + i\Lambda_\sigma(\xi)]\widehat{U_\sigma}(\xi,t) = c \sum_{\mu,\nu\in\mathcal{I}_0} \int_{\mathbb{R}^3} m_{\sigma;\mu,\nu}(\xi,\eta)\widehat{U_\mu}(\xi-\eta,t)\widehat{U_\nu}(\eta,t)\,d\eta \tag{5.1}$$

for $\sigma \in \{i, e, b\}$. For any $\mu \in \mathcal{I}_0$, let $\iota_\mu \in \{+, -\}$ denote its sign and let $\sigma_\mu \in \{i, e, b\}$ denote its component, i.e.,

$$\begin{aligned} &\iota_{i+} = \iota_{e+} = \iota_{b+1} = \iota_{b+2} = \iota_{b+3} := +,\\ &\iota_{i-} = \iota_{e-} = \iota_{b-1} = \iota_{b-2} = \iota_{b-3} := -,\\ &\sigma_{i+} = \sigma_{i-} := i, \qquad \sigma_{e+} = \sigma_{e-} := e,\\ &\sigma_{b+1} = \sigma_{b+2} = \sigma_{b+3} = \sigma_{b-1} = \sigma_{b-2} = \sigma_{b-3} := b. \end{aligned} \tag{5.2}$$

Let

$$V_\sigma(t) := e^{it\Lambda_\sigma} U_\sigma(t), \qquad \sigma \in \{i, e, b\},$$
$$\widetilde{\Lambda}_\mu := \iota_\mu \Lambda_{\sigma_\mu}, \qquad V_\mu(t) := e^{it\widetilde{\Lambda}_\mu} U_\mu(t), \qquad \mu \in \mathcal{I}_0.$$

Equation (5.1) is equivalent to

$$\begin{aligned} \frac{d}{dt}[\widehat{V_\sigma}(\xi,t)] &= c \sum_{\mu,\nu\in\mathcal{I}_0} \int_{\mathbb{R}^3} e^{it[\Lambda_\sigma(\xi)-\widetilde{\Lambda}_\mu(\xi-\eta)-\widetilde{\Lambda}_\nu(\eta)]} m_{\sigma;\mu,\nu}(\xi,\eta)\widehat{V_\mu}(\xi-\eta,t)\widehat{V_\nu}(\eta,t)\,d\eta \\ &= c \sum_{\mu,\nu\in\mathcal{I}_0} \mathcal{F}[Q_t^{\sigma;\mu,\nu}(V_\mu(t), V_\nu(t))](\xi) \end{aligned} \tag{5.3}$$

where, by definition,

$$\mathcal{F}[Q_s^{\sigma;\mu,\nu}(f,g)](\xi) := \int_{\mathbb{R}^3} e^{is[\Lambda_\sigma(\xi)-\widetilde{\Lambda}_\mu(\xi-\eta)-\widetilde{\Lambda}_\nu(\eta)]} m_{\sigma;\mu,\nu}(\xi,\eta)\widehat{f}(\xi-\eta)\widehat{g}(\eta)\,d\eta. \tag{5.4}$$

Therefore, for any $t \in [0, T_0]$ and $\sigma \in \{i, e, b\}$,

$$\begin{aligned}\widehat{V_\sigma}(\xi, t) - \widehat{V_\sigma}(\xi, 0) \\ = c \sum_{\mu,\nu \in \mathcal{I}_0} \int_0^t \int_{\mathbb{R}^3} e^{is[\Lambda_\sigma(\xi) - \widetilde{\Lambda}_\mu(\xi-\eta) - \widetilde{\Lambda}_\nu(\eta)]} m_{\sigma;\mu,\nu}(\xi, \eta) \widehat{V_\mu}(\xi - \eta, s) \widehat{V_\nu}(\eta, s)\, d\eta ds.\end{aligned} \tag{5.5}$$

The desired bound (4.13) is equivalent to proving that

$$\|V_\sigma(t) - V_\sigma(0)\|_Z \lesssim \delta_1^2 \tag{5.6}$$

for any $t \in [0, T_0]$ and any $\sigma \in \{i, e, b\}$. Given $t \in [0, T_0]$, we fix a suitable decomposition of the function $\mathbf{1}_{[0,t]}$; i.e., we fix functions $q_0, \dots, q_{L+1} : \mathbb{R} \to [0, 1]$, $|L - \log_2(2 + t)| \le 2$, with the properties

$$\begin{aligned}&\sum_{m=0}^{L+1} q_m(s) = \mathbf{1}_{[0,t]}(s), \quad \mathbf{1}_{[0,1]} \le q_0 \le \mathbf{1}_{[0,2]}, \\ &\mathbf{1}_{[t-1,t]} \le q_{L+1} \le \mathbf{1}_{[t-2,t]}, \quad \operatorname{supp} q_m \subseteq [2^{m-1}, 2^{m+1}], \\ &q_m \in C^1(\mathbb{R}) \qquad \text{and} \qquad \int_0^t |q_m'(s)|\, ds \lesssim 1 \qquad \text{for } m = 1, \dots, L.\end{aligned} \tag{5.7}$$

Recall the conclusions of Lemma 3.3. Using also Lemma A.1 and the formula (5.5), for (5.6) it suffices to prove the following proposition.

PROPOSITION 5.1. *Assume $t \in [0, T_0]$ is fixed, and define the functions q_m as in* (5.7). *For any $\sigma \in \{i, e, b\}$, $\mu, \nu \in \mathcal{I}_0$, we define the bilinear operators $T_m^{\sigma;\mu,\nu}$ by*

$$\mathcal{F}\big[T_m^{\sigma;\mu,\nu}(f, g)\big](\xi) := \int_{\mathbb{R}} \int_{\mathbb{R}^3} e^{is[\Lambda_\sigma(\xi) - \widetilde{\Lambda}_\mu(\xi-\eta) - \widetilde{\Lambda}_\nu(\eta)]} q_m(s) \cdot \widehat{f}(\xi - \eta, s) \widehat{g}(\eta, s)\, d\eta ds. \tag{5.8}$$

For any $\mu \in \mathcal{I}_0$, we define functions $f_\mu : \mathbb{R}^3 \times [0, T_0] \to \mathbb{C}$,

$$f_\mu := \delta_1^{-1} Q_\mu V_\mu, \tag{5.9}$$

where $Q_\mu f := \mathcal{F}^{-1}(q_\mu \cdot \widehat{f})$ for some $q_\mu \in \mathcal{S}^{100}$ with $\|q_\mu\|_{\mathcal{S}^{100}} \le 1$. We decompose

$$f_\mu = \sum_{k' \in \mathbb{Z}} \sum_{j' \ge \max(-k', 0)} P_{[k'-2, k'+2]}(\widetilde{\varphi}_{j'}^{(k')} \cdot P_{k'} f_\mu) = \sum_{(k', j') \in \mathcal{J}} f_{k', j'}^\mu. \tag{5.10}$$

For any $k \in \mathbb{Z}$, let

$$k_i := \min(k, 0), \qquad k_e = k_b := 0.$$

Then

$$\sum_{(k_1,j_1),(k_2,j_2) \in \mathcal{J}} (1 + 2^k) 2^{k_\sigma} \Big\| \widetilde{\varphi}_j^{(k)} \cdot P_k T_m^{\sigma;\mu,\nu}(f_{k_1,j_1}^\mu, f_{k_2,j_2}^\nu) \Big\|_{B_{k,j}} \lesssim 2^{-\beta^4 m} \tag{5.11}$$

for any fixed

$$\sigma \in \{i, e, b\}, \quad \mu, \nu \in \mathcal{I}_0, \quad (k, j) \in \mathcal{J}, \quad m \in \{0, \dots, L + 1\}. \tag{5.12}$$

This formulation and, in particular, the introduction of Q_μ in (5.9) and the factor $(1+2^k)2^{k_\sigma}$ in (5.11) are based on the structure of the multiplier in (3.23)–(3.26) and the fact our norms are invariant under the action of Calderón–Zygmund operators, as seen from Lemma A.1.

It follows from the definition that

$$
\begin{aligned}
T_m^{\sigma;\mu,\nu}(f,g) &= \int_{\mathbb{R}} q_m(s)\widetilde{T}_s^{\sigma;\mu,\nu}(f(s),g(s))\,ds,\\
\mathcal{F}\big[\widetilde{T}_s^{\sigma;\mu,\nu}(f',g')\big](\xi) &:= \int_{\mathbb{R}^3} e^{is[\Lambda_\sigma(\xi)-\widetilde{\Lambda}_\mu(\xi-\eta)-\widetilde{\Lambda}_\nu(\eta)]}\cdot\widehat{f'}(\xi-\eta)\widehat{g'}(\eta)\,d\eta.
\end{aligned}
\tag{5.13}
$$

For $\sigma\in\{i,e,b\}$ and $\mu,\nu\in\mathcal{I}_0$, we define also the functions $\Phi^{\sigma;\mu,\nu}:\mathbb{R}^3\times\mathbb{R}^3\to\mathbb{R}$ and $\Xi^{\mu,\nu}:\mathbb{R}^3\times\mathbb{R}^3\to\mathbb{R}^3$,

$$
\begin{aligned}
\Phi^{\sigma;\mu,\nu}(\xi,\eta) &:= \Lambda_\sigma(\xi)-\widetilde{\Lambda}_\mu(\xi-\eta)-\widetilde{\Lambda}_\nu(\eta) = \Lambda_\sigma(\xi)-\iota_\mu\Lambda_{\sigma_\mu}(\xi-\eta)-\iota_\nu\Lambda_{\sigma_\nu}(\eta),\\
\Xi^{\mu,\nu}(\xi,\eta) &:= (\nabla_\eta\Phi^{\sigma;\mu,\nu})(\xi,\eta) = -\iota_\mu(\nabla\Lambda_{\sigma_\mu})(\eta-\xi)-\iota_\nu(\nabla\Lambda_{\sigma_\nu})(\eta).
\end{aligned}
\tag{5.14}
$$

In view of Lemma A.1 and the main hypothesis (4.12), we have

$$
\sup_{t\in[0,T_0]}\|f_\mu(t)\|_{H^{N_0}\cap Z}\lesssim 1
\tag{5.15}
$$

for functions f_μ defined as in (5.9). Letting

$$
Ef^\mu_{k',j'}(s) := e^{-is\widetilde{\Lambda}_\mu}f^\mu_{k',j'}(s),
\tag{5.16}
$$

it follows from Lemma A.5 that for any $\mu\in\mathcal{I}_0$ and $s\in[0,T_0]$,

$$
\begin{aligned}
&\sum_{j'\ge\max(-k',0)}(\|Ef^\mu_{k',j'}(s)\|_{L^2}+\|f^\mu_{k',j'}(s)\|_{L^2})\lesssim\min(2^{-(N_0-1)k'},2^{(1+\beta-\alpha)k'}),\\
&\sum_{j'\ge\max(-k',0)}\|Ef^\mu_{k',j'}(s)\|_{L^\infty}\lesssim\min(2^{-6k'},2^{(1/2-\beta-\alpha)k'})(1+s)^{-1-\beta},\\
&\sup_{\xi\in\mathbb{R}^3}\Big|D^\rho_\xi\widehat{f^\mu_{k',j'}}(\xi,s)\Big|\lesssim_{|\rho|}(2^{\alpha k'}+2^{10k'})^{-1}\cdot 2^{-(1/2-\beta)\widetilde{k'}}2^{|\rho|j'}.
\end{aligned}
\tag{5.17}
$$

Sometimes, we will also need the more precise bounds

$$
\|Ef^\mu_{k',j'}(s)\|_{L^2}+\|f^\mu_{k',j'}(s)\|_{L^2}\lesssim(2^{\alpha k'}+2^{10k'})^{-1}2^{2\beta\widetilde{k'}}2^{-(1-\beta)j'}
\tag{5.18}
$$

and

$$
\|Ef^\mu_{k',j'}(s)\|_{L^\infty}\lesssim\min(2^{\beta k'},2^{-6k'})(1+s)^{-(5/4-10\beta)}2^{(1/4-11\beta)j'}
\tag{5.19}
$$

for any $(k',j')\in\mathcal{J}$. The last bound follows using (A.21)–(A.25) and recalling that $\alpha\in[0,\beta]$.

To integrate by parts in time (the method of normal forms) we need suitable information on the derivatives $\partial_s f^{\mu}_{k',j'}$. It follows from (5.3) and Lemma A.6 that, for any $(k',j')\in\mathcal{J}$, $\mu\in\mathcal{I}_0$, and $s\in[0,T_0]$,

$$\|(\partial_s f^{\mu}_{k',j'})(s)\|_{L^2}\lesssim 2^{k'_\sigma}\min[(1+s)^{-1-\beta},2^{3k'/2}]\cdot\min[1,2^{-(N_0-5)k'}]. \tag{5.20}$$

Moreover,

$$\text{if}\quad 2^{k'}\in[2^{-D},2^D]\text{ and }\sigma\in\{e,b\}\quad\text{ or }\quad 2^{k'}\in(0,2^D]\text{ and }\sigma=i, \tag{5.21}$$

then

$$\|(\widehat{\partial_s f^{\mu}_{k',j'}})(s)\|_{L^\infty}\lesssim(1+s)^{-1+\beta/10}2^{-k'}. \tag{5.22}$$

5.2. *Proof of Proposition* 5.1. We will prove the key bound (5.11) in several steps. The main ingredients in the proof are the estimates (5.15)–(5.20) above. In this subsection we start by considering some of the easier cases. In particular, we estimate all the interactions that are not space-time resonant and reduce significantly the range of the main parameters m,j,k,k_1,j_1,k_2,j_2. The goal is to reduce matters to proving Proposition 5.9. In all the cases analyzed in this subsection we can control the stronger norm $B^{\sigma,1}_{k,j}$; see Definition 4.1.

In the first two lemmas we use Sobolev regularity to estimate the contributions that correspond to one of the frequencies k,k_1,k_2 being larger than the parameter j.

LEMMA 5.2. *With $D=D(\varepsilon,T,C_b)$ sufficiently large, the estimate*

$$\sum_{(k_1,j_1),(k_2,j_2)\in\mathcal{J}}(1+2^k)2^{k_\sigma}\Big\|\widetilde{\varphi}^{(k)}_j\cdot P_kT^{\sigma;\mu,\nu}_m(f^{\mu}_{k_1,j_1},f^{\nu}_{k_2,j_2})\Big\|_{B^1_{k,j}}\lesssim 2^{-\beta^4m} \tag{5.23}$$

holds if

$$j\le\beta m/2+N_0'k_++D^2,\qquad\text{where}\qquad N_0':=2N_0/3-10. \tag{5.24}$$

Proof of Lemma 5.2. We observe that, in view of Definition 4.1,

$$\|\widetilde{\varphi}^{(k)}_j\cdot P_kh\|_{B^1_{k,j}}\lesssim(2^{\alpha k}+2^{10k})\cdot 2^{3j/2}2^{(1/2-\beta)\widetilde{k}}\|\widetilde{\varphi}^{(k)}_j\cdot P_kh\|_{L^2}. \tag{5.25}$$

Therefore, it suffices to prove that

$$\sum_{(k_1,j_1),(k_2,j_2)\in\mathcal{J}}(1+2^k)(2^{\alpha k}+2^{10k})2^{3j/2}2^{(1/2-\beta)\widetilde{k}}\Big\|P_kT^{\sigma;\mu,\nu}_m(f^{\mu}_{k_1,j_1},f^{\nu}_{k_2,j_2})\Big\|_{L^2}\lesssim 2^{-\beta^4m}. \tag{5.26}$$

Recalling the definition (5.16), it is easy to see that

$$\mathcal{F}\big[P_kT^{\sigma;\mu,\nu}_m(f^{\mu}_{k_1,j_1},f^{\nu}_{k_2,j_2})\big](\xi)=\int_{\mathbb{R}}\int_{\mathbb{R}^3}\varphi_k(\xi)e^{is\Lambda_\sigma(\xi)}q_m(s)\widehat{Ef^{\mu}_{k_1,j_1}}(\xi-\eta,s)\widehat{Ef^{\nu}_{k_2,j_2}}(\eta,s)\,d\eta ds.$$

Therefore, using Plancherel theorem,

(5.27)

$$\Big\|P_kT_m^{\sigma;\mu,\nu}(f^\mu_{k_1,j_1},f^\nu_{k_2,j_2})\Big\|_{L^2}\lesssim \min\Big(\int_{\mathbb{R}}q_m(s)\|Ef^\mu_{k_1,j_1}(s)\|_{L^2}\|Ef^\nu_{k_2,j_2}(s)\|_{L^\infty}\,ds,\\ \int_{\mathbb{R}}q_m(s)\|Ef^\mu_{k_1,j_1}(s)\|_{L^\infty}\|Ef^\nu_{k_2,j_2}(s)\|_{L^2}\,ds\Big).$$

Using now (5.17) and recalling the properties of the functions q_m (see (5.7)),

$$\sum_{(k_1,j_1),(k_2,j_2)\in\mathcal{J}}(1+2^k)\Big\|P_kT_m^{\sigma;\mu,\nu}(f^\mu_{k_1,j_1},f^\nu_{k_2,j_2})\Big\|_{L^2}\lesssim 2^{-(N_0-4)k_+}2^{-\beta m}. \tag{5.28}$$

It follows that the left-hand side of (5.26) is dominated by

$$2^{-\beta m}2^{(1/2-\beta+\alpha)k}2^{3j/2}$$

when $k\leq 0$ and by

$$2^{-(N_0-15)k}2^{-\beta m}2^{3j/2}$$

when $k\geq 0$. The bound (5.26) follows if $j\leq \beta m/2+(2N_0/3-10)k_++D^2$, as desired. □

Lemma 5.3. *Assume that*

$$j\geq \beta m/2+N_0'k_++D^2. \tag{5.29}$$

Then, with the same notation as before,

(5.30)

$$\sum_{(k_1,j_1),(k_2,j_2)\in\mathcal{J},\ \max(k_1,k_2)\geq j/N_0'}(1+2^k)2^{k_\sigma}\Big\|\widetilde{\varphi}^{(k)}_j\cdot P_kT_m^{\sigma;\mu,\nu}(f^\mu_{k_1,j_1},f^\nu_{k_2,j_2})\Big\|_{B^1_{k,j}}\\ \lesssim 2^{-\beta^4 m},$$

(5.31)

$$\sum_{(k_1,j_1),(k_2,j_2)\in\mathcal{J},\,\min(k_1,k_2)\leq -10j}(1+2^k)2^{k_\sigma}\Big\|\widetilde{\varphi}^{(k)}_j\cdot P_kT_m^{\sigma;\mu,\nu}(f^\mu_{k_1,j_1},f^\nu_{k_2,j_2})\Big\|_{B^1_{k,j}}$$

(5.32)

$$\lesssim 2^{-\beta^4 m},$$

and

(5.33)

$$\sum_{(k_1,j_1),(k_2,j_2)\in\mathcal{J},\,\max(j_1,j_2)\geq 10j}(1+2^k)2^{k_\sigma}\Big\|\widetilde{\varphi}^{(k)}_j\cdot P_kT_m^{\sigma;\mu,\nu}(f^\mu_{k_1,j_1},f^\nu_{k_2,j_2})\Big\|_{B^1_{k,j}}\\ \lesssim 2^{-\beta^4 m}.$$

Proof of Lemma 5.3. Using (5.17), (5.25), and (5.27), the left-hand side of (5.30) is dominated by

$$\sum_{(k_1,j_1),(k_2,j_2)\in\mathcal{J},\ \max(k_1,k_2)\geq j/N_0'} (1+2^k)(2^{\alpha k}+2^{10k}) \cdot 2^{3j/2}2^{(1/2-\beta)\widetilde{k}}\left\|P_kT_m^{\sigma;\mu,\nu}(f^\mu_{k_1,j_1},f^\nu_{k_2,j_2})\right\|_{L^2} \lesssim 2^{-\beta m}2^{-(N_0-6)j/N_0'}2^{3j/2}2^{(1/2-\beta)\widetilde{k}},$$

which clearly suffices, in view of (5.29). Similarly, the left-hand side of (5.31) is dominated by

$$\sum_{(k_1,j_1),(k_2,j_2)\in\mathcal{J},\ \min(k_1,k_2)\leq -10j} (1+2^k)(2^{\alpha k}+2^{10k}) \cdot 2^{3j/2}2^{(1/2-\beta)\widetilde{k}}\left\|P_kT_m^{\sigma;\mu,\nu}(f^\mu_{k_1,j_1},f^\nu_{k_2,j_2})\right\|_{L^2} \lesssim 2^{-\beta m}2^{-3j}\cdot(2^{\alpha k}+2^{10k})2^{3j/2}2^{(1/2-\beta)\widetilde{k}},$$

which clearly suffices. Finally, using the more precise bound (5.18), the left-hand side of (5.33) is dominated by

$$\sum_{(k_1,j_1),(k_2,j_2)\in\mathcal{J},\ \max(j_1,j_2)\geq 10j} (1+2^k)(2^{\alpha k}+2^{10k}) \cdot 2^{3j/2}2^{(1/2-\beta)\widetilde{k}}\left\|P_kT_m^{\sigma;\mu,\nu}(f^\mu_{k_1,j_1},f^\nu_{k_2,j_2})\right\|_{L^2} \lesssim 2^{-\beta m}2^{-3j}\cdot(2^{\alpha k}+2^{10k})2^{3j/2}2^{(1/2-\beta)\widetilde{k}},$$

which clearly suffices. □

We examine the conclusions of Lemmas 5.2 and 5.3, and we notice that Proposition 5.1 follows from Proposition 5.4 below.

PROPOSITION 5.4. *With the same notation as in Proposition* 5.1, *we have*

$$(1+2^k)2^{k_\sigma}\left\|\widetilde{\varphi}^{(k)}_j\cdot P_kT_m^{\sigma;\mu,\nu}(f^\mu_{k_1,j_1},f^\nu_{k_2,j_2})\right\|_{B_{k,j}}\lesssim 2^{-\beta^4(m+j)} \tag{5.34}$$

for any fixed $\mu,\nu\in\mathcal{I}_0$, $(k,j),(k_1,j_1),(k_2,j_2)\in\mathcal{J}$, *and* $m\in[0,L+1]\cap\mathbb{Z}$, *satisfying*

$$j\geq \beta m/2+N_0'k_++D^2,\qquad -10j\leq k_1,\ k_2\leq j/N_0',\qquad \max(j_1,j_2)\leq 10j. \tag{5.35}$$

5.3. *Proof of Proposition* 5.4. In this subsection we will show that proving Proposition 5.4 can be further reduced to proving Proposition 5.9 below. The arguments are more complicated than before, and we need to examine our bilinear operators more carefully; however, in all cases discussed in this subsection we can still control the stronger $B^1_{k,j}$ norms.

We notice that we are looking to prove the bound (5.34) for *fixed* $k, j, k_1, j_1, k_2, j_2, m$. We will consider several cases, depending on the relative sizes of these parameters. First we use the qualitative fact that the speed of propagation is uniformly bounded to discard regions where $|x| \gg t$.

LEMMA 5.5. *The bound* (5.34) *holds provided that* (5.35) *holds and*, *in addition*,

$$j \geq \max(m + D, -k(1 + \beta^2) + D). \tag{5.36}$$

Proof of Lemma 5.5. Using definition (4.4) it suffices to prove that

$$\begin{aligned}(1 + 2^k)(2^{\alpha k} + 2^{10k}) \cdot 2^{(1+\beta)j} \Big\|\widetilde{\varphi}_j^{(k)} \cdot P_k T_m^{\sigma;\mu,\nu}(f^\mu_{k_1,j_1}, f^\nu_{k_2,j_2})\Big\|_{L^2} \\ + (1 + 2^k)(2^{\alpha k} + 2^{10k}) \cdot 2^{(1/2-\beta)\widetilde{k}}\Big\|\mathcal{F}[\widetilde{\varphi}_j^{(k)} \cdot P_k T_m^{\sigma;\mu,\nu}(f^\mu_{k_1,j_1}, f^\nu_{k_2,j_2})]\Big\|_{L^\infty} \\ \lesssim 2^{-\beta^4(m+j)}.\end{aligned} \tag{5.37}$$

Assume first that

$$\min(j_1, j_2) \leq (1 - \beta^2)j. \tag{5.38}$$

By symmetry, we may assume that $j_1 \leq (1 - \beta^2)j$ and write

$$\begin{aligned}&\widetilde{\varphi}_j^{(k)}(x) \cdot P_k T_m^{\sigma;\mu,\nu}(f^\mu_{k_1,j_1}, f^\nu_{k_2,j_2})(x) \\ &\quad = c\widetilde{\varphi}_j^{(k)}(x) \int_{\mathbb{R}^3}\int_{\mathbb{R}}\int_{\mathbb{R}^3} \varphi_k(\xi) e^{ix\cdot\xi} e^{is[\Lambda_\sigma(\xi) - \widetilde{\Lambda}_\mu(\xi-\eta) - \widetilde{\Lambda}_\nu(\eta)]} q_m(s) \\ &\quad\quad \cdot \widehat{f^\mu_{k_1,j_1}}(\xi - \eta, s)\widehat{f^\nu_{k_2,j_2}}(\eta, s)\, d\eta ds d\xi.\end{aligned}$$

We examine the integral in ξ in the formula above. We recall the assumptions (5.35), (5.36), and (5.38), and the last bound in (5.17). Notice that, using only the assumption (5.36) and the definition (3.12) (see also Lemma A.4),

$$\Big|\nabla_\xi\big[x\cdot\xi + s[\Lambda_\sigma(\xi) - \widetilde{\Lambda}_\mu(\xi-\eta) - \widetilde{\Lambda}_\nu(\eta)]\big]\Big| \geq |x| - s\Big|\nabla_\xi[\Lambda_\sigma(\xi) - \widetilde{\Lambda}_\mu(\xi-\eta)]\Big| \geq 2^{j-10}.$$

We apply Lemma A.2 (with $K \approx 2^j$, $\epsilon \approx \min(2^{-j_1}, 2^k)$) to conclude that

$$\Big|\widetilde{\varphi}_j^{(k)}(x) \cdot P_k T_m^{\sigma;\mu,\nu}(f^\mu_{k_1,j_1}, f^\nu_{k_2,j_2})(x)\Big| \lesssim 2^{-10j}|\widetilde{\varphi}_j^{(k)}(x)|,$$

and the desired bounds (5.37) follow easily.

Assume now that

$$\min(j_1, j_2) \geq (1 - \beta^2)j. \tag{5.39}$$

By symmetry, we may assume that $k_1 \le k_2$. We prove first the bound on the second term in the left-hand side of (5.37): using (5.18) we estimate

$$\begin{aligned}
&(1+2^k)(2^{\alpha k}+2^{10k})\cdot 2^{(1/2-\beta)\widetilde{k}}\|\mathcal{F}[\widetilde{\varphi}^{(k)}_j\cdot P_kT_m^{\sigma;\mu,\nu}(f^\mu_{k_1,j_1},f^\nu_{k_2,j_2})]\Big\|_{L^\infty}\\
&\lesssim (1+2^k)(2^{\alpha k}+2^{10k})2^{(1/2-\beta)\widetilde{k}}\cdot 2^m\sup_{s\in[2^{m-1},2^{m+1}]}\|f^\mu_{k_1,j_1}(s)\|_{L^2}\|f^\nu_{k_2,j_2}(s)\|_{L^2}\\
&\lesssim (1+2^k)(2^{\alpha k}+2^{10k})2^{(1/2-\beta)\widetilde{k}}2^j\cdot(2^{\alpha k_1}+2^{10k_1})^{-1}2^{2\beta\widetilde{k_1}}2^{-(1-\beta)j_1}\\
&\quad\cdot(2^{\alpha k_2}+2^{10k_2})^{-1}2^{2\beta\widetilde{k_2}}2^{-(1-\beta)j_2}\\
&\lesssim (1+2^k)2^j\cdot 2^{-\alpha k_1}\min(2^{(1+\beta)k_1},2^{-(1-\beta-\beta^2)j})\cdot 2^{-(1-\beta-\beta^2)j}.
\end{aligned}$$

This suffices to prove the desired bound in (5.37), as it can be easily seen by considering the cases $k_1 \le -j$ and $k_1 \ge -j$.

Some more care is needed to prove the bound on the first term in the left-hand side of (5.37). We recall that

$$f^\mu_{k_1,j_1}=P_{[k_1-2,k_1+2]}(\widetilde{\varphi}^{(k_1)}_{j_1}\cdot P_{k_1}f_\mu)\quad\text{and}\quad f^\nu_{k_2,j_2}=P_{[k_2-2,k_2+2]}(\widetilde{\varphi}^{(k_2)}_{j_2}\cdot P_{k_2}f_\nu).$$

Since $\|\widetilde{\varphi}^{(k_1)}_{j_1}\cdot P_{k_1}f_\mu(s)\|_{B_{k_1,j_1}}+\|\widetilde{\varphi}^{(k_2)}_{j_2}\cdot P_{k_2}f_\nu(s)\|_{B_{k_2,j_2}}\lesssim 1$, see (5.15), we use (4.6)–(4.9) to decompose

$$\begin{aligned}
&\widetilde{\varphi}^{(k_1)}_{j_1}\cdot P_{k_1}f_\mu(s)=(2^{\alpha k_1}+2^{10k_1})^{-1}[g^\mu_{k_1,j_1}(s)+h^\mu_{k_1,j_1}(s)],\\
&g^\mu_{k_1,j_1}(s)=g^\mu_{k_1,j_1}(s)\cdot\widetilde{\varphi}^{(k_1)}_{[j_1-2,j_1+2]},\qquad h^\mu_{k_1,j_1}(s)=h^\mu_{k_1,j_1}(s)\cdot\widetilde{\varphi}^{(k_1)}_{[j_1-2,j_1+2]},\\
&2^{(1+\beta)j_1}\|g^\mu_{k_1,j_1}(s)\|_{L^2}+2^{(1/2-\beta)\widetilde{k_1}}\|\widehat{g^\mu_{k_1,j_1}}(s)\|_{L^\infty}\lesssim 1,\\
&2^{(1-\beta)j_1}\|h^\mu_{k_1,j_1}(s)\|_{L^2}+\|\widehat{h^\mu_{k_1,j_1}}(s)\|_{L^\infty}+2^{\gamma j_1}\|\widehat{h^\mu_{k_1,j_1}}(s)\|_{L^1}\lesssim 2^{-8|k_1|}
\end{aligned}\tag{5.40}$$

and

$$\begin{aligned}
&\widetilde{\varphi}^{(k_2)}_{j_2}\cdot P_{k_2}f_\nu(s)=(2^{\alpha k_2}+2^{10k_2})^{-1}[g^\nu_{k_2,j_2}(s)+h^\nu_{k_2,j_2}(s)],\\
&g^\nu_{k_2,j_2}(s)=g^\nu_{k_2,j_2}(s)\cdot\widetilde{\varphi}^{(k_2)}_{[j_2-2,j_2+2]},\qquad h^\nu_{k_2,j_2}(s)=h^\nu_{k_2,j_2}(s)\cdot\widetilde{\varphi}^{(k_2)}_{[j_2-2,j_2+2]},\\
&2^{(1+\beta)j_2}\|g^\nu_{k_2,j_2}(s)\|_{L^2}+2^{(1/2-\beta)\widetilde{k_2}}\|\widehat{g^\nu_{k_2,j_2}}(s)\|_{L^\infty}\lesssim 1,\\
&2^{(1-\beta)j_2}\|h^\nu_{k_2,j_2}(s)\|_{L^2}+\|\widehat{h^\nu_{k_2,j_2}}(s)\|_{L^\infty}+2^{\gamma j_2}\|\widehat{h^\nu_{k_2,j_2}}(s)\|_{L^1}\lesssim 2^{-8|k_2|}.
\end{aligned}\tag{5.41}$$

Using these decompositions and recalling the definition (5.13), to prove the desired bound on the first term in the left-hand side of (5.37), it suffices to prove that for any $s\in[2^{m-1},2^{m+1}]$,

(5.42)
$$
\begin{aligned}
&(1+2^k)(2^{\alpha k}+2^{10k})2^{(1+\beta)j}\cdot(2^{\alpha k_1}+2^{10k_1})^{-1}(2^{\alpha k_2}+2^{10k_2})^{-1}2^m\\
&\qquad\Big[\Big\|\widetilde{\varphi}_j^{(k)}\cdot P_k\widetilde{T}_s^{\sigma;\mu,\nu}(P_{[k_1-2,k_1+2]}g^\mu_{k_1,j_1}(s),P_{[k_2-2,k_2+2]}g^\nu_{k_2,j_2}(s))\Big\|_{L^2}\\
&\qquad+\Big\|\widetilde{\varphi}_j^{(k)}\cdot P_k\widetilde{T}_s^{\sigma;\mu,\nu}(P_{[k_1-2,k_1+2]}g^\mu_{k_1,j_1}(s),P_{[k_2-2,k_2+2]}h^\nu_{k_2,j_2}(s))\Big\|_{L^2}\\
&\qquad+\Big\|\widetilde{\varphi}_j^{(k)}\cdot P_k\widetilde{T}_s^{\sigma;\mu,\nu}(P_{[k_1-2,k_1+2]}h^\mu_{k_1,j_1}(s),P_{[k_2-2,k_2+2]}g^\nu_{k_2,j_2}(s))\Big\|_{L^2}\\
&\qquad+\Big\|\widetilde{\varphi}_j^{(k)}\cdot P_k\widetilde{T}_s^{\sigma;\mu,\nu}(P_{[k_1-2,k_1+2]}h^\mu_{k_1,j_1}(s),P_{[k_2-2,k_2+2]}h^\nu_{k_2,j_2}(s))\Big\|_{L^2}\Big]\\
&\quad\lesssim 2^{-\beta^4(m+j)}.
\end{aligned}
$$

Recall that we assumed $k_1\le k_2$; therefore we may also assume that $k\le k_2+4$. Using (5.40)–(5.41) and recalling (5.39), we estimate

$$
\begin{aligned}
&\Big\|P_k\widetilde{T}_s^{\sigma;\mu,\nu}(P_{[k_1-2,k_1+2]}g^\mu_{k_1,j_1}(s),P_{[k_2-2,k_2+2]}g^\nu_{k_2,j_2}(s))\Big\|_{L^2}\\
&\qquad\lesssim\|\mathcal{F}(P_{[k_1-2,k_1+2]}g^\mu_{k_1,j_1})(s)\|_{L^1}\|g^\nu_{k_2,j_2}(s)\|_{L^2}\\
&\qquad\lesssim 2^{3k_1/2}2^{-(1+\beta)j_1}2^{-(1+\beta)j_2}\\
&\qquad\lesssim 2^{3k_1/2}2^{-(2+2\beta)(1-\beta^2)j},\\
&\Big\|P_k\widetilde{T}_s^{\sigma;\mu,\nu}(P_{[k_1-2,k_1+2]}h^\mu_{k_1,j_1}(s),P_{[k_2-2,k_2+2]}h^\nu_{k_2,j_2}(s))\Big\|_{L^2}\\
&\qquad\lesssim\|\widehat{h^\mu_{k_1,j_1}}(s)\|_{L^1}\|\widehat{h^\nu_{k_2,j_2}}(s)\|_{L^2}\\
&\qquad\lesssim 2^{-\gamma j_1}2^{-8|k_1|}2^{-(1-\beta)j_2}2^{-8|k_2|}\\
&\qquad\lesssim 2^{-8|k_1|}2^{-(2+2\beta)(1-\beta^2)j},\\
&\Big\|P_k\widetilde{T}_s^{\sigma;\mu,\nu}(P_{[k_1-2,k_1+2]}h^\mu_{k_1,j_1}(s),P_{[k_2-2,k_2+2]}g^\nu_{k_2,j_2}(s))\Big\|_{L^2}\\
&\qquad\lesssim\|\widehat{h^\mu_{k_1,j_1}}(s)\|_{L^1}\|\widehat{g^\nu_{k_2,j_2}}(s)\|_{L^2}\\
&\qquad\lesssim 2^{-\gamma j_1}2^{-8|k_1|}2^{-(1+\beta)j_2}\\
&\qquad\lesssim 2^{-8|k_1|}2^{-(2+2\beta)(1-\beta^2)j},
\end{aligned}
$$

and

$$
\begin{aligned}
&\Big\|P_k\widetilde{T}_s^{\sigma;\mu,\nu}(P_{[k_1-2,k_1+2]}g^\mu_{k_1,j_1}(s),P_{[k_2-2,k_2+2]}h^\nu_{k_2,j_2}(s))\Big\|_{L^2}\\
&\qquad\lesssim\min\Big(2^{3k_1/2}\|\widehat{g^\mu_{k_1,j_1}}(s)\|_{L^2}\|\widehat{h^\nu_{k_2,j_2}}(s)\|_{L^2},\|\widehat{g^\mu_{k_1,j_1}}(s)\|_{L^2}\|\widehat{h^\nu_{k_2,j_2}}(s)\|_{L^1}\Big)\\
&\qquad\lesssim 2^{-(1+\beta)j_1}2^{-8|k_2|}\min\Big(2^{-(1-\beta)j_2}2^{3k_1/2},2^{-\gamma j_2}\Big)\\
&\qquad\lesssim 2^{-(2+2\beta)(1-\beta^2)j}2^{3k_1/4}2^{-8|k_2|}.
\end{aligned}
$$

Since $2^m \lesssim 2^j$ and $(2^{\alpha k}+2^{10k})(2^{\alpha k_2}+2^{10k_2})^{-1} \lesssim 1$, the left-hand side of (5.42) is dominated by

$$C(1+2^k)2^{(1+\beta)j}\cdot(2^{\alpha k_1}+2^{10k_1})^{-1}2^j\cdot 2^{-(2+2\beta)(1-\beta^2)j}(2^{3k_1/2}+2^{3k_1/4}2^{-8|k_2|}) \lesssim 2^{-2\beta j/3}(1+2^k),$$

which suffices since $2^k \lesssim 2^{j/N_0'}$. This completes the proof of the lemma. □

We estimate now the contribution at very low frequencies. Here we use also the null form structure at small frequencies of the multipliers $m_{i;\mu,\nu}$.

Lemma 5.6. *The bound* (5.34) *holds provided that* (5.35) *holds and*, *in addition*,

$$\max(m+D,j) \leq -k(1+\beta^2)+D. \tag{5.43}$$

Proof of Lemma 5.6. In view of the restrictions (5.43) and (5.35), we may assume that $k \leq -D^2/2$. Using the definition, it is easy to see that

$$\|\widetilde{\varphi}_j^{(k)}\cdot P_kh\|_{B^1_{k,j}} \lesssim (2^{\alpha k}+2^{10k})2^{(1+\beta)j}2^{3k/2}\|\widehat{P_kh}\|_{L^\infty}. \tag{5.44}$$

Therefore, it suffices to prove that

$$2^{k_\sigma}2^{\alpha k}2^{(1+\beta)j}2^{3k/2}\Big\|\mathcal{F}P_kT_m^{\sigma;\mu,\nu}(f^\mu_{k_1,j_1},f^\nu_{k_2,j_2})\Big\|_{L^\infty} \lesssim 2^{-\beta^4(m+j)}. \tag{5.45}$$

Using (5.18) and recalling $\alpha \leq 2\beta$, we estimate

$$\begin{aligned}\Big\|\mathcal{F}P_kT_m^{\sigma;\mu,\nu}(f^\mu_{k_1,j_1},f^\nu_{k_2,j_2})\Big\|_{L^\infty} &\lesssim \int_{\mathbb{R}} q_m(s)\|f^\mu_{k_1,j_1}(s)\|_{L^2}\|f^\nu_{k_2,j_2}(s)\|_{L^2}\,ds\\ &\lesssim \|q_m\|_{L^1}\cdot(2^{\alpha k_1}+2^{10k_1})^{-1}2^{2\beta\widetilde{k_1}}2^{-(1-\beta)j_1}\cdot(2^{\alpha k_2}+2^{10k_2})^{-1}2^{2\beta\widetilde{k_2}}2^{-(1-\beta)j_2}\\ &\lesssim \|q_m\|_{L^1}\min(1,2^{-5k_1})2^{-(1-\beta)j_1}\cdot\min(1,2^{-5k_2})2^{-(1-\beta)j_2}.\end{aligned}$$

Recalling the definitions (4.1) and the assumptions, the desired bound (5.45) follows if

$$\sigma = i \quad\text{or}\quad m = L+1 \quad\text{or}\quad m \leq (1-\beta)(j_1+j_2)-(1/2-\beta)k.$$

It remains to prove the bound (5.45) in the case

$$\sigma\in\{e,b\} \quad\text{and}\quad m\in[1,L]\cap\mathbb{Z} \quad\text{and}\quad m \geq -(1/2-\beta)k+(1-\beta)(j_1+j_2). \tag{5.46}$$

Since $j_1+k_1 \geq 0$, $j_2+k_2 \geq 0$, and $k \leq -D^2/2$, the conditions (5.43) and (5.46) show that $k_1,k_2 \geq k/3$ and $|k_1-k_2| \leq 10$. Using also (5.43), for (5.45) it suffices to prove that, assuming (5.46),

$$\Big\|\mathcal{F}P_kT_m^{\sigma;\mu,\nu}(f^\mu_{k_1,j_1},f^\nu_{k_2,j_2})\Big\|_{L^\infty} \lesssim 2^{-k(1/2+\alpha-\beta-2\beta^2)}. \tag{5.47}$$

Recall the definitions

$$\begin{aligned}(5.48)\qquad &\mathcal{F}P_kT_m^{\sigma;\mu,\nu}(f^{\mu}_{k_1,j_1},f^{\nu}_{k_2,j_2})(\xi)\\ &\quad=\varphi_k(\xi)\int_{\mathbb{R}}\int_{\mathbb{R}^3}e^{is\Phi^{\sigma;\mu,\nu}(\xi,\eta)}q_m(s)\widehat{f^{\mu}_{k_1,j_1}}(\xi-\eta,s)\widehat{f^{\nu}_{k_2,j_2}}(\eta,s)\,d\eta ds,\end{aligned}$$

where

$$\Phi^{\sigma;\mu,\nu}(\xi,\eta)=\Lambda_\sigma(\xi)-\widetilde{\Lambda}_\mu(\xi-\eta)-\widetilde{\Lambda}_\nu(\eta).$$

To prove (5.47) we would like to integrate by parts in η and s in formula (5.48). We decompose

$$\begin{aligned}&\mathcal{F}P_kT_m^{\sigma;\mu,\nu}(f^{\mu}_{k_1,j_1},f^{\nu}_{k_2,j_2})(\xi)=G(\xi)+H(\xi),\\ &G(\xi):=\varphi_k(\xi)\int_{\mathbb{R}}\int_{\mathbb{R}^3}e^{is\Phi^{\sigma;\mu,\nu}(\xi,\eta)}\\ &\qquad\cdot\varphi(2^{20D}(1+2^{k_2})\Phi^{\sigma;\mu,\nu}(\xi,\eta))q_m(s)\widehat{f^{\mu}_{k_1,j_1}}(\xi-\eta,s)\widehat{f^{\nu}_{k_2,j_2}}(\eta,s)\,d\eta ds,\\ &H(\xi):=\varphi_k(\xi)\int_{\mathbb{R}}\int_{\mathbb{R}^3}e^{is\Phi^{\sigma;\mu,\nu}(\xi,\eta)}\\ &\qquad\cdot[1-\varphi(2^{20D}(1+2^{k_2})\Phi^{\sigma;\mu,\nu}(\xi,\eta))]q_m(s)\widehat{f^{\mu}_{k_1,j_1}}(\xi-\eta,s)\widehat{f^{\nu}_{k_2,j_2}}(\eta,s)\,d\eta ds.\end{aligned}$$

The function H can be estimated using integration by parts in s, (5.20), the assumptions (5.7), and the bounds (5.17). Indeed,

$$\begin{aligned}|H(\xi)|&\lesssim(1+2^{k_2})\sup_{s\in[2^{m-1},2^{m+1}]}\Big[\big\|\widehat{f^{\mu}_{k_1,j_1}}(s)\big\|_{L^2}\big\|\widehat{f^{\nu}_{k_2,j_2}}(s)\big\|_{L^2}\\ &\qquad+2^m\big\|(\partial_s\widehat{f^{\mu}_{k_1,j_1}})(s)\big\|_{L^2}\big\|\widehat{f^{\nu}_{k_2,j_2}}(s)\big\|_{L^2}+2^m\big\|\widehat{f^{\mu}_{k_1,j_1}}(s)\big\|_{L^2}\big\|(\partial_s\widehat{f^{\nu}_{k_2,j_2}})(s)\big\|_{L^2}\Big]\\ &\lesssim\min(1,2^{-(N_0-10)k_2}).\end{aligned}$$

Therefore, for (5.47) it suffices to prove that

$$(5.49)\qquad \big\|G\big\|_{L^\infty}\lesssim 2^{-k(1/2+\alpha-\beta-2\beta^2)}.$$

Recall the definitions (5.14),

$$(5.50)\qquad \Xi^{\mu,\nu}(\xi,\eta)=(\nabla_\eta\Phi^{\sigma;\mu,\nu})(\xi,\eta)=-\iota_1\nabla\Lambda_{\sigma_1}(\eta-\xi)-\iota_2\nabla\Lambda_{\sigma_2}(\eta),$$

where

$$\mu=(\sigma_1\iota_1),\qquad \nu=(\sigma_2\iota_2),\qquad \sigma_1,\sigma_2\in\{i,e,b\},\qquad \iota_1,\iota_2\in\{+,-\}.$$

For $l\in\mathbb{Z}$, let

$$\begin{aligned}(5.51)\qquad &G_{\leq l}(\xi):=\varphi_k(\xi)\int_{\mathbb{R}}\int_{\mathbb{R}^3}\varphi_{(-\infty,l]}(\Xi^{\mu,\nu}(\xi,\eta))\cdot e^{is\Phi^{\sigma;\mu,\nu}(\xi,\eta)}\\ &\qquad\cdot\varphi(2^{20D}(1+2^{k_2})\Phi^{\sigma;\mu,\nu}(\xi,\eta))q_m(s)\widehat{f^{\mu}_{k_1,j_1}}(\xi-\eta,s)\widehat{f^{\nu}_{k_2,j_2}}(\eta,s)\,d\eta ds.\end{aligned}$$

Let $G_l:=G_{\leq l}-G_{\leq l-1}$. In proving (5.49) we may assume that $j_1\leq j_2$. If $l\geq l_0=-20D-4\max(k_2,0)$, then we integrate by parts in η, using Lemma A.2

with $K \approx 2^{m+l}$ and $\epsilon^{-1} \approx 2^{j_2} + 2^{-\min(l,0)-\min(k_2,0)} + 2^{k_2}$. Using also the last bound in (5.17), (5.43), and (5.46) to ensure $\epsilon K \geq 2^{\beta^2 m}$ and $(2^{j_1}+2^{j_2})^\rho \leq \epsilon^{-\rho}$, it follows that

$$\sum_{l \geq l_0+1} \|G_l\|_{L^\infty} \lesssim (1+2^{5k_2})^{-1}. \tag{5.52}$$

It remains to estimate $\|G_{\leq l_0}\|_{L^\infty}$. Since $\sigma \neq i$, it follows from Proposition B.2 that $G_{\leq l_0} \equiv 0$. This completes the proof of the lemma. □

We estimate now the contributions coming from large input parameters j_1 or j_2.

LEMMA 5.7. *The bound* (5.34) *holds provided that* (5.35) *holds and*, *in addition*,

$$j \leq m + D \quad \text{and} \quad \max(j_1, j_2) \geq (1-\beta/10)m + k_\sigma, \tag{5.53}$$

or

$$j \leq m + D \quad \text{and} \quad \min(k_1, k_2) \leq -9m/10. \tag{5.54}$$

Proof of Lemma 5.7. Assume first that (5.54) holds. We estimate, assuming $k_1 \leq k_2$ and using (5.17),

$$\begin{aligned}(1+2^k)&2^{k_\sigma}(2^{\alpha k}+2^{10k})2^{(1+\beta)j}2^{3k/2}\Big\|\mathcal{F}P_kT_m^{\sigma;\mu,\nu}(f^\mu_{k_1,j_1}, f^\nu_{k_2,j_2})\Big\|_{L^\infty}\\ &\lesssim 2^{(1+\beta)j}2^m(1+2^{11k})2^{3k_2/2} \sup_{s\in[2^{m-1},2^{m+1}]} \|\widehat{f^\mu_{k_1,j_1}}(s)\|_{L^1}\|\widehat{f^\nu_{k_2,j_2}}(s)\|_{L^\infty}\\ &\lesssim 2^{(2+\beta)m}(1+2^{11k})2^{3k_2/2}\cdot 2^{5k_1/2}2^{-k_2/2}.\end{aligned}$$

The desired bound (5.34) follows using also (5.44).

Assume now that (5.53) holds. Using definition (4.4), it suffices to prove that

$$\begin{aligned}&(1+2^k)2^{k_\sigma}(2^{\alpha k}+2^{10k})\cdot 2^{(1+\beta)j}\Big\|\widetilde{\varphi}^{(k)}_j\cdot P_kT_m^{\sigma;\mu,\nu}(f^\mu_{k_1,j_1}, f^\nu_{k_2,j_2})\Big\|_{L^2}\\ &+(1+2^k)2^{k_\sigma}(2^{\alpha k}+2^{10k})\cdot 2^{(1/2-\beta)\widetilde{k}}\Big\|\mathcal{F}[\widetilde{\varphi}^{(k)}_j\cdot P_kT_m^{\sigma;\mu,\nu}(f^\mu_{k_1,j_1}, f^\nu_{k_2,j_2})]\Big\|_{L^\infty}\\ &\lesssim 2^{-\beta^4(m+j)}.\end{aligned} \tag{5.55}$$

By symmetry, we may assume $k_1 \leq k_2$. We prove first the bounds (5.55) in the case

$$k_1 \leq -5m/6. \tag{5.56}$$

Using (5.17), for any $s \in [0,t]$,

$$\|\widehat{f^\mu_{k_1,j_1}}(s)\|_{L^1} \lesssim 2^{3k_1}\|\widehat{f^\mu_{k_1,j_1}}(s)\|_{L^\infty} \lesssim 2^{(5/2-\alpha+\beta)k_1}.$$

Therefore, using (5.17) again, it follows that

$$\begin{aligned}\Big\|\mathcal{F}T_m^{\sigma;\mu,\nu}(f^\mu_{k_1,j_1},f^\nu_{k_2,j_2})\Big\|_{L^2}&\lesssim 2^m\sup_{s\in[2^{m-1},2^{m+1}]}\|\widehat{f^\mu_{k_1,j_1}}(s)\|_{L^1}\|\widehat{f^\nu_{k_2,j_2}}(s)\|_{L^2}\\&\lesssim 2^m2^{(5/2-\alpha+\beta)k_1}\min(2^{-(N_0-1)k_2},2^{(1+\beta-\alpha)k_2})\end{aligned}$$

and

(5.57)

$$\begin{aligned}\Big\|\mathcal{F}T_m^{\sigma;\mu,\nu}(f^\mu_{k_1,j_1},f^\nu_{k_2,j_2})\Big\|_{L^\infty}&\lesssim 2^m\sup_{s\in[2^{m-1},2^{m+1}]}\|\widehat{f^\mu_{k_1,j_1}}(s)\|_{L^1}\|\widehat{f^\nu_{k_2,j_2}}(s)\|_{L^\infty}\\&\lesssim 2^m2^{(5/2-\alpha+\beta)k_1}\cdot(2^{\alpha k_2}+2^{10k_2})^{-1}2^{-(1/2-\beta)\widetilde{k_2}}.\end{aligned}$$

Therefore, recalling (5.56), if $k\le 0$, then the left-hand side of (5.55) is dominated by

$$C2^{(2+\beta)m}2^{(5/2-\alpha+\beta)k_1}\lesssim 2^{(-1/12+5\alpha/6+\beta/6)m},$$

which suffices. Similarly, if $k\ge 0$, then the left-hand side of (5.55) is dominated by

$$\begin{aligned}C2^{(2+\beta)m}2^{(5/2-\alpha+\beta)k_1}2^{-(N_0-15)k}+C2^{k_2}2^m2^{(5/2-\alpha+\beta)k_1}\\\lesssim 2^{k_2}2^{(-1/12+5\alpha/6+\beta/6)m},\end{aligned}$$

which also suffices.

To prove the bound (5.55) when $-5m/6\le k_1\le k_2$, we decompose, as in (5.40)–(5.41), for any $s\in[2^{m-1},2^{m+1}]$,

$$(5.58)\qquad\begin{aligned}&\widetilde{\varphi}^{(k_1)}_{j_1}\cdot P_{k_1}f_\mu(s)=(2^{\alpha k_1}+2^{10k_1})^{-1}[g^\mu_{k_1,j_1}(s)+h^\mu_{k_1,j_1}(s)],\\&g^\mu_{k_1,j_1}(s)=g^\mu_{k_1,j_1}(s)\cdot\widetilde{\varphi}^{(k_1)}_{[j_1-2,j_1+2]},\qquad h^\mu_{k_1,j_1}(s)=h^\mu_{k_1,j_1}(s)\cdot\widetilde{\varphi}^{(k_1)}_{[j_1-2,j_1+2]},\\&2^{(1+\beta)j_1}\|g^\mu_{k_1,j_1}(s)\|_{L^2}+2^{(1/2-\beta)\widetilde{k_1}}\|\widehat{g^\mu_{k_1,j_1}}(s)\|_{L^\infty}\lesssim 1,\\&2^{(1-\beta)j_1}\|h^\mu_{k_1,j_1}(s)\|_{L^2}+\|\widehat{h^\mu_{k_1,j_1}}(s)\|_{L^\infty}+2^{\gamma j_1}\|\widehat{h^\mu_{k_1,j_1}}(s)\|_{L^1}\lesssim 2^{-8|k_1|},\end{aligned}$$

and

$$(5.59)\qquad\begin{aligned}&\widetilde{\varphi}^{(k_2)}_{j_2}\cdot P_{k_2}f_\nu(s)=(2^{\alpha k_2}+2^{10k_2})^{-1}[g^\nu_{k_2,j_2}(s)+h^\nu_{k_2,j_2}(s)],\\&g^\nu_{k_2,j_2}(s)=g^\nu_{k_2,j_2}(s)\cdot\widetilde{\varphi}^{(k_2)}_{[j_2-2,j_2+2]},\qquad h^\nu_{k_2,j_2}(s)=h^\nu_{k_2,j_2}(s)\cdot\widetilde{\varphi}^{(k_2)}_{[j_2-2,j_2+2]},\\&2^{(1+\beta)j_2}\|g^\nu_{k_2,j_2}(s)\|_{L^2}+2^{(1/2-\beta)\widetilde{k_2}}\|\widehat{g^\nu_{k_2,j_2}}(s)\|_{L^\infty}\lesssim 1,\\&2^{(1-\beta)j_2}\|h^\nu_{k_2,j_2}(s)\|_{L^2}+\|\widehat{h^\nu_{k_2,j_2}}(s)\|_{L^\infty}+2^{\gamma j_2}\|\widehat{h^\nu_{k_2,j_2}}(s)\|_{L^1}\lesssim 2^{-8|k_2|}.\end{aligned}$$

We will prove now the L^2 bound

(5.60)

$$(1+2^k)2^{k_\sigma}(2^{\alpha k}+2^{10k})\cdot 2^{(2+\beta)m}\Big\|P_k\widetilde{T}^{\sigma;\mu,\nu}_s(f^\mu_{k_1,j_1}(s),f^\nu_{k_2,j_2}(s))\Big\|_{L^2}\lesssim 2^{-2\beta^4 m}$$

for any $s \in [2^{m-1}, 2^{m+1}]$; see (5.13) for the definition of the bilinear operators $\widetilde{T}_s^{\sigma;\mu,\nu}$. In view of the assumption (5.53) this would clearly imply the desired L^2 bound in (5.55).

Assume first that $\min(j_1, j_2) \leq (1-15\beta)m$, i.e.,

$$\min(j_1,j_2) \leq (1-15\beta)m,\ \ \max(j_1,j_2) \geq (1-\beta/10)m+k_\sigma,\ \ k_2 \geq k_1 \geq -5m/6. \tag{5.61}$$

Using (5.18) and (5.19),

$$\begin{aligned}
&\|P_k\widetilde{T}_s^{\sigma;\mu,\nu}(f^\mu_{k_1,j_1}(s), f^\nu_{k_2,j_2}(s))\big\|_{L^2}\\
&\qquad\lesssim \min(\|Ef^\mu_{k_1,j_1}(s)\|_{L^\infty}\|Ef^\nu_{k_2,j_2}(s)\|_{L^2}, \|Ef^\mu_{k_1,j_1}(s)\|_{L^2}\|Ef^\nu_{k_2,j_2}(s)\|_{L^\infty})\\
&\qquad\lesssim \min(2^{\beta k_1}, 2^{-6k_1})\min(2^{\beta k_2}, 2^{-6k_2})\\
&\qquad\quad\cdot 2^{-m(5/4-10\beta)}2^{(1/4-11\beta)\min(j_1,j_2)}2^{-(1-\beta)\max(j_1,j_2)}\\
&\qquad\lesssim 2^{-k_\sigma}(1+2^{k_2})^{-6}2^{-(2+3\beta/2)m},
\end{aligned}$$

which suffices to prove (5.60).

Assume now that $\min(j_1, j_2) \geq (1-15\beta)m$, i.e.,

$$\min(j_1,j_2) \geq (1-15\beta)m,\ \ \max(j_1,j_2) \geq (1-\beta/10)m+k_\sigma,\ \ k_2 \geq k_1 \geq -5m/6. \tag{5.62}$$

We recall that

$$\begin{aligned}
f^\mu_{k_1,j_1} &= P_{[k_1-2,k_1+2]}(\widetilde{\varphi}^{(k_1)}_{j_1}\cdot P_{k_1}f_\mu)\\
&= (2^{\alpha k_1}+2^{10k_1})^{-1}[P_{[k_1-2,k_1+2]}g^\mu_{k_1,j_1} + P_{[k_1-2,k_1+2]}h^\mu_{k_1,j_1}],\\
f^\nu_{k_2,j_2} &= P_{[k_2-2,k_2+2]}(\widetilde{\varphi}^{(k_2)}_{j_2}\cdot P_{k_2}f_\nu)\\
&= (2^{\alpha k_2}+2^{10k_2})^{-1}[P_{[k_2-2,k_2+2]}g^\nu_{k_2,j_2} + P_{[k_2-2,k_2+2]}h^\nu_{k_2,j_2}],
\end{aligned} \tag{5.63}$$

and we use the bounds in (5.58)–(5.59). Then we estimate, using also (5.62),

$$\begin{aligned}
&\Big\|P_k\widetilde{T}_s^{\sigma;\mu,\nu}(P_{[k_1-2,k_1+2]}h^\mu_{k_1,j_1}(s), P_{[k_2-2,k_2+2]}h^\nu_{k_2,j_2}(s))\Big\|_{L^2}\\
&\qquad\lesssim \|\widehat{h^\mu_{k_1,j_1}}(s)\|_{L^1}\|\widehat{h^\nu_{k_2,j_2}}(s)\|_{L^2}\\
&\qquad\lesssim 2^{-\gamma j_1}2^{-(1-\beta)j_2}2^{-8|k_1|}2^{-8|k_2|}\\
&\qquad\lesssim 2^{-(\gamma+1-25\beta)m}2^{-6|k_1|}2^{-8|k_2|},\\
&\Big\|P_k\widetilde{T}_s^{\sigma;\mu,\nu}(P_{[k_1-2,k_1+2]}h^\mu_{k_1,j_1}(s), P_{[k_2-2,k_2+2]}g^\nu_{k_2,j_2}(s))\Big\|_{L^2}\\
&\qquad\lesssim \|\widehat{h^\mu_{k_1,j_1}}(s)\|_{L^1}\|\widehat{g^\nu_{k_2,j_2}}(s)\|_{L^2}\\
&\qquad\lesssim 2^{-\gamma j_1}2^{-(1+\beta)j_2}2^{-8|k_1|}\\
&\qquad\lesssim 2^{-m(\gamma+1-25\beta)}2^{-6|k_1|},
\end{aligned}$$

$$\begin{aligned}\Big\|P_k\widetilde{T}_s^{\sigma;\mu,\nu}(P_{[k_1-2,k_1+2]}g^\mu_{k_1,j_1}(s),P_{[k_2-2,k_2+2]}h^\nu_{k_2,j_2}(s))\Big\|_{L^2}&\\ \lesssim \|\widehat{g^\mu_{k_1,j_1}}(s)\|_{L^2}\|\widehat{h^\nu_{k_2,j_2}}(s)\|_{L^1}&\\ \lesssim 2^{-(1+\beta)j_1}2^{-\gamma j_2}2^{-8|k_2|}&\\ \lesssim 2^{-m(\gamma+1-25\beta)}2^{-8|k_2|}2^{-k_\sigma}&\end{aligned}$$

and, using also (A.21)–(A.25) (compare with the bounds (5.19)),

$$\begin{aligned}&\Big\|P_k\widetilde{T}_s^{\sigma;\mu,\nu}(P_{[k_1-2,k_1+2]}g^\mu_{k_1,j_1}(s),P_{[k_2-2,k_2+2]}g^\nu_{k_2,j_2}(s))\Big\|_{L^2}\\ &\quad\lesssim \min\Big(\|e^{-is\widetilde{\Lambda}_\mu}P_{[k_1-2,k_1+2]}(g^\mu_{k_1,j_1}(s))\|_{L^\infty}\|g^\nu_{k_2,j_2}(s)\|_{L^2},\\ &\qquad\qquad \|g^\mu_{k_1,j_1}(s)\|_{L^2}\|e^{-is\widetilde{\Lambda}_\nu}P_{[k_2-2,k_2+2]}(g^\nu_{k_2,j_2}(s))\|_{L^\infty}\Big)\\ &\quad\lesssim 2^{-(1+\beta)\max(j_1,j_2)}\cdot 2^{-m(5/4-10\beta)}2^{(1/4-11\beta)\min(j_1,j_2)}(1+2^{4k_2})\\ &\quad\lesssim 2^{-k_\sigma}(1+2^{4k_2})2^{-(2+19\beta/10)m}.\end{aligned}$$

Therefore, using also $\alpha\in[0,\beta/2]$ and $k_1\geq -5m/6$, the left-hand side of (5.60) is dominated by

$$C(1+2^{5k_2})2^{-\alpha k_1}2^{-9\beta m/10}\lesssim (1+2^{5k_2})2^{-29m\beta/60}.$$

This completes the proof of (5.60).

To complete the proof of (5.55) it remains to prove the L^∞ bound. This would follow from the estimate

(5.64)
$$(1+2^k)2^{k_\sigma}(2^{\alpha k}+2^{10k})\cdot 2^{(1/2-\beta)\widetilde{k}}2^m\Big\|\mathcal{F}P_k\widetilde{T}_s^{\sigma;\mu,\nu}(f^\mu_{k_1,j_1}(s),f^\nu_{k_2,j_2}(s))\Big\|_{L^\infty}\lesssim 2^{-2\beta^4 m}$$

for all $s\in[2^{m-2},2^{m+2}]$. If $k_1\leq -2m/5$ then, as in (5.57),

$$\begin{aligned}\Big\|\mathcal{F}P_k\widetilde{T}_s^{\sigma;\mu,\nu}(f^\mu_{k_1,j_1}(s),f^\nu_{k_2,j_2}(s))\Big\|_{L^\infty}&\lesssim \|\widehat{f^\mu_{k_1,j_1}}(s)\|_{L^1}\|\widehat{f^\nu_{k_2,j_2}}(s)\|_{L^\infty}\\ &\lesssim 2^{(5/2-\alpha+\beta)k_1}(2^{\alpha k_2}+2^{10k_2})^{-1}2^{-(1/2-\beta)\widetilde{k}_2},\end{aligned}$$

and therefore the left-hand side of (5.64) is dominated by

$$C(1+2^k)2^{k_\sigma}\cdot(2^{\alpha k}+2^{10k})(2^{\alpha k_2}+2^{10k_2})^{-1}\cdot 2^{(1/2-\beta)(\widetilde{k}-\widetilde{k}_2)}\cdot 2^m2^{(5/2-\alpha+\beta)k_1},$$

which is sufficient.

We now assume that $-2m/5\leq k_1\leq k_2$, and we decompose $f^\mu_{k_1,j_1},f^\nu_{k_2,j_2}$ as in (5.58), (5.59), (5.63). If $j_1\leq j_2$, we estimate

$$\begin{aligned}\Big\|\mathcal{F}P_k\widetilde{T}_s^{\sigma;\mu,\nu}(P_{[k_1-2,k_1+2]}(g^\mu_{k_1,j_1}(s)+h^\mu_{k_1,j_1}(s)),P_{[k_2-2,k_2+2]}g^\nu_{k_2,j_2}(s))\Big\|_{L^\infty}&\\ \lesssim\Big(\|\widehat{g^\mu_{k_1,j_1}}(s)\|_{L^2}+\|\widehat{h^\mu_{k_1,j_1}}(s)\|_{L^2}\Big)\|\widehat{g^\nu_{k_2,j_2}}(s)\|_{L^2}&\\ \lesssim 2^{(1+\beta)\widetilde{k}_1}2^{-(1+\beta)j_2}&\end{aligned}$$

and

$$\begin{aligned}\Big\|\mathcal{F}P_k\widetilde{T}_s^{\sigma;\mu,\nu}(P_{[k_1-2,k_1+2]}(g^{\mu}_{k_1,j_1}(s)+h^{\mu}_{k_1,j_1}(s)),P_{[k_2-2,k_2+2]}h^{\nu}_{k_2,j_2}(s))\Big\|_{L^\infty}\\ \lesssim \big(\|\widehat{g^{\mu}_{k_1,j_1}}(s)\|_{L^\infty}+\|\widehat{h^{\mu}_{k_1,j_1}}(s)\|_{L^\infty}\big)\|\widehat{h^{\nu}_{k_2,j_2}}(s)\|_{L^1}\\ \lesssim 2^{-(1/2-\beta)\widetilde{k_1}}\cdot 2^{-8|k_2|}2^{-\gamma j_2}.\end{aligned}$$

Since $-\widetilde{k_1}\le 2m/5$, $\alpha\le\beta$ and $2^{j_2}\gtrsim 2^{m(1-\beta/10)}2^{k_\sigma}$, it follows that if $j_1\le j_2$, then

$$\begin{aligned}&\Big\|\mathcal{F}P_k\widetilde{T}_s^{\sigma;\mu,\nu}(f^{\mu}_{k_1,j_1}(s),f^{\nu}_{k_2,j_2}(s))\Big\|_{L^\infty}\\ &\lesssim 2^{-(1+\beta)k_\sigma}2^{-(1+\beta)(1-\beta/10)m}\cdot(2^{\alpha k_1}+2^{10k_1})^{-1}(2^{\alpha k_2}+2^{10k_2})^{-1}.\end{aligned}\tag{5.65}$$

Similarly, if $j_1\ge j_2$, we estimate

$$\begin{aligned}\Big\|\mathcal{F}P_k\widetilde{T}_s^{\sigma;\mu,\nu}(P_{[k_1-2,k_1+2]}g^{\mu}_{k_1,j_1}(s),P_{[k_2-2,k_2+2]}(g^{\nu}_{k_2,j_2}(s)+h^{\nu}_{k_2,j_2}(s)))\Big\|_{L^\infty}\\ \lesssim\|\widehat{g^{\mu}_{k_1,j_1}}(s)\|_{L^2}\big(\|\widehat{g^{\nu}_{k_2,j_2}}(s)\|_{L^2}+\|\widehat{h^{\nu}_{k_2,j_2}}(s)\|_{L^2}\big)\\ \lesssim 2^{-(1+\beta)j_1}2^{(1+\beta)\widetilde{k}_2}\end{aligned}$$

and

$$\begin{aligned}\Big\|\mathcal{F}P_k\widetilde{T}_s^{\sigma;\mu,\nu}(P_{[k_1-2,k_1+2]}h^{\mu}_{k_1,j_1}(s),P_{[k_2-2,k_2+2]}(g^{\nu}_{k_2,j_2}(s)+h^{\nu}_{k_2,j_2}(s)))\Big\|_{L^\infty}\\ \lesssim\|\widehat{h^{\mu}_{k_1,j_1}}(s)\|_{L^1}\big(\|\widehat{g^{\nu}_{k_2,j_2}}(s)\|_{L^\infty}+\|\widehat{h^{\nu}_{k_2,j_2}}(s)\|_{L^\infty}\big)\\ \lesssim 2^{-\gamma j_1}2^{-6|k_1|}.\end{aligned}$$

Since $2^{j_1}\gtrsim 2^{m(1-\beta/10)}2^{k_\sigma}$, it follows that if $j_1\ge j_2$, then

$$\begin{aligned}&\Big\|\mathcal{F}P_k\widetilde{T}_s^{\sigma;\mu,\nu}(f^{\mu}_{k_1,j_1}(s),f^{\nu}_{k_2,j_2}(s))\Big\|_{L^\infty}\\ &\lesssim 2^{-(1+\beta)k_\sigma}2^{-(1+\beta)(1-\beta/10)m}\cdot(2^{\alpha k_1}+2^{10k_1})^{-1}(2^{\alpha k_2}+2^{10k_2})^{-1}.\end{aligned}\tag{5.66}$$

Using (5.65) and (5.66), the left-hand side of (5.64) is dominated by

$$C(1+2^k)2^{-\alpha k_1}2^{-4\beta m/5},$$

which suffices. This completes the proof of the lemma. □

Now that we have identified m as the largest parameter, we may remove the nonresonant part of the nonlinearity. For any $\kappa\in(0,2^{D/10}]$, we define

(5.67)

$$
\begin{aligned}
T_m^{\sigma;\mu,\nu}(f,g) &= R_{m,\kappa}^{\sigma;\mu,\nu}(f,g) + N_m^{1;\sigma;\mu,\nu}(f,g) + N_{m,\kappa}^{2;\sigma;\mu,\nu}(f,g),\\
\mathcal{F}\big[N_m^{1;\sigma;\mu,\nu}(f,g)\big](\xi) &:= \int_{\mathbb{R}}\int_{\mathbb{R}^3} e^{is\Phi^{\sigma;\mu,\nu}(\xi,\eta)}\chi_T^{\sigma;\mu,\nu}(\xi,\eta)q_m(s)\\
&\qquad\cdot \widehat{f}(\xi-\eta,s)\widehat{g}(\eta,s)\,d\eta ds,\\
\chi_T^{\sigma;\mu,\nu}(\xi,\eta) &:= \varphi_{[1,\infty)}(2^{D^2+\max(0,k_1,k_2)}\Phi^{\sigma;\mu,\nu}(\xi,\eta)),\\
\mathcal{F}\big[N_{m,\kappa}^{2;\sigma;\mu,\nu}(f,g)\big](\xi) &:= \int_{\mathbb{R}}\int_{\mathbb{R}^3} e^{is\Phi^{\sigma;\mu,\nu}(\xi,\eta)}\chi_S^{\sigma;\mu,\nu}(\xi,\eta)q_m(s)\\
&\qquad\cdot \widehat{f}(\xi-\eta,s)\widehat{g}(\eta,s)\,d\eta ds,\\
\chi_S^{\sigma;\mu,\nu}(\xi,\eta) &:= \varphi(2^{D^2+\max(0,k_1,k_2)}\Phi^{\sigma;\mu,\nu}(\xi,\eta))\varphi_{[1,\infty)}(|\Xi^{\mu,\nu}(\xi,\eta)|/\kappa),\\
\mathcal{F}\big[R_{m,\kappa}^{\sigma;\mu,\nu}(f,g)\big](\xi) &:= \int_{\mathbb{R}}\int_{\mathbb{R}^3} e^{is\Phi^{\sigma;\mu,\nu}(\xi,\eta)}\chi_R^{\sigma;\mu,\nu}(\xi,\eta)q_m(s)\\
&\qquad\cdot \widehat{f}(\xi-\eta,s)\widehat{g}(\eta,s)\,d\eta ds,\\
\chi_R^{\sigma;\mu,\nu}(\xi,\eta) &:= \varphi(2^{D^2+\max(0,k_1,k_2)}\Phi^{\sigma;\mu,\nu}(\xi,\eta))\varphi(|\Xi^{\mu,\nu}(\xi,\eta)|/\kappa).
\end{aligned}
$$

Our last lemma in this section shows that only the resonant part of the interaction $R_{m,\kappa}^{\sigma;\mu,\nu}$ may produce more problematic outputs not in $B_{j,k}^1$.

LEMMA 5.8. *Assume that* $\sigma \in \{i,e,b\}$, $\mu,\nu \in \mathcal{I}_0$, $(k,j),(k_1,j_1),(k_2,j_2) \in \mathcal{J}$, $m \in [0,L+1]\cap\mathbb{Z}$, *and*

$$
\begin{aligned}
&-9m/10 \le k_1,k_2 \le j/N_0', \quad \max(j_1,j_2) \le (1-\beta/10)m + k_\sigma,\\
&\beta m/2 + N_0'k_+ + D^2 \le j \le m+D, \quad m \ge -k(1+\beta^2).
\end{aligned}
\tag{5.68}
$$

Then, assuming $m \in [0,L]\cap\mathbb{Z}$,

$$
\begin{aligned}
&(1+2^k)\|\widetilde{\varphi}_j^{(k)} \cdot P_k N_m^{1;\sigma;\mu,\nu}(f_{k_1,j_1}^{\mu}, f_{k_2,j_2}^{\nu})\|_{B_{k,j}^1} \lesssim 2^{-2\beta^4 m},\\
&(1+2^k)\|\widetilde{\varphi}_j^{(k)} \cdot P_k N_{m,\kappa}^{2;\sigma;\mu,\nu}(f_{k_1,j_1}^{\mu}, f_{k_2,j_2}^{\nu})\|_{B_{k,j}^1} \lesssim 2^{-2\beta^4 m}
\end{aligned}
\tag{5.69}
$$

for any $\kappa \in (0,2^{D/10}]$ *satisfying*

$$
2^m\kappa \ge 2^{\beta^2 m}2^{\max(j_1,j_2)}, \qquad 2^m\kappa \ge 2^{\beta^2 m}\kappa^{-1}2^{-\min(k_1,k_2,0)}2^{-D}. \tag{5.70}
$$

Moreover, for $m = L+1$,

$$
(1+2^k)\|\widetilde{\varphi}_j^{(k)} \cdot P_k T_{L+1}^{\sigma;\mu,\nu}(f_{k_1,j_1}^{\mu}, f_{k_2,j_2}^{\nu})\|_{B_{k,j}^1} \lesssim 2^{-2\beta^4 L}. \tag{5.71}
$$

Proof of Lemma 5.8. To prove the second inequality in (5.69) we can apply Lemma A.2 with

$$
K = \kappa 2^m, \qquad \epsilon^{-1} = 2^{j_1} + 2^{j_2} + \kappa^{-1}2^{-\min(0,k_1,k_2)}, \qquad \epsilon K \ge 2^{\beta^2 m}
$$

and the assumptions (5.70) to show that

$$
|\mathcal{F}\big[N_{m,\kappa}^{2;\sigma;\mu,\nu}(f_{k_1,j_1}^{\mu}, f_{k_2,j_2}^{\nu})\big](\xi)| \lesssim 2^{-10m}. \tag{5.72}
$$

The second inequality in (5.69) follows easily using (5.68).

To prove the first inequality in (5.69) when $m \leq L$, we first integrate by parts in s and obtain that

$$\mathcal{F}\Big[N_m^{1;\sigma;\mu,\nu}(f,g)\Big](\xi) = -\int_{\mathbb{R}}\int_{\mathbb{R}^3} e^{is\Phi^{\sigma;\mu,\nu}(\xi,\eta)}\frac{\chi_T^{\sigma;\mu,\nu}(\xi,\eta)}{i\Phi^{\sigma;\mu,\nu}(\xi,\eta)} \cdot \partial_s\left[q_m(s)\widehat{f}(\xi-\eta,s)\widehat{g}(\eta,s)\right]\,d\eta ds. \tag{5.73}$$

Therefore,

$$\begin{aligned}
\mathcal{F}\Big[N_m^{1;\sigma;\mu,\nu}(f^\mu_{k_1,j_1},f^\nu_{k_2,j_2})\Big] &= i\left[N_{11}+N_{12}+N_{13}\right],\\
N_{11}(\xi) &:= \int_{\mathbb{R}}\int_{\mathbb{R}^3} e^{is\Phi^{\sigma;\mu,\nu}(\xi,\eta)}\frac{\chi_T^{\sigma;\mu,\nu}(\xi,\eta)}{\Phi^{\sigma;\mu,\nu}(\xi,\eta)}q_m'(s)\\
&\qquad\cdot\widehat{f^\mu_{k_1,j_1}}(\xi-\eta,s)\widehat{f^\nu_{k_2,j_2}}(\eta,s)\,d\eta ds,\\
N_{12}(\xi) &:= \int_{\mathbb{R}}\int_{\mathbb{R}^3} e^{is\Phi^{\sigma;\mu,\nu}(\xi,\eta)}\frac{\chi_T^{\sigma;\mu,\nu}(\xi,\eta)}{\Phi^{\sigma;\mu,\nu}(\xi,\eta)}q_m(s)\\
&\qquad\cdot(\partial_s\widehat{f^\mu_{k_1,j_1}})(\xi-\eta,s)\widehat{f^\nu_{k_2,j_2}}(\eta,s)\,d\eta ds,\\
N_{13}(\xi) &:= \int_{\mathbb{R}}\int_{\mathbb{R}^3} e^{is\Phi^{\sigma;\mu,\nu}(\xi,\eta)}\frac{\chi_T^{\sigma;\mu,\nu}(\xi,\eta)}{\Phi^{\sigma;\mu,\nu}(\xi,\eta)}q_m(s)\\
&\qquad\cdot\widehat{f^\mu_{k_1,j_1}}(\xi-\eta,s)(\partial_s\widehat{f^\nu_{k_2,j_2}})(\eta,s)\,d\eta ds.
\end{aligned}$$

We show first that

$$(1+2^k)(2^{\alpha k}+2^{10k})2^{(1+\beta)m}\|P_kN_m^{1;\sigma;\mu,\nu}(f^\mu_{k_1,j_1},f^\nu_{k_2,j_2})\|_{L^2}\lesssim 2^{-2\beta^4 m}. \tag{5.74}$$

We may assume $k_1\leq k_2$. Using integration by parts, it is easy to see that

$$\left\|\mathcal{F}^{-1}\left[\frac{\chi_T^{\sigma;\mu,\nu}(\xi,\eta)}{\Phi^{\sigma;\mu,\nu}(\xi,\eta)}\varphi_{[k-4,k+4]}(\xi)\varphi_{[k_1-4,k_1+4]}(\xi-\eta)\varphi_{[k_2-4,k_2+4]}(\eta)\right]\right\|_{L^1(\mathbb{R}^3\times\mathbb{R}^3)} \lesssim 2^{20\max(0,k_2)}. \tag{5.75}$$

Using the decomposition (5.73), Lemma A.3, and (5.75), we see that

$$\begin{aligned}
&\|P_kN_m^{1;\sigma;\mu,\nu}(f^\mu_{k_1,j_1},f^\nu_{k_2,j_2})\|_{L^2}\\
&\lesssim 2^{20\max(0,k_2)}\sup_{s\in[2^{m-2},2^{m+2}]}\Big[\min\Big\{\|Ef^\mu_{k_1,j_1}(s)\|_{L^\infty}\|f^\nu_{k_2,j_2}(s)\|_{L^2},\\
&\qquad\qquad \|f^\mu_{k_1,j_1}(s)\|_{L^2}\|Ef^\nu_{k_2,j_2}(s)\|_{L^\infty}\Big\}\\
&\quad+2^m\|Ef^\mu_{k_1,j_1}(s)\|_{L^\infty}\|(\partial_sf^\nu_{k_2,j_2})(s)\|_{L^2}+2^m\|(\partial_sf^\mu_{k_1,j_1})(s)\|_{L^2}\|Ef^\nu_{k_2,j_2}(s)\|_{L^\infty}\Big].
\end{aligned} \tag{5.76}$$

It follows from (5.17) and (5.20) that

$$2^m \|Ef^{\mu}_{k_1,j_1}(s)\|_{L^\infty} \|(\partial_s f^{\nu}_{k_2,j_2})(s)\|_{L^2} + 2^m \|(\partial_s f^{\mu}_{k_1,j_1})(s)\|_{L^2} \|Ef^{\nu}_{k_2,j_2}(s)\|_{L^\infty} \lesssim 2^{-6\max(k_2,0)} 2^{-(1+2\beta)m}.$$

Moreover, using (5.17)–(5.18),

$$\min\left\{\|Ef^{\mu}_{k_1,j_1}(s)\|_{L^\infty} \|f^{\nu}_{k_2,j_2}(s)\|_{L^2}, \|f^{\mu}_{k_1,j_1}(s)\|_{L^2} \|Ef^{\nu}_{k_2,j_2}(s)\|_{L^\infty}\right\} \lesssim 2^{-6\max(k_2,0)} 2^{-(1+\beta)m} 2^{-(1-\beta)\max(j_1,j_2)}.$$

Finally, if $\max(j_1, j_2) \leq 2\beta m$ then, using (5.18) and (5.19),

$$\min\left\{\|Ef^{\mu}_{k_1,j_1}(s)\|_{L^\infty} \|f^{\nu}_{k_2,j_2}(s)\|_{L^2}, \|f^{\mu}_{k_1,j_1}(s)\|_{L^2} \|Ef^{\nu}_{k_2,j_2}(s)\|_{L^\infty}\right\} \lesssim 2^{-6\max(k_2,0)} 2^{-(5/4-15\beta)m}.$$

It follows from the last three bounds and (5.76) that

$$\|P_k N^{1;\sigma;\mu,\nu}_m (f^{\mu}_{k_1,j_1}, f^{\nu}_{k_2,j_2})\|_{L^2} \lesssim 2^{15\max(k_2,0)} 2^{-(1+2\beta)m},$$

and the desired bound (5.74) follows since $2^k \lesssim 2^{k_2} \lesssim 2^{m/N_0'}$.

We show now that

$$(5.77)\quad (1+2^k) 2^{(1/2-\beta)\tilde{k}} (2^{\alpha k} + 2^{10k}) \|\mathcal{F} P_k N^{1;\sigma;\mu,\nu}_m (f^{\mu}_{k_1,j_1}, f^{\nu}_{k_2,j_2})\|_{L^\infty} \lesssim 2^{-2\beta^4 m}.$$

We may assume $k_1 \leq k_2$ and use the Cauchy-Schwartz inequality, (5.17), and (5.20) to see that

$$\begin{aligned} &\|N_{12}\|_{L^\infty} + \|N_{13}\|_{L^\infty} \\ &\quad \lesssim 2^{\max(0,k_2)} 2^m \sup_{s \in [2^{m-2}, 2^{m+2}]} \Big[\|(\partial_s \widehat{f}^{\mu}_{k_1,j_1})(s)\|_{L^2} \|\widehat{f}^{\nu}_{k_2,j_2}(s)\|_{L^2} \\ &\qquad\qquad + \|\widehat{f}^{\mu}_{k_1,j_1}(s)\|_{L^2} \|(\partial_s \widehat{f}^{\nu}_{k_2,j_2})(s)\|_{L^2} \Big] \\ &\quad \lesssim 2^{-\beta m} (1 + 2^{(N_0-10)k_2})^{-1}. \end{aligned}$$

This implies that N_{12} and N_{13} give acceptable contributions to (5.77). Proceeding as above, using (5.18) we also get

$$\|N_{11}\|_{L^\infty} \lesssim (1+2^{k_2}) 2^{\beta \widetilde{k}_1} 2^{-(1-\beta)j_1} \min(2^{-(N_0-5)k_2}, 2^{-(1-\beta)j_2}).$$

Therefore, this gives an acceptable contribution to (5.77) unless

$$(5.78)\qquad |k| + |k_1| + |k_2| + j_1 + j_2 \leq \beta^2 m.$$

Assuming that (5.78) holds, we need to strenghten the L^∞ bound on N_{11} slightly. We decompose

$$N_{11} = N_{11;1} + N_{11;2},$$
$$N_{11;1}(\xi) := \int_{\mathbb{R}}\int_{\mathbb{R}^3} e^{is\Phi^{\sigma;\mu,\nu}(\xi,\eta)} \frac{\chi_T^{\sigma;\mu,\nu}(\xi,\eta)}{\Phi^{\sigma;\mu,\nu}(\xi,\eta)} \varphi(\delta^{-1}|\Xi^{\mu,\nu}(\xi,\eta)|) q_m'(s) \cdot \widehat{f_{k_1,j_1}^{\mu}}(\xi-\eta,s)\widehat{f_{k_2,j_2}^{\nu}}(\eta,s)\, d\eta ds,$$
$$N_{11;2}(\xi) := \int_{\mathbb{R}}\int_{\mathbb{R}^3} e^{is\Phi^{\sigma;\mu,\nu}(\xi,\eta)} \frac{\chi_T^{\sigma;\mu,\nu}(\xi,\eta)}{\Phi^{\sigma;\mu,\nu}(\xi,\eta)} [1-\varphi(\delta^{-1}|\Xi^{\mu,\nu}(\xi,\eta)|)] q_m'(s) \cdot \widehat{f_{k_1,j_1}^{\mu}}(\xi-\eta,s)\widehat{f_{k_2,j_2}^{\nu}}(\eta,s)\, d\eta ds,$$

with $\delta := 2^{-m/3}$. Applying Lemma A.2 with $K = 2^{2m/3}$, $\epsilon = 2^{-m/3}$, it is easy to see that

$$|N_{11;2}(\xi)| \lesssim 2^{-10m},$$

provided that (5.78) holds, which is clearly sufficient. On the other hand, using the definition (5.14) and the bounds (A.5), we observe that

$$\begin{aligned}|\Xi^{\mu,\nu}(\xi,\eta)| &\gtrsim |\nabla\widetilde{\Lambda}_\nu(\eta)| \cdot \min\Big(\Big|(\xi-\eta)/|\xi-\eta| - \eta/|\eta|\Big|, \Big|(\xi-\eta)/|\xi-\eta| + \eta/|\eta|\Big|\Big) \\ &\gtrsim 2^{-\beta m} \min\Big(\Big|(\xi-\eta)/|\xi-\eta| - \eta/|\eta|\Big|, \Big|(\xi-\eta)/|\xi-\eta| + \eta/|\eta|\Big|\Big).\end{aligned}$$

Consequently, if $|\xi| \in [2^{k-2}, 2^{k+2}]$, $|\xi-\eta| \in [2^{k_1-2}, 2^{k_1+2}]$ and $|\eta| \in [2^{k_2-2}, 2^{k_2+2}]$, and $|\Xi^{\mu,\nu}(\xi,\eta)| \lesssim 2^{-m/3}$, then

$$\min\big(|\eta/|\eta| - \xi/|\xi||, |\eta/|\eta| + \xi/|\xi||\big) \lesssim 2^{-m/4}.$$

Then, a simple estimate using the L^∞ bounds in (5.17) gives $|N_{11;1}(\xi)| \lesssim 2^{-m/6}$, which is sufficient to finish the proof of (5.77). The first bound in (5.69) follows from (5.74) and (5.77).

The bound (5.71) follows by a similar (in fact easier) argument; since $\|q_{L+1}\|_{L^1} \lesssim 1$, one does not need to integrate by parts in s and one can simply estimate the appropriate L^2 and L^∞ norms in the same way we estimated the contributions of the function N_{11} in the argument above. □

We examine now the conclusions of Lemmas 5.5, 5.6, 5.7, and 5.8. We notice that to complete the proof of Proposition 5.4, it suffices to prove Proposition 5.9 below.

PROPOSITION 5.9. *Assume* $\sigma \in \{i,e,b\}$, $\mu,\nu \in \mathcal{I}_0$, $(k,j),(k_1,j_1),(k_2,j_2) \in \mathcal{J}$, $m \in [1,L]\cap\mathbb{Z}$, *and*

$$\begin{aligned}&-9m/10 \le k_1,k_2 \le j/N_0', \quad \max(j_1,j_2) \le (1-\beta/10)m + k_\sigma, \\ &\beta m/2 + N_0' k_+ + D^2 \le j \le m + D, \quad m \ge -k(1+\beta^2).\end{aligned} \tag{5.79}$$

Then there is

$$\kappa \in (0, 2^{D/10}], \quad \kappa \geq \max\left(2^{(\beta^2 m - m)/2} 2^{-\min(k_1,k_2,0)/2} 2^{-D/2}, 2^{\beta^2 m - m} 2^{\max(j_1,j_2)}\right),$$

such that

$$(1+2^k)2^{k\sigma} \left\| \widetilde{\varphi}_j^{(k)} \cdot P_k R_{m,\kappa}^{\sigma;\mu,\nu}(f_{k_1,j_1}^{\mu}, f_{k_2,j_2}^{\nu}) \right\|_{B_{k,j}} \lesssim 2^{-2\beta^4 m}. \tag{5.80}$$

We prove this proposition in the next three sections. We consider several types of resonant interactions, which involve input and output frequencies located on spheres or at the origin, as well as the different phase functions $\Phi^{\sigma;\mu,\nu}$. We classify these interactions into three basic types (see Proposition B.2) and analyze the contributions separately in the next three sections. The optimal value of κ for which we prove (5.80) depends, of course, on all the other parameters.

6. **Proof of Proposition 4.3, II: Case A resonant interactions**

In the following, given a set $\mathcal{S}$, we write $\Phi^{\sigma;\mu,\nu} \in\in \mathcal{S}$ if $\Phi^{\sigma;\mu,\nu} \in \mathcal{S}$ or $\Phi^{\sigma;\nu,\mu} \in \mathcal{S}$. In this section we consider type A interactions (see Proposition B.2) and prove the following proposition:

PROPOSITION 6.1. *Assume that* $(k,j),(k_1,j_1),(k_2,j_2) \in \mathcal{J}$, $m \in [1,L] \cap \mathbb{Z}$,

$$\begin{aligned}\Phi^{\sigma;\mu,\nu} \in \mathcal{T}_A' := \{&\Phi^{i;e+,i-}, \Phi^{i;b+,i-}, \Phi^{i;b-,e+}, \Phi^{i;b+,e-}, \Phi^{e;e+,i+}, \Phi^{e;b+,i+}, \Phi^{e;b+,i-},\\ &\Phi^{e;b+,e-}, \Phi^{b;e+,i+}, \Phi^{b;b+,i+}, \Phi^{b;e+,e+}, \Phi^{b;b+,e+}, \Phi^{b;b+,e-}\},\end{aligned} \tag{6.1}$$

and

$$\begin{aligned}&-D/2 \leq k, k_1, k_2 \leq D/2, \quad \max(j_1,j_2) \leq (1-\beta/10)m,\\ &\beta m/2 + N_0' k_+ + D^2 \leq j \leq m + D.\end{aligned} \tag{6.2}$$

Then there is $\kappa \in (0,1]$, $\kappa \geq \max\left(2^{(\beta^2 m - m)/2}, 2^{\beta^2 m - m} 2^{\max(j_1,j_2)}\right)$, *such that*

$$\left\| \widetilde{\varphi}_j^{(k)} \cdot P_k R_{m,\kappa}^{\sigma;\mu,\nu}(f_{k_1,j_1}^{\mu}, f_{k_2,j_2}^{\nu}) \right\|_{B_{k,j}} \lesssim 2^{-2\beta^4 m}. \tag{6.3}$$

The phases in the set $\mathcal{T}_A'$ are the same as the phases in the set $\mathcal{T}_A$, after interchanging the last two indices. Without loss of generality, we may assume that $\Phi^{\sigma;\mu,\nu} \in \mathcal{T}_A'$ instead of $\Phi^{\sigma;\mu,\nu} \in\in \mathcal{T}_A$.

The rest of the section is concerned with the proof of Proposition 6.1. The interactions corresponding to Case A are among the most difficult to control. In particular, they produce outputs that fail to belong to the "strong" $B_{k,j}^1$ spaces. A key element we need is a precise description of the sizes of the various elements close to the resonant set. This is made possible by the fact that the Hessian of the phases is nondegenerate. We refer to the introduction of [26] for more details.

Recall that $\Xi^{\mu,\nu}(\xi,\eta) = \nabla_\eta \Phi^{\sigma;\mu,\nu}(\xi,\eta)$. We define first the interaction functions for the space-resonant phases in $\mathcal{T}_A'$ given in (6.1), the functions $p^{\sigma;\mu,\nu}$ and $q^{\mu,\nu}$ defined below. They help us characterize the vanishing set for $\Xi^{\mu,\nu}$ through the equality (6.4). Only the functions $p^{\sigma;\mu,\nu}$ play an essential role, but the functions $q^{\mu,\nu}$ appear as simpler intermediate functions. Our goal is to define these functions such that

$$\Xi^{\mu,\nu}(q^{\mu,\nu}(\eta),\eta) = 0 = \Xi^{\mu,\nu}(\xi, p^{\sigma;\mu,\nu}(\xi)). \tag{6.4}$$

The first equality holds for all η, and the second equality holds for all ξ where $p^{\sigma;\mu,\nu}(\xi)$ is well defined.

For this, we first define

$$p^{b;e+,e+}(\xi) := \xi/2, \quad q^{e+,e+}(\eta) := 2\eta, \qquad t^{e,e}(r) := r. \tag{6.5}$$

The other functions require a little more care. We first define $q^{\mu,\nu}$ and then invert the process. We define the real-valued functions $t^{ei}, t^{bi}, t^{be} : [0,\infty) \to [0,\infty)$ by the relation

$$\lambda_e'(t^{ei}(r)) = \lambda_b'(t^{bi}(r)) = \lambda_i'(r), \quad \lambda_b'(t^{be}(r)) = \lambda_e'(r).$$

Since λ_e' and λ_b' are injective (see Lemma A.4) and using also (B.14), these functions are well defined. We can directly see that $t^{bi}(r) \le t^{ei}(r)$, $t^{be}(r) \le r$, and since

$$\lambda_i'(r) \in [\lambda_i'(r_*), \lambda_i'(0)] \subseteq \Big[\lambda_i'(r_*), \frac{\sqrt{1+T}}{\sqrt{1+\varepsilon}}\Big] \qquad \text{for any } r \in [0,\infty),$$

we get from Lemma A.4 that

$$\begin{aligned} \sqrt{\varepsilon}\lambda_i'(r_*)/(2T) \le t^{ei}(r) \le \sqrt{3\varepsilon/T}, \quad \sqrt{\varepsilon}\lambda_i'(r_*)/C_b \le t^{bi}(r) \le \sqrt{\varepsilon/C_b},\\ 0 \le t^{be}(r) \le \frac{\sqrt{T(1+\varepsilon)}}{\sqrt{C_b^2 - TC_b}}. \end{aligned} \tag{6.6}$$

More precisely, we have

$$t^{bi}(r) = \frac{\sqrt{\varepsilon(1+\varepsilon)}}{\sqrt{C_b}} \frac{\lambda_i'(r)}{\sqrt{C_b - \varepsilon(\lambda_i'(r))^2}}, \quad t^{be}(r) = \frac{\sqrt{\varepsilon(1+\varepsilon)}}{\sqrt{C_b}} \frac{\lambda_e'(r)}{\sqrt{C_b - \varepsilon(\lambda_e'(r))^2}},$$

while t^{ei} has a similar behavior. Note, in particular, that $T(t^{ei}(r))^2 \le 3\varepsilon$, $C_b(t^{bi}(r))^2 \le \varepsilon$, $C_b(t^{be}(r))^2 \le T(1+\varepsilon)/(C_b - T)$. Let

$$(\partial t^{\sigma_1\sigma_2})(r) := \frac{dt^{\sigma_1\sigma_2}(r)}{dr}, \qquad (\sigma_1,\sigma_2) \in \{(e,e),(e,i),(b,i),(b,e)\}.$$

Using Lemma A.4,

(6.7)
$$\begin{aligned}\big|(\partial t^{ei})(r)\big| &= \Big|\frac{\lambda_i''(r)}{\lambda_e''(t^{ei}(r))}\Big| \\ &\le |\lambda_i''(r)|\frac{\varepsilon^{1/2}(1+Tt^{ei}(r)^2)^{3/2}}{T(1-\sqrt{\varepsilon})} \le \frac{8\sqrt{2}\sqrt{\varepsilon}(1+3\varepsilon)^{3/2}T}{(1-\sqrt{\varepsilon})T} \le \frac{1}{2},\\ \big|(\partial t^{bi})(r)\big| &= \Big|\frac{\lambda_i''(r)}{\lambda_b''(t^{bi}(r))}\Big| \le |\lambda_i''(r)|\frac{\varepsilon^{1/2}(1+\varepsilon+C_bt^{bi}(r)^2)^{3/2}}{C_b(1+\varepsilon)}\\ &\le \frac{8\sqrt{2}\sqrt{\varepsilon}T(1+2\varepsilon)^{3/2}}{C_b(1+\varepsilon)} \le \frac{1}{2},\\ C_{C_b,\varepsilon}^{-1} \le (\partial t^{be})(r) &= \frac{\lambda_e''(r)}{\lambda_b''(t^{be}(r))}\\ &\le \frac{(1+\sqrt{\varepsilon})T}{(1+Tr^2)^{3/2}}\frac{(1+\varepsilon+C_bt^{be}(r)^2)^{3/2}}{C_b(1+\varepsilon)} \le \frac{(1+4\sqrt{\varepsilon})TC_b^{1/2}}{(C_b-T)^{3/2}} \le \frac{1}{2}.\end{aligned}$$

We now define $q^{\mu,\nu}$ when $(\sigma_1,\sigma_2)\in\{(e,i),(b,i),(b,e)\}$ by the formula

$$q^{\mu,\nu}(\eta) := \eta + (\iota_1\cdot\iota_2)t^{\sigma_1\sigma_2}(|\eta|)\frac{\eta}{|\eta|} = \tilde{t}^{\mu,\nu}(|\eta|)\frac{\eta}{|\eta|},$$

such that $\Xi^{\mu,\nu}(q^{\mu,\nu}(\eta),\eta)=0$. Then we define the function $r^{\mu,\nu}(s)$ as the inverse function of $\widetilde{t}^{\mu,\nu}(r) := r + (\iota_1\cdot\iota_2)t^{\sigma_1\sigma_2}(r)$. Therefore,

$$r^{\mu,\nu} : [\iota_1\iota_2 t^{\sigma_1\sigma_2}(0),\infty) \to [0,\infty)$$

is a well-defined increasing function, and

$$(\partial_s r^{\mu,\nu})(s) = \frac{1}{1+\iota_1\iota_2(\partial t^{\sigma_1\sigma_2})(r^{\mu,\nu}(s))}, \qquad s\in[\iota_1\iota_2 t^{\sigma_1\sigma_2}(0),\infty). \tag{6.8}$$

We can now finally define the functions $p^{\sigma;\mu,\nu}$ and $\chi_A^{\sigma;\mu,\nu}:[0,\infty)\to[0,1]$:

(a) if $\Phi^{\sigma;\mu,\nu}\in\mathcal{T}_A'\setminus\{\Phi^{e;b+,i-}\}$, then we define

$$\begin{aligned}&I^{\sigma;\mu,\nu} := [t^{\sigma_1\sigma_2}(0),\infty), \quad p^{\sigma;\mu,\nu}(\xi) := r^{\mu,\nu}(|\xi|)\xi/|\xi| \text{ for } |\xi|\in I^{\sigma;\mu,\nu},\\ &\chi_A^{\sigma;\mu,\nu} := \mathbf{1}_{(t^{\sigma_1\sigma_2}(0)+2^{-2D},\infty)};\end{aligned} \tag{6.9}$$

(b) if $\Phi^{\sigma;\mu,\nu}=\Phi^{e;b+,i-}$, then we define

$$\begin{aligned}&I^{\sigma;\mu,\nu} := [0,t^{bi}(0)], \quad p^{\sigma;\mu,\nu}(\xi) := -r^{\mu,\nu}(-|\xi|)\xi/|\xi| \text{ for } |\xi|\in I^{\sigma;\mu,\nu},\\ &\chi_A^{\sigma;\mu,\nu} := \mathbf{1}_{(0,t^{bi}(0)-2^{-2D})}.\end{aligned} \tag{6.10}$$

In both cases we also define

$$r^{\sigma;\mu,\nu}(|\xi|) := p^{\sigma;\mu,\nu}(\xi)\cdot\xi/|\xi|. \tag{6.11}$$

The functions $p^{\sigma;\mu,\nu}$ are not defined (and not needed) outside the range specified above, since we will use them only to study resonant interactions.

These functions are the key to an efficient analysis of Case A through the use of the following lemma:

LEMMA 6.2. *Assume* $\Phi^{\sigma;\mu,\nu}\in\mathcal{T}'_A$ *(see* (6.1)*) and* $-D/2\leq k,k_1,k_2\leq D/2$.

(i) *Assume that* $\delta\in[0,2^{-100D}]$, *and assume that* $(\xi,\eta)\in\mathbb{R}^3\times\mathbb{R}^3$ *is a point such that*

$$|\xi|\in[2^{k-4},2^{k+4}],\qquad |\eta|\in[2^{k_2-4},2^{k_2+4}],\qquad |\xi-\eta|\in[2^{k_1-4},2^{k_1+4}],$$
$$|\Xi^{\mu,\nu}(\xi,\eta)|\leq\delta,\qquad |\Phi^{\sigma;\mu,\nu}(\xi,\eta)|\leq 2^{-100D}. \tag{6.12}$$

Then

$$\chi_A^{\sigma;\mu,\nu}(|\xi|)=1,\quad \Big|\eta-p^{\sigma;\mu,\nu}(\xi)\Big|\leq 2^{40D}\delta\quad \text{and}\quad \Xi^{\mu,\nu}(\xi,p^{\sigma;\mu,\nu}(\xi))=0, \tag{6.13}$$

and

$$\min\Big(|(\partial_s r^{\sigma;\mu,\nu})(|\xi|)|,|1-(\partial_s r^{\sigma;\mu,\nu})(|\xi|)|\Big)\geq 2^{-4D},$$
$$|(D_s^\rho r^{\sigma;\mu,\nu})(|\xi|)|\leq 2^{20D},\qquad \rho=0,1,\dots 4. \tag{6.14}$$

Moreover, if $\sigma_2=i$, *then*

$$\Big||\eta|-r_*\Big|\gtrsim_{C_b,\varepsilon}1. \tag{6.15}$$

(ii) *Let* $\Psi^{\sigma;\mu,\nu}:I^{\sigma;\mu,\nu}\to\mathbb{R}$ *be defined by*

$$\begin{aligned}\Psi^{\sigma;\mu,\nu}(s)&:=\Phi^{\sigma;\mu,\nu}(se,r^{\sigma;\mu,\nu}(s)e)\\&=\lambda_\sigma(s)-\iota_1\lambda_{\sigma_1}(|r^{\sigma;\mu,\nu}(s)-s|)-\iota_2\lambda_{\sigma_2}(|r^{\sigma;\mu,\nu}(s)|)\end{aligned} \tag{6.16}$$

for some $e\in\mathbb{S}^2$. *(The definition, of course, does not depend on the choice of* e.*) Then there is some constant* $\widetilde{c}=\widetilde{c}(\sigma,\mu,\nu)\in\{-1,1\}$ *with the following property*:

$$\text{the set } \widetilde{I}_k^{\sigma;\mu,\nu}:=\{s\in[2^{k-4},2^{k+4}]\cap I^{\sigma;\mu,\nu}:|\Psi^{\sigma;\mu,\nu}(s)|\leq 2^{-110D}\}\text{ is an interval;}$$
$$\widetilde{c}\cdot(\partial_s\Psi^{\sigma;\mu,\nu})(s)\geq 2^{-20D}\text{ for any } s\in\widetilde{I}_k^{\sigma;\mu,\nu}. \tag{6.17}$$

Proof of Lemma 6.2. Since $\Phi^{\sigma;\mu,\nu}\in\mathcal{T}'_A$, $q^{\mu,\nu}$ is well defined. We start from the elementary formula

$$\begin{aligned}|\Xi^{\mu,\nu}(\xi,\eta)|&=|\Xi^{\mu,\nu}(\xi,\eta)-\Xi^{\mu,\nu}(q^{\mu,\nu}(\eta),\eta)|\\&\approx_{C_b,\varepsilon}|\lambda'_{\sigma_1}(|\xi-\eta|)-\lambda'_{\sigma_1}(|q^{\mu,\nu}(\eta)-\eta|)|\\&\quad+\max(\lambda'_{\sigma_1}(|\xi-\eta|),\lambda'_{\sigma_1}(|q^{\mu,\nu}(\eta)-\eta|))\Big|\frac{\xi-\eta}{|\xi-\eta|}-\frac{q^{\mu,\nu}(\eta)-\eta}{|q^{\mu,\nu}(\eta)-\eta|}\Big|.\end{aligned}$$

Since $\lambda'_{\sigma_1}(r)\geq 2^{-2D}$ and $\lambda''_{\sigma_1}(r)\geq 2^{-2D}(1+r)^{-3}$ if $r\geq 2^{-D/2-10}$, the condition $|\Xi^{\mu,\nu}(\xi,\eta)|\leq\delta$ shows that

$$\Big||\xi-\eta|-|q^{\mu,\nu}(\eta)-\eta|\Big|+\Big|\frac{\xi-\eta}{|\xi-\eta|}-\frac{q^{\mu,\nu}(\eta)-\eta}{|q^{\mu,\nu}(\eta)-\eta|}\Big|\leq 2^{10D}\delta.$$

This shows that

$$|\xi - q^{\mu,\nu}(\eta)| \leq 2^{20D}\delta \quad \text{and} \quad |f^{\sigma;\mu,\nu}(|\eta|)| \leq 2^{30D}, \tag{6.18}$$

where $f^{\sigma;\mu,\nu} : [0,\infty) \to \mathbb{R}$ is defined by

$$f^{\sigma;\mu,\nu}(r) := \Phi^{\sigma;\mu,\nu}(q^{\mu,\nu}(re), re) = \lambda_\sigma(|\widetilde{t}^{\mu,\nu}(r)|) - \iota_1\lambda_{\sigma_1}(t^{\sigma_1\sigma_2}(r)) - \iota_2\lambda_{\sigma_2}(r). \tag{6.19}$$

We turn now to the proof of the lemma. We observe first that (6.14) follows from the formula (6.8) and the bounds (6.15), (6.13), and (6.7). We note also that the conclusion that $\widetilde{I}_k^{\sigma;\mu,\nu}$ is a closed interval in the first line of (6.17) is a consequence of the existence of a constant $\widetilde{c}$ satisfying the inequality $\widetilde{c}(\partial_s\Psi^{\sigma;\mu,\nu})(s) \geq 2^{-20D}$ for any $s \in \widetilde{I}_k^{\sigma;\mu,\nu}$ in the second line of (6.17).

We prove the claims in the lemma by analyzing several cases.

Case 1. $\Phi^{\sigma;\mu,\nu} \in \{\Phi^{b;b+,e+}, \Phi^{b;e+,e+}\}$. In this case we have

$$\begin{aligned}
&t^{\sigma_1\sigma_2}(0) = 0,\ \ \chi_A^{\sigma;\mu,\nu} = \mathbf{1}_{(2^{-2D},\infty)},\ \ \widetilde{t}^{\mu,\nu}(r) = r + t^{\sigma_1\sigma_2}(r),\ \ r^{\mu,\nu}(s) \in [0,s],\\
&f^{\sigma;\mu,\nu}(r) = \lambda_b(r + t^{\sigma_1\sigma_2}(r)) - \lambda_{\sigma_1}(t^{\sigma_1\sigma_2}(r)) - \lambda_e(r),\\
&\Psi^{\sigma;\mu,\nu}(s) = \lambda_b(s) - \lambda_{\sigma_1}(s - r^{\mu,\nu}(s)) - \lambda_e(r^{\mu,\nu}(s)),\\
&(\partial_s\Psi^{\sigma;\mu,\nu})(s) = \lambda_b'(s) - \lambda_e'(r^{\mu,\nu}(s)).
\end{aligned} \tag{6.20}$$

The claims (6.13) and (6.17) with $\widetilde{c} = 1$ follow easily (using, for example, (B.5)), and the claim (6.15) is trivial.

Case 2. $\Phi^{\sigma;\mu,\nu} \in \{\Phi^{e;e+,i+}, \Phi^{e;b+,i+}, \Phi^{b;e+,i+}, \Phi^{b;b+,i+}\}$. In this case we have

$$\begin{aligned}
t^{\sigma_1\sigma_2}(0) &\approx_{C_b,\varepsilon} 1, \quad \chi_A^{\sigma;\mu,\nu} = \mathbf{1}_{(t^{\sigma_1\sigma_2}(0)+2^{-2D},\infty)},\\
\widetilde{t}^{\mu,\nu}(r) &= r + t^{\sigma_1\sigma_2}(r), \quad r^{\mu,\nu}(s) \in [0,s],\\
f^{\sigma;\mu,\nu}(r) &= \lambda_\sigma(r + t^{\sigma_1\sigma_2}(r)) - \lambda_{\sigma_1}(t^{\sigma_1\sigma_2}(r)) - \lambda_i(r),\\
\Psi^{\sigma;\mu,\nu}(s) &= \lambda_\sigma(s) - \lambda_{\sigma_1}(s - r^{\mu,\nu}(s)) - \lambda_i(r^{\mu,\nu}(s)),\\
(\partial_s\Psi^{\sigma;\mu,\nu})(s) &= \lambda_\sigma'(s) - \lambda_i'(r^{\mu,\nu}(s)).
\end{aligned} \tag{6.21}$$

Notice that

$$(\partial_r f^{\sigma;\mu,\nu})(r) = [1 + (\partial t^{\sigma_1\sigma_2})(r)][\lambda_\sigma'(r + t^{\sigma_1\sigma_2}(r)) - \lambda_{\sigma_1}'(t^{\sigma_1\sigma_2}(r))].$$

Therefore, using also (6.7) and Lemma A.4, $(\partial_r f^{\sigma;\mu,\nu})(r) \geq_{C_b,\varepsilon} r(1+r^2)^{-3/2}$ and $f^{\sigma;\mu,\nu}(0) \geq 0$ if $\Phi^{\sigma;\mu,\nu} \in \{\Phi^{e;e+,i+}, \Phi^{b;e+,i+}, \Phi^{b;b+,i+}\}$. Therefore, the inequality $|f^{\sigma;\mu,\nu}(|\eta|)| \leq 2^{-20D}$ in (6.18) cannot be verified in these cases for any (ξ,η) as in (6.12), and the conclusions of the lemma are trivial.

On the other hand, if $\Phi^{\sigma;\mu,\nu} = \Phi^{e;b+,i+}$, then the claims in (6.13) follow easily, using (6.18) and the hypothesis of the lemma. To prove the remaining claims we show first that

$$|\eta| \le 3T^{-1/2}/4 \le 3r_*/4. \tag{6.22}$$

Indeed, starting from the inequalities $|f^{e;b+,i+}(|\eta|)| \le 2^{-20D}$ and $t^{bi}(r) \le \sqrt{\varepsilon/C_b}$ (see (6.6)), and using also (A.7), it follows that

$$\begin{aligned} 2^{-20D} &\ge \lambda_e(|\eta|) - \varepsilon^{-1/2}\sqrt{1+2\varepsilon} - \lambda_i(|\eta|) \\ &\ge \varepsilon^{-1/2}\big(\sqrt{1+T|\eta|^2} - \sqrt{1+2\varepsilon}\big) - \sqrt{(T+1)(\varepsilon+1)}|\eta|. \end{aligned}$$

The desired bound (6.22) follows. This clearly implies the bound (6.15).

Finally, to prove (6.17), we calculate

(6.23)
$$\begin{aligned} \Psi^{e;b+,i+}(t^{bi}(0)) &= \lambda_e(t^{bi}(0)) - \lambda_b(t^{bi}(0)) \le -C^{-1}_{C_b,\varepsilon}, \\ (\partial_s\Psi^{e;b+,i+})(t^{bi}(0)) &= \lambda'_e(t^{bi}(0)) - \lambda'_i(0) = \lambda'_e(t^{bi}(0)) - \lambda'_b(t^{bi}(0)) \le -C^{-1}_{C_b,\varepsilon}, \\ (\partial_s^2\Psi^{e;b+,i+})(s) &= \lambda''_e(s) - (\partial_s r^{b+,i+})(s)\lambda''_i(r^{b+,i+}(s)). \end{aligned}$$

Therefore, $(\partial_s^2\Psi^{e;b+,i+})(s) \ge C^{-1}_{C_b,\varepsilon}$ for all $s \in [t^{bi}(0),\infty)$ for which $r^{b+,i+}(s) \le r_*$. On the other hand, as in the proof of (6.22), if $s \in [2^{k-4}, 2^{k+4}]$ has the property that $|\Psi^{e;b+,i+}(s)| \le 2^{-20D}$, then $r^{b+,i+}(s) \le 4r_*/5$. The desired conclusion (6.17) follows with $\widetilde{c} = 1$ by combining the inequalities in (6.23).

Case 3. $\Phi^{\sigma;\mu,\nu} \in \{\Phi^{i;b-,e+}, \Phi^{i;b+,e-}, \Phi^{e;b+,e-}, \Phi^{b;b+,e-}\}$. In this case we have

(6.24)
$$\begin{aligned} &t^{\sigma_1\sigma_2}(0) = 0,\ \ \chi_A^{\sigma;\mu,\nu} = \mathbf{1}_{(2^{-2D},\infty)},\ \ \widetilde{t}^{\mu,\nu}(r) = r - t^{\sigma_1\sigma_2}(r),\ \ r^{\mu,\nu}(s) \in [s,\infty), \\ &f^{\sigma;\mu,\nu}(r) = \lambda_\sigma(r - t^{\sigma_1\sigma_2}(r)) - \iota_1\lambda_b(t^{\sigma_1\sigma_2}(r)) - \iota_2\lambda_e(r), \\ &\Psi^{\sigma;\mu,\nu}(s) = \lambda_\sigma(s) - \iota_1\lambda_b(r^{\mu,\nu}(s) - s) - \iota_2\lambda_e(r^{\mu,\nu}(s)), \\ &(\partial_s\Psi^{\sigma;\mu,\nu})(s) = \lambda'_\sigma(s) - \iota_2\lambda'_e(r^{\mu,\nu}(s)). \end{aligned}$$

The claims in (6.13) follow easily, using the hypothesis and (6.18). The claim (6.15) is trivial. The conclusion (6.17) also follows from the formulas above if $\iota_2 = -$, with $\widetilde{c} = 1$.

It remains to prove (6.17) when $\Phi^{\sigma;\mu,\nu} = \Phi^{i;b-,e+}$, in which case we set $\widetilde{c} = -1$. For $s \ge r_*$, we estimate

$$(\partial_s\Psi^{i;b-,e+})(s) = \lambda'_i(s) - \lambda'_e(r^{\mu,\nu}(s)) \le 1 - \lambda'_e(r_*) \le -1,$$

which gives the desired conclusion (6.17) when $s \ge r_*$. On the other hand, we calculate

$$\begin{aligned} \Psi^{i;b-,e+}(0) &= \lambda_i(0) + \lambda_b(0) - \lambda_e(0) = 0, \\ (\partial_s\Psi^{i;b-,e+})(0) &= \lambda'_i(0) - \lambda'_e(0) \ge C^{-1}_{C_b,\varepsilon}, \\ (\partial_s^2\Psi^{i;b-,e+})(s) &= \lambda''_i(s) - (\partial_s r^{b-,e+})(s)\lambda''_e(r^{b-,e+}(s)). \end{aligned}$$

Therefore, $(\partial_s^2\Psi^{i;b-,e+})(s) \leq -C_{C_b,\varepsilon}^{-1}$ for $s \in [0, r_*]$, and the desired conclusion (6.17) with $\widetilde{c} = -1$ follows in this range as well.

Case 4. $\Phi^{\sigma;\mu,\nu} \in \{\Phi^{i;e+,i-}, \Phi^{i;b+,i-}\}$. In this case we have

$$
\begin{aligned}
&t^{\sigma_1\sigma_2}(0) \approx_{C_b,\varepsilon} 1, \quad \chi_A^{\sigma;\mu,\nu} = \mathbf{1}_{(t^{\sigma_1\sigma_2}(0)+2^{-2D},\infty)},\\
&\widetilde{t}^{\mu,\nu}(r) = r - t^{\sigma_1\sigma_2}(r), \quad r^{\mu,\nu}(s) \in [s,\infty),\\
&f^{\sigma;\mu,\nu}(r) = \lambda_i(|r - t^{\sigma_1\sigma_2}(r)|) - \lambda_{\sigma_1}(t^{\sigma_1\sigma_2}(r)) + \lambda_i(r),\\
&\Psi^{\sigma;\mu,\nu}(s) = \lambda_i(s) - \lambda_{\sigma_1}(r^{\mu,\nu}(s) - s) + \lambda_i(r^{\mu,\nu}(s)),\\
&(\partial_s\Psi^{\sigma;\mu,\nu})(s) = \lambda_i'(s) + \lambda_i'(r^{\mu,\nu}(s)).
\end{aligned}
\tag{6.25}
$$

Recalling that $t^{\sigma_1 i}(r) \leq \sqrt{3\varepsilon/T}$ and $\lambda_i(r) \leq \sqrt{1+r^2}$ for any $r \in [0,\infty)$ (see (6.6) and (A.4)), we estimate

$$\lambda_i(|r - t^{\sigma_1\sigma_2}(r)|) - \lambda_{\sigma_1}(t^{\sigma_1\sigma_2}(r)) + \lambda_i(r) \leq -\varepsilon^{-1/2} + 2\sqrt{1+r^2}$$

for any $r \in [0,\infty)$. The inequality $|f^{\sigma;\mu,\nu}(|\eta|)| \leq 2^{-20D}$ (see (6.18)) then shows that $|\eta| \geq (3\varepsilon)^{-1/2}$. Therefore, $|q^{\mu,\nu}(\eta)| = |\eta| - t^{\sigma_1 i}(|\eta|) \geq |\eta|/2$, and the conclusions in (6.13) follow using also (6.18). The claim (6.15) follows from $|\eta| \geq (3\varepsilon)^{-1/2}$. Finally, the conclusion (6.17) with $\widetilde{c} = 1$ follows from the last formula in (6.25).

Case 5. $\Phi^{\sigma;\mu,\nu} = \Phi^{e;b+,i-}$. In this case we have

$$
\begin{aligned}
&t^{\sigma_1\sigma_2}(0) \approx_{C_b,\varepsilon} 1, \quad \chi_A^{\sigma;\mu,\nu} = \mathbf{1}_{(0,t^{\sigma_1\sigma_2}(0)-2^{-2D})},\\
&\widetilde{t}^{\mu,\nu}(r) = r - t^{bi}(r), \quad r^{\mu,\nu}(s) \in [s,\infty),\\
&f^{\sigma;\mu,\nu}(r) = \lambda_e(|r - t^{bi}(r)|) - \lambda_b(t^{bi}(r)) + \lambda_i(r),\\
&\Psi^{\sigma;\mu,\nu}(s) = \lambda_e(s) - \lambda_b(r^{\mu,\nu}(-s) + s) + \lambda_i(r^{\mu,\nu}(-s)),\\
&(\partial_s\Psi^{\sigma;\mu,\nu})(s) = \lambda_e'(s) - \lambda_b'(r^{\mu,\nu}(-s) + s).
\end{aligned}
\tag{6.26}
$$

Clearly, $-f^{\sigma;\mu,\nu}(0) \gtrsim_{C_b,\varepsilon} 1$. Extending λ_e as an even function on $\mathbb{R}$ we calculate, for $r \geq 0$,

$$
\begin{aligned}
(\partial_r f^{\sigma;\mu,\nu})(r) &= (1 - (\partial t^{bi})(r))\lambda_e'(r - t^{bi}(r)) - (\partial t^{bi})(r)\lambda_b'(t^{bi}(r)) + \lambda_i'(r)\\
&= [1 - (\partial t^{bi})(r)][\lambda_b'(t^{bi}(r)) + \lambda_e'(r - t^{bi}(r))].
\end{aligned}
$$

Let $r_0 \in [0,\infty)$ denote the unique number with the property that $r_0 = t^{bi}(r_0)$. In view of (6.6), $r_0 \leq \sqrt{\varepsilon/C_b} \leq r_*/2$. Moreover, $r - t^{bi}(r) \geq 0$ if $r \geq r_0$ and $r - t^{bi}(r) \leq 0$ if $r \leq r_0$. Therefore, $(\partial_r f^{\sigma;\mu,\nu})(r) \gtrsim_{C_b,\varepsilon} 1$ if $r \geq r_0$ and $(\partial_r f^{\sigma;\mu,\nu})(r) \gtrsim_{C_b,\varepsilon} r$ if $r \in [0, r_0]$. Moreover,

$$
\begin{aligned}
f^{\sigma;\mu,\nu}(r_0) &= \lambda_e(0) - \lambda_b(r_0) + \lambda_i(r_0) = \int_0^{r_0} [\lambda_i'(\rho) - \lambda_b'(\rho)]\, d\rho\\
&\geq r_0\lambda_i'(r_0) - r_0\lambda_b'(r_0) + \int_0^{r_0} [\lambda_b'(r_0) - \lambda_b'(\rho)]\, d\rho \gtrsim_{C_b,\varepsilon} 1.
\end{aligned}
$$

Therefore, the strictly increasing function $f^{\sigma;\mu,\nu}$ has a unique zero in the interval $(2^{-D/2}, r_0 - 2^{-D/2})$. It follows from (6.18) that if $\eta = re$, then $r \in (2^{-D}, r_0 - 2^{-D})$ and $|\xi - (r - t^{bi}(r))e| \leq 2^{20D}\delta$. The conclusions in (6.13) follow. The conclusion (6.15) follows using also $r_0 \leq r_*/2$. The inequality (6.17) follows using, for example, (B.5). □

Remark. The analysis in Case 2 in the proof of Lemma 6.2 shows that the phases $\Phi^{e;e+,i+}$, $\Phi^{b;e+,i+}$, and $\Phi^{b;b+,i+}$ are, in fact, nonresonant, in the sense that there are no points $(\xi,\eta) \in \mathbb{R}^3 \times \mathbb{R}^3$ satisfying (6.12). Therefore, in Proposition 6.1 we may assume that

$$\Phi^{\sigma;\mu,\nu} \in \mathcal{T}''_A := \{\Phi^{i;e+,i-}, \Phi^{i;b+,i-}, \Phi^{i;b-,e+}, \Phi^{i;b+,e-}, \Phi^{e;b+,i+}, \Phi^{e;b+,i-}, \Phi^{e;b+,e-}, \Phi^{b;e+,e+}, \Phi^{b;b+,e+}, \Phi^{b;b+,e-}\}. \tag{6.27}$$

6.1. *Proof of Proposition* 6.1. Once the functions $p^{\sigma;\mu,\nu}$ have been created, the rest of the analysis follows similar lines to the analysis of [26, §4]. The main ingredients we need come from the refined $B_{k,j}$ norms and additional L^2 orthogonality arguments. We prove Proposition 6.1 in two steps (see Lemmas 6.3 and 6.4 below) depending on the maximum in the definition of κ.

Lemma 6.3. *The bound* (6.3) *holds provided that* (6.2) *and* (6.27) *hold and*, *in addition*,

$$\max(j_1, j_2) \leq (m - \beta^2 m)/2, \tag{6.28}$$

with

$$\kappa := 2^{(\beta^2 m - m)/2}. \tag{6.29}$$

Proof of Lemma 6.3. For simplicity of notation, let

$$\begin{aligned} G(\xi) &:= \mathcal{F}[P_k R^{\sigma;\mu,\nu}_{m,\kappa}(f^\mu_{k_1,j_1}, f^\nu_{k_2,j_2})](\xi), \\ &= \varphi_k(\xi)\chi^{\sigma;\mu,\nu}_A(|\xi|)\int_{\mathbb{R}}\int_{\mathbb{R}^3} e^{is\Phi^{\sigma;\mu,\nu}(\xi,\eta)}\chi^{\sigma;\mu,\nu}_R(\xi,\eta)q_m(s) \\ &\quad \cdot \widehat{f^\mu_{k_1,j_1}}(\xi-\eta,s)\widehat{f^\nu_{k_2,j_2}}(\eta,s)\,d\eta ds, \end{aligned} \tag{6.30}$$

where $\chi^{\sigma;\mu,\nu}_A$ was defined in Lemma 6.2 and, as before,

$$\chi^{\sigma;\mu,\nu}_R(\xi,\eta) = \varphi(2^{D^2+\max(0,k_1,k_2)}\Phi^{\sigma;\mu,\nu}(\xi,\eta))\varphi(|\Xi^{\mu,\nu}(\xi,\eta)|/\kappa).$$

Using the L^∞ bounds in (5.17) and (6.13) (with $\delta = 4\kappa$), we see easily that

$$\|G\|_{L^\infty} \lesssim \kappa^3 \cdot 2^m \lesssim 2^{-m/2}2^{3\beta^2 m/2}. \tag{6.31}$$

This suffices to prove (6.3) if, for example, $j \leq m(1/2 - 4\beta)$. To cover the entire range $j \leq m + D$ we integrate by parts in s.

In the argument below we may assume that $G \neq 0$; in particular, this guarantees that the main assumption (6.12) is satisfied. With $\Psi^{\sigma;\mu,\nu}(|\xi|) =$

$\Phi^{\sigma;\mu,\nu}(\xi,p^{\sigma;\mu,\nu}(\xi))$, defined as in (6.16), assume that

$$2^m|\Psi^{\sigma;\mu,\nu}(|\xi|)|\in[2^l,2^{l+1}],\ l\in[\beta m,\infty)\cap\mathbb{Z}. \tag{6.32}$$

Then, using Lemma 6.2, we see that if $|\eta-p^{\sigma;\mu,\nu}(\xi)|\leq 2^{50D}\kappa$, then

$$\begin{aligned}|\Phi^{\sigma;\mu,\nu}(\xi,\eta)-\Psi^{\sigma;\mu,\nu}(|\xi|)|&\leq|\eta-p^{\sigma;\mu,\nu}(\xi)|\cdot\sup_{|\zeta-p^{\sigma;\mu,\nu}(\xi)|\leq 2^{50D}\kappa}|\Xi^{\mu,\nu}(\xi,\zeta)|\\&\leq 2^{60D}\kappa|\eta-p^{\sigma;\mu,\nu}(\xi)|,\end{aligned}$$

since $\Xi^{\mu,\nu}(\xi,p^{\mu,\nu}(\xi))=0$. Therefore,

$$2^m|\Phi^{\sigma;\mu,\nu}(\xi,\eta)|\in[2^{l-3},2^{l+4}]\qquad\text{if}\qquad\chi_R^{\sigma;\mu,\nu}(\xi,\eta)\neq 0.$$

After integration by parts in s it follows that

$$\begin{aligned}|G(\xi)|\lesssim 2^{m-l}|\varphi_k(\xi)|\int_{\mathbb{R}}\int_{\mathbb{R}^3}&|\chi_R^{\sigma;\mu,\nu}(\xi,\eta)|\,|q_m'(s)|\,|\widehat{f^\mu_{k_1,j_1}}(\xi-\eta,s)|\,|\widehat{f^\nu_{k_2,j_2}}(\eta,s)|\\&+|\chi_R^{\sigma;\mu,\nu}(\xi,\eta)|\,|q_m(s)|\,|(\partial_s\widehat{f^\mu_{k_1,j_1}})(\xi-\eta,s)|\,|\widehat{f^\nu_{k_2,j_2}}(\eta,s)|\\&+|\chi_R^{\sigma;\mu,\nu}(\xi,\eta)|\,|q_m(s)|\,|\widehat{f^\mu_{k_1,j_1}}(\xi-\eta,s)|\,|(\partial_s\widehat{f^\nu_{k_2,j_2}})(\eta,s)|\,d\eta ds.\end{aligned}$$

We use now (5.7), the last bound in (5.17), (5.22), and Lemma 6.2. It follows that

$$|G(\xi)|\lesssim 2^{m-l}|\varphi_k(\xi)|\chi_A^{\sigma;\mu,\nu}(|\xi|)\cdot\kappa^3\lesssim|\varphi_k(\xi)|\chi_A^{\sigma;\mu,\nu}(|\xi|)\cdot 2^{-l}2^{-m/2}2^{\beta m/5} \tag{6.33}$$

provided that (6.32) holds.

We can now prove the desired bound (6.3). To make use of (6.32)–(6.33) we need a good description of the level sets of the functions $\Psi^{\sigma;\mu,\nu}$. Let

$$\begin{aligned}l_0&:=\lfloor\beta m+2\rfloor,\\D_{l_0}&:=\{\xi\in\mathbb{R}^3:2^m|\Psi^{\sigma;\mu,\nu}(|\xi|)|\leq 2^{l_0}\ \text{and}\ |\varphi_k(\xi)|\chi_A^{\sigma;\mu,\nu}(|\xi|)\neq 0\},\\D_l&:=\{\xi\in\mathbb{R}^3:2^m|\Psi^{\sigma;\mu,\nu}(|\xi|)|\in(2^{l-1},2^l]\\&\qquad\text{and}\ |\varphi_k(\xi)|\chi_A^{\sigma;\mu,\nu}(|\xi|)\neq 0\},\quad l\in[l_0+1,m-100D]\cap\mathbb{Z},\\G&=\sum_{l=l_0}^{m-100D}G_l,\qquad G_l(\xi):=G(\xi)\cdot\mathbf{1}_{D_l}(\xi).\end{aligned}$$

For (6.3) it remains to prove that for any $l\in[l_0,m-100D]\cap\mathbb{Z}$,

$$\big\|\widetilde{\varphi}_j^{(k)}\cdot\mathcal{F}^{-1}(G_l)\|_{B_{k,j}}\lesssim 2^{-3\beta^4 m}. \tag{6.34}$$

Using (6.17) in Lemma 6.2, it follows that there is

$$\theta^{\sigma;\mu,\nu}=\theta^{\sigma;\mu,\nu}(\mu,\nu,\sigma,k,k_1,k_2,l)\in[2^{-D},\infty)$$

with the property that

$$D_l\subseteq\{\xi\in\mathbb{R}^3:\big||\xi|-\theta^{\sigma;\mu,\nu}\big|\lesssim 2^{l-m}\}. \tag{6.35}$$

Therefore, using also (6.33) if $l \geq l_0 + 1$ and (6.31) if $l = l_0$,

$$\begin{aligned}\big\|\widetilde{\varphi}_j^{(k)} \cdot \mathcal{F}^{-1}(G_l)\|_{B^1_{k,j}} &\lesssim 2^{(1+\beta)j}\|G_l\|_{L^2} + \|G_l\|_{L^\infty}\\ &\lesssim 2^{\beta m}2^{-l}2^{-m/2}2^{\beta m/5} \cdot \big(2^{(1+\beta)j}2^{(l-m)/2} + 1\big)\\ &\lesssim 2^{j-m}2^{-l/2}2^{11\beta m/5} + 2^{-l}2^{-m/2}2^{6\beta m/5}.\end{aligned}$$

This clearly suffices to prove (6.34) if $l \geq 6\beta m$ or $j \leq m - 3\beta m$.

It remains to prove (6.34) in the remaining case

$$l \in [l_0, 6\beta m] \cap \mathbb{Z} \qquad \text{and} \qquad j \in [m - 3\beta m, m + D] \cap \mathbb{Z}. \tag{6.36}$$

For this we need to use the norms $B^2_{k,j}$ defined in (4.5). Assume first that $l \geq l_0 + 1$. As before we estimate easily

$$\begin{aligned}2^{(1-\beta)j}\|G_l\|_{L^2} + \|G_l\|_{L^\infty} &\lesssim 2^{-l}2^{-m/2}2^{\beta m/5} \cdot \big(2^{(1-\beta)m}2^{(l-m)/2} + 1\big)\\ &\lesssim 2^{-l/2}2^{-4\beta m/5} + 2^{-l}2^{-m/2}2^{\beta m/5}.\end{aligned}$$

Therefore, for (6.34), it suffices to prove that

$$2^{\gamma j} \sup_{R \in [2^{-j}, 2^k],\, \xi_0 \in \mathbb{R}^3} R^{-2}\Big\|\mathcal{F}\big[\widetilde{\varphi}_j^{(k)} \cdot \mathcal{F}^{-1}(G_l)\big]\Big\|_{L^1(B(\xi_0,R))} \lesssim 2^{-3\beta^4 m}. \tag{6.37}$$

Since $\Big|\mathcal{F}(\widetilde{\varphi}_j^{(k)})(\xi)\Big| \lesssim 2^{3j}(1 + 2^j|\xi|)^{-6}$, it follows from (6.33) that

$$\begin{aligned}\Big|\mathcal{F}\big[\widetilde{\varphi}_j^{(k)} \cdot \mathcal{F}^{-1}(G_l)\big](\xi)\Big| &\lesssim \int_{\mathbb{R}^3} |G_l(\xi - \eta)| \cdot 2^{3j}(1 + 2^j|\eta|)^{-6}\, d\eta\\ &\lesssim 2^{-l}2^{-m/2}2^{\beta m/5} \int_{\mathbb{R}^3} \mathbf{1}_{D_l}(\xi - \eta) \cdot 2^{3j}(1 + 2^j|\eta|)^{-6}\, d\eta.\end{aligned}$$

Therefore, using now (6.35), for any $R \in [2^{-j}, 2^k]$ and $\xi_0 \in \mathbb{R}^3$,

$$R^{-2}\Big\|\mathcal{F}\big[\widetilde{\varphi}_j^{(k)} \cdot \mathcal{F}^{-1}(G_l)\big]\Big\|_{L^1(B(\xi_0,R))} \lesssim 2^{-l}2^{-m/2}2^{\beta m/5} \cdot 2^{l-m} \lesssim 2^{-3m/2}2^{\beta m/5},$$

and the bound (6.37) follows.

Similarly, using (6.31) and (6.35),

$$2^{(1-\beta)j}\|G_{l_0}\|_{L^2} + \|G_{l_0}\|_{L^\infty} \lesssim 2^{(1-\beta)(j-m)}2^{-\beta m + l_0/2 + 3\beta^2 m} + 2^{-m/4} \lesssim 2^{-3\beta^4 m}$$

and

$$\begin{aligned}\Big|\mathcal{F}\big[\widetilde{\varphi}_j^{(k)} \cdot \mathcal{F}^{-1}(G_{l_0})\big](\xi)\Big| &\lesssim \int_{\mathbb{R}^3} |G_{l_0}(\xi - \eta)| \cdot 2^{3j}(1 + 2^j|\eta|)^{-6}\, d\eta\\ &\lesssim 2^{-m/2}2^{3\beta^2 m} \int_{\mathbb{R}^3} \mathbf{1}_{D_{l_0}}(\xi - \eta) \cdot 2^{3j}(1 + 2^j|\eta|)^{-6}\, d\eta,\end{aligned}$$

from where we conclude that, for any $R \in [2^{-j}, 2^k]$ and $\xi_0 \in \mathbb{R}^3$,

$$R^{-2}\Big\|\mathcal{F}\big[\widetilde{\varphi}_j^{(k)} \cdot \mathcal{F}^{-1}(G_{l_0})\big]\Big\|_{L^1(B(\xi_0,R))} \lesssim 2^{-m/2}2^{3\beta^2 m} \cdot 2^{l_0 - m} \lesssim 2^{-3m/2}2^{2\beta m}.$$

The desired bound (6.34) follows when $l = l_0$, which completes the proof of the lemma. □

Lemma 6.4. *The bound* (6.3) *holds provided that* (6.2) *and* (6.27) *hold and*, *in addition*,

$$\max(j_1, j_2) \geq (m - \beta^2 m)/2, \tag{6.38}$$

with

$$\kappa := 2^{\beta^2 m} 2^{\max(j_1,j_2)-m}. \tag{6.39}$$

Proof of Lemma 6.4. Using definition (4.4), it suffices to prove that

$$\begin{aligned} 2^{(1+\beta)j} \Big\| \widetilde{\varphi}_j^{(k)} \cdot P_k R_{m,\kappa_1}^{\sigma;\mu,\nu}(f_{k_1,j_1}^\mu, f_{k_2,j_2}^\nu) \Big\|_{L^2} \\ + \Big\| \mathcal{F}[\widetilde{\varphi}_j^{(k)} \cdot P_k R_{m,\kappa_1}^{\sigma;\mu,\nu}(f_{k_1,j_1}^\mu, f_{k_2,j_2}^\nu)] \Big\|_{L^\infty} \lesssim 2^{-2\beta^4 m}. \end{aligned} \tag{6.40}$$

Let $G = \mathcal{F} P_k R_{m,\kappa_1}^{\sigma;\mu,\nu}(f_{k_1,j_1}^\mu, f_{k_2,j_2}^\nu)$ be given as in (6.30). In proving (6.40) we may assume that $G \neq 0$; in particular, this guarantees that the main assumptions (6.12) of Lemma 6.2 are satisfied. We prove first the L^∞ bound in (6.40). Assume that $j_1 \leq j_2$. (The case $j_1 \geq j_2$ is similar.) Then (see (5.17) and (4.6)–(4.8))

$$\begin{aligned} &\|\widehat{f_{k_1,j_1}^\mu}(s)\|_{L^\infty} \lesssim 1, \\ &\sup_{\xi_0 \in \mathbb{R}^3} \|\widehat{f_{k_2,j_2}^\nu}(s)\|_{L^1(B(\xi_0,R))} \lesssim 2^{-(1+\beta)j_2} R^{3/2} \qquad \text{for any } R \leq 1. \end{aligned}$$

Using (6.13) in Lemma 6.2 it follows that

$$\|G\|_{L^\infty} \lesssim 2^m \cdot 2^{-(1+\beta)j_2} \kappa^{3/2} \lesssim 2^{-m/2} 2^{2\beta^2 m} 2^{(1/2-\beta)j_2} \lesssim 2^{-2\beta^4 m},$$

as desired.

To get the L^2 bound in (6.40) it suffices to show that

$$2^{(2+2\beta)m} \|G\|_{L^2}^2 \lesssim 2^{-4\beta^4 m}. \tag{6.41}$$

To prove this we need first an orthogonality argument. The point of this argument is to show that the space-time resonant contributions coming from different values s and s' in the function G (see definition (6.30)) are essentially orthogonal, provided that $|s - s'| \gtrsim 2^m \kappa$. More precisely, let $\chi : \mathbb{R} \to [0,1]$ denote a smooth function supported in the interval $[-2, 2]$ with the property that

$$\sum_{n \in \mathbb{Z}} \chi(x - n) = 1 \qquad \text{for any } x \in \mathbb{R}. \tag{6.42}$$

We define the smooth function $\chi' : \mathbb{R}^3 \to [0,1]$, $\chi'(x,y,z) := \chi(x)\chi(y)\chi(z)$. Recall the functions $\Psi^{\sigma;\mu,\nu}$ defined in (6.16). For any $v \in \mathbb{Z}^3$ and $n \in \mathbb{Z}$, we define

$$G_{v,n}(\xi) := \chi'(\kappa^{-1}\xi - v)\varphi_k(\xi) \int_{\mathbb{R}}\int_{\mathbb{R}^3} e^{is\Phi^{\sigma;\mu,\nu}(\xi,\eta)}\chi_R^{\sigma;\mu,\nu}(\xi,\eta)\chi(2^{-m}\kappa^{-1}s-n)q_m(s) \cdot \widehat{f_{k_1,j_1}^{\mu}}(\xi-\eta,s)\widehat{f_{k_2,j_2}^{\nu}}(\eta,s)\,d\eta ds, \tag{6.43}$$

and we notice that $G = \sum_{v\in\mathbb{Z}^3}\sum_{n\in\mathbb{Z}} G_{v,n}$. In view of Lemma 6.2(i) we notice also that the functions $\widetilde{G}_{v,s}$ are trivial unless

$$v \in Z_\kappa^{\sigma;\mu,\nu} := \{w \in \mathbb{Z}^3 : \kappa|w| \in [2^{k-4}, 2^{k+4}] \cap [t^{\sigma_1\sigma_2}(0) + 2^{-4D}, \infty),\ |\Psi^{\sigma;\mu,\nu}(\kappa|w|)| \le 2^{-200D}\}. \tag{6.44}$$

We show now that

$$\|G\|_{L^2}^2 \lesssim \sum_{v\in Z_\kappa^{\sigma;\mu,\nu}}\sum_{n\in\mathbb{Z}} \|G_{v,n}\|_{L^2}^2 + 2^{-10m}. \tag{6.45}$$

This additional orthogonality in time allows us a crucial gain of $\kappa^{1/2}$ in the time integration, compared to the trivial bound. To prove this bound we estimate

$$\|G\|_{L^2}^2 \lesssim \sum_{v\in Z_\kappa^{\sigma;\mu,\nu}} \Big\|\sum_{n\in\mathbb{Z}} G_{v,n}\Big\|_{L^2}^2 \lesssim \sum_{v\in Z_\kappa^{\sigma;\mu,\nu}}\sum_{n_1,n_2\in\mathbb{Z}} |\langle G_{v,n_1}, G_{v,n_2}\rangle|.$$

Therefore, for (6.45) it suffices to prove that

$$|\langle G_{v,n_1}, G_{v,n_2}\rangle| \lesssim 2^{-20m} \qquad \text{if } v \in Z_\kappa^{\sigma;\mu,\nu} \text{ and } |n_1-n_2| \ge 2^{100D}. \tag{6.46}$$

We may rewrite

$$(\mathcal{F}^{-1}G_{v,n})(x) := \int_{\mathbb{R}}\int_{\mathbb{R}^3\times\mathbb{R}^3} e^{i[x\cdot\xi + s\Phi^{\sigma;\mu,\nu}(\xi,\eta)]}\chi(2^{-m}\kappa^{-1}s-n)q_m(s)$$
$$\chi_R^{\sigma;\mu,\nu}(\xi,\eta)\chi'(\kappa^{-1}\xi - v)\varphi_k(\xi)\widehat{f_{k_1,j_1}^{\mu}}(\xi-\eta,s)\widehat{f_{k_2,j_2}^{\nu}}(\eta,s)\,d\eta ds$$

and notice that

$$\begin{aligned}\nabla_\xi\,[x\cdot\xi + s\Phi^{\sigma;\mu,\nu}(\xi,\eta)] &= x + n\kappa 2^m\nabla_\xi\Phi(\kappa v, p^{\sigma;\mu,\nu}(\kappa v)) \\ &\quad + (s\nabla_\xi\Phi(\xi,\eta) - n\kappa 2^m\nabla_\xi\Phi(\kappa v, p^{\sigma;\mu,\nu}(\kappa v))), \\ &= x + n\kappa 2^m\nabla_\xi\Phi(\kappa v, p^{\sigma;\mu,\nu}(\kappa v)) + O(2^m\kappa).\end{aligned}$$

Let $w_n := n\kappa 2^m \cdot (\Psi^{\sigma;\mu,\nu})'(\kappa|v|)\cdot v/|v|$, and integrate by parts in ξ using Lemma A.2 with

$$K \approx |x + w_n|, \qquad \epsilon^{-1} \approx 2^{\max(j_1,j_2)}.$$

It follows that, for any $n \in \mathbb{Z}$,

$$|\mathcal{F}^{-1}(G_{v,n})(x)| \lesssim |x+w_n|^{-200} \qquad \text{if} \qquad |x+w_n| \ge 2^{50D}\kappa 2^m.$$

Moreover, using Lemma 6.2 and (6.44) we conclude that $|(\Psi^{\sigma;\mu,\nu})'(\kappa|v|)| \ge 2^{-20D}$. Therefore, if $|n_1-n_2| \ge 2^{100D}$, then $|w_{n_1} - w_{n_2}| \ge 2^{70D}\kappa 2^m$ and the bound (6.46) follows. This completes the proof of (6.45).

In view of (6.45), for (6.41) it remains to prove that

$$2^{(2+2\beta)m} \sum_{v\in Z_\kappa^{\sigma;\mu,\nu},\, n\in[2^{-10}\kappa^{-1},2^{10}\kappa^{-1}]} \|G_{v,n}\|_{L^2}^2 \lesssim 2^{-4\beta^4 m}. \tag{6.47}$$

Let

$$\begin{aligned}\widetilde{G}_{v,s}(\xi) := \chi'(\kappa^{-1}\xi - v)\varphi_k(\xi)\int_{\mathbb{R}^3} e^{is\Phi^{\sigma;\mu,\nu}(\xi,\eta)}\chi_R^{\sigma;\mu,\nu}(\xi,\eta)\\ \cdot \widehat{f_{k_1,j_1}^{\mu}}(\xi-\eta,s)\widehat{f_{k_2,j_2}^{\nu}}(\eta,s)\,d\eta,\end{aligned} \tag{6.48}$$

such that

$$G_{v,n}(\xi) = \int_{\mathbb{R}} \widetilde{G}_{v,s}(\xi)\chi(2^{-m}\kappa^{-1}s - n)q_m(s)\,ds.$$

Therefore, for any (v,n),

$$\|G_{v,n}\|_{L^2}^2 \lesssim 2^m\kappa \int_{\mathbb{R}} \|\widetilde{G}_{v,s}\|_{L^2}^2 \chi(2^{-m}\kappa^{-1}s-n)q_m(s)\,ds.$$

Therefore, for (6.47) it suffices to prove that for any $s\in[2^{m-1},2^{m+1}]$,

$$2^{(4+2\beta)m}\kappa \sum_{v\in Z_\kappa^{\sigma;\mu,\nu}} \|\widetilde{G}_{v,s}\|_{L^2}^2 \lesssim 2^{-4\beta^4 m}. \tag{6.49}$$

Assuming $v\in Z_\kappa^{\sigma;\mu,\nu}$ fixed, the variables in the definition of the function $\widetilde{G}_{v,s}$ are naturally restricted as follows:

$$|\xi-\kappa v|\lesssim\kappa, \qquad |\eta - p^{\sigma;\mu,\nu}(\kappa v)|\lesssim\kappa,$$

where $p^{\sigma;\mu,\nu}$ is defined as in Lemma 6.2. More precisely, we define the functions f_1^v and f_2^v by the formulas

$$\begin{aligned}\widehat{f_1^v}(\theta,s) &:= \varphi(2^{-50D}\kappa^{-1}(\theta-\kappa v + p^{\sigma;\mu,\nu}(\kappa v)))\cdot\widehat{f_{k_1,j_1}^{\mu}}(\theta,s),\\ \widehat{f_2^v}(\theta,s) &:= \varphi(2^{-50D}\kappa^{-1}(\theta - p^{\sigma;\mu,\nu}(\kappa v))\cdot\widehat{f_{k_2,j_2}^{\nu}}(\theta,s).\end{aligned} \tag{6.50}$$

Since

$$|p^{\sigma;\mu,\nu}(\kappa v_1) - p^{\sigma;\mu,\nu}(\kappa v_2)| \geq 2^{-80D}\kappa$$

and

$$\Big|[\kappa v_1 - p^{\sigma;\mu,\nu}(\kappa v_1)] - [\kappa v_2 - p^{\sigma;\mu,\nu}(\kappa v_2)]\Big| \geq 2^{-80D}\kappa$$

whenever $|v_1-v_2|\gtrsim 1$ (these inequalities are consequences of the lower bounds in the first line of (6.14) in Lemma 6.2), it follows by the fact that the support of the functions have finite overlap that, for any $s\in\mathbb{R}$,

$$\begin{aligned}\sum_{v\in Z_\kappa^{\sigma;\mu,\nu}} \|f_1^v(s)\|_{L^2}^2 \lesssim \|f_{k_1,j_1}^{\mu}(s)\|_{L^2}^2 \lesssim 2^{-2j_1+2\beta j_1},\\ \sum_{v\in Z_\kappa^{\sigma;\mu,\nu}} \|f_2^v(s)\|_{L^2}^2 \lesssim \|f_{k_2,j_2}^{\nu}(s)\|_{L^2}^2 \lesssim 2^{-2j_2+2\beta j_2}.\end{aligned} \tag{6.51}$$

For any $v \in \mathbb{R}^3$ and $g_1, g_2 \in L^2(\mathbb{R}^3)$, let

$$\begin{aligned} A_v(g_1,g_2)(\xi) := \chi'(\kappa^{-1}\xi - v)\varphi_k(\xi)\int_{\mathbb{R}^3}\chi_R^{\sigma;\mu,\nu}(\xi,\eta) \\ \cdot \mathcal{F}(P_{[k_1-4,k_1+4]}g_1)(\xi-\eta)\mathcal{F}(P_{[k_2-4,k_2+4]}g_2)(\eta)\,d\eta. \end{aligned} \tag{6.52}$$

We observe that

$$\widetilde{G}_{v,s}(\xi) = e^{is\Lambda_\sigma(\xi)}A_v[Ef_1^v(s), Ef_2^v(s)](\xi),$$
$$Ef_1^v(s) = e^{-is\widetilde{\Lambda}_\mu}f_1^v(s), \qquad Ef_2^v(s) = e^{-is\widetilde{\Lambda}_\nu}f_2^v(s).$$

Therefore, for (6.49) it suffices to prove that, for any $s \in [2^{m-1}, 2^{m+1}]$,

$$2^{(4+2\beta)m}\kappa\sum_{v\in Z_\kappa^{\sigma;\mu,\nu}}\|A_v(Ef_1^v(s), Ef_2^v(s))\|_{L^2}^2 \lesssim 2^{-4\beta^4 m}. \tag{6.53}$$

We notice now that if $p, q \in [2,\infty]$, $1/p + 1/q = 1/2$, then

$$\|A_v(g_1,g_2)\|_{L^2} \lesssim \|g_1\|_{L^p}\|g_2\|_{L^q}. \tag{6.54}$$

Indeed, as in the proof of Lemma A.3, we write

$$\mathcal{F}^{-1}(A_v(g_1,g_2))(x) = c\int_{\mathbb{R}^3\times\mathbb{R}^3} g_1(y)g_2(z)K_v(x;y,z)\,dydz,$$

where

$$\begin{aligned} K_v(x;y,z) := \int_{\mathbb{R}^3\times\mathbb{R}^3} e^{i(x-y)\cdot\xi}e^{i(y-z)\cdot\eta}\chi'(\kappa^{-1}\xi - v)\varphi(\kappa^{-1}\Xi^{\mu,\nu}(\xi,\eta)) \\ \cdot\varphi_k(\xi)\varphi(2^{D^2+\max(0,k_1,k_2)}\Phi^{\sigma;\mu,\nu}(\xi,\eta))\varphi_{[k_1-4,k_1+4]}(\xi-\eta)\varphi_{[k_2-4,k_2+4]}(\eta)\,d\xi d\eta. \end{aligned}$$

Recall that $k, k_1, k_2 \in [-D/2, D/2]$, and integrate by parts in ξ and η. Using Lemma 6.2, it follows that

$$|K_v(x;y,z)| \lesssim \kappa^3(1+\kappa^{-1}|x-y|)^{-4}\cdot\kappa^3(1+\kappa^{-1}|y-z|)^{-4},$$

and the desired estimate (6.54) follows.

We can now prove the main estimate (6.53). Assume first that

$$\max(j_1,j_2) \le (3/5-\beta)m. \tag{6.55}$$

By symmetry, we may assume again that $j_1 \le j_2$ and estimate using (6.50)

$$\|Ef_1^v(s)\|_{L^\infty} \lesssim \|\widehat{f_1^v}(s)\|_{L^1} \lesssim \kappa^3.$$

Therefore, using (6.54) and (6.51), the left-hand side of (6.53) is dominated by

$$\begin{aligned} C2^{4m+2\beta m}\kappa\sum_{v\in Z_\kappa^{\sigma;\mu,\nu}}\kappa^6\|Ef_2^v(s)\|_{L^2}^2 \\ \lesssim 2^{4m+2\beta m}\kappa^7\cdot 2^{-2j_2+2\beta j_2} \lesssim 2^{-3(1-\beta)m}2^{(5+2\beta)j_2}, \end{aligned}$$

and the desired bound (6.53) follows provided that (6.55) holds.

Assume now that

$$\max(j_1,j_2) \ge (3/5-\beta)m, \quad \max(j_1,j_2) - \min(j_1,j_2) \ge 8\beta m. \tag{6.56}$$

By symmetry, we may assume that $j_1 \le j_2$ and estimate, using (6.15), (5.17), and either (A.37), (A.42), (A.49) or (A.55),

$$\|Ef_1^v(s)\|_{L^\infty} \lesssim 2^{-3m/2}2^{(1/2+\beta)j_1}.$$

Therefore, using (6.54) and (6.51), the left-hand side of (6.53) is dominated by

$$\begin{aligned} C2^{4m+2\beta m}\kappa \sum_{v\in Z_\kappa^{\sigma;\mu,\nu}} 2^{-3m}2^{(1+2\beta)j_1}\|Ef_2^v(s)\|_{L^2}^2 &\lesssim 2^{m+2\beta m}\kappa\cdot 2^{(1+2\beta)j_1}2^{-2j_2+2\beta j_2} \\ &\lesssim 2^{j_1-j_2}2^{3\beta m}2^{2\beta j_1}2^{2\beta j_2}, \end{aligned}$$

and the desired bound (6.53) follows provided that (6.56) holds.

Finally, assume that

$$\max(j_1,j_2)-\min(j_1,j_2)\le 8\beta m \qquad \text{and} \qquad \max(j_1,j_2)\ge (3/5-\beta)m. \tag{6.57}$$

In this case we need the more refined decomposition in (4.6)–(4.8). More precisely, using the definitions, for fixed $s\in[2^{m-1},2^{m+1}]$, we decompose

$$f^\mu_{k_1,j_1}(s) = P_{[k_1-2,k_1+2]}(g_1+h_1), \qquad f^\nu_{k_2,j_2}(s) = P_{[k_2-2,k_2+2]}(g_2+h_2),$$

where

$$g_1 = g_1\cdot \widetilde{\varphi}^{(k_1)}_{[j_1-2,j_1+2]}, \quad g_2 = g_2\cdot \widetilde{\varphi}^{(k_2)}_{[j_2-2,j_2+2]} \tag{6.58}$$

and

$$\begin{aligned} &2^{(1+\beta)j_1}\|g_1\|_{L^2} + 2^{(1-\beta)j_1}\|h_1\|_{L^2} \\ &\qquad + 2^{\gamma j_1}\sup_{R\in[2^{-j_1},2^{k_1}],\theta_0\in\mathbb{R}^3} R^{-2}\|\widehat{h_1}\|_{L^1(B(\theta_0,R))} \lesssim 1, \\ &2^{(1+\beta)j_2}\|g_2\|_{L^2} + 2^{(1-\beta)j_2}\|h_2\|_{L^2} \\ &\qquad + 2^{\gamma j_2}\sup_{R\in[2^{-j_2},2^{k_2}],\theta_0\in\mathbb{R}^3} R^{-2}\|\widehat{h_2}\|_{L^1(B(\theta_0,R))} \lesssim 1. \end{aligned} \tag{6.59}$$

Then, we define the functions $g_1^v, h_1^v, g_2^v, h_2^v$ by the formulas (compare with (6.50))

$$\begin{aligned} \widehat{g_1^v}(\theta) &:= \varphi(2^{-50D}\kappa^{-1}(\theta-\kappa v+p^{\sigma;\mu,\nu}(\kappa v)))\cdot\mathcal{F}(P_{[k_1-2,k_1+2]}g_1)(\theta), \\ \widehat{h_1^v}(\theta) &:= \varphi(2^{-50D}\kappa^{-1}(\theta-\kappa v+p^{\sigma;\mu,\nu}(\kappa v)))\cdot\mathcal{F}(P_{[k_1-2,k_1+2]}h_1)(\theta), \\ \widehat{g_2^v}(\theta) &:= \varphi(2^{-50D}\kappa^{-1}(\theta-p^{\sigma;\mu,\nu}(\kappa v)))\cdot\mathcal{F}(P_{[k_2-2,k_2+2]}g_2)(\theta), \\ \widehat{h_2^v}(\theta) &:= \varphi(2^{-50D}\kappa^{-1}(\theta-p^{\sigma;\mu,\nu}(\kappa v)))\cdot\mathcal{F}(P_{[k_2-2,k_2+2]}h_2)(\theta). \end{aligned} \tag{6.60}$$

As in (6.51), using L^2 orthogonality and (6.59), we have

$$\begin{aligned} \sum_{v\in Z_\kappa^{\sigma;\mu,\nu}}\|g_1^v\|_{L^2}^2 &\lesssim 2^{-2j_1-2\beta j_1}, &\quad \sum_{v\in Z_\kappa^{\sigma;\mu,\nu}}\|h_1^v\|_{L^2}^2 &\lesssim 2^{-2j_1+2\beta j_1}, \\ \sum_{v\in Z_\kappa^{\sigma;\mu,\nu}}\|g_2^v\|_{L^2}^2 &\lesssim 2^{-2j_2-2\beta j_2}, &\quad \sum_{v\in Z_\kappa^{\sigma;\mu,\nu}}\|h_2^v\|_{L^2}^2 &\lesssim 2^{-2j_2+2\beta j_2}. \end{aligned} \tag{6.61}$$

Let $E^\mu_s f = e^{-is\widetilde{\Lambda}_\mu} f$. Using (6.15), and either (A.37), (A.42), (A.49) or (A.55) together with (6.58)–(6.59), we derive the L^∞ bounds

$$\begin{aligned} &\|E^\mu_s g^v_1\|_{L^\infty} \lesssim 2^{-3m/2}\|g^v_1\|_{L^1} \lesssim 2^{-3m/2}2^{(1/2-\beta)j_1},\\ &\|E^\mu_s h^v_1\|_{L^\infty} \lesssim \|\widehat{h^v_1}\|_{L^1} \lesssim \kappa^2 2^{-\gamma j_1},\\ &\|E^\nu_s g^v_2\|_{L^\infty} \lesssim 2^{-3m/2}\|g^v_2\|_{L^1} \lesssim 2^{-3m/2}2^{(1/2-\beta)j_2},\\ &\|E^\nu_s h^v_2\|_{L^\infty} \lesssim \|\widehat{h^v_2}\|_{L^1} \lesssim \kappa^2 2^{-\gamma j_2} \end{aligned} \tag{6.62}$$

for any $v \in Z^{\sigma;\mu,\nu}_\kappa$. Using (6.54) and (6.61)–(6.62), we estimate, assuming $j_1 \le j_2$,

$$\begin{aligned} 2^{4m+2\beta m}\kappa &\sum_{v\in Z^{\sigma;\mu,\nu}_\kappa} \Big[\|A_v(E^\mu_s g^v_1, E^\nu_s g^v_2)\|^2_{L^2} + \|A_v(E^\mu_s h^v_1, E^\nu_s g^v_2)\|^2_{L^2}\Big]\\ &\lesssim 2^{4m+2\beta m}\kappa \sum_{v\in Z^{\sigma;\mu,\nu}_\kappa} \|g^v_2\|^2_{L^2}(\|E^\nu_s g^v_1\|^2_{L^\infty} + \|E^\nu_s h^v_1\|^2_{L^\infty})\\ &\lesssim 2^{4m+2\beta m}\kappa\cdot 2^{-2j_2-2\beta j_2}\cdot[2^{-3m}2^{(1-2\beta)j_1} + \kappa^4 2^{-2\gamma j_1}]\\ &\lesssim 2^{3m}2^{(2\beta+\beta^2)m}2^{-(1+2\beta)j_2}\cdot 2^{-3m}2^{(1-2\beta)j_2}\\ &\lesssim 2^{-\beta^3 m}. \end{aligned}$$

Similarly, we estimate

$$\begin{aligned} 2^{4m+2\beta m}\kappa &\sum_{v\in Z^{\sigma;\mu,\nu}_\kappa} \Big[\|A_v(E^\mu_s g^v_1, E^\nu_s h^v_2)\|^2_{L^2} + \|A_v(E^\mu_s h^v_1, E^\nu_s h^v_2)\|^2_{L^2}\Big]\\ &\lesssim 2^{4m+2\beta m}\kappa \sum_{v\in Z^{\sigma;\mu,\nu}_\kappa} \|E^\nu_s h^v_2\|^2_{L^\infty}(\|E^\nu_s g^v_1\|^2_{L^2} + \|E^\nu_s h^v_1\|^2_{L^2})\\ &\lesssim 2^{4m+2\beta m}\kappa\cdot\kappa^4 2^{-2\gamma j_2}\cdot 2^{-2j_1+2\beta j_1}\\ &\lesssim 2^{-m/10}. \end{aligned}$$

The desired estimate (6.53) follows from the last two bounds and the restriction (6.57). This completes the proof of the lemma. □

7. **Proof of Proposition 4.3, III: Case B resonant interactions**

In light of Proposition 5.9, in this section we consider type B interactions (see Proposition B.2) and prove the following proposition:

Proposition 7.1. *Assume that* $(k,j),(k_1,j_1),(k_2,j_2) \in \mathcal{J}$, $m \in [1,L]\cap\mathbb{Z}$,

$$\Phi^{\sigma;\mu,\nu} \in\in \mathcal{T}_B = \{\Phi^{e;i+,e+}, \Phi^{e;i-,e+}, \Phi^{b;i+,b+}, \Phi^{b;i-,b+}\}, \tag{7.1}$$

and

$$(7.2)\qquad \begin{aligned} &-9m/10 \le k_1, k_2 \le j/N_0', \quad \max(j_1, j_2) \le (1-\beta/10)m,\\ &\beta m/2 + N_0' k_+ + D^2 \le j \le m + D,\\ &k_1 \le -D/3, \qquad k \ge -D/4, \qquad |k - k_2| \le 10. \end{aligned}$$

Then there is $\kappa \in (0,1]$, $\kappa \ge \max\Big(2^{(\beta^2 m - m)/2} 2^{-k_1/2}, 2^{\beta^2 m - m} 2^{\max(j_1,j_2)}\Big)$, *such that*

$$(7.3)\qquad (1+2^k)\Big\|\widetilde{\varphi}_j^{(k)} \cdot P_k R_{m,\kappa}^{\sigma;\mu,\nu}(f_{k_1,j_1}^{\mu}, f_{k_2,j_2}^{\nu})\Big\|_{B^1_{k,j}} \lesssim 2^{-2\beta^4 m}.$$

The rest of the section is concerned with the proof of Proposition 7.1. We have assumed, without loss of generality, that $k_1 \le k_2$. As in Case A, the proof of the proposition relies on a careful analysis of resonant interactions. For this analysis, we need to understand well the geometry of almost resonant sets.

For $\sigma \in \{e, b\}$, let R_σ denote the unique solutions in $(0,\infty)$ of the equations

$$(7.4)\qquad \lambda_\sigma'(R_\sigma) = \lambda_i'(0) = \sqrt{(1+T)/(1+\varepsilon)}.$$

The numbers R_σ are well defined, in view of Lemma A.4, and $R_\sigma \approx_{\varepsilon, C_b} 1$.

For $(\mu,\nu) \in \{(i+,e+),(i-,e+),(i+,b+),(i-,b+)\}$, $\mu = (i\iota_1)$, $\nu = (\sigma_2+)$, $\sigma_2 \in \{e,b\}$, we define the functions $r^{\mu,\nu} : (R_{\sigma_2} - 2^{-D/5}, R_{\sigma_2} + 2^{-D/5}) \to (R_{\sigma_2} - 2^{-D/10}, R_{\sigma_2} + 2^{-D/10})$ as the unique solutions of the equations

$$(7.5)\qquad \lambda_{\sigma_2}'(r^{\mu,\nu}(s)) - \lambda_i'(s - r^{\mu,\nu}(s)) = 0.$$

These functions are the analogues of the functions defined in Section 6 above (6.8) for $\Phi^{\sigma;\mu,\nu} \in\in \mathcal{T}_A$. Notice that these functions are well defined for $s \in (R_{\sigma_2} - 2^{-D/5}, R_{\sigma_2} + 2^{-D/5})$, since the functions $r \to \lambda_{\sigma_2}'(r) - \lambda_i'(s-r)$ are strictly increasing and vanish in the appropriate ranges, as a consequence of Lemma A.4(i) and the observation that $\lambda_i''(0) = 0$. Moreover,

$$(7.6)\qquad |(\partial_s r^{\mu,\nu})(s)| \approx_{C_b,\varepsilon} |s - r^{\mu,\nu}(s)| \qquad \text{for any } s \in (R_{\sigma_2} - 2^{-D/5}, R_{\sigma_2} + 2^{-D/5}).$$

LEMMA 7.2. *Assume that* $\mu = (i\iota_1)$, $\iota_1 \in \{+,-\}$, $\nu = (\sigma_2+)$, $\sigma_2 \in \{e,b\}$, $k, k_1, k_2 \in \mathbb{Z}$, $k_1 \le -D/3$, $k \ge -D/4$, $|k - k_2| \le 10$, *and* $\delta \in [0, 2^{-10D}]$. *Assume that there is a point* $(\xi,\eta) \in \mathbb{R}^3 \times \mathbb{R}^3$ *satisfying*

$$(7.7)\qquad |\xi| \in [2^{k-4}, 2^{k+4}], \ |\eta| \in [2^{k_2-4}, 2^{k_2+4}], \ |\xi-\eta| \in [2^{k_1-4}, 2^{k_1+4}], \ |\Xi^{\mu,\nu}(\xi,\eta)| \le \delta.$$

(i) *Then*

$$(7.8)\qquad k, k_2 \in [-D/100, D/100] \quad \textit{and} \quad \Big||\xi| - R_{\sigma_2}\Big| + \Big||\eta| - R_{\sigma_2}\Big| \lesssim_{C_b,\varepsilon} 2^{k_1} + \delta.$$

More precisely, if $\xi = se$ *for some* $s > 0$ *and some unit vector* $e \in \mathbb{S}^2$*, then*

$$
\begin{aligned}
&|s - R_{\sigma_2}| \lesssim_{C_b,\varepsilon} 2^{k_1} + \delta,\\
&\eta = re + \eta', \qquad |r - r^{\mu,\nu}(s)| \lesssim_{C_b,\varepsilon} \delta,\\
&|r - s| \approx_{C_b,\varepsilon} 2^{k_1}, \qquad \iota_1|s - r| = s - r,\\
&\eta' \cdot e = 0, \qquad |\eta'| \lesssim_{C_b,\varepsilon} 2^{k_1}\delta.
\end{aligned}
\tag{7.9}
$$

(ii) *If, in addition,* $\delta \le 2^{k_1 - D/10}$*, then*

$$
\begin{aligned}
&\text{if } \iota_1 = +, \quad \text{then } s - R_{\sigma_2} \approx_{C_b,\varepsilon} 2^{k_1} \text{ and } R_{\sigma_2} - r^{\mu,\nu}(s) \approx_{C_b,\varepsilon} 2^{2k_1},\\
&\text{if } \iota_1 = -, \quad \text{then } R_{\sigma_2} - s \approx_{C_b,\varepsilon} 2^{k_1} \text{ and } R_{\sigma_2} - r^{\mu,\nu}(s) \approx_{C_b,\varepsilon} 2^{2k_1}.
\end{aligned}
\tag{7.10}
$$

Proof of Lemma 7.2. (i) We start from the formula

$$
\Xi^{\mu,\nu}(\xi,\eta) = -\iota_1\lambda_i'(|\eta - \xi|)\frac{\eta - \xi}{|\eta - \xi|} - \lambda_{\sigma_2}'(|\eta|)\frac{\eta}{|\eta|}.
\tag{7.11}
$$

Since $\Big|\lambda_i'(|\eta - \xi|) - \lambda_i'(0)\Big| \lesssim_{\varepsilon,C_b} 2^{2k_1}$, the condition $|\Xi^{\mu,\nu}(\xi,\eta)| \le \delta$ and the estimates in Lemma A.4(i) show that $\Big||\eta| - R_{\sigma_2}\Big| \lesssim_{\varepsilon,C_b} 2^{2k_1} + \delta$. The desired bounds in (7.8) follow.

We prove now the claims in (7.9). Letting $\xi = se$ for some $s > 0, e \in \mathbb{S}^2$ and $\eta = re + \eta'$, $r \in \mathbb{R}$, $\eta' \cdot e = 0$, the condition $|\Xi^{\mu,\nu}(\xi,\eta)| \le \delta$ and the formula (7.11) show that

$$
\begin{aligned}
&\Big| - \iota_1\lambda_i'(\sqrt{(r-s)^2 + |\eta'|^2})\frac{r - s}{\sqrt{(r-s)^2 + |\eta'|^2}} - \lambda_{\sigma_2}'(\sqrt{r^2 + |\eta'|^2})\frac{r}{\sqrt{r^2 + |\eta'|^2}}\Big| \le \delta,\\
&\Big| - \iota_1\lambda_i'(\sqrt{(r-s)^2 + |\eta'|^2})\frac{\eta'}{\sqrt{(r-s)^2 + |\eta'|^2}} - \lambda_{\sigma_2}'(\sqrt{r^2 + |\eta'|^2})\frac{\eta'}{\sqrt{r^2 + |\eta'|^2}}\Big| \le \delta.
\end{aligned}
\tag{7.12}
$$

Recall that $\lambda_i'(0) > 0$ and that $\lambda_e'(0) = \lambda_b'(0) = 0$. Recalling also the assumptions (7.7) and the bounds (7.8), the second equation in (7.12) shows that $|\eta'| \lesssim_{C_b,\varepsilon} 2^{k_1}\delta$ as desired. In addition, $|s - r| \approx_{C_b,\varepsilon} 2^{k_1}$, therefore

$$
\Big|s - R_{\sigma_2}\Big| + \Big|r - R_{\sigma_2}\Big| \lesssim_{C_b,\varepsilon} 2^{k_1} + \delta.
$$

The first equation in (7.12) now gives

$$
\Big|\iota_1\lambda_i'(|r - s|)\frac{r - s}{|r - s|} + \lambda_{\sigma_2}'(r)\Big| \le 2\delta.
\tag{7.13}
$$

Since $\lambda_{\sigma_2}'(r) \approx_{C_b,\varepsilon} 1$, it follows that $\iota_1(r - s) = -|r - s|$ and, therefore,

$$
\Big| - \lambda_i'(s - r) + \lambda_{\sigma_2}'(r)\Big| \le 2\delta.
$$

Finally, we notice that the derivative of the map $r \to -\lambda_i'(s-r) + \lambda_{\sigma_2}'(r)$ in $\approx_{C_b,\varepsilon} 1$ in the appropriate ranges of r, s, therefore $|r - r^{\mu,\nu}(s)| \lesssim_{C_b,\varepsilon} \delta$. This completes the proof of (7.12).

(ii) If $\delta \leq 2^{k_1 - D/10}$ then, using (7.9), $|s - r^{\mu,\nu}(s)| \approx_{C_b,\varepsilon} 2^{k_1}$. Therefore, using Lemma A.4(i), $\lambda_i'(0) - \lambda_i'(s - r^{\mu,\nu}(s)) \approx_{C_b,\varepsilon} 2^{2k_1}$. Using the definitions (7.4)–(7.5) it follows that $R_{\sigma_2} - r^{\mu,\nu}(s) \approx_{C_b,\varepsilon} 2^{2k_1}$. Therefore, $|r - R_{\sigma_2}| \lesssim_{C_b,\varepsilon} 2^{2k_1} + \delta$. The remaining bounds in (7.10) now follow from the identity $\iota_1|s-r| = s - r$ (see (7.9)) and the assumption $\delta + 2^{2k_1} \leq 2^{k_1 - D/10}$. □

7.1. *Proof of Proposition* 7.1. We further divide the proof into several lemmas.

LEMMA 7.3. *The bound* (7.3) *holds if* (7.2) *holds and*, *in addition*,

$$\begin{aligned} &\Phi^{\sigma;\mu,\nu} \in \{\Phi^{e;e+,i+}, \Phi^{e;e+,i-}, \Phi^{b;b+,i+}, \Phi^{b;b+,i-}\} \text{ or} \\ &k \notin [-D/100, D/100] \text{ or } k_2 \notin [-D/100, D/100], \end{aligned} \tag{7.14}$$

with

$$\kappa := 2^{-10D}. \tag{7.15}$$

Proof of Lemma 7.3. In any of these cases we have $P_k R^{\sigma;\mu,\nu}_{m,\kappa}(f^\mu_{k_1,j_1}, f^\nu_{k_2,j_2}) = 0$, using either Lemma A.4(i) (which shows that $\lambda_i'(r) \approx_{C_b,\varepsilon} 1$, $r \in [0,\infty)$, and $\lambda_e'(r) \approx \lambda_b'(r) \approx_{C_b,\varepsilon} r$, $r \in [0,1]$) or Lemma 7.2. □

LEMMA 7.4. *The bound* (7.3) *holds if* (7.2) *holds and*, *in addition*,

$$\begin{aligned} &\Phi^{\sigma;\mu,\nu} \in \{\Phi^{e;i+,e+}, \Phi^{e;i-,e+}, \Phi^{b;i+,b+}, \Phi^{b;i-,b+}\}, \\ &k, k_2 \in [-D/100, D/100], \qquad \max(j_1, j_2) \leq (m - \beta^2 m)/2 - k_1/2, \end{aligned} \tag{7.16}$$

with

$$\kappa := 2^{(\beta^2 m - m)/2 - k_1/2}. \tag{7.17}$$

Proof of Lemma 7.4. Let

$$\begin{aligned} G(\xi) :&= \varphi_k(\xi) \cdot \mathcal{F}\big[R^{\sigma;\mu,\nu}_{m,\kappa}(f^\mu_{k_1,j_1}, f^\nu_{k_2,j_2})\big](\xi) \\ &= \varphi_k(\xi) \int_{\mathbb{R}} \int_{\mathbb{R}^3} e^{is\Phi^{\sigma;\mu,\nu}(\xi,\eta)} \chi_R^{\sigma;\mu,\nu}(\xi,\eta) q_m(s) \\ &\quad \cdot \widehat{f^\mu_{k_1,j_1}}(\xi-\eta,s) \widehat{f^\nu_{k_2,j_2}}(\eta,s)\, d\eta ds \end{aligned} \tag{7.18}$$

where, as before,

$$\chi_R^{\sigma;\mu,\nu}(\xi,\eta) = \varphi(2^{D^2 + \max(0,k_1,k_2)} \Phi^{\sigma;\mu,\nu}(\xi,\eta)) \varphi(|\Xi^{\mu,\nu}(\xi,\eta)|/\kappa).$$

Recalling (4.4), it suffices to prove that

$$2^{(1+\beta)j} \|\widetilde{\varphi}^{(k)}_j \cdot \mathcal{F}^{-1}(G)\|_{L^2} + \|G\|_{L^\infty} \lesssim 2^{-2\beta^4 m}. \tag{7.19}$$

Using Lemma 7.2 and the L^∞ bounds in (5.17), for any $\xi \in \mathbb{R}^3$, we have

(7.20)
$$\begin{aligned}|G(\xi)| &\lesssim 2^m \cdot 2^{2k_1}\kappa^2 \min(2^{k_1},\kappa)\cdot 2^{-k_1/2}\cdot \mathbf{1}_{[-2^D(2^{k_1}+\kappa),2^D(2^{k_1}+\kappa)]}(|\xi|-R_{\sigma_2})\\ &\lesssim 2^{2\beta^2 m}2^{k_1/2}\min(2^{k_1},\kappa)\cdot \mathbf{1}_{[-2^D(2^{k_1}+\kappa),2^D(2^{k_1}+\kappa)]}(|\xi|-R_{\sigma_2}).\end{aligned}$$

The L^∞ bound in (7.19) follows.

To prove the L^2 bound in (7.19) we notice first that we may assume that $2^j \lesssim 2^{\beta^2 m}2^m(\kappa+2^{k_1})$, which is stronger than the assumption $j \le m+D$ in (7.2). Indeed, assuming that $\xi = se$, $\eta = re+\eta'$ satisfy (7.7) with $\delta = 2\kappa$ and using that $\sigma = \sigma_2$, we estimate

$$(7.21)\quad \Big|(\nabla_\xi \Phi^{\sigma;\mu,\nu})(\xi,\eta)\Big| = \Big|-(\nabla_\eta \Phi^{\sigma;\mu,\nu})(\xi,\eta)+[\nabla\Lambda_\sigma(\xi)-\nabla\Lambda_\sigma(\eta)]\Big| \lesssim 2^{k_1}+\kappa.$$

Therefore, we make the change of variables $\eta = \xi-\theta$ in (7.18) and rewrite

$$\begin{aligned}(\mathcal{F}^{-1}G)(x) = c\int_{\mathbb{R}}\int_{\mathbb{R}^3\times\mathbb{R}^3} & e^{i[x\cdot\xi+s\Phi^{\sigma;\mu,\nu}(\xi,\xi-\theta)]}\varphi_k(\xi)\chi_R^{\sigma;\mu,\nu}(\xi,\xi-\theta)q_m(s)\\ &\cdot \widehat{f^\mu_{k_1,j_1}}(\theta,s)\widehat{f^\nu_{k_2,j_2}}(\xi-\theta,s)\,d\xi d\theta ds.\end{aligned}$$

Integrating by parts in ξ using (5.17) and Lemma A.2 with $K \approx 2^{\beta^2 m}2^m(\kappa+2^{k_1})$, $\epsilon^{-1}\approx 2^m(\kappa+2^{k_1})$, it follows that

$$2^{(1+\beta)j}\|\widetilde{\varphi}^{(k)}_j\cdot\mathcal{F}^{-1}(G)\|_{L^2}\lesssim 2^{-m}\qquad \text{if}\qquad 2^j \ge 2^{\beta^2 m}2^m(\kappa+2^{k_1}).$$

Therefore, for (7.19) it suffices to prove that

$$(7.22)\qquad 2^{(1+\beta)m}(\kappa+2^{k_1})^{1+\beta}\|G\|_{L^2}\lesssim 2^{-2\beta^2 m}.$$

Case 1. It follows from (7.20) that the left-hand side of (7.22) is dominated by

$$C2^{(1+\beta)m}2^{2\beta^2 m}2^{k_1/2}2^{k_1}\kappa(2^{k_1}+\kappa)^{1/2}\lesssim 2^{(1/2+2\beta)m}(2^{3k_1/2}+2^{k_1}\kappa^{1/2}).$$

The desired bound (7.22) follows from (7.17) if $k_1 \le -m/3-4\beta m$.

Case 2. Assume now that

$$(7.23)\qquad -m/3+\beta m \le k_1 \le -D/3.$$

In this case we need to improve on the bound (7.20). We use Lemma 7.2 with $\delta = 2\kappa$ and notice that, as a consequence of (7.23), $\delta \le 2^{k_1-D/10}$. Assuming $\xi = se$ and $\eta = re+\eta'$ satisfy (7.7), we estimate, using also Lemma A.4, Lemma 7.2, and (7.13),

(7.24)
$$\begin{aligned}\Phi^{\sigma;\mu,\nu}(\xi,\eta) &= \lambda_{\sigma_2}(s) - \iota_1\lambda_i(\sqrt{(r-s)^2+|\eta'|^2}) - \lambda_{\sigma_2}(\sqrt{r^2+|\eta'|^2})\\ &= \lambda_{\sigma_2}(s) - \iota_1\lambda_i(|r-s|) - \lambda_{\sigma_2}(r) + O_{C_b,\varepsilon}(\kappa^2)\\ &= \lambda_{\sigma_2}(s) - \lambda_{\sigma_2}(r) - \lambda_i(s-r) + O_{C_b,\varepsilon}(\kappa^2)\\ &= \lambda_{\sigma_2}(s) - \lambda_{\sigma_2}(R_{\sigma_2}) - \lambda_i(s-R_{\sigma_2}) + O_{C_b,\varepsilon}(\kappa^2+2^{3k_1})\\ &= \lambda_{\sigma_2}(s) - \lambda_{\sigma_2}(R_{\sigma_2}) - \lambda'_{\sigma_2}(R_{\sigma_2})\cdot(s-R_{\sigma_2}) + O_{C_b,\varepsilon}(\kappa^2+2^{3k_1})\\ &\approx_{C_b,\varepsilon} 2^{2k_1}.\end{aligned}$$

More precisely, we use (7.9) in the second equality, (7.10) and (7.4) in the fourth equality, and (7.4) together with $\lambda_i''(0)=0$ in the fifth equality. We can now integrate by parts in s in the formula (7.18) to conclude that

(7.25)
$$\begin{aligned}|G(\xi)| \lesssim 2^{-2k_1}|\varphi_k(\xi)|\\ \int_{\mathbb{R}}\int_{\mathbb{R}^3} &|\varphi(|\Xi^{\mu,\nu}(\xi,\eta)|/\kappa)|\,|q'_m(s)|\,|\widehat{f^{\mu}_{k_1,j_1}}(\xi-\eta,s)|\,|\widehat{f^{\nu}_{k_2,j_2}}(\eta,s)|\\ &+|\varphi(|\Xi^{\mu,\nu}(\xi,\eta)|/\kappa)|\,|q_m(s)|\,|(\partial_s\widehat{f^{\mu}_{k_1,j_1}})(\xi-\eta,s)|\,|\widehat{f^{\nu}_{k_2,j_2}}(\eta,s)|\\ &+|\varphi(|\Xi^{\mu,\nu}(\xi,\eta)|/\kappa)|\,|q_m(s)|\,|\widehat{f^{\mu}_{k_1,j_1}}(\xi-\eta,s)|\,|(\partial_s\widehat{f^{\nu}_{k_2,j_2}})(\eta,s)|\,d\eta ds.\end{aligned}$$

We use now (5.7), the last bound in (5.17), and the bound (5.22). In view of Lemma 7.2, the volume of integration is $\approx (2^{k_1}\kappa)^2\kappa$ and it follows that

$$\begin{aligned}|G(\xi)| &\lesssim 2^{-2k_1}\mathbf{1}_{[0,2^D]}(2^{-k_1}|s-R_{\sigma_2}|)\cdot 2^{2k_1}\kappa^3\cdot 2^{\beta m/10}2^{-k_1}\\ &\lesssim \mathbf{1}_{[0,2^D]}(2^{-k_1}|s-R_{\sigma_2}|)\cdot 2^{-3m/2}2^{\beta m/5}2^{-5k_1/2}.\end{aligned} \tag{7.26}$$

Therefore, the left-hand side of (7.22) is dominated by

$$2^{(1+\beta)m}2^{k_1}\cdot 2^{-3m/2}2^{\beta m/5}2^{-2k_1} \lesssim 2^{-k_1}2^{-m/2}2^{2\beta m},$$

and the desired bound (7.22) follows using also (7.23).

Case 3. It remains to prove the bound (7.22) in the case

(7.27)
$$-m/3-4\beta m \le k_1 \le -m/3+\beta m \quad \text{ and } \quad 2^{-m/3-\beta m} \le \kappa \le 2^{-m/3+3\beta m}.$$

We define

$$\begin{aligned}G'(\xi) := \varphi_k(\xi)\int_{\mathbb{R}}\int_{\mathbb{R}^3} &e^{is\Phi^{\sigma;\mu,\nu}(\xi,\eta)}\varphi(2^{D^2+\max(0,k_1,k_2)}\Phi^{\sigma;\mu,\nu}(\xi,\eta))q_m(s)\\ &\cdot\widehat{f^{\mu}_{k_1,j_1}}(\xi-\eta,s)\widehat{f^{\nu}_{k_2,j_2}}(\eta,s)\,d\eta ds\end{aligned}$$

and notice that, using integration by parts in η as in the proof of Lemma 5.8,

$$\|G-G'\|_{L^2} \lesssim 2^{-10m}.$$

Moreover, using Lemma A.3 and the L^∞ bounds (7.57) below,

$$\begin{aligned}\|G'\|_{L^2} &\lesssim 2^m \sup_{s\in[2^{m-1},2^{m+1}]} \min\Big\{\|Ef^{\mu}_{k_1,j_1}(s)\|_{L^\infty}\|f^{\nu}_{k_2,j_2}(s)\|_{L^2},\\ &\qquad\qquad \|f^{\mu}_{k_1,j_1}(s)\|_{L^2}\|Ef^{\nu}_{k_2,j_2}(s)\|_{L^\infty}\Big\}\\ &\lesssim 2^m\cdot 2^{-3m/2}2^{-\max(j_1,j_2)(1/2-2\beta)}.\end{aligned}$$

The desired bound (7.22) follows if $\max(j_1,j_2)\geq m/2$, using also (7.27).

Finally, assume that

$$\begin{aligned}&-m/3-4\beta m\leq k_1\leq -m/3+\beta m,\\ &2^{-m/3-\beta m}\leq \kappa\leq 2^{-m/3+3\beta m},\qquad \max(j_1,j_2)\leq m/2.\end{aligned}\tag{7.28}$$

In this case we need to improve slightly on the pointwise bound (7.20). Assuming $\xi=r'e$, $r'\in(0,\infty)$, $e\in\mathbb{S}^2$ and letting $\eta=re+\eta'$, $\eta'\cdot e=0$, we define, for any $l\in\mathbb{Z}$,

$$\begin{aligned}G'_{\leq l}(\xi):=\varphi_k(r'e)\int_{\mathbb{R}}\int_{\mathbb{R}^2}\int_{\mathbb{R}} &e^{is\Phi^{\sigma;\mu,\nu}(r'e,re+\eta')}\\ &\cdot\varphi(2^{D^2+\max(0,k_1,k_2)}\Phi^{\sigma;\mu,\nu}(r'e,re+\eta'))q_m(s)\\ &\cdot\varphi((r-r^{\mu,\nu}(r'))/2^l\kappa)\varphi(\eta'/(2^{D+k_1}\kappa))\\ &\cdot\widehat{f^{\mu}_{k_1,j_1}}(r'e-re-\eta',s)\widehat{f^{\nu}_{k_2,j_2}}(re+\eta',s)\,drd\eta'ds.\end{aligned}$$

Clearly, $\|G-G'_{\leq D}\|_{L^2}\lesssim 2^{-10m}$; see (7.9). Estimating as in (7.20),

$$\begin{aligned}|G'_{\leq l}(\xi)| &\lesssim 2^m\cdot 2^{2k_1}\kappa^2 2^l\kappa\cdot 2^{-k_1/2}\cdot\mathbf{1}_{[-2^D(2^{k_1}+\kappa),2^D(2^{k_1}+\kappa)]}(|\xi|-R_{\sigma_2})\\ &\lesssim 2^l 2^{2\beta^2 m-m/2}\cdot\mathbf{1}_{[-2^D(2^{k_1}+\kappa),2^D(2^{k_1}+\kappa)]}(|\xi|-R_{\sigma_2}).\end{aligned}$$

Therefore, setting $l_0:=-\lfloor 8\beta m\rfloor$ we estimate

$$\begin{aligned}2^{(1+\beta)m}(\kappa+2^{k_1})^{1+\beta}\|G'_{\leq l_0}\|_{L^2} &\lesssim 2^{m+\beta m}2^{-m/3+3\beta m}\cdot 2^{l_0}2^{2\beta^2 m-m/2}2^{-m/6+3\beta m/2}\\ &\lesssim 2^{-\beta^2 m}.\end{aligned}\tag{7.29}$$

Finally, for (r,η') in the support of integration of $G_{\leq l}(\xi)$, we have from Lemma 7.2 and (7.5) that

$$\begin{aligned}&\Big|\frac{d}{dr}\Phi^{\sigma;\mu,\nu}(r'e,re+\eta')\Big|\\ &=\Big|-\iota_1\lambda_i'(\sqrt{(r-r')^2+|\eta'|^2})\frac{r-r'}{\sqrt{(r-r')^2+|\eta'|^2}}-\lambda'_{\sigma_2}(\sqrt{r^2+|\eta'|^2})\frac{r}{\sqrt{r^2+|\eta'|^2}}\Big|\\ &=\Big|\frac{\iota_1(r'-r)}{|r-r'|}\lambda_i'(r'-r)-\lambda'_{\sigma_2}(r)\Big|+O_{C_b,\varepsilon}(\kappa^2+2^{2k_1})\\ &\gtrsim_{C_b,\varepsilon} 2^l\kappa,\end{aligned}$$

and we may notice that we can integrate by parts in r and use Lemma A.2 with $K \approx 2^m 2^l \kappa$ and $\epsilon^{-1} \approx 2^{\beta m}[2^{\max(j_1,j_2)} + 2^{-l}(2^{k_1}+\kappa)^{-1}]$ to show that

$$|G_{\leq l+1}(\xi) - G_{\leq l}(\xi)| \lesssim 2^{-10m} \tag{7.30}$$

if $l \in [l_0, D]$. Indeed, it follows from (7.28) that $K\epsilon \gtrsim 2^{\beta m}$. The desired estimate (7.22) follows using also (7.29). □

LEMMA 7.5. *The bound* (7.3) *holds if* (7.2) *holds and*, *in addition*,

$$\begin{aligned}&\Phi^{\sigma;\mu,\nu} \in \{\Phi^{e;i+,e+}, \Phi^{e;i-,e+}, \Phi^{b;i+,b+}, \Phi^{b;i-,b+}\},\\ &k, k_2 \in [-D/100, D/100], \qquad \max(j_1, j_2) \geq (m - \beta^2 m)/2 - k_1/2,\end{aligned} \tag{7.31}$$

with

$$\kappa := 2^{\max(j_1,j_2)+\beta^2 m - m}. \tag{7.32}$$

Proof of Lemma 7.5. We define the function G as in (7.18); it suffices to prove that

$$2^{(1+\beta)j}\|\widetilde{\varphi}_j^{(k)} \cdot \mathcal{F}^{-1}(G)\|_{L^2} + \|G\|_{L^\infty} \lesssim 2^{-2\beta^4 m}. \tag{7.33}$$

The L^∞ bound in (7.33) is easy: if $j_1 \leq j_2$, then we use the bounds

$$\begin{aligned}&\|\widehat{f^\mu_{k_1,j_1}}(s)\|_{L^\infty} \lesssim 2^{-k_1/2},\\ &|\varphi_k(\xi)| \int_{\mathbb{R}^3} \Big|\widehat{f^\nu_{k_2,j_2}}(\eta, s)\Big| \Big|\varphi(|\Xi^{\mu,\nu}(\xi,\eta)|/\kappa)\Big| \mathbf{1}_{[2^{k_1-4},2^{k_1+4}]}(|\xi-\eta|)\, d\eta\\ &\qquad\qquad\qquad\qquad\qquad\lesssim 2^{-(1+\beta)j_2}(\kappa^3 2^{2k_1})^{1/2},\end{aligned}$$

which follow from Lemma 7.2, the bounds (5.15), and Definition 4.1. Therefore, in this case,

$$\|G\|_{L^\infty} \lesssim 2^m \cdot 2^{-k_1/2} 2^{-(1+\beta)j_2}(\kappa^3 2^{2k_1})^{1/2} \lesssim 2^{-\beta m/4} 2^{(j_2-m)/8},$$

which suffices by (7.2). Similarly, if $j_1 \geq j_2$, then we use the bounds

$$\begin{aligned}&\|\widehat{f^\nu_{k_2,j_2}}(s)\|_{L^\infty} \lesssim 1,\\ &|\varphi_k(\xi)| \int_{\mathbb{R}^3} \Big|\widehat{f^\mu_{k_1,j_1}}(\xi-\eta, s)\Big| \Big|\varphi(|\Xi^{\mu,\nu}(\xi,\eta)|/\kappa)\Big| \mathbf{1}_{[2^{k_2-4},2^{k_2+4}]}(|\eta|)\, d\eta\\ &\qquad\qquad\qquad\qquad\qquad\lesssim 2^{-(1+\beta)j_1}(\kappa^3 2^{2k_1})^{1/2},\end{aligned}$$

and the desired L^∞ bound on G follows as before.

The L^2 bound in (7.33) is more complicated. We notice first that the same argument as in the proof of Lemma 7.4, using the estimate (7.21), shows that

$$2^{(1+\beta)j}\|\widetilde{\varphi}_j^{(k)} \cdot \mathcal{F}^{-1}(G)\|_{L^2} \lesssim 2^{-m} \qquad \text{if} \qquad 2^j \geq 2^{\beta^2 m} 2^m(\kappa + 2^{k_1}). \tag{7.34}$$

To continue we consider three cases.

Case 1. Assume first that

$$2^{k_1-D} \le \kappa. \tag{7.35}$$

In view of (7.34), in this case it remains to prove that

$$2^{(1+\beta)m}\kappa^{1+\beta}\|G\|_{L^2} \lesssim 2^{-2\beta^2 m}. \tag{7.36}$$

We argue as in the proof of Lemma 5.7. We define first

$$G'(\xi) := \varphi_k(\xi)\int_{\mathbb{R}}\int_{\mathbb{R}^3} e^{is\Phi^{\sigma;\mu,\nu}(\xi,\eta)}\varphi(2^{D^2+\max(0,k_1,k_2)}\Phi^{\sigma;\mu,\nu}(\xi,\eta))q_m(s) \cdot \widehat{f^{\mu}_{k_1,j_1}}(\xi-\eta,s)\widehat{f^{\nu}_{k_2,j_2}}(\eta,s)\,d\eta ds$$

and notice that, using integration by parts in η and Lemma A.2 with $K \approx 2^m\kappa$, $\epsilon^{-1} \approx 2^{\max(j_1,j_2)}$,

$$\|G-G'\|_{L^2} \lesssim 2^{-10m}. \tag{7.37}$$

Since $\|\widetilde{\varphi}^{(k_1)}_{j_1}\cdot P_{k_1}f_\mu(s)\|_{B_{k_1,j_1}} + \|\widetilde{\varphi}^{(k_2)}_{j_2}\cdot P_{k_2}f_\nu(s)\|_{B_{k_2,j_2}} \lesssim 1$ (see (5.15)), we use (4.6)–(4.9) to decompose

$$\begin{aligned}
&\widetilde{\varphi}^{(k_1)}_{j_1}\cdot P_{k_1}f_\mu(s) = 2^{-\alpha k_1}[g^{\mu}_{k_1,j_1}(s)+h^{\mu}_{k_1,j_1}(s)],\\
&2^{(1+\beta)j_1}\|g^{\mu}_{k_1,j_1}(s)\|_{L^2} + 2^{(1/2-\beta)\widetilde{k_1}}\|\widehat{g^{\mu}_{k_1,j_1}}(s)\|_{L^\infty} \lesssim 1,\\
&2^{(1-\beta)j_1}\|h^{\mu}_{k_1,j_1}(s)\|_{L^2} + \|\widehat{h^{\mu}_{k_1,j_1}}(s)\|_{L^\infty} + 2^{\gamma j_1}\|\widehat{h^{\mu}_{k_1,j_1}}(s)\|_{L^1} \lesssim 2^{-8|k_1|}
\end{aligned} \tag{7.38}$$

and

$$\begin{aligned}
&\widetilde{\varphi}^{(k_2)}_{j_2}\cdot P_{k_2}f_\nu(s) = [g^{\nu}_{k_2,j_2}(s)+h^{\nu}_{k_2,j_2}(s)],\\
&2^{(1+\beta)j_2}\|g^{\nu}_{k_2,j_2}(s)\|_{L^2} + \|\widehat{g^{\nu}_{k_2,j_2}}(s)\|_{L^\infty} \lesssim 1,\\
&2^{(1-\beta)j_2}\|h^{\nu}_{k_2,j_2}(s)\|_{L^2} + \|\widehat{h^{\nu}_{k_2,j_2}}(s)\|_{L^\infty} + 2^{\gamma j_2}\|\widehat{h^{\nu}_{k_2,j_2}}(s)\|_{L^1} \lesssim 1.
\end{aligned} \tag{7.39}$$

For $f,g\in L^2(\mathbb{R}^3)$, $\xi\in\mathbb{R}^3$, and $s\in[2^{m-1},2^{m+1}]$, let

$$\begin{aligned}
\widetilde{G}'_s(f,g)(\xi) := \varphi_k(\xi)\int_{\mathbb{R}^3} & e^{is\Phi^{\sigma;\mu,\nu}(\xi,\eta)}\varphi(2^{D^2+\max(0,k_1,k_2)}\Phi^{\sigma;\mu,\nu}(\xi,\eta))\\
&\cdot\widehat{f}(\xi-\eta)\widehat{g}(\eta)\,d\eta.
\end{aligned} \tag{7.40}$$

Using also (7.37), for (7.36) it suffices to prove that, for any $s\in[2^{m-1},2^{m+1}]$,

$$2^{2\beta^2 m}2^{-\alpha k_1}2^{(2+\beta)m}\kappa^{1+\beta}\|\widetilde{G}'_s(f,g)\|_{L^2} \lesssim 1, \tag{7.41}$$

where

$$\begin{aligned}
&f\in\{P_{[k_1-2,k_1+2]}g^{\mu}_{k_1,j_1}(s), P_{[k_1-2,k_1+2]}h^{\mu}_{k_1,j_1}(s)\},\\
&g\in\{P_{[k_2-2,k_2+2]}g^{\nu}_{k_2,j_2}(s), P_{[k_2-2,k_2+2]}h^{\nu}_{k_2,j_2}(s)\}.
\end{aligned}$$

Using Lemma A.3 and (3.16) it follows that

$$\|\widetilde{G}'_s(f,g)\|_{L^2} \lesssim \min\big(\|E^{\mu}_s f\|_{L^2}\|E^{\nu}_s g\|_{L^\infty}, \|E^{\mu}_s f\|_{L^\infty}\|E^{\nu}_s g\|_{L^2}\big), \tag{7.42}$$

where $E^\mu_s f := e^{-is\widetilde{\Lambda}_\mu} f$ and $E^\nu_s g := e^{-is\widetilde{\Lambda}_\nu} g$. In view of Lemma A.5 (see also (5.19)),

$$\begin{aligned} \|E^\mu_s f\|_{L^\infty} &\lesssim 2^{\beta k_1} 2^{-m(5/4-10\beta)} 2^{j_1(1/4-11\beta)}, \\ \|E^\nu_s g\|_{L^\infty} &\lesssim 2^{-m(5/4-10\beta)} 2^{j_2(1/4-11\beta)} \end{aligned} \tag{7.43}$$

for

$$\begin{aligned} f &\in \{P_{[k_1-2,k_1+2]} g^\mu_{k_1,j_1}(s), P_{[k_1-2,k_1+2]} h^\mu_{k_1,j_1}(s)\}, \\ g &\in \{P_{[k_2-2,k_2+2]} g^\nu_{k_2,j_2}(s), P_{[k_2-2,k_2+2]} h^\nu_{k_2,j_2}(s)\}. \end{aligned}$$

If $|j_1 - j_2| \geq 10\beta m$, then we use (7.42)–(7.43), together with the estimate $2^{\max(j_1,j_2)} \approx \kappa 2^m 2^{-\beta^2 m}$ and the L^2 bounds $\|f\|_{L^2} \lesssim 2^{2\beta k_1} 2^{-(1-\beta) j_1}$, $\|g\|_{L^2} \lesssim 2^{-(1-\beta) j_2}$, to estimate

$$\begin{aligned} \|\widetilde{G}'_s(f,g)\|_{L^2} &\lesssim 2^{\beta k_1} 2^{-m(5/4-10\beta)} 2^{\min(j_1,j_2)(1/4-11\beta)} \cdot 2^{-(1-\beta)\max(j_1,j_2)} \\ &\lesssim 2^{\beta k_1} 2^{-m(5/4-10\beta)} 2^{-10\beta m(1/4-11\beta)} (\kappa 2^m 2^{-\beta^2 m})^{-3/4-10\beta} \\ &\lesssim 2^{\beta k_1} \kappa^{-1} 2^{-2m} 2^{-5\beta m/4} \end{aligned}$$

for

$$\begin{aligned} f &\in \{P_{[k_1-2,k_1+2]} g^\mu_{k_1,j_1}(s), P_{[k_1-2,k_1+2]} h^\mu_{k_1,j_1}(s)\}, \\ g &\in \{P_{[k_2-2,k_2+2]} g^\nu_{k_2,j_2}(s), P_{[k_2-2,k_2+2]} h^\nu_{k_2,j_2}(s)\}. \end{aligned}$$

The desired bound (7.41) follows in this case.

On the other hand, if $|j_1 - j_2| \leq 10\beta m$, then we estimate, using (7.38)–(7.39) and (7.42)–(7.43),

$$\begin{aligned} &\|\widetilde{G}'_s(P_{[k_1-2,k_1+2]} g^\mu_{k_1,j_1}(s), P_{[k_2-2,k_2+2]} g^\nu_{k_2,j_2}(s))\|_{L^2} \\ &\qquad \lesssim 2^{-m(5/4-10\beta)} 2^{\min(j_1,j_2)(1/4-11\beta)} \cdot 2^{-(1+\beta)\max(j_1,j_2)} \\ &\qquad \lesssim 2^{-m(5/4-10\beta)} (\kappa 2^m 2^{-\beta^2 m})^{-3/4-12\beta} \\ &\qquad \lesssim \kappa^{-1} 2^{-(2+2\beta)m} 2^{\beta^2 m}. \end{aligned}$$

Moreover, for $g \in \{P_{[k_2-2,k_2+2]} g^\nu_{k_2,j_2}(s), P_{[k_2-2,k_2+2]} h^\nu_{k_2,j_2}(s)\}$, we estimate, using also the assumption (7.35),

$$\begin{aligned} \|\widetilde{G}'_s(P_{[k_1-2,k_1+2]} h^\mu_{k_1,j_1}(s), g)\|_{L^2} &\lesssim \|\widehat{h^\mu_{k_1,j_1}}(s)\|_{L^1} \|g\|_{L^2} \lesssim 2^{-\gamma j_1} 2^{8k_1} 2^{-(1-\beta) j_2} \\ &\lesssim 2^{8k_1} 2^{16\beta m} (\kappa 2^m)^{-(1-\beta+\gamma)} \lesssim 2^{k_1} 2^{-m(1-17\beta+\gamma)}. \end{aligned}$$

Finally,

$$\begin{aligned}
&\|\widetilde{G}'_s(P_{[k_1-2,k_1+2]}g^{\mu}_{k_1,j_1}(s), P_{[k_2-2,k_2+2]}h^{\nu}_{k_2,j_2}(s))\|_{L^2}\\
&\quad\lesssim \min\Big[\|\widehat{g^{\mu}_{k_1,j_1}}(s)\|_{L^2}\|\widehat{h^{\nu}_{k_2,j_2}}(s)\|_{L^1}, 2^{3k_1/2}\|\widehat{g^{\mu}_{k_1,j_1}}(s)\|_{L^2}\|\widehat{h^{\nu}_{k_2,j_2}}(s)\|_{L^2}\Big]\\
&\quad\lesssim \min\Big[2^{-(1+\beta)j_1}2^{-\gamma j_2}, 2^{3k_1/2}2^{-(1+\beta)j_1}2^{-(1-\beta)j_2}\Big]\\
&\quad\lesssim \min\Big[2^{16\beta m}(\kappa 2^m)^{-(1+\beta+\gamma)}, \kappa^{3/2}2^{16\beta m}(\kappa 2^m)^{-2}\Big]\\
&\quad\lesssim 2^{16\beta m}\min\Big[2^{-(1+\gamma)m}\kappa^{-(1+\gamma)}, 2^{-2m}\kappa^{-1/2}\Big].
\end{aligned}$$

The desired bound (7.41) follows from these last three estimates, which completes the proof in Case 1.

Case 2. Assume now that

$$\kappa \leq 2^{k_1-D} \quad\text{and}\quad \kappa \leq 2^{-m(1/3+\beta/2)}. \tag{7.44}$$

In view of (7.34), in this case it remains to prove that

$$2^{(1+\beta)m}2^{(1+\beta)k_1}\|G\|_{L^2} \lesssim 2^{-2\beta^2 m}. \tag{7.45}$$

As in Lemma 7.4 (see (7.24)–(7.26)), we estimate pointwise

$$\begin{aligned}
|G(\xi)| &\lesssim 2^{-2k_1}\cdot 2^{2k_1}\kappa^3\cdot 2^{\beta m/10}2^{-k_1}\cdot \mathbf{1}_{[-2^{k_1+D},2^{k_1+D}]}(|\xi| - R_{\sigma_2})\\
&\lesssim \kappa^3 2^{\beta m/10}2^{-k_1}\cdot \mathbf{1}_{[-2^{k_1+D},2^{k_1+D}]}(|\xi| - R_{\sigma_2}).
\end{aligned}$$

Therefore,

$$2^{(1+\beta)m}2^{(1+\beta)k_1}\|G\|_{L^2} \lesssim 2^{(1+5\beta/4)m}\kappa^3,$$

and the desired estimate (7.45) follows since $\kappa \leq 2^{-m(1/3+\beta/2)}$.

Case 3. Finally assume that

$$2^{-m(1/3+\beta/2)} \leq \kappa \leq 2^{k_1-D}. \tag{7.46}$$

In view of (7.34), in this case it remains to prove that

$$2^{(2+2\beta)m}2^{(2+2\beta)k_1}\|G\|^2_{L^2} \lesssim 2^{-4\beta^2 m}. \tag{7.47}$$

Step 1. We need first a suitable decomposition and an orthogonality argument, as in the proof of Lemma 6.4. Let $\chi : \mathbb{R} \to [0,1]$ denote the cutoff function satisfying (6.42), and let $\chi'(x,y,z) := \chi(x)\chi(y)\chi(z)$. We define, for any $v \in \mathbb{Z}^3$ and $n \in \mathbb{Z}$,

$$\begin{aligned}
G_{v,n}(\xi) := \chi'(\kappa^{-1}\xi - v)\varphi_k(\xi)\int_{\mathbb{R}}\int_{\mathbb{R}^3} & e^{is\Phi^{\sigma;\mu,\nu}(\xi,\eta)}\\
&\cdot \chi_R^{\sigma;\mu,\nu}(\xi,\eta)\chi(2^{k_1-m}\kappa^{-1}s - n)q_m(s)\\
&\cdot \widehat{f^{\mu}_{k_1,j_1}}(\xi-\eta,s)\widehat{f^{\nu}_{k_2,j_2}}(\eta,s)\,d\eta ds,
\end{aligned} \tag{7.48}$$

and we notice that $G = \sum_{v\in\mathbb{Z}^3}\sum_{n\in\mathbb{Z}} G_{v,n}$.

We show now that

$$\|G\|_{L^2}^2 \lesssim \sum_{v\in\mathbb{Z}^3}\sum_{n\in\mathbb{Z}} \|G_{v,n}\|_{L^2}^2 + 2^{-10m}. \tag{7.49}$$

Indeed, we clearly have

$$\|G\|_{L^2}^2 \lesssim \sum_{v\in\mathbb{Z}^3}\Big\|\sum_{n\in\mathbb{Z}} G_{v,n}\Big\|_{L^2}^2 \lesssim \sum_{v\in\mathbb{Z}^3}\sum_{n_1,n_2\in\mathbb{Z}} |\langle G_{v,n_1}, G_{v,n_2}\rangle|.$$

Therefore, for (7.49) it suffices to prove that

$$|\langle G_{v,n_1}, G_{v,n_2}\rangle| \lesssim 2^{-20m} \qquad \text{if } v\in\mathbb{Z}^3 \text{ and } |n_1-n_2|\geq 2^{100D}. \tag{7.50}$$

To prove this we need to estimate $|\mathcal{F}^{-1}(G_{v,n})(x)|$. We would like to integrate by parts in the formula (7.48). Using Lemmas 7.2 and A.4(i), for $\xi = se$, $\eta = re+\eta'$ satisfying (7.7) with $\delta = 2\kappa$ and $|\xi-\kappa v|\lesssim\kappa$, we estimate

$$\begin{aligned}(\nabla_\xi\Phi^{\sigma;\mu,\nu})(\xi,\eta) &= -(\nabla_\eta\Phi^{\sigma;\mu,\nu})(\xi,\eta) + [\nabla\Lambda_\sigma(\xi)-\nabla\Lambda_\sigma(\eta)]\\ &= \lambda'_{\sigma_2}(s)e - \lambda'_{\sigma_2}(r^{\mu,\nu}(s))e + O_{C_{b,\varepsilon}}(\kappa)\\ &= [\lambda'_{\sigma_2}(\kappa|v|) - \lambda'_{\sigma_2}(r^{\mu,\nu}(\kappa|v|))]\cdot v/|v| + O_{C_{b,\varepsilon}}(\kappa).\end{aligned}$$

In particular, $|\nabla_\xi\Phi^{\sigma;\mu,\nu}(\xi,\eta)|\approx 2^{k_1}$. After repeated integration by parts in ξ (see (6.43)–(6.47) for a similar argument), it follows that

$$\begin{aligned}|\mathcal{F}^{-1}(G_{v,n})(x)| &\lesssim |x+w_n|^{-200} \quad \text{if} \quad |x+w_n|\geq 2^{50D}2^m\kappa,\\ w_n &:= n\kappa 2^{m-k_1}[\lambda'_{\sigma_2}(\kappa|v|) - \lambda'_{\sigma_2}(r^{\mu,\nu}(\kappa|v|))]\cdot v/|v|\end{aligned}$$

for any $n\in\mathbb{Z}$. Therefore, if $|n_1-n_2|\geq 2^{100D}$ then $|w_{n_1}-w_{n_2}|\geq 2^{70D}\kappa 2^m$ and the desired bound (7.50) follows. This completes the proof of (7.49).

In view of (7.49) and Lemma 7.2, for (7.47) it remains to prove that

$$2^{(2+2\beta)m}2^{(2+2\beta)k_1}\sum_{(v,n)\in\mathbb{Z}^3\times\mathbb{Z}} \|G_{v,n}\|_{L^2}^2 \lesssim 2^{-4\beta^2 m}. \tag{7.51}$$

Let

$$\begin{aligned}G_n(\xi) &:= \sum_{v\in\mathbb{Z}^3} G_{v,n}(\xi)\\ &= \varphi_k(\xi)\int_{\mathbb{R}}\int_{\mathbb{R}^3} e^{is\Phi^{\sigma;\mu,\nu}(\xi,\eta)}\chi_R^{\sigma;\mu,\nu}(\xi,\eta)\chi(2^{k_1-m}\kappa^{-1}s-n)q_m(s)\\ &\quad\cdot \widehat{f_{k_1,j_1}^{\mu}}(\xi-\eta,s)\widehat{f_{k_2,j_2}^{\nu}}(\eta,s)\,d\eta ds\end{aligned}$$

and[10]

$$G'_n(\xi) := \varphi_k(\xi)\int_{\mathbb{R}}\int_{\mathbb{R}^3} e^{is\Phi^{\sigma;\mu,\nu}(\xi,\eta)}\varphi(2^{D^2+\max(0,k_1,k_2)}\Phi^{\sigma;\mu,\nu}(\xi,\eta)) \\ \times \chi(2^{k_1-m}\kappa^{-1}s-n)q_m(s)\cdot \widehat{f^{\mu}_{k_1,j_1}}(\xi-\eta,s)\widehat{f^{\nu}_{k_2,j_2}}(\eta,s)\,d\eta ds.$$

Notice that

$$\sum_{v\in\mathbb{Z}^3}\|G_{v,n}\|^2_{L^2}\lesssim\|G_n\|^2_{L^2},\qquad \|G_n-G'_n\|_{L^2}\lesssim 2^{-10m} \tag{7.52}$$

for any $n\in\mathbb{Z}$. Since $G'_n\equiv 0$ unless $n\in[2^{k_1-4}\kappa^{-1},2^{k_1+4}\kappa^{-1}]$, for (7.51) it suffices to prove that

$$\sup_{n\in[2^{k_1-4}\kappa^{-1},2^{k_1+4}\kappa^{-1}]} 2^{(1+\beta)m}2^{(1+\beta)k_1}2^{k_1/2}\kappa^{-1/2}\|G'_n\|_{L^2}\lesssim 2^{-2\beta^2 m}. \tag{7.53}$$

For $f,g\in L^2(\mathbb{R}^3)$, $\xi\in\mathbb{R}^3$, and $s\in[2^{m-1},2^{m+1}]$ let, as in (7.40),

$$\widetilde{G}'_s(f,g)(\xi)=\varphi_k(\xi)\int_{\mathbb{R}^3} e^{is\Phi^{\sigma;\mu,\nu}(\xi,\eta)}\varphi(2^{D^2+\max(0,k_1,k_2)}\Phi^{\sigma;\mu,\nu}(\xi,\eta)) \\ \cdot\widehat{f}(\xi-\eta)\widehat{g}(\eta)\,d\eta.$$

The left-hand side of (7.53) is dominated by

$$2^{(1+\beta)m}2^{(1+\beta)k_1}2^{k_1/2}\kappa^{-1/2}\cdot 2^m\kappa 2^{-k_1}\sup_{s\in[2^{m-1},2^{m+1}]}\|\widetilde{G}'_s(f^{\mu}_{k_1,j_1},f^{\nu}_{k_2,j_2})\|_{L^2}.$$

Therefore, it remains to prove that

$$2^{(2+\beta)m}2^{(1/2+\beta)k_1}\kappa^{1/2}\sup_{s\in[2^{m-1},2^{m+1}]}\|\widetilde{G}'_s(f^{\mu}_{k_1,j_1}(s),f^{\nu}_{k_2,j_2}(s))\|_{L^2}\lesssim 2^{-2\beta^2 m}. \tag{7.54}$$

Step 2. We decompose $\widetilde{\varphi}^{(k_1)}_{j_1}\cdot P_{k_1}f_\mu(s)=2^{-\alpha k_1}[g^{\mu}_{k_1,j_1}(s)+h^{\mu}_{k_1,j_1}(s)]$ and $\widetilde{\varphi}^{(k_2)}_{j_2}\cdot P_{k_2}f_\nu(s)=[g^{\nu}_{k_2,j_2}(s)+h^{\nu}_{k_2,j_2}(s)]$ as in (7.38)–(7.39). In this proof we will also need the stronger bounds (4.8) on the functions $h^{\mu}_{k_1,j_1}(s)$ and $h^{\nu}_{k_2,j_2}(s)$,

$$\begin{aligned} &2^{\gamma j_1}\sup_{R\in[2^{-j_1},2^{k_1}],\,\xi_0\in\mathbb{R}^3}R^{-2}\|\widehat{h^{\mu}_{k_1,j_1}}(s)\|_{L^1(B(\xi_0,R))}\lesssim 2^{10k_1},\\ &2^{\gamma j_2}\sup_{R\in[2^{-j_2},2^{k_2}],\,\xi_0\in\mathbb{R}^3}R^{-2}\|\widehat{h^{\nu}_{k_2,j_2}}(s)\|_{L^1(B(\xi_0,R))}\lesssim 1, \end{aligned} \tag{7.55}$$

[10]In some arguments that involve the use of Lemma A.3 it is necessary to pass to operators that contain "smooth" symbols, such as the symbol $(\xi,\eta)\to\varphi(2^{D^2+\max(0,k_1,k_2)}\Phi^{\sigma;\mu,\nu}(\xi,\eta))$ in the operators $\widetilde{G}'_s$ below. Lemma A.3 is not directly compatible with "rough" symbols such as $(\xi,\eta)\to\chi^{\sigma;\mu,\nu}_R(\xi,\eta)$ since the L^1 norm of the inverse Fourier transform of such symbols is very large.

and the support properties (4.7). Recall the L^2 bounds

$$\begin{aligned}&\|g^\mu_{k_1,j_1}(s)\|_{L^2}\lesssim 2^{-(1+\beta)j_1}, &&\|h^\mu_{k_1,j_1}(s)\|_{L^2}\lesssim 2^{8k_1}2^{-(1-\beta)j_1},\\ &\|g^\nu_{k_2,j_2}(s)\|_{L^2}\lesssim 2^{-(1+\beta)j_2}, &&\|h^\nu_{k_2,j_2}(s)\|_{L^2}\lesssim 2^{-(1-\beta)j_2}.\end{aligned}\tag{7.56}$$

With $E^\mu_s f=e^{-is\widetilde{\Lambda}_\mu}f$ and $E^\nu_s g=e^{-is\widetilde{\Lambda}_\nu}g$ as in the proof in Case 1, we use the kernel bounds (A.49), (A.37), and (A.42) (as in the proof of Lemma A.5) to conclude that

$$\begin{aligned}&\|E^\mu_s P_{[k_1-2,k_1+2]}(g^\mu_{k_1,j_1}(s))\|_{L^\infty}\lesssim 2^{k_1/2}2^{-3m/2}2^{j_1(1/2-\beta)},\\ &\|E^\mu_s P_{[k_1-2,k_1+2]}(h^\mu_{k_1,j_1}(s))\|_{L^\infty}\lesssim 2^{8k_1}\min[2^{-3m/2}2^{j_1(1/2+\beta)},2^{-\gamma j_1}]\\ &\|E^\nu_s P_{[k_2-2,k_2+2]}(g^\nu_{k_2,j_2}(s))\|_{L^\infty}\lesssim 2^{-3m/2}2^{j_2(1/2-\beta)},\\ &\|E^\nu_s P_{[k_2-2,k_2+2]}(h^\nu_{k_2,j_2}(s))\|_{L^\infty}\lesssim \min[2^{-3m/2}2^{j_2(1/2+\beta)},2^{-\gamma j_2}]\end{aligned}\tag{7.57}$$

for any $s\in[2^{m-1},2^{m+1}]$. We combine these bounds and Lemma A.3. It follows from (7.57) that

$$\|Ef^\mu_{k_1,j_1}(s)\|_{L^\infty}\lesssim 2^{-3m/2}2^{j_1(1/2+\beta)}\quad\text{and}\quad \|Ef^\nu_{k_2,j_2}(s)\|_{L^\infty}\lesssim 2^{-3m/2}2^{j_2(1/2+\beta)}.$$

Recalling that $2^{\max(j_1,j_2)}\approx\kappa 2^m 2^{-\beta^2 m}$ (see (7.32)), we have

$$\begin{aligned}\|\widetilde{G}'_s(f^\mu_{k_1,j_1}(s),f^\nu_{k_2,j_2}(s))\|_{L^2}&\lesssim 2^{-3m/2}2^{\min(j_1,j_2)(1/2+\beta)}\cdot 2^{-(1-\beta)\max(j_1,j_2)}\\ &\lesssim 2^{-|j_1-j_2|(1/2+\beta)}2^{-3m/2}2^{(-1/2+2\beta)\max(j_1,j_2)}\\ &\lesssim 2^{-|j_1-j_2|(1/2+\beta)}2^{-2m}\kappa^{-1/2}\cdot 2^{\beta^2 m}2^{2\beta m}\end{aligned}$$

for any $s\in[2^{m-1},2^{m+1}]$. The desired bound (7.54) follows if $2^{-|j_1-j_2|}2^{k_1}\leq 2^{-6\beta m}$.

It remains to prove (7.54) in the case

$$2^{|j_1-j_2|}2^{-k_1}\leq 2^{6\beta m}.\tag{7.58}$$

We start by using the bounds (7.56)–(7.57) more carefully. We estimate

$$\begin{aligned}\|\widetilde{G}'_s(f,g)\|_{L^2}&\lesssim 2^{-3m/2}2^{\min(j_1,j_2)(1/2-\beta)}\cdot 2^{-(1+\beta)\max(j_1,j_2)}\\ &\lesssim 2^{-(2+2\beta)m}\kappa^{-1/2}\kappa^{-2\beta}2^{\beta^2 m}\end{aligned}$$

if $(f,g)=\big(P_{[k_1-2,k_1+2]}(g^\mu_{k_1,j_1}(s)),P_{[k_2-2,k_2+2]}(g^\nu_{k_2,j_2}(s))\big)$. This is consistent with the desired bound (7.54) if we recall that $\kappa^{-1}\lesssim 2^{m/3+\beta m/2}$ (see (7.46)). Therefore, it remains the prove that

$$2^{(2+\beta)m}2^{k_1/2}\kappa^{1/2}\sup_{s\in[2^{m-1},2^{m+1}]}\|\widetilde{G}'_s(f,g)\|_{L^2}\lesssim 2^{-2\beta^2 m},\tag{7.59}$$

if (7.58) holds, and

$$(f,g)\in\Big\{\big(P_{[k_1-2,k_1+2]}(g^{\mu}_{k_1,j_1}(s)),P_{[k_2-2,k_2+2]}(h^{\nu}_{k_2,j_2}(s))\big),\\ \big(P_{[k_1-2,k_1+2]}(h^{\mu}_{k_1,j_1}(s)),P_{[k_2-2,k_2+2]}(g^{\nu}_{k_2,j_2}(s))\big),\\ \big(P_{[k_1-2,k_1+2]}(h^{\mu}_{k_1,j_1}(s)),P_{[k_2-2,k_2+2]}(h^{\nu}_{k_2,j_2}(s))\big)\Big\}. \tag{7.60}$$

One could try arguing as before: recalling that $\gamma = 3/2 - 4\beta$ and using (7.58), for (f,g) as in (7.60), we estimate

$$\begin{aligned}\|\widetilde{G}'_s(f,g)\|_{L^2} &\lesssim 2^{-\gamma\min(j_1,j_2)}\cdot 2^{-(1-\beta)\max(j_1,j_2)}\\ &\lesssim 2^{\gamma|j_1-j_2|}2^{-(\gamma+1-\beta)\max(j_1,j_2)}\\ &\lesssim 2^{-5m/2+15\beta m}\kappa^{-5/2+5\beta}2^{3\beta^2 m}.\end{aligned}$$

Therefore, the left-hand side of (7.59) is dominated by

$$C2^{k_1/2}2^{(2+\beta)m}\kappa^{1/2}\cdot 2^{3\beta^2 m}2^{-5m/2+15\beta m}\kappa^{-5/2+5\beta}\lesssim 2^{3\beta^2 m}2^{-m(1/2-16\beta)}\kappa^{-(2-5\beta)}.$$

The desired bound (7.59) follows if κ^{-1} is sufficiently small, say $\kappa^{-1}\le 2^{m/6}$, but not in the full range $\kappa^{-1}\le 2^{m(1/3+\beta/2)}$ (see (7.46)). To cover the full range we need an additional argument that uses the stronger bounds (7.55).

Step 3. We prove now (7.59). We reinsert first the cutoff function $\chi_R^{\sigma;\mu,\nu}$, i.e., we define

$$\begin{aligned}\widetilde{G}''_s(f,g)(\xi) &:= \varphi_k(\xi)\int_{\mathbb{R}^3} e^{is\Phi^{\sigma;\mu,\nu}(\xi,\eta)}\chi_R^{\sigma;\mu,\nu}(\xi,\eta)\cdot\widehat{f}(\xi-\eta)\widehat{g}(\eta)\,d\eta.\\ \chi_R^{\sigma;\mu,\nu}(\xi,\eta) &= \varphi(2^{D^2+\max(0,k_1,k_2)}\Phi^{\sigma;\mu,\nu}(\xi,\eta))\varphi(|\Xi^{\mu,\nu}(\xi,\eta)|/\kappa),\end{aligned} \tag{7.61}$$

where (f,g) are as in (7.60). As before, integrating by parts in η, we notice that $\|\widetilde{G}''_s(f,g)-\widetilde{G}'_s(f,g)\|_{L^2}\lesssim 2^{-10m}$. Then we decompose

$$\begin{aligned}\widetilde{G}''_s(f,g) &= \sum_{v\in\mathbb{Z}^3}\widetilde{G}''_{v,s}(f,g),\\ \widetilde{G}''_{v,s}(f,g) &:= \chi'(\kappa^{-1}\xi-v)\varphi_k(\xi)\int_{\mathbb{R}^3} e^{is\Phi^{\sigma;\mu,\nu}(\xi,\eta)}\chi_R^{\sigma;\mu,\nu}(\xi,\eta)\cdot\widehat{f}(\xi-\eta)\widehat{g}(\eta)\,d\eta,\end{aligned} \tag{7.62}$$

where χ' is as before. For (7.59), it remains to prove that

$$2^{(4+2\beta)m}2^{k_1}\kappa\sum_{v\in\mathbb{Z}^3}\|\widetilde{G}''_{v,s}(f,g)\|_{L^2}^2\lesssim 2^{-4\beta^2 m} \tag{7.63}$$

for any $s\in[2^{m-1},2^{m+1}]$ and (f,g) as in (7.60).

In view of Lemma 7.2, the variables in the definition of the function $\widetilde{G}''_{v,s}(f,g)$ are naturally restricted as follows:

$$
\begin{aligned}
&v \in \mathbb{Z}^3, \qquad \Big|\kappa|v| - R_{\sigma_2}\Big| \lesssim_{C_b,\varepsilon} 2^{k_1},\\
&\xi = a\widehat{v} + \xi', \qquad \xi'\cdot\widehat{v} = 0, \qquad |\xi'| \lesssim_{C_b,\varepsilon} \kappa, \qquad \Big|a - \kappa|v|\Big| \lesssim_{C_b,\varepsilon} \kappa,\\
&\xi - \eta = b\widehat{v} + \theta', \qquad \theta'\cdot\widehat{v} = 0, \qquad |\theta'| \lesssim_{C_b,\varepsilon} 2^{k_1}\kappa,\\
&\Big|b - (\kappa|v| - r^{\mu,\nu}(\kappa|v|))\Big| \lesssim_{C_b,\varepsilon} \kappa,
\end{aligned}
$$

where $\widehat{v} = v/|v|$. More precisely, for any fixed v, we define the functions f^v and g^v by the formulas

$$
\begin{aligned}
\widehat{f^v}(\theta) &:= \varphi[|\theta'|/(\kappa 2^{k_1+D})]\varphi\Big[[\rho - \kappa|v| + r^{\mu,\nu}(\kappa|v|)]/(\kappa 2^D)\Big]\cdot\widehat{f}(\theta),\\
\widehat{g^v}(\theta) &:= \varphi(|\theta'|/(\kappa 2^D))\varphi\Big[[\rho - r^{\mu,\nu}(\kappa|v|)]/(\kappa 2^D)\Big]\cdot\widehat{g}(\theta),
\end{aligned}
\tag{7.64}
$$

where $\theta = \rho\widehat{v} + \theta'$, $\rho\in\mathbb{R}$, $\theta'\cdot\widehat{v} = 0$. In view of Lemma 7.2 and (7.6), the functions $\widehat{f^v}$ (respectively $\widehat{g^v}$) have essentially pairwise disjoint supports, i.e.,

$$
\sum_{v\in\mathbb{Z}^3}\|\widehat{f^v}\|_{L^2}^2 \lesssim \|\widehat{f}\|_{L^2}^2, \qquad \sum_{v\in\mathbb{Z}^3}\|\widehat{g^v}\|_{L^2}^2 \lesssim 2^{-k_1}\|\widehat{g}\|_{L^2}^2.
\tag{7.65}
$$

Moreover, they suffice to determine the functions $\widetilde{G}''_{v,s}(f,g)$, i.e.,

$$
\widetilde{G}''_{v,s}(f,g)(\xi) = \chi'(\kappa^{-1}\xi - v)\varphi_k(\xi)\int_{\mathbb{R}^3} e^{is\Phi^{\sigma;\mu,\nu}(\xi,\eta)}\chi_R^{\sigma;\mu,\nu}(\xi,\eta)\cdot\widehat{f^v}(\xi-\eta)\widehat{g^v}(\eta)\,d\eta.
$$

We use (7.55), (7.56), and (7.65). For

$$
(f,g) = \Big(P_{[k_1-2,k_1+2]}(g^{\mu}_{k_1,j_1}(s)), P_{[k_2-2,k_2+2]}(h^{\nu}_{k_2,j_2}(s))\Big)
$$

or

$$
(f,g) = \Big(P_{[k_1-2,k_1+2]}(h^{\mu}_{k_1,j_1}(s)), P_{[k_2-2,k_2+2]}(h^{\nu}_{k_2,j_2}(s))\Big),
$$

we estimate

$$
\begin{aligned}
\sum_{v\in\mathbb{Z}^3}\|\widetilde{G}''_{v,s}(f,g)\|_{L^2}^2 &\lesssim \sum_{v\in\mathbb{Z}^3}\|\widehat{f^v}\|_{L^2}^2\|\widehat{g^v}\|_{L^1}^2\\
&\lesssim \|\widehat{f}\|_{L^2}^2\sup_{v\in\mathbb{Z}^3}\|\widehat{g^v}\|_{L^1}^2 \lesssim 2^{-2j_1+2\beta j_1}2^{-2\gamma j_2}\kappa^4.
\end{aligned}
$$

For $(f,g) = \Big(P_{[k_1-2,k_1+2]}(h^{\mu}_{k_1,j_1}(s)), P_{[k_2-2,k_2+2]}(g^{\nu}_{k_2,j_2}(s))\Big)$, we estimate

$$
\begin{aligned}
\sum_{v\in\mathbb{Z}^3}\|\widetilde{G}''_{v,s}(f,g)\|_{L^2}^2 &\lesssim \sum_{v\in\mathbb{Z}^3}\|\widehat{f^v}\|_{L^1}^2\|\widehat{g^v}\|_{L^2}^2\\
&\lesssim 2^{-k_1}\|\widehat{g}\|_{L^2}^2\sup_{v\in\mathbb{Z}^3}\|\widehat{f^v}\|_{L^1}^2 \lesssim 2^{-2j_2+2\beta j_2}2^{-2\gamma j_1}\kappa^4.
\end{aligned}
$$

Therefore, using also the assumption (7.58), the left-hand side of (7.63) is dominated by

$$C2^{(4+2\beta)m}\kappa\cdot\kappa^4 2^{-2\gamma\min(j_1,j_2)}2^{-(2-2\beta)\max(j_1,j_2)}$$
$$\lesssim 2^{(4+2\beta)m}\kappa^5 2^{2\gamma|j_1-j_2|}2^{-(2\gamma+2-2\beta)\max(j_1,j_2)}\lesssim 2^{-\beta m},$$

and the desired bound (7.63) follows. This completes the proof of the lemma. □

8. Proof of Proposition 4.3, IV: Case C resonant interactions

PROPOSITION 8.1. *Assume that* $(k,j),(k_1,j_1),(k_2,j_2)\in\mathcal{J}$, $m\in[1,L]\cap\mathbb{Z}$,

(8.1)
$$\Phi^{\sigma;\mu,\nu}\in\in\mathcal{T}_C=\{\Phi^{i;i+,i+},\Phi^{i;i+,i-},\Phi^{i;i-,i-},\Phi^{i;e+,e-},\Phi^{i;e+,b-},\Phi^{i;e-,b+},\Phi^{i;b+,b-}\},$$

and

(8.2)
$$-9m/10\le k_1,k_2\le j/N_0',\ \ \max(j_1,j_2)\le(1-\beta/10)m+k,\ \ m\ge -k(1+\beta^2),$$
$$\beta m/2+N_0'k_++D^2\le j\le m+D,\ \ k\le -D/4.$$

Then there is $\kappa\in(0,2^{D/10}]$,

$$\kappa\ge\kappa_0:=\max\big(2^{(\beta^2m-m)/2}2^{-\min(k_1,k_2,0)/2}2^{-D/2},2^{\beta^2m-m}2^{\max(j_1,j_2)}\big),$$

such that

$$2^k\Big\|\widetilde{\varphi}_j^{(k)}\cdot P_kR_{m,\kappa}^{\sigma;\mu,\nu}(f^{\mu}_{k_1,j_1},f^{\nu}_{k_2,j_2})\Big\|_{B^1_{k,j}}\lesssim 2^{-2\beta^4m}. \tag{8.3}$$

The rest of the section is concerned with the proof of this proposition. We decompose

(8.4)
$$R_{m,\kappa}^{\sigma;\mu,\nu}(f,g)=R_{m,\kappa,1}^{\sigma;\mu,\nu}(f,g)+R_{m,\kappa,2}^{\sigma;\mu,\nu}(f,g),$$
$$\mathcal{F}\big[R_{m,\kappa,l}^{\sigma;\mu,\nu}(f,g)\big](\xi):=\int_{\mathbb{R}}\int_{\mathbb{R}^3}e^{is\Phi^{\sigma;\mu,\nu}(\xi,\eta)}\chi_{R,l}^{\sigma;\mu,\nu}(\xi,\eta)q_m(s)\widehat{f}(\xi-\eta,s)\widehat{g}(\eta,s)\,d\eta ds,$$

where $\chi_{R,1}^{\sigma;\mu,\nu}:=\chi_R^{\sigma;\mu,\nu}\varphi(2^{-k+D}\Phi^{\sigma;\mu,\nu})$ and $\chi_{R,2}^{\sigma;\mu,\nu}:=\chi_R^{\sigma;\mu,\nu}\varphi_{[1,\infty)}(2^{-k+D}\Phi^{\sigma;\mu,\nu})$ (compare with (5.67)).

The proposition follows from Lemmas 8.2, 8.3, 8.4, 8.5, 8.6 below. We estimate first the easier contribution of the operators $R_{m,\kappa,2}^{\sigma;\mu,\nu}$.

LEMMA 8.2. *Assume that* (8.2) *holds. Then for any* $\kappa\in[\kappa_0,2^{D/10}]$, *we have*

$$2^k\Big\|\widetilde{\varphi}_j^{(k)}\cdot P_kR_{m,\kappa,2}^{\sigma;\mu,\nu}(f^{\mu}_{k_1,j_1},f^{\nu}_{k_2,j_2})\Big\|_{B^1_{k,j}}\lesssim 2^{-2\beta^4m}. \tag{8.5}$$

Proof of Lemma 8.2. This is similar to the proof of the bound (5.69) in Lemma 5.8 since, on the support of integration, we have that $|\Phi^{\sigma;\mu,\nu}(\xi,\eta)| \gtrsim 2^k$. After integration by parts in s, we obtain that

$$2^k\mathcal{F}[R^{i;\mu,\nu}_{m,\kappa,2}(f^\mu_{k_1,j_1},f^\nu_{k_2,j_2})] = i\,[R_{21}+R_{22}+R_{23}]\,,$$

$$\begin{aligned} R_{21}(\xi) &:= \int_{\mathbb{R}}\int_{\mathbb{R}^3} e^{is\Phi^{i;\mu,\nu}(\xi,\eta)}\frac{2^k\chi^{i;\mu,\nu}_{R,2}(\xi,\eta)}{\Phi^{i;\mu,\nu}(\xi,\eta)}q'_m(s)\\ &\qquad\cdot\widehat{f^\mu_{k_1,j_1}}(\xi-\eta,s)\widehat{f^\nu_{k_2,j_2}}(\eta,s)\,d\eta ds,\\ R_{22}(\xi) &:= \int_{\mathbb{R}}\int_{\mathbb{R}^3} e^{is\Phi^{i;\mu,\nu}(\xi,\eta)}\frac{2^k\chi^{i;\mu,\nu}_{R,2}(\xi,\eta)}{\Phi^{i;\mu,\nu}(\xi,\eta)}q_m(s)\\ &\qquad\cdot(\partial_s\widehat{f^\mu_{k_1,j_1}})(\xi-\eta,s)\widehat{f^\nu_{k_2,j_2}}(\eta,s)\,d\eta ds,\\ R_{23}(\xi) &:= \int_{\mathbb{R}}\int_{\mathbb{R}^3} e^{is\Phi^{i;\mu,\nu}(\xi,\eta)}\frac{2^k\chi^{i;\mu,\nu}_{R,2}(\xi,\eta)}{\Phi^{i;\mu,\nu}(\xi,\eta)}q_m(s)\\ &\qquad\cdot\widehat{f^\mu_{k_1,j_1}}(\xi-\eta,s)(\partial_s\widehat{f^\nu_{k_2,j_2}})(\eta,s)\,d\eta ds. \end{aligned} \tag{8.6}$$

Recall Definition 4.1. We first show that

$$2^{(1/2-\beta+\alpha)k}\cdot 2^k\|\varphi_k\cdot\mathcal{F}R^{i;\mu,\nu}_{m,\kappa,2}(f^\mu_{k_1,j_1},f^\nu_{k_2,j_2})\|_{L^\infty}\lesssim 2^{-2\beta^4 m}. \tag{8.7}$$

Indeed, using Cauchy-Schwartz inequality, (5.17), and (5.20), we see that

$$\begin{aligned} \|\varphi_k\cdot R_{22}\|_{L^\infty} &\lesssim 2^m\sup_{s\in[2^{m-2},2^{m+2}]}\|(\partial_s\widehat{f^\mu_{k_1,j_1}}(s))\|_{L^2}\|\widehat{f^\nu_{k_2,j_2}}(s)\|_{L^2}\\ &\lesssim 2^{-\beta m}2^{-10\max(k_1,k_2,0)},\\ \|\varphi_k\cdot R_{23}\|_{L^\infty} &\lesssim 2^m\sup_{s\in[2^{m-2},2^{m+2}]}\|\widehat{f^\mu_{k_1,j_1}}(s)\|_{L^2}\|(\partial_s\widehat{f^\nu_{k_2,j_2}}(s))\|_{L^2}\\ &\lesssim 2^{-\beta m}2^{-10\max(k_1,k_2,0)}, \end{aligned} \tag{8.8}$$

and this gives acceptable contributions. Proceeding as above, using (5.18) we get

$$\|\varphi_k\cdot R_{21}\|_{L^\infty}\lesssim 2^{-(1-\beta)(j_1+j_2)}(1+2^{k_1})^{-10}(1+2^{k_2})^{-10}. \tag{8.9}$$

Therefore, this gives an acceptable contribution to (8.7) unless

$$|k|+|k_1|+|k_2|+j_1+j_2\le\beta^2 m. \tag{8.10}$$

Now, assuming that (8.10) holds, we can strengthen the L^∞ bound. We observe that

$$|\Xi^{\mu,\nu}(\xi,\eta)|\gtrsim 2^{-\beta m}\min\Big(\Big|(\xi-\eta)/|\xi-\eta|-\eta/|\eta|\Big|,\Big|(\xi-\eta)/|\xi-\eta|+\eta/|\eta|\Big|\Big).$$

Consequently, if $|\xi| \in [2^{k-2}, 2^{k+2}]$, $|\xi-\eta| \in [2^{k_1-2}, 2^{k_1+2}]$, $|\eta| \in [2^{k_2-2}, 2^{k_2+2}]$, and $|\Xi^{\mu,\nu}(\xi,\eta))| \lesssim 2^{-m/3}$, then

$$\min\Big(\Big|(\xi-\eta)/|\xi-\eta| - \eta/|\eta|\Big|, \Big|(\xi-\eta)/|\xi-\eta| + \eta/|\eta|\Big|\Big) \lesssim 2^{-m/4}.$$

A simple estimate using the L^∞ bounds in (5.17) then gives $\|\varphi_k \cdot R_{21}\|_{L^\infty} \lesssim 2^{-m/6}$, which suffices to finish the proof of (8.7).

For (8.5), it remains to prove that

$$2^{(1+\alpha)k} 2^{(1+\beta)m} \|P_k R^{i;\mu,\nu}_{m,\kappa,2}(f^\mu_{k_1,j_1}, f^\nu_{k_2,j_2})\|_{L^2} \lesssim 2^{-2\beta^4 m}. \tag{8.11}$$

We use again the decomposition (8.6) and notice that

$$\begin{aligned} &\frac{2^k \chi^{i;\mu,\nu}_{R,2}(\xi,\eta)}{\Phi^{i;\mu,\nu}(\xi,\eta)} \\ &\quad = \varphi(|\Xi^{\mu,\nu}(\xi,\eta)|/\kappa) \frac{\varphi_{[1,\infty)}(2^{-k+D}\Phi^{i;\mu,\nu}(\xi,\eta))\varphi(2^{D^2+\max(k_1,k_2,0)}\Phi^{i;\mu,\nu}(\xi,\eta))}{2^{-k}\Phi^{i;\mu,\nu}(\xi,\eta)} \\ &\quad = \varphi(|\Xi^{\mu,\nu}(\xi,\eta)|/\kappa) \int_{\mathbb{R}} e^{i\lambda\Phi^{i;\mu,\nu}(\xi,\eta)} \psi(\lambda)\, d\lambda, \end{aligned} \tag{8.12}$$

where, as a consequence of the Fourier inversion formula

$$\psi(\lambda) = C \int_{\mathbb{R}} e^{-i\lambda x} \frac{\varphi_{[1,\infty)}(2^{-k+D}x)\varphi(2^{D^2+\max(k_1,k_2,0)}x)}{2^{-k}x}\, dx. \tag{8.13}$$

Simple integration by parts estimates show that, for any integer $N \geq 2$,

$$|\psi(\lambda)| \lesssim_N 2^{\beta^4 m} 2^k (1 + 2^k|\lambda|)^{-N}. \tag{8.14}$$

Let

$$\begin{aligned} R_{21}(\xi,\lambda) &:= \int_{\mathbb{R}}\int_{\mathbb{R}^3} e^{i(s+\lambda)\Phi^{i;\mu,\nu}(\xi,\eta)} \varphi(|\Xi^{\mu,\nu}(\xi,\eta)|/\kappa) q_m'(s) \\ &\qquad \cdot \widehat{f^\mu_{k_1,j_1}}(\xi-\eta,s)\widehat{f^\nu_{k_2,j_2}}(\eta,s)\, d\eta ds, \\ R_{22}(\xi,\lambda) &:= \int_{\mathbb{R}}\int_{\mathbb{R}^3} e^{i(s+\lambda)\Phi^{i;\mu,\nu}(\xi,\eta)} \varphi(|\Xi^{\mu,\nu}(\xi,\eta)|/\kappa) q_m(s) \\ &\qquad \cdot (\partial_s\widehat{f^\mu_{k_1,j_1}})(\xi-\eta,s)\widehat{f^\nu_{k_2,j_2}}(\eta,s)\, d\eta ds, \\ R_{23}(\xi,\lambda) &:= \int_{\mathbb{R}}\int_{\mathbb{R}^3} e^{i(s+\lambda)\Phi^{i;\mu,\nu}(\xi,\eta)} \varphi(|\Xi^{\mu,\nu}(\xi,\eta)|/\kappa) q_m(s) \\ &\qquad \cdot \widehat{f^\mu_{k_1,j_1}}(\xi-\eta,s)(\partial_s\widehat{f^\nu_{k_2,j_2}})(\eta,s)\, d\eta ds. \end{aligned} \tag{8.15}$$

The formulas (8.12) and (8.6) show that, for $l \in \{1,2,3\}$,

$$R_{2l}(\xi) = \int_{\mathbb{R}} R_{2l}(\xi,\lambda)\psi(\lambda)\, d\lambda.$$

Recall also that $2^{m+k} \geq 2^{\beta^2 m/2}$; see (8.2). In view of the rapid decay in (8.14), for (8.11) it suffices to prove that, for $l \in \{1,2,3\}$,

$$\begin{aligned} &2^{\alpha k}2^{(1+\beta)m}\|\varphi_k(\xi)R_{2l}(\xi,\lambda)\|_{L^2_\xi} \lesssim 2^{-3\beta^4 m} && \text{if} && |\lambda| \leq 2^{m-10}, \\ &2^{\alpha k}2^{(1+\beta)m}\|\varphi_k(\xi)R_{2l}(\xi,\lambda)\|_{L^2_\xi} \lesssim 2^{4m} && \text{if} && |\lambda| \geq 2^{m-10}. \end{aligned} \tag{8.16}$$

The bound in the second line of (8.16) follows easily using L^2 bounds, as in (8.8)–(8.9). For the first bound, we define the functions $\widetilde{R}_{2l}$, $l \in \{1,2,3\}$ as in (8.15), but without the factor $\varphi(|\Xi^{\mu,\nu}(\xi,\eta)|/\kappa)$. As in the proof of Lemma 5.8 (see (5.72)), integration by parts in η shows that the difference between R_{2l} and $\widetilde{R}_{2l}$ is rapidly decreasing in m. It remains to prove that

$$2^{\alpha k}2^{(1+\beta)m}\|\varphi_k(\xi)\widetilde{R}_{2l}(\xi,\lambda)\|_{L^2_\xi} \lesssim 2^{-3\beta^4 m} \quad \text{if} \quad |\lambda| \leq 2^{m-10} \text{ and } l \in \{1,2,3\}. \tag{8.17}$$

We use the L^2 bounds

$$\begin{aligned} &\|f^\mu_{k_1,j_1}(s)\|_{L^2} \lesssim (2^{\alpha k_1}+2^{10k_1})^{-1}2^{-(1-\beta)j_1}2^{2\beta k_1}, \\ &\|f^\nu_{k_2,j_2}(s)\|_{L^2} \lesssim (2^{\alpha k_2}+2^{10k_2})^{-1}2^{-(1-\beta)j_2}2^{2\beta k_2}, \\ &\|(\partial_s f^\mu_{k_1,j_1})(s)\|_{L^2} + \|(\partial_s f^\nu_{k_2,j_2})(s)\|_{L^2} \lesssim 2^{-m(1+\beta)} \end{aligned} \tag{8.18}$$

(see (5.18) and (5.20)) and the L^∞ bounds

$$\begin{aligned} &\|e^{-i(s+\lambda)\widetilde{\Lambda}_\mu}f^\mu_{k_1,j_1}(s)\|_{L^\infty} + \|e^{-i(s+\lambda)\widetilde{\Lambda}_\nu}f^\nu_{k_2,j_2}(s)\|_{L^\infty} \lesssim 2^{-m(1+\beta)}, \\ &\|e^{-i(s+\lambda)\widetilde{\Lambda}_\mu}f^\mu_{k_1,j_1}(s)\|_{L^\infty} \lesssim 2^{-m(5/4-10\beta)}2^{j_1(1/4-11\beta)}, \\ &\|e^{-i(s+\lambda)\widetilde{\Lambda}_\nu}f^\nu_{k_2,j_2}(s)\|_{L^\infty} \lesssim 2^{-m(5/4-10\beta)}2^{j_2(1/4-11\beta)}. \end{aligned} \tag{8.19}$$

These L^∞ bounds are similar to the bounds (5.17) and (5.19), once we recall that $|s+\lambda| \approx 2^m$. Using Lemma A.3, it follows that

$$\begin{aligned} &2^{\alpha k}2^{(1+\beta)m}\|\widetilde{R}_{21}(\xi,\lambda)\|_{L^2_\xi} \\ &\qquad \lesssim 2^{(1+\beta)m}\cdot 2^{-m(5/4-10\beta)}2^{\min(j_1,j_2)(1/4-11\beta)}2^{-(1-\beta)\max(j_1,j_2)} \lesssim 2^{-\beta m}, \\ &2^{\alpha k}2^{(1+\beta)m}\big[\|\widetilde{R}_{22}(\xi,\lambda)\|_{L^2_\xi} + \|\widetilde{R}_{23}(\xi,\lambda)\|_{L^2_\xi}\big] \\ &\qquad \lesssim 2^{(1+\beta)m}\cdot 2^m 2^{-m(1+\beta)}2^{-m(1+\beta)} \lesssim 2^{-\beta m}, \end{aligned}$$

and the desired bound (8.17) follows. □

We estimate now the contribution of the operators $R^{\sigma;\mu,\nu}_{m,\kappa,1}$, starting with some of the easier cases.

Lemma 8.3. *Assume that* (8.2) *holds,*

$$\Phi^{\sigma;\mu,\nu} \in\in \{\Phi^{i;i-,i-}, \Phi^{i;e+,e-}, \Phi^{i;e+,b-}, \Phi^{i;e-,b+}, \Phi^{i;b+,b-}\}, \tag{8.20}$$

and $\kappa = \kappa_0$. *Then*

$$2^k \Big\| \widetilde{\varphi}_j^{(k)} \cdot P_k R^{\sigma;\mu,\nu}_{m,\kappa,1}(f^{\mu}_{k_1,j_1}, f^{\nu}_{k_2,j_2}) \Big\|_{B^1_{k,j}} \lesssim 2^{-2\beta^4 m}. \tag{8.21}$$

Proof of Lemma 8.3. Clearly, $P_k R^{\sigma;\mu,\nu}_{m,\kappa,1}(f^{\mu}_{k_1,j_1}, f^{\nu}_{k_2,j_2}) = 0$ if

$$\Phi^{\sigma;\mu,\nu} = \Phi^{i;i-,i-}$$

or if

$$\Phi^{\sigma;\mu,\nu} \in\in \{\Phi^{i;e+,e-}, \Phi^{i;e+,b-}, \Phi^{i;e-,b+}, \Phi^{i;b+,b-}\} \text{ and } 2\max(k_1,k_2) \leq k - D/10.$$

Indeed, in this case, since $\lambda_e(0) = \lambda_b(0)$ and $\lambda_e'(0) = \lambda_b'(0) = 0$, we see that

$$|\Phi^{i;\mu,\nu}(\xi,\eta)| \geq \Lambda_i(\xi) - |\Lambda_{\sigma_1}(\xi-\eta) - \Lambda_{\sigma_2}(\eta)| \gtrsim |\xi| - C_{C_b,\varepsilon}(|\xi-\eta|^2 + |\eta|^2) \gtrsim 2^k.$$

It remains to prove the lemma in the case

$$\Phi^{\sigma;\mu,\nu} \in\in \{\Phi^{i;e+,e-}, \Phi^{i;e+,b-}, \Phi^{i;e-,b+}, \Phi^{i;b+,b-}\} \text{ and } k \leq 2\max(k_1,k_2) + D/10. \tag{8.22}$$

In this case, since $k \leq -D/4$ from (8.2), we remark that $k \leq \min(k_1,k_2) - 20$, $|k_1 - k_2| \leq 8$, and that

$$|\Xi^{\mu,\nu}(\xi,\eta)| \gtrsim_{C_b,\varepsilon} \begin{cases} |\xi|(1+|\xi-\eta|+|\eta|)^{-3} & \text{if } \sigma_1 = \sigma_2, \\ (|\eta|+|\xi-\eta|)(1+|\xi-\eta|+|\eta|)^{-3} & \text{if } \sigma_1 \neq \sigma_2. \end{cases} \tag{8.23}$$

These inequalities follow from Lemma A.4. Consequently, we see that if

$$\begin{aligned} k &\geq 2\beta m - m/2 - \min(k_1,k_2,0)/2 \quad \text{if} \quad \sigma_1 = \sigma_2, \\ \max(k_1,k_2) &\geq 2\beta m - m/2 - \min(k_1,k_2,0)/2 \quad \text{if} \quad \sigma_1 \neq \sigma_2, \end{aligned} \tag{8.24}$$

then $P_k R^{i;\mu,\nu}_{m,\kappa,1} = 0$. (Recall that

$$\kappa = \max(2^{-(1-\beta^2)m/2} 2^{-\min(k_1,k_2,0)/2} 2^{-D/2}, 2^{\max(j_1,j_2)-(1-\beta^2)m})$$

and $\max(j_1,j_2) \leq (1-\beta/10)m + k$; see (8.2).) The desired bound (8.3) becomes trivial in this case.

Independently, using Lemma A.3 and (A.37), (A.42), we directly see that

$$\begin{aligned} &\|P_k T^{i;\mu,\nu}_m(f^{\mu}_{k_1,j_1}, f^{\nu}_{k_2,j_2})\|_{L^2} \\ &\qquad \lesssim 2^m \sup_{s\in[2^{m-4},2^{m+4}]} \min(\|E f^{\mu}_{k_1,j_1}(s)\|_{L^\infty} \|f^{\nu}_{k_2,j_2}(s)\|_{L^2}, \\ &\qquad\qquad\qquad \|f^{\mu}_{k_1,j_1}(s)\|_{L^2} \|E f^{\nu}_{k_2,j_2}(s)\|_{L^\infty}) \\ &\qquad \lesssim 2^m 2^{-3m/2}, \end{aligned}$$

from which we deduce that

$$2^{(1+\alpha)k} 2^{(1+\beta)j} \|P_k T^{i;\mu,\nu}_m(f^{\mu}_{k_1,j_1}, f^{\nu}_{k_2,j_2})\|_{L^2} \lesssim 2^{(1+\alpha)k} 2^{(1/2+\beta)m}.$$

In addition

$$\begin{aligned}2^{(3/2-\beta+\alpha)k}&\|\mathcal{F}P_kT_m^{i;\mu,\nu}(f^\mu_{k_1,j_1},f^\nu_{k_2,j_2})\|_{L^\infty}\\&\lesssim 2^{(3/2-\beta+\alpha)k}2^m\sup_{s\in[2^{m-4},2^{m+4}]}\|f^\mu_{k_1,j_1}(s)\|_{L^2}\|f^\nu_{k_2,j_2}(s)\|_{L^2}\\&\lesssim 2^{(3/2-\beta+\alpha)k}2^m2^{(k_1+k_2)/2}.\end{aligned}$$

Therefore,

$$2^k\|\widetilde{\varphi}^{(k)}_j\cdot P_kT_m^{i;\mu,\nu}(f^\mu_{k_1,j_1},f^\nu_{k_2,j_2})\|_{B^1_{k,j}}\lesssim 2^{(1+\alpha)k}2^{(1/2+\beta)m}+2^{(3/2-\beta+\alpha)k}2^m2^{(k_1+k_2)/2}. \tag{8.25}$$

Recall also that $T_m^{i;\mu,\nu}=N_m^{1;i;\mu,\nu}+N_{m,\kappa}^{2;i;\mu,\nu}+R_{m,\kappa,1}^{i;\mu,\nu}+R_{m,\kappa,2}^{i;\mu,\nu}$; see (5.67) and (8.4). The operators $N_m^{1;i;\mu,\nu}$, $N_{m,\kappa}^{2;i;\mu,\nu}$, and $R_{m,\kappa,2}^{i;\mu,\nu}$ have already been bounded in Lemmas 5.8 and 8.2. Therefore, using also (8.25),

$$2^k\|\widetilde{\varphi}^{(k)}_j\cdot P_kR_{m,\kappa,1}^{i;\mu,\nu}(f^\mu_{k_1,j_1},f^\nu_{k_2,j_2})\|_{B^1_{k,j}}\lesssim 2^{-2\beta^4m}+2^{(1+\alpha)k}2^{(1/2+\beta)m}+2^{(3/2-\beta+\alpha)k}2^m2^{(k_1+k_2)/2}. \tag{8.26}$$

This gives the desired bound (8.21) if $\sigma_1\neq\sigma_2$ and (8.24) does not hold, using also (8.22). If $\sigma_1=\sigma_2$ and $k\leq -3m/4$, then (8.21) follows also from (8.26). On the other hand, if $k\geq -3m/4$, since Λ_{σ_1} is smooth when $\sigma_1\in\{e,b\}$, we observe that

$$\forall\ |\rho|\geq 2,\ \sigma_1\in\{e,b\},\qquad |D^\rho_\eta\Phi^{i;\sigma_1+,\sigma_1-}(\xi,\eta)|\lesssim_\rho 2^k,$$

as long as $|\xi|\leq 2^{k+4}$. Besides, from (8.2), we have that $\max(j_1,j_2)\leq(1-\beta/10)m+k$. Therefore, we recall (8.23) and use Lemma A.2 with $K\approx 2^{(1-\beta/20)m+k}$, $\epsilon=2^{-\max(j_1,j_2)}$ to conclude that $|P_kT_m^{i;\mu,\nu}(f^\mu_{k_1,j_1},f^\nu_{k_2,j_2})(\xi)|\lesssim 2^{-4m}$, from which the desired inequality (8.21) follows easily. □

We consider now the remaining two phases. A key observation is the weak ellipticity bound

$$|\Phi^{i,i\pm,i\pm}(\xi,\eta)|\gtrsim\frac{|\xi||\xi-\eta||\eta|}{1+|\xi|^2+|\eta|^2}\qquad\text{when}\quad\min\{|\xi|,|\xi-\eta|,|\eta|\}\leq 1. \tag{8.27}$$

This follows from the bound

$$\lambda_i(a)+\lambda_i(b)-\lambda_i(a+b)\gtrsim_{C_b,\varepsilon}a\min(1,b)^2\qquad\text{if } 0\leq a\leq b\text{ and } a\in[0,2^{-D/20}]. \tag{8.28}$$

Indeed, using Lemma A.4, if $b\leq r_*/2$, then

$$\begin{aligned}\lambda_i(a)+\lambda_i(b)-\lambda_i(a+b)&=\int_0^a[\lambda'(r)-\lambda'(b+r)]\,dr\\&=\int_0^a\int_0^b-\lambda_i''(r+s)\,drds\approx_{C_b,\varepsilon}ab^2.\end{aligned}$$

On the other hand, if $b \geq r_*/2$, then

$$\lambda_i(a)+\lambda_i(b)-\lambda_i(a+b)=\int_0^a[\lambda'(r)-\lambda'(b+r)]\,dr \geq \int_0^a[q_i(r)+rq_i'(r)-q_i(b+r)]\,dr\approx_{C_b,\varepsilon} a,$$

and the desired lower bound (8.28) follows.

We prove first the required L^∞ bounds.

Lemma 8.4. *Assume that* (8.2) *holds and*

$$\Phi^{\sigma;\mu,\nu}\in\in\{\Phi^{i;i+,i+},\Phi^{i;i+,i-}\}.$$

Then, for any $\kappa\in[\kappa_0,2^{D/10}]$, *we have*

$$2^{(3/2+\alpha-\beta)k}\|\mathcal{F}P_kR^{i;\mu,\nu}_{m,\kappa,1}(f^\mu_{k_1,j_1},f^\nu_{k_2,j_2})\|_{L^\infty}\lesssim 2^{-2\beta^4m}. \tag{8.29}$$

Proof of Lemma 8.4. Integration by parts in s, as in Lemma 8.2, gives

$$\begin{aligned}\mathcal{F}[R^{i;\mu,\nu}_{m,\kappa,1}(f^\mu_{k_1,j_1},f^\nu_{k_2,j_2})]&=i\,[R_{11}+R_{12}+R_{13}]\,,\\ R_{11}(\xi)&:=\int_{\mathbb{R}}\int_{\mathbb{R}^3}e^{is\Phi^{i;\mu,\nu}(\xi,\eta)}\frac{\chi^{i;\mu,\nu}_{R,1}(\xi,\eta)}{\Phi^{i;\mu,\nu}(\xi,\eta)}q_m'(s)\\ &\qquad\cdot\widehat{f^\mu_{k_1,j_1}}(\xi-\eta,s)\widehat{f^\nu_{k_2,j_2}}(\eta,s)\,d\eta ds,\\ R_{12}(\xi)&:=\int_{\mathbb{R}}\int_{\mathbb{R}^3}e^{is\Phi^{i;\mu,\nu}(\xi,\eta)}\frac{\chi^{i;\mu,\nu}_{R,1}(\xi,\eta)}{\Phi^{i;\mu,\nu}(\xi,\eta)}q_m(s)\\ &\qquad\cdot(\partial_s\widehat{f^\mu_{k_1,j_1}})(\xi-\eta,s)\widehat{f^\nu_{k_2,j_2}}(\eta,s)\,d\eta ds,\\ R_{33}(\xi)&:=\int_{\mathbb{R}}\int_{\mathbb{R}^3}e^{is\Phi^{i;\mu,\nu}(\xi,\eta)}\frac{\chi^{i;\mu,\nu}_{R,1}(\xi,\eta)}{\Phi^{i;\mu,\nu}(\xi,\eta)}q_m(s)\\ &\qquad\cdot\widehat{f^\mu_{k_1,j_1}}(\xi-\eta,s)(\partial_s\widehat{f^\nu_{k_2,j_2}})(\eta,s)\,d\eta ds.\end{aligned}\tag{8.30}$$

First, using (8.27), (5.17), and (5.20), we see that

$$\begin{aligned}&2^{(3/2+\alpha-\beta)k}\|\varphi_k\cdot R_{12}\|_{L^\infty}\\ &\lesssim 2^{(3/2+\alpha-\beta)k}2^{-k-\widetilde{k_1}-\widetilde{k_2}}2^m\sup_{s\in[2^{m-4},2^{m+4}]}\|(\partial_s\widehat{f^\mu_{k_1,j_1}})(s)\|_{L^2}\|\widehat{f^\nu_{k_2,j_2}}(s)\|_{L^2}\\ &\lesssim 2^{(3/2+\alpha-\beta)k}2^{-k-\widetilde{k_1}-\widetilde{k_2}}2^m\cdot 2^{\widetilde{k_1}}2^{-(1+\beta)m}2^{(1+\beta-\alpha)\widetilde{k_2}}\\ &\lesssim 2^{-\beta^3m}.\end{aligned}$$

Similarly, $2^{(3/2+\alpha-\beta)k}\|\varphi_k\cdot R_{13}\|_{L^\infty}\lesssim 2^{-\beta^3m}$. Moreover, assuming that

$$|k|+|k_1|+|k_2|+|j_1|+|j_2|\geq\beta^2m$$

and using Hölder's inequality, we find that

$$\begin{aligned}
&2^{(3/2+\alpha-\beta)k}\|\varphi_k\cdot R_{11}\|_{L^\infty}\\
&\lesssim 2^{(3/2+\alpha-\beta)k}2^{-k-\widetilde{k_1}-\widetilde{k_2}}\sup_{s\in[2^{m-4},2^{m+4}]}\|\widehat{f^\mu_{k_1,j_1}}(s)\|_{L^2}\|\widehat{f^\nu_{k_2,j_2}}(s)\|_{L^2}\\
&\lesssim 2^{(3/2+\alpha-\beta)k}2^{-k-\widetilde{k_1}-\widetilde{k_2}}2^{2\beta(\widetilde{k_1}+\widetilde{k_2})}(2^{\alpha k_1}+2^{10k_1})^{-1}(2^{\alpha k_2}+2^{10k_2})^{-1}2^{-(1-\beta)(j_1+j_2)}\\
&\lesssim 2^{-2\beta^4 m}.
\end{aligned}$$

On the other hand, if

$$|k|+|k_1|+|k_2|+|j_1|+|j_2|\le\beta^2 m,$$

then we can proceed as in the proof of the bound (8.7) in Lemma 8.2 to estimate also $2^{(3/2+\alpha-\beta)k}\|\varphi_k\cdot R_{11}\|_{L^\infty}\lesssim 2^{-\beta^3 m}$. The desired bound (8.29) follows. ☐

We prove now the weighted L^2 bounds in two steps.

Lemma 8.5. *Assume that* (8.2) *holds and*

$$\Phi^{\sigma;\mu,\nu}\in\in\{\Phi^{i;i+,i+},\Phi^{i;i+,i-}\}.$$

Then the L^2 bound

$$2^{(1+\alpha)k}2^{(1+\beta)j}\|\widetilde{\varphi}^{(k)}_j\cdot P_kR^{i;\mu,\nu}_{m,\kappa,1}(f^\mu_{k_1,j_1},f^\nu_{k_2,j_2})\|_{L^2}\lesssim 2^{-2\beta^4 m} \tag{8.31}$$

holds for any $\kappa\in[\kappa_0,2^{D/10}]$, provided that either

$$\max(k_1,k_2)\ge -D/10, \tag{8.32}$$

or

$$\begin{gathered}\max(k_1,k_2)\le -D/10,\quad 2k+\min(k_1,k_2)\le(\beta^2m-m)+2D,\\ \min(k_1,k_2)\le k-10,\end{gathered} \tag{8.33}$$

or

$$\begin{gathered}\max(k_1,k_2)\le -D/10,\quad 2k+\min(k_1,k_2)\le(\beta^2m-m)+2D,\\ k-10\le\min(k_1,k_2)\le -3\beta m.\end{gathered} \tag{8.34}$$

Proof of Lemma 8.5. Using (8.27), $P_kR^{i;\mu,\nu}_{m,\kappa,1}(f^\mu_{k_1,j_1},f^\nu_{k_2,j_2})=0$ if (8.32) is satisfied. Assume now that (8.33) holds. In this case $k\ge\max(k_1,k_2)-4$ and necessarily $\max(j_1,j_2)\ge m/8$ by (8.33). We may assume that $k_2\le k_1$. If $j_1\le j_2-6\beta m$ then, using Plancherel, (5.18) and (A.49),

$$\begin{aligned}
\|T^{i;\mu,\nu}_m(f^\mu_{k_1,j_1},f^\nu_{k_2,j_2})\|_{L^2}&\lesssim 2^m\sup_{s\in[2^{m-4},2^{m+4}]}\|Ef^\mu_{k_1,j_1}(s)\|_{L^\infty}\|f^\nu_{k_2,j_2}(s)\|_{L^2}\\
&\lesssim 2^m\cdot 2^{-3/2m}2^{k_1/2}2^{(1/2+\beta)j_1}2^{-(1-\beta)j_2}2^{3/2\beta(k_1+k_2)}
\end{aligned}$$

and, therefore,

$$2^{(1+\alpha)k}2^{(1+\beta)j}\|T_m^{i;\mu,\nu}(f^\mu_{k_1,j_1},f^\nu_{k_2,j_2})\|_{L^2}\\ \lesssim 2^{(1+\alpha)[k+k_2/2]}2^{(1/2+\beta)m}2^{-(1/2+\beta/4)(j_2-j_1)}2^{3/4\beta j_1}\lesssim 2^{-\beta^3 m}.$$

The desired bound (8.31) follows using also (5.69) and (8.5). If $j_1\geq j_2-6\beta m$ and $\max(j_1,j_2)\geq m/8$, then

$$(1/2+\beta)j_2-(1-\beta)j_1\leq -(1/2-2\beta)j_1+3\beta m+6\beta^2 m\leq -4\beta m.$$

Using Plancherel, (5.18) and (A.49), we estimate

$$\|T_m^{i;\mu,\nu}(f^\mu_{k_1,j_1},f^\nu_{k_2,j_2})\|_{L^2}\lesssim 2^m\sup_{s\in[2^{m-4},2^{m+4}]}\|f^\mu_{k_1,j_1}(s)\|_{L^2}\|Ef^\nu_{k_2,j_2}(s)\|_{L^\infty}\\ \lesssim 2^m\cdot 2^{-3/2m}2^{k_2/2}2^{(1/2+\beta)j_2}2^{-(1-\beta)j_1}2^{3/2\beta(k_1+k_2)}$$

and, therefore,

$$2^{(1+\alpha)k}2^{(1+\beta)j}\|T_m^{i;\mu,\nu}(f^\mu_{k_1,j_1},f^\nu_{k_2,j_2})\|_{L^2}\\ \lesssim 2^{(1+\alpha)[k+k_2/2]}2^{(1/2+\beta)m}2^{-4\beta m}2^{3/2\beta(k_1+k_2/2)}\lesssim 2^{-\beta^3 m}.$$

The desired bound (8.31) follows using also (5.69) and (8.5).

Finally, assume that (8.34) holds, so $|k_1-k_2|\leq 20$. In this case, we may simply use Plancherel, (5.18), and (A.49) to estimate (assuming for example $j_2\leq j_1$)

(8.35)
$$\|T_m^{i;\mu,\nu}(f^\mu_{k_1,j_1},f^\nu_{k_2,j_2})\|_{L^2}\lesssim 2^m\sup_{s\in[2^{m-4},2^{m+4}]}\|f^\mu_{k_1,j_1}(s)\|_{L^2}\|Ef^\nu_{k_2,j_2}(s)\|_{L^\infty}\\ \lesssim 2^m\cdot 2^{-3/2m}2^{k_2/2}2^{(1/2+\beta)j_2}2^{-(1-\beta)j_1}2^{3/2\beta(k_1+k_2)}.$$

The desired bound follows as before, using also (5.69) and (8.5). □

We complete now the proof of the weighted L^2 bounds, with the remaining cases.

LEMMA 8.6. *Assume that* (8.2) *holds,* $\Phi^{\sigma;\mu,\nu}\in\in\{\Phi^{i;i+,i+},\Phi^{i;i+,i-}\}$, *and*

$$k_2\leq k_1\leq -D/10.$$

Then the L^2 *bound*

$$2^{(1+\alpha)k}2^{(1+\beta)j}\|\widetilde{\varphi}^{(k)}_j\cdot P_kR^{i;\mu,\nu}_{m,\kappa,1}(f^\mu_{k_1,j_1},f^\nu_{k_2,j_2})\|_{L^2}\lesssim 2^{-2\beta^4 m} \tag{8.36}$$

holds provided that either

$$2k+k_2\leq(\beta^2 m-m)+2D,\quad k_2\geq -3\beta m,\quad \textit{and}\quad \kappa=2^{D/10}, \tag{8.37}$$

or

(8.38)
$$2k+k_2\geq(\beta^2 m-m)+2D,\quad j\leq m+k_2(1/2+4\beta)+D,\quad \textit{and}\quad \kappa=2^{D/10},$$

or

$$2k+k_2 \geq (\beta^2 m - m) + 2D, \quad j \leq j_1 + \beta^2 m + D, \quad \textit{and} \quad \kappa = 2^{D/10}, \tag{8.39}$$

or

$$\begin{aligned} &2k+k_2 \geq (\beta^2 m - m) + 2D,\\ &j \geq \max(m + k_2(1/2+4\beta), j_1 + \beta^2 m) + D, \ \textit{and}\ \kappa = 2^{k-D/4}. \end{aligned} \tag{8.40}$$

Proof of Lemma 8.6. We will often use the decomposition (8.30) and the inequality (8.27). As in the proof of Lemma 8.2, we also notice that if $|\xi| \in [2^{k-2}, 2^{k+2}]$, $|\eta| \in [2^{k_2-2}, 2^{k_2+2}]$, $|\xi-\eta| \in [2^{k_1-2}, 2^{k_1+2}]$, then

$$\begin{aligned} \frac{\chi_{R,1}^{i;\mu,\nu}(\xi,\eta)}{\Phi^{i;\mu,\nu}(\xi,\eta)} &= \varphi(|\Xi^{\mu,\nu}(\xi,\eta)|/\kappa)\frac{\varphi(2^{-k+D}\Phi^{i;\mu,\nu}(\xi,\eta))\varphi(2^{D^2}\Phi^{i;\mu,\nu}(\xi,\eta))}{\Phi^{i;\mu,\nu}(\xi,\eta)}\\ &\quad \cdot \varphi_{[1,\infty)}(2^{-k-k_1-k_2+D}\Phi^{i;\mu,\nu}(\xi,\eta))\\ &= \varphi(|\Xi^{\mu,\nu}(\xi,\eta)|/\kappa)\int_{\mathbb{R}} e^{i\lambda\Phi^{i;\mu,\nu}(\xi,\eta)}\psi_2(\lambda)\,d\lambda \end{aligned} \tag{8.41}$$

where, as a consequence of the Fourier inversion formula,

$$\psi_2(\lambda) = C\int_{\mathbb{R}} e^{-i\lambda x}\frac{\varphi(2^{-k+D}x)\varphi(2^{D^2}x)}{x}\varphi_{[1,\infty)}(2^{-k-k_1-k_2+D}x)\,dx. \tag{8.42}$$

Simple integration by parts estimates show that, for any integer $N \geq 2$,

$$|\psi_2(\lambda)| \lesssim_N 2^{\beta^4 m}(1+2^{k+k_1+k_2}|\lambda|)^{-N}. \tag{8.43}$$

Let

$$\begin{aligned} R_{11}(\xi,\lambda) &:= \varphi_k(\xi)\int_{\mathbb{R}}\int_{\mathbb{R}^3} e^{i(s+\lambda)\Phi^{i;\mu,\nu}(\xi,\eta)}\varphi(|\Xi^{\mu,\nu}(\xi,\eta)|/\kappa)q_m'(s)\\ &\quad \cdot \widehat{f_{k_1,j_1}^{\mu}}(\xi-\eta,s)\widehat{f_{k_2,j_2}^{\nu}}(\eta,s)\,d\eta ds,\\ R_{12}(\xi,\lambda) &:= \varphi_k(\xi)\int_{\mathbb{R}}\int_{\mathbb{R}^3} e^{i(s+\lambda)\Phi^{i;\mu,\nu}(\xi,\eta)}\varphi(|\Xi^{\mu,\nu}(\xi,\eta)|/\kappa)q_m(s)\\ &\quad \cdot (\partial_s\widehat{f_{k_1,j_1}^{\mu}})(\xi-\eta,s)\widehat{f_{k_2,j_2}^{\nu}}(\eta,s)\,d\eta ds,\\ R_{13}(\xi,\lambda) &:= \varphi_k(\xi)\int_{\mathbb{R}}\int_{\mathbb{R}^3} e^{i(s+\lambda)\Phi^{i;\mu,\nu}(\xi,\eta)}\varphi(|\Xi^{\mu,\nu}(\xi,\eta)|/\kappa)q_m(s)\\ &\quad \cdot \widehat{f_{k_1,j_1}^{\mu}}(\xi-\eta,s)(\partial_s\widehat{f_{k_2,j_2}^{\nu}})(\eta,s)\,d\eta ds. \end{aligned} \tag{8.44}$$

The formulas (8.30) and (8.41) show that, for $l \in \{1,2,3\}$,

$$\varphi_k(\xi)R_{1l}(\xi) = \int_{\mathbb{R}} R_{1l}(\xi,\lambda)\psi_2(\lambda)\,d\lambda.$$

In view of the rapid decay in (8.43), for (8.36) it suffices to prove that, for $l \in \{1,2,3\}$,

(8.45)
$$2^{-(k+k_1+k_2)}2^{(1+\alpha)k}2^{(1+\beta)j}\|\widetilde{\varphi}^{(k)}_j\cdot\mathcal{F}^{-1}[R_{1l}(.,\lambda)]\|_{L^2}\lesssim 2^{-3\beta^4 m}\qquad\text{if}\quad |\lambda|\le 2^{\beta^2m/2}2^{-(k+k_1+k_2)},$$
$$2^{(1+\alpha)k}2^{(1+\beta)m}\|R_{1l}(\xi,\lambda)\|_{L^2_\xi}\lesssim 2^{4m}\qquad\text{if}\quad |\lambda|\ge 2^{\beta^2m/2}2^{-(k+k_1+k_2)}.$$

Assume first that (8.37) holds. If $k\le -2m/3$, then the desired bound follows from (8.35). On the other hand, if $k\ge -2m/3$, then $-(k+k_1+k_2)\le 3m/4$. Notice also that $\varphi(|\Xi^{\mu,\nu}(\xi,\eta)|/\kappa)=1$ in the support of the integrals in (8.44). Therefore, if $|\lambda|\le 2^{\beta^2m/2}2^{-(k+k_1+k_2)}$, then $|s+\lambda|\approx 2^m$ and

$$\begin{aligned}\|\mathcal{F}^{-1}[R_{12}(.,\lambda)]\|_{L^2}&\lesssim 2^m\sup_{s\in[2^{m-4},2^{m+4}]}\|(\partial_s f^\mu_{k_1,j_1})(s)\|_{L^2}\|e^{-i(s+\lambda)\widetilde{\Lambda}_\nu}f^\nu_{k_2,j_2}(s)\|_{L^\infty}\\&\lesssim 2^{-(1+2\beta)m}2^{k_1}2^{(1/2-2\beta)k_2},\end{aligned}$$

using (5.17) and (5.20). Similarly, using also (A.49) and recalling that $k_2\le k_1$,

$$\|\mathcal{F}^{-1}[R_{13}(.,\lambda)]\|_{L^2}+\|\mathcal{F}^{-1}[R_{11}(.,\lambda)]\|_{L^2}\lesssim 2^{-(1+\beta+\beta^2)m}2^{k_1+k_2}2^{-(1/2+4\beta)k_2}.$$

Therefore,

$$\sum_{l\in\{1,2,3\}}\|\mathcal{F}^{-1}[R_{1l}(.,\lambda)]\|_{L^2}\lesssim 2^{-(1+\beta+\beta^2)m}2^{k_1+k_2}2^{-(1/2+4\beta)k_2}.\tag{8.46}$$

The desired bound in the first line of (8.45) follows since $j\le m+D$. The bound in the second line follows as well, since in this case one can simply estimate $\|R_{1l}(\xi,\lambda)\|_{L^\infty_\xi}\lesssim 1$ for $l\in\{1,2,3\}$.

The proof is similar in the case described in (8.38), since the bound (8.46) still holds and $2^{(1+\beta)j}\lesssim 2^{(1+\beta)m}2^{k_2(1/2+5\beta)}$, in view of the stronger assumption on j. On the other hand, in the case described in (8.39), one can assume $m+k_2\le j\le j_1+\beta^2m+D$ and prove the stronger bound

$$2^{(1+\alpha)k}2^{(1+\beta)j}\|P_kT^{i;\mu,\nu}_m(f^\mu_{k_1,j_1},f^\nu_{k_2,j_2})\|_{L^2}\lesssim 2^{-3\beta^4m},$$

by estimating as in the proof of (5.60).

Finally, assume that (8.40) holds. Notice first that the choice $\kappa=2^{k-D/4}$ is acceptable in this case, i.e., $\kappa\ge\kappa_0$. We may assume $k_2\le k_1$ and use the formula (8.4),

(8.47)
$$\begin{aligned}P_kR^{i;\mu,\nu}_{m,\kappa,1}(f^\mu_{k_1,j_1},f^\nu_{k_2,j_2})(x)&=C\int_{\mathbb{R}^3}\int_{\mathbb{R}}\int_{\mathbb{R}^3}\varphi_k(\xi)e^{i[x\cdot\xi+s\Phi^{i;\mu,\nu}(\xi,\eta)]}\varphi(|\Xi^{\mu,\nu}(\xi,\eta)|/\kappa)\\&\cdot\varphi(2^{-k+D}\Phi^{i;\mu,\nu}(\xi,\eta))\varphi(2^{D^2}\Phi^{i;\mu,\nu}(\xi,\eta))q_m(s)\widehat{f^\mu_{k_1,j_1}}(\xi-\eta,s)\widehat{f^\nu_{k_2,j_2}}(\eta,s)\,d\eta ds d\xi.\end{aligned}$$

We will show that if (ξ,η) is in the support of the integral in (8.47), then

$$\left|\nabla_\xi\Phi^{i;\mu,\nu}(\xi,\eta)\right|\lesssim 2^{k_2}.\tag{8.48}$$

Assuming this bound and (8.40), one would get rapid decay, i.e.,

$$|P_k R^{i;\mu,\nu}_{m,\kappa,1}(f^{\mu}_{k_1,j_1}, f^{\nu}_{k_2,j_2})(x)| \lesssim 2^{-4m}$$

if $|x| \geq 2^{j-4}$, using integration by parts in ξ and Lemma A.2 with $K \approx 2^j$ and $\epsilon \approx \min(2^{-j_1}, 2^{2k-D/2})$.

It remains to prove (8.48). We will use repeatedly the following observation from the sine law in the triangle formed by ξ, $\xi-\eta$, η:

$$\frac{\sin(\angle(\xi,\eta))}{|\xi-\eta|} = \frac{\sin(\angle(\xi,\xi-\eta))}{|\eta|} = \frac{\sin(\angle(\xi-\eta,\eta))}{|\xi|}. \tag{8.49}$$

Recall the formulas

$$\begin{aligned} \Phi^{i;\mu,\nu}(\xi,\eta) &= \lambda_i(|\xi|) - \iota_\mu\lambda_i(|\xi-\eta|) - \iota_\nu\lambda_i(|\eta|),\\ \Xi^{\mu,\nu}(\xi,\eta) &= \iota_\mu\lambda_i'(|\xi-\eta|)\frac{\xi-\eta}{|\xi-\eta|} - \iota_\nu\lambda_i'(|\eta|)\frac{\eta}{|\eta|},\\ \nabla_\xi\Phi^{i;\mu,\nu}(\xi,\eta) &= \lambda_i'(|\xi|)\frac{\xi}{|\xi|} - \iota_\mu\lambda_i'(|\xi-\eta|)\frac{\xi-\eta}{|\xi-\eta|}, \end{aligned} \tag{8.50}$$

where $\iota_\mu, \iota_\nu \in \{+,-\}$. Since $|\lambda_i'(|\xi|) - \lambda_i'(|\xi-\eta|)| \lesssim 2^{k_2}$ (see Lemma A.4(i)), we may assume that $|\Phi^{i;\mu,\nu}(\xi,\eta)| \leq 2^{k-D+1}$ and $|\Xi^{\mu,\nu}(\xi,\eta)| \leq 2^{k-D+1}$, and it suffices to prove that

$$|\angle(\xi, \iota_\mu(\xi-\eta))| \lesssim 2^{k_2}. \tag{8.51}$$

The condition $|\Xi^{\mu,\nu}(\xi,\eta)| \leq 2^{k-D+1} \ll 1$ together with Lemma A.4(i) show that

$$|\angle(\iota_\mu(\xi-\eta), \iota_\nu\eta)| \leq 2^{k-D/2}. \tag{8.52}$$

In particular, using also the identity (8.49),

$$\sin(\angle(\xi-\eta,\eta)) \leq 2^{k-D/2}, \qquad \sin(\angle(\xi,\xi-\eta)) \leq 2^{k_2-D/4}. \tag{8.53}$$

The desired inequality (8.51) follows if $\iota_\mu = +$ and $k_1 - k_2 \geq 10$. Notice also that (8.51) is trivial if $\iota_\mu = -$ and $k_1 - k_2 \geq 10$, since in this case the condition $|\Phi^{i;\mu,\nu}(\xi,\eta)| \leq 2^{k-D+1}$ cannot be satisfied.

It remains to prove (8.51) in the case $0 \leq k_1 - k_2 \leq 10$. If $(\iota_\mu,\iota_\nu) = (+,+)$, then (8.51) follows easily from (8.52) and (8.49), as before. If $(\iota_\mu,\iota_\nu) = (+,-)$, then we use the inequality

$$\Phi^{i;i+,i-}(\xi,\eta) = \lambda_i(|\xi|) + \lambda_i(|\eta|) - \lambda_i(|\xi-\eta|) \geq \lambda_i(|\xi|+|\eta|) - \lambda_i(|\xi-\eta|)$$

(see (8.27)) together with the assumption $|\Phi^{i;i+,i-}(\xi,\eta)| \leq 2^{k-D+1}$ to conclude that $\max(|\xi|,|\eta|) \leq |\xi-\eta|$. Using also (8.53) it follows that the vectors ξ and $\xi-\eta$ are almost aligned and pointing in the same direction, so (8.51) follows from (8.53). The proof in the case $(\iota_\mu,\iota_\nu) = (-,+)$ is similar, and this completes the proof of the lemma. □

Appendix A. General estimates and the functions $\Lambda_i, \Lambda_e, \Lambda_b$

In this section we summarize the linear and the bilinear estimates we use in the paper. We also provide precise descriptions of the eigenvalues $\Lambda_i, \Lambda_e, \Lambda_b$ defined in (3.12).

We note first that Calderón–Zygmund operators are compatible with the spaces constructed in Definition 4.1. More precisely,

LEMMA A.1. *Assume $q \in \mathcal{S}^{10}$ (see (3.23)) and $T_q f := \mathcal{F}^{-1}(q \cdot \widehat{f})$. Then*

$$\|T_q f\|_Z \lesssim \|q\|_{\mathcal{S}^{10}} \|f\|_Z \qquad \text{for any } f \in Z.$$

We omit the proof of this lemma, since it is identical to the proof of Lemma 5.1 in [26]. The following general oscillatory integral estimate is used often. (See Lemma 5.4 in [26] for the simple proof.)

LEMMA A.2. *Assume that $0 < \epsilon \leq 1/\epsilon \leq K$, $N \geq 1$ is an integer, and $f, g \in C^N(\mathbb{R}^n)$. Then*

$$\Big| \int_{\mathbb{R}^n} e^{iKf} g \, dx \Big| \lesssim_N (K\epsilon)^{-N} \Big[\sum_{|\rho| \leq N} \epsilon^{|\rho|} \|D_x^\rho g\|_{L^1} \Big], \tag{A.1}$$

provided that f is real-valued,

$$|\nabla_x f| \geq \mathbf{1}_{\mathrm{supp}\, g}, \qquad \text{and} \qquad \|D_x^\rho f \cdot \mathbf{1}_{\mathrm{supp}\, g}\|_{L^\infty} \lesssim_N \epsilon^{1-|\rho|}, \ 2 \leq |\rho| \leq N. \tag{A.2}$$

We will often use the following simple bilinear estimate. (See, for example, [27, Lemma 5.2] for the proof.)

LEMMA A.3. *Assume $p, q \in [2, \infty]$ satisfy $1/p + 1/q = 1/2$, and $m \in L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$. Then, for any $f, g \in L^2(\mathbb{R}^3)$,*

$$\Big\| \int_{\mathbb{R}^3} m(\xi, \eta) \cdot \widehat{f}(\xi - \eta) \widehat{g}(\eta) \, d\eta \Big\|_{L^2_\xi} \lesssim \|\mathcal{F}^{-1} m\|_{L^1(\mathbb{R}^3 \times \mathbb{R}^3)} \|f\|_{L^p} \|g\|_{L^q}. \tag{A.3}$$

Recall the functions $\Lambda_i, \Lambda_e, \Lambda_b$ defined in (3.12). Let $\lambda_i, \lambda_e, \lambda_b : [0, \infty) \to [0, \infty)$,

$$\begin{aligned}
\lambda_i(r) &:= \varepsilon^{-1/2} \sqrt{\frac{(1+\varepsilon) + (T+\varepsilon) r^2 - \sqrt{\big((1-\varepsilon) + (T-\varepsilon) r^2\big)^2 + 4\varepsilon}}{2}}, \\
\lambda_e(r) &:= \varepsilon^{-1/2} \sqrt{\frac{(1+\varepsilon) + (T+\varepsilon) r^2 + \sqrt{\big((1-\varepsilon) + (T-\varepsilon) r^2\big)^2 + 4\varepsilon}}{2}}, \\
\lambda_b(r) &:= \varepsilon^{-1/2} \sqrt{1 + \varepsilon + C_b r^2},
\end{aligned} \tag{A.4}$$

such that $\Lambda_\sigma(\xi) = \lambda_\sigma(|\xi|)$, $\sigma \in \{i, e, b\}$. We also define

$$c_\sigma = \lim_{r \to +\infty} \lambda'_\sigma(r), \quad c_i = 1, \ c_e = \sqrt{T/\varepsilon}, \ c_b = \sqrt{C_b/\varepsilon}.$$

The graphs in Figure 1 below illustrate the qualitative features of the dispersion relations. The pictures are obtained for the range of parameters $\varepsilon = 0.1$, $T = 1$ and $C_b = 6$ that is outside of the range considered in (1.16) but, as proved in Lemma A.4 below, the qualitative behavior of the functions λ_i, λ_e, λ_b remains similar, except that the functions become more "separated" (and do not fit in a common plot). In Figure 2, we also add a graph showing

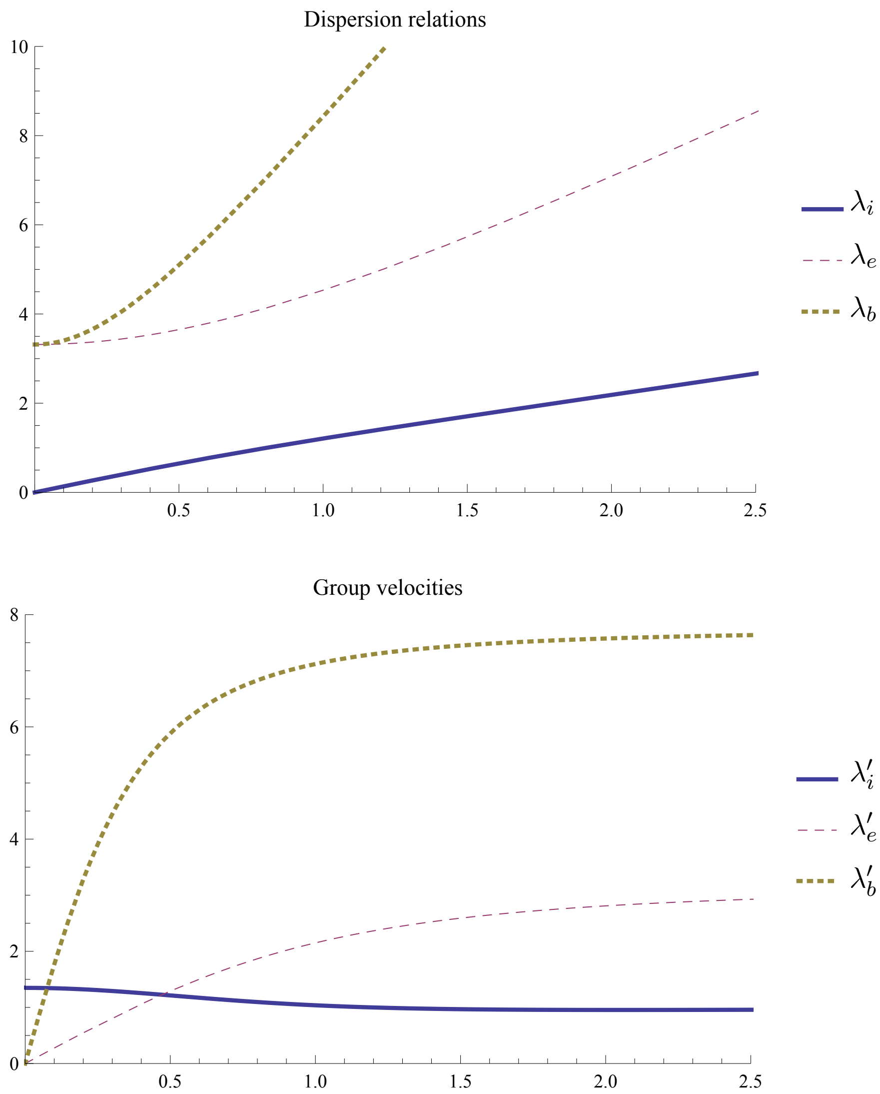

Figure 1. The graphs above show the dispersion relations and group velocities that we are considering. The graphs correspond to the parameters $\varepsilon = 0.1$, $T = 1$ and $C_b = 6$ other values of the parameters lead to qualitatively similar graphs.

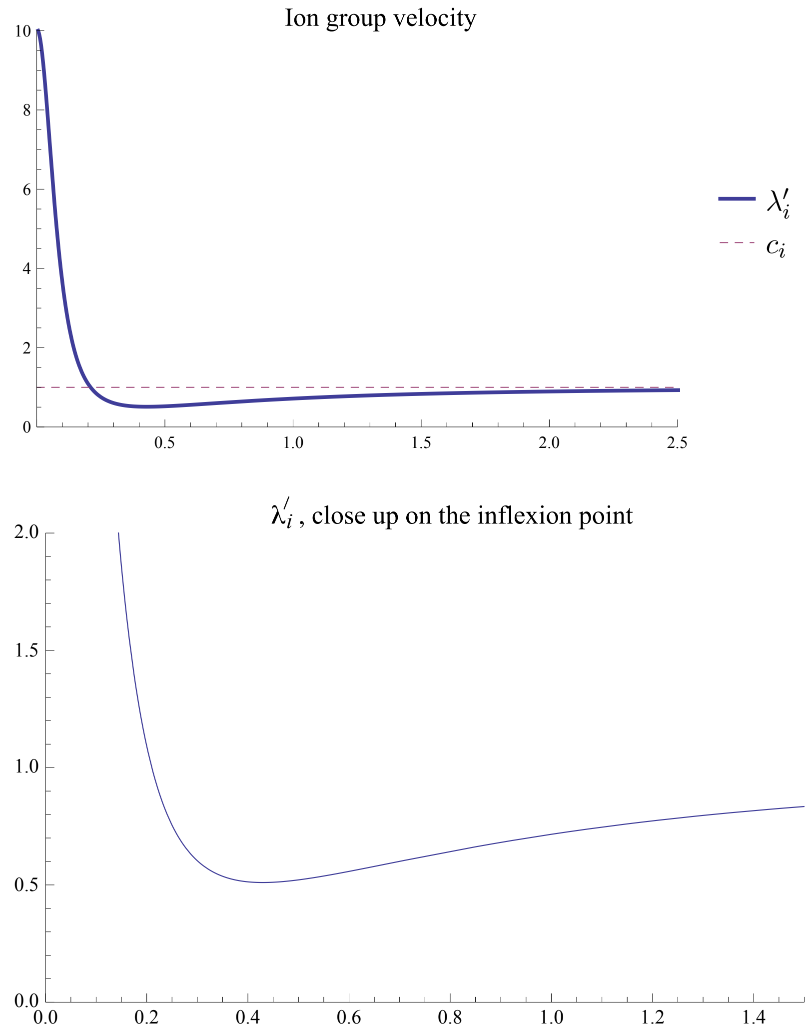

Figure 2. The ion group velocity λ_i' when $\varepsilon = 0.001$, $T = 100$. Note, in particular, the inflection point at r_* and the fact that λ_i' attains its maximum at 0.

λ_i' for an admissible set of parameters ($\varepsilon = 10^{-3}$, $T = 1$, $C_b = 6$), with a zoom on the region around r_* where λ_i has an inflection point.

Lemma A.4. (i) *The functions $\lambda_i, \lambda_e, \lambda_b$ are smooth on $[0, \infty)$ and satisfy*

(A.5)
$$\begin{aligned}
&\lambda_i(0) = 0,\ \ \lambda_i''(0) = 0,\ \ \lambda_i'''(0) \approx_{C_b,\varepsilon} -1,\ \ \lambda_i'(r) \approx_{C_b,\varepsilon} 1 \ \ \textit{for any}\ \ r \in [0,\infty),\\
&\lambda_e(0) = \sqrt{\varepsilon^{-1}+1},\ \ \lambda_e'(0) = 0,\ \ \lambda_e''(r) \approx_{C_b,\varepsilon} (1+r^2)^{-3/2} \ \ \textit{for any}\ \ r \in [0,\infty),\\
&\lambda_b(0) = \sqrt{\varepsilon^{-1}+1},\ \ \lambda_b'(0) = 0,\ \ \lambda_b''(r) \approx_{C_b,\varepsilon} (1+r^2)^{-3/2} \ \ \textit{for any}\ \ r \in [0,\infty).
\end{aligned}$$

In addition, there is a constant $r_* \in (T^{-1/2}, 4T^{-1/2} + 4T^{-1/4})$ *such that*

$$\lambda_i''(r_*) = 0, \quad |\lambda_i''(r)| \approx_{C_{b,\varepsilon}} \min(r, r^{-3}) \text{ if } |r - r_*| \ge 2^{-D}, \\ |\lambda_i'''(r)| \approx_{C_{b,\varepsilon}} 1 \text{ if } |r - r_*| \le 2^{-D/2}. \tag{A.6}$$

Moreover,

$$r \le \lambda_i(r) \le \sqrt{(T+1)(\varepsilon+1)}r, \quad r \ge 0, \\ \max(\lambda_\sigma(0), c_\sigma r) \le \lambda_\sigma(r) \le \lambda_\sigma(0) + c_\sigma r, \quad \sigma \in \{e, b\}, \ r \ge 0. \tag{A.7}$$

(ii) *Letting* $h_\varepsilon(r) := \varepsilon^{-1/2}\sqrt{1 + Tr^2}$, *we have*

$$|D_r^\rho(\lambda_e - h_\varepsilon)(r)| \le \sqrt{\varepsilon}|D_r^\rho h_\varepsilon(r)|, \quad 0 \le \rho \le 2. \tag{A.8}$$

(iii) *We have* $\lambda_i(r) = rq_i(r)$ *for some* $1 \le q_i(r) \le \sqrt{(1+T)/(1+\varepsilon)}$, $q_i(r) \to 1$ *as* $r \to +\infty$ *such that*

$$q_i'(r) \le -\frac{1}{2}\frac{T^2 r}{[1 + T + Tr^2]^2}. \tag{A.9}$$

Moreover,

$$|\lambda_i''(r)| \le 8\sqrt{2}T \qquad \text{and} \qquad \lambda_i''(r) \le 10r^{-3} \text{ for } r \ge 4T^{-1/2} + 4T^{-1/4}. \tag{A.10}$$

Proof of Lemma A.4. (i) Recall the assumptions (1.16), which are used implicitly many times in this lemma. The claims in (A.5) and (A.7) are straightforward consequences of the definitions. To prove (A.6), we use first the formula

$$\sqrt{\varepsilon}\lambda_e(r)\lambda_i(r) = r\left[1 + T + Tr^2\right]^{1/2}$$

to see that one can extend $\lambda_i(r)$ into a smooth odd function of r. Starting from the relation

$$\lambda_i^2(r) = \frac{1}{2\varepsilon}\left[1 + \varepsilon + (T+\varepsilon)r^2 - \sqrt{u^2 + 4\varepsilon}\right], \quad u = 1 - \varepsilon + (T - \varepsilon)r^2, \tag{A.11}$$

and taking up to three derivatives, we find that

$$\begin{aligned} 2\lambda_i(r)\lambda_i'(r) &= \frac{T+\varepsilon}{\varepsilon}r - \frac{T-\varepsilon}{\varepsilon}ru(u^2 + 4\varepsilon)^{-1/2}, \\ 2(\lambda_i'(r))^2 + 2\lambda_i(r)\lambda_i''(r) &= \frac{T+\varepsilon}{\varepsilon} - \frac{T-\varepsilon}{\varepsilon}u(u^2 + 4\varepsilon)^{-1/2} \\ &\quad - 8(T-\varepsilon)^2 r^2 (u^2 + 4\varepsilon)^{-3/2}, \\ 6\lambda_i'(r)\lambda_i''(r) + 2\lambda_i(r)\lambda_i^{(3)}(r) &= -\frac{24(T-\varepsilon)^2 r}{[u^2 + 4\varepsilon]^{\frac{5}{2}}}\left[(1+\varepsilon)^2 - (T-\varepsilon)^2 r^4\right] := A(r). \end{aligned} \tag{A.12}$$

In particular, $\lambda_i' > 0$. Since λ_i is odd, its even derivatives vanish at 0. Dividing by r and letting $r \to 0$ in the first and third lines gives

$$(\lambda_i'(0))^2 = (1+T)/(1+\varepsilon), \quad 8\lambda_i'(0)\lambda_i^{(3)}(0) = -24(T-\varepsilon)^2(1+\varepsilon)^{-3}.$$

Since $\lambda_i'(0) > 0$, we see that $\lambda_i'' < 0$ on some interval $(0,\delta)$. Let $R_A = \sqrt{(1+\varepsilon)/(T-\varepsilon)}$ be the positive root of $A(r)$. We claim that $\lambda_i'' < 0$ on $(0,R_A)$. Indeed, we see from (A.12) that, on this interval, so long as $\lambda_i''(r) \geq A(r)/(12\lambda_i'(r))$, $\lambda_i^{(3)} < 0$ and λ_i'' is decreasing. Hence $\lambda_i''(R_A) \leq 0$. If $\lambda_i''(R_A) = 0$, using (A.12), we see that $\lambda_i^{(3)}(R_A) = 0$ and R_A is a single root for $\lambda_i^{(3)}$ and a double root for λ_i''. Dividing by $r - R_A$ and letting $r \to R_A$, we therefore find that

$$2\lambda_i(R_A)\lambda_i^{(4)}(R_A) = \lim_{r\to R_A} \frac{A(r)}{r-R_A} > 0.$$

But then $\lambda_i''(r) > 0$ for some $r < R_A$, a contradiction.

It is clear that the argument above can be made quantitative, and we prove that

> for any $\delta > 0$, there is $\delta' = \delta'(\delta,T,\varepsilon) > 0$ such that $\lambda_i''(r) \leq -\delta'$ for any $r \in [\delta, R_A]$.

This suffices to prove the desired claim (A.6) for $r \in [0,R_A]$.

We now claim that λ_i'' vanishes exactly once on $(R_A,+\infty)$. Indeed, using again (A.12), we see that if $\lambda_i''(r_*) = 0$, then

$$\lambda_i^{(3)}(r_*) = A(r_*)/(2\lambda_i(r_*)) > 0.$$

Let r_{**} be the next zero of λ_i''. Since $\lambda_i'' \geq 0$ on (r_*,r_{**}), we have that $\lambda_i^{(3)}(r_{**}) \leq 0$. Plugging $r = r_{**}$ in the third line of (A.12) gives a contradiction. Finally, we remark that there exists such r_* since we will show below that $\lambda_i'' > 0$ for r large enough.

Indeed, using the second equation in (A.12),

(A.13)
$$\lambda_i(r)\lambda_i''(r) = 1-(\lambda_i'(r))^2 + \frac{T-\varepsilon}{2\varepsilon}[1-u(u^2+4\varepsilon)^{-1/2}] - 4(T-\varepsilon)^2r^2(u^2+4\varepsilon)^{-3/2}.$$

Therefore, using (A.11) and (A.12) and letting $v := (1-u(u^2+4\varepsilon)^{-1/2})/(2\varepsilon)$,

$$\begin{aligned}(\lambda_i(r))^3\lambda_i''(r) &= (\lambda_i(r))^2 - (\lambda_i(r))^2(\lambda_i'(r))^2 \\ &\quad + (\lambda_i(r))^2(T-\varepsilon)v - 4(\lambda_i(r))^2(T-\varepsilon)^2r^2(u^2+4\varepsilon)^{-3/2} \\ &= r^2 + 1 - v(u^2+4\varepsilon)^{1/2} - r^2(1+(T-\varepsilon)v)^2 \\ &\quad + (\lambda_i(r))^2(T-\varepsilon)v - 4(\lambda_i(r))^2(T-\varepsilon)^2r^2(u^2+4\varepsilon)^{-3/2}.\end{aligned}$$

Notice that $v \le u^{-2} \le (T-\varepsilon)^{-2}r^{-4}$, $(u^2+4\varepsilon)^{1/2} \le (T-\varepsilon)r^2+1+\varepsilon$, and $(\lambda_i(r))^2 \in [r^2, r^2+1]$. Therefore, if $(T-\varepsilon)r^2 \ge 10(1+\sqrt{T})$, then

$$(\lambda_i(r))^3\lambda_i''(r) \in [1/10, 1], \tag{A.14}$$

and the desired conclusion (A.6) follows.

(ii) We calculate

$$h_\varepsilon'(r) = \varepsilon^{-1/2}Tr(1+Tr^2)^{-1/2}, \qquad h_\varepsilon''(r) = \varepsilon^{-1/2}T(1+Tr^2)^{-3/2}.$$

Start from the formula

$$\lambda_e^2(r) = \frac{1}{\varepsilon}\left[1+Tr^2+\frac{\sqrt{u^2+4\varepsilon}-u}{2}\right]$$

where, as before, $u = 1-\varepsilon+(T-\varepsilon)r^2$. Therefore, $\lambda_e(r) \ge h_\varepsilon(r)$ and

$$\lambda_e(r) - h_\varepsilon(r) = \frac{\lambda_e^2(r)-h_\varepsilon^2(r)}{\lambda_e(r)+h_\varepsilon(r)} \le \frac{\sqrt{u^2+4\varepsilon}-u}{4\varepsilon h_\varepsilon(r)} \le \frac{1}{2uh_\varepsilon(r)}. \tag{A.15}$$

The desired bound (A.8) follows for $\rho = 0$.

Using the formulas above, we also calculate

$$2\lambda_e(r)\lambda_e'(r) = \frac{2Tr}{\varepsilon} - \frac{(T-\varepsilon)r}{\varepsilon}\frac{\sqrt{u^2+4\varepsilon}-u}{\sqrt{u^2+4\varepsilon}}.$$

Therefore, $\lambda_e'(r) \le h_\varepsilon'(r)$ and, using also (A.15),

(A.16)

$$h_\varepsilon'(r) - \lambda_e'(r) = \frac{2h_\varepsilon(r)h_\varepsilon'(r) - 2\lambda_e(r)\lambda_e'(r) + 2h_\varepsilon'(r)(\lambda_e(r)-h_\varepsilon(r))}{2\lambda_e(r)} \le \frac{2Tr}{u^2\lambda_e}.$$

The desired bound (A.8) follows for $\rho = 1$.

Finally, we calculate

$$2\lambda_e(r)\lambda_e''(r) + 2(\lambda_e'(r))^2 = \frac{2T}{\varepsilon} + E,$$

where

$$E := -\frac{(T-\varepsilon)}{\varepsilon}\frac{\sqrt{u^2+4\varepsilon}-u}{\sqrt{u^2+4\varepsilon}} + \frac{8(T-\varepsilon)^2r^2}{(u^2+4\varepsilon)^{3/2}}.$$

Therefore, using also (A.15) and (A.16),

$$\begin{aligned}|\lambda_e''(r) - h_\varepsilon''(r)| &\le \frac{|E| + 2|\lambda_e(r)-h_\varepsilon(r)|h_\varepsilon''(r) + 2|(h_\varepsilon'(r))^2 - (\lambda_e'(r))^2|}{2\lambda_e} \\ &\le \frac{20(T+1)}{u^2\lambda_e}.\end{aligned}$$

The desired bound (A.8) follows for $\rho = 2$.

(iii) Starting from the formula

$$\sqrt{\varepsilon}\lambda_e(r)\lambda_i(r) = r\sqrt{1+T+Tr^2},$$

we calculate

$$\text{(A.17)}\qquad q_i(r)=\frac{\sqrt{1+T+Tr^2}}{\sqrt{\varepsilon}\lambda_e}=\Big[1-\frac{T}{1+T+Tr^2}+\frac{\sqrt{u^2+4\varepsilon}-u}{2(1+T+Tr^2)}\Big]^{-1/2}$$

and, with $v=(1-u(u^2+4\varepsilon)^{-1/2})/(2\varepsilon)$ as before,

$$\begin{aligned}q_i'(r)=&-\Big[1-\frac{T}{1+T+Tr^2}+\frac{\sqrt{u^2+4\varepsilon}-u}{2(1+T+Tr^2)}\Big]^{-3/2}\\&\cdot\Big[\frac{T^2r-Tr\varepsilon v\sqrt{u^2+4\varepsilon}-(1+T+Tr^2)(T-\varepsilon)r\varepsilon v}{(1+T+Tr^2)^2}\Big].\end{aligned}$$

This suffices to prove (A.9).

The second bound in (A.10) follows from (A.14). To prove the first bound in (A.10), we notice that it follows from part (i) that there are two values $r_{\min}\in(0,r_*)$ and $r_{\max}\in(r_*,\infty)$ such that

$$\lambda_i''(r)\in[\lambda_i''(r_{\min}),\lambda_i''(r_{\max})]\qquad\text{for any } r\in[0,\infty).$$

Using the identity in the second line of (A.12), it follows that $\lambda_i(r)\lambda_i''(r)\leq 1$ for any $r\geq\sqrt{(1+\varepsilon)/(T-\varepsilon)}$. Since $r_{\max}\geq r_*\geq\sqrt{(1+\varepsilon)/(T-\varepsilon)}$, it follows that

$$\text{(A.18)}\qquad \lambda_i''(r_{\max})\leq 1/\lambda_i(r_{\max})\leq 1/r_{\max}\leq\sqrt{T}.$$

To estimate $|\lambda_i''(r_{\min})|$, we use (A.13) and the observation $|\lambda_i'(r)|\leq q_i(r)\leq\sqrt{(1+T)/(1+\varepsilon)}$ to write

$$\begin{aligned}\lambda_i(r)\lambda_i''(r)&\geq 1-\frac{1+T}{1+\varepsilon}+\frac{T-\varepsilon}{2\varepsilon}[1-u(u^2+4\varepsilon)^{-1/2}]\\&\quad-4(T-\varepsilon)^2r^2(u^2+4\varepsilon)^{-3/2}\\&\geq\frac{T-\varepsilon}{1+\varepsilon}\Big[-1+\frac{1+\varepsilon}{2\varepsilon}(1-u(u^2+4\varepsilon)^{-1/2})\Big]-4(T-\varepsilon)^{3/2}r\\&\geq-8(T-\varepsilon)^{3/2}r.\end{aligned}$$

Moreover, since $\lambda_i^{(3)}(r_{\min})=0$ and $\lambda_i''(r_{\min})\leq 0$, it follows from the identity in the last line of (A.12) that $r_{\min}\leq R_A=\sqrt{(1+\varepsilon)/(T-\varepsilon)}$. Therefore, using also the fact that q_i is decreasing on $[0,\infty)$ (see (A.9)) and the identity (A.17), it follows that

$$-\lambda_i''(r_{\min})\leq\frac{8(T-\varepsilon)^{3/2}r_{\min}}{\lambda_i(r_{\min})}\leq\frac{8(T-\varepsilon)^{3/2}}{q_i(R_A)}\leq 8\sqrt{2}(T-\varepsilon).$$

The desired estimate in (A.10) follows using also (A.18). □

LEMMA A.5. *Assume* $\|f\|_Z\leq 1$, $t\in\mathbb{R}$, $(k,j)\in\mathcal{J}$, *and let* $\widetilde{k}=\min(k,0)$ *and*

$$f_{k,j}:=P_{[k-2,k+2]}[\widetilde{\varphi}_j^{(k)}\cdot P_kf].$$

(i) *Then*

$$\|f_{k,j}\|_{L^2} \lesssim (2^{\alpha k} + 2^{10k})^{-1} \cdot 2^{2\beta\widetilde{k}} 2^{-(1-\beta)j} \tag{A.19}$$

and

$$\sup_{\xi\in\mathbb{R}^3} \left| D^\rho_\xi \widehat{f_{k,j}}(\xi) \right| \lesssim_{|\rho|} (2^{\alpha k} + 2^{10k})^{-1} \cdot 2^{-(1/2-\beta)\widetilde{k}} 2^{|\rho| j}. \tag{A.20}$$

Moreover, if $k \leq 0$ *and* $\sigma \in \{e, b\}$, *then*

$$\left\| e^{it\Lambda_\sigma} f_{k,j} \right\|_{L^\infty} \lesssim 2^{(3/2-\alpha)k} 2^{-(1+\beta)j} (1 + 2^{-j}|t|2^k)^{-3/2+10\beta}. \tag{A.21}$$

If $k \geq 0$ *and* $\sigma \in \{e, b\}$, *then*

$$\left\| e^{it\Lambda_\sigma} f_{k,j} \right\|_{L^\infty} \lesssim 2^{-6k} 2^{-(1+\beta)j} (1 + 2^{-j}|t|)^{-3/2+10\beta}. \tag{A.22}$$

With r_* *defined as in* (A.6), *let* $k_* := \log_2 r_*$. *If* $k \leq k_* - 3$ *and* $\sigma = i$, *then*

$$\left\| e^{it\Lambda_\sigma} f_{k,j} \right\|_{L^\infty} \lesssim 2^{(3/2-\alpha)k} 2^{-(1+\beta)j} (1 + 2^{-j}|t|2^{2k/3})^{-3/2+10\beta}. \tag{A.23}$$

If $k \in [k_* - 3, k_* + 3]$ *and* $\sigma = i$, *then*

$$\left\| e^{it\Lambda_\sigma} f_{k,j} \right\|_{L^\infty} \lesssim 2^{-(1+\beta)j} (1 + 2^{-j}|t|)^{-5/4+10\beta}. \tag{A.24}$$

If $k \geq k_* + 3$ *and* $\sigma = i$, *then*

$$\left\| e^{it\Lambda_\sigma} f_{k,j} \right\|_{L^\infty} \lesssim 2^{-6k} 2^{-(1+\beta)j} (1 + 2^{-j}|t|)^{-3/2+10\beta}. \tag{A.25}$$

(ii) *As a consequence,*

$$\sum_{j\geq \max(-k,0)} \|f_{k,j}\|_{L^2} \lesssim \min(2^{(1+\beta-\alpha)k}, 2^{-10k}) \tag{A.26}$$

and,[11] *for any* $\sigma \in \{i, e, b\}$,

$$\sum_{j\geq \max(-k,0)} \left\| e^{it\Lambda_\sigma} f_{k,j} \right\|_{L^\infty} \lesssim \min(2^{(1/2-\beta-\alpha)k}, 2^{-6k})(1 + |t|)^{-1-\beta}. \tag{A.27}$$

Proof of Lemma A.5. We start by decomposing, as in (4.6)–(4.8),

$$\begin{gathered} \widetilde{\varphi}^{(k)}_j \cdot P_k f = (2^{\alpha k} + 2^{10k})^{-1} (g_{1,j} + g_{2,j}), \\ g_{1,j} = g_{1,j} \cdot \widetilde{\varphi}^{(k)}_{[j-2,j+2]}, \qquad g_{2,j} = g_{2,j} \cdot \widetilde{\varphi}^{(k)}_{[j-2,j+2]}, \end{gathered} \tag{A.28}$$

such that

$$2^{(1+\beta)j} \|g_{1,j}\|_{L^2} + 2^{(1/2-\beta)\widetilde{k}} \|\widehat{g_{1,j}}\|_{L^\infty} \lesssim 1, \tag{A.29}$$

[11]In many places we will be able to use the simpler bound (A.27), instead of the more precise bounds (A.21)–(A.25).

and

(A.30)
$$2^{(1-\beta)j}\|g_{2,j}\|_{L^2}+\|\widehat{g_{2,j}}\|_{L^\infty}+2^{\gamma j}\sup_{R\in[2^{-j},2^k],\,\xi_0\in\mathbb{R}^3}R^{-2}\|\widehat{g_{2,j}}\|_{L^1(B(\xi_0,R))}\lesssim 2^{-10|k|}.$$

The bound (A.19) follows easily.

To prove (A.20) we use the formulas in (A.28) to write, for $\mu=1,2$,

$$\widehat{g_{\mu,j}}(\xi)=c\int_{\mathbb{R}^3}\widehat{g_{\mu,j}}(\eta)\mathcal{F}(\widetilde{\varphi}^{(k)}_{[j-2,j+2]})(\xi-\eta)\,d\eta.$$

Therefore,

$$D^\rho_\xi\widehat{g_{\mu,j}}(\xi)=c\int_{\mathbb{R}^3}\widehat{g_{\mu,j}}(\eta)\mathcal{F}(x^\rho\cdot\widetilde{\varphi}^{(k)}_{[j-2,j+2]})(\xi-\eta)\,d\eta. \tag{A.31}$$

The desired bounds (A.20) follow using the bounds $\|\widehat{g_{\mu,j}}\|_{L^\infty}\lesssim 2^{-(1/2-\beta)\widetilde{k}}$; see (A.29)–(A.30).

We consider now the L^∞ bounds (A.21)–(A.25). Using (A.28)–(A.30), we have

$$\|\widehat{f_{k,j}}\|_{L^1(B(\xi_0,R))}\lesssim(2^{\alpha k}+2^{10k})^{-1}\min(R^3 2^{-(1/2-\beta)\widetilde{k}},R^{3/2}2^{-(1+\beta)j})$$

for any $\xi_0\in\mathbb{R}^3$ and $R\le 2^k$. Therefore, for any $k\in\mathbb{Z}$ and $\sigma\in\{i,e,b\}$,

$$\left\|e^{it\Lambda_\sigma}f_{k,j}\right\|_{L^\infty}\lesssim(2^{\alpha k}+2^{10k})^{-1}\cdot 2^{3k/2}2^{-(1+\beta)j}. \tag{A.32}$$

Step 1. We consider first the simplest case

$$\sigma\in\{e,b\},\qquad k\le 0,\qquad |t|\ge 2^{j-k+D} \tag{A.33}$$

and prove that, for any $x\in\mathbb{R}^3$,

(A.34)
$$\left|\int_{\mathbb{R}^3}e^{ix\cdot\xi}e^{it\Lambda_\sigma(\xi)}\widehat{f_{k,j}}(\xi)\,d\xi\right|\lesssim 2^{(3/2-\alpha)k}2^{-(1+\beta)j}\rho_1^{3/2-10\beta},\qquad \rho_1:=2^{j-k}|t|^{-1}.$$

The bound (A.21) would clearly follow from (A.32) and (A.34). Using the decomposition (A.28), it suffices to prove that, for $\mu\in\{1,2\}$,

$$\begin{gathered}\left\|g_{\mu,j}*K^\sigma_{k,t}\right\|_{L^\infty}\lesssim 2^{3k/2}2^{-(1+\beta)j}\rho_1^{3/2-10\beta},\\ K^\sigma_{k,t}(x):=\int_{\mathbb{R}^3}e^{ix\cdot\xi}e^{it\Lambda_\sigma(\xi)}\varphi_{[k-2,k+2]}(\xi)\,d\xi.\end{gathered} \tag{A.35}$$

Recall that the kernel of the operator on $\mathbb{R}^3$ defined by the radial multiplier $\xi\to p(|\xi|)$ is

$$K(x)=c\int_0^\infty p(s)s^2\frac{e^{is|x|}-e^{-is|x|}}{s|x|}\,ds. \tag{A.36}$$

We show that

$$\|K^\sigma_{k,t}\|_{L^\infty}\lesssim|t|^{-3/2}. \tag{A.37}$$

In view of (A.36) it suffices to prove that

$$\text{(A.38)} \qquad \Big|\int_0^\infty s^2\varphi^1_{[k-2,k+2]}(s)e^{it\lambda_\sigma(s)}\frac{e^{isr}-e^{-isr}}{sr}\,ds\Big|\lesssim |t|^{-3/2}$$

for any $r\in(0,\infty)$. Recall the assumption (A.33); in particular, $|t|\geq 2^{D-2k}$. Since $\lambda'_\sigma(s)\approx\min(s,1)$ (see (A.5)), the bound (A.38) follows by integration by parts unless $r\approx|t|2^k$. On the other hand, if $r\approx|t|2^k$ then the bound (A.38) follows by stationary phase, using $\lambda''_\sigma(s)\approx(1+s^2)^{-3/2}$; see (A.5).

In view of (A.37) and the assumptions (A.28)–(A.30), it follows that

$$\big\|g_{1,j}*K^\sigma_{k,t}\big\|_{L^\infty}\lesssim\|g_{1,j}\|_{L^1}\|K^\sigma_{k,t}\|_{L^\infty}\lesssim 2^{3j/2}2^{-(1+\beta)j}|t|^{-3/2}\lesssim 2^{3k/2}2^{-(1+\beta)j}\rho_1^{3/2}$$

and

$$\begin{aligned}\big\|g_{2,j}*K^\sigma_{k,t}\big\|_{L^\infty}&\lesssim\|g_{2,j}\|_{L^1}\|K^\sigma_{k,t}\|_{L^\infty}\\&\lesssim 2^{3j/2}2^{-(1-\beta)j}2^{2\beta k}|t|^{-3/2}\lesssim 2^{3k/2}2^{-(1+\beta)j}\rho_1^{3/2}2^{2\beta(j+k)}.\end{aligned}$$

The bounds (A.35) follow if $\mu=1$ or if $\mu=2$ and $2^{j+k}\leq\rho_1^{-5}$. On the other hand, if $\mu=2$ and $2^{j+k}\geq\rho_1^{-5}$ then, using the L^1 bounds on $\widehat{g_{2,j}}$ in (A.30),

$$\begin{aligned}\big\|g_{2,j}*K^\sigma_{k,t}\big\|_{L^\infty}\lesssim\big\|\widehat{g_{2,j}}\big\|_{L^1}&\lesssim 2^{3k/2}2^{-(1+\beta)j}2^{-(\gamma-\beta-1)(j+k)}\\&\lesssim 2^{3k/2}2^{-(1+\beta)j}\rho_1^{5(\gamma-\beta-1)},\end{aligned}$$

which suffices to prove (A.35) in this case as well.

Step 2. We consider now the case

$$\text{(A.39)} \qquad \sigma\in\{e,b\},\qquad k\geq 0,\qquad |t|\geq 2^{j+k+D}$$

and prove that, for any $x\in\mathbb{R}^3$,

$$\text{(A.40)} \quad \Big|\int_{\mathbb{R}^3}e^{ix\cdot\xi}e^{it\Lambda_\sigma(\xi)}\widehat{f_{k,j}}(\xi)\,d\xi\Big|\lesssim 2^{-6k}2^{-(1+\beta)j}\rho_2^{3/2-10\beta},\qquad \rho_2:=2^j|t|^{-1}.$$

The bound (A.22) would clearly follow from (A.32) and (A.40). Using the decomposition (A.28), it suffices to prove that, for $\mu\in\{1,2\}$,

$$\text{(A.41)} \qquad \big\|g_{\mu,j}*K^\sigma_{k,t}\big\|_{L^\infty}\lesssim 2^{4k}2^{-(1+\beta)j}\rho_2^{3/2-10\beta},$$

where $K^\sigma_{k,t}$ is defined as in (A.35).

As before, we show that

$$\text{(A.42)} \qquad \|K^\sigma_{k,t}\|_{L^\infty}\lesssim|t|^{-3/2}2^{3k}.$$

In view of (A.36), it suffices to prove that

$$\text{(A.43)} \qquad \Big|\int_0^\infty s^2\varphi^1_{[k-2,k+2]}(s)e^{it\lambda_\sigma(s)}\frac{e^{isr}-e^{-isr}}{sr}\,ds\Big|\lesssim|t|^{-3/2}2^{3k}$$

for any $r\in(0,\infty)$. Recall the assumption (A.39), in particular, $|t|\geq 2^{D+k}$. Since $\lambda'_\sigma(s)\approx\min(s,1)$ (see (A.5)), the bound (A.43) follows by integration by parts unless $r \approx |t|$. On the other hand, if $r \approx |t|$, then the bound (A.43) follows by stationary phase, using $\lambda''_\sigma(s) \approx (1+s^2)^{-3/2}$; see (A.5).

In view of (A.42) and the assumptions (A.28)–(A.30), it follows that

$$\left\|g_{1,j} * K^\sigma_{k,t}\right\|_{L^\infty} \lesssim \|g_{1,j}\|_{L^1}\|K^\sigma_{k,t}\|_{L^\infty} \lesssim 2^{3j/2}2^{-(1+\beta)j}2^{3k}|t|^{-3/2} \lesssim 2^{3k}2^{-(1+\beta)j}\rho_2^{3/2}$$

and

$$\begin{aligned}\left\|g_{2,j} * K^\sigma_{k,t}\right\|_{L^\infty} \lesssim \|g_{2,j}\|_{L^1}\|K^\sigma_{k,t}\|_{L^\infty} &\lesssim 2^{3j/2}2^{-(1-\beta)j}2^{3k}|t|^{-3/2}\\ &\lesssim 2^{3k}2^{-(1+\beta)j}\rho_2^{3/2}2^{2\beta j}.\end{aligned}$$

The bounds (A.41) follow if $\mu = 1$ or if $\mu = 2$ and $2^j \leq \rho_2^{-5}$. On the other hand, if $\mu = 2$ and $2^j \geq \rho_2^{-5}$ then, using the L^1 bounds on $\widehat{g_{2,j}}$ in (A.30),

$$\left\|g_{2,j} * K^\sigma_{k,t}\right\|_{L^\infty} \lesssim \left\|\widehat{g_{2,j}}\right\|_{L^1} \lesssim 2^{-(1+\beta)j}2^{-(\gamma-\beta-1)j} \lesssim 2^{-(1+\beta)j}\rho_2^{5(\gamma-\beta-1)},$$

which suffices to prove (A.41) in this case as well.

An identical argument shows that

$$\text{if} \quad k \geq k_* + 3, \qquad |t| \geq 2^{j+k+D} \tag{A.44}$$

then, for any $x \in \mathbb{R}^3$,

$$\left|\int_{\mathbb{R}^3} e^{ix\cdot\xi}e^{it\Lambda_i(\xi)}\widehat{f_{k,j}}(\xi)\,d\xi\right| \lesssim 2^{-6k}2^{-(1+\beta)j}(2^j/|t|)^{3/2-10\beta}. \tag{A.45}$$

The bound (A.25) clearly follows from (A.32) and (A.45).

Step 3. We consider now the case

$$k \leq k_* - 3, \qquad |t| \geq 2^{j-2k/3+D} \tag{A.46}$$

and prove that, for any $x \in \mathbb{R}^3$,

$$\left|\int_{\mathbb{R}^3} e^{ix\cdot\xi}e^{it\Lambda_i(\xi)}\widehat{f_{k,j}}(\xi)\,d\xi\right| \lesssim 2^{(3/2-\alpha)k}2^{-(1+\beta)j}\rho_3^{3/2-10\beta}, \qquad \rho_3 := 2^{j-2k/3}|t|^{-1}. \tag{A.47}$$

The bound (A.23) would clearly follow from (A.32) and (A.47). Using the decomposition (A.28), it suffices to prove that, for $\mu \in \{1,2\}$,

$$\begin{aligned}&\left\|g_{\mu,j} * K^i_{k,t}\right\|_{L^\infty} \lesssim 2^{3k/2}2^{-(1+\beta)j}\rho_3^{3/2-10\beta},\\ &K^i_{k,t}(x) := \int_{\mathbb{R}^3} e^{ix\cdot\xi}e^{it\Lambda_i(\xi)}\varphi_{[k-2,k+2]}(\xi)\,d\xi.\end{aligned} \tag{A.48}$$

As before, we show that

$$\|K^i_{k,t}\|_{L^\infty} \lesssim 2^{k/2}|t|^{-3/2}. \tag{A.49}$$

In view of (A.36), it suffices to prove that

$$\left|\int_0^\infty s^2\varphi^1_{[k-2,k+2]}(s)e^{it\lambda_i(s)}\frac{e^{isr}-e^{-isr}}{sr}\,ds\right| \lesssim 2^{k/2}|t|^{-3/2} \tag{A.50}$$

for any $r \in (0,\infty)$. Recall the assumption (A.33); in particular, $|t| \geq 2^{D-5k/3}$. Since $\lambda_i'(s) \approx 1$ (see (A.5)), the bound (A.50) follows by integration by parts unless $r \approx |t|$. On the other hand, if $r \approx |t|$, then the bound (A.50) follows by stationary phase, using $\lambda_\sigma''(s) \approx s$; see (A.6).

As before, we can now prove (A.48). Using (A.37) and (A.28)–(A.30), it follows that

$$\left\| g_{1,j} * K^i_{k,t} \right\|_{L^\infty} \lesssim \|g_{1,j}\|_{L^1} \|K^i_{k,t}\|_{L^\infty} \lesssim 2^{3j/2} 2^{-(1+\beta)j} 2^{k/2} |t|^{-3/2} \\ \lesssim 2^{3k/2} 2^{-(1+\beta)j} \rho_3^{3/2}$$

and

$$\left\| g_{2,j} * K^i_{k,t} \right\|_{L^\infty} \lesssim \|g_{2,j}\|_{L^1} \|K^i_{k,t}\|_{L^\infty} \lesssim 2^{3j/2} 2^{-(1-\beta)j} 2^{2\beta k} 2^{k/2} |t|^{-3/2} \\ \lesssim 2^{3k/2} 2^{-(1+\beta)j} \rho_3^{3/2} 2^{2\beta(j+k)}.$$

The bounds (A.48) follow if $\mu = 1$ or if $\mu = 2$ and $2^{j+k} \leq \rho_3^{-5}$. On the other hand, if $\mu = 2$ and $2^{j+k} \geq \rho_3^{-5}$ then, using the L^1 bounds on $\widehat{g_{2,j}}$ in (A.30),

$$\left\| g_{2,j} * K^i_{k,t} \right\|_{L^\infty} \lesssim \left\| \widehat{g_{2,j}} \right\|_{L^1} \lesssim 2^{3k/2} 2^{-(1+\beta)j} 2^{-(\gamma-\beta-1)(j+k)} \\ \lesssim 2^{3k/2} 2^{-(1+\beta)j} \rho_3^{5(\gamma-\beta-1)},$$

which suffices to prove (A.48) in this case as well.

Step 4. Finally, we consider the case

$$k \in [k_* - 3, k_* + 3], \qquad |t| \geq 2^{j+4D} \tag{A.51}$$

and prove that, for any $x \in \mathbb{R}^3$,

$$\left| \int_{\mathbb{R}^3} e^{ix\cdot\xi} e^{it\Lambda_i(\xi)} \widehat{f_{k,j}}(\xi)\, d\xi \right| \lesssim 2^{-(1+\beta)j} \rho_2^{5/4-10\beta}, \qquad \rho_2 = 2^j |t|^{-1}. \tag{A.52}$$

The bound (A.24) would clearly follow from (A.32) and (A.52).

Using the decomposition (A.28)–(A.30), it suffices to prove that, for $\mu \in \{1,2\}$ and $x \in \mathbb{R}^3$,

$$\left| \int_{\mathbb{R}^3} e^{ix\cdot\xi} e^{it\Lambda_i(\xi)} \varphi_{[k-2,k+2]}(\xi) \big[1 - \varphi[(|\xi| - r_*)/\rho_2^{1/2}] \big] \widehat{g_{\mu,j}}(\xi)\, d\xi \right| \lesssim 2^{-(1+\beta)j} \rho_2^{5/4-10\beta} \tag{A.53}$$

and

$$\left| \int_{\mathbb{R}^3} e^{ix\cdot\xi} e^{it\Lambda_i(\xi)} \varphi[(|\xi| - r_*)/\rho_2^{1/2}] \widehat{g_{\mu,j}}(\xi)\, d\xi \right| \lesssim 2^{-(1+\beta)j} \rho_2^{5/4-10\beta}. \tag{A.54}$$

Letting

$$K_{k,t;\delta}(x) := \int_{\mathbb{R}^3} e^{ix\cdot\xi} e^{it\Lambda_i(\xi)} \varphi_{[k-2,k+2]}(\xi) \big[1 - \varphi[(|\xi| - r_*)/\delta] \big]\, d\xi$$

and arguing as in the proof of (A.49), it is easy to see that

$$\|K_{k,t;\delta}\|_{L^\infty} \lesssim |t|^{-3/2} \delta^{-1/2} \tag{A.55}$$

provided that $\delta \in [|t|^{-1/2}, 2^{-D}]$. As before, this suffices to prove the bounds (A.53).

To prove (A.54), we may assume, without loss of generality, that $x/t = (-z_1, 0, 0)$ for some $z_1 \in [0, \infty)$. The formula (A.31) together with the bounds in (A.29) and (A.30) show that

(A.56)
$$\begin{aligned}
&2^{(1+\beta)j}\|D^\rho_\xi \widehat{g_{1,j}}(\xi)\|_{L^2} + \|D^\rho_\xi \widehat{g_{1,j}}(\xi)\|_{L^\infty} \lesssim 2^{|\rho|j},\\
&2^{(1-\beta)j}\|D^\rho_\xi \widehat{g_{2,j}}(\xi)\|_{L^2} + \|D^\rho_\xi \widehat{g_{2,j}}(\xi)\|_{L^\infty}\\
&\qquad\qquad + 2^{\gamma j}\sup_{R\in[2^{-j},1],\,\xi_0\in\mathbb{R}^3} R^{-2}\|D^\rho_\xi \widehat{g_{2,j}}(\xi)\|_{L^1(B(\xi_0,R))} \lesssim 2^{|\rho|j}.
\end{aligned}$$

With $\xi = (\xi_1, \xi_2, \xi_3) = (\xi_1, \xi')$ and $l \in \mathbb{Z} \cap (-\infty, D/10]$, we define

$$\begin{aligned}
I^\mu_{\le l} &:= \int_{\mathbb{R}^3} \varphi(|\xi'|/2^l) e^{it(\Lambda_i(\xi) - z_1\xi_1)} \varphi[(|\xi| - r_*)/\rho_2^{1/2}] \widehat{g_{\mu,j}}(\xi)\, d\xi,\\
I^\mu_l &:= I^\mu_{\le l} - I^\mu_{\le l-1}.
\end{aligned}$$

We fix $l_0 \in \mathbb{Z}$ such that

$$2^{l_0} \le \rho_2 + |t|^{-1/2} \le 2^{l_0+1}.$$

We use (A.56) with $|\rho| = 0$ to estimate, if $2^j \ge |t|^{1/2}$,

$$\begin{aligned}
|I^1_{\le l_0}| &\lesssim 2^{l_0}\rho_2^{1/4}\|\widehat{g_{1,j}}\|_{L^2} \lesssim 2^{-(1+\beta)j}2^{l_0}\rho_2^{1/4},\\
|I^2_{\le l_0}| &\lesssim \frac{\rho_2^{1/2}}{2^{l_0}} \cdot 2^{2l_0}2^{-\gamma j} \lesssim 2^{-(1+\beta)j}2^{l_0}\rho_2^{1/2}.
\end{aligned}$$

On the other hand, if $2^j \le |t|^{1/2}$, then

$$|I^1_{\le l_0}| + |I^2_{\le l_0}| \lesssim 2^{2l_0}\rho_2^{1/2}(\|\widehat{g_{1,j}}\|_{L^\infty} + \|\widehat{g_{2,j}}\|_{L^\infty}) \lesssim 2^{2l_0}\rho_2^{1/2}.$$

Therefore, in both cases,

$$|I^1_{\le l_0}| + |I^2_{\le l_0}| \lesssim 2^{-(1+\beta)j}\rho_2^{5/4}. \tag{A.57}$$

To estimate $|I^\mu_l|$ for $l \ge l_0+1$, we integrate by parts in ξ', using Lemma A.2 with $K \approx |t|2^l$ and $\epsilon^{-1} \approx 2^j + 2^{-l} + 2^l\rho_2^{-1/2}$. Arguing as before, we estimate, if $2^j = \max(2^j, 2^{-l}, 2^l\rho_2^{-1/2})$,

$$\begin{aligned}
|I^1_l| &\lesssim 2^{2j}/(|t|^2 2^{2l}) \cdot 2^l\rho_2^{1/4}2^{-(1+\beta)j} \lesssim \rho_2^{5/4}2^{-(1+\beta)j} \cdot \rho_2 2^{-l},\\
|I^2_l| &\lesssim 2^{2j}/(|t|^2 2^{2l}) \cdot 2^l(2^l + \rho_2^{1/2})2^{-\gamma j} \lesssim 2^{-\gamma j}\rho_2^2 + \rho_2^{3/2}2^{-(1+\beta)j} \cdot \rho_2 2^{-l}.
\end{aligned}$$

On the other hand, if $2^{-l} = \max(2^j, 2^{-l}, 2^l\rho_2^{-1/2})$, then

$$|I^1_l| + |I^2_l| \lesssim 2^{-2l}/(|t|^2 2^{2l}) \cdot 2^{2l}\rho_2^{1/2} \lesssim 2^{-2l}|t|^{-2}\rho_2^{1/2}.$$

Finally, if $2^l\rho_2^{-1/2} = \max(2^j, 2^{-l}, 2^l\rho_2^{-1/2})$, then

$$|I^1_l| + |I^2_l| \lesssim (2^{2l}\rho_2^{-1})/(|t|^2 2^{2l}) \cdot 2^{2l}\rho_2^{1/2} \lesssim 2^{2l}|t|^{-2}\rho_2^{-1/2}.$$

Therefore,

$$\sum_{l\geq l_0+1} |I_l^1| + |I_l^2| \lesssim 2^{-(1+\beta)j}\rho_2^{5/4}. \tag{A.58}$$

The desired bound (A.54) follows from (A.57) and (A.58). □

Our last lemma in this section is a bilinear estimate. Recall the operators $Q_s^{\sigma;\mu,\nu}$ defined in (5.4),

$$\mathcal{F}[Q_s^{\sigma;\mu,\nu}(f,g)](\xi) = \int_{\mathbb{R}^3} e^{is[\Lambda_\sigma(\xi)-\widetilde{\Lambda}_\mu(\xi-\eta)-\widetilde{\Lambda}_\nu(\eta)]} m_{\sigma;\mu,\nu}(\xi,\eta)\widehat{f}(\xi-\eta)\widehat{g}(\eta)\,d\eta.$$

LEMMA A.6. *Assume* $s\in\mathbb{R}$, $\sigma\in\{i,e,b\}$, $\mu,\nu\in\mathcal{I}_0$, *and*

$$\|f\|_{Z\cap H^{N_0}} \leq 1, \qquad \|g\|_{Z\cap H^{N_0}} \leq 1. \tag{A.59}$$

Then, for any $k'\in\mathbb{Z}$,

$$\|P_{k'}Q_s^{\sigma;\mu,\nu}(f,g)\|_{L^2} \lesssim 2^{k'_\sigma}\min[(1+s)^{-1-\beta}, 2^{3k'/2}]\cdot\min[1,2^{-(N_0-5)k'}]. \tag{A.60}$$

Moreover,

$$\text{if}\quad 2^{k'}\in[2^{-D},2^D] \text{ and } \sigma\in\{e,b\} \quad\text{or}\quad 2^{k'}\in(0,2^D] \text{ and } \sigma=i, \tag{A.61}$$

then

$$\|\mathcal{F}P_{k'}Q_s^{\sigma;\mu,\nu}(f,g)\|_{L^\infty} \lesssim (1+s)^{-1+\beta/10}2^{-k'}. \tag{A.62}$$

Proof of Lemma A.6. Clearly, the left-hand side of (A.60) is dominated by

(A.63)
$$C\sum_{k_1,k_2\in\mathbb{Z}} \Big\|\varphi_{k'}(\xi)\int_{\mathbb{R}^3} e^{-is[\widetilde{\Lambda}_\mu(\xi-\eta)+\widetilde{\Lambda}_\nu(\eta)]} m_{\sigma;\mu,\nu}(\xi,\eta)\widehat{P_{k_1}f}(\xi-\eta,s)\widehat{P_{k_2}g}(\eta,s)\,d\eta\Big\|_{L^2_\xi}.$$

Using (A.26)–(A.27) and the assumption (A.59),

(A.64)
$$\begin{aligned}
\|P_{k''}f\|_{L^2} + \|P_{k''}g\|_{L^2} &\lesssim \min(2^{(1+\beta-\alpha)k''}, 2^{-N_0k''}),\\
\|e^{-is\widetilde{\Lambda}_\mu}P_{k''}f\|_{L^\infty} + \|e^{-is\widetilde{\Lambda}_\nu}P_{k''}g\|_{L^\infty} &\lesssim \min(2^{(1/2-\beta-\alpha)k''}, 2^{-6k''})(1+s)^{-1-\beta}
\end{aligned}$$

for any $k''\in\mathbb{Z}$. Using (A.64) and the description of the symbols $m_{\sigma;\mu,\nu}$ in Lemma 3.3, the expression in (A.63) is dominated by

$$\begin{aligned}
C2^{k'_\sigma}\sum_{k_1,k_2\in\mathbb{Z},\,k_1\leq k_2,\,k'\leq k_2+4} &\min(2^{(1+\beta-\alpha)k_2}, 2^{-(N_0-2)k_2})\\
&\cdot\min(2^{(1/2-\beta-\alpha)k_1}, 2^{-6k_1})(1+s)^{-1-\beta}\\
&\lesssim 2^{k'_\sigma}(1+s)^{-1-\beta}\min(1,2^{-(N_0-5)k'}).
\end{aligned}$$

Moreover, if $k' \leq 0$, we use again (A.64) and Lemma 3.3 to estimate the expression in (A.63) by

$$
\begin{aligned}
&C\sum_{k_1,k_2\in\mathbb{Z}} 2^{3k'/2}\Big\|\int_{\mathbb{R}^3} e^{-is[\widetilde{\Lambda}_\mu(\xi-\eta)+\widetilde{\Lambda}_\nu(\eta)]} m_{\sigma;\mu,\nu}(\xi,\eta)\widehat{P_{k_1}f}(\xi-\eta,s)\widehat{P_{k_2}g}(\eta,s)\,d\eta\Big\|_{L^\infty_\xi}\\
&\lesssim 2^{3k'/2}2^{k'_\sigma}\sum_{k_1,k_2\in\mathbb{Z}} \min(2^{(1+\beta-\alpha)k_1},2^{-(N_0-2)k_1})\cdot\min(2^{(1+\beta-\alpha)k_2},2^{-(N_0-2)k_2})\\
&\lesssim 2^{3k'/2}2^{k'_\sigma}.
\end{aligned}
$$

The desired bound (A.60) follows.

To prove (A.62) we use first Lemma 3.3 and Lemma A.1 and decompose the functions f, g in suitable atoms. It suffices to prove that if

$$
\|h_1\|_{Z\cap H^{N_0}} + \|h_2\|_{Z\cap H^{N_0}} \leq 1, \tag{A.65}
$$

and we decompose

$$
h_i = \sum_{(k_i,j_i)\in\mathcal{J}} h^i_{k_i,j_i}, \qquad h^i_{k_i,j_i} := P_{[k_i-2,k_i+2]}(\widetilde{\varphi}^{(k_i)}_{j_i}\cdot P_{k_i}h_i), \qquad i=1,2,
$$

then

$$
\begin{aligned}
\sum_{(k_1,j_1),(k_2,j_2)\in\mathcal{J}} 2^{k'}\Big|\varphi_{k'}(\xi)\int_{\mathbb{R}^3} e^{is[\Lambda_\sigma(\xi)-\widetilde{\Lambda}_\mu(\xi-\eta)-\widetilde{\Lambda}_\nu(\eta)]}\widehat{h^1_{k_1,j_1}}(\xi-\eta)\widehat{h^2_{k_2,j_2}}(\eta)\,d\eta\Big|&\\
\lesssim (1+s)^{-1+\beta/10}2^{-k'}&
\end{aligned}
\tag{A.66}
$$

for any $\xi\in\mathbb{R}^3$, $\mu,\nu\in\mathcal{I}_0$, $s\in\mathbb{R}$, and k',σ as in (A.61).

We use first only the L^2 bounds

$$
\begin{aligned}
\|h^1_{k_1,j_1}\|_{L^2} &\lesssim \min(2^{-N_0k_1}, 2^{(2\beta-\alpha)\widetilde{k_1}}2^{-(1-\beta)j_1}),\\
\|h^2_{k_2,j_2}\|_{L^2} &\lesssim \min(2^{-N_0k_2}, 2^{(2\beta-\alpha)\widetilde{k_2}}2^{-(1-\beta)j_2});
\end{aligned}
\tag{A.67}
$$

see (A.65) and (A.19). The full bound (A.66) follows easily if $s^{1-\beta/10}\leq 2^{D^2}2^{-2k'}$. Assuming $s^{1-\beta/10}\geq 2^{D^2}2^{-2k'}$, we estimate easily

$$
\begin{aligned}
\sum_{((k_1,j_1),(k_2,j_2))\in J_1} 2^{k'}\Big|\varphi_{k'}(\xi)\int_{\mathbb{R}^3} e^{is[\Lambda_\sigma(\xi)-\widetilde{\Lambda}_\mu(\xi-\eta)-\widetilde{\Lambda}_\nu(\eta)]}\widehat{h^1_{k_1,j_1}}(\xi-\eta)\widehat{h^2_{k_2,j_2}}(\eta)\,d\eta\Big|&\\
\lesssim s^{-1}2^{-k'},&
\end{aligned}
$$

where

$$
\begin{aligned}
J_1 := \{((k_1,j_1),(k_2,j_2))\in\mathcal{J}\times\mathcal{J} : 2^{\max(k_1,k_2)} &\geq s^{2/N_0}\\
\text{or } 2^{\max(j_1,j_2)} &\geq s^{1+4\beta}2^{k'}\}.
\end{aligned}
$$

Let

$$J_2 := \{((k_1,j_1),(k_2,j_2)) \in \mathcal{J}\times\mathcal{J} : 2^{\max(k_1,k_2)} \le s^{2/N_0} \text{ and } 2^{\max(j_1,j_2)} \le s^{1+4\beta}2^{k'}\},$$

and notice that J_2 has at most $C(\ln s)^4$ elements. Therefore, for (A.66) it suffices to prove that

$$\left|\varphi_{k'}(\xi)\int_{\mathbb{R}^3} e^{-is[\widetilde{\Lambda}_\mu(\xi-\eta)+\widetilde{\Lambda}_\nu(\eta)]}\widehat{h^1_{k_1,j_1}}(\xi-\eta)\widehat{h^2_{k_2,j_2}}(\eta)\,d\eta\right| \lesssim 2^{-2k'}s^{-1+\beta/11}, \tag{A.68}$$

provided that $\xi\in\mathbb{R}^3$, $\mu,\nu\in\mathcal{I}_0$, $k'\in\mathbb{Z}\cap(-\infty,D]$, $s^{1-\beta/10}\ge 2^{D^2}2^{-2k'}$, and $((k_1,j_1),(k_2,j_2))\in J_2$.

Assume first that

$$2^{\max(j_1,j_2)} \ge 2^{-D^2}s^{1-\beta/11}2^{k'}. \tag{A.69}$$

Without loss of generality, in proving (A.68) we may assume that $j_1\le j_2$. Then, using (4.6), (4.9) and the assumption (A.65), we have

$$\|\widehat{h^2_{k_2,j_2}}\|_{L^1} \lesssim 2^{-(1+\beta)j_2}2^{3k_2/2}(2^{\alpha k_2}+2^{10k_2})^{-1}. \tag{A.70}$$

Using (A.20), $\|\widehat{h^1_{k_1,j_1}}\|_{L^\infty}\lesssim 2^{-\widetilde{k_1}/2}$. Using also (A.67), we estimate the left-hand side of (A.68) by

$$\begin{aligned}C\min(\|\widehat{h^1_{k_1,j_1}}\|_{L^\infty}\|\widehat{h^2_{k_2,j_2}}\|_{L^1}, \|\widehat{h^1_{k_1,j_1}}\|_{L^2}\|\widehat{h^2_{k_2,j_2}}\|_{L^2})\\ \lesssim \min(2^{-\widetilde{k_1}/2}2^{-(1+\beta)j_2}, 2^{\widetilde{k_1}(1+\beta-\alpha)}2^{-(1-\beta)j_2}) \lesssim 2^{-j_2}.\end{aligned}$$

The desired bound (A.68) follows if we assume (A.69).

Assume now that

$$2^{2\min(k_1,k_2)} \le 2^{D^2}2^{-2k'}s^{-1+\beta/11}. \tag{A.71}$$

Without loss of generality, in proving (A.68) we may assume that $k_2\le k_1$. Then, using (A.70) we estimate the left-hand side of (A.68) by

$$C\|\widehat{h^1_{k_1,j_1}}\|_{L^\infty}\|\widehat{h^2_{k_2,j_2}}\|_{L^1} \lesssim 2^{-k_1/2}2^{5k_2/2}\lesssim 2^{2k_2},$$

as desired

Finally, it remains to prove (A.68) assuming that

$$2^{\max(j_1,j_2)} \le 2^{-D^2}s^{1-\beta/11}2^{k'} \quad\text{and}\quad 2^{2\min(k_1,k_2)} \ge 2^{D^2}2^{-2k'}s^{-1+\beta/11}. \tag{A.72}$$

In this case we would like to integrate by parts in η to estimate the integral in (A.68). Using the bounds (A.1) and (A.20),

$$\left|\int_{\mathbb{R}^3}[1-\varphi_{\le 0}(\delta^{-1}\Xi^{\mu,\nu}(\xi,\eta))]e^{-is[\widetilde{\Lambda}_\mu(\xi-\eta)+\widetilde{\Lambda}_\nu(\eta)]}\widehat{h^1_{k_1,j_1}}(\xi-\eta)\widehat{h^2_{k_2,j_2}}(\eta)\,d\eta\right| \lesssim s^{-2} \tag{A.73}$$

as long as

$$\delta \in (0,1], \quad s\delta \geq s^{\beta^2} 2^{\max(j_1,j_2)}, \quad s\delta \geq s^{\beta^2}\delta^{-1}2^{-\min(k_1,k_2,0)}. \tag{A.74}$$

Therefore, letting

$$\begin{aligned} D(\xi,\delta) := \{\eta \in \mathbb{R}^3 : |\eta| \in [2^{k_2-4}, 2^{k_2+4}], \\ |\xi-\eta| \in [2^{k_1-4}, 2^{k_1+4}],\ |\Xi^{\mu,\nu}(\xi,\eta)| \leq 2\delta\}, \end{aligned} \tag{A.75}$$

for (A.68) it remains to prove that, for some δ satisfying (A.74),

$$\left|\varphi_{k'}(\xi)\int_{\mathbb{R}^3} \mathbf{1}_{D(\xi,\delta)}(\eta)\left|\widehat{h^1_{k_1,j_1}}(\xi-\eta)\right|\left|\widehat{h^2_{k_2,j_2}}(\eta)\right| d\eta\right| \lesssim 2^{-2k'}s^{-1+\beta/11}, \tag{A.76}$$

provided that $\xi \in \mathbb{R}^3$, $\mu,\nu \in \mathcal{I}_0$, $k' \in \mathbb{Z}\cap(-\infty,D]$, and $((k_1,j_1),(k_2,j_2)) \in J_2$ satisfies (A.72). Without loss of generality, we may assume that $k_2 \leq k_1$.

We examine now the sets $D(\xi,\delta)$ defined in (A.75). Assume that $\mu = (\sigma_1\iota_1)$, $\nu = (\sigma_2\iota_2)$, $\sigma_1,\sigma_2 \in \{i,e,b\}$, $\iota_1,\iota_2 \in \{+,-\}$. Notice that

$$\Xi^{\mu,\nu}(\xi,\eta) = -\iota_1\frac{\lambda'_{\sigma_1}(|\eta-\xi|)}{|\eta-\xi|}(\eta-\xi) - \iota_2\frac{\lambda'_{\sigma_2}(|\eta|)}{|\eta|}\eta = A(\eta-\xi)+B\eta,$$

where

$$A := -\iota_1\frac{\lambda'_{\sigma_1}(|\eta-\xi|)}{|\eta-\xi|}, \qquad B := -\iota_2\frac{\lambda'_{\sigma_2}(|\eta|)}{|\eta|}.$$

In view of Lemma A.4(i), we have $\min(|A|,|B|) \gtrsim_{C_b,\varepsilon} 2^{-\max(k_1,0)}$. Letting $\xi = se$, $e \in \mathbb{S}^2$, $s \in [2^{k'-2}, 2^{k'+2}]$, and $\eta = re+\eta'$, $r \in \mathbb{R}$, $\eta'\cdot e = 0$, and assuming $\eta \in D(\xi,\delta)$, it follows that

$$|(A+B)\eta'| \leq 2\delta, \qquad |(A+B)r - As| \leq 2\delta.$$

We let $\delta := \max(s^{\beta^2-1}2^{\max(j_1,j_2)}, s^{(\beta^2-1)/2}2^{-\min(k_2,0)/2})$, such that (A.74) is satisfied. In view of (A.72), it follows that $|(A+B)r| \gtrsim_{C_b,\varepsilon} 2^{-\max(k_1,0)}2^{k'}$, therefore $|A+B| \gtrsim_{C_b,\varepsilon} 2^{-\max(k_1,0)}2^{k'}2^{-k_2}$. This shows that

$$|\eta'| \lesssim 2^{\max(k_1,0)}2^{-k'}2^{k_2}\delta.$$

In other words, we proved that if $\xi = se$, $e \in \mathbb{S}^2$, $s \in [2^{k'-2}, 2^{k'+2}]$, then

$$\begin{aligned} D(\xi,\delta) \subseteq \{\eta = re+\eta' \in \mathbb{R}^3 : |r| + |\eta'|^2 \leq 2^{k_2+4}, \\ \eta'\cdot e = 0,\ |\eta'| \lesssim 2^{\max(k_1,0)}2^{-k'}2^{k_2}\delta\}. \end{aligned} \tag{A.77}$$

Using (A.77) and the L^∞ bounds $\|\widehat{h^1_{k_1,j_1}}\|_{L^\infty} \lesssim 2^{-k_1/2}$, $\|\widehat{h^2_{k_2,j_2}}\|_{L^\infty} \lesssim 2^{-k_2/2}$, we can bound the left-hand side of (A.76) by

$$C2^{-k_1/2}2^{-k_2/2}\cdot(2^{\max(k_1,0)}2^{-k'}2^{k_2}\delta)^2 2^{k_2} \lesssim 2^{2\min(k_2,0)}\delta^2 2^{-2k'}s^{8/N_0}.$$

This suffices to prove (A.76) if $2^{\max(j_1,j_2)}2^{\min(k_2,0)} \leq s^{1/2}$. On the other hand, if $j_1 \leq j_2$ and $2^{j_2}2^{\min(k_2,0)} \geq s^{1/2}$, then we estimate the left-hand side of (A.76) by

$$C2^{-k_1/2}\|\mathbf{1}_{D(\xi,\delta)}(\eta) \cdot \widehat{h^2_{k_2,j_2}}(\eta)\|_{L^1_\eta} \lesssim 2^{-k_1/2} \cdot 2^{k_2/2}2^{\max(k_1,0)}2^{-k'}2^{k_2}\delta \cdot 2^{-j_2} \lesssim 2^{-k'}s^{-1+\beta/11},$$

which also suffices to prove (A.76). Finally, if $j_1 \geq j_2$ and $2^{j_1}2^{\min(k_2,0)} \geq s^{1/2}$, then we estimate the left-hand side of (A.76) by

$$C2^{-k_2/2}\|\mathbf{1}_{D(\xi,\delta)}(\eta) \cdot \widehat{h^1_{k_1,j_1}}(\xi-\eta)\|_{L^1_\eta} \lesssim 2^{-k_2/2} \cdot 2^{k_2/2}2^{\max(k_1,0)}2^{-k'}2^{k_2}\delta \cdot 2^{-j_1} \lesssim 2^{-k'}s^{-1+\beta/11},$$

which also suffices to prove (A.76). This completes the proof of the lemma. □

Appendix B. **Classification of resonances**

We define the order $i < e < b$. Recall that we introduced a large number $D \gg (\varepsilon^{-1} + C_b)^{10}$, depending only on ε and C_b. For $\sigma \in \{i, e, b\}$ and $\mu, \nu \in \mathcal{I}_0$ (see definition (3.21)),

(B.1) $$\mu = (\sigma_1\iota_1), \qquad \nu = (\sigma_2\iota_2), \qquad \sigma_1, \sigma_2 \in \{i, e, b\}, \qquad \iota_1, \iota_2 \in \{+, -\}.$$

Recall the definitions of the smooth functions $\Lambda_\sigma : \mathbb{R}^3 \to (0,\infty)$, $\Phi^{\sigma;\mu,\nu} : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ and $\Xi^{\mu,\nu} : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$,

(B.2) $$\begin{aligned}\Phi^{\sigma;\mu,\nu}(\xi,\eta) &= \Lambda_\sigma(\xi) - \iota_1\Lambda_{\sigma_1}(\xi-\eta) - \iota_2\Lambda_{\sigma_2}(\eta),\\ \Xi^{\mu,\nu}(\xi,\eta) &= (\nabla_\eta\Phi^{\sigma;\mu,\nu})(\xi,\eta) = -\iota_1\nabla\Lambda_{\sigma_1}(\eta-\xi) - \iota_2\nabla\Lambda_{\sigma_2}(\eta).\end{aligned}$$

In this subsection we prove several lemmas describing the structure of almost resonant sets, which are the sets where both $|\Phi^{\sigma;\mu,\nu}(\xi,\eta)|$ and $|\Xi^{\mu,\nu}(\xi,\eta)|$ are small. Recall the sets

(B.3)
$$\begin{aligned}\mathcal{L}^{\sigma;\mu,\nu}_{k,k_1,k_2;\delta_1,\delta_2} = \{(\xi,\eta) \in& \mathbb{R}^3 \times \mathbb{R}^3 : |\xi| \in [2^{k-4}, 2^{k+4}],\ |\xi-\eta| \in [2^{k_1-4}, 2^{k_1+4}],\\ &|\eta| \in [2^{k_2-4}, 2^{k_2+4}], |\Xi^{\mu,\nu}(\xi,\eta)| \leq \delta_1,\ |\Phi^{\sigma;\mu,\nu}(\xi,\eta)| \leq \delta_2\}\end{aligned}$$

defined for $\sigma \in \{i, e, b\}$, $\mu, \nu \in \mathcal{I}_0$, $k, k_1, k_2 \in \mathbb{Z}$, $\delta_1, \delta_2 \in (0,\infty)$. We define also

$$\begin{aligned}\mathcal{L}_{k,k_1,k_2} := \{(\xi,\eta) \in \mathbb{R}^3 \times \mathbb{R}^3 :\ & |\xi| \in [2^{k-4}, 2^{k+4}],\\ & |\xi-\eta| \in [2^{k_1-4}, 2^{k_1+4}],\ |\eta| \in [2^{k_2-4}, 2^{k_2+4}]\}.\end{aligned}$$

Given a phase $\Phi^{\sigma;\mu,\nu}$ and a set of phases $\mathcal{T}$, we denote $\Phi^{\sigma;\mu,\nu} \in\in \mathcal{T}$ if either $\Phi^{\sigma;\mu,\nu} \in \mathcal{T}$ or $\Phi^{\sigma;\nu,\mu} \in \mathcal{T}$, and $\Phi^{\sigma\mu,\nu} \notin\notin \mathcal{T}$ if neither possibility holds.

We show first that some phases do not contribute in the analysis of resonant interactions. We define the 39 strongly elliptic phases (which do not vanish in $\mathbb{R}^3 \times \mathbb{R}^3$),

$$\begin{aligned}\mathcal{T}_{\mathrm{Sell}} := \{&\Phi^{i;i+,e+}, \Phi^{i,i+,e-}, \Phi^{i;i+,b+}, \Phi^{i;i+,b-}, \Phi^{i,i-,e-}, \Phi^{i;i-,b-}, \Phi^{i,e+,e+},\\ &\Phi^{i;e+,b+}, \Phi^{i,e-,e-}, \Phi^{i;e-,b-}, \Phi^{i;b+,b+}, \Phi^{i;b-,b-}, \Phi^{e;i+,i-}, \Phi^{e;i+,e-},\\ &\Phi^{e;i+,b-}, \Phi^{e;i-,i-}, \Phi^{e;i-,e-}, \Phi^{e;i-,b-}, \Phi^{e;e+,e+}, \Phi^{e;e+,e-} \Phi^{e;e+,b+},\\ &\Phi^{e;e+,b-}, \Phi^{e;e-,e-}, \Phi^{e;e-,b-}, \Phi^{e;b+,b+}, \Phi^{e;b-,b-}, \Phi^{b;i+,i-}, \Phi^{b;i+,e-},\\ &\Phi^{b;i+,b-}, \Phi^{b;i-,i-}, \Phi^{b;i-,e-}, \Phi^{b;i-,b-}, \Phi^{b;e+,e-}, \Phi^{b;e+,b-}, \Phi^{b;e-,e-},\\ &\Phi^{b;e-,b-}, \Phi^{b;b+,b+}, \Phi^{b;b+,b-}, \Phi^{b;b-,b-}\}.\end{aligned}$$

We define four additional nonresonant phases (for which $|\Phi^{\sigma;\mu,\nu}| + |\nabla_\eta \Phi^{\sigma;\mu,\nu}|$ does not vanish in $\mathbb{R}^3 \times \mathbb{R}^3$):

$$\mathcal{T}_{\mathrm{NR}} = \{\Phi^{e;i+,i+}, \Phi^{e;b+,b-}, \Phi^{b;i+,i+}, \Phi^{b;i-,e+}\}.$$

LEMMA B.1. *Assume that* $\Phi^{\sigma;\mu,\nu} \in\in \mathcal{T}_{\mathrm{Sell}} \cup \mathcal{T}_{\mathrm{NR}}$. *If*

$$\delta_1 \leq 2^{-D} 2^{-4\max(k_1,k_2,0)}, \ \delta_2 \leq 2^{-D} 2^{-\max(k_1,k_2,0)},$$

then $\mathcal{L}^{\sigma;\mu,\nu}_{k,k_1,k_2;\delta_1,\delta_2} = \emptyset$.

Proof of Lemma B.1. We claim that if $\Phi^{\sigma;\mu,\nu} \in\in \mathcal{T}_{\mathrm{Sell}}$, then we have

$$|\Phi^{\sigma;\mu,\nu}(\xi,\eta)| \gtrsim_{\varepsilon,C_b} 2^{-\max(k_1,k_2,0)}. \tag{B.4}$$

This would clearly suffice to prove the claim for the strongly elliptic phases.

If $(\iota_1,\iota_2) = (-,-)$, the proof of (B.4) is a direct consequence of the fact that $\lambda_b \geq \lambda_e \geq 1$. To deal with the remaining 22 phases in $\mathcal{T}_{\mathrm{Sell}}$ we observe that, as a consequence of Lemma A.4, we have, for any $r \in [0,\infty)$,

$$r \leq \lambda_i(r) \leq \lambda_e(r) \leq \lambda_b(r), \quad \lambda_b(r) - \lambda_e(r) \gtrsim_{\varepsilon,C_b} r, \quad \lambda_e(r) - \lambda_i(r) \gtrsim_{\varepsilon,C_b} 1 + r. \tag{B.5}$$

In addition, for any $r_1, r_2 \in [0,\infty)$,

$$\begin{aligned}&\lambda_i(r_1) + \lambda_i(r_2) - \lambda_i(r_1+r_2) \geq 0,\\ &\lambda_e(r_1) + \lambda_e(r_2) - \lambda_e(r_1+r_2) \gtrsim_{\varepsilon,C_b} (1+\min(r_1,r_2))^{-1},\\ &\lambda_b(r_1) + \lambda_b(r_2) - \lambda_b(r_1+r_2) \gtrsim_{\varepsilon,C_b} (1+\min(r_1,r_2))^{-1}.\end{aligned} \tag{B.6}$$

Indeed, the first bound in (B.6) follows from the formula $\lambda_i(r) = rq_i(r)$ in Lemma A.4 and the fact that q_i is decreasing. For the second bound in (B.6), we use the fact that the function $r \to \lambda_e(r) - h_\varepsilon(r)$ is nonnegative and decreasing on $[0,\infty)$ (see (A.15)), and therefore

$$\begin{aligned}\lambda_e(r_1) + \lambda_e(r_2) - \lambda_e(r_1+r_2) &\geq h_\varepsilon(r_1) + h_\varepsilon(r_2) - h_\varepsilon(r_1+r_2)\\ &\gtrsim_{\varepsilon,C_b} (1+\min(r_1,r_2))^{-1}.\end{aligned}$$

The third bound follows directly from the definition.

Using (B.5), (B.6), and the monotonicity of the functions $\lambda_i, \lambda_e, \lambda_b$ on $[0,\infty)$, we can now prove lower bounds for the absolute values of the 22 phases in $\mathcal{T}_{\rm Sell}$. If $(\iota_1,\iota_2)=(+,+)$ and $\rho,\tau\in\{e,b\}$, then

$$\begin{aligned}
-\Phi^{i;i+,\tau+}(\xi,\eta) &= [-\Lambda_i(\xi)+\Lambda_i(\xi-\eta)+\Lambda_i(\eta)]+[\Lambda_\tau(\eta)-\Lambda_i(\eta)] \gtrsim_{\varepsilon,C_b} 1,\\
-\Phi^{i;\rho+,\tau+}(\xi,\eta) &= [-\Lambda_i(\xi)+\Lambda_i(\xi-\eta)+\Lambda_i(\eta)]\\
&\quad +[\Lambda_\rho(\xi-\eta)-\Lambda_i(\xi-\eta)]+[\Lambda_\tau(\eta)-\Lambda_i(\eta)] \gtrsim_{\varepsilon,C_b} 1,\\
-\Phi^{e;\rho+,\tau+}(\xi,\eta) &= [-\Lambda_e(\xi)+\Lambda_e(\xi-\eta)+\Lambda_e(\eta)]\\
&\quad +[\Lambda_\rho(\xi-\eta)-\Lambda_e(\xi-\eta)]+[\Lambda_\tau(\eta)-\Lambda_e(\eta)]\\
&\gtrsim_{\varepsilon,C_b} 2^{-\max(k_1,k_2,0)},\\
-\Phi^{b;b+,b+}(\xi,\eta) &= -\Lambda_b(\xi)+\Lambda_b(\xi-\eta)+\Lambda_b(\eta) \gtrsim_{\varepsilon,C_b} 2^{-\max(k_1,k_2,0)}.
\end{aligned}$$

On the other hand, if $(\iota_1,\iota_2)=(+,-)$ and $\rho,\tau\in\{e,b\}$, then

$$\begin{aligned}
\Phi^{i;i+,\tau-}(\xi,\eta) &= [\Lambda_i(\xi)-\Lambda_i(\xi-\eta)+\Lambda_i(\eta)]+[\Lambda_\tau(\eta)-\Lambda_i(\eta)] \gtrsim_{\varepsilon,C_b} 1,\\
\Phi^{\rho;i+,i-}(\xi,\eta) &= [\Lambda_i(\xi)-\Lambda_i(\xi-\eta)+\Lambda_i(\eta)]+[\Lambda_\rho(\xi)-\Lambda_i(\xi)] \gtrsim_{\varepsilon,C_b} 1,\\
\Phi^{\rho;i+,\tau-}(\xi,\eta) &= [\Lambda_i(\xi)-\Lambda_i(\xi-\eta)+\Lambda_i(\eta)]\\
&\quad +[\Lambda_\rho(\xi)-\Lambda_i(\xi)]+[\Lambda_\tau(\eta)-\Lambda_i(\eta)] \gtrsim_{\varepsilon,C_b} 1,\\
\Phi^{\rho;e+,\tau-}(\xi,\eta) &= [\Lambda_e(\xi)-\Lambda_e(\xi-\eta)+\Lambda_e(\eta)]\\
&\quad +[\Lambda_\rho(\xi)-\Lambda_e(\xi)]+[\Lambda_\tau(\eta)-\Lambda_e(\eta)] \gtrsim_{\varepsilon,C_b} 2^{-\max(k_1,k_2,0)},\\
\Phi^{b;b+,b-}(\xi,\eta) &= [\Lambda_b(\xi)-\Lambda_b(\xi-\eta)+\Lambda_b(\eta)] \gtrsim_{\varepsilon,C_b} 2^{-\max(k_1,k_2,0)}.
\end{aligned}$$

The desired lower bound (B.4) follows for all phases $\Phi^{\sigma;\mu,\nu}\in\mathcal{T}_{\rm Sell}$.

We now consider the phases $\Phi^{\sigma;\mu,\nu}\in\in\mathcal{T}_{\rm NR}$. Assume first $\sigma_1=\sigma_2=i$. Then, since $\lambda_i'\gtrsim_{C_b,\varepsilon}1$, we see that smallness of $|\Xi^{\mu,\nu}(\xi,\eta)|$ implies $(\xi-\eta)\cdot\eta\geq 0$. But then $|\xi|\geq\max(|\xi-\eta|,|\eta|)$ and, using (A.7),

$$\begin{aligned}
\min[\Phi^{e;i+,i+}(\xi,\eta),\Phi^{b;i+,i+}(\xi,\eta)] &\geq \lambda_e(|\xi|)-2\lambda_i(|\xi|)\\
&\geq h_\varepsilon(|\xi|)-2\sqrt{(T+1)(\varepsilon+1)}|\xi| \gtrsim_{\varepsilon,C_b} 1,
\end{aligned}$$

and the desired conclusion $\mathcal{L}^{\sigma;i+,i+}_{k,k_1,k_2;\delta_1,\delta_2}=\emptyset$, $\sigma\in\{e,b\}$, follows.

Assume now that $\Phi^{\sigma;\mu,\nu}\in\in\{\Phi^{e;b+,b-}\}$. If $\max(k_1,k_2)\leq -D/10$ then, using (A.5), $\Phi^{e;b+,b-}(\xi,\eta)\geq 1$. On the other hand, if $\max(k_1,k_2)\geq -D/10$, we see from smallness of $|\Xi^{b+,b-}(\xi,\eta)|$ that $|\xi|\leq 2^{-D}$. But in this case,

$$|\lambda_b(|\xi-\eta|)-\lambda_b(|\eta|)|\lesssim_{\varepsilon,C_b}|\xi|\leq\lambda_e(0)/2,$$

hence $\Phi^{e;b+,b-}(\xi,\eta)\geq 1$.

Finally, assume that $\Phi^{\sigma;\mu,\nu} \in\in \{\Phi^{b;i-,e+}\}$. By symmetry we may assume that

$$\Phi^{\sigma;\mu,\nu}(\xi,\eta) = \Phi^{b;i-,e+}(\xi,\eta) = \lambda_b(|\xi|) + \lambda_i(|\xi-\eta|) - \lambda_e(|\eta|).$$

The condition $|\Xi^{\mu,\nu}(\xi,\eta)| \leq 2^{-D}$ and Lemma A.4(i) show that $2^{k_2} \gtrsim_{C_b,\varepsilon} 1$. The condition $|\Phi^{\sigma;\mu,\nu}(\xi,\eta)| \leq 2^{-D}$ then shows that $|\eta| \geq |\xi|$. Since

$$\lambda_i'(r) \leq \frac{\sqrt{1+T}}{\sqrt{1+\varepsilon}} \qquad \text{and} \qquad \lambda_e'(r) \geq \frac{(1-\sqrt{\varepsilon})Tr}{\sqrt{\varepsilon}\sqrt{1+Tr^2}},$$

for any $r \in [0,\infty)$ (see Lemma A.4(ii) and (iii)), the restriction $|\Xi^{\mu,\nu}(\xi,\eta)| \leq 2^{-D}$ shows that $|\eta| \leq \sqrt{\varepsilon/T}$. Therefore, $|\xi| \leq \sqrt{\varepsilon/T}$ and $|\xi-\eta| \leq 2\sqrt{\varepsilon/T}$. Since $r_* \geq T^{-1/2}$ (see Lemma A.4(i)), it follows that $|\xi-\eta| \leq r_*/2$. Therefore, λ_i' is decreasing on the interval $[0,|\xi-\eta|]$, and we estimate, recalling that $2^{-D} \geq \left|\Xi^{\mu,\nu}(\xi,\eta)\right| \geq \left|\lambda_i'(|\xi-\eta|) - \lambda_e'(|\eta|)\right|$,

$$\begin{aligned}\Phi^{\sigma;\mu,\nu}(\xi,\eta) &= \int_0^{|\xi-\eta|} \lambda_i'(s)\,ds + \int_0^{|\xi|} \lambda_b'(s)\,ds - \int_0^{|\eta|} \lambda_e'(s)\,ds \\ &\geq C_{C_b,\varepsilon}^{-1} + |\xi-\eta|\lambda_i'(|\xi-\eta|) - (|\eta|-|\xi|)\lambda_e'(|\eta|) \\ &\gtrsim_{C_b,\varepsilon} 1.\end{aligned}$$

This provides the contradiction. □

We consider now the remaining 20 phases, and define three sets of phases

(B.7)

$$\begin{aligned}\mathcal{T}_A :=& \{\Phi^{i;i-,e+}, \Phi^{i;i-,b+}, \Phi^{i;e+,b-}, \Phi^{i;e-,b+}, \Phi^{e;i+,e+}, \Phi^{e;i+,b+}, \Phi^{e;i-,b+}, \Phi^{e;e-,b+}, \\ &\Phi^{b;i+,e+}, \Phi^{b;i+,b+}, \Phi^{b;e+,e+}, \Phi^{b;e+,b+}, \Phi^{b;e-,b+}\}, \\ \mathcal{T}_B :=& \{\Phi^{e;i+,e+}, \Phi^{e;i-,e+}, \Phi^{b;i+,b+}, \Phi^{b;i-,b+}\}, \\ \mathcal{T}_C :=& \{\Phi^{i;i+,i+}, \Phi^{i;i+,i-}, \Phi^{i;i-,i-}, \Phi^{i;e+,e-}, \Phi^{i;e+,b-}, \Phi^{i;e-,b+}, \Phi^{i;b+,b-}\}.\end{aligned}$$

Notice that some phases, such as $\Phi^{e;i+,e+}$, belong to more than one set. The set $\mathcal{T}_A$ corresponds to phases having nondegenerate stationary points on spheres, while the sets $\mathcal{T}_B, \mathcal{T}_C$ consist of phases with degenerate behavior around 0 (in η, $\xi-\eta$, or ξ). More precisely,

Proposition B.2. *Assume $k, k_1, k_2 \in \mathbb{Z}$ and that there is a point $(\xi,\eta) \in \mathcal{L}_{k,k_1,k_2}$ satisfying*

$$\begin{aligned}|\Xi^{\mu,\nu}(\xi,\eta)| &\leq \delta_1 = 2^{-10D}2^{-4\max(0,k_1,k_2)}, \\ |\Phi^{\sigma;\mu,\nu}(\xi,\eta)| &\leq \delta_2 = 2^{-10D}2^{-\max(0,k_1,k_2)}.\end{aligned}$$

Then one of the three following possibilities holds:

Case A: $-D/2 \leq k, k_1, k_2 \leq D/2$ *and* $\Phi^{\sigma;\mu,\nu} \in\in \mathcal{T}_A$.

Case B: $\min(k_1,k_2) \leq -D/3$, $k \geq -D/4$ *and* $\Phi^{\sigma;\mu,\nu} \in\in \mathcal{T}_B$.

Case C: $k \leq -D/4$ *and* $\Phi^{\sigma;\mu,\nu} \in\in \mathcal{T}_C$.

Proof of Proposition B.2. We divide the proof in several steps.

Step 1. Assume that

$$\Phi^{\sigma;\mu,\nu} \in\in \{\Phi^{i;i+,i+}, \Phi^{i;i+,i-}, \Phi^{i;i-,i-}\}. \tag{B.8}$$

These phases are only in the set $\mathcal{T}_C$, and we have to prove that if $\mathcal{L}^{\sigma;\mu,\nu}_{k,k_1,k_2;\delta_1,\delta_2} \neq \emptyset$, then

$$k \le -D/2. \tag{B.9}$$

Assume for contradiction that $k \ge -D/2$. Using Lemma A.4(iii) it is easy to see that if $\Phi^{\sigma;\mu,\nu} \in\in \{\Phi^{i;i-,i-}\}$, then

$$|\Phi^{\sigma;\mu,\nu}(\xi,\eta)| \gtrsim_{\varepsilon,C_b} \max(|\xi|,|\eta|,|\xi-\eta|) \gtrsim_{\varepsilon,C_b} 2^{-D/2},$$

which is not possible. On the other hand, using Lemma A.4(iii), for any $r,s\in[0,\infty)$,

$$\begin{aligned}\lambda_i(r)+\lambda_i(s)-\lambda_i(r+s) &= r(q_i(r)-q_i(r+s))+s(q_i(s)-q_i(r+s))\\ &\gtrsim_{\varepsilon,C_b} \frac{\min(r,s)(r+s)^2}{(1+(r+s)^2)(1+\min(r,s)^2)}.\end{aligned}$$

Therefore, if $\Phi^{\sigma;\mu,\nu} \in\in \{\Phi^{i;i+,i+}, \Phi^{i;i+,i-}\}$, $k \ge -D/2$, and $\mathcal{L}^{\sigma;\mu,\nu}_{k,k_1,k_2;\delta_1,\delta_2} \neq \emptyset$, then $\min(k_1,k_2) \le -5D$. On the other hand, if $\min(k_1,k_2) \le -5D$ and $\max(k_1,k_2) \ge -D/2-10$, then we use the bound

$$\lambda_i'(r)-\lambda_i'(s) \ge q_i(0) - C_{C_b,\varepsilon} r - q_i(s) \ge 2^{-2D},$$

whenever $0\le r\le 2^{-4D}$ and $s\ge 2^{-D}$, which is a consequence of Lemma A.4(iii). Therefore, in this case $|\Xi^{\mu,\nu}(\xi,\eta)| \ge 2^{-2D}$ for all $(\xi,\eta)\in\mathcal{L}_{k,k_1,k_2}$, which provides the contradiction.

Step 2. Assume that

$$\Phi^{\sigma;\mu,\nu} \in\in \{\Phi^{i;e+,e-}, \Phi^{i;b+,b-}\}. \tag{B.10}$$

These phases are only in the set $\mathcal{T}_C$, and it suffices to prove that if $\mathcal{L}^{\sigma;\mu,\nu}_{k,k_1,k_2;\delta_1,\delta_2} \neq \emptyset$, then

$$k \le -D/2. \tag{B.11}$$

Assume for contradiction that $k \ge -D/2$. Since $\lambda_\sigma'(0)=0$, $\sigma\in\{e,b\}$, the restriction $|\Xi^{\mu,\nu}(\xi,\eta)| \le \delta_1$ shows that $\min(k_1,k_2)\ge -D$. On the other hand, if $\min(k,k_1,k_2)\ge -D$ then, for any $(\xi,\eta)\in\mathcal{L}_{k,k_1,k_2}$, we have

$$\begin{aligned}|\Xi^{\mu,\nu}(\xi,\eta)| &\gtrsim_{C_b,\varepsilon} \Big|\lambda_\sigma'(|\eta-\xi|)-\lambda_\sigma'(|\eta|)\Big|\\ &\quad + \max[\lambda_\sigma'(|\eta-\xi|),\lambda_\sigma'(|\eta|)]\Big|\frac{\eta-\xi}{|\eta-\xi|}-\frac{\eta}{|\eta|}\Big| \ge 2^D\delta_1,\end{aligned}$$

which provides a contradiction.

Step 3. Assume that

(B.12) $$\Phi^{\sigma;\mu,\nu} \in\in \{\Phi^{i;e+,b-}, \Phi^{i;e-,b+}\}.$$

These phases are in the sets $\mathcal{T}_C$ and $\mathcal{T}_A$, and we have to prove that if

$$\mathcal{L}^{\sigma;\mu,\nu}_{k,k_1,k_2;\delta_1,\delta_2} \neq \emptyset,$$

then

$$\text{either} \qquad k \leq -D/4 \qquad \text{or} \qquad -D/2 \leq k, k_1, k_2 \leq D/2.$$

This is equivalent to proving that

(B.13)
$$\text{if} \quad k \geq -D/4 \quad \text{and} \quad \mathcal{L}^{\sigma;\mu,\nu}_{k,k_1,k_2;\delta_1,\delta_2} \neq \emptyset, \quad \text{then} \quad -D/2 \leq k, k_1, k_2 \leq D/2.$$

Since

(B.14) $$\lim_{r\to\infty} \lambda_i'(r) = 1, \qquad \lim_{r\to\infty} \lambda_e'(r) = \sqrt{T/\varepsilon}, \qquad \lim_{r\to\infty} \lambda_b'(r) = \sqrt{C_b/\varepsilon},$$

it is easy to see that the condition $\mathcal{L}^{\sigma;\mu,\nu}_{k,k_1,k_2;\delta_1,\delta_2} \neq \emptyset$ implies that $\max(k_1, k_2, k_3) \leq D/4$. Since $k \geq -D/4$, it follows that $\max(k_1, k_2) \geq -D/4-10$. Recall that $\lambda_\sigma'(0) = 0$ and $\lambda_\sigma''(r) \approx_{C_b,\varepsilon} (1 + r^2)^{-3/2}$, $\sigma \in \{e, b\}$ (see Lemma A.4(i)). Since $|\Xi^{\mu,\nu}(\xi, \eta)| \leq \delta_1$ for some $(\xi, \eta) \in \mathcal{L}_{k,k_1,k_2}$, it follows that $\min(k_1, k_2) \geq -D/2$, as desired.

Step 4. Assume that

(B.15) $$\Phi^{\sigma;\mu,\nu} \in\in \{\Phi^{e;i-,e+}, \Phi^{b;i-,b+}\}.$$

These phases are only in the set $\mathcal{T}_B$, and we have to prove that if $\mathcal{L}^{\sigma;\mu,\nu}_{k,k_1,k_2;\delta_1,\delta_2} \neq \emptyset$, then

(B.16) $$k \geq -D/4 \quad \text{and} \quad \min(k_1, k_2) \leq -D/3.$$

It is easy to see that $\max(k_1, k_2, k) \geq -D/10$; otherwise, $|\Xi^{\mu,\nu}(\xi, \eta)| \geq 2^{-2D}$ for all $(\xi, \eta) \in \mathcal{L}_{k,k_1,k_2}$, in view of the fact that $\lambda_e'(0) = \lambda_b'(0) = 0$ and $\lambda_i'(0) \approx_{C_b,\varepsilon} 1$. Therefore, it remains to prove that if $\mathcal{L}^{\sigma;\mu,\nu}_{k,k_1,k_2;\delta_1,\delta_2} \neq \emptyset$, then

(B.17) $$\min(k_1, k_2) \leq -D/3.$$

Assume, for contradiction, that (B.17) fails. We may assume, without loss of generality, that

$$\Phi^{\sigma;\mu,\nu}(\xi, \eta) = \Phi^{\sigma;i-,\sigma+}(\xi, \eta) = \lambda_\sigma(|\xi|) + \lambda_i(|\xi - \eta|) - \lambda_\sigma(|\eta|)$$

for $\sigma \in \{e, b\}$. We argue as in the proof of Lemma B.1, $\Phi^{\sigma;\mu,\nu} = \Phi^{b;i-,e+} \in \mathcal{T}_{\text{NR}}$. The conditions $|\Phi^{\sigma;\mu,\nu}(\xi, \eta)| \leq 2^{-10D}$ and $k_1 \geq -D/3$ show that $|\eta| \geq |\xi|$. Since

$$\lambda_i'(r) \leq \frac{\sqrt{1+T}}{\sqrt{1+\varepsilon}} \qquad \text{and} \qquad \lambda_b'(r) \geq \lambda_e'(r) \geq \frac{(1-\sqrt{\varepsilon})Tr}{\sqrt{\varepsilon}\sqrt{1+Tr^2}},$$

for any $r \in [0,\infty)$ (see Lemma A.4(ii) and (iii)), the restriction $|\Xi^{\mu,\nu}(\xi,\eta)| \leq 2^{-D}$ shows that $|\eta| \leq \sqrt{\varepsilon/T}$. Therefore, $|\xi| \leq \sqrt{\varepsilon/T}$ and $|\xi-\eta| \leq 2\sqrt{\varepsilon/T}$. Since $r_* \geq T^{-1/2}$ (see Lemma A.4(i)), it follows that $|\xi-\eta| \leq r_*/2$. Therefore, λ_i' is decreasing on the interval $[0,|\xi-\eta|]$ and we estimate, recalling that $2^{-10D} \geq \Big|\Xi^{\mu,\nu}(\xi,\eta)\Big| \geq \Big|\lambda_i'(|\xi-\eta|) - \lambda_\sigma'(|\eta|)\Big|$ and $|\xi-\eta| \geq 2^{-D/2}$,

$$\begin{aligned}\Phi^{\sigma;\mu,\nu}(\xi,\eta) &= \int_0^{|\xi-\eta|} \lambda_i'(s)\,ds - \int_{|\xi|}^{|\eta|} \lambda_\sigma'(s)\,ds \\ &\geq 2^{-2D} + |\xi-\eta|\lambda_i'(|\xi-\eta|) - (|\eta|-|\xi|)\lambda_\sigma'(|\eta|) \geq 2^{-4D}.\end{aligned}$$

This provides the contradiction.

Step 5. Assume that

$$\Phi^{\sigma;\mu,\nu} \in\in \{\Phi^{e;i+,e+}, \Phi^{b;i+,b+}\}. \tag{B.18}$$

These phases are in the sets $\mathcal{T}_B$ and $\mathcal{T}_A$, and we have to prove that if $\mathcal{L}^{\sigma;\mu,\nu}_{k,k_1,k_2;\delta_1,\delta_2} \neq \emptyset$, then

$$\text{either}\quad k \geq -D/4,\ \min(k_1,k_2) \leq -D/3 \quad\text{or}\quad -D/2 \leq k,k_1,k_2 \leq D/2. \tag{B.19}$$

It is easy to see that $\max(k_1,k_2,k) \geq -D/10$; otherwise, $|\Xi^{\mu,\nu}(\xi,\eta)| \geq 2^{-2D}$ for all $(\xi,\eta) \in \mathcal{L}_{k,k_1,k_2}$, in view of the fact that $\lambda_e'(0) = \lambda_b'(0) = 0$ and $\lambda_i'(0) \approx_{C_b,\varepsilon} 1$. Therefore, for (B.19) it suffices to prove that

$$\text{if}\quad \min(k_1,k_2) \geq -D/3, \quad\text{then}\quad -D/2 \leq k,k_1,k_2 \leq D/2. \tag{B.20}$$

In view of (B.14), it is clear that $\min(k_1,k_2) \leq D/10$; otherwise, $|\Xi^{\mu,\nu}(\xi,\eta)| \geq 2^{-2D}$ for all $(\xi,\eta) \in \mathcal{L}_{k,k_1,k_2}$. On the other hand, if $k \geq D/4$, then $\max(k_1,k_2) \geq D/4 - 10$, and one can use (B.14) again to see easily that this in contradiction with the assumption $\mathcal{L}^{\sigma;\mu,\nu}_{k,k_1,k_2;\delta_1,\delta_2} \neq \emptyset$. Therefore, $\max(k,k_1,k_2) \leq D/2$, as desired.

Finally, for (B.20) it remains to prove that $k \geq -D/2$. Assuming, for contradiction, that $k \leq -D/2$, and recalling that $\max(k_1,k_2,k) \geq -D/10$, it follows that $\max(k_1,k_2) \geq -D/10$, $|k_1-k_2| \leq 10$. Therefore, $|\Phi^{\sigma;\mu,\nu}(\xi,\eta)| \gtrsim_{C_b,\varepsilon} 2^{-2D}$, which provides a contradiction.

Step 6. Assume that

$$\Phi^{\sigma;\mu,\nu} \in\in \{\Phi^{e;e-,b+}, \Phi^{b;e+,e+}, \Phi^{b;e+,b+}, \Phi^{b;e-,b+}\}. \tag{B.21}$$

These phases are only in the set $\mathcal{T}_A$, and we have to prove that if $\mathcal{L}^{\sigma;\mu,\nu}_{k,k_1,k_2;\delta_1,\delta_2} \neq \emptyset$, then

$$-D/2 \leq k,k_1,k_2 \tag{B.22}$$

and

$$k,k_1,k_2 \leq D/2. \tag{B.23}$$

We prove first (B.22). We notice that $\max(k_1, k_2) \geq -D/10$; otherwise, $|\Phi^{\sigma;\mu,\nu}(\xi,\eta)| \gtrsim_{C_b,\varepsilon} 1$ for any $(\xi,\eta) \in \mathcal{L}_{k,k_1,k_2}$, since $\lambda_e(0) = \lambda_b(0) = \sqrt{1+\varepsilon^{-1}}$. This implies that $\min(k_1,k_2) \geq -D/4$; otherwise, $|\Xi^{\mu,\nu}(\xi,\eta)| \gtrsim_{C_b,\varepsilon} 2^{-D}$ for any $(\xi,\eta) \in \mathcal{L}_{k,k_1,k_2}$, since $\lambda_e'(r) \approx_{C_b,\varepsilon} \min(r,1)$ and $\lambda_b'(r) \approx_{C_b,\varepsilon} \min(r,1)$.

To complete the proof of (B.22), assume, for contradiction, that $k \leq -D/2$ and therefore $\max(k_1,k_2) \geq -D/10$, $|k_1 - k_2| \leq 10$. If $\Phi^{\sigma;\mu,\nu} \in\in \{\Phi^{b;e+,e+}, \Phi^{b;e+,b+}\}$, then $|\Phi^{\sigma;\mu,\nu}(\xi,\eta)| \gtrsim_{C_b,\varepsilon} 1$ for any $(\xi,\eta) \in \mathcal{L}_{k,k_1,k_2}$, in contradiction with the assumption $\mathcal{L}^{\sigma;\mu,\nu}_{k,k_1,k_2;\delta_1,\delta_2} \neq \emptyset$. On the other hand, if $\Phi^{\sigma;\mu,\nu} \in\in \{\Phi^{e;e-,b+}, \Phi^{b;e-,b+}\}$, then $|\Xi^{\mu,\nu}(\xi,\eta)| \gtrsim_{C_b,\varepsilon} 2^{-2D}$ for any $(\xi,\eta) \in \mathcal{L}_{k,k_1,k_2}$, which is again in contradiction with the assumption $\mathcal{L}^{\sigma;\mu,\nu}_{k,k_1,k_2;\delta_1,\delta_2} \neq \emptyset$. This last bound is a consequence of the estimate

$$\lambda_b'(r) - \lambda_e'(r) \gtrsim_{C_b,\varepsilon} \min(1,r) \qquad \text{for any } r \geq 0, \tag{B.24}$$

which follows from Lemma A.4(ii). This completes the proof of (B.22).

We prove now (B.23). We notice first that $\min(k,k_1,k_2) \leq D/10$; otherwise, either $|\Xi^{\mu,\nu}(\xi,\eta)| \gtrsim_{C_b,\varepsilon} 1$ or $|\Phi^{\sigma;\mu,\nu}(\xi,\eta)| \gtrsim_{C_b,\varepsilon} 1$ for any $(\xi,\eta) \in \mathcal{L}_{k,k_1,k_2}$, using (B.14). Assuming, for contradiction, that (B.23) fails, we need to consider two cases:

$$\begin{aligned} &\text{either} \quad k \leq D/10,\ \max(k_1,k_2) \geq D/2,\ |k_1 - k_2| \leq 10, \\ &\quad\ \text{or} \quad \min(k_1,k_2) \leq D/10,\ \max(k,k_1,k_2) \geq D/2,\ |k - \max(k_1,k_2)| \leq 10. \end{aligned} \tag{B.25}$$

In the first case, we use (B.14) to see that $|\Xi^{\mu,\nu}(\xi,\eta)| \gtrsim_{C_b,\varepsilon} 1$ for any $(\xi,\eta) \in \mathcal{L}_{k,k_1,k_2}$ if

$$\Phi^{\sigma;\mu,\nu} \in\in \{\Phi^{e;e-,b+}, \Phi^{b;e+,b+}, \Phi^{b;e-,b+}\}.$$

We also notice that $|\Xi^{\mu,\nu}(\xi,\eta)| \gtrsim_{C_b,\varepsilon} 1$ for any $(\xi,\eta) \in \mathcal{L}_{k,k_1,k_2}$ if $\Phi^{\sigma;\mu,\nu} \in\in \{\Phi^{b;e+,e+}\}$, which completes the contradiction in this case.

Assume now that the inequalities in the second line of (B.25) hold. By symmetry we may assume that $k_1 = \min(k_1,k_2)$. In view of (B.14), it is clear that $|\Xi^{\mu,\nu}(\xi,\eta)| \gtrsim_{C_b,\varepsilon} 1$ if

$$(\xi,\eta) \in \mathcal{L}_{k,k_1,k_2}$$

and

$$\Phi^{\sigma;\mu,\nu} \in \{\Phi^{e;e-,b+}, \Phi^{b;e+,b+}, \Phi^{b;e-,b+}\}.$$

Also, using (B.5) it is clear that $|\Phi^{\sigma;\mu,\nu}(\xi,\eta)| \gtrsim_{C_b,\varepsilon} 1$ if $(\xi,\eta) \in \mathcal{L}_{k,k_1,k_2}$ and $\Phi^{\sigma;\mu,\nu} \in \{\Phi^{e;,b+,e-}, \Phi^{b;e+,e+}, \Phi^{b;b+,e+}, \Phi^{b;b+,e-}\}$. This completes the contradiction in this case as well, and the desired bound (B.23) follows.

Step 7. Assume that

$$\Phi^{\sigma;\mu,\nu} \in\in \{\Phi^{i;i-,e+}, \Phi^{i;i-,b+}, \Phi^{e;i+,b+}, \Phi^{e;i-,b+}, \Phi^{b;i+,e+}\}. \tag{B.26}$$

These phases are only in the set $\mathcal{T}_A$, and we have to prove that if $\mathcal{L}^{\sigma;\mu,\nu}_{k,k_1,k_2;\delta_1,\delta_2} \neq \emptyset$, then

$$-D/2 \le k, k_1, k_2 \tag{B.27}$$

and

$$k, k_1, k_2 \le D/2. \tag{B.28}$$

To prove (B.28), assume, for contradiction, that $\max(k_1, k_2) \ge D/4$. By symmetry, we may assume also $k_1 \le k_2$. Using (B.14) it is easy to see that $|\Xi^{\mu,\nu}(\xi,\eta)| \gtrsim_{C_b,\varepsilon} 1$ if $(\xi,\eta) \in \mathcal{L}_{k,k_1,k_2}$ and

$$\Phi^{\sigma;\mu,\nu} \in \{\Phi^{i;i-,e+}, \Phi^{i;i-,b+}, \Phi^{e;i+,b+}, \Phi^{e;i-,b+}, \Phi^{b;i+,e+}\}.$$

On the other hand, if

$$\Phi^{\sigma;\mu,\nu} \in \{\Phi^{i;,e+,i-}, \Phi^{i;b+,i-}, \Phi^{e;b+,i+}, \Phi^{e;b+,i-}, \Phi^{b;e+,i+}\}$$

then, using again (B.14) and the smallness of $|\Xi^{\mu,\nu}(\xi,\eta)|$, we necessarily have $k_1 \le D/10$, $|k-k_2| \le 10$. In this case, however, $|\Phi^{\sigma;\mu,\nu}(\xi,\eta)| \gtrsim_{C_b,\varepsilon} 1$, due to (B.14). This completes the proof of the contradiction.

We prove now (B.27). We notice that $\max(k_1,k_2) \ge -D/10$; otherwise, $|\Xi^{\mu,\nu}(\xi,\eta)| \gtrsim_{C_b,\varepsilon} 1$ for any $(\xi,\eta) \in \mathcal{L}_{k,k_1,k_2}$, since $\lambda'_e(0) = \lambda'_b(0) = 0$, $\lambda'_i(0) \approx_{C_b,\varepsilon} 1$. Assume, for contradiction, that (B.27) fails. We may assume by symmetry that $k_1 \le k_2$, and we need to consider two cases:

$$\begin{aligned} &\text{either} \qquad k_1 \le -D/2, \quad k_2 \ge -D/10, \quad |k-k_2| \le 10, \\ &\text{or} \qquad k \le -D/2, \quad k_2 \ge -D/10, \quad |k_1-k_2| \le 10. \end{aligned} \tag{B.29}$$

Assume first that the inequalities in the first line of (B.29) hold. Since $\lambda'_i(r) \approx_{C_b,\varepsilon} 1$ and $\lambda'_b(0) = \lambda'_e(0) = 0$, it is easy to see that $|\Xi^{\mu,\nu}(\xi,\eta)| \gtrsim_{C_b,\varepsilon} 2^{-2D}$ if $(\xi,\eta) \in \mathcal{L}_{k,k_1,k_2}$ and

$$\Phi^{\sigma;\mu,\nu} \in \{\Phi^{i;e+,i-}, \Phi^{i;b+,i-}, \Phi^{e;b+,i+}, \Phi^{e;b+,i-}, \Phi^{b;e+,i+}, \}.$$

On the other hand, using (B.5), $|\Phi^{\sigma;\mu,\nu}(\xi,\eta)| \gtrsim_{C_b,\varepsilon} 2^{-2D}$ if $(\xi,\eta) \in \mathcal{L}_{k,k_1,k_2}$ and

$$\Phi^{\sigma;\mu,\nu} \in \{\Phi^{i;i-,e+}, \Phi^{i;i-,b+}, \Phi^{e;i+,b+}, \Phi^{e;i-,b+}, \Phi^{b;i+,e+}\}.$$

The desired contradiction follows in this case.

Finally, assume that the inequalities in the second line of (B.29) hold. Using (B.5) it is easy to see that $|\Phi^{\sigma;\mu,\nu}(\xi,\eta)| \gtrsim_{C_b,\varepsilon} 2^{-2D}$ if $(\xi,\eta) \in \mathcal{L}_{k,k_1,k_2}$ and $\Phi^{\sigma;\mu,\nu} \in\in \{\Phi^{i;i-,e+}, \Phi^{i;i-,b+}, \Phi^{e;i+,b+}, \Phi^{b;i+,e+}\}$. On the other hand, if $\Phi^{\sigma;\mu,\nu} \in\in \{\Phi^{e;i-,b+}\}$, then the contradiction follows by the same argument as in Step 4. This completes the proof of the proposition. □

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(Received: March 12, 2013)
(Revised: August 21, 2015)

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