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Kähler–Einstein metrics with edge singularities

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Abstract

This article considers the existence and regularity of Kähler–Einstein metrics on a compact Kähler manifold M with edge singularities with cone angle $2\pi\beta$ along a smooth divisor D. We prove existence of such metrics with negative, zero and some positive cases for all cone angles $2\pi\beta \leq 2\pi$. The results in the positive case parallel those in the smooth case. We also establish that solutions of this problem are polyhomogeneous, i.e., have a complete asymptotic expansion with smooth coefficients along D for all $2\pi\beta < 2\pi$.

1. Introduction

Let $D \subset M$ be a smooth divisor in a compact Kähler manifold. A Kähler edge metric on M with angle $2\pi\beta$ along D is a Kähler metric on $M\backslash D$ that is asymptotically equivalent at D to the model edge metric

$$g_\beta := |z_1|^{2\beta-2}|dz_1|^2 + \sum_{j=2}^{n} |dz_j|^2;$$

here $z_1, z_2, \ldots, z_n$ are holomorphic coordinates such that $D = \{z_1 = 0\}$ locally. We always assume that $0 < \beta \leq 1$.

Of particular interest is the existence and geometry of metrics of this type that are also Einstein. The existence of Kähler–Einstein (KE) edge metrics was first conjectured by Tian in the mid 1990's [60]. In fact, Tian conjectured the existence of KE metrics with 'crossing' edge singularities when D has simple normal crossings. One motivation was his observation that these metrics could be used to prove various inequalities in algebraic geometry; in particular, the Bogomolov–Miyaoka–Yau inequality could be proved by deforming the cone angle of Kähler–Einstein edge metrics with negative curvature to 2π. Furthermore, these metrics can be used to bound the degree of immersed curves in general type varieties. He also anticipated that the complete Tian–Yau KE metric on the complement of a divisor should be the limit of the Kähler–Einstein edge metrics as the angle $2\pi\beta$ tends to 0. Recently, Donaldson [24] proposed using these metrics in a similar way to construct smooth Kähler–Einstein metrics on Fano manifolds by deforming the cone angle of Kähler–Einstein metrics of positive curvature, and more generally to relate this approach to the much-studied obstructions to existence of smooth Kähler–Einstein metrics.

One of the main results in this article is a proof of Tian's conjecture on the existence of Kähler–Einstein edge metrics when D is smooth. In a sequel to this article we shall prove the general case [46]; this involves substantial additional complications due to the singularities of the divisor.

In the lowest dimensional setting, M is a Riemann surface and the problem is to find constant curvature metrics with prescribed conic singularities (with cone angle less than 2π) at a finite collection of points. This was accomplished in general by McOwen and Troyanov [49], [67]; as part of this, Troyanov found some interesting restrictions on the cone angles necessary for the existence of spherical cone metrics. Later, Luo and Tian [41] established the uniqueness of these metrics. For the problem in higher dimensions, we focus only on the case where D is smooth, unless explicitly stated. A preliminary study of the case of Kähler–Einstein edge metrics with negative curvature appeared in the thesis of the first named author [33], where it was already suggested that some of the *a priori* estimates of Aubin and Yau [2], [70] should carry over to this setting when $\beta \in (0, \frac{1}{2}]$. An announcement for the existence in that negative case with $\beta \in (0, \frac{1}{2}]$ was made over ten years ago by the first and second named authors [45]. There were several analytic issues described in that announcement that seemed to complicate the argument substantially, and details never appeared.

Recently there has been a renewed interest in these problems stemming from an important advance by Donaldson [24], alluded to just above, whose insightful observations make it possible to establish good linear estimates. He proves a deformation theorem, showing that the set of attainable cone angles for KE edge metrics is open. The key to his work is the identification of a function space in the space of bounded functions on which the linearized Monge–Ampère equation is solvable.

We realized, immediately following the appearance of [24], that the change of perspective suggested by his advance makes it possible to apply the theory of elliptic edge operators from [43] so as to circumvent the difficulties surrounding the openness part of the argument proposed in [45]. Indeed, we show that estimates equivalent to those of Donaldson (but on slightly different function spaces) follow from some of the basic results in that theory, and we explain this at some length in this paper. This alternate approach to the linear theory allows us to go somewhat further, and we use it to show that solutions are polyhomogeneous, i.e., have complete asymptotic expansions in possibly noninteger powers of the distance to the divisor and positive integer powers of the log of this distance function, with all coefficients smooth along the divisor. This was announced in [45] and speculated on in [24], and the existence of this higher regularity should be very helpful in the further study of these metrics.

What we achieve here is the following. We prove existence of Kähler–Einstein edge (KEE) metrics with cone angle $2\pi\beta$ that have negative, zero and positive curvature, as appropriate, for all cone angles $2\pi\beta \leq 2\pi$, when D is smooth. Existence in the positive case is proved under the condition that the twisted Mabuchi K-energy is proper, in parallel to Tian's result in the smooth case [61]. Next, we prove that solutions of a general class of complex Monge–Ampère equations are polyhomogeneous, i.e., have complete asymptotic expansions with smooth coefficients. We provide a sharper identification of the function space defined by Donaldson for his deformation result. As we have briefly noted above, there are two slightly different scales of Hölder spaces that play a role in this type of problem. One, used in [24], we call the wedge Hölder spaces; the other, from [43], are the edge Hölder spaces. Functions in the wedge Hölder spaces are slightly more regular, which is crucial in certain parts of the argument; on the other hand, the edge Hölder spaces are invariant with respect to the dilation structure inherent in this problem, which makes the linear theory, and certain parts of the nonlinear theory, more transparent.

We shall employ these spaces at various points in the argument. What makes it possible to go from the edge spaces back to the wedge spaces is Tian's regularity argument from Appendix B, which shows that any solution to the Monge–Ampère equation that is bounded along with its Laplacian is automatically in a wedge Hölder space. The results in Section 4 then show that the solution is polyhomogeneous.

The key new ingredient for deriving the nonlinear *a priori* estimates is the new *Ricci continuity method*, which can be considered as a continuity method analogue of the Ricci flow. This was introduced in the context of the Ricci iteration by the third named author [52], and one point of this article is to show that it is perhaps the best suited for proving existence of Kähler–Einstein metrics. Indeed, we derive our estimates also for more classical continuity paths studied in the literature and at the appropriate junctures indicate how these break down unless β is in the restricted "orbifold range" $(0, \frac{1}{2}]$, while this new continuity method works for all $\beta \in (0, 1]$.

In proving the *a priori* estimates we have made an effort to extend various classical arguments and bounds to this singular setting with minimal assumptions on the background geometry. In particular, the Ricci continuity method together with the Chern–Lu inequality allows us to obtain the *a priori* estimate on the Laplacian assuming only that the reference edge metric has bisectional curvature bounded above. We then explain how the Evans–Krylov theory together with our asymptotic expansion imply *a priori* Hölder bounds on the second derivatives for all cone angles with no further curvature assumptions. Reducing the dependence of the estimates for the existence of a Kähler–Einstein metric to only an upper bound on the bisectional curvature of the reference metric does not seem to have been observed previously even in the smooth setting, where traditionally a lower bound on the bisectional curvature is required, or at least an upper bound on the bisectional curvature together with a lower bound on some curvature. Thus, as a by-product, we also obtain a new and unified proof of the classical results of Aubin, Yau and Tian, on existence of KE metrics on smooth compact Kähler manifolds. Finally, in the case of positive curvature, we show how to control the Sobolev constant and infimum of the Green function, which are both needed for the uniform estimate. In an appendix it is shown that the bisectional curvature of one reference metric is bounded from above on $M\backslash D$ whenever $\beta \in (0,1]$. The difficult calculations to establish this were obtained by the third named author and Chi Li, and this appendix constitutes yet another necessary component of this work.

Before stating our results, let us mention some other recent articles concerning existence. It was expected that if one were to have linear estimates such as the ones obtained by Donaldson [24], and if $\beta \in (0, \frac{1}{2}]$, so that the curvature of the reference metric is bounded, then it should be possible to adapt the classical Aubin–Yau *a priori* estimates and hence obtain existence when $\mu \leq 0$. This was carried out in [15]. Another quite different approach to existence for $\beta \in (0, \frac{1}{2}]$ and $\mu \leq 0$ but allowing divisors with simple normal crossings, based on approximation by smooth metrics (and thus avoiding the linear estimates), is due to Campana, Guenancia and Păun [16]. Both [15] and [16] appeared around the same time as the present article. Finally, in a different direction, Berman [9] showed how to bypass the linear estimates and produce KE metrics whose volume form is asymptotic to that of an edge metric using a variational approach. However, neither of these methods give good information about the regularity or the geometry of the solution metric near the divisor.

We now state our main results more precisely. Since some of the terminology in these two theorems is perhaps unfamiliar in complex geometry, we recall the notion of polyhomogeneity described briefly earlier in this introduction. The existence of a polyhomogeneous expansion should be regarded as an optimal regularity statement for a solution, and it is the natural and unavoidable replacement for smoothness for these types of degenerate problems. Just as with the Taylor expansions for smooth functions, the asymptotic expansions we use in this paper are rarely convergent. We refer to Sections 2 and 3 for more on this and for all relevant notation.

THEOREM 1 (Asymptotic expansion of solutions). *Let ω be a polyhomogeneous Kähler edge metric with angle $2\pi\beta \in (0, 2\pi]$. Suppose that, for some Hölder exponent $\gamma \in (0, 1)$, $u \in \mathcal{D}_s^{0,\gamma} \cap \mathrm{PSH}(M, \omega)$, $s = w$ or e, is a solution of the complex Monge–Ampère equation*

$$\omega_u^n = \omega^n e^{f - su}, \qquad \text{on } M \setminus D, \tag{1}$$

where $\omega_u = \omega + \sqrt{-1}\,\partial\bar\partial u$ and $f \in \mathcal{A}^0_{\mathrm{phg}}(X)$. Then u is polyhomogeneous, i.e., $u \in \mathcal{A}^0_{\mathrm{phg}}(X)$.

This result admits a straightforward generalization if the exponential on the right-hand side is replaced by a function $F(z, u)$ that is polyhomogeneous in its arguments and is such that if $u \in \mathcal{A}^0_{\mathrm{phg}}$, then $F(z, u) \in \mathcal{A}^0_{\mathrm{phg}}$.

THEOREM 2 (Kähler–Einstein edge metrics). *Let (M, ω_0) be a compact Kähler manifold with $D \subset M$ a smooth divisor, and suppose $\mu[\omega_0] + (1-\beta)[D] = c_1(M)$, where $\beta \in (0, 1]$ and $\mu \in \mathbb{R}$. If $\mu > 0$, suppose in addition that the twisted K-energy E_0^β is proper. Then there exists a Kähler–Einstein edge metric $\omega_{\varphi_{\mathrm{KE}}}$ with Ricci curvature μ and with angle $2\pi\beta$ along D. This metric is unique when $\mu < 0$, unique in its Kähler class when $\mu = 0$ and unique up to automorphisms that preserve D when $\mu > 0$. This metric is polyhomogeneous; namely, φ_{KE} admits a complete asymptotic expansion with smooth coefficients as $r \to 0$ of the form*

$$\varphi_{\mathrm{KE}}(r, \theta, Z) \sim \sum_{j,k \geq 0} \sum_{\ell=0}^{N_{j,k}} a_{jk\ell}(\theta, Z) r^{j + k/\beta} (\log r)^\ell, \tag{2}$$

where $r = |z_1|^\beta/\beta$ and $\theta = \arg z_1$, and with each $a_{jk\ell} \in \mathcal{C}^\infty$. There are no terms of the form $r^\zeta (\log r)^\ell$ with $\ell > 0$ if $\zeta \leq 2$. In particular, φ_{KE} has infinite conormal regularity and a precise Hölder regularity as measured relative to the reference edge metric ω, which is encoded by $\varphi_{\mathrm{KE}} \in \mathcal{A}^0 \cap \mathcal{D}_w^{0,\gamma}$.

We refer to Proposition 4.4 for the determination of the first several terms in the expansion (2).

To clarify the conclusions about regularity in these theorems, we first prove infinite 'conormal' regularity ($\varphi \in \mathcal{A}^0$), which means simply that the solution is tangentially smooth and also infinitely differentiable with respect to the vector field $r\partial_r$; we then establish Hölder continuity of some second derivatives with respect to the model metric ($\varphi \in \mathcal{D}_w^{0,\gamma}$); finally, we prove the existence of an asymptotic expansion in powers of the distance to the edge ($\varphi \in \mathcal{A}^0_{\mathrm{phg}}$). This expansion also leads to the precise asymptotics of the curvature tensor and its covariant derivatives. For example, when $\beta \leq \frac{1}{2}$, we have $\varphi_{\mathrm{KE}} \in \mathcal{C}_w^{2, \frac{1}{\beta} - 2}$, all third derivatives of the form $(\varphi_{\mathrm{KE}})_{i\bar{j}k}$ belong to $\mathcal{C}_w^{0, \frac{1}{\beta} - 2}$, and therefore so do all Christoffel symbols, and the curvature tensor of $\omega_{\varphi_{\mathrm{KE}}}$ is

Hölder continuous. However, assuming only that $\beta \le 1$, we have $\Delta_\omega \varphi_{\mathrm{KE}} \in \mathcal{C}^{0,\gamma}_w$ for some $\gamma \in (0, \frac{1}{\beta} - 1]$, but in general the curvature tensor does not lie in L^∞. (This follows readily from the calculations of the appendix.) Again, we refer to Proposition 4.4 for more precise information.

Theorem 2 is the generalization to the edge setting of the classical theorems of Aubin, Yau ($\mu \le 0$) and Tian ($\mu > 0$) on existence of KE metrics in the compact smooth setting [2], [70], [61]. Its proof gives a new and unified treatment for all μ even in the smooth setting. It is also a satisfactory generalization of Troyanov's theorem on the existence of constant curvature metrics with conic singularities on Riemann surfaces [67] inasmuch as the cone angle restrictions that appear in his work arise only in the positive curvature case, and they are the same as the properness of the twisted K-energy in that setting. Just as for the smooth setting [61], the properness assumption should be a necessary condition for existence in the absence of holomorphic vector fields that are tangent to D.

Finally, consider the special case that M is Fano and D is a smooth anticanonical divisor. (The existence of such a divisor is related to the so-called Elephant Conjectures in algebraic geometry and is known when $n \le 3$ by work of Shokurov and others.) Then, as noted by Berman [9], the twisted K-energy is proper for small $\mu = \beta$. Theorem 2 thus gives the following corollary conjectured by Donaldson [23].

COROLLARY 1. *Let M be a Fano manifold, and suppose that there exists a smooth anticanonical divisor $D \subset M$. Then there exists some $\beta_0 \in (0, 1]$ such that for all $\beta \in (0, \beta_0)$, there exists a* KEE *metric with angle $2\pi\beta$ along D and with positive Ricci curvature equal to β.*

Added in revision: There has been substantial work in this area in the years following the initial appearance of this article; cf., in particular, the papers [22], [51], [17], [18], [19], [63]. We refer the reader to the survey [53] for further references and background. In both [18] and [63], the construction of a smooth KE metric is carried out by studying the deformations of a KEE metric as the cone angle increases.

Our original proof of the $\mathcal{D}^{0,\gamma}_w$ estimate had an error, now corrected by Appendix B. The paper [18] contains a different approach to this estimate.

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2. Preliminaries

We set the stage for the rest of the article with a collection of facts and results needed later. First consider the flat model situation, where $M = \mathbb{C}^n$ with linear coordinates $(z_1, \ldots, z_n)$ and D is the linear subspace $\{z_1 = 0\}$. For brevity we often write

$$Z = (z_2, \ldots, z_n).$$

The model singular Kähler form and singular Kähler metric are given by

$$\omega_\beta = \frac{1}{2}\sqrt{-1}\left(|z_1|^{2\beta-2} dz^1 \wedge \overline{dz^1} + \sum_{j=2}^{n} dz^j \wedge \overline{dz^j}\right) \tag{3}$$

and

$$g_\beta = |z_1|^{2\beta-2}|dz^1|^2 + \sum_{j=2}^{n} |dz^j|^2. \tag{4}$$

This is the product of a flat one complex dimensional conic metric with cone angle $2\pi\beta$ with $\mathbb{C}^{n-1}$. We always assume that $0 < \beta \le 1$; the expressions above make sense for any real β, but their geometries are quite different for β outside of this range.

Now suppose that M is a compact Kähler manifold and D a smooth divisor. Fix $\beta \in (0, 1]$ and $\mu \in \mathbb{R}$, and assume that there is a Kähler class $\Omega = \Omega_{\mu,\beta}$ such that

$$\mu\Omega + 2\pi(1-\beta)c_1(L_D) = 2\pi c_1(M). \tag{5}$$

Here, L_D is the line bundle associated to D. Thus, $c_1(M) - (1-\beta)c_1(L_D)$ is a positive or negative class if $\mu > 0$ or $\mu < 0$. If $\mu = 0$, then Ω is an arbitrary Kähler class.

Let g be any Kähler metric that is smooth (or of some fixed finite regularity) on $M \setminus D$. We shall say that g is a Kähler edge metric with angle $2\pi\beta$ if, in any local holomorphic coordinate system near D where $D = \{z_1 = 0\}$ and $z_1 = \rho e^{\sqrt{-1}\theta}$,

$$g_{1\bar{1}} = F\rho^{2\beta-2}, \; g_{1\bar{j}} = g_{i\bar{1}} = O(\rho^{\beta-1+\eta'}), \text{ and all other } g_{i\bar{j}} = O(1), \tag{6}$$

for some $\eta' > 0$, where F is a bounded nonvanishing function that is at least continuous at D (and that will have some specified regularity). If this is the case, we say that g is asymptotically equivalent to g_β and that its associated Kähler form ω (which by abuse of terminology we sometimes also refer to as a metric) is asymptotically equivalent to ω_β. There are slightly weaker hypotheses under which it is reasonable to say that g has angle $2\pi\beta$ at D, but the definition we have given here is sufficient for our purposes. We denote by $\operatorname{Ric}\omega$ the Ricci current (on M) associated to ω, namely, in local coordinates $\operatorname{Ric}\omega = -\sqrt{-1}\partial\bar\partial \log\det[g_{i\bar j}]$ if $\omega = \sqrt{-1}g_{i\bar j}dz^i\wedge\overline{dz^j}$. Thus, $\operatorname{Ric}\omega - 2\pi(1-\beta)[D]$ is a $(1,1)$ current on M with a continuous potential, where $[D]$ is the current associated to integration along D.

Definition 2.1. With all notation as above, a Kähler current ω, with associated singular Kähler metric g, is called a Kähler–Einstein edge current, respectively metric, with angle $2\pi\beta$ along D and Ricci curvature μ if ω and g are asymptotically equivalent to ω_β and g_β and if

$$\operatorname{Ric}\omega - 2\pi(1-\beta)[D] = \mu\omega. \tag{7}$$

In this section we present some preliminary facts about the geometry and analysis of the class of Kähler edge metrics. We first review some different coordinate charts near the edge D used extensively below. Many calculations in this article are most easily done in a singular real coordinate chart, although when the complex structure is particularly relevant to a calculation, we use certain adapted complex coordinate charts. While all of this is quite elementary, there are some identifications that can be confusing, so it is helpful to make all of this very explicit. We calculate the curvature tensor for any one such metric g, assuming it is sufficiently regular. We then introduce the relevant class of Kähler edge potentials and describe the continuity method that will be used for the existence theory. As we recall, this particular continuity method is closely related to the Ricci iteration that, naturally, we also treat simultaneously in this article. We conclude the section with a fairly lengthy description of the various function spaces that will be used later. Rather than a purely technical matter, this discussion gets to the heart of some of the more important analytic and geometric issues that must be faced here. There are two rather different choices of Hölder spaces; one is naturally associated to this class of Kähler edge metrics and was employed, albeit in a slightly different guise, by Donaldson [24], while the other, from [43], is well adapted to this edge geometry because of its naturality under dilations and has been used in many other analytic and geometric problems where edges appear. Use of these latter function spaces is central to our method.

2.1. *Coordinate systems.* Fix local complex coordinates $(z_1,\ldots,z_n) = (z_1, Z)$ with $D = \{z_1 = 0\}$ locally. There are two other coordinate systems that are quite useful for certain purposes. The first is a singular holomorphic coordinate chart, where we replace z_1 by $\zeta = z_1^\beta/\beta$. Of course, ζ is multi-valued, but we can work locally in the logarithmic Riemann surface that uniformizes this variable. Thus if $z_1 = \rho e^{\sqrt{-1}\theta}$, then $\zeta = re^{\sqrt{-1}\tilde\theta}$, where $r = \rho^\beta/\beta$ and $\tilde\theta = \beta\theta$. The second is the real cylindrical coordinate system (r,θ,y) around D, where

$r = |\zeta|$ as above, θ is the argument of z_1, and $(y_1, \ldots, y_{2n-2}) = (\operatorname{Re} Z, \operatorname{Im} Z)$. Note that $re^{\sqrt{-1}\theta} = z_1 |z_1|^{\beta-1}/\beta$. We use either (z_1, Z) or (ζ, Z) in situations where the formalism of complex analysis is useful, and (r, θ, y) elsewhere. For later purposes, note that

$$(8) \quad d\zeta = z_1^{\beta-1} dz_1 \Leftrightarrow dz_1 = (\beta\zeta)^{\frac{1}{\beta}-1} d\zeta, \qquad \frac{\partial \zeta}{\partial z_1} = z_1^{\beta-1} \Leftrightarrow \frac{\partial z_1}{\partial \zeta} = (\beta\zeta)^{\frac{1}{\beta}-1}.$$

One big advantage of either of these other coordinate systems is that they make the model metric g_β appear less singular. Indeed,

$$(9) \qquad g_\beta = |d\zeta|^2 + |dZ|^2 = dr^2 + \beta^2 r^2 d\theta^2 + |dy|^2.$$

In either case, one may regard the coordinate change as encoding the singularity of the metric via a singular coordinate system. This is only possible for edges of real codimension two, and there are many places, both in [24] and here, where we take advantage of this special situation. For edges of higher codimension, one cannot conceal the singular geometry so easily; see [43]. The expression for g_β in cylindrical coordinates makes clear that for any β, β', we have $C_1 g_\beta \le g_{\beta'} \le C_2 g_\beta$; the corresponding inequality in the original z coordinates must be stated slightly differently, as $C_1 g_\beta \le \Phi^* g_{\beta'} \le C_2 g_\beta$, where $\Phi(z_1, \ldots, z_n) = (z_1^{\beta'/\beta}, z_2, \ldots, z_n)$.

We now compute the complex derivatives in these coordinates. We have

$$(10) \ \partial_{z^1} = \frac{1}{2} e^{-\sqrt{-1}\theta} \left(\partial_\rho - \frac{\sqrt{-1}}{\rho} \partial_\theta \right) = \frac{1}{2} e^{-\sqrt{-1}\theta} (\beta r)^{1-\frac{1}{\beta}} \left(\partial_r - \frac{\sqrt{-1}}{\beta r} \partial_\theta \right),$$

and then

$$(11) \qquad \partial^2_{z^1 \overline{z^1}} = (\beta r)^{2-\frac{2}{\beta}} \left(\partial_r^2 + \frac{1}{r} \partial_r + \frac{1}{\beta^2 r^2} \partial_\theta^2 \right).$$

The other mixed complex partials $\partial^2_{z^1 \overline{z^j}}$, $\partial^2_{z^i \overline{z^1}}$ and $\partial^2_{z^i \overline{z^j}}$ are compositions of the operator in (10) and its conjugate and certain combinations of the ∂_{y_ℓ}. From this we obtain that

$$(12) \qquad \Delta_{g_\beta} u = \sum_{i,j=1}^n (g_\beta)^{i\bar{j}} u_{i\bar{j}} = \left(\partial_r^2 + \frac{1}{r} \partial_r + \frac{1}{\beta^2 r^2} \partial_\theta^2 + \Delta_y \right) u,$$

since $(g_\beta)^{1\bar{1}} = \rho^{2-2\beta} = (\beta r)^{\frac{2}{\beta}-2}$ and $(g_\beta)^{1\bar{j}}, (g_\beta)^{i\bar{1}} = 0$ and all other $(g_\beta)^{i\bar{j}} = \delta^{ij}$.

As already described, we shall work with the class of Kähler metrics g that satisfy condition (6), and which we call asymptotically equivalent to g_β. If g is of this type, then

$$(13) \quad g^{1\bar{1}} = F^{-1} \rho^{2-2\beta}, \quad g^{1\bar{j}}, g^{i\bar{1}} = O(\rho^{\eta'+1-\beta}), \quad \text{and all other } g^{i\bar{j}} = O(1)$$

for some $\eta' > 0$, hence

$$\Delta_g = F^{-1}\left(\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{\beta^2 r^2}\partial_\theta^2 + \sum_{r,s=1}^{2n-2} c_{rs}(r,\theta,y)\partial^2_{y_r y_s}\right) + E, \tag{14}$$

where

$$E := r^{\eta-2} \sum_{i+j+|\mu|\le 2} a_{ij\mu}(r,\theta,y)(r\partial_r)^i \partial_\theta^j (r\partial_y)^\mu.$$

Here $\eta > 0$ is determined from η' and β, and all coefficients have some specified regularity down to $r = 0$. In particular, the coefficient matrix (c_{rs}) is positive definite, with $c_{rs}(0,\theta,y)$ independent of θ, and the coefficients $a_{ij\mu}$ are bounded as $r \to 0$. Thus there are no cross-terms to leading order, and the $1\bar{1}$ part of the operator Δ_g is 'standard' once we multiply the entire operator by F.

One way that this asymptotic structure will be used is as follows. Fundamental to this work is the role of the family of dilations $S_\lambda : (r,\theta,y) \mapsto (\lambda r, \theta, \lambda y)$ centered at some point $p \in D$ corresponding to $y = 0$. If we push forward this operator by S_λ, which has the effect of expanding a very small neighbourhood of p, then the principal part scales approximately like λ^2 while E scales like $\lambda^{2-\eta}$. Hence, after a linear change of the y coordinates,

$$A\lambda^{-2}(S_\lambda)_* \Delta_g \longrightarrow \Delta_{g_\beta} \quad \text{as} \quad \lambda \to \infty, \tag{15}$$

where $A = F(p)$. In particular, E scales away completely in this limit.

One important comment is that if the derivatives $u_{i\bar{j}}$ are all bounded, and if g satisfies these asymptotic conditions, then so does $\tilde{g}$, where $\tilde{g}_{i\bar{j}} = g_{i\bar{j}} + u_{i\bar{j}}$.

A key point in the treatment below, exploited by Donaldson [24], is that for any Kähler metric g, Δ_g only involves combinations of the following second-order operators:

$$\begin{aligned}
P_{1\bar{1}} &= \left(\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{\beta^2 r^2}\partial_\theta^2\right),\\
P_{1\bar{\ell}} &= \left(\partial_r - \frac{\sqrt{-1}}{\beta r}\partial_\theta\right)\partial_{\overline{z_\ell}},\\
P_{\ell\bar{1}} &= \left(\partial_r + \frac{\sqrt{-1}}{\beta r}\partial_\theta\right)\partial_{z_\ell}, \quad \ell = 2,\ldots,n, \text{ and}\\
P_{\ell\bar{k}} &= \partial^2_{z_\ell \overline{z_k}}, \quad \ell, k = 2,\ldots,n.
\end{aligned}$$

Regularity properties in certain function spaces considered below involve precisely these derivatives, while others are less sensitive about the decomposition of ∂_{z_j} and $\partial_{\overline{z_j}}$ into their $(1,0)$ and $(0,1)$ parts. We therefore introduce the following collections of differential operators:

$$\begin{aligned}
\mathcal{Q} &= \{\partial_r, r^{-1}\partial_\theta, \partial_{y_\ell}, \partial^2_{ry_\ell}, \partial^2_{r\theta}, \partial^2_{\theta y_\ell}\},\\
\mathcal{Q}^* &= \mathcal{Q} \sqcup \{\partial^2_{y_k y_\ell}, P_{1\bar{1}}\}.
\end{aligned} \tag{16}$$

The reason for singling out the extra operators in $\mathcal{Q}^* \setminus \mathcal{Q}$ is that the relevant boundedness properties are more subtle for these.

As a final note, let us record the form of the complex Monge–Ampère operator in these coordinates for any Kähler metric that satisfies the decay assumptions above. We have

$$(\omega + \sqrt{-1}\,\partial\bar\partial u)^n/\omega^n = \frac{\det(g_{i\bar\jmath} + \sqrt{-1}u_{i\bar\jmath})}{\det g_{i\bar\jmath}} = \det(\delta_i^j + \sqrt{-1}u_i^j),$$

where $u_i^j = u_{i\bar k} g^{j\bar k}$. Using the calculations above, we have

$$\begin{aligned}
u_1{}^1 &= F^{-1}P_{1\bar 1}u + O(r^\eta)u_{1\bar\jmath},\\
u_1{}^j &= e^{-\sqrt{-1}\,\theta}(\beta r)^{1-\frac{1}{\beta}} g^{j\bar k}P_{1\bar k}u + O(r^{\eta+\frac{1}{\beta}-1})P_{1\bar 1}u,\\
u_i{}^1 &= F^{-1}e^{\sqrt{-1}\,\theta}(\beta r)^{\frac{1}{\beta}-1}P_{i\bar 1}u + O(r^{\eta+\frac{1}{\beta}-1})u_{i\bar\jmath},\\
u_i{}^j &= g^{j\bar k}P_{i\bar k}u + O(r^\eta)u_{i\bar 1}.
\end{aligned}$$

This means that if we multiply every column but the first in $(\delta_i{}^j + \sqrt{-1}u_i{}^j)$ by $e^{\sqrt{-1}\theta}(\beta r)^{1/\beta-1}$ and every row but the first by $e^{-\sqrt{-1}\theta}(\beta r)^{1-1/\beta}$, then the determinant remains the same, and we have shown that

(17)
$$\frac{\det(g_{i\bar\jmath} + \sqrt{-1}u_{i\bar\jmath})}{\det g_{i\bar\jmath}} = \det\begin{pmatrix} 1 + F^{-1}P_{1\bar 1}\,u & F^{-1}P_{2\bar 1}\,u & \dots & F^{-1}P_{n\bar 1}u\\ \vdots & \vdots & \vdots & \vdots\\ g^{n\bar k}P_{1\bar k}u & \dots & \dots & 1 + g^{n\bar\ell}P_{n\bar\ell}\,u\end{pmatrix} + R,$$

where $R = r^\eta R_0(u_{p\bar q})$, with R_0 polynomial in its entries.

2.2. *Kähler edge potentials.* Fix a smooth Kähler form ω_0 with $[\omega_0] \in \Omega \equiv \Omega_{\mu,\beta}$. Consider the space of all Kähler potentials relative to ω_0, asymptotically equivalent to the model metric,

$$\begin{aligned}(18)\quad \mathcal{H}_{\omega_0} := \{\varphi \in \mathcal{C}^\infty(M\setminus D) \cap \mathcal{C}^0(M) \,:\, \omega_\varphi := \omega_0 + \sqrt{-1}\partial\bar\partial\varphi > 0 \text{ on } M\\ \text{and } \omega_\varphi \text{ asymptotically equivalent to } \omega_\beta\}.\end{aligned}$$

Note that in our notation, $\mathcal{H}_{\omega_0} \cong \mathcal{H}_\eta$ for any smooth η cohomologous to ω_0 but not for any $\eta \in \mathcal{H}_{\omega_0}$. The first observation is that such Kähler edge metrics exist.

LEMMA 2.2. *Let $\beta \in (0,1]$. Then $\mathcal{H}_{\omega_0}$ is nonempty.*

Proof. Let h be a smooth Hermitian metric on L_D, and let s be a global holomorphic section of L_D so that $D = s^{-1}(0)$. We claim that for $c > 0$ sufficiently small, the function

$$\phi_0 := c|s|_h^{2\beta} = c(|s|_h^2)^\beta \tag{19}$$

belongs to $\mathcal{H}_{\omega_0}$. To prove this, it suffices to consider $p \in M \setminus D$ near D. Use a local holomorphic frame e for L_D and local holomorphic coordinates $\{z_i\}_{i=1}^n$ valid in a neighborhood of p, such that $s = z_1 e$, so that locally D is cut out by z_1. Let

$$a := |e|_h^2, \tag{20}$$

and set $H := a^\beta$, so $|s|_h^{2\beta} = H|z_1|^{2\beta}$. Note that H is smooth and positive. Then

$$\begin{aligned}\sqrt{-1}\,\partial\bar\partial|s|_h^{2\beta} = {}& \beta^2 H|z_1|^{2\beta-2}\sqrt{-1}\,dz_1\wedge\overline{dz_1} \\ & + 2\beta\mathrm{Re}(|z_1|^{2\beta}{z_1}^{-1}\sqrt{-1}\,dz_1\wedge\bar\partial H) + |z_1|^{2\beta}\sqrt{-1}\,\partial\bar\partial H.\end{aligned} \tag{21}$$

For $c > 0$ small, the form $\omega_0 + \sqrt{-1}\partial\bar\partial\phi_0$ is positive definite and satisfies the conditions of (6) and hence is asymptotically equivalent to g_β. $\square$

It is useful to record the form of ω_{ϕ_0} in the (ζ, Z) coordinates as well. First note that if ψ_0 is a Kähler potential for ω_0, then using (8),

$$\begin{aligned}\sqrt{-1}\,\partial\bar\partial\psi_0 = {}& (\psi_0)_{z_1\bar z_1}|\beta\zeta|^{\frac{2}{\beta}-2}\sqrt{-1}\,d\zeta\wedge\overline{d\zeta} \\ & + \sum_{j>1} 2\mathrm{Re}((\psi_0)_{z_1\bar z_j}(\beta\zeta)^{\frac{1}{\beta}-1}\sqrt{-1}\,d\zeta\wedge\overline{dz_j}) \\ & + \sum_{i,j>1}(\psi_0)_{z_i\bar z_j}\sqrt{-1}\,dz_i\wedge\overline{dz_j}.\end{aligned} \tag{22}$$

Next, $|s|_h^{2\beta} = \beta^2 H|\zeta|^2$, hence

$$\sqrt{-1}\,\partial\bar\partial(c|s|_h^{2\beta}) = c\beta^2\Big(\sqrt{-1}\,Hd\zeta\wedge\overline{d\zeta} + 2\mathrm{Re}(\bar\zeta\sqrt{-1}\,d\zeta\wedge\bar\partial H) + |\zeta|^2\sqrt{-1}\,\partial\bar\partial H\Big). \tag{23}$$

From these two expressions, it is clear once again that $\phi_0 \in \mathcal{H}_{\omega_0}$ when c is sufficiently small. Putting these expressions together shows that $\omega_0+\sqrt{-1}\,\partial\bar\partial\phi_0$ is locally equal to

$$\begin{aligned}&\Big(|\beta\zeta|^{\frac{2}{\beta}-2}(\psi_0)_{z_1\overline{z_1}} + c\beta^2 H + c|\beta\zeta|^{\frac{2}{\beta}}H_{z_1\overline{z_1}} + 2c\beta^{\frac{1}{\beta}+1}\mathrm{Re}(\bar\zeta^{\frac{1}{\beta}}H_{\overline{z_1}})\Big)\sqrt{-1}\,d\zeta\wedge\overline{d\zeta} \\ &\quad + 2\mathrm{Re}\sum_{j>1}\Big((\beta\zeta)^{\frac{1}{\beta}-1}(\psi_0)_{z_1\overline{z_j}} + c\beta^2\bar\zeta H_{\overline{z^j}} + c\beta^{\frac{1}{\beta}+1}\zeta^{\frac{1}{\beta}}\bar\zeta H_{z_1\overline{z^j}}\Big)\sqrt{-1}\,d\zeta\wedge\overline{dz^j} \\ &\quad + \sqrt{-1}\,\partial_Z\bar\partial_Z\psi_0 + |\zeta|^2\sqrt{-1}\,\partial_Z\bar\partial_Z H.\end{aligned} \tag{24}$$

The reason for writing the derivatives of ψ_0 and H with respect to z_1 rather than ζ is because we know that both of these functions are smooth in the original z coordinates, and hence so are its derivatives with respect to z.

We now use this expression to deduce some properties of the curvature tensor of g. This turns out to be simple in this singular holomorphic coordinate system. The coefficients of the $(0,4)$ curvature tensor are given by

$$R_{i\bar{j}k\bar{l}} = -g_{i\bar{j},k\bar{l}} + g^{s\bar{t}}\, g_{i\bar{t},k}\, g_{s\bar{j},\bar{l}}, \tag{25}$$

where the indices after a comma indicate differentiation with respect to a variable. In the following, contrary to previous notation, we temporarily use the subscripts 1 and $\bar{1}$ to denote components of the metric or derivatives with respect to ζ and $\bar{\zeta}$, not z_1 or $\overline{z_1}$.

LEMMA 2.3. *The curvature tensor of* $\omega = \omega_0 + \sqrt{-1}\,\partial\bar{\partial}\phi_0$ *is uniformly bounded on* $M\backslash D$ *provided* $\beta \in (0, \frac{1}{2}]$.

Proof. Since in the (ζ, Z) coordinates, $cI < [g_{i\bar{j}}] < CI$, it suffices to show that $|R_{i\bar{j}k\bar{l}}| < C$. From (24),

$$\begin{aligned} &g_{1\bar{1},1} = O\left(|\zeta|^{\frac{2}{\beta}-3}\right), && g_{1\bar{j},1} = O\left(|\zeta|^{\frac{1}{\beta}-2}\right), && g_{i\bar{j},1} = O\left(|\zeta|^{\frac{1}{\beta}-1}\right),\\ &g_{1\bar{1},k} = O(1), && g_{1\bar{j},k} = O\left(|\zeta| + |\zeta|^{\frac{1}{\beta}-1}\right), && g_{i\bar{j},k} = O(1). \end{aligned}$$

Similarly, $|g_{i\bar{j},k\bar{l}}| \le C\left(1 + |\zeta|^{\frac{1}{\beta}-2} + |\zeta|^{\frac{2}{\beta}-4}\right)$. □

As conjectured by Donaldson, there seem to be genuine cohomological obstructions to finding reference edge metrics with bounded curvature when $\beta > 1/2$. Nevertheless, in Proposition A.1 it is shown that the bisectional curvature of ω is bounded from above on $M\backslash D$ provided $\beta \le 1$. This fact comes out of the proof as some kind of miracle, yet it would be enlightening to have a more geometric explanation for it. In a related vein, we remark in passing that it is not difficult to write down local expressions (near D) for metrics equivalent to the model metric that have bisectional curvature unbounded from above or below or both. For instance, when $n = 1$, the curvature of $(|z_1|^{2\beta-2} - 1)|dz_1|^2$ equals $(1-\beta)^2\rho^{-2\beta}/(1-\rho^{2-2\beta})^2$ (here $\rho = |z_1|$) and hence tends to $+\infty$ as $\rho \searrow 0$. More generally, one can easily choose ψ polyhomogeneous but not smooth so that the curvature of the metric $|z_1|^{2\beta-2}e^{\psi}|dz_1|^2$ is unbounded, either above or below or both. In other words, the asymptotics of the curvature depend on the higher order terms in the expansion of the metric.

2.3. *The twisted Ricci potential.* From now on (except when otherwise stated) we denote

$$\omega := \omega_0 + \sqrt{-1}\,\partial\bar{\partial}\phi_0 \in \mathcal{H}_{\omega_0}, \tag{26}$$

with ϕ_0 given by (19). In the remainder of this article, we refer to ω as the reference metric. Define f_ω by

$$\sqrt{-1}\partial\bar{\partial} f_\omega = \operatorname{Ric}\omega - 2\pi(1-\beta)[D] - \mu\omega, \tag{27}$$

where $[D]$ denotes the current of integration along D, and with the normalization

$$\frac{1}{V}\int_M e^{f_\omega}\omega^n = 1, \quad \text{where } V := \int_M \omega^n. \tag{28}$$

We call this the twisted Ricci potential (see Lemma 4.5 for precise regularity of f_ω); this terminology refers to the fact that the adjoint bundle $K_M + (1-\beta)L_D$ takes the place of the canonical bundle K_M. Alternatively, one can also think of $\operatorname{Ric}\omega - 2\pi(1-\beta)[D]$ as a kind of Bakry-Émery Ricci tensor.

2.4. *The Ricci continuity method for the twisted Kähler–Einstein equation.* The existence of Kähler–Einstein metrics asymptotically equivalent to g_β is governed by the Monge–Ampère equation

$$\omega_\varphi^n = e^{f_\omega - \mu\varphi}\omega^n. \tag{29}$$

We seek a solution $\varphi \in \mathcal{H}_{\omega_0}$, and we shall do so using a particular continuity method. We consider a continuity path in the space of metrics $\mathcal{H}_\omega$ (with some specified regularity) obtained from the Ricci flow via a backwards Euler discretization, as first suggested in [52]. Alternatively, it can be obtained essentially by concatenating (and extending) two previously studied paths, one by Aubin [3] in the positive case and the other by Tian–Yau [65] in the negative case. The path is given by

$$\omega_\varphi^n = \omega^n e^{f_\omega - s\varphi}, \quad s \in (-\infty, \mu], \tag{30}$$

where $\varphi(-\infty) = 0$ and $\omega_{\varphi(-\infty)} = \omega$. We call this the *Ricci continuity path.* Adapting the proof of a result of Wu [68], we prove later that there exists a solution $\varphi(s)$ for $s \ll -1$ of the form $s^{-1}f_\omega + o(1/s)$. A key feature of this continuity path is that

$$\operatorname{Ric}\omega_\varphi = s\omega_\varphi + (\mu - s)\omega + 2\pi(1-\beta)[D], \tag{31}$$

which implies the very useful property that for all solutions $\varphi(s)$ along this path, the Ricci curvature is bounded below on $M\setminus D$, i.e., $\operatorname{Ric}\omega_\varphi > s\omega_\varphi$. As we explain in Section 6.3, another important property is that the Mabuchi K-energy is monotone along this path.

Much of the remainder of this article is directed toward analyzing this family of Monge–Ampère equations: Section 3 describes the linear analysis needed to understand the openness part of the continuity argument as well as the regularity theory; Section 4 uses this linear analysis to prove that solutions are automatically 'smooth' at D, by which we mean that they are polyhomogeneous (see below); the *a priori* estimates needed to obtain the closedness of the continuity argument are derived in the remaining sections of the article, and the proof is concluded in Section 9.

We will pursue a somewhat parallel development of this proof using two different scales of Hölder spaces since we hope to illustrate the relative merits of each of these classes of function spaces, with future applications in mind. Certain aspects of the proof work much more easily in one setting rather than the other, but we give a complete proof of the *a priori* estimates in either framework. The proof of higher regularity, which shows that these two approaches are ultimately equivalent and facilitates the continuity argument, relies directly on only one of these scales of spaces.

Remark 2.4. The continuity path (30) has several useful properties, some already noted above, which are necessary for the proof of Theorem 2 when $\beta > 1/2$. However, we also consider the two-parameter family of equations

$$\omega_\varphi^n = e^{tf_\omega + c_t - s\varphi}\omega^n, \quad c_t := -\log \frac{1}{V}\int_M e^{tf_\omega}\omega^n, \quad (s,t) \in A, \tag{32}$$

where $A := (-\infty, 0] \times [0,1] \ \cup [0,\mu] \times \{1\}$. This incorporates the continuity path $s = \mu, 0 \le t \le 1$, which is the common one in the literature. The analysis required to study this two-parameter family requires little extra effort and has been included since it may be useful elsewhere. It provides an opportunity to use the Chern–Lu inequality in its full generality (see Section 7). In addition, we have already noted that one cannot obtain openness for (30) at $s = -\infty$ directly but must produce a solution for s (very) negative by some other method. Wu [68] accomplishes this by a perturbation argument; the augmented continuity path (32) gives yet another means to do this, but it works only when $\beta \le 1/2$. We refer to Section 9 for more details.

We emphasize that our proof of Theorem 2 when $\beta > 1/2$ or when $\mu > 0$ requires the path (30) (i.e., fixing $t = 1$).

2.5. *The twisted Ricci iteration.* The idea of using the particular continuity method (30) to prove the existence of Kähler–Einstein metrics for all μ (independently of sign) was suggested in [52, p. 1533]. As explained there and recalled below, this path arises from discretizing the Ricci flow via the Ricci iteration. After treating this continuity path we will be in a position to prove smooth edge convergence of the (twisted) Ricci iteration to the Kähler–Einstein edge metric.

One Kähler–Ricci flow in our setting is

$$\frac{\partial\omega(t)}{\partial t} = -\mathrm{Ric}\ \omega(t) + 2\pi(1-\beta)[D] + \mu\omega(t), \quad \omega(0) = \omega \in \mathcal{H}_{\omega_0}.$$

Let $\tau \in (0,\infty)$. The (time τ) Ricci iteration, introduced in [52], is the sequence $\{\omega_{k\tau}\}_{k\in\mathbb{N}} \subset \mathcal{H}_\omega$, satisfying the equations

$$\omega_{k\tau} = \omega_{(k-1)\tau} + \tau\mu\omega_{k\tau} - \tau\mathrm{Ric}\,\omega_{k\tau} + \tau 2\pi(1-\beta)[D], \qquad \omega_{0\tau} = \omega,$$

for each $k \in \mathbb{N}$ for which a solution exists in $\mathcal{H}_\omega$. This is the backwards Euler discretization of the Kähler–Ricci flow. Equivalently, let $\omega_{k\tau} = \omega_{\psi_{k\tau}}$, with $\psi_{k\tau} = \sum_{l=1}^{k} \varphi_{l\tau}$. Then,

$$\omega^n_{\psi_{k\tau}} = \omega^n e^{f_\omega - \mu\psi_{k\tau} + \frac{1}{\tau}\varphi_{k\tau}}. \tag{33}$$

Since the first step is simply $\omega^n_{\varphi_\tau} = \omega^n e^{f_\omega + (\frac{1}{\tau} - \mu)\varphi_\tau}$, the Ricci iteration exists (uniquely) once a solution exists (uniquely) for (30) for $s = \mu - \frac{1}{\tau}$. Thus, much like for the Ricci flow, a key point is to prove uniformity of the *a priori* estimates as k tends to infinity. The convergence to the Kähler–Einstein metric then follows essentially by the monotonicity of the twisted K-energy if the Kähler–Einstein metric is unique.

As noted above, our choice of the particular continuity path (30) allows us to treat the continuity method and the Ricci iteration in a unified manner. When $\mu \le 0$, our estimates for (30), the arguments of [52] and the higher regularity developed in Section 4 imply the uniqueness, existence and edge smooth convergence of the iteration for all τ. When $\mu > 0$, the uniqueness of the (twisted) Ricci iteration was proven recently by Berndtsson [12], and it follows from his result that whenever the twisted K-energy E_0^β is proper, then also the Kähler–Einstein edge metric must be unique. Given this, our analysis here and in [52] immediately implies smooth (in the edge spaces) convergence of the iteration for large enough times steps, more specifically, provided $\tau > 1/\mu$ and E_0^β is proper, or else provided $\tau > 1/\alpha_{\Omega,\omega}$, and $\alpha_{\Omega,\omega} > \mu$, where $\alpha_{\Omega,\omega}$ is Tian's invariant defined in Section 6.3. (Note that by Lemma 6.11 this assumption implies E_0^β is proper.) As pointed out to us by Berman, given the results of [52], the remaining cases follow immediately in the same manner by using one additional very useful pluripotential estimate contained in [11, Lemma 6.4], stated explicitly in [9] and recalled in Lemma 6.10 below. As already observed in [10] this estimate gives, in an elegant manner, a uniform estimate on the oscillation of solutions along the iteration and is used in [10] to prove convergence of the twisted Ricci iteration and flow in general singular settings, smoothly away from the singular set, and global $\mathcal{C}^0$ convergence on the level of potentials. Our result below, in the case $\mu > 0$, is complementary to theirs since it shows how to use their uniform estimate and our analysis to obtain smooth convergence near the edge. We thank Berman for his encouragement to include this result here, prior to the appearance of [10].

To summarize, we have the following statement.

THEOREM 2.5. *Under the assumptions of Theorem* 2, *the Ricci iteration* (33) *exists uniquely and subconverges in* $\mathcal{D}_w^{0,\gamma} \cap \mathcal{A}^0$ *to a Kähler–Einstein edge metric in* $\mathcal{H}_\omega$. *Whenever the* KEE *metric is unique*, *in particular when there are no holomorphic vector fields tangent to* D, *the iteration itself converges.*

These function spaces encode the strongest possible convergence for this problem and are defined next. The proof of Theorem 2.5 is given in Section 9.

2.6. *Function spaces.* To conclude this section of preliminary material, we review the various function spaces used below. These are the 'wedge and edge' Hölder spaces, as well as the spaces of conormal and polyhomogeneous functions necessary for our treatment of the higher regularity theory. The wedge Hölder spaces are the ones used in [24] and are naturally associated to the incomplete edge geometry. The edge spaces, introduced in [43], are also naturally associated to this geometry and have some particularly favorable properties stemming from their invariance under dilations. Using this, certain parts of the proofs below become quite simple. The wedge Hölder spaces, on the other hand, are closer to standard Hölder spaces and indeed reduce to them when $\beta = 1$. They impose stronger regularity conditions. Since we use both types of spaces here, we describe many of the proofs below in both settings. This is important for applications and should also give the reader a better sense of their relative advantages.

Before giving any of the formal definitions below, let us recall that a Hölder space is naturally associated to a distance function d via the Hölder seminorm

$$[u]_{d;0,\gamma} := \sup_{\substack{p \neq p' \\ d(p,p') \leq 1}} \frac{|u(p) - u(p')|}{d(p,p')^\gamma}.$$

We only need to take the supremum over points with distance at most 1 apart, since if $d(p,p') > 1$, then this quotient is bounded by $2 \sup |u|$. The two different spaces below differ simply through the different choices of distance function d.

2.6.1. *Wedge Hölder spaces.* First consider the distance function d_1 associated to the model metric g_β; note that it is clearly equivalent to replace the actual g_β distance function with any other function on $M \times M$ that is uniformly equivalent, and it is simplest to use the one defined in the coordinates (r, θ, y) by

$$d_1((r,\theta,y),(r',\theta',y')) = \sqrt{|r-r'|^2 + (r+r')^2|\theta-\theta'|^2 + |y-y'|^2}.$$

Note that the angle parameter β does not appear explicitly in this formula, but if we were to have included it, there would be a factor of β^2 before $(r+r')^2|\theta-\theta'|^2$. This changes d_1 at most by a factor, so we may as well omit it altogether.

Now define the wedge Hölder space $\mathcal{C}_w^{0,\gamma} \equiv \mathcal{C}_w^{0,\gamma}(M)$ to consist of all functions u on $M \setminus D$ for which

$$||u||_{w;0,\gamma} := \sup |u| + [u]_{d_1;0,\gamma} < \infty.$$

The spaces with higher regularity are defined using differentiations with respect to unit length vector fields with respect to g_β; these vector fields are spanned by ∂_r, $r^{-1}\partial_\theta$ and ∂_{y_j}. Thus

$$\mathcal{C}^{k,\gamma}_w(M) = \{u : \partial_r^i (r^{-1}\partial_\theta)^j \partial_y^\mu u \in \mathcal{C}^{0,\gamma}_w(M) \quad \forall\ i + j + |\mu| \le k\}.$$

There a few potentially confusing points about these spaces. The first is that the spaces $\mathcal{C}^{k,\gamma}_w$ with $k > 0$ seem to depend on the choice of coordinates or choice of frame. It is not hard to untangle the dependence or lack thereof, but since we only use these spaces when $k = 0$, this discussion is relegated to another paper. Second, it is worth comparing this definition with the equivalent one given in [24]. As is evident from the definition above, the space $\mathcal{C}^{0,\gamma}_w$ above does *not* depend on the cone angle parameter β (at least so long as β stays bounded away from 0 and ∞). However, suppose we consider the (apparently) fixed function $f = |z_1|^a = \rho^a$ for some $a > 0$ in terms of the original holomorphic coordinates. In terms of the cylindrical coordinates (r, θ, y), we have $f = \beta^{a/\beta} r^{a/\beta}$, and hence $f \in \mathcal{C}^{0,\gamma}_w$ if and only if $a/\beta \ge \gamma$, i.e., $a \ge \beta\gamma$. Inequalities of this type appear in [24]. This seems inconsistent with the claim that the Hölder space is independent of β; the discrepancy between these statements is explained by observing that the singular coordinate change *does* depend on β, and while the function ρ^a is independent of β, its composition with this coordinate change is not. Equivalently, if we pull back the function space $\mathcal{C}^{0,\gamma}_w$ via this coordinate change, then we get a varying family of function spaces on M. We prefer, however, to think of $M \setminus D$ as a fixed but singular geometric object, with smooth structure determined by the coordinates (r, θ, y) and with a single scale of naturally associated Hölder spaces.

2.6.2. *Edge Hölder spaces.* Now consider the distance function d_2 associated to the *complete* metric

$$\hat{g}_\beta := r^{-2} g_\beta = \frac{dr^2 + |dy|^2}{r^2} + \beta^2 d\theta^2.$$

As before, the distance d_2 is replaced by the metric

$$d_2((r,\theta,y),(r',\theta',y')) = (r + r')^{-1}\sqrt{(r - r')^2 + (r + r')^2(\theta - \theta')^2 + |y - y'|^2},$$

which is uniformly equivalent to it. It suffices to consider only $r, r' \le C$. As before, no factor of β is included.

The Hölder norm $||u||_{e;0,\gamma}$ is now defined using the seminorm associated to d_2. The higher Hölder norms are defined using unit length vector fields with respect to $\hat{g}_\beta$, which are spanned by $\{r\partial_r, \partial_\theta, r\partial_y\}$. The corresponding spaces of functions for which these norms are finite are denoted $\mathcal{C}^{k,\gamma}_e \equiv \mathcal{C}^{k,\gamma}_e(M)$.

The key property of this distance function is that it is invariant with respect to the scaling

$$(r, \theta, y) \mapsto (\lambda r, \theta, \lambda y)$$

for any $\lambda > 0$. The vector fields $r\partial_r$, ∂_θ and $r\partial_y$ are also invariant with respect to these dilations. This means that if $u_{\lambda,y_0}(r,\theta,y) = u(\lambda^{-1}r, \theta, \lambda^{-1}y + y_0)$, then $||u_{\lambda,y_0}||_{e;k,\gamma} = ||u||_{e;k,\gamma}$. (We assume, of course, that both (r,θ,y) and $(\lambda^{-1}r, \theta, \lambda^{-1}y + y_0)$ lie in the domain of u.) One way to interpret this is as follows. Consider the annular region

$$B_{\lambda,y_0} := \{(r,\theta,y) : 0 < \lambda < r < 2\lambda,\ |y - y_0| < \lambda\}$$

for λ small. The image of this annulus under translation by y_0 and dilation by λ^{-1} is the standard annulus $B_{1,0}$. Hence if u is supported in B_{λ,y_0}, then u_{λ,y_0} is defined in $B_{1,0}$ and $||u||_{e;k,\gamma} = ||u_{\lambda,y_0}||_{e;k,\gamma}$.

For any $\nu \in \mathbb{R}$, we also define weighted edge Hölder spaces

$$r^\nu C_e^{k,\gamma}(M) = \{u = r^\nu v : v \in \mathcal{C}_e^{k,\gamma}(M)\}.$$

Although $\mathcal{C}_e^{0,\gamma}(M) \subset L^\infty(M)$, elements of $\mathcal{C}_e^{0,\gamma}(M)$ need not be continuous at $r = 0$; an easy example is the function $\sin\log r$, which lies in $\mathcal{C}_e^{k,\gamma}$ for all k. On the other hand, elements of $r^\nu\mathcal{C}_e^{0,\gamma}$ are continuous and vanish at D if $\nu > 0$.

2.6.3. *Comparison between the wedge and edge Hölder spaces.* Let us now comment on the relationship between these spaces. Since $r, r' \le C$, we have $d_1 \le C^{-1}d_2$, and hence

$$||u||_{e;k,\gamma} \le C^{-\gamma}||u||_{w;k,\gamma}$$

or, equivalently,

$$\mathcal{C}_w^{k,\gamma} \subset \mathcal{C}_e^{k,\gamma}. \tag{34}$$

Elements in the wedge Hölder space are more regular than those in the edge Hölder space. For example, unlike elements of $\mathcal{C}_e^{0,\gamma}$, elements of $\mathcal{C}_w^{0,\gamma}$ are continuous up to D. Moreover, if $u \in \mathcal{C}_w^{0,\gamma}$, then $u(0,\theta,y)$ is independent of θ and lies in $\mathcal{C}^{0,\gamma}(D)$; by contrast, if $u \in \mathcal{C}_e^{0,\gamma}$, then the 'tangential' difference quotient $|u(r,\theta,y) - u(r,\theta,y')|/|y - y'|^\gamma$ is bounded by $Cr^{-\gamma}$.

However, there is a direct relationship between the two spaces. Define

$$\mathcal{C}_w^{0,\gamma}(M)_0 = \{u \in \mathcal{C}_w^{0,\gamma}(M) : u|_D = 0\}.$$

Next, if $u \in \mathcal{C}_w^{0,\gamma}(M)$, write $u_0 = u|_D \in \mathcal{C}^{0,\gamma}(D)$. There exists an extension operator $\mathcal{C}^{0,\gamma}(D) \ni u_0 \mapsto E(u_0) = U \in \mathcal{C}_w^{0,\gamma}$. Fixing an identification of a neighborhood $\mathcal{V}$ of D with a bundle of truncated cones over D and collapsing the S^1 cross-sections of these conic fibers yields a map $\mathcal{V} \to D \times [0, r_0)$. Requiring any local (r,θ,y) coordinates to be coherent with this extension, we may choose U to be independent of θ and to equal the 'ordinary' harmonic extension of u_0 in the (r,y) coordinates, i.e., $(\partial_r^2 + \Delta_y)U = 0$. Actually, the only properties of U needed later are that $U \in \mathcal{C}^\infty(M \setminus D)$ and

$$|\partial_r U| + |\partial_y U| \le Cr^{-1+\gamma}. \tag{35}$$

For the harmonic extension, these bounds are a classical characterization of Hölder spaces, and this characterization of Hölder spaces is explained carefully in [55, Chap. V, §4.2]. We note that it is straightforward to choose such an extension in a less ad hoc way using the theory of edge Poisson operators developed in [48].

Now decompose any $u \in \mathcal{C}_w^{0,\gamma}$ as

$$u = U + \tilde{u}, \qquad \tilde{u} \in \mathcal{C}_w^{0,\gamma}(M)_0.$$

This is useful because the two components have different characterizations. We have already explained the relevant regularity properties of $U = E(u_0)$. As for the other component, we assert that

$$\mathcal{C}_w^{0,\gamma}(M)_0 = r^\gamma \mathcal{C}_e^{0,\gamma}(M). \tag{36}$$

To explain this, note that if u lies in the space on the left, then $|u(r,y,\theta)| \leq Cr^\gamma$, so the function $v = r^{-\gamma}u$ is at least bounded. The proof of (36) is an elementary calculation checking that $||u||_{w;0,\gamma} \leq C||v||_{e;0,\gamma}$ and $||v||_{e,0,\gamma} \leq C||u||_{w;0,\gamma}$.

There are certain advantages to using the edge Hölder spaces. First observe that if $\mu \in (0,1)$, then $r^\mu \in \mathcal{C}_w^{0,\gamma}$ only when $\mu \leq \gamma$, while $r^\mu \in \mathcal{C}_e^{0,\gamma}$ for all $\gamma \in (0,1)$. Furthermore, $r^\mu \notin \mathcal{C}_w^{k,\gamma}$ for any $k \geq 1$, but since $(r\partial_r)^j r^\mu = \mu^j r^\mu$, we see that $r^\mu \in \mathcal{C}_e^{k,\gamma}$ for all $k \geq 0$. In other words, the edge spaces more naturally accommodate noninteger exponents. This is important when dealing with singular elliptic equations because solutions of such equations typically involve noninteger powers of r, and it is quite reasonable to think of these solutions as being infinitely differentiable in a suitable sense. One final point is that basic Hölder regularity theory for elliptic differential edge operators is phrased in terms of the edge spaces; these are scale-invariant estimates. The pseudodifferential parametrices in the edge calculus, discussed in Section 3 below, are most easily shown to be bounded on edge spaces; their boundedness on the wedge spaces is a consequence of that result.

In the remainder of this article, whenever our discussion applies to both of these spaces, we refer to the 'generic' singular Hölder space $\mathcal{C}_s^{k,\gamma}$, where

s equals either w or e.

This s should not be confused either with the parameter s along the continuity path (30) nor with the holomorphic section s defined in Lemma 2.2.

2.6.4. *Conormal and polyhomogeneous functions.* The final set of spaces we define are the spaces of conormal and polyhomogeneous functions.

Definition 2.6. For any $\nu \in \mathbb{R}$, define $\mathcal{A}^\nu$, the space of conormal functions of weight ν, to consist of all functions $u = r^\nu v$ where v and all of its derivatives with respect to the vector fields $r\partial_r$, ∂_θ, ∂_y are bounded; see (50).

Next, we say that $u \in \mathcal{A}^\nu$ is polyhomogeneous, and write $u \in \mathcal{A}^\nu_{\mathrm{phg}}$, if it has an expansion of the form

$$u \sim \sum_{j=0}^{\infty} \sum_{p=0}^{N_j} a_{jp}(\theta, y) r^{\sigma_j} (\log r)^p,$$

where the coefficients a_{jp} are all $\mathcal{C}^\infty$, and $\{\sigma_j\}$ is a discrete sequence of complex numbers such that $\operatorname{Re}\sigma_j \to \infty$, with $\operatorname{Re}\sigma_j \geq \nu$ for all j and $N_j = 0$ if $\operatorname{Re}\sigma_j = \nu$. This expansion can be differentiated arbitrarily many times with the corresponding differentiated remainder. We say that u has a nonnegative index set if $u \in \mathcal{A}^0_{\mathrm{phg}}$, and if any exponent σ in its expansion has $\operatorname{Re}\sigma = 0$, then $\sigma = 0$. Note finally that if $u \in \mathcal{A}^0_{\mathrm{phg}}$, then u is bounded, and if any such u has nonnegative index set, then u is continuous up to the boundary.

These function spaces accomodate behavior typical for solutions of degenerate elliptic edge problems, e.g., functions like $r^\sigma(\log r)^p a(\theta, y)$ where a is smooth, p is a nonnegative integer and $\sigma \in \mathbb{C}$. We remark that these spaces are the correct analogues of the spaces of infinitely differentiable functions in this context. Note that when $\beta = 1$, $\mathcal{A}^0$ *does not* correspond to $\mathcal{C}^\infty(M)$. In this setting, we make a distinction between functions that are infinitely differential (conormal) and those that have 'Taylor series' expansions (i.e., are polyhomogeneous). We remark also that the expansions of polyhomogeneous functions are rarely convergent but only give 'order of vanishing' type estimates. It is usually difficult to control the size of the neighborhood on which such an expansion provides a good approximation.

3. **Linear analysis**

We now present the key facts about the linear elliptic theory needed to handle the existence, deformation and regularity theory for canonical edge metrics. We discuss this from two points of view, reviewing the estimates outlined by Donaldson in the wedge Hölder spaces, and also describing how to obtain analogous estimates in the edge Hölder spaces. These latter estimates are obtained through the use of edge pseudodifferential operators, as developed in [43]. This methodology, part of the general framework of geometric microlocal analysis, yields the most incisive results for the class of degenerate elliptic operators that arise here, and as we shall see, there are numerous places below where the more delicate parts of the linear analysis needed to prove our main results here require this full theory. In other words, the use of the edge calculus in this paper is an essential feature, rather than simply a more systematic way of rephrasing estimates analogous to those described by Donaldson.

Fix a Kähler edge metric g on M with cone angle $2\pi\beta$ along the smooth divisor D; we initially suppose that the metric g is polyhomogeneous along D, though this will be relaxed later. For the rest of this section, we consider the operator $L = \Delta_g + V$ where V is polyhomogeneous with nonnegative index set (and hence is bounded); in certain places below we extend certain results to the case where g and V are not polyhomogeneous but have some given Hölder regularity.

Our method is based on the realization that the Schwartz kernel of the Green operator for the Friedrichs extension of L has a fairly simple polyhomogeneous structure, and knowing this structure, one can read off the estimates we need. This Green function is a pseudodifferential edge operator. The article [43] contains a detailed development of this class of operators, their mapping properties and the elliptic parametrix construction in this calculus. We review various aspects of this theory now, at all times maintaining focus on the particular problem at hand. We give specific references to the appropriate results and sections of [43] so as to guide the interested reader to the details of the proofs of the results we need. We also recall Donaldson's estimates, explain the essential differences between his and the ones obtained here through the edge theory and describe the differences between these two approaches to proving these estimates. Our approach gives an alternative proof of his estimates.

3.1. *Edge structures and edge operators.* We have already indicated that it can be advantageous to think of M with a Kähler edge metric as being a singular object, but it is more convenient to formulate the edge theory via structures on the manifold with boundary obtained by taking the real blowup of M along D.

The general notion of an edge structure on a manifold with boundary X is defined in terms of a space of vector fields $\mathcal{V}_e(X)$ on that manifold, where we assume that ∂X is the total space of a fibration $\pi : \partial X \to Y$ with fiber F. The space $\mathcal{V}_e(X)$ consists of all smooth vector fields on X that are unconstrained in the interior but that lie tangent to the fibers at the boundary. In our setting, the manifold X is obtained by taking the real blowup of M around D, so ∂X is the unit normal circle bundle $\mathrm{SN}D$ over D. To be more specific, the real blowup $X := [M; D]$ is by definition the disjoint union $(M \setminus D) \sqcup \mathrm{SN}D$, endowed with the unique smallest topological and differential structure so that the lifts of smooth functions on M and polar coordinates around D are smooth. There is a natural smooth blowdown map $X \to M$.

Before proceeding, we note a subtlety here related to the fact that there are actually two natural smooth structures: one is induced by the holomorphic coordinates $(z_1, \ldots, z_n)$, where $D = \{z_1 = 0\}$ locally, and the other by the cylindrical coordinate system (r, θ, y) defined earlier. Indeed, since $r = |z_1|^\beta/\beta$, functions smooth with respect to z are not necessarily smooth with respect to (r, θ, y) and vice versa. These structures are, of course, equivalent via the coordinate transformation. However, perhaps the correct perspective is that it is *not* the smooth structure on X but rather the 'polyhomogeneous structure,' i.e., the ring of polyhomogeneous functions, that is fundamental. Indeed, the polyhomogeneous structure is preserved by this coordinate change. At any rate, for $X = [M, D]$, $\partial X = \{\rho = 0\}$, where $\rho = |z_1|$ (or equivalently, $\{r = 0\}$ where r is defined as above), and the S^1 fibers of ∂X are the level sets $\{y = \text{const.}\}$. Functions on X are polyhomogeneous if and only if they are polyhomogeneous with respect to either of the coordinate systems (r, θ, y) or (ρ, θ, y). Finally, and here the difference between ρ and r is important, we define $\mathcal{V}_e(X)$ to be generated by the vector fields $r\partial_r$, ∂_θ and $r\partial_y$.

Next, the space of differential edge operators $\mathrm{Diff}_e^*(X)$ consists of all operators that can be written locally as finite sums of products of elements of $\mathcal{V}_e(X)$. Thus again for $X = [M, D]$, if $m \geq 0$, then the typical element of $\mathrm{Diff}_e^m(X)$ has the form

$$A = \sum_{j+k+|\mu|\leq m} a_{jk\mu}(r, \theta, y)(r\partial_r)^j \partial_\theta^k (r\partial_y)^\mu. \tag{37}$$

We now restrict attention exclusively to the case $X = [M; D]$ and $m = 2$, though there are suitable versions of all of the main linear results below in the general edge setting.

If g is an incomplete edge metric on M with cone angle β, then $L = \Delta_g + V$ can be written as in (12), as the sum of a principal part and an error term E. However, it is $A = r^2 L$ that is an edge operator in the sense we have just defined.

A differential edge operator is called elliptic if it is an 'elliptic combination' of elements of $\mathcal{V}_e(X)$, for example a sum of squares of a generating set of sections plus lower order terms. This is the case for the operator A here; we refer to [43, §2] for the coordinate invariant formulation of edge ellipticity and for more on edge vector fields and their dual one-forms.

3.2. *Normal and indicial operators.* If A is an elliptic edge operator, its mapping properties are governed not only by its ellipticity, but also by two model operators, the indicial and normal operators $I(A)$ and $N(A)$, respectively, which are defined at each point of D. While these may be defined invariantly, let us simply record here that for $A = r^2 L$, with $L = \Delta_g + V$, and after a certain natural identification that we explain below,

$$N(A) = (s\partial_s)^2 + \beta^{-2}\partial_\theta^2 + s^2 \Delta_w \quad \text{and} \quad I(A) = (s\partial_s)^2 + \beta^{-2}\partial_\theta^2, \tag{38}$$

where (s, w) are global affine coordinates on a half-space $\mathbb{R}_s^+ \times \mathbb{R}_w^{2n-2}$, $\Delta_w := \sum_{i=1}^{2n-2} \frac{\partial^2}{\partial w_i^2}$ and $\theta \in S_{2\pi}^1$ (the circle of radius 2π). Note that

$$N(A) = s^2 L_\beta, \quad \text{where } L_\beta = \partial_s^2 + \frac{1}{s}\partial_s + \frac{1}{\beta^2 s^2}\partial_\theta^2 + \Delta_w$$

is the Laplacian of the flat model metric g_β.

Informally, $N(A)$ is obtained by dropping the error term r^2E, freezing coordinates at a given point $y_0 \in D$ and replacing the local coordinates (r, y) by global affine coordinates $(s, w) \in \mathbb{R}^+ \times \mathbb{R}^{2n-2}$. More invariantly, $N(A)$ is the limit of rescalings of A by the group of dilations based at a point $y_0 \in Y$, and it acts on functions defined on the inward-pointing normal bundle of the fiber of ∂X over y_0. The indicial operator $I(A)$ is even simpler: it is defined by dropping the terms in $N(A)$ that have the property that they map any function $s^a v(\theta, w)$ (with v smooth) to a function that vanishes faster than s^a. The only term in the operator $N(A)$ above that is discarded for this reason is $s^2\Delta_w$.

In general, both $N(A)$ and $I(A)$ could depend on y_0 (for example, if the cone angle were to vary along D). Fortunately, in our case of interest, this dependence is quite simple, and as we have indicated above, it can effectively be normalized away. Indeed, from (6), the term $\partial^2_{z_1\bar{z}_1}$ is multiplied by the factor F^{-1}, which depends on all variables but is independent of θ at $r = 0$. The normal operator of Fr^2L, obtained by this rescaling procedure above, has the form (38), but initially the terms involving derivatives in w are a second-order constant coefficient elliptic operator on $\mathbb{R}^{2n-2}$, multiplied by s^2. A linear change of variables in w, depending smoothly on y_0, puts this into standard form $s^2\Delta_w$. Thus the correct statement is that the normal operators at different points y_0 can be identified with one another, and similarly for the indicial operator.

A number $a \in \mathbb{C}$ is called an indicial root of A (and also of L) if there exists a nontrivial function $\psi(\theta)$ such that $I(A)s^a\psi(\theta) = 0$; thus, for $A = r^2L$,

$$I(A)s^a\psi(\theta) = (\beta^{-2}\partial_\theta^2 + a^2)\psi = 0 \Longleftrightarrow \begin{cases} a \in \{j/\beta : j \in \mathbb{Z}\}, \\ \psi_j(\theta) = a_j \cos j\theta + b_j \sin j\theta,\ j \geq 1. \end{cases}$$

The case $j = 0$ here is special since 0 is a ‘double’ indicial root, so $\psi_0(\theta) = 1$ and both $I(A)(s^0) = I(A)(s^0 \log s) = 0$. This is special to the case that D has codimension two. These indicial roots are just the square roots of the eigenvalues of $-\beta^{-2}\partial_\theta^2$, which leads to the observation that it is quite important that θ lies on a compact manifold (namely, S^1), since otherwise the spectrum, and hence the set of indicial roots, would not be discrete. Note also that for any $a \in \mathbb{C}$ and $\psi(\theta, y) \in \mathcal{C}^\infty$, it is always true that $A(r^a\psi(\theta, y)) = O(r^a)$, but a is an indicial root if and only if $A(r^a\psi_j) = O(r^{a+1})$ and $\psi(\theta, y) = a(y)\psi_j(\theta)$, where $a(y)$ is essentially arbitrary.

3.3. *Mapping properties and the Friedrichs domain.* We next describe the basic mapping properties of L on weighted Hölder spaces; these are the content of [43, Cor. 6.4] applied to the operator $A = r^2L$.

Proposition 3.1. *The mapping*

$$L : r^\nu \mathcal{C}_e^{\ell+2,\gamma} \longrightarrow r^{\nu-2}\mathcal{C}_e^{\ell,\gamma} \tag{39}$$

has closed range if and only if $\nu \notin \{\frac{j}{\beta}, j \in \mathbb{Z}\}$.

The indicial roots are excluded as weights here because for these values, (39) does not have closed range.

Although this proposition, and indeed the emphasis in all of [43], is on the Fredholm (and semi-Fredholm) theory of operators such as $A = r^2 L$, it is more relevant for us to focus on L and its action as an unbounded operator acting on a space with a fixed weight, rather than between two differently weighted spaces. The main new issue from this point of view is to select a self-adjoint extension; we assume throughout that the term of order 0 is real-valued so that L is a symmetric operator on the core domain $\mathcal{C}_0^\infty(M \setminus D)$. Rather than reviewing the well-known classical theory of self-adjoint extensions, we recall simply that since L is semibounded, there is always a distinguished self-adjoint realization called the *Friedrichs* extension, which is defined using the coercive quadratic form

$$\langle u, v\rangle = \int_M \left(\nabla u \cdot \overline{\nabla v} - V u \overline{v}\right)\, dV_g. \tag{40}$$

We can identify the domain $\mathcal{D}_{\mathrm{Fr}}(L)$ of this Friedrichs extension explicitly. It can be shown, see [43, §7], that any $u \in \mathcal{D}_{\mathrm{Fr}}(L)$ has a 'weak' partial expansion $u \sim u_0(y) + \tilde{u}$, where $\tilde{u} = O(r^\mu)$ for some $\mu > 0$ and u_0 may be a distribution of negative order, but is independent of r; this expansion is called weak because it only becomes an asymptotic expansion in the usual sense (in particular, with decaying remainder) provided both sides are paired with a test function $\chi(y)$ (depending only on y). Thus $u \in \mathcal{D}_{\mathrm{Fr}}(L)$ if and only if

$$(r, \theta) \mapsto \langle u(r, \theta, \cdot), \chi(y)\rangle = \langle u_0(y), \chi(y)\rangle + \mathcal{O}(r^\mu)$$

for any $\chi \in \mathcal{C}^\infty(Y)$. To distinguish this from behavior of more general solutions, it is also proved in [43, §7] that if u is any L^2 solution to $Lu = f$ with $f \in L^2$, then this expansion could contain an extra term $\langle u_{01}(y), \chi(y)\rangle \log r$ on the right. Hence the Friedrichs domain is characterized by the requirement that the coefficient u_{01} of $\log r$ vanish. We note that a principal source of the difficulties reported in [45] revolved around some technicalities encountered when working with these weak expansions.

Henceforth we work exclusively with the Friedrichs extension of L, and we denote it simply by L. It is straightforward to deduce using Hardy-type estimates that the domain $\mathcal{D}_{\mathrm{Fr}}(L)$ is compactly contained in L^2, which proves that L has discrete spectrum as an operator on this space. Its nullspace is finite dimensional, with every element bounded and polyhomogeneous. Thus there is a uniquely defined generalized inverse G determined by

$$LG = GL = \mathrm{Id} - \Pi,$$

where Π is the finite rank orthogonal projector onto the nullspace. Essentially by definition, if K is the L^2 nullspace of L, then $\mathcal{D}_{\mathrm{Fr}}(L) = G(L^2(M, dV_g)) \oplus K$.

We now shift to the analogous but less-standard discussion for L acting between Hölder spaces. Proceeding by analogy with these L^2 definitions, we define the Hölder-Friedrichs domains

$$\mathcal{D}_s^{0,\gamma}(L) := \{u \in \mathcal{C}_s^{0,\gamma} : Lu \in \mathcal{C}_s^{0,\gamma}\} \quad \text{for } s = w \text{ or } e.$$

We claim that

$$\mathcal{D}_e^{0,\gamma} \subsetneq \mathcal{C}_e^{2,\gamma}, \quad \text{and} \quad \mathcal{D}_w^{0,\gamma} \supsetneq \mathcal{C}_w^{2,\gamma}.$$

To see these inclusions, note first that if $u \in L^\infty$ and $Lu \in \mathcal{C}_e^{0,\gamma}$, then a basic edge regularity theorem, proved using the mapping properties of the Green function G, see [43, proposition §3.7], gives that $u \in \mathcal{C}_e^{2,\gamma}$. However, if $u \in \mathcal{C}_e^{2,\gamma}$, then Lu is usually not bounded, and in fact typically we only have $Lu \in r^{-2}\mathcal{C}_e^{2,\gamma}$. On the other hand, we explain below that $G : \mathcal{C}_w^{0,\gamma} \to \mathcal{C}_w^{0,\gamma}$, so $\mathcal{D}_w^{0,\gamma} \subset \mathcal{C}_w^{0,\gamma}$. Moreover, functions in this domain lie in $\mathcal{C}^{2,\gamma}(X \setminus \partial X)$. However, as we describe more carefully below, $\mathcal{D}_w^{0,\gamma}$ contains the function $v = r^{1/\beta}e^{i\theta}$, and hence if $\beta > 1/2$ then, for example, $\partial_r^2 r^{1/\beta}e^{i\theta} \notin L^\infty$. In order to accomodate functions with these fractional exponents in the wedge spaces, we henceforth assume that

$$\text{if } s = w, \text{ then } \gamma \in (0,1) \cap (0, \tfrac{1}{\beta} - 1]. \tag{41}$$

Note that this guarantees at least that $r^{1/\beta}e^{i\theta} \in \mathcal{C}_w^{1,\gamma}$.

The mapping

$$L : \mathcal{D}_s^{0,\gamma}(L) \longrightarrow \mathcal{C}_s^{0,\gamma} \tag{42}$$

is invertible up to a possible finite dimensional nullspace. We need to obtain a more explicit characterization of these singular 'Hölder-Friedrichs' domains. The first step in this direction uses the Green function G exactly as in the L^2 theory:

Proposition 3.2. *The nullspace K of L in $L^2(M, dV_g)$ coincides with the nullspace of L in $\mathcal{C}_s^{2,\gamma}$, and we have*

$$\mathcal{D}_s^{0,\gamma}(L) = G(\mathcal{C}_s^{0,\gamma}) \oplus K = \{u = Gf : f \in \mathcal{C}_s^{0,\gamma}\} \oplus K.$$

Proof. To prove the first assertion about nullspaces, apply [43, Prop. 7.17] to see that an element of either nullspace is polyhomogeneous and lies in both L^2 and $\mathcal{C}_s^{2,\gamma}$. Since $\mathcal{C}_s^{0,\gamma} \subset L^\infty(M) \subset L^2(M, dV_g)$, the space on the right in the displayed equation is well defined. If u is in the space on the right, then clearly $Lu = f \in \mathcal{C}_s^{0,\gamma}$. Conversely, if $u \in \mathcal{C}_s^{0,\gamma}$, $f \in \mathcal{C}_s^{0,\gamma}$ and $Lu = f$ distributionally, then u is in the L^2 Friedrichs domain. Clearly $L(u - Gf) = 0$, and since both u and Gf are in L^2, we can write $u - Gf = v$ for some $v \in K$. □

3.4. *Finer properties of functions in the Hölder-Friedrichs domain.* This last proposition sets the stage for the more detailed study of the regularity of functions in these domains. In this subsection we first recall Donaldson's estimates, which characterize $\mathcal{D}_w^{0,\gamma}(L)$, and then state the corresponding results for $\mathcal{D}_e^{0,\gamma}(L)$, with the proofs deferred to the next subsection. We include some auxiliary regularity results that are used later.

The characterizations of the domains $\mathcal{D}_s^{0,\gamma}$ will be given in terms of which derivatives lie in $\mathcal{C}_s^{0,\gamma}$. We also show that either of these domains are independent of the operator L in the sense that they remain the same if we replace the polyhomogeneous Kähler edge metric ω by any metric ω_u, where the Kähler potential u itself only lies in $\mathcal{D}_w^{0,\gamma}$, and $V \in \mathcal{C}_s^{0,\gamma}$.

To gain a sense of where we are headed, recall that on a closed smooth manifold M, the L^2 Friedrichs domain of the Laplacian is equal to the Sobolev space $W^{2,2}(M)$. This follows from the basic elliptic estimates, of course, but is also a consequence of the boundedness on L^2 of the Riesz potential operator $\nabla^2 \circ \Delta^{-1}$. The corresponding Schwartz kernels are pseudodifferential operators of order 0, and we can appeal to the general boundedness properties of this class of operators. Pseudodifferential theory has its origins in attempts to answer questions of this type.

We follow a similar route here. The Green operator G represents Δ^{-1}, and the problem becomes one of determining which second derivatives applied to G yield 'Riesz potential' operators that are bounded on $\mathcal{C}_s^{0,\gamma}$. As we now describe, if $u \in \mathcal{D}_s^{0,\gamma}(L)$, then *not* every second derivative term appearing in the operator L, written as a real operator, applied to u lands in $\mathcal{C}_s^{0,\gamma}$. Donaldson's simple yet crucial observation [24] is that this is not necessary! As described in Section 2, the Monge–Ampère operator decomposes into the sum of the individual $(1,1)$-type terms $g^{i\bar{j}}u_{i\bar{j}}$, each of which involve particular combinations of real second derivatives that in the notation of Section 2 are the expressions $P_{i\bar{j}}u$. The next proposition shows that these simple 'monomial' operators characterize $\mathcal{D}_w^{0,\gamma}$ in the sense that they provide an equivalent norm on $\mathcal{D}_w^{0,\gamma}(L)$ for each fixed operator L associated to an edge metric; see also Corollary 3.5 below.

Proposition 3.3. *Let γ satisfy* (41), *and recall the set of operators $\mathcal{Q}^*$ in* (16). *Then $\mathcal{D}_w^{0,\gamma}(L) = \{u \in \mathcal{C}_w^{0,\gamma} : \ Q_i u \in \mathcal{C}_w^{0,\gamma}, \ \ \forall\, Q_i \in \mathcal{Q}^*\}$. Equivalently, each of the maps*

$$Q_i \circ G : \mathcal{C}_w^{0,\gamma} \longrightarrow \mathcal{C}_w^{0,\gamma}, \quad Q_i \in \mathcal{Q}^*, \tag{43}$$

is bounded.

The proof of Proposition 3.3 is described in Section 3.5.

Remark 3.4. (i) This is essentially the same as the result alluded to in [24] that if $u \in \mathcal{D}_w^{0,\gamma}(L)$, then

$$\sum_{i\bar{\jmath}} ||g^{i\bar{\jmath}}u_{i\bar{\jmath}}||_{w;0,\gamma} \leq C(||Lu||_{w;0,\gamma} + ||u||_{\mathcal{C}^0}),$$

where the important point is that on the left we have a sum of norms rather than a norm of the sum.

Another useful way to phrase this involves the norms

$$||u||_{\mathcal{D}_w^{0,\gamma}} = ||u||_{w;0,\gamma} + \sum_{Q_i \in \mathcal{Q}^*} ||Q_i u||_{w;0,\gamma}. \tag{44}$$

Later we also use the seminorm $[\,\cdot\,]_{\mathcal{D}_s^{0,\gamma}}$ defined by omitting the initial $||\cdot||_{s;0,\gamma}$ term. Proposition 3.3 implies that $||\cdot||_{\mathcal{D}_w^{0,\gamma}}$ is a Banach norm on $\mathcal{D}_w^{0,\gamma}$. The space $\mathcal{D}_w^{0,\gamma}$ is the same as the space $\mathcal{C}^{2,\gamma,\beta}$ introduced in [24].

There is an equivalence of norms:

$$C_1||u||_{\mathcal{D}_w^{0,\gamma}} \leq ||u||_{\mathcal{C}^0} + ||Lu||_{w;0,\gamma} \leq C_2||u||_{\mathcal{D}_w^{0,\gamma}},$$

where the constants C_1 and C_2 depend on the coefficients $g^{i\bar{\jmath}}$. In our application below, these metric coefficients are determined by the solution φ of the Monge–Ampère equation. We will prove a uniform $\mathcal{C}^0$ bound on $\Delta\varphi$ that ensures that these constants remain uniform across the family of edge metrics that arise in the continuity argument.

Since the complex operators $P_{i\bar{\jmath}}$ are sums of the real operators Q_i, one direction of Proposition 3.3 is trivial: if u and every $Q_i u$ lie in $\mathcal{C}_w^{0,\gamma}$, then trivially $Lu \in \mathcal{C}_w^{0,\gamma}$ since Lu is just a sum of these terms with coefficients in $\mathcal{C}_w^{0,\gamma}$ (or better). The other direction is proved by showing that the compositions $Q_i \circ G$ are bounded operators. This is accomplished by Donaldson for the model problem by direct scaling methods. Our proof here uses that each of these Riesz operators are pseudodifferential edge operators of order 0 and then invokes basic boundedness results for such operators. We explain this more carefully in the next subsection.

(ii) It is at this point that the theory in edge and wedge Hölder spaces differs significantly. Indeed, it turns out that it is *not* true that certain of the Riesz potentials $Q_i \circ G$ are bounded on $\mathcal{C}_e^{0,\gamma}$; in particular, this boundedness fails when $Q_i = \partial^2_{y_j y_\ell}$. This can be seen by a specific example in local coordinates: the function $u = y_k y_\ell \log(r^2 + |y|^2)$ lies in $\mathcal{C}_e^{0,\gamma}$, and it is not hard to check that when $k \neq \ell$, then $Lu \in \mathcal{C}_e^{0,\gamma}$ as well. However, $\partial^2_{y_k y_\ell} u \sim \log(r^2 + |y|^2) \notin \mathcal{C}_e^{0,\gamma}$. It turns out that $\mathcal{C}_e^{0,\gamma}$ is a borderline space for this boundedness. Note that we could equally well have replaced the $y_i y_j$ prefactor in u by $\operatorname{Re} z_i \overline{z_j}$; this is still harmonic, so $Lu \in \mathcal{C}_e^{0,\gamma}$, and it is also still true that $\partial^2_{i\bar{\jmath}} u \sim \log r$.

Despite this defect, the spaces $\mathcal{C}_e^{k,\gamma}$ still serve some important roles in the arguments in the rest of this paper.

The following result is the key to the higher regularity theory. Recall that $\phi \in \mathrm{PSH}(M, \omega)$ means that $\omega_\phi > 0$.

COROLLARY 3.5. *Suppose that* $\phi \in \mathcal{D}_w^{0,\gamma}(L) \cap \mathrm{PSH}(M,\omega)$ *is a limit in the topology of* $\mathcal{D}_w^{0,\gamma}(L)$ *of a sequence of polyhomogeneous potentials, and let* $L_\phi := \Delta_{\omega_\phi} + V$ *for some* $V \in \mathcal{C}_s^{0,\gamma}$. *(To make the notation coherent, write* $L_0 = L$.*) Then*

$$\mathcal{D}_w^{0,\gamma}(L_0) = \mathcal{D}_w^{0,\gamma}(L_\phi). \tag{45}$$

We only claim this result when ϕ is a limit in the appropriate Hölder norm of polyhomogeneous functions, but not for an arbitrary element of $\mathcal{D}_w^{0,\gamma}(L) \cap \mathrm{PSH}(M,\omega)$. This is the classical distinction between the 'little' and 'big' Hölder spaces, and it is adequate in our setting since we shall only need to apply this result when ϕ lies along the continuity path, and hence is a limit in this sense. This raises an interesting analytic question on which we comment after the proof.

Proof. Observe that $L_\phi u = f$ can be rewritten as

$$(\Delta_{\omega_\phi} - 1)u = f - (V+1)u \in \mathcal{C}_w^{0,\gamma},$$

so we may as well assume that $V \equiv -1$, which is a convenient choice because $\Delta_{\omega_\phi} - 1$ is invertible. Letting $G_\phi = L_\phi^{-1}$, then the assertion is equivalent to the fact that the range of G_ϕ is independent of ϕ (in the allowable space of functions).

When ϕ is polyhomogeneous, then the inverse G_ϕ is a pseudodifferential edge operator and (45) follows from Proposition 3.3.

To prove the assertion for ϕ that is a limit of polyhomogeneous functions, note first that the inclusion $\subseteq$ is obvious. Indeed, if $u \in \mathcal{D}_w^{0,\gamma}(L_0)$, then $Qu \in \mathcal{C}_w^{0,\gamma}$ for every $Q \in \mathcal{Q}^*$. Now write $(g_\phi)_{i\bar{\jmath}} = g_{i\bar{\jmath}} + \sqrt{-1}\phi_{i\bar{\jmath}}$, so that $(g_\phi)^{i\bar{\jmath}} = g^{i\bar{\jmath}} + \eta^{i\bar{\jmath}}$ for some $\eta^{i\bar{\jmath}} \in \mathcal{C}_w^{0,\gamma}$. Then $L_\phi u \in \mathcal{C}_w^{0,\gamma}$ as well, i.e., $u \in \mathcal{D}_w^{0,\gamma}(L_\phi)$.

These two facts together show that $\mathcal{D}_w^{0,\gamma}(L_\phi)$ remains the same when ϕ varies in the dense set of polyhomogeneous functions, but it might potentially jump up when ϕ is a limit of polyhomogeneous potentials.

For the converse, we claim that there is an *a priori* estimate

$$\sum_{Q \in \mathcal{Q}^*} ||Qu||_{w;0,\gamma} \le C\left(||L_\phi u||_{w;0,\gamma} + ||u||_{w;0,\gamma}\right), \tag{46}$$

which holds *only* for functions $u \in \mathcal{D}_w^{0,\gamma}(L)$ (but *not* $\mathcal{D}_s^{0,\gamma}(L_\phi)$), where the constant C is locally uniform in ϕ. To prove this, note that this estimate is true for L_ϕ when $\phi = 0$ and $u \in \mathcal{D}_w^{0,\gamma}(L)$. Freezing coefficients of a more general L_ϕ locally near any point $q \in M$, we can approximate this operator by one with polyhomogeneous coefficients, with an error term that has coefficients small in $\mathcal{C}_w^{0,\gamma}$. We prove the estimate in small coordinate charts for the nearby operator and then by perturbation for L_ϕ itself, absorbing the small coefficients into the left-hand side. These local estimates can then be pasted together with a partition of unity. This method makes clear that the constant C depends only on the $\mathcal{C}_w^{0,\gamma}$ norm of the coefficients $\eta^{i\bar{\jmath}}$.

To complete the argument, we must prove that for any $f \in \mathcal{C}_w^{0,\gamma}$, the unique solution u in $\mathcal{D}_w^{0,\gamma}(L_\phi)$ to $L_\phi u = f$ necessarily lies in $\mathcal{D}_w^{0,\gamma}(L_0)$. Note that another way to phrase this is that we must prove that $L_\phi : \mathcal{D}_w^{0,\gamma}(L_0) \to \mathcal{C}_w^{0,\gamma}$ is surjective.

Fix $f \in \mathcal{C}_w^{0,\gamma}$, and let $u \in \mathcal{D}_w^{0,\gamma}(L_\phi)$ solve $L_\phi u = f$. By local elliptic regularity (equivalently, boundedness of G on edge Hölder spaces), we also know that $u \in \mathcal{C}_e^{2,\gamma}$. We must show that $u \in \mathcal{D}_w^{0,\gamma}(L_0)$.

Choose a sequence ϕ_j of polyhomogeneous functions that converge to ϕ in $\mathcal{D}_w^{0,\gamma}(L)$. For each j, there is a unique $u_j \in \mathcal{D}_w^{0,\gamma}(L)$ with $L_{\phi_j} u_j = f$. Applying (46) (with L_{ϕ_j}) gives

$$\sum_{Q \in \mathcal{Q}^*} ||Qu_j||_{w;0,\gamma} \leq C.$$

There is a subsequence of the u_j such that each Qu_j converges in some weaker norm $\mathcal{C}_w^{0,\gamma'}$ and, furthermore, the $\mathcal{C}_w^{0,\gamma}$ norms of these Qu_j are uniformly bounded. There is a limiting function $u \in \mathcal{C}_w^{0,\gamma}$ (even though the limit takes place in a weaker topology), and moreover each Qu lies in $\mathcal{C}_w^{0,\gamma}$, so in fact $u \in \mathcal{D}_w^{0,\gamma}(L)$ and $L_\phi u = f$, as desired.

This proves that L_ϕ restricted to $\mathcal{D}_w^{0,\gamma}(L)$ is surjective and hence finally that $\mathcal{D}_w^{0,\gamma}(L_0) = \mathcal{D}_w^{0,\gamma}(L_\phi)$. $\square$

Remark 3.6. This result is equivalent to the assertion that $L_\phi : \mathcal{D}_w^{0,\gamma}(L_0) \to \mathcal{C}_w^{0,\gamma}$ is surjective. The latter statement is clearly an open condition for $\phi \in \mathcal{D}_w^{0,\gamma}(L_0)$, which gives the stronger conclusion that the result actually holds not just for ϕ lying in the closed subspace in $\mathcal{D}_w^{0,\gamma}(L)$ consisting of limits of polyhomogeneous functions, but for all ϕ in some open neighbourhood of this subspace. This suggests, of course, that the result might be true for all $\phi \in \mathcal{D}_w^{0,\gamma}(L) \cap \mathrm{PSH}(M, \omega)$. We do not have a proof of this, but in any case this extension is not needed here.

3.5. *Pseudodifferential edge operators and their boundedness.* We describe the proof of Proposition 3.3 in this subsection. The main point is to describe the structure of the Green operator G or, more specifically, the precise pointwise structure of its Schwartz kernel $G(z, z')$. This structure is then used to bound the integrals

$$Q_i u(z) = \int_X Q_i G(z, z') f(z')\, dV_g(z'), \quad f = Lu \in \mathcal{C}_w^{0,\gamma}. \tag{47}$$

The fact that makes this work is that the operators $Q_i \circ G$ are pseudodifferential edge operators; most of these compositions are of weakly positive type (cf. Definition 3.10), and [43, Prop. 3.27] gives their boundedness on the edge Hölder space $\mathcal{C}_e^{0,\gamma}$. We provide an extension of that argument to prove that each $Q_i \circ G$ is bounded on $\mathcal{C}_w^{0,\gamma}$ as well. While this replicates the results of [24], the refined structure of these operators proved here is an important ingredient in the higher regularity theory.

More broadly, we describe why the Schwartz kernel G has a polyhomogeneous structure and show how one can deduce from this that most of the Riesz potentials $Q_i \circ G$ are in the edge calculus and of weakly positive type. The boundedness of such weakly positive edge operators on edge and wedge Hölder spaces is a basic feature of the edge calculus. Donaldson derives the polyhomogeneous structure of the Green function just for the flat model problem G_β by explicit calculation and then proves the Hölder estimates on the wedge spaces in that setting by hand. The edge calculus is a systematization of the perturbation arguments that allow one to pass from this flat model to the actual curved problem, but one which yields, in particular, the polyhomogeneous structure of the Green function for the curved problem, which plays a significant role for the higher regularity theory.

The edge calculus $\Psi_e^*(X)$ is a space of pseudodifferential operators on X, elements of which have degeneracies at ∂X similar to the ones exhibited by differential edge operators as in (37). We use X systematically now rather than M since it is more natural for the descriptions below to work on a manifold with boundary. This space of operators is large enough to contain not only all differential edge operators A, but also parametrices and generalized inverses for the elliptic operators in this category, as well as for incomplete elliptic edge operators like $L = r^{-2}A$. The term 'calculus' (rather than algebra) is used to indicate that $\Psi_e^*(X)$ is almost closed under composition, with the caveat that not every pair of elements may be composed due to growth properties of Schwartz kernels in the incoming and outgoing variables that prevent the corresponding integrals from converging.

An element $B \in \Psi_e^*(X)$ is characterized by specific regularity properties of its Schwartz kernel $B(z, z')$ as a distribution on $X \times X = X^2$; the superscript $*$ is a placeholder for a set of indices that indicate the singularity structure of this distribution in various geometric regimes in X^2. By definition, any such $B(z, z')$ is the pushforward of a distribution K_B defined on a space X_e^2, called the edge double space, that is a resolution of X^2 obtained by performing a (real) blow-up of the fiber diagonal (defined below) of $(\partial X)^2$. This distribution K_B has a standard pseudodifferential singularity along the lifted diagonal (by which we mean a polyhomogeneous expansion in powers of the distance to this submanifold), as well as polyhomogeneous expansions at all boundary hypersurfaces of X_e^2 and product-type expansions at the higher codimension corners. We have defined polyhomogeneity on manifolds with boundary earlier, and we will extend this to manifolds with corners below. The detailed notation $B \in \Psi_e^{m,k,E_{\mathrm{rf}},E_{\mathrm{lf}}}(X)$ records the pseudodifferential order m along the diagonal and the exponent sets in the expansions at the various boundary faces. We explain this in more detail now, but all of this is described fully in [43, §2–3].

We first construct the blowup X_e^2. The product $X^2 = X \times X$ is a manifold with corners up to codimension two. The corner $(\partial X)^2$ has a distinguished submanifold, denoted $\mathrm{fdiag}_{\partial X}$, which is the fiber diagonal. This consists of the set of points (p,p') such that $\pi(p) = \pi(p')$. This is blown up normally, resulting in a space $[X^2, \mathrm{fdiag}_{\partial X}]$ that by definition is the edge double space X_e^2. Using local coordinates (r,θ,y) on the first factor of X and an identical copy (r',θ',y') on the second copy, the corner is the submanifold $\{r = r' = 0\}$, and $\mathrm{fdiag}_{\partial X} = \{r = r' = 0, y = y'\}$. The blowup may be thought of as introducing polar coordinates around this submanifold:

$$\begin{aligned}&R = |(r,r',y-y')| \geq 0,\\ &\omega = R^{-1}(r,r',y-y') \in S_+^{2n-1} = \{\omega = (\omega_1,\omega_2,\tilde{\omega}) : |\omega| = 1, \omega_1,\omega_2 \geq 0\},\end{aligned}$$

supplemented by y', θ, θ' to make a full coordinate system. Thus X_e^2 has a new boundary hypersurface, $\{R = 0\}$, called the 'front face' ff, and the lifts of the two original boundary hypersurfaces, $\{\omega_1 = 0\}$ and $\{\omega_2 = 0\}$, called the right and left faces, rf and lf, respectively. We write defining functions for these faces as ρ_{ff}, ρ_{rf} and ρ_{lf}. The diagonal of X^2 lifts to the submanifold $\mathrm{diag}_e = \{\omega = (1/\sqrt{2}, 1/\sqrt{2}, 0), \theta = \theta', R \geq 0\}$.

Here are some motivations for this construction. First, Schwartz kernels of pseudodifferential operators are singular along the diagonal in X^2, but the fact that this diagonal intersects the corner nontransversely makes these singularities hard to describe near this intersection. By contrast, the lifted diagonal diag_e intersects the boundary of X^2 only in the interior of ff, and this intersection is transversal; this turns out to allow for a simpler description of the singularity of the Schwartz kernel there. Another point is that X_e^2 captures the homogeneity under dilations inherent in this problem. The flat model operator

$$L_\beta = \partial_r^2 + r^{-1}\partial_r + (\beta r)^{-2}\partial_\theta^2 + \Delta_y,$$

on the product space $[0,\infty)_r \times S_\theta^1 \times \mathbb{R}_y^{2n-2}$ is homogeneous of order -2 with respect to the dilations $(r,\theta,y) \mapsto (\lambda r, \theta, \lambda y)$ and is also translation invariant in y. It follows that the Schwartz kernel $G_\beta(z,z')$ of the inverse for the Friedrichs extension of L_β commutes with translations in y, thus depends only on the difference $y - y'$ rather than y and y' individually, and is homogeneous of order $-2n+2$ in the sense that

$$G_\beta(\lambda r, \lambda r', \lambda(y-y'), \theta, \theta') = \lambda^{-2n+2} G_\beta(r, r', y-y', \theta, \theta'). \tag{48}$$

In the polar coordinate system above, this simply says that

$$G_\beta(r, r', y-y', \theta, \theta') = \mathcal{G}_\beta(\omega, \theta, \theta') R^{-2n+2} \tag{49}$$

or, equivalently, that G_β lifts to the double space $(\mathbb{R}^+ \times S^1 \times \mathbb{R}^{2n-2})^2_e$ and decomposes as a product of the simple factor R^{-2n+2} and the 'angular part' $\mathcal{G}_\beta$. A further analysis shows that $\mathcal{G}_\beta$ has a singularity at $\omega = (1/\sqrt{2}, 1/\sqrt{2}, 0)$ and polyhomogeneous expansions along the side faces $\{\omega_1 = 0\}$ and $\{\omega_2 = 0\}$.

We now recall the general definition of polyhomogeneity on manifolds with corners. We do this only on the model orthant $\mathcal{O} = (\mathbb{R}^+)^k \times \mathbb{R}^\ell$, with linear coordinates $(x_1, \ldots, x_k, y_1, \ldots, y_\ell)$, but this definition is coordinate-invariant and hence translates immediately to arbitrary manifolds with corners. First, let

$$\mathcal{V}_b(\mathcal{O}) := \mathrm{span}_{\mathcal{C}^\infty}\{x_1\partial_{x_1}, \ldots, x_k\partial_{x_k}, \partial_{y_1}, \ldots, \partial_{y_\ell}\}$$

be the space of all smooth vector fields tangent to all boundaries of this space. We may as well assume that all distributions are supported in a ball $\{|x|^2 + |y|^2 \leq 1\}$. If $\nu = (\nu_1, \ldots, \nu_k) \in \mathbb{R}^k$, then u is conormal of order ν, $u \in \mathcal{A}^\nu(\mathcal{O})$, if

$$V_1 \ldots V_j u \in x^\nu L^\infty(\mathcal{O}) \ \forall\, j \geq 0 \text{ and for all } V_i \in \mathcal{V}_b(\mathcal{O}). \tag{50}$$

Next, u is polyhomogeneous if near the origin in $\mathcal{O}$, u has an expansion of the form

$$u \sim \sum_\ell \sum_{|p| \leq N_j} a_{\ell,p}(y) x^{\gamma^{(\ell)}} (\log x)^p,$$

where $\{\gamma^{(\ell)} = (\gamma_1^{(\ell)}, \ldots, \gamma_k^{(\ell)})\}\}$ is a sequence of k-tuples in $\mathbb{C}^k$ with $\mathrm{Re}\gamma_j^{(\ell)} \to \infty$ as $\ell \to \infty$, $x^\gamma = x_1^{\gamma_1} \ldots x_k^{\gamma_k}$ and $(\log x)^p := (\log x_1)^{p_1} \cdots (\log x_k)^{p_k}$, with each $p \in \mathbb{N}_o^k$. The coefficients $a_{\ell,p}(y)$ are smooth. As with polyhomogeneous expansions for functions near a codimension one boundary, these sums are not usually convergent, but may still be differentiated term-by-term, etc. If u is polyhomogeneous in this sense, then each coefficient of its expansion at any one of the boundary hypersurfaces or corners is polyhomogeneous on that face. We associate to such an expansion an index family $E = \{E^{(\ell)}\}$, $\ell = 1, \ldots, k$, consisting of all pairs of multi-indices $\{(\gamma, p)\}$ of exponents that occur in this expansion, and we denote by $\mathcal{A}^E_{\mathrm{phg}}$ the space of all such distributions. As in the codimension one case, we say that $u \in \mathcal{A}^0_{\mathrm{phg}}$ if u is polyhomogeneous and if each index set $E^{(\ell)}$ is greater than or equal to 0 in the sense of Definition 2.6. We also write the simple index set $\{(\gamma + \ell, 0) : \ell \in \mathbb{N}_0\}$ simply as (γ); thus $\mathcal{A}^{(\gamma)}_{\mathrm{phg}} = x^\gamma \mathcal{C}^\infty$, i.e., $u = x^\gamma v$ where v is $\mathcal{C}^\infty$ up to that face. (Even more specifically, a function that is smooth in the traditional sense up to the boundary and corners has index set (0).)

We now define the space of pseudodifferential edge operators on X.

Definition 3.7. We say that $B \in \Psi_e^{m,\eta,E_{\mathrm{rf}},E_{\mathrm{lf}}}(X)$ if the Schwartz kernel of B is the pushforward from X^2_e to X^2 of a distribution K_B on X^2_e that has the following properties. K_B decomposes as a sum $K_B^{(1)} + K_B^{(2)}$ where $K_B^{(1)} = \rho_{\mathrm{ff}}^{-2n+\eta} K_B'$ is supported in a neighbourhood of diag_e that does not intersect the side faces, where K_B' has a classical pseudodifferential singularity of order m along this lifted diagonal that is smoothly extendible across ff. (This simply says that K'_B is smooth up to ff away from diag_e and that the singularity along this lifted diagonal extends across ff so that it remains conormal on the 'continuation' of the diagonal across this face.) The term $K_B^{(2)}$ is required to be polyhomogeneous on X_e^2 with index sets $(\eta - 2n)$ at ff, E_{rf} at rf and E_{lf} at lf, and it vanishes in a neighbourhood of the lifted diagonal.

This decomposition of K_B into two terms isolates that part of K_B that contains the diagonal singularity, and it emphasizes the key fact that this singularity is uniform up to ff. The shift of the order at the front face by $-2n$ is an artifact of a normalization: indeed, the volume form dV_g is uniformly equivalent to $rdrd\theta dy$, so the Schwartz kernel of the identity operator relative to this measure is a smooth nonvanishing multiple of

$$r^{-1}\delta(r-r')\delta(\theta-\theta')\delta(y-y').$$

Since $\delta(r-r')$ and $\delta(y-y')$ are homogeneous of degrees -1 and $2-2n$, respectively, this Schwartz kernel is homogeneous of order $-2n$, and we simply want this to match with the fact that Id is an operator of order 0.

Finally, we may state the basic structure theorem for the Green function of L.

PROPOSITION 3.8. *Let g be a polyhomogeneous edge metric with angle β along D and $L = -\Delta_g + V$ where V is polyhomogeneous and bounded on X, and suppose that G is the generalized inverse to the Friedrichs extension of L. Then $G \in \Psi_e^{-2,2,E,E}(X)$, where the index set E is determined by the indicial roots of L_β and by the index sets of g and V. In particular, if g and V are smooth* (*i.e.*, *both have index set* (0) *at* ∂X), *then*

$$E \subset \{(j/\beta + k, \ell) : j, k, \ell \in \mathbb{N}_0 \text{ and } \ell = 0 \text{ for } j+k \le 1,\ (j,k,\ell) \neq (0,1,0)\}. \tag{51}$$

Moreover, if g and V are polyhomogeneous with index set contained in the index set (51), *then the index set E for G is also contained in the index set* (51).

Remark 3.9. The fact that G has the same index set E at the left and right faces is natural since G is symmetric. The index set E may be slightly more complicated when $\beta \in \mathbb{Q}$ since in that case j/β can equal a positive integer for certain j, and this creates extra logarithmic factors in the expansion (i.e., elements of E of the form $(k,1)$), but these all occur sufficiently high in the expansion – in $\mathrm{Re}\,\zeta \ge 1$ – and hence do not affect the considerations below. These log terms are absent if g is an orbifold metric.

Despite the seemingly elaborate language needed to state this result, this structure theorem for G includes the one given by Donaldson [24] for the model Green function, but the key advantage is that we have this same refined structure for the curved operator G too.

This result is one of the main conclusions of [43]: it is simply the elliptic parametrix construction in the edge calculus, modified slightly to accomodate the minor differences for Laplacians of incomplete rather than complete edge metrics. As with any parametrix construction, the first main step is to obtain detailed information about the solution operator for the model problem Δ_{g_β} or, in other words, about the model Green function G_β. This is the technical core, and the rest of the argument uses pseudodifferential calculus to write the Green function for L as a perturbation of G_β. The specific information we need to obtain, then, is that the Schwartz kernel of G_β has the same polyhomogeneous structure as in the statement of Proposition 3.8. This may be approached in a few ways. The first, appearing in [43, §§4, 5], is to take the Fourier transform in y, thus reducing Δ_β to the family of operators

$$\widehat{\Delta}_\beta = \partial_r^2 + r^{-1}\partial_r + (\beta r)^{-2}\partial_\theta^2 - |\eta|^2$$

on $\mathbb{R}^+ \times S^1$ where η is the variable dual to y. To keep track of the dependence on η, set $s = r|\eta|$ to convert this to

$$|\eta|^2 \left(\partial_s^2 + \frac{1}{s}\partial_s + \frac{1}{\beta^s s^2}\partial_\theta^2 - 1 \right).$$

This can be analyzed explicitly by separation of variables. Chasing back through these transformations yields a tractable expression for G_β. The equivalent approach in [24] is to write G_β as an integral over $0 < t < \infty$ of the heat kernel $\exp(t\Delta_\beta)$. This heat kernel is the product of the heat kernel on the model two dimensional cone with cone angle $2\pi\beta$ and the Euclidean heat kernel on $\mathbb{R}^{2n-2}$. The former of these is known classically, albeit as an infinite sum involving Bessel functions (see [24] and [50]) while the latter is the standard Gaussian. Either method requires about the same amount of work.

A minor point in the statement of Proposition 3.8 that turns out to be important below is the fact that the index set E does not contain the element $(1, 0)$; in other words, the monomials r and r' do not appear in the expansion of G at the left and right faces, respectively. This can be explained as follows. As a distribution, $G(r, \theta, y, r', \theta', y')$ satisfies

$$LG = r^{-1}\delta(r - r')\delta(\theta - \theta')\delta(y - y').$$

Restricting to the interior of rf, away from the front face, we see that $LG = 0$ there. Since we know at this point that G is polyhomogeneous, we can calculate formally, i.e. letting L act on the series expansion and collecting terms with the same powers. It is then easy to see that L cannot annihilate the term $a(\theta, y)r$; indeed, referring to (14), the only possible problematic term in $L(a(\theta, y)r)$ is $(\partial_r^2 + r^{-1}\partial_r + \beta^{-2}r^{-2}\partial_\theta^2)(ar) = r^{-1}(\beta^{-2}\partial_\theta^2 + 1)a$. However, since $\beta < 1$, there is no other term in the expansion that could cancel this, and it is impossible for $\beta^{-2}a_{\theta\theta} + a$ to vanish unless $a \equiv 0$. This proves the claim.

Definition 3.10. An index set E is called nonnegative if, for any $(\gamma, p) \in E$, $\mathrm{Re}\gamma \geq 0$ and if $\mathrm{Re}\,\gamma = 0$, then $(\gamma, p) = (0,0)$.

An operator $B \in \Psi_e^{m,\eta,E,E'}(X)$ is said to be of nonnegative type if $m \leq 0$, $\eta \geq 0$, and both E and E' are nonnegative. It is called weakly positive if, in addition, either $\eta > 0$ or $E > 0$. (Thus the excluded case is when $\eta = 0$ and E contains $(0,0)$.) We shall always assume that if the leading exponent at lf is 0, then the corresponding coefficient does not depend on θ.

We now state the basic boundedness theorem needed in the proof of Proposition 3.3.

PROPOSITION 3.11. *Let $B \in \Psi_e^{m,r,E,E'}(X)$. If B is of weakly positive type so, in particular, $m \leq 0$, and if the first nonzero element of the index set for B at* lf *is greater than* 1, *then*

$$B : \mathcal{C}_w^{\ell,\gamma}(X) \longrightarrow \mathcal{C}_w^{\ell,\gamma}(X)$$

is bounded for any $\ell \in \mathbb{N}_0$.

Proof. The order of the singularity along the diagonal is not the key issue here, so we assume that $m = 0$. We proceed in a series of increasingly general steps. For the first, suppose that the index set of B at lf is strictly positive and $\mu \geq 0$ is strictly smaller than all elements in this index set, and consider boundedness on weighted edge Hölder spaces. The result here is [43, Prop. 3.27], which asserts that

$$B : r^\mu \mathcal{C}_e^{\ell,\gamma}(X) \longrightarrow r^\mu \mathcal{C}_e^{\ell,\gamma}(X) \tag{52}$$

is bounded for every $\ell \geq 0$.

Let us recall how (52) is proved. Decompose B into a sum of two operators, $B_1 + B_2$, where the Schwartz kernel of B_1 is supported near the lifted diagonal $\mathrm{diag}_e \subset X_e^2$ and carries the full pseudodifferential singularity, while that of B_2 is polyhomogeneous on X_e^2 and has the same index sets as B at lf, ff and rf. The boundedness of $B_1 : r^\mu \mathcal{C}_e^{\ell,\gamma} \to r^{\mu+\eta}\mathcal{C}_e^{\ell,\gamma} \hookrightarrow r^\mu \mathcal{C}_e^{\ell,\gamma}$ is a consequence of the approximate dilation invariance of both the Schwartz kernel of B_1 and of the edge Hölder norm, as well as the standard local boundedness on Hölder spaces for (ordinary, nondegenerate) pseudodifferential operators. Rigorously, this is done using a Whitney cube decomposition and scaling arguments. This uses only the conditions $m \leq 0$ and $\eta \geq 0$.

As for the other term, noting that $(r\partial_r)^i (r\partial_y)^j \partial_\theta^\ell B_2$ has the same structural properties and index sets as B_2 itself, we see that it suffices to prove that if $\tilde{f} \in r^\mu \mathcal{C}_e^{\ell,\gamma}$, then $|B_2 \tilde{f}| \leq C r^\mu$, since then every $|(r\partial_r)^i (r\partial_y)^j \partial_\theta^k B_2 \tilde{f}| \leq C r^\mu$ as well.

For convenience, and since slight variants of this calculation are used several places below, here is a precise statement of a special case of the pushforward theorem, [43, Prop. A.18]. Suppose that $H \in \Psi_e^{-\infty,\eta,E,E'}$ has Schwartz kernel that is pointwise nonnegative and where $E' + \mu > -2$. Suppose the smallest element in the index set E is $(\lambda_0, 0)$. Then

$$H(r^\mu) = \int H(r,\tilde{r},y,\tilde{y},\theta,\tilde{\theta})\tilde{r}^\mu\, \tilde{r}d\tilde{r}d\tilde{y}d\tilde{\theta}$$

is polyhomogeneous with leading term $ar^{\hat{\lambda}}$, where $\hat{\lambda} = \min\{\lambda_0, \mu+\eta\}$, provided $\mu + \eta \neq \lambda_0$, or with leading term $ar^{\lambda_0}\log r$ if $\lambda_0 = \mu + \eta$. More generally, if $|f| \leq Cr^\mu$, then $|Hf| \leq Cr^{\hat{\lambda}}$ when $\lambda_0 \neq \mu + \eta$ and $|Hf| \leq Cr^{\lambda_0}\log r$ when $\lambda_0 = \mu + \eta$.

Returning to the problem above, applying this result shows that because of the assumption about the index set of B and the fact that $\mu \geq 0$, we obtain that $|B_2(r^\mu)| \leq Cr^\mu$ as desired.

Next, when the leading exponent in the expansion of B at lf is 0, then the sharp statement is that

$$B : r^\mu \mathcal{C}_e^{\ell,\gamma} \longrightarrow \mathcal{C}_w^{0,\gamma} \cap \mathcal{C}_e^{\ell,\gamma}.$$

It suffices to consider just the contribution from B_2 and to show that $B_2\tilde{f} \in \mathcal{C}_w^{0,\gamma}$. For this particular argument, it is actually enough to assume that B is of nonnegative type, so let us assume that the leading exponent in the expansion of B_2 at ff is 0. As a first step note that since $|\tilde{f}| \leq Cr^\gamma$, then by the pushforward theorem, $|B_2\tilde{f}| \leq C$. To improve this, note that $r\partial_r$ annihilates the leading term a_0r^0 of B_2 at lf, so $r\partial_r B_2$ is weakly positive with leading exponent at lf strictly greater than γ. Applying the pushforward theorem for this operator gives $|r\partial_r B_2\tilde{f}| \leq Cr^\gamma$ or, equivalently, $|\partial_r B_2\tilde{f}| \leq Cr^{\gamma-1}$. We next observe that $r\partial_y B_2$ has index set greater than or equal to 1 at lf, so $|r\partial_y B_2\tilde{f}| \leq Cr^\gamma$, i.e., $\partial_y B_2\tilde{f}| \leq r^{\gamma-1}$. Finally, by hypothesis, $r^{-1}\partial_\theta B_2$ has positive index set at lf and has order $\eta = -1$ at ff, hence $|r^{-1}\partial_\theta B_2\tilde{f} \leq Cr^{\gamma-1}$. These estimates on the derivatives yield, by a standard argument, that $B_2\tilde{f} \in \mathcal{C}_w^{0,\gamma}$.

Finally, let us turn to our actual goal, when $f \in \mathcal{C}_w^{\ell,\gamma}$. First suppose that $\ell = 0$, and fix $f \in \mathcal{C}_w^{0,\gamma}$. We wish to prove that $Bf \in \mathcal{C}_w^{0,\gamma}$ as well. Since B is of weakly positive type, we have that either the index set E at lf is greater than γ, or else the order η at ff is positive. In our application, $\eta \geq 1$, but in fact any $\eta > \gamma$ is sufficient for this estimate. The key to this argument is the decomposition $f = \hat{f} + \tilde{f}$ from Section 2.6.3, where $\hat{f}$ is the harmonic extension of $f_0 = f|_D$ and $\tilde{f} \in r^\gamma \mathcal{C}_e^{0,\gamma}$. We have just proved that $B\tilde{f} \in \mathcal{C}_w^{0,\gamma}$, so the remaining issue is to study the behavior of $\hat{u} = B\hat{f}$, assuming just that B is of weakly positive type. Recalling now the bounds (35), we first estimate that $\partial_y(B\hat{f}) = B\partial_y\hat{f} + [\partial_y, B]\hat{f}$. (Because $\hat{f}$ is smooth in the interior, we no longer need to isolate the diagonal contribution of B.) By [43, Prop. 3.30], the commutator $[\partial_y, B]$ is an operator of the same type and order as B so, in particular, $|[\partial_y, B]\hat{f}| \leq C|\log r|$. On the other hand, using (35), $|B\partial_y \hat{f}| \leq |B(\tilde{r}^{\gamma-1})| \leq Cr^{\gamma-1}$. For the r derivative, we reintroduce the decomposition $B = B_1 + B_2$. We have $r\partial_r B_1 \hat{f} = B_1 r\partial_r \hat{f} + [r\partial_r, B_1]\hat{f}$. Now B_1 is of strictly positive type, i.e., it is of order 0, vanishes (to all orders) at lf and to order $\eta > \gamma$ at ff; by [43, Prop. 3.30] again, the commutator $[r\partial_r, B_1]$ has the same properties. Therefore $|r\partial_r B_1 \hat{f}| \leq Cr^{\gamma}$. Finally, $r\partial_r B_2$ vanishes to order greater than γ at lf and ff, hence $|r\partial_r B_2 \hat{f}| \leq Cr^{\gamma}$ too. Altogether, $|\partial_r B\hat{f}| \leq Cr^{\gamma-1}$, as required. The analogous estimate for $r^{-1}\partial_\theta B\hat{f}$ is similar but simpler. These estimates together imply that $B\hat{f} \in \mathcal{C}^{0,\gamma}_w$.

Suppose at last that $\ell > 0$. We use a simple commutator argument again, noting that $\partial_y Bf = B\partial_y f + [\partial_y, B]f$, so by induction we obtain that $f \in \mathcal{C}^{\ell,\gamma}_w \Rightarrow Bf \in \mathcal{C}^{\ell,\gamma}_w$. □

Implicit in this argument is that when $\eta = 0$ and $E \geq 0$, Bf may have logarithmic growth as $r \to 0$ when $f \in \mathcal{C}^{0,\gamma}_e$. To see this, observe first that $B_1 f$ is well behaved as before. In addition, $r\partial_r B_2$ is weakly positive with positive index set at lf, so $r\partial_r B_2 f \in \mathcal{C}^{\ell,\gamma}_e$ for all ℓ. Using only that $|r\partial_r Bf| \leq C$ and integrating from 1 to r gives $|Bf| \leq C(1 + |\log r|)$. It is for this reason that we have to treat certain of the operators in $\mathcal{Q}^*$ separately.

Proof of Proposition 3.3. Proposition 3.11 provides the main step. The key point is that $Q_i \circ G$ is of weakly positive type for all $Q_i \in \mathcal{Q}$; we then reduce to this case for the remaining operators $Q_i \in \mathcal{Q}^* \setminus \mathcal{Q}$.

Suppose then that $Q_i \in \mathcal{Q}$; we show that $Q_i G$ is of weakly positive type. Observe that each vector field $r\partial_r$, ∂_θ and $r\partial_{y_j}$ on X lift smoothly to X^2_e via the blowdown map $\pi : X^2_e \to X$; indeed, each of these lifts is a vector field on this blown up space that is tangent to all boundary faces. Thus $\rho_{\mathrm{ff}}\pi^*\partial_{y_\ell}$ is tangent to all faces, as is $\rho_{\mathrm{ff}}\pi^* r^{-1}\partial_\theta$, while $\rho_{\mathrm{ff}}\pi^*\partial_r$ differentiates transversely to the left face, but is tangent to all other faces. Therefore, each operator $Q_i \in \mathcal{Q}^*$ lifts to an operator on X^2_e of the form $\rho_{\mathrm{ff}}^{-k}\tilde{Q}_i$ where $\tilde{Q}_i$ acts tangentially along ff and where $k = 1$ or 2. When $k = 1$, the composition $Q_i \circ G$ has order 1 at ff, so it is of weakly positive type. When $k = 2$, then it is necessary that Q_i annihilate the leading coefficient of G at lf so that $Q_i \circ G$ has positive index set there, and this is precisely what determines the subcollection $\mathcal{Q} \subset \mathcal{Q}$. Note that we are using here that the coefficient a_0 of r^0 at lf is independent of θ, which is the case since this term is annihilated by the indicial operator. We have now proved that $Q_i \circ G : \mathcal{C}^{0,\gamma}_w \to \mathcal{C}^{0,\gamma}_w$ for all $Q_i \in \mathcal{Q}$.

Next consider the operator $Q = \partial^2_{y_i y_j}$. From the considerations in the previous paragraph, $Q \circ G$ is nonnegative, but not strictly positive, so if $f \in \mathcal{C}^{0,\gamma}_w$, then it takes an extra step to show that $Q \circ G(f) \in \mathcal{C}^{0,\gamma}_w$. Decompose $G = G_1 + G_2$ with G_1 supported near the diagonal and vanishing near the left and right faces, and so that G_2 has no diagonal singularity. Then $Q \circ G_1$ has order 0 but has empty index set near lf, hence is weakly positive, so Proposition 3.11 shows that $Q \circ G_1 f \in \mathcal{C}^{0,\gamma}_w$.

For the other term, recall the decomposition $f = \hat{f} + \tilde{f}$, as in the proof of Proposition 3.11, where $\tilde{f} \in r^\gamma \mathcal{C}^{0,\gamma}_e$ and $\hat{f}$ satisfies (35). The previous proof shows that $Q \circ G_2(\tilde{f}) \in r^\gamma \mathcal{C}^{0,\gamma}_e \subset \mathcal{C}^{0,\gamma}_w$; cf. (36). Next, write

$$Q \circ G_2(\hat{f}) = (\partial_{y_i} \circ G_2)(\partial_{y_j} \hat{f}) + \partial_{y_i}[\partial_{y_j}, G_2]\hat{f}.$$

Observe that $\partial_{y_i} \circ G_2 \in \Psi^{-\infty,1,E,E}_e$. (Here E is the same index set as in the characterization of G.) The other operator is of the same type; indeed, by [43, Prop. 3.30], the commutator $[\partial_{y_j}, G_2] \in \Psi^{-\infty,2,E,E}_e$ so $\partial_{y_i}[\partial_{y_j}, G] \in \Psi^{-\infty,1,E,E}_e$ as well. Now using the pushforward theorem together with the estimates $|\hat{f}| \leq C$, $|\partial_{y_j} \hat{f}| \leq C r^{-1+\gamma}$, we see that $|Q \circ G_2(\hat{f})| \leq C$. To show that these terms are actually in $\mathcal{C}^{0,\gamma}_w$, note that $\partial_r \circ (\partial_{y_i} \circ G_2)$, $\partial_{y_\ell} \circ (\partial_{y_i} \circ G_2)$ and $r^{-1}\partial_\theta \circ (\partial_{y_i} \circ G_2)$ all lie in $\Psi^{-\infty,0,E',E}_e$, where E' is a nonnegative index set, so by the pushforward again, applying these to a function bounded by $r^{-1+\gamma}$ produces a function that is again bounded by a multiple of $r^{-1+\gamma}$. (For the second term, we even have a much stronger estimate, but this is unimportant here.) We have proved that $|\nabla \circ Q \circ G_2(\hat{f})| \leq C r^{-1+\gamma}$, and hence that $Q \circ G_2(\hat{f}) \in \mathcal{C}^{0,\gamma}_w$.

The final operator to consider is $P_{1\bar{1}}$. We use the same trick as [24], noting that $P_{1\bar{1}} = \Delta_g - \sum a_i Q_i$, where the Q_i are all operators of the type considered above, with coefficients in $\mathcal{C}^{0,\gamma}_w$, and from this the corresponding bound is clear. $\square$

We remark that the proof gives slightly more. Namely, in case a_2, the coefficient of r^2 in the expansion of G at lf is independent of θ, which is the case for solutions of the Monge–Ampère equation (see Proposition 4.4), and $\beta < 1/2$, then $r^{-2}\partial^2_\theta \circ G$ is of weakly positive type, hence both $r^{-2}\partial^2_\theta \circ G$ and $(\partial^2_r + r^{-1}\partial_r) \circ G$ are bounded on $\mathcal{C}^{0,\gamma}_w$.

3.6. *A comparison of methods.* The previous subsections provide a review of the terminology and basic results about edge operators. The point of including this is to show that Proposition 3.3 follows directly from this general existing theory. Since Donaldson's approach [24] has the appearance of being more elementary, it is worth saying a bit more about the similarities and differences between the approaches, as well as the advantages of the one here.

The two slightly different methods for constructing the model kernel G_β are equivalent, and there is little to recommend one method over the other. The other two steps of the argument in [24] are inverted relative to the development here. The edge parametrix construction provides a systematic way to pass from the polyhomogeneous structure of the model inverse G_β to the corresponding structure for the actual inverse G. Once that is known, the Hölder boundedness for G and G_β are then deduced from the general result about boundedness of edge pseudodifferential operators acting on edge Hölder spaces, the proof of which reduces by scaling to little more than the boundedness of standard pseudodifferential operators on ordinary Hölder spaces. Donaldson, by contrast, first establishes the Hölder estimates for the model operator G_β using related scaling arguments and then observes that these estimates can be patched together to obtain the Hölder boundedness for the differentiated kernels $P_{i\bar{j}}G$. In other words, the patching (or transition from the model to the actual inverse) is done at the level of the parametrix in our approach, but at the level of *a priori* estimates in a particular function space in Donaldson's. The disadvantage of this latter approach is that one is too closely tied to the function space on which the model *a priori* estimates were obtained. This makes that method harder to apply when proving the higher regularity estimates, for example, and this higher regularity turns out to be key in the existence theory. Thus, while the edge theory requires a certain amount of technical overhead, it provides a number of substantial benefits. These become even more apparent in the generalization of this theory to the case of divisors with simple normal crossings.

4. Higher regularity for solutions of the Monge–Ampère equation

We now use the machinery of the last section to prove one of our main results, that under reasonable initial hypotheses, solutions of the complex Monge-Ampere equation are polyhomogeneous (Theorem 1). This type of proof has appeared in many places by now. One origin is the proof of polyhomogeneity for *complete* Bergman and Kähler–Einstein metrics on strictly pseudoconvex domains by Lee and Melrose [37]; that argument was clarified and recast into something near the present form in [44], where polyhomogeneity of solutions of the singular Yamabe problem (or obstructions to such polyhomogeneity) was determined. This regularity result was announced in [45].

We turn to the proof of Theorem 1. There are three main steps. The first is to show that $u \in \mathcal{C}_e^{k,\gamma}$ for every $k \in \mathbb{N}$; the second is to improve this to full conormality, i.e., to show that $u \in \mathcal{A}^0$; in the last, we improve this conormality to the existence of a polyhomogeneous expansion. The first step is equivalent to standard higher elliptic regularity for Monge-Ampere equations; this uses the dilation invariance properties of the edge Hölder spaces in a crucial way. The second step then breaks this dilation invariance by showing that we may also differentiate arbitrarily many times along D. The final step uses an iteration to show that u has a longer and longer partial polyhomogeneous expansion.

We begin, then, by quoting a consequence of the Evans–Krylov–Safonov theory concerning solutions of Monge–Ampère equations [36], [26], [35], or rather its extension to the complex Monge–Ampère equation [54].

THEOREM 4.1. *Let ω be a smooth Kähler metric in a ball $B \subset \mathbb{C}^n$ and $F \in \mathcal{C}^\infty(B \times \mathbb{R})$. Suppose that $u \in \mathcal{C}^2(B)$ is a solution of $\omega_u^n/\omega^n = F(z,u)$ on B. Then for any $k \geq 2$, there is a constant C depending on F, k, ω, $\sup_B |u|$ and $\sup_B |\Delta_\omega u|$ such that if B' is a ball with the same center as B but with half the radius, then*

$$||u||_{\mathcal{C}^{k,\gamma}(B')} \leq C.$$

The constant C depends uniformly on the $\mathcal{C}^{k+3}(B)$ norm of the coefficients of ω.

To be precise, the Evans–Krylov theorem gives the $\mathcal{C}^{2,\gamma}(B)$ estimate. The higher regularity is obtained by a straightforward bootstrap, since differentiating the equation with respect to any coordinate vector field W gives a linear equation for Wu with coefficients depending on at most the second derivatives of u, to which we can apply ordinary Schauder estimates since using the $\mathcal{C}^{2,\gamma}(B)$ estimate, the coefficients in the resulting equation are Hölder continuous.

To adapt this to our setting, we first observe that the Monge–Ampère equation is invariant with respect to the scaling $S_\lambda(r,\theta,y) = (\lambda r, \theta, \lambda y)$, which in the original complex coordinates takes the form

$$(z_1, \ldots, z_n) \to (\lambda^{1/\beta} z_1, \lambda z_2, \ldots, \lambda z_n).$$

To see this, let $\tilde{\omega}$ be any polyhomogeneous Kähler edge current, i.e. an element of $\mathcal{H}_{\omega_0}$ (this was denoted by ω in the statement of Theorem 1, but we use the tilde here to avoid confusion with the reference metric ω (26)), and let $\tilde{g}$ denote its associated Kähler metric. We see from (6) (and polyhomogeneity) that as $\lambda \to \infty$,

$$\lambda^2 S_{1/\lambda}^* \tilde{g} \longrightarrow c g_\beta,$$

for some constant $c > 0$, where g_β is the flat model edge metric on $\mathbb{C}^n$.

Now let B be the ball of radius $r_0/2$ centered at some point (r_0, y_0) in the coordinates (r,y), where r_0 is small, let B' be the ball of half this radius and consider the sets $B \times S^1$ and $B' \times S^1$. Choosing coordinates so that $y_0 = 0$, we obtain the family of metrics

$$g_{r_0} := r_0^{-2} S_{r_0}^* \left(\tilde{g}|_{B \times S^1} \right),$$

which we regard as defined on $\tilde{B} \times S^1$, where $\tilde{B}$ is a ball of radius $1/2$ centered at $(1,0)$. Let $\tilde{B}'$ be the ball of radius $1/4$ centered at this same point. Finally, consider the family of functions $u_{r_0}(r,\theta,y) = S_{r_0}^* u(r,\theta,y) = u(r_0 r, \theta, r_0 y)$, also defined on $\tilde{B} \times S^1$.

By pulling back the original Monge–Ampère equation (1) from the ball B to $\tilde{B}$, we see that for each $r_0 < 1$, u_{r_0} satisfies the Monge–Ampère equation with respect to the metric g_{r_0}. Applying the Evans-Krylov estimate and bootstrapping in this standard ball then gives that

$$||u_{r_0}||_{k,\gamma;\tilde{B}' \times S^1} \leq C,$$

where C depends on g_{r_0}, $\sup|u_{r_0}|$ and $\sup|\Delta_{g_{r_0}} u_{r_0}|$. Since g_{r_0} converges smoothly in this region, $\sup|u_{r_0}|$ is uniformly controlled, and using that $\Delta_{g_{r_0}} u_{r_0} = r_0^2(\Delta_{\tilde{g}} u)_{r_0}$, this last term is also uniformly bounded as $r_0 \searrow 0$, we conclude that u_{r_0} is uniformly bounded in any $\mathcal{C}^{k,\gamma}$ norm in $\tilde{B}' \times S^1$.

The last step is to recall that the edge Hölder norms are invariant under these rescalings. In other words,

$$|| \, u|_{B'} \, ||_{e;k,\gamma} = || \, u_{r_0}|_{\tilde{B}'} \, ||_{e;k,\gamma}.$$

The global $\mathcal{C}^{k,\gamma}_e$ norm of u is the supremum of these norms over all such balls B', and hence this too is finite for all $k \geq 0$. We have proved that $u \in \mathcal{C}^{k,\gamma}_e$ for any k.

We have proved that $(r\partial_r)^j (r\partial_y)^k \partial_\theta^\ell u$ is bounded for any $j, k, \ell \geq 0$, thus *a priori* we only know that $\partial_y^\alpha u$ may blow up like $r^{-|\alpha|}$. We now address this and show that these tangential derivatives are bounded too. Write the Monge–Ampère equation as

$$\log\det(g_{i\bar{j}} + u_{i\bar{j}}) = \log\det(g_{i\bar{j}}) + \log F(z,u).$$

(As explained just after the statement of Theorem 1, the result holds when the usual exponential on the right-hand side is replaced by a function $F(z,u)$ satisfying a few properties.) Applying ∂_y to both sides and using the standard formula for the derivative of a logarithmic determinant, we find that

$$(\Delta_{\tilde{g}} - V)\partial_y u = f, \tag{53}$$

where $\tilde{g}_{i\bar{j}} = g_{i\bar{j}} + u_{i\bar{j}}$, $V = F_u(z,u)/F(z,u)$ and $f = \partial_y \log\det(g_{i\bar{j}}) - \Delta_{\tilde{g}}\partial_y\phi + F_z(z,u)/F(z,u)$, where ϕ is a local Kähler potential for the reference metric g, i.e., such that $\phi_{i\bar{j}} = g_{i\bar{j}}$.

Recall that even if the initial assumption is that $u \in \mathcal{D}^{0,\gamma}_e$, we immediately know from Theorem B.1 that $u \in \mathcal{D}^{0,\gamma}_w$, and this implies that both V and f lie in $\mathcal{C}^{0,\gamma}_w$. Since $\partial_y \in Q$, Proposition 3.3 implies that $\partial_y u$ is bounded. We now wish to apply Corollary 3.5 to improve this regularity, but to do so, we must show that u — and hence the Kähler potential for the metric $\tilde{g}$ — is the limit in $\mathcal{D}^{0,\gamma}_w$ of polyhomogeneous functions. Granting this for the moment, this corollary implies that $\partial_y u \in \mathcal{D}^{0,\gamma}_w$.

This same argument goes on to show that $\partial_y^k u \in \mathcal{D}^{0,\gamma}_e$ for any k. Indeed, suppose inductively that for some $k \geq 2$, $\partial_y^j u \in \mathcal{D}^{0,\gamma}_w$ for all $j \leq k-1$, and

$$(\Delta_{\tilde{g}} - V)\partial_y^{k-1} u = f_0^{(k-1)} + H^{(k-1)}(z, u, \partial_y u, \ldots, \partial_y^{k-1} u, Qu, \ldots, Q\partial_y^{k-2} u) \in \mathcal{C}^{0,\gamma}_e,$$

where $f_0^{(k-1)} \in \mathcal{A}^0_{\mathrm{phg}}$ and $H^{(k-1)}$ is a smooth function of its arguments $\partial_y^j u$ and $Q_i \partial_y^j u$, $Q_i \in \mathcal{Q}$. (In our example, $H^{(k-1)}$ is an algebraic function of these arguments.) Differentiating yields

$$(\Delta_{\tilde g} - V)\partial_y^k u = \partial_y f_0^{(k-1)} \\ + \partial_y H^{(k-1)}(z, u, \partial_y u, \dots, \partial_y^{k-1} u, Qu, \dots, Q\partial_y^{k-2} u) - [\partial_y, \Delta_{\tilde g} - V]\partial_y^{k-1} u.$$

Since $\partial_y^{k-1} u \in \mathcal{D}_w^{0,\gamma}$, we conclude first that $\partial_y^k u \in \mathcal{C}_w^{0,\gamma}$ and in addition, by a straightforward calculation, that the right side lies in $\mathcal{C}_w^{0,\gamma}$. (Note that we may as well assume that $[\partial_y, Q] = 0$.) Hence applying Corollary 3.5 with precisely the same operator as before shows that $\partial_y^k u$ lies in $\mathcal{D}_w^{0,\gamma}$ and satisfies an equation with correct structure. This completes the inductive step. Recalling that we already proved that $(r\partial_r)^i (r\partial_y)^j \partial_\theta^\ell u \in \mathcal{C}_e^{0,\gamma}$ for every $i, j, \ell \geq 0$, an almost identical induction proves that $(r\partial_r)^i \partial_y^j \partial_\theta^\ell u \in \mathcal{D}_w^{0,\gamma}$ for every $i, j, \ell \geq 0$. This proves, altogether, that $u \in \mathcal{A}^0$.

We now address the claim that the Kähler potential for $\tilde g$ is a limit of polyhomogeneous functions. Prior to this inductive argument, we only know that $u \in \mathcal{D}_e^{0,\gamma}$ (or more precisely, that $u \in \mathcal{D}_e^{k,\gamma}$ for every $k \geq 0$). Theorem 8.1, which rests on Tian's Theorem B.1 as stated and proved in Appendix B, asserts that if u is a solution to this Monge–Ampère equation such that u and $\Delta_{\tilde g} u$ are simply bounded, then necessarily $u \in \mathcal{D}_w^{0,\gamma}$. The claim is implied by the fact that if the Hölder exponent γ is replaced by a slightly smaller value $\gamma' \in (0, \gamma)$, then u can be approximated by polyhomogeneous functions in the topology of $\mathcal{D}_w^{0,\gamma'}$. In the interior, away from the edge, this is the familiar fact that the closure of $\mathcal{C}^\infty$ in $\mathcal{C}^{0,\gamma'}$ contains $\mathcal{C}^{0,\gamma}$ for any $0 < \gamma' < \gamma < 1$, which can be proved by mollification. Near the edge, it is possible to use a similar mollification argument in a fixed local coordinate system, but let us explain a more systematic approach using the heat kernel.

Lemma 4.2. *If $0 < \gamma' < \gamma < 1$, then $\mathcal{A} := \mathcal{A}^0_{\mathrm{phg}} \cap \mathcal{D}_w^{0,\gamma}$ is dense in $\mathcal{D}_w^{0,\gamma'}$*

Proof. Consider the heat kernel $e^{t\Delta}$ associated to the L^2 Friedrichs extension of Δ_g, where g is any fixed (smooth or polyhomogeneous) edge metric. The Schwartz kernel of $e^{t\Delta}$ is constructed in [47], and it is proved there that if $t > 0$, then $f_t := e^{t\Delta} f \in \mathcal{A}^0_{\mathrm{phg}}$ for any $f \in L^2$, in particular, for $f \in \mathcal{D}_w^{0,\gamma}$. In addition, $\partial_t f_t$ is polyhomogeneous with nonnegative index set for any $t > 0$, and $\partial_t f_t = \Delta f_t$, so $f_t \in \mathcal{D}_w^{0,\gamma}$ too, hence $f_t \in \mathcal{A}$. Next, since Δ commutes with $e^{t\Delta}$, it follows that $f_t \to f$ and $e^{t\Delta}\Delta f = \Delta e^{t\Delta} f = \Delta f_t \to \Delta f$ in L^2. This already implies that $\mathcal{A}$ is dense in $\mathcal{D}_{\mathrm{Fr}}(\Delta)$ in the L^2 graph topology; we shall need this fact later in Section 6.

To prove the corresponding Hölder space result, note that using the same commutation, it suffices to prove that $f_t \to f$ in $\mathcal{C}_w^{0,\gamma'}$, since the same argument also gives $\Delta f_t \to \Delta f$ in $\mathcal{C}_w^{0,\gamma'}$. This convergence is proved by noting that by standard heat kernel arguments, $f_t \to f$ in $\mathcal{C}^0$ and, moreover, $||f_t||_{\mathcal{C}_w^{0,\gamma}} \leq C$

uniformly in t. (This last fact can be proved using very similar arguments to the ones in the proof of Proposition 3.11.) It is then a simple exercise in real analysis to conclude that $f_t \to f$ in the slightly weaker norm $||\cdot||_{\mathcal{C}_w^{0,\gamma'}}$. □

We come to the final step, that u is polyhomogeneous. This requires two more boundedness properties of edge pseudodifferential operators, namely that this class of operators preserves the spaces of conormal and of polyhomogeneous functions. In particular, if B is any weakly positive pseudodifferential edge operator, then

$$B : \mathcal{A}^0(X) \longrightarrow \mathcal{A}^0(X) \quad \text{and} \qquad B : \mathcal{A}^0_{\mathrm{phg}}(X) \longrightarrow \mathcal{A}^0_{\mathrm{phg}}(X) \tag{54}$$

are both bounded. The pseudodifferential order m is irrelevant at this point since we are applying B to functions that are infinitely differentiable (with respect to the edge vector fields) anyway. The improvement in the argument below relies on a refinement of (54). For the following argument, introduce the notation $\mathcal{A}^{\nu-}(X) = \cap_{\varepsilon>0}\mathcal{A}^{\nu-\varepsilon}$.

Lemma 4.3. *Let $B \in \Psi_e^{m,2,E,E'}(X)$, where E and E' are nonnegative. Then*

$$B : \mathcal{A}^0(X) \longrightarrow \mathcal{A}^0_{\mathrm{phg}}(X) + \mathcal{A}^{2-}(X)$$

and, more generally, if $\nu \geq 0$,

$$B : \mathcal{A}^0_{\mathrm{phg}}(X) + \mathcal{A}^{\nu-}(X) \longrightarrow \mathcal{A}^0_{\mathrm{phg}}(X) + \mathcal{A}^{(\nu+2)-}(X).$$

More concretely,

$$\begin{aligned} u &\sim \sum_{0 \leq Re\,\gamma < \nu} a_{\gamma,p} r^\gamma (\log r)^p + \mathcal{O}(r^{\nu-}) \\ \Longrightarrow Bu &\sim \sum_{0 \leq Re\,\gamma < \nu+2} b_{\gamma,p} r^\gamma (\log r)^p + \mathcal{O}(r^{(\nu+2)-}), \end{aligned}$$

where the errors on each side are conormal, $\mathcal{O}(r^{\nu-})$ denotes an error that decays like $r^{\nu-\varepsilon}$ for all $\varepsilon > 0$, and $a_{\gamma,p} = b_{\gamma,p} = 0$ if $Re\,\gamma = 0$ and $p \geq 1$.

Proof. The second assertion is an easy consequence of the first. To prove this first assertion, if B has index set with all exponents greater than or equal to 2 at the left ($r \to 0$) face, then since B vanishes to order 2 at the front face, we can write $B = r^2\tilde{B}$, where $\tilde{B}$ is nonnegative. Hence in that case, $B : \mathcal{A}^0 \to \mathcal{A}^{2-}$.

Now suppose that the exponents in the expansion of B at the left face of X_e^2 that lie in the range $[0, 2)$ are $\gamma_1, \ldots, \gamma_N$, and assume that there are no log terms in these expansions for simplicity. Then

$$B^{(N)} := (r\partial_r - \gamma_1)(r\partial_r - \gamma_2)\cdots(r\partial_r - \gamma_N)B \in \Psi_e^{m+N,2,E(2),E'}(X),$$

where $E(2)$ is some new index set derived from E that has all elements greater than or equal to 2. Thus we can apply the previous observation to see that $B^{(N)} : \mathcal{A}^0 \to \mathcal{A}^{2-}$ or, said slightly differently, if f is bounded and conormal, so $f \in \mathcal{A}^0$, then $B^{(N)} f = u^{(N)}$ is of the form $r^{2-\varepsilon} v_\varepsilon$ for any $\varepsilon > 0$ where v_ε is bounded and conormal. Now we can integrate the ODE $(r\partial_r - \gamma_1) \cdots (r\partial_r - \gamma_N)$ to see that $u = Bf$ has a partial polyhomogeneous expansion with all terms of the form r^{γ_j}, $j = 1, \ldots, N$, since each of these terms are killed by $r\partial_r - \gamma_j$. $\square$

We wish to apply this lemma when G is the Green function for $\Delta_g + V$, where g and V are polyhomogeneous. It is straightforward to extend this result slightly to show that it remains valid for some fixed ν provided both g and V only lie in $\mathcal{A}^0_{\mathrm{phg}} + \mathcal{A}^\nu$. We leave details of this extension to the reader.

Finally, let us apply this to the equation $L\partial_y u = f$; cf. (53). We know initially that $f \in \mathcal{A}^0$, hence at the first step, $\partial_y u \in \mathcal{A}^0_{\mathrm{phg}} + \mathcal{A}^{2-}$. But this now gives that f and the coefficients of L lie in $\mathcal{A}^0_{\mathrm{phg}} + \mathcal{A}^{2-}$, hence $\partial_y u \in \mathcal{A}^0_{\mathrm{phg}} + \mathcal{A}^{4-}$. Continuing on in this manner gives a complete expansion for $\partial_y u$, and from this we deduce also that u is polyhomogeneous. This concludes the proof of Theorem 1.

Let us remark what is really going on in this proof. Once we have established that u is conormal, i.e., that it is infinitely differentiable with respect to $r\partial_r$, ∂_θ and ∂_y, then we can treat the Monge–Ampère equation satisfied by u as an ODE in the r direction; all dependence in the other directions can be treated parametrically and, in particular, y and θ directions are harmless. Thus the important step is going from $u \in \cap \mathcal{C}^{k,\gamma}_e$ to $\partial^\ell_y u \in \cap \mathcal{C}^{k,\gamma}_e$ for all $\ell \geq 0$.

While this sort of iteration method was already mentioned in [24], it is less awkward to use edge spaces here. The reason is that the different scales in this problem make it necessary to work with functions involving integer powers of both r and $r^{1/\beta}$, and these are only finitely differentiable in the wedge spaces but infinitely differentiable in the edge spaces.

Determination of leading terms. For various applications below, in particular the determination of the asymptotics of the metric and curvature, we must determine the first few terms of the expansion of a solution of the Monge–Ampère equation.

PROPOSITION 4.4. *Let φ be a solution of the Monge–Ampère equation* (30). *Suppose that $\varphi \in \mathcal{D}^{0,\gamma}_w$, and hence by Theorem* 1, *$\varphi \in \mathcal{A}^0_{\mathrm{phg}}$. Then the asymptotic expansion of φ takes the form*

$$\varphi(r, \theta, y) \sim \sum_{j,k,\ell \geq 0} a_{jk\ell}(\theta, y) r^{j + \frac{k}{\beta}} (\log r)^\ell \tag{55}$$

as $r \searrow 0$. Certain coefficients are always absent; *for example, $a_{00\ell} = 0$ for $\ell > 0$ and $a_{10\ell} \equiv 0$ for all ℓ. If $a_{jk\ell} = 0$ for some j, k for all $\ell > 0$, then we* *write this coefficient simply as* a_{jk}. *When* $0 < \beta < 1/2$,

$$\varphi(r,\theta,y) \sim a_{00}(y)+a_{20}(y)r^2+(a_{01}(y)\sin\theta+b_{01}(y)\cos\theta)r^{\frac{1}{\beta}}+a_{40}(y)r^4+O(r^{4+\varepsilon}) \tag{56}$$

for some $\varepsilon = \varepsilon(\beta) > 0$; *when* $\beta = 1/2$, *the asymptotic sum on the right includes an extra term* $(a_{02}(y)\sin 2\theta + b_{02}(y)\cos 2\theta)r^4$; *finally, if* $1/2 < \beta < 1$, *then*

$$\varphi(r,\theta,y) = a_{00}(y) + (a_{01}(y)\sin\theta + b_{01}(y)\cos\theta)r^{\frac{1}{\beta}} + a_{20}(y)r^2 + O(r^{2+\varepsilon}) \tag{57}$$

for some $\varepsilon = \varepsilon(\beta) > 0$.

We begin with a lemma.

LEMMA 4.5. *The twisted Ricci potential* f_ω *can be expressed as*

$$f_\omega = \sum_{k=-1}^{n-1} c_{0k} r^{2k+\frac{2}{\beta}} + \sum_{k=0}^{n-1} (c_{1k} + c_{2k} r\cos\theta + c_{3k} r\sin\theta) r^{2k},$$

where each c_{jk} *is a smooth function of* $r^{\frac{1}{\beta}}, \theta$, *and* y.

Remark 4.6. We may, of course, Taylor expand the coefficients c_{ik} to obtain an asymptotic sum involving the terms $r^{2k+(2+\ell)/\beta}$ and $r^{2k+\ell/\beta}$, respectively, with coefficients depending only on y and θ.

Proof. By (21),

$$\begin{aligned} \omega^n/(n!(\sqrt{-1})^{n^2} d\mathbf{z}\wedge d\overline{\mathbf{z}}) &= \det\Big[\frac{\partial^2(\psi_0+\phi_0)}{\partial z^i \partial \overline{z^j}}\Big] \\ &= \sum_{k=0}^{n} f_{0k}|z_1|^{2k\beta} + \sum_{k=1}^{n}(f_{1k} + f_{2k}z_1 + f_{3k}\overline{z_1})|z_1|^{2k\beta-2}, \end{aligned} \tag{58}$$

where all f_{jk} are smooth functions of $(z_1, \ldots, z_n)$, and $d\mathbf{z} := dz_1 \wedge \cdots \wedge dz_n$. It follows that

$$\begin{aligned} \frac{\omega^n}{|s|_h^{2\beta-2}\omega_0^n} &= \frac{(\omega_0+\sqrt{-1}\partial\bar\partial\phi_0)^n}{|s|_h^{2\beta-2}\omega_0^n} \\ &= \sum_{k=-1}^{n-1} \tilde f_{00} r^{2k+\frac{2}{\beta}} + \sum_{k=0}^{n-1}(\tilde f_{1k} + \tilde f_{2k} r\cos\theta + \tilde f_{3k} r\sin\theta)) r^{2k}, \end{aligned} \tag{59}$$

where each $\tilde f_{jk}$ is a smooth function of the arguments $r^{\frac{1}{\beta}}\cos\theta, r^{\frac{1}{\beta}}\sin\theta$ and y. In addition, we have already noted that $\phi_0 = r^2\Phi_0$, where Φ_0 is also smooth as a function of $r^{\frac{1}{\beta}}\cos\theta, r^{\frac{1}{\beta}}\sin\theta$ and y. The result now follows directly from the equation

$$e^{-f_\omega} = \frac{(\omega_0+\sqrt{-1}\partial\bar\partial\phi_0)^n}{|s|_h^{2\beta-2}\omega_0^n} e^{\mu\phi_0 - F_{\omega_0}}, \tag{60}$$

where F_{ω_0} is defined by $\sqrt{-1}\partial\bar\partial F_{\omega_0} = \operatorname{Ric}\omega_0 - \mu\omega_0 + (1-\beta)\sqrt{-1}\partial\bar\partial \log a$ (where a is defined in (20)), and the equation itself, together with (28), fixes a normalization for F_{ω_0}, and again F_{ω_0} is smooth in these same arguments. $\square$

Proof of Proposition 4.4. The idea is quite simple. Since we now know that φ has an asymptotic expansion, we simply substitute a 'general' expansion into the equation

$$\omega_\varphi^n/\omega^n = F(z,\varphi) \tag{61}$$

and determine the unknown exponents and coefficients. Since our main case of interest is when $F(z,\varphi) = e^{f_\omega - s\varphi}$, we shall explain the argument for this special function, but it should be clear that the same type of argument works in general.

Using the precise form of the expansion for f_ω determined above, the index set for φ must be contained in

$$\Gamma := \{(j + k/\beta, \ell) : j,k,\ell \in \mathbb{N}_0\}.$$

In other words, the only terms that appear are of the form $a_{jk\ell}(\theta,y)r^{j+\frac{k}{\beta}}(\log r)^\ell$. This is done inductively. Supposing that we know that this is true for all j,k such that $j + k/\beta \le A$, then we only need consider the action of $P_{1\bar 1}$ on the next term in the series $a_{\gamma\ell}r^\gamma(\log r)^\ell$. This must either be annihilated by $P_{1\bar 1}$, i.e., γ is an integer multiple of $1/\beta$, or else it must match a previous term in the expansion, i.e., $\gamma - 2 = j' + k'/\beta$. In either case, the form of the expansion propagates one step further.

Since the solution φ is bounded, there are no terms $a_{00\ell}(\log r)^\ell$ with $\ell > 0$, so using the convention in the statement of the theorem, the leading term is simply $a_{00}r^0$. Note further that a_{00} depends only on y but not on θ. This can be seen by substituting in the equation. If a_{00} were to depend nontrivially on θ, then the term $P_{1\bar 1}\varphi$ would contain $r^{-2}\partial_\theta^2 a_{00}$, and this is not cancelled by any other term in the equation. Hence $a_{00} = a_{00}(y)$.

Similar reasoning can be applied to the next few terms in the expansion. We use discreteness of the set of exponents to progressively isolate the most singular terms after we substitute the putative expansion for φ into the equation. Since a_{00} is independent of θ and r, $P_{1\bar 1}a_{00}$, and $P_{1\bar j}a_{00}$ and $P_{i\bar 1}a_{00}$ are all bounded (in fact, zero). Hence if the next term in the expansion is $a_{\gamma\ell}r^\gamma(\log r)^\ell$ with $\gamma \le 2$, then applying $P_{1\bar 1}$ to it produces as its most singular term $r^{\gamma-2}(\log r)^\ell(\gamma^2 + \partial_\theta^2)a_{\gamma\ell}$. This shows immediately that either γ must be an indicial root, i.e., $\gamma = 1/\beta$ if $\beta > 1/2$ with $a_{\gamma\ell}$ a linear combination of $\cos\theta$ and $\sin\theta$, or else $\gamma = 2$. Note that this also shows that $a_{10\ell} \equiv 0$ for all $\ell \ge 0$.

Assuming $\gamma < 2$ and $\ell > 0$, then using the leading order cancellation, the next most singular term in $P_{1\bar 1}a_{01\ell}r^{\frac{1}{\beta}}(\log r)^\ell$ is $\gamma r^{\gamma-2}(\log r)^{\ell-1}a_{01\ell}$ with no other term to cancel it. This is impossible, so we have ruled out all such terms with $\ell > 0$. If $\gamma = 2$ and $\ell > 0$, there is no longer a leading order cancellation, but we are left with the singular term $a_{20\ell}(\log r)^\ell$, so $a_{20\ell} = 0$ when $\ell > 0$.

Now consider what happens to the term $a_{20}r^2$. It interacts with the leading order terms a_{00} in φ and c_{00} in f_ω only. Neither of these depend on θ, so we find that a_{20} is a function of y alone.

We can continue this same reasoning further. Applying $P_{1\bar{1}}$ to the next term in the expansion $a_{\gamma\ell}r^\gamma(\log r)^\ell$ beyond $a_{20}r^2$ produces a leading order term that is a nonzero multiple of $a_{\gamma\ell}r^{\gamma-2}(\log r)^\ell$ if $\ell > 0$. Even though this term is bounded now, there are no other log terms at the level $r^{\gamma-2}$ in (61). On the other hand, if $\ell = 0$, then we end up with a term $r^{\gamma-2}(\gamma^2 + \partial_\theta^2)a_{\gamma 0}$, and there are no terms in (61) to cancel it either. Hence γ must be one of the two indicial roots k/β, $k = 1$ or 2, and the coefficient must be a linear combination of $\cos k\theta$ and $\sin k\theta$.

We comment further on the cases $\beta = 1/2$ or $\beta = 1/4$. In the former, one might suspect that one would need a term $r^2 \log r a_{021}$ because applying $P_{1\bar{1}}$ to this should match the r^0 term coming from the leading coefficients of φ and f_ω. However, those coefficients do not depend on θ, whereas a_{021} would be a combination of $\cos 2\theta$ and $\sin 2\theta$, as above, so there is no interaction, hence no log terms at this location. This is also true for $\beta = 1/4$. □

Remark 4.7. It is worth noting explicitly that while both the reference and solution metrics have expansions, the solution metric may have more terms in its expansion than the reference metric. One consequence of this is that the computations in the appendix do *not* apply to the solutions $\omega_{\varphi(s)}$, $s > -\infty$; in particular, one cannot conclude that the bisectional curvatures of the solution metrics are bounded when $\beta > 1/2$, and indeed, they are not!

Using Theorem 1 and Proposition 4.4, we obtain the following regularity statement.

COROLLARY 4.8. *Let φ be a solution of the Monge–Ampère equation* (30), *with $\varphi \in \mathcal{D}_e^{0,\gamma}$. Then φ is polyhomogeneous, and there exists some $\varepsilon > 0$ depending only on β such that $\varphi \in \mathcal{D}_w^{0,\gamma'}$ for every $\gamma' \in [0, \varepsilon(\beta)]$.*

Remark 4.9. As noted in the introduction, Proposition 4.4 sheds light on the distinction between the easier 'orbifold regime' $\beta \in (0, \frac{1}{2}]$ and the case $\beta \in (\frac{1}{2}, 1)$. In particular, we see that one should not expect uniform estimates even on the third derivatives $\varphi_{i\bar{j}k}$ when $\beta > \frac{1}{2}$. This is one reason why we study the Hölder norms of second derivatives in Section 8 rather than considering the third order estimates as in the classical approach of Aubin and Yau.

5. Maximum principle and the uniform estimate

We now recall the formulation of the maximum principle in this singular setting. The main issue is to find barrier functions that allow one to reduce to the classical maximum principle on $M \backslash D$. These barrier functions were used already in [33].

Lemma 5.1. *Let f be continuous on M and satisfy $|f(r,\theta,y)-a(y)| \leq Cr^\gamma$ for some $a \in \mathcal{C}^0(D)$ and $0<\gamma<1$. Then for ε sufficiently small,*

(i) *if $C>0$, then $f+C|s|_h^\varepsilon$ achieves its maximum in $M \backslash D$;*
(ii) *if $c>0$ is small enough, then $c|s|_h^\varepsilon \in \mathrm{PSH}(M,\omega)$.*

Proof. (i) The function $|s|_h^\varepsilon$ is comparable to $r^{\varepsilon/\beta}$, so for $C>0$, $r \mapsto f(r,\theta,y)+C|s|_h^\varepsilon$ strictly increases, hence it cannot reach its maximum at $r=0$.

(ii) Let h be a smooth Hermitian metric on L_D with global holomorphic section s so that $D=s^{-1}(0)$. For any $b \geq 0$, we have $\sqrt{-1}\partial\bar\partial b \geq b\sqrt{-1}\partial\bar\partial \log b$. Setting $b:=|s|_h^\varepsilon$ gives

$$\sqrt{-1}\partial\bar\partial b \geq \sqrt{-1}\,\varepsilon|s|_h^\varepsilon\partial\bar\partial\log|s|_h = -\frac{1}{2}\varepsilon|s|_h^\varepsilon R(h) > -C\omega, \tag{62}$$

where C depends only on the choice of $\omega, h, s, \varepsilon$. Thus $C^{-1}b \in \mathrm{PSH}(M,\omega)$. □

The assumption on f above holds, in particular, for $f \in \mathcal{C}_w^{0;\gamma}$, and for f and $\Delta_\omega f$ when $f \in \mathcal{D}_w^{0;\gamma}$.

This lemma is used as follows. Replacing $|s|_h^\varepsilon$ by $c|s|_h^\varepsilon$ and letting c tend to 0, we obtain estimates that are the same as those one would expect from the maximum principle on $M \backslash D$. See the proofs of Lemmas 5.2 and 7.2 below for more on this. The uniqueness and *a priori* $\mathcal{C}^0$ estimate when $\mu \leq 0$ are now immediate consequences.

Lemma 5.2. *Solutions to the Monge–Ampère equation* (32) *with $s \leq 0$ are unique* (*when $s=0$, only unique up to a constant*) *in $\mathcal{D}_w^{0;\gamma} \cap \mathrm{PSH}(M,\omega)$ and satisfy*

$$||\varphi(s,t)||_{\mathcal{C}^0(M)} \leq C = C(||f_\omega||_{\mathcal{C}^0(M)}, M, \omega). \tag{63}$$

Proof. Uniqueness when $s<0$ is proved in [33]; that argument carries over directly to this Monge–Ampère equation and either of the types of function spaces we are using here, because of Lemma 5.1. Finally, when $s=0$, the result of Błocki [14] gives uniqueness in $L^\infty(M)$ up to a constant, and that constant can be chosen by requiring that $\sup\varphi(0,t) = \lim_{s\to 0^-}\sup\varphi(s,t)$.

The same argument also shows that $||\varphi(s,t)||_{\mathcal{C}^0(M)} \leq -2s^{-1}||f_\omega||_{\mathcal{C}^0(M)}$ for each $s<0$. One can then obtain a uniform estimate for all $s \leq 0$ as follows. First, by the above, we may assume that $s>S$, for some $S<0$. With respect to the fixed smooth Kähler form ω_0, (32) can be rewritten as

$$\omega_\varphi^n = \omega_0^n F|s|_h^{2\beta-2} e^{tf_\omega + c_t - s\varphi},$$

where $F \in \mathcal{C}^0(M)$. By the previous estimate, $||e^{tf_\omega + c_t - s\varphi}||_{\mathcal{C}^0(M)} \le C$ uniformly in s. It follows that $||F|s|_h^{2\beta-2} e^{tf_\omega + c_t - s\varphi}||_{L^p(M,\omega_0^n)} \le C_p$, for all $p \in (1, 1/(1-\beta))$, with C_p independent of $s \le 0$. Assuming this, by Kołodziej's estimate [34], $\operatorname{osc}\varphi(s,t) \le C$, with $C > 0$ independent of s, t, and since by (32) $\varphi(s,t)$ changes sign, then also $|\varphi(s,t)| \le C$. □

6. The uniform estimate in the positive case

In contrast to the nonpositive curvature cases, when $\mu > 0$, there are well-known obstructions to the existence of an *a priori* $\mathcal{C}^0$ estimate along the continuity path. In this section we review the standard theory due to Tian and others [61], [62] along with the necessary modifications to adapt it to our setting. For an alternative variational approach that can be applied to more general classes of plurisubharmonic functions, we refer to [9].

6.1. *Poincaré and Sobolev inequalities.* In this subsection we show that along the continuity path (30) one has uniform Poincaré and Sobolev inequalities.

We first prove that a uniform Poincaré inequality holds as soon as $s > \varepsilon > 0$. The following argument is the analogue of [62, Lemma 6.12] in this edge setting and also generalizes [41, Lemma 3] to higher dimensions. The second part is the same assertion as [24, Prop. 8]. The proof here takes advantage of the fine regularity results for solutions available to us.

LEMMA 6.1. *Denote by $\Delta_{\omega_{\varphi(s)}}$ the Friedrichs extension of the Laplacian associated to $\omega_{\varphi(s)}$.*

(i) *For any $s \in (0,\mu)$, $\lambda_1(-\Delta_{\omega_{\varphi(s)}}) > s$.*
(ii) *For $s = \mu$, $\lambda_1(-\Delta_{\omega_{\varphi(\mu)}}) \ge \mu$. If $(\Delta_{\omega_{\varphi(\mu)}} + \mu)\psi = 0$, then $\nabla^{1,0}_{g_{\varphi(\mu)}}\psi$ is a holomorphic vector field tangent to D.*

Proof. (i) Let ψ be an eigenfunction of $\Delta_{\omega_{\varphi(s)}}$ with eigenvalue $-\lambda_1$. Since $\varphi(s)$ is polyhomogeneous, then the eigenfunctions of $\Delta_{\omega_{\varphi(s)}}$ are also polyhomogeneous. This is a special case of the main regularity theorem for linear elliptic differential edge operators from [43]. The proof uses the same pseudodifferential machinery described in Section 3 (although for this particular result, it is possible to give a more elementary proof). The key fact is that $\psi \sim a_0 r^0 + a_1 r^{\frac{1}{\beta}} + a_2 r^2 + O(r^{2+\eta})$ for some $\eta > 0$ and, in particular, there is no $\log r$ in this expansion, since we are using the Friedrichs extension.

The Bochner–Weitzenböck formula states that on $M \setminus D$,

$$\frac{1}{2}\Delta_g|\nabla_g f|_g^2 = \mathrm{Ric}\,(\nabla_g f, \nabla_g f) + |\nabla^2 f|_g^2 + \nabla f \cdot \nabla(\Delta_g f).$$

Since $\Delta_g = 2\Delta_\omega$ and $|\nabla^2 f|_g^2 = 2|\nabla^{1,0}\nabla^{1,0} f|^2 + 2(\Delta_\omega f)^2$, this becomes

$$\Delta_\omega|\nabla^{1,0}\psi|_g^2 = 2\mathrm{Ric}\,(\nabla^{1,0}\psi, \nabla^{0,1}\psi) + 2|\nabla^{1,0}\nabla^{1,0}\psi|^2 + 2\lambda_1^2\psi^2 - 4\lambda_1|\nabla^{1,0}\psi|_\omega^2. \tag{64}$$

We now claim that

$$\int_M \Delta_{\omega_\varphi}|\nabla^{1,0}\psi|_{\omega_\varphi}^2 \omega_\varphi^n = 0. \tag{65}$$

This follows directly from the expansion of ψ, since the worst term in the expansion of $\nabla^{1,0}\psi$ is $r^{\frac{1}{\beta}-1}$. Hence if we integrate over $r \geq \varepsilon$, then the boundary term is of order $\varepsilon^{\frac{2}{\beta}-2}$ (taking into account the measure $rd\theta dy$ on this boundary), and this tends to 0 with ε. This proves the claim. Thus integrating (64) and using that $\mathrm{Ric}\,\omega(s) > s\omega(s)$ when $s < \mu$, we see that $\lambda_1 > s$.

(ii) When $s = \mu$, this same argument yields $\lambda_1 \geq \mu$. Moreover, equality holds precisely when $\nabla^{1,0}\nabla^{1,0}\psi = 0$ on $M \setminus D$; i.e., $\nabla^{1,0}\psi$ is a holomorphic vector field on $M \setminus D$. Using the asymptotic expansion, $\nabla^{1,0}\psi$ is continuous up to D and hence extends holomorphically to M. Now, the coefficient of $\frac{\partial}{\partial\zeta}$ equals $g^{1\bar{j}}\psi_{\bar{j}}$. By (13) $g^{1\bar{j}} = O(r^{\eta'})$ and hence vanishes on D for $j \neq 1$ (and ψ is infinitely differentiable in the $j \neq 1$ directions), while although $g^{1\bar{1}}$ is uniformly positive, $\psi_{\bar{1}} = O(r^{\frac{1}{\beta}-1})$, so this terms also vanishes on D. In conclusion, the $\frac{\partial}{\partial\zeta}$ component of $\nabla^{1,0}\psi$ vanishes at D, so this vector field is tangent to D. □

We now estimate the Sobolev constant. First observe that the Sobolev inequality holds for the model edge metric g_β, i.e., since $\dim M = 2n$,

$$||f||_{L^{\frac{2n}{n-1}}(M,g_\beta)} \leq C_S||f||_{W^{1,2}(M,g_\beta)}, \tag{66}$$

and hence also for any metric uniformly equivalent to it. One way to prove this is to note that it suffices to prove this inequality locally, in the neighbourhood of any point; away from D this is just the standard Sobolev inequality, while in a neighbourhood of any point $p \in D$ we can use the (ζ, Z) coordinate system to reduce to the standard Euclidean case. An alternate proof relies on the well-known equivalence of the Sobolev inequality with the fact that the heat kernel for the scalar Laplacian blows up like t^{-n} as $t \searrow 0$ (since the overall dimension is $2n$). Since g_β is a product of a cone with a Euclidean space, this, in turn, reduces to the fact that the heat kernel on a two-dimensional cone blows up like t^{-1}, which can be verified by direct computation; see, e.g., [24].

As an aside, observe that using (66), the standard Moser iteration proof of the $\mathcal{C}^0$ estimate for $s = 0$ [62] goes through exactly as in the smooth case and hence can be used instead of Kołodziej's estimate to prove Lemma 5.2.

Next, we derive a uniform Sobolev inequality when $s > \varepsilon$. Our approach follows Bakry [4] closely and relies on the general theory of diffusive semigroups. The following result is essentially a special case of [4, Th. 6.10].

PROPOSITION 6.2. *Let $\varepsilon \in (0,1)$ and $s \in (\varepsilon, 1]$. There exists a uniform constant $C > 0$, depending only on $(M,\omega), n$ and ε, so that for any $f \in W^{1,2}(M, \omega_{\varphi(s)})$,*

$$||f||_{L^{\frac{2n}{n-1}}(M,\omega_{\varphi(s)})} \leq C||f||_{W^{1,2}(M,\omega_{\varphi(s)})}.$$

Proof. Let $L = \Delta_{\omega_\varphi}$. Proposition 2.1 in [4] (which holds for substantially more general operators L) asserts that if $\mathcal{A} \subset \mathcal{D}_{\mathrm{Fr}}(L)$ (recall (40)) is a subspace preserved by L and e^{tL} and dense in L^2, then it is also dense in $\mathcal{D}_{\mathrm{Fr}}(L)$ with respect to the graph norm $||f||_{L^2} + ||Lf||_{L^2}$. We can also verify this directly in our setting and, in fact, have already done so in the proof of Lemma 4.2 above; but cf. also the discussion in [4, p. 35].

Now, for any two functions $f, g \in \mathcal{A}$, define the quantities

$$2\Gamma(f,g) := L(fg) - fLg - gLf \tag{67}$$

and

$$2\Gamma_2(f,g) := L\Gamma(f,g) - \Gamma(f,Lg) - \Gamma(g,Lf). \tag{68}$$

Note that on the smooth part $M \setminus D$,

$$\Gamma(f,f) = |\nabla f|^2 \tag{69}$$

(the gradient and norm are taken with respect to ω_φ) and

$$\Gamma_2(f,f) = \frac{1}{2}L|\nabla f|^2 - \nabla f.\nabla Lf = \operatorname{Ric}\omega_\varphi(\nabla f, \nabla f) + |\nabla^2 f|^2. \tag{70}$$

Since $f \in \mathcal{A}$, (69) holds on all of M as a $W^{1,2}$ distribution. Furthermore, by virtue of (31), (70) implies that

$$\Gamma_2(f,f) \geq C\varepsilon\,\Gamma(f,f) + \frac{1}{2n}(Lf)^2 \tag{71}$$

in the sense of distributions on all of M, where $C = C(n)$ is a universal constant. Both of these assertions can be checked easily using that f is polyhomogeneous with an expansion $f = a_0(y) + r^{\frac{1}{\beta}}(a_1(y)\cos\theta + a_2(y)\sin\theta) + a_2(y)r^2 + O(r^{2+c})$ for some $c > 0$.

Following the definition and notation of [4, p. 93], we have proved that L satisfies the "uniform curvature-dimension condition" $\mathrm{CD}(C\varepsilon, 2n)$. We can then follow the general argument in [4] to obtain uniform Sobolev bounds [4, Th. 6.10]; cf. also [5, Th. 1]. This procedure also leads to a uniform Poincaré estimate; however, an examination of the proof of [4, Prop. 6.3] shows that this is essentially equivalent to the one given above in Lemma 6.1(i). In any case, we now sketch Bakry's reasoning and explain in detail why it applies here.

The first point is that it suffices to prove the uniform Sobolev inequality only for functions in $\mathcal{A}$. Indeed, using the density of $\mathcal{A}$ in the graph norm, we must show that both sides in the Sobolev inequality are continuous in this topology. For the right-hand side, this is obvious. Since there is *some* (not necessarily uniform) Sobolev inequality, cf. the paragraph containing (66), $W^{1,2}$ is contained in $L^{\frac{2n}{n-1}}$ and so a sequence converging in $W^{1,2}$ converges weakly in $L^{\frac{2n}{n-1}}$, and so the left-hand side is also continuous in the appropriate sense.

We now show uniform control of the Sobolev constant. Fix $2 < p < \frac{2n}{n-1}$ and $\delta > 0$, and let $f_k^{(p)} \in \mathcal{A}$ be a sequence which converges towards the supremum of the ratio

$$\frac{||F||_p^2 - (1+\delta)||F||_2^2}{\Gamma(F,F)} \tag{72}$$

over $F \in \mathcal{D}_{\mathrm{Fr}}(L)$. Denote this supremum by γ_p. As usual, we can assume that $f_k^{(p)} \geq 0$ and $||f_k^{(p)}||_2 = 1$. Using the compactness of $W^{1,2}$ in L^p (which is a consequence of the existence of a Sobolev inequality; for a general semigroup, this compactness is not automatic and is proved in [4, Th. 4.11]), we can extract a subsequence converging weakly in $\mathcal{D}_{\mathrm{Fr}}(L)$ and strongly in L^p to a nontrivial limit function $f^{(p)} \geq 0$, which we call f for simplicity. This satisfies $||f||_p^2 = (1+\delta) + \gamma_p \Gamma(f,f)$. Assuming that we have normalized the measure associated to $\omega^n_{\varphi(s)}$ to have unit volume, then f must be nonconstant since $\delta > 0$. Since f maximizes (72), the usual argument in the calculus of variations gives $||f||_p^{2-p}\langle f^{p-1}, g\rangle = (1+\delta)\langle f, g\rangle + \gamma_p\Gamma(f,g) = \langle f, (1+\delta)g - \gamma_p Lg\rangle$, for any $g \in \mathcal{D}_{\mathrm{Fr}}(L)$ or, equivalently, $||f||_p^{2-p}\langle f^{p-1}, R_\lambda(h)\rangle = \langle f, \gamma_p h\rangle$. Here $\lambda := (1+\delta)/\gamma_p$ and $g = R_\lambda(h)$ where $R_\lambda = \int_0^\infty e^{-\lambda t}e^{tL}dt = (\lambda I - L)^{-1}$ is the resolvent of L. This shows that $f = R_\lambda(||f||_p^{2-p} f^{p-1})/\gamma_p$, or equivalently

$$||f||_p^{2-p} f^{p-1} = ((1+\delta) - \gamma_p L)f. \tag{73}$$

Following [1, §3.2], the solution to this subcritical Yamabe-type equation must be polyhomogeneous. Then, by a determination of the leading terms in the expansion of f, it readily follows that $f \in \mathcal{D}_w^{0,\gamma}$. Both these assertions are simpler analogues of Theorem 1 and Proposition 4.4, and their proof follows similar, but simpler, arguments since this is a quasilinear equation and not a fully nonlinear one. Thus, $f \in \mathcal{A}$.

LEMMA 6.3. *The constant γ_p associated to the embedding $W^{1,2} \subset L^p$ satisfies*

$$\gamma_p \leq \frac{(2n-1)(p-2)(1+\delta)}{2nC\varepsilon}. \tag{74}$$

Proof. Fix $a \in \mathbb{R}$. We let g be such that $g^a = f \equiv f^{(p)}$. We then divide (73) by f and then substitute $f = g^a$ to get

$$\begin{aligned} ||f||_p^{2-p} g^{a(p-2)} &= 1 + \delta - \gamma_p g^{-a}[a g^{a-1} Lg + a(a-1) g^{a-2}\Gamma(g,g)] \\ &= 1 + \delta - a\gamma_p[g^{-1}Lg + (a-1)g^{-2}\Gamma(g,g)], \end{aligned} \tag{75}$$

Now, following Bakry, we multiply this by $-gLg$ and integrate:

$$\begin{aligned} &-||f||_p^{2-p} \left\langle g^{1+a(p-2)}, Lg \right\rangle \\ &\qquad = (1+\delta)\Gamma(g,g) + a\gamma_p ||Lg||^2 + a\gamma_p(a-1)\left\langle \frac{Lg}{g}, \Gamma(g,g)\right\rangle. \end{aligned}$$

The left-hand side can be rewritten as

$$||f||_p^{2-p}\Gamma(g^{1+a(p-2)}, g) = C_p(1 + a(p-2))\langle g^{a(p-2)}, \Gamma(g,g)\rangle.$$

This can be rewritten using (75) as

$$(1 + a(p-2))\langle 1 + \delta - a\gamma_p[g^{-1}Lg + (a-1)g^{-2}\Gamma(g,g)], \Gamma(g,g)\rangle.$$

Altogether, we have

$$\begin{aligned} &(1 + a(p-2))\left\langle 1 + \delta - a\gamma_p[g^{-1}Lg + (a-1)g^{-2}\Gamma(g,g)], \Gamma(g,g)\right\rangle \\ &\qquad = (1+\delta)\Gamma(g,g) + a\gamma_p ||Lg||^2 + a\gamma_p(a-1)\left\langle \frac{Lg}{g}, \Gamma(g,g)\right\rangle. \end{aligned}$$

So the constant $||f||_p^{2-p}$ disappears; from this point on we follow Bakry, and as in [4, (6.37)], we obtain

$$\begin{aligned} &\frac{1+\delta}{\gamma_p}(p-2)||\Gamma(g,g)||_1 \\ &= ||Lg||_2^2 + a(p-1)\langle Lg/g, \Gamma(g,g)\rangle + (a-1)(1 + a(p-2))||\Gamma(g,g)/g||_2^2. \end{aligned}$$

We now invoke a consequence of (71), which holds by the chain rule [4, (6.38)]: for any $b \in \mathbb{R}$,

$$\Gamma_2(g,g) + b\Gamma(g,\Gamma(g,g))/g + b^2(\Gamma(g,g)/g)^2 \geq C\varepsilon\Gamma(g,g) + \frac{1}{2n}(Lg + b\Gamma(g,g)/g)^2.$$

Integrating gives

$$\begin{aligned} &\left(\frac{1+\delta}{\gamma_p}(p-2) - \frac{2n}{2n-1}C\varepsilon\right)\Gamma(g,g) \\ &\qquad \geq ((a-1)(1 + a(p-2)) - b(b + 2n/(2n+1)))\, ||\Gamma(g,g)/g||_2^2. \end{aligned}$$

Choosing a, b appropriately as in [4, p. 110], we see that the right-hand side is nonnegative, which implies a uniform bound on γ_p since $\Gamma(g,g) \geq 0$. $\square$

Letting $p \nearrow 2n/(n-1)$, and using the fact that there is a Sobolev inequality at the critical exponent, we see that this Sobolev constant has the upper bound (74) with $p = 2n/(n-1)$. This concludes the proof of Proposition 6.2. $\square$

Remark 6.4. In fact, [5, Th. 3] shows that we can find a uniform bound for the diameter of $(M, \omega_{\varphi(s)})$ from Proposition 6.2. Indeed, define

$$D(\Gamma) := \sup\{|f(x) - f(y)| \,:\, x, y \in M, f \in A, ||\Gamma(f,f)||_{L^\infty(M)} \leq 1\},$$

and apply the Sobolev inequality to the functions $(1 + \lambda f)^{1-\frac{n}{2}}$ for any $f \in \mathcal{A}$. (One must check that such functions are once again in $\mathcal{A}$.) It then follows that $D(\Gamma) \leq C\varepsilon^{-1/2}$.

Remark 6.5. There are other possible approaches to the estimation of the Sobolev constants. One approach, suggested in a remark in the first version of this article, is to approximate $\omega_{\varphi(s)}$ by smooth Kähler metrics with a uniform positive lower bound on the Ricci curvature. This has been carried out in detail in [63], [18]. Another approach is to show that as a metric-measure space, the completion of $(M \setminus D, \omega_\varphi, {\omega_\varphi}^n)$ satisfies a uniform (generalized) doubling property. The arguments of [30], [32] show that the Poincaré inequality implies a Sobolev inequality. This was described in detail in an earlier version of this paper, but for brevity we have replaced this by the semigroup approach above.

6.2. *Energy functionals.* Unlike in the previous cases, there are well-known obstructions to obtaining a $\mathcal{C}^0$ estimate in the positive case. The existence of such an estimate is then described in terms of the behavior of certain energy functionals. For more background, we refer to [3], [7], [56], [62].

The energy functionals I, J, introduced by Aubin [3], are defined by

$$I(\omega, \omega_\varphi) = \frac{1}{V}\int_M \sqrt{-1}\partial\varphi \wedge \bar{\partial}\varphi \wedge \sum_{l=0}^{n-1} \omega^{n-1-l} \wedge {\omega_\varphi}^l = \frac{1}{V}\int_M \varphi(\omega^n - \omega_\varphi^n),$$

$$J(\omega, \omega_\varphi) = \frac{V^{-1}}{n+1}\int_M \sqrt{-1}\partial\varphi \wedge \bar{\partial}\varphi \wedge \sum_{l=0}^{n-1}(n-l)\omega^{n-l-1} \wedge {\omega_\varphi}^l.$$

This definition certainly makes sense for pairs of smooth Kähler forms, and by the continuity of the mixed Monge–Ampère operators on $\mathrm{PSH}(M, \omega_0) \cap \mathcal{C}^0(M)$ [8, Prop. 2.3], these functionals can be uniquely extended to pairs $(\omega_0, \omega_\varphi)$, with ω_0 smooth and $\omega_\varphi \in \mathcal{H}_{\omega_0}$, and hence also to $\mathcal{H}_\omega \times \mathcal{H}_\omega$, where now by ω we mean the reference metric given by (26). These functionals are nonnegative and equivalent:

$$\frac{1}{n}J \leq I - J \leq \frac{n}{n+1}I \leq nJ. \tag{76}$$

One use of these functionals is in deriving a conditional $\mathcal{C}^0$ estimate.

LEMMA 6.6. *Let* $s \in (0, \mu)$. *Any* $\mathcal{C}^0(M) \cap \mathrm{PSH}(M, \omega)$ *solution* $\varphi(s)$ *to* (30) *is unique. Moreover, if* $\varphi(s) \in \mathcal{D}_s^{0,\gamma}$, *then* $||\varphi(s)||_{\mathcal{C}^0(M)} \leq C(1 + I(\omega, \omega_{\varphi(s)}))$ *for all* $s \in (\varepsilon, \mu)$.

Proof. The uniqueness is due to Berndtsson [12]. We now prove the estimate. Using the uniform estimates on the Poincaré and Sobolev constants, the arguments proceed much as in the smooth case [62, Lemma 6.19].

First, let G_ω be the Green function of $-\Delta_\omega$; i.e., $-\Delta_\omega G_\omega = -G_\omega \Delta_\omega = \mathrm{Id} - \Pi$, where Π is the orthogonal projector onto the constants. (Note that this is contrary to our previous sign convention for G, but it conforms with the usual convention for this estimate.) Necessarily, $\int_M G_\omega(\cdot, \tilde{z})\omega^n(\tilde{z}) = 0$. We claim that $A_\omega := -\inf_{M\times M} G_\omega < \infty$. Assuming this for the moment, we can write

$$\varphi_s(z) = V^{-1} \int_M \varphi_s \omega^n - \int_M G_\omega(x, y)\Delta_\omega \varphi_s(y)\omega^n(y).$$

Hence, since $-n < \Delta_\omega \varphi_s$,

$$\sup \varphi(s) \leq \frac{1}{V} \int_M \varphi(s)\omega^n + nVA_\omega. \tag{77}$$

To prove this claim about the Green function, recall that

$$G(z, \tilde{z}) = \int_0^\infty (H(t, z, \tilde{z}) - \Pi(z, \tilde{z}))\, dt,$$

where H is the heat kernel associated to this (Friedrichs) Laplacian and $\Pi(z, \tilde{z})$ is the Schwartz kernel of this rank one projector. This integral converges absolutely for any $z \neq \tilde{z}$. We rewrite this as

$$G(z, \tilde{z}) = \int_0^1 H(t, z, \tilde{z})\, dt - \Pi(z, \tilde{z}) + \int_1^\infty (H(t, z, \tilde{z}) - \Pi(z, \tilde{z}))\, dt. \tag{78}$$

It follows easily from standard estimates that the integral from 1 to ∞ converges to a bounded function. On the other hand, by the maximum principle, $H > 0$, so the first term on the right is nonpositive. Finally, $\Pi(z, \tilde{z}) = V^{-1}$ is just a constant, so G is bounded below.

To conclude the proof, it suffices to prove $-\inf \varphi(s) \leq -\frac{C}{V} \int_M \varphi(s)\omega^n_{\varphi(s)}$. (Indeed, $\varphi(s)$ changes sign by the normalization (27) of f_ω, so $||\varphi(s)||_{\mathcal{C}^0(M)} \leq \operatorname{osc} \varphi(s)$.) This can be shown in one of two ways. The first is by noting that Bando–Mabuchi's Green's function lower bound [7] extends to our present setting, and thus $A_{\omega(s)} < C$ uniformly in s and $-\inf \varphi(s) \leq -\frac{1}{V} \int_M \varphi(s)\omega^n_{\varphi(s)} + nVC$. Indeed, the proof of their bound relies on an estimate of Cheng–Li [20] of the heat kernel $H_{\omega(s)}(t, z, \tilde{z}) - V^{-1} \leq Ct^{-n}$ with C depending only on terms of the Poincaré and Sobolev constants, and hence independent of $s > \varepsilon$. Thus, by (78) $A_{\omega(s)} < C$, as desired. The second uses Moser iteration, as in [56]. $\square$

6.3. *Mabuchi's K-energy and Tian's invariants.* Define the twisted Mabuchi K-energy functional by integration over paths $\{\omega_{\varphi_t}\} \subset \mathcal{H}_{\omega_0}$ smooth in t,

$$E_0^\beta(\omega, \omega_\varphi) := -\frac{1}{V}\int_{M\times[0,1]} \dot\varphi_t \Delta_{\varphi_t} f_{\varphi_t} \omega_{\varphi_t}^n \wedge dt. \tag{79}$$

Its critical points are Kähler–Einstein edge metrics. The following is an extension of a formula of Tian [59, p. 254], [60, (5.12)] (cf. [9], [38]) to the twisted setting. In particular it shows that E_0^β is well defined on $\mathcal{H}_{\omega_0} \times \mathcal{H}_{\omega_0}$. The proof, as others in this subsection, are straightforward extensions of their counterparts from the smooth setting and are included for the reader's convenience.

Lemma 6.7. *One has*

$$E_0^\beta(\omega, \omega_\varphi) = \frac{1}{V}\int_M \log\frac{\omega_\varphi^n}{\omega^n}\omega_\varphi^n - \mu(I-J)(\omega,\omega_\varphi) + \frac{1}{V}\int_M f_\omega(\omega^n - \omega_\varphi^n). \tag{80}$$

Proof. For any smooth (in t) path $\{\omega_{\varphi_t}\} \subset \mathcal{H}_\omega$ connecting ω and ω_φ [62, p. 70],

$$(I-J)(\omega, \omega_{\varphi_1}) = -\frac{1}{V}\int_{M\times[0,1]} \varphi_t \Delta_{\varphi_t}\dot\varphi_t \omega_{\varphi_t} \wedge dt. \tag{81}$$

Hence the variation of the right-hand side of (80) equals

$$\int_M \Delta_\varphi \dot\varphi \left(\log\frac{\omega_\varphi^n}{\omega^n} + 1 + \mu\varphi - f_\omega\right)\omega_\varphi^n,$$

and this coincides with $dE_0^\beta(\dot\varphi)$ since $f_{\omega_\varphi} = f_\omega - \mu\varphi - \log\frac{\omega_\varphi^n}{\omega^n} + c_\varphi$ with c_φ a constant. The formula then follows since both sides vanish when $\omega_\varphi = \omega$. □

As noted in the introduction, a key property of the continuity path (30) is the monotonicity of E_0^β. Monotonicity of similar twisted K-energy functionals was noted, e.g., in [52], and the following is the analogue of [52, Lemma 9.3].

Lemma 6.8. *E_0^β is monotonically decreasing along the continuity path* (30).

Proof. By (31), $\sqrt{-1}\partial\bar\partial f_{\omega_\varphi} = -(\mu - s)\sqrt{-1}\partial\bar\partial\varphi$, and from (30), we have $(\Delta_\varphi + s)\dot\varphi = -\varphi$. It follows that

$$\frac{d}{ds}E_0^\beta(\omega, \omega_{\varphi(s)}) = -\frac{\mu - s}{V}\int_M \dot\varphi\Delta_\varphi(\Delta_\varphi + s)\dot\varphi\omega_\varphi^n,$$

and this is nonpositive by the positivity of $\Delta_\varphi^2 + s\Delta_\varphi$, which is immediate for $s \le 0$, and follows from Lemma 6.1, when $s \in (0, \mu)$. □

Following Tian [61], we say that E_0^β is proper if $\lim_{j\to\infty}(I-J)(\omega,\omega_j) = \infty$ implies that necessarily, $\lim_{j\to\infty} E_0^\beta(\omega, \omega_j) = \infty$. From Lemmas 6.6 and 6.8, we have

Corollary 6.9. *Let $\varphi(s) \in \mathcal{D}_s^{0,\gamma} \cap \mathrm{PSH}(M,\omega)$. If E_0^β is proper, then $||\varphi(s)||_{\mathcal{C}^0(M)} \le C$, independently of $s \in (\varepsilon, \mu)$.*

We also note that, as observed by Berman [9], an alternative proof of Corollary 6.9 follows by combining Kołodziej's estimate [34] and the following result contained in [11, Lemma 6.4] and [9]. (Note that $\varphi(s)$ change sign.)

Lemma 6.10 ([11], [9]). *Suppose $J(\omega, \omega_\varphi) \le C$. Then for each $t > 0$, there exists $C' = C'(C, M, \omega, t)$ such that $\int_M e^{-t(\varphi - \sup\varphi)}\omega^n \le C'$.*

We next recall the definition of Tian's invariants [56], [58]:

$$\begin{aligned}
\alpha_{\Omega,\chi} &:= \sup\Big\{\, a \,:\, \sup_{\varphi\in \mathrm{PSH}\cap\mathcal{C}^\infty(M,\omega_0)} \int_M e^{-a(\varphi-\sup\varphi)}\chi^n < \infty \Big\},\\
\alpha(M) &:= \alpha_{c_1(M),\omega_0},\\
\beta_{\Omega,\omega} &:= \sup\{\, b \,:\, \mathrm{Ric}\,\chi \ge b\chi \text{ for some } \chi \in \mathcal{H}_\omega\},\\
\beta(M) &:= \sup\{\, b \,:\, \mathrm{Ric}\,\lambda \ge b\lambda \text{ for some } \lambda \in \mathcal{H}_{c_1}^\infty\},
\end{aligned}$$

where the measure χ^n is assumed to have density in $L^p(M, \omega_0^n)$, for some $p > 1$, and where, for emphasis, $\mathcal{H}_\omega$ is given by (18) and, when M is Fano, $\mathcal{H}_{c_1}^\infty$ denotes the space of smooth Kähler forms representing $c_1(M)$ (and finally, as always, $\Omega = \frac{1}{\mu}c_1(M) - \frac{1-\beta}{\mu}c_1(L_D)$ with $\Omega = [\omega_0] = [\omega]$, ω_0 a smooth Kähler form, and $\omega = \omega(\beta)$ the reference Kähler edge current). These invariants are always positive as shown by Tian when χ^n is smooth, and hence also in general by the Hölder inequality. For some relations between $\alpha_{\Omega,\omega}$ and α_{Ω,ω_0}, we refer to [9] where such invariants for singular measures were studied in depth.

Lemma 6.11. *Suppose that $\alpha_{\Omega,\omega} - \frac{n\mu}{n+1} > \varepsilon$. Then $E_0^\beta \ge \varepsilon I - C$ for some $C \ge 0$.*

Proof. Again we follow the classical argument [60, p. 164], [62, p. 95]. Note that for any $a \in (0, \alpha_\omega)$, there exists a constant C_a such that $\frac{1}{V}\int_M \log\frac{\omega_\varphi^n}{\omega^n}\omega_\varphi^n \ge aI(\omega,\omega_\varphi) - C_a$. Indeed, by (77) and Jensen's inequality,

$$\begin{aligned}
e^{C_a} &\ge \frac{1}{V}\int_M e^{-a(\varphi - \frac{1}{V}\int_M \varphi\omega^n)}\omega^n\\
&= \frac{1}{V}\int_M e^{-\log\frac{\omega_\varphi^n}{\omega^n} - a(\varphi - \frac{1}{V}\int_M\varphi\omega^n)}\omega_\varphi^n \ge e^{-\frac{1}{V}\int_M (\log\frac{\omega_\varphi^n}{\omega^n} + a(\varphi - \frac{1}{V}\int_M \varphi\omega^n))\omega_\varphi^n}.
\end{aligned}$$

By (76) and (80) it then follows that $E_0^\beta \ge (a - \frac{n\mu}{n+1})I - C$. □

Corollary 6.12. *For all $s \in (-\infty, \frac{n+1}{n}\alpha_{\Omega,\omega}) \cap (-\infty, \mu]$, we have*

$$||\varphi(s)||_{\mathcal{C}^0(M)} \le C,$$

with C independent of s.

Proof. When $\alpha_{\Omega,\omega} > \frac{n\mu}{n+1}$, the result follows from Lemma 6.11 and Corollary 6.9. (Note, as explained in Section 9, that there is no difficulty in treating the interval $s \in (0, \varepsilon)$ for some $\varepsilon > 0$.) In general, the classical derivation [62] carries over. □

This conditional $\mathcal{C}^0$ estimate implies of course, given the other ingredients of the proof of Theorem 2, that $\beta_{\Omega,\omega} \geq \min\{\mu, \frac{n+1}{n}\alpha_{\Omega,\omega}\}$, just as in the smooth setting. We remark that Donaldson [23] conjectured that when $D \subset M$ is a smooth anticanonical divisor of a Fano manifold, then $\beta(M) = \sup\{\beta : (29)$ admits a solution with $\mu = \beta\}$. Note that the Calabi–Yau theorem shows that the left-hand side is positive, while Corollary 1 shows that the right-hand side is positive. Our results have further direct bearings on this problem, which we hope to discuss elsewhere.

7. The Laplacian estimate

Let $f : M \to N$ be a holomorphic map between two complex manifolds. The Chern–Lu inequality was originally used by Lu to bound $|\partial f|^2$ when the target manifold has negative bisectional curvature [40] under some technical assumptions. This inequality was later used by Yau [69] together with his maximum principle to greatly generalize the result to the case where (M, ω) is a complete Kähler manifold with a lower bound C_1 on the Ricci curvature, and (N, η) is a Hermitian manifold whose bisectional curvature is bounded above by a negative constant $-C_3$. These results lead to Yau's Schwarz lemma, which says that the map f decreases distances in a manner depending only on $C_3 > 0$ and C_1.

In a related direction, the use of the Chern–Lu inequality to prove a Laplacian estimate for complex Monge–Ampère equations seems to go back in print at least to Bando–Kobayashi [6], who considered the case $\operatorname{Ric}\omega \geq -C_2\eta$ and C_3 arbitrary (not necessarily positive) in the context of constructing a Ricci flat metric on the complement of a divisor. Next, the case $\operatorname{Ric}\omega \geq -C_1\omega$ (and again C_3 arbitrary) was used in proving the *a priori* Laplacian estimate for the Ricci iteration [52].

The point of Proposition 7.1 below is to state the Chern–Lu inequality in a unified manner that applies to a wide range of Monge–Ampère equations that appear naturally in Kähler geometry. It makes the Laplacian estimate in these settings slightly simpler and the explicit dependence on the geometry more transparent. In addition, the Chern–Lu inequality applies in some situations where the standard derivation [2], [70], [54] of the Aubin–Yau Laplacian estimate may fail (as in the case of the Ricci iteration) or give an estimate with different dependence on the geometry (which will be crucial in our setting). While Proposition 7.1 below should be folklore among experts, it seems that it is less well known than it deserves to be. In particular, we are not aware of a treatment of the Aubin–Yau or Calabi–Yau theorems that uses it.

7.1. *The Chern–Lu inequality.* Let (M,ω), (N,η) be compact Kähler manifolds and let $f : M \to N$ be a holomorphic map with $\partial f \neq 0$. The Chern–Lu inequality [21], [40] is

$$\Delta_\omega \log |\partial f|^2 \geq \frac{\operatorname{Ric}\omega \otimes \eta(\partial f, \bar\partial f)}{|\partial f|^2} - \frac{\omega \otimes R^N(\partial f, \bar\partial f, \partial f, \bar\partial f)}{|\partial f|^2} \quad \text{on } M. \tag{82}$$

Since the original statement [39, (7.13)], [40, (4.13)] contains a misprint, we include a direct and slightly simplified derivation (since we restrict to the Kähler setting) for completeness. We note also that (82) can be obtained as a special case of a formula of Eells–Sampson [25, (16)] on the Laplacian of the energy density of a harmonic map.

Write $\partial f : T^{1,0}M \to T^{1,0}N$. Then ∂f is a section of $T^{1,0\star}M \otimes T^{1,0}N$ given in local holomorphic coordinates by $\partial f = \frac{\partial f^i}{\partial z^j} dz^j \otimes \frac{\partial}{\partial w^i}$. With respect to the metric induced by ω and η on the product bundle above,

$$u := |\partial f|^2 = g^{i\bar l} h_{j\bar k} \frac{\partial f^j}{\partial z^i} \overline{\frac{\partial f^k}{\partial z^l}}. \tag{83}$$

Compute in normal coordinates at a point

$$\begin{aligned}\Delta_\omega u = \sum_p u_{p\bar p} &= -\sum_{i,l,p} g_{l\bar i, p\bar p} h_{j\bar k} f^j_{,i} f^{\bar k}_{,\bar l} + \sum_{i,p} h_{j\bar k, d\bar m} f^d_{,p} f^{\bar m}_{,\bar p} f^j_{i} f^{\bar k}_{,\bar i} + \sum_{i,j,p} f^j_{,ip} f^{\bar j}_{,\bar i\bar p} \\ &= \operatorname{Ric}\omega \otimes \eta(\partial f, \bar\partial f) - \omega \otimes R^N(\partial f, \bar\partial f, \partial f, \bar\partial f) + \sum_{i,j,p} f^j_{,ip} f^{\bar j}_{,\bar i\bar p}.\end{aligned}$$

By the Cauchy–Schwarz inequality,

$$u \sum_{i,j,p} f^j_{,ip} f^{\bar j}_{,\bar i\bar p} \geq \sum_k u_k u_{\bar k},$$

and since $\Delta_\omega \log u = \Delta_\omega u/u - \sum_k u_k u_{\bar k}/u^2$, the desired inequality follows.

One particularly useful form of the Chern–Lu inequality is when f is the identity map.

Proposition 7.1. *In the above, let* $f = \mathrm{id} : (M,\omega) \to (M,\eta)$ *be the identity map, and assume that* $\operatorname{Ric}\omega \geq -C_1\omega - C_2\eta$ *and that* $\operatorname{Bisec}_\eta \leq C_3$ *for some* $C_1, C_2, C_3 \in \mathbb{R}$. *Then,*

$$\Delta_\omega \log |\partial f|^2 \geq -C_1 - (C_2 + 2C_3)|\partial f|^2. \tag{84}$$

In particular, if $\omega = \eta + \sqrt{-1}\partial\bar\partial\varphi$, *then*

$$\Delta_\omega\Big(\log \operatorname{tr}_\omega\eta - (C_2 + 2C_3 + 1)\varphi\Big) \geq -C_1 - (C_2 + 2C_3 + 1)n + \operatorname{tr}_\omega\eta. \tag{85}$$

Hence, $\omega \geq C\eta$ *for some* $C > 0$ *depending only on* C_1, C_2, C_3, n *and* $||\varphi||_{\mathcal{C}^0(M)}$.

Proof. By (83), $u = \mathrm{tr}_\omega \eta$. The assumption on $\mathrm{Ric}\,\omega$ implies that

$$\begin{aligned}\mathrm{Ric}\,\omega \otimes \eta(\partial f, \bar\partial f) &= g^{i\bar l} g^{k\bar j} R_{i\bar j} h_{k\bar l} \geq -C_1 g^{i\bar l} g^{k\bar j} g_{i\bar j} h_{k\bar l} - C_2 g^{i\bar l} g^{k\bar j} h_{i\bar j} h_{k\bar l} \\ &= -C_1 \mathrm{tr}_\omega \eta - C_2 (\eta,\eta)_\omega \geq -C_1 \mathrm{tr}_\omega \eta - C_2 (\mathrm{tr}_\omega \eta)^2,\end{aligned}$$

where the last inequality follows from the identity [13, Lemma 2.77]

$$(\eta,\eta)_\omega = (\mathrm{tr}_\omega \eta)^2 - n(n-1)\frac{\eta\wedge\eta\wedge\omega^{n-2}}{\omega^n}.$$

Similarly, we also have

$$\begin{aligned}-\omega\otimes R^N(\partial f, \bar\partial f, \partial f, \bar\partial f) &= -g^{i\bar j} g^{k\bar l} R^N_{i\bar j k\bar l} \\ &\geq -C_3 g^{i\bar j} g^{k\bar l}(h_{i\bar j} h_{k\bar l} + h_{i\bar l} h_{k\bar j}) \geq -2C_3 (\mathrm{tr}_\omega \eta)^2.\end{aligned}$$

Thus, (84) follows from (82). Since $\mathrm{tr}_\omega \eta = n - \Delta_\omega \varphi$, equation (85) follows from (84). □

7.2. *The Laplacian estimate in the singular Hölder spaces.* We now apply the Chern–Lu inequality to obtain an *a priori* Laplacian estimate for the continuity method (32). For solutions of (30), it gives a bound depending, in addition to the uniform norm, on an upper bound on the bisectional curvature of the reference metric; in contrast, the well-known Aubin–Yau bound depends on a lower bound for the bisectional curvature [2], [70], [54].

LEMMA 7.2. *Suppose that there exists a reference metric $\omega \in \mathcal{H}_{\omega_0}$ with $\mathrm{Ric}\,\omega \geq -C_2\omega$ and $\mathrm{Bisec}_\omega \leq C_3$ on $M\setminus D$, where $C_2 \in \mathbb{R}\cup\{\infty\}$, $C_3 \in \mathbb{R}$. Let $s > S$. Solutions to (32) in $\mathcal{D}^{0,\gamma}_s \cap \mathcal{C}^4(M\setminus D)\cap \mathrm{PSH}(M,\omega)$ satisfy*

(86)
$$||\Delta_\omega \varphi(s,t)||_{\mathcal{C}^0(M)} \leq C = C(||f_\omega||_{\mathcal{C}^0(M)}, ||\varphi(s,t)||_{\mathcal{C}^0(M)}, S, (1-t)C_2, C_3),$$

where $(1-t)C_2$ is understood to be 0 when $t=1$. Moreover, $\frac{1}{C}\omega \leq \omega_{\varphi(s)} \leq C\omega$.

Proof. Along the continuity path (32),

$$\mathrm{Ric}\,\omega_\varphi = (1-t)\mathrm{Ric}\,\omega + s\omega_\varphi + (\mu t - s)\omega + 2\pi(1-\beta)[D] \geq S\omega_{\varphi_{s,t}} - (1-t)C_2\omega.$$

Hence, the assumptions of Proposition 7.1 are satisfied (we take $\mathrm{id} : (M,\omega_\varphi) \to (M,\omega)$), and the desired estimates follow directly from (85) if the maximum of (the bounded function) $\log \mathrm{tr}_{\omega_\varphi}\omega - A\varphi$ takes place in $M\setminus D$.

Next, suppose the maximum is attained on D. We claim that $\log \mathrm{tr}_{\omega_\varphi}\omega \in \mathcal{C}^{0,\tilde\gamma}_s$ for any $\tilde\gamma \leq \min\{\frac{1}{\beta}-1, \gamma\}$. Indeed, in the local coordinates $z_1,\ldots,z_n$, $g^{i\bar j}_\varphi = \frac{1}{\det g_\varphi} A_{i\bar j}$, where A is the cofactor matrix of $[g_\varphi]$. Since $A_{i\bar j}$ is a polynomial in the components $g_{\varphi\, k\bar\ell}$, it too lies in $\mathcal{C}^{0,\gamma}_s$. In addition, $1/\det g_\varphi = e^{-f_\omega - c_t + s\varphi}/\det g_\omega = |z_1|^{2-2\beta}F$ for some $F \in \mathcal{C}^{0,\gamma}_s$, hence this lies in $\mathcal{C}^{0,\tilde\gamma}_s$ for $\tilde\gamma \leq \frac{1}{\beta}-1$. Hence $\mathrm{tr}_{\omega_\varphi}\omega = g^{i\bar j}_\varphi g_{i\bar j} \in \mathcal{C}^{0,\tilde\gamma}_s$, proving the claim.

Now by Lemma 5.1 applied to $f := \log \operatorname{tr}_{\omega_\varphi}\omega - A\varphi$, we have that $f + |s|_h^\varepsilon$ achieves its maximum away from D for $\varepsilon < \beta\gamma$. (When $s = e$, we use that $\varphi \in \mathcal{A}_{\mathrm{phg}}$ by Theorem 1.) By (85) and Lemma 5.1(ii) (and in particular (62)), for all sufficiently large $N > 1$, we have

$$\Delta_\varphi(f + N^{-1}|s|_h^\varepsilon) \geq -C_1 - (2C_3 + 1)n + (1 - C/N)\operatorname{tr}_{\omega_\varphi}\omega.$$

The maximum principle thus implies $\operatorname{tr}_{\omega_\varphi}\omega \leq C = C(C_1, C_3, ||\varphi||_{\mathcal{C}^0(M)}, \omega)$, and so $\omega_\varphi \geq C\omega$. Going back to (30) we have $\omega_\varphi^n \leq C\omega^n$ (with C depending on $||f_\omega||_{\mathcal{C}^0(M)}$ and $||\varphi(s,t)||_{\mathcal{C}^0(M)}$), and so also $\omega_\varphi \leq C\omega$. □

8. Hölder estimates for second derivatives

In the interior of $M \setminus D$, the Evans–Krylov regularity theory for Monge–Ampère equations (Theorem 4.1) may be applied to obtain the *a priori* interior $\mathcal{C}^{2,\gamma}$ estimate for a solution φ on any ball B' depending on $\mathcal{C}^0$ estimates for φ and $\Delta_\omega\varphi$ on a slightly larger ball. This depends heavily, of course, on the uniform ellipticity of the Laplacian, and hence it does not apply directly for balls arbitrarily close to D.

We now explain how to obtain *a priori* estimates in $\mathcal{D}_w^{0,\gamma}$ using the *a priori* Laplacian estimate from the last section. The proof uses an old argument due to Tian.

THEOREM 8.1. *Let $\varphi(s) \in \mathcal{D}_w^{0,0} \cap \mathcal{C}^4(M \setminus D) \cap \mathrm{PSH}(M, \omega)$ be a solution to* (30) *with $s > S$ and $0 < \beta \leq 1$. Then*

$$||\varphi(s)||_{\mathcal{D}_w^{0,\gamma}} \leq C,$$

where $C = C(S, \beta, \omega, n, ||\Delta_\omega\varphi||_{L^\infty(M)}, ||\varphi||_{L^\infty(M)})$.

Proof. Let $\mathcal{U}$ be a neighborhood in M. According to Theorem B.1, if ω_φ is locally represented by $u_{i\bar{j}}dz^i \wedge \overline{dz^j}$ on $\mathcal{U} \setminus D$, then for some fixed $\gamma, r_0 > 0$, every $a \in (0, r_0)$ and all x such that $B_a(x) \subset U$, we have $||\nabla u_{i\bar{j}}||^2_{B_a(x)} \leq C\, a^{2n-2+2\gamma}$. The constants γ, r_0, C are all uniformly controlled. The Poincaré inequality gives $||u_{i\bar{j}} - C_{x,a}||^2_{B_a(x)} \leq C\, a^{2n+2\gamma}$, where $C_{x,a} = \int_{B_a(x)} u_{i\bar{j}}\omega^n / \int_{B_a(x)} \omega^n$. Using the integral characterization of Hölder spaces, see [31, Th. 3.1] for example, patching up the estimates over a finite cover, and using that we already have uniform bounds on $\Delta_\omega\varphi$ itself, it follows that $||\Delta_\omega\varphi||_{C_w^{0,\gamma}} \leq C$. □

The proof of Theorem B.1 requires the following lemma that is perhaps of independent interest. Let ψ be a fixed Kähler potential for ω valid in a neighborhood of $y_0 \in D$. For each pair of parameters (s, t), consider the function $h = h(s, t)$ defined by the equality

$$\log h := \log F + \log\det[\psi_{i\bar{j}}] = tf_\omega + c_t - s\varphi + \log\det[\psi_{i\bar{j}}]. \tag{87}$$

By Lemma 2.2 (see (24)), $\psi \in \mathcal{D}_w^{0,\gamma}$ for any $\gamma \in (0, \frac{1}{\beta} - 1]$.

We now state a collection of estimates for h, but we note that only the Lipschitz bound (iv) is used later in the proof of Theorem B.1.

LEMMA 8.2. *Define $h = h(s,t)$ by (87), with $s > S$. Then the following estimates hold with constants independent of t, s:*

(i) *For $\beta \le 1$, $||h(s,t)||_{C^0(M)} \le C(S, M, \omega, \beta, ||\varphi(s,t)||_{C^0(M)})$.*
(ii) *For $\beta \le 1/2$, $||h(s,t)||_{\mathcal{D}^{0,0}_w} \le C = C(S, M, \omega, \beta, ||\Delta_\omega \varphi(s,t)||_{\mathcal{C}^0})$.*
(iii) *For $\beta \le 2/3$, $||h(s,t)||_{w;0,1} \le C = C(S, M, \omega, \beta, ||\varphi(s,t)||_{\mathcal{C}^{0,1}_w})$.*
(iv) *For $\beta \le 1$,*

$$||h(s,1)||_{w;1,\frac{1}{\beta}-1} \le C = C(S, M, \omega, \beta, ||\varphi(s,1)||_{\mathcal{C}_w^{1,\frac{1}{\beta}-1}}). \tag{88}$$

Moreover, $||h(s,1)_{k\bar{l}}||_{C^0(M)} \le C(S, M, \omega, \beta, ||\Delta_\omega \varphi(s,1)||_{\mathcal{C}^0})$.

Proof. By (58) and (8), near D,

$$\begin{aligned}\det[\psi_{i\bar{j}}] &= \beta^2 |\zeta|^{\frac{2}{\beta}-2} \det\Big[\frac{\partial^2(\psi_0+\phi_0)}{\partial z^i \partial \overline{z^j}}\Big] \\ &= \beta^2 \sum_{k=0}^{n} f_{0k} |\zeta|^{2k-2+\frac{2}{\beta}} + \beta^2 \sum_{k=0}^{n-1} (f_{1k} + f_{2k}\zeta^{\frac{1}{\beta}} + f_{3k}\overline{\zeta^{\frac{1}{\beta}}}) |\zeta|^{2k},\end{aligned}$$

with f_{jk} smooth functions of $(z_1, \dots, z_n)$. Thus, if $\beta \in (0, 2/3]$, then $\log\det[\psi_{i\bar{j}}]$ is in $\mathcal{C}^{0,1}_w$. Moreover, if $\beta \in (0, 1/2]$, then $\log\det[\psi_{i\bar{j}}]$ is in $\mathcal{C}^{1,1}_w$; for that it suffices to remark that

$$\partial_r f_{jk} = \frac{\partial f_{jk}}{\partial \rho}\frac{\partial \rho}{\partial r} = \frac{\partial f_{jk}}{\partial \rho}(\beta r)^{\frac{1}{\beta}-1} \in C^{0,1}_w$$

if $\beta \in (0, 1/2]$. Next, by Lemma 4.5, and the same reasoning as above, it follows that when $\beta \in (0, 1/2]$, $f_\omega \in C^{1,1}_w$ and that when $\beta \in (0, 2/3]$, $f_\omega \in C^{0,1}_w$. Therefore, (ii) and (iii) follow. Note also that the above computations show that $f_\omega, \det[\psi_{i\bar{j}}] \in L^\infty(M)$ for all $\beta \in (0, 1]$, proving (i).

Now, assume $t = 1$. Denote, as before, $d\mathbf{z} = dz_1 \wedge \cdots \wedge dz_n$. By (60),

$$\begin{aligned} f_\omega + \log\det[\psi_{i\bar{j}}] &= f_\omega + \log \frac{(\omega_0 + \sqrt{-1}\partial\bar{\partial}\psi_0)^n}{(\sqrt{-1})^{n^2} d\zeta \wedge d\bar{\zeta} \wedge dz_2 \wedge d\bar{z_2} \wedge \cdots \wedge dz_n \wedge d\bar{z_n}} \\ &= f_\omega + \log \frac{(\omega_0 + \sqrt{-1}\partial\bar{\partial}\psi_0)^n}{|z_1|^{2\beta-2}(\sqrt{-1})^{n^2} d\mathbf{z} \wedge d\bar{\mathbf{z}}} \\ &= \log \frac{|s|_h^{2\beta-2}\omega_0^n e^{F_{\omega_0}-\mu\phi_0}}{|z_1|^{2\beta-2}(\sqrt{-1})^{n^2} d\mathbf{z} \wedge d\bar{\mathbf{z}}} \\ &= (2\beta-2)\log a + F_{\omega_0} - \mu\phi_0 + \log\det\Big[\frac{\partial^2 \psi_0}{\partial z^i \partial \bar{z^j}}\Big], \end{aligned}$$

where a is defined in (20). Thus $f_\omega + \log\det[\psi_{i\bar{j}}]$ can be written as a sum $\Phi_1 - \mu\phi_0 = \Phi_1 - \mu r^2 \Phi_0$ with Φ_i, $i = 0, 1$, smooth functions of $(z_1, \overline{z_1}, \dots, z_n, \overline{z_n})$. Hence, by the reasoning above, $f_\omega + \log\det[\psi_{i\bar{j}}] \in C_w^{1,\frac{1}{\beta}-1}$, and therefore $h(s,1)$

belongs to $C_w^{1,\frac{1}{\beta}-1}$ as soon as $\varphi(s,1)$ does. The statement about the $(1,1)$-part of the Hessian of $h(s,1)$ follows in the same way. This concludes the proof of (iv). $\square$

9. Existence of Kähler–Einstein edge metrics

We now conclude the proof of Theorem 2 on the existence of Kähler–Einstein edge metrics, as well as of the convergence of the twisted Ricci iteration (Theorem 2.5). We then describe the additional regularity properties as stated in Theorem 2.

Starting the continuity path. Intuitively, the Ricci continuity path (30) has the trivial solution $\omega(-\infty)=\omega$ at $s=-\infty$. Even if one could make rigorous sense of this, one could not apply the implicit function theorem directly to obtain solutions for large negative finite values of s. Indeed, reparametrizing (30) by setting $\sigma=-1/s$, then the linearization of the Monge-Ampère equation at σ equals $\sigma\Delta_{\varphi(-1/\sigma)}-1$, and this degenerates at $\sigma=0$. It is therefore necessary to find a different way to produce a solution of (30) for sufficiently negative, but finite, values of s. Once this has been accomplished, we can then proceed with the rest of the continuity method.

When $\beta\in(0,1/2]$, this difficulty can be circumvented by using the two-parameter family; see Remark 2.4. Indeed, as described in Section 2.4, the original continuity path (30) embeds into the two-parameter family (32), and it is trivial that solutions exist for the finite parameter values $(s,0)$. Unfortunately, the *a priori* estimates needed to carry out the rest of the continuity argument for the two-parameter family hold only when $\beta\le 1/2$. Thus, to handle the general case, we must use another method to obtain a solution of (30) for some large negative value of s. Wu [68, Prop. 7.3] used a Newton iteration argument to obtain such a solution in a different setting. However, his argument requires a lower Ricci curvature bound on the reference metric (see [68, p. 431]), which we lack. In other words, no small multiple of f_ω belongs to $\mathcal{H}_\omega$. What follows is an adaptation of Wu's argument that requires no curvature control on the reference metric.

Reformulate the original complex Monge–Ampère equation in terms of the operator

$$N_\sigma:\mathcal{D}_w^{0,\gamma}\to\mathcal{C}_w^{0,\gamma},\quad N_\sigma(\Phi):=\log(\omega_{\sigma\Phi}^n/e^{f_\omega}\omega^n)-\Phi.$$

As we remark at the end of this argument, the following argument works equally well in the edge spaces and leads to the same conclusion. Observe that $DN_\sigma|_\Phi=\sigma\Delta_{\sigma\Phi}-\mathrm{Id}$. Now, suppose that $\sigma\Phi\in\mathcal{H}_\omega\cap\mathcal{A}_{\mathrm{phg}}$. By Proposition 3.2, $DN_\sigma|_\Phi:\mathcal{D}_w^{0,\gamma}\to\mathcal{C}_w^{0,\gamma}$ is Fredholm of index 0, and by the maximum principle and Lemma 5.1, its nullspace K is trivial when $s<0$. Hence, this operator is an isomorphism from $\mathcal{D}_w^{0,\gamma}$ to $\mathcal{C}_w^{0,\gamma}$, with

$$||u||_{\mathcal{D}_w^{0,\gamma}} \leq C||DN_\sigma u||_{\mathcal{C}_w^{0,\gamma}}. \tag{89}$$

Denote by $DN_\sigma|_\Phi^{-1}$ the inverse of this map on $\mathcal{C}_w^{0,\gamma}$.

We now set up the iteration method that will converge to a solution (Newton iteration for N_σ). Define a sequence of elements $\Phi_k \in \mathcal{D}_e^{0,\gamma}$ by setting $\Phi_0 = 0$ and then

$$\Phi_k = (\mathrm{Id} - DN_\sigma|_{\Phi_{k-1}}^{-1} \circ N_\sigma)(\Phi_{k-1}), \quad k \in \mathbb{N}$$

or, equivalently,

$$\Phi_k - \Phi_{k-1} - DN_\sigma|_{\Phi_{k-1}}^{-1} N_\sigma(\Phi_{k-1}) = 0. \tag{90}$$

When $\Phi \in \mathcal{A}^0_{\mathrm{phg}}$, $DN_\sigma|_\Phi^{-1}$ preserves polyhomogeneity, so each of the successive Φ_k are polyhomogeneous. Since $N_\sigma(-f_\omega) = 0$ when $\sigma = 0$, it might seem more natural to set $\Phi_0 = -f_\omega$. However, this would cause a problem at the very next step since, as already observed two paragraphs above, $\sqrt{-1}\partial\bar\partial f_\omega$ blows up at $r = 0$ when $\beta > 1/2$.

Next, observe that

$$\begin{aligned} N_\sigma(\Phi_k) &= N_\sigma(\Phi_k) - N_\sigma(\Phi_{k-1}) - DN_\sigma|_{\Phi_{k-1}}(\Phi_k - \Phi_{k-1}) \\ &= \int_0^1 (1-c) D^2 N_\sigma|_{c\Phi_k + (1-c)\Phi_{k-1}}(\Phi_k - \Phi_{k-1}, \Phi_k - \Phi_{k-1})\, dc. \end{aligned} \tag{91}$$

This will be estimated using the equality $D^2N_\sigma|_\Phi(a,b) = \sigma^2(\partial\bar\partial a, \partial\bar\partial b)_{\omega_{\sigma\Phi}}$, which holds provided $\sigma\Phi \in \mathcal{H}_\omega$. We now deduce inductively the sequences of estimates

$$||\Phi_j - \Phi_{j-1}||_{\mathcal{D}_w^{0,\gamma}} \leq C_1||N_\sigma(\Phi_{j-1})||_{\mathcal{C}_w^{0,\gamma}} \tag{92}$$

and

$$||N_\sigma(\Phi_j)||_{\mathcal{C}_w^{0,\gamma}} \leq C_2\sigma^2||\Phi_j - \Phi_{j-1}||^2_{\mathcal{D}_w^{0,\gamma}} \tag{93}$$

for every $j \geq 1$ with constants C_1 and C_2 independent of j. Suppose then that these hold for every $j \leq k$. We shall prove that they hold also for $j = k+1$ with the same constants C_i.

First note that $\Phi_k = \Phi_k - \Phi_0 = \sum_{j=1}^k (\Phi_j - \Phi_{j-1})$. Using (92) and (93) iteratively gives

$$||\Phi_j - \Phi_{j-1}||_{\mathcal{D}_w^{0,\gamma}} \leq C_1C_2\sigma^2||\Phi_{j-1} - \Phi_{j-2}||^2_{\mathcal{D}_w^{0,\gamma}} \leq (C_1C_2\sigma^2)^{2^{j-1}-1}||\Phi_1||^{2^{j-1}}_{\mathcal{D}_w^{0,\gamma}},$$

and then $||N_\sigma(\Phi_j)||_{\mathcal{C}_w^{0,\gamma}} \leq C_2\sigma^2(C_1C_2\sigma^2)^{2^{j-1}-1}||\Phi_1||^{2^{j-1}}_{\mathcal{D}_w^{0,\gamma}}$. We conclude that

$$||\Phi_k||_{\mathcal{D}_w^{0,\gamma}} \leq \sum_{j=1}^k ||\Phi_j - \Phi_{j-1}||_{\mathcal{D}_w^{0,\gamma}} \leq 2||\Phi_1||_{\mathcal{D}_w^{0,\gamma}},$$

provided $C_1C_2\sigma^2||\Phi_1||_{\mathcal{D}_w^{0,\gamma}} \le 1/2$. Thus if σ is sufficiently small, then $||\sigma\Phi_k||_{\mathcal{D}_w^{0,\gamma}} \le \eta$ for some fixed $\eta > 0$. So, $\sigma\Phi_k \in \mathcal{H}_\omega$, and if we let C_1 denote the supremum of the norm of $DN_\sigma|_\Phi^{-1}$ among all Φ with $||\Phi||_{\mathcal{D}_w^{0,\gamma}} \le \eta$, then (92) holds with k replaced by $k+1$ and the same C_1.

To obtain the final estimate, note that $\sigma(c\Phi_k + (1-c)\Phi_{k-1})$ lies in the same ball of radius η for $0 \le c \le 1$, hence also in $\mathcal{H}_\omega$, which means that (91) with k replaced by $k+1$ can be estimated as before; C_2 is the constant needed to estimate this integral, which is uniform so long as $c\Phi_k + (1-c)\Phi_{k-1}$ remain in the fixed η ball. This proves (93) with k replaced by $k+1$.

We may now conclude that $\Phi_\infty := \lim_{k\to\infty}\Phi_k = \sum_{k=0}^\infty(\Phi_{k+1} - \Phi_k)$ exists and lies in the same η ball in $\mathcal{D}_w^{0,\gamma}$, so $\sigma\Phi_\infty \in \mathcal{H}_\omega$.

Finally, by Theorem 1, Φ_∞ is polyhomogeneous.

Openness. Define $M_{s,t} : \mathcal{D}_w^{0,\gamma} \to C_w^{0,\gamma}$ by

$$M_{s,t}(\varphi) := \log\frac{\omega_\varphi^n}{\omega^n} - tf_\omega + s\varphi, \quad (s,t) \in A = (-\infty,0]\times[0,1] \,\cup [0,\mu]\times\{1\}.$$

Note that $M_{s,0}(0) = 0$. If $\varphi(s,t) \in \mathcal{D}_w^{0,\gamma} \cap \mathrm{PSH}(M,\omega)$ is a solution of (32), we claim that its linearization

$$DM_{s,t}|_{\varphi(s,t)} = \Delta_{\varphi(s,t)} + s : \mathcal{D}_w^{0,\gamma} \to \mathcal{C}_w^{0,\gamma}, \quad (s,t)\in A, \tag{94}$$

is an isomorphism when $s \ne 0$. If $s = 0$, this map is an isomorphism if we restrict on each side to the codimension one subspace of functions with integral equal to 0. Furthermore, we also claim that $\mathcal{D}_w^{0,\gamma} \times A \ni (\varphi,s,t) \mapsto M_{s,t}(\varphi) \in \mathcal{C}_w^{0,\gamma}$ is a $\mathcal{C}^1$ mapping. Given these claims, the Implicit Function Theorem then guarantees the existence of a solution $\varphi(\tilde{s},\tilde{t}) \in \mathcal{D}_w^{0,\gamma}$ for all $(\tilde{s},\tilde{t}) \in A$ sufficiently close to (s,t).

Proposition 3.2 asserts that (94) is Fredholm of index 0 for any $(s,t)\in A$, and by Proposition 3.3,

$$||u||_{\mathcal{D}_w^{0,\gamma}} \le C(||DM_{s,t}u||_{\mathcal{C}_w^{0,\gamma}} + ||u||_{\mathcal{C}^0}), \tag{95}$$

Its nullspace K is clearly trivial when $s < 0$, and also by Lemma 6.1 for $(s,1)$ with $s \in (0,\mu)$; finally, when $s = 0$ it consists of constants. Thus $DM_{s,t}$ is an isomorphism when $s \ne 0$ and is an isomorphism on the L^2 orthogonal complement to the constants when $s = 0$. This proves the first claim.

The second claim follows from (94) and Corollary 3.5, which shows that the domains of these linearizations at different φ are all the same. The smooth dependence on (s,t) is obvious.

Note finally that using (95), nearby solutions remain in $\mathrm{PSH}(M,\omega)$.

We have written this out explicitly for the wedge spaces, but note that all of these arguments go through verbatim for the edge spaces. Observe, however, that using the results of Section 4, the nearby solutions are necessarily polyhomogeneous.

Closedness. Fix some $S < 0$, and denote $A_S := \{(s,t) \in A : s \in (S,0]\}$. Let $\{(s_j,t_j)\}$ be a sequence in $\operatorname{int} A_S$ converging to $(s,t) \in \overline{A_S}$, and let $\varphi(s_j,t_j) \in \mathcal{D}^{0,\gamma}_w \cap \mathrm{PSH}(M,\omega)$ be solutions to (32). Under the assumptions of Theorem 2, the results of Sections 5, 7 and 8 imply that $||\varphi(s_j,t_j)||_{\mathcal{D}^{0,\gamma}_w} \le C$, where C depends on S, a lower bound on the Ricci curvature of ω times $(1-\min_j t_j)$, and an upper bound on its bisectional curvature, both over $M\backslash D$; alternatively, the Aubin–Yau Laplacian estimate [2], [70], [54] gives a bound depending on S and a lower bound on the bisectional curvature of ω over $M\backslash D$. Thus, when $\beta \in (0, \frac{1}{2}] \cup \{1\}$, Lemma 2.3 implies that either type of bounds give a uniform estimate $||\varphi(s,t)||_{\mathcal{D}^{0,\gamma}_w} \le C$ for all $(s,t) \in A_S$. In general, restrict to the path (30) (i.e., let $t_j = 1$ for all j) and then $||\varphi(s,1)||_{\mathcal{D}^{0,\gamma}_w} \le C$ for all $s \in (S,0]$ by Proposition A.1 and the results of Sections 5, 7 and 8. Thus, for any $\gamma' \in (0,\gamma)$, there is a subsequence that converges in $\mathcal{D}^{0,\gamma'}_w$ such that the limit function $\varphi(s,1)$ lies in $\mathcal{D}^{0,\gamma}_w$. Observe that $M_{s,1}(\varphi(s,1)) = 0$ and $\varphi(s,1) \in \mathcal{C}^\infty(M \setminus D)$.

Letting $S \to -\infty$, we obtain a solution for all $(s,t) \in A_\infty$ in the case $\beta \in (0, \frac{1}{2}] \cup \{1\}$ and for all $(s,1) \in (-\infty, 0] \times \{1\}$ in the general case. Now by openness in A about the solution at $(0,1)$ (cf. [3], [7]), there exist solutions also for $[0,\varepsilon) \times \{1\} \subset A$. Then by Corollary 6.9 and the previous arguments we obtain solutions for all $(s,t) \in A$ when $\beta \in (0, \frac{1}{2}] \cup \{1\}$ and for all $(s,1) \in (-\infty, \mu] \times \{1\}$ in the general case. By Theorem 1, these solutions are polyhomogeneous. Finally, $\varphi(s,t) \in \mathrm{PSH}(M,\omega)$. (This follows from a continuity argument, observing that the right-hand side of the Monge–Ampère equation is positive.)

Regularity. Using the steps above, we obtain a solution $\varphi := \varphi(\mu,1) \in \mathcal{A}^0_{\mathrm{phg}} \cap \mathrm{PSH}(M,\omega)$ to (30). Denote by g_μ the associated Kähler–Einstein edge metric. Using Proposition 4.4 and (6), g_μ is asymptotically equivalent to the reference metric g and, moreover, by the explicit form of the expansion and the fact that $P_{1\bar{1}}$ annihilates the r^0 and $r^{\frac{1}{\beta}}$ terms, we obtain that $\varphi \in \mathcal{A}^0 \cap \mathcal{D}^{0,\varepsilon(\beta)}_w$, where $\varepsilon(\beta)$ is determined by Proposition 4.4 (see Corollary 4.8). This completes the proof of Theorem 2.

Convergence of the Ricci iteration. We use the notation of Section 2.5. As noted there, $\mu - \frac{1}{\tau}$ plays the role of s. Consider first the case $\mu \le 0$. By the earlier analysis of (30), for any $\tau > 0$, the iteration exists uniquely and $\{\psi_{k\tau}\}_{k\in\mathbb{N}} \subset \mathcal{D}^{0,\gamma}_w$. By Lemma 5.1, the inductive maximum principle argument of [52] yields $|\psi_{k\tau}| \le C$. Along the iteration, just as for the path (30), the Ricci curvature is bounded from below by $\mu - \frac{1}{\tau}$, hence Proposition 7.1 and Lemma 5.1 show that $|\Delta_{\omega_{k\tau}} \psi_{k\tau}| \le C$. (We consider the maps $\mathrm{id} : (M, \omega_{k\tau}) \to (M,\omega)$.) Going back to equation (33) and using the $\mathcal{C}^0$ estimate then shows that $|\Delta_\omega \psi_{k\tau}| \le C$, hence by Theorem 8.1, $|\psi_{k\tau}|_{\mathcal{D}^{0,\gamma}_w} \le C$.

Thus a subsequence converges (as explained above for the continuity method) to an element ψ_∞ of $\mathcal{D}_w^{0,\gamma} \cap \mathcal{C}^\infty(M\backslash D)$. Since each step in the iteration follows a continuity path of the form (30) with ω replaced by $\omega_{k\tau}$, Lemma 6.8 implies that $E_0^\beta(\omega_{(k-1)\tau}, \omega_{k\tau}) < 0$ (unless ω was already Kähler–Einstein). Since E_0^β is an exact energy functional, i.e., satisfies a cocyle condition [42], then $E_0^\beta(\omega, \omega_{k\tau}) = \sum_{j=1}^k E_0^\beta(\omega_{(j-1)\tau}, \omega_{j\tau}) < 0$. Therefore, ψ_∞ is a fixed point of E_0^β, hence a Kähler–Einstein edge metric. By Lemma 5.2 such Kähler–Einstein metrics are unique; we conclude that the original iteration converges to ψ_∞ both in $\mathcal{A}_0$ and in $\mathcal{D}_w^{0,\gamma'}$ for each $\gamma' \in (0, \gamma)$.

Next, consider the case $\mu > 0$, and take $\mu = 1$ for simplicity. By the properness assumption, Corollary 6.9 implies the iteration exists (uniquely by Lemma 6.6) for each $\tau \in (0, \infty)$ and then the monotonicity of E_0^β implies that $J(\omega, \omega_{k\tau}) \le C$. To obtain a uniform estimate on $\operatorname{osc} \psi_{k\tau}$ we will employ the argument of [10] as explained to us by Berman. By Lemma 6.10, have $\int_M e^{-p(\psi_{k\tau} - \sup \psi_{k\tau})} \omega^n \le C$, where $p/3 = \max\{1 - \frac{1}{\tau}, \frac{1}{\tau}\}$. Now rewrite (33) as

$$\omega_{\psi_{k\tau}}^n = \omega^n e^{f_\omega - (1-\frac{1}{\tau})\psi_{k\tau} - \frac{1}{\tau}\psi_{(k-1)\tau}}. \tag{96}$$

Using Kołodziej's estimate and the Hölder inequality this yields the uniform estimate $\operatorname{osc} \psi_{k\tau} \le C$. Unlike for solutions of (30), the functions $\psi_{k\tau}$ need not be changing signs. Therefore we let $\tilde{\psi}_{k\tau} := \psi_{k\tau} - \frac{1}{V}\int_M \psi_{k\tau}\omega^n$. As in the previous paragraph we obtain a uniform estimate $\operatorname{tr}_{\omega_{k\tau}} \omega \le C$. However, to conclude that $\operatorname{tr}_\omega \omega_{k\tau} \le C$ from (96) we must show that $|(1 - \frac{1}{\tau})\psi_{k\tau} - \frac{1}{\tau}\psi_{(k-1)\tau}| \le C$. This is shown in [52, p. 1543]. Thus, as before, we conclude that $\{\tilde{\psi}_{k\tau}\}$ subconverges to the potential of a Kähler–Einsteinedge metric. Whenever it is unique, the iteration itself necessarily converges. Berndtsson's generalized Bando–Mabuchi Theorem [7], [12] shows uniqueness of Kähler–Einstein edge metrics up to an automorphism (which must preserve D by (7) or Lemma 6.1). This concludes the proof of Theorem 2.5.

Appendix A. Upper bound on the bisectional curvature of the reference metric

By Chi Li[1] and Yanir A. Rubinstein

Proposition A.1. *Let $\beta \in (0, 1]$, and let $\omega = \omega_0 + \sqrt{-1}\partial\bar{\partial}|s|_h^{2\beta}$ be given by* (26). *The bisectional curvature of ω is bounded from above on $M\backslash D$.*

We denote throughout by $\hat{g}, g$ the Kähler metrics associated to ω_0, ω, respectively. As in [66], to simplify the calculation and estimates we need

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a lemma to choose an appropriate local holomorphic frame and coordinate system, whose elementary proof we include for the reader's convenience. We thank Gang Tian for pointing out to us the calculations in [66] that were helpful in writing this appendix.

LEMMA A.2 ([66, p. 599]). *There exists* $\varepsilon_0 > 0$ *such that if* $0 < \operatorname{dist}_{\hat{g}}(p, D) \le \varepsilon_0$, *then we can choose a local holomorphic frame* e *of* L_D *and local holomorphic coordinates* $\{z_i\}_{i=1}^n$ *valid in a neighborhood of* p, *such that*

(i) $s = z_1 e$, *and* $a := |e|_h^2$ *satisfies* $a(p) = 1$, $da(p) = 0$, $\frac{\partial^2 a}{\partial z_i \partial z_j} a(p) = 0$; *and*

(ii) $\hat{g}_{i\bar{j},k}(p) = \frac{\partial}{\partial z^k}\omega_0(\frac{\partial}{\partial z^i}, \frac{\partial}{\overline{\partial z^j}})|_p = 0$, *whenever* $j \neq 1$.

Proof. (i) Fix any point $q \in D$, and choose a local holomorphic frame e' and holomorphic coordinates $\{w_i\}_{i=1}^n$ in $B_{\hat{g}}(q, \varepsilon(q))$ for $0 < \varepsilon(q) \ll 1$. Let $s = f'e'$ with f' a holomorphic function and $|e'|_h^2 = c$. Let $e = Fe'$ for some nonvanishing holomorphic function F to be specified later. Then $a = |Fe'|_h^2 = |F|^2 c$. Now fix any point $p \in B_{\hat{g}}(q, \varepsilon(q)) \setminus \{q\}$. In order for a to satisfy the vanishing properties with respect to the variables $\{w_i\}_{i=1}^n$ at a point p, we can just choose F such that $F(p) = c(p)^{-1/2}$, and

$$\begin{aligned}
\partial_{w^i} F(p) &= -c^{-1} F \partial_{w^i} c(p) = -c^{-3/2} \partial_{w^i} c(p) \\
\partial_{w^i}\partial_{w^j} F(p) &= -c^{-1}(F\partial_{w^i}\partial_{w^j} c + \partial_{w^j} c \partial_{w^i} F + \partial_{w^i} c \partial_{w^j} F)(p) \\
&= -c^{-3/2}\partial_{w^i}\partial_{w^j} c(p) + 2c^{-5/2}\partial_{w^i} c \partial_{w^j} c(p).
\end{aligned}$$

Since $c = |e'|_h^2$ is never zero, when $\varepsilon(q)$ is small, which implies $|w - w(p)|$ is small, we can assume $F \neq 0$ in $B_{\hat{g}}(q, \varepsilon(q))$. Now $s = fe = f'e'$ with $f = f'F^{-1}$ a holomorphic function. Since $D = \{s = 0\}$ is a smooth divisor, we can assume $\partial_{w^1} f(q) \neq 0$, and choosing $\varepsilon(q)$ sufficiently small, we can assume that $\partial_{w^1} f \neq 0$ in $B_{\hat{g}}(q, \varepsilon(q))$. Thus by the inverse function theorem, $z_1 = f(w_1, \ldots, w_n), z_2 = w_2, \ldots, z_n = w_n$ are holomorphic coordinates in $B_{\hat{g}}(q, \varepsilon(q)/2)$ and now $s = f(w)e = z_1 e$. By the chain rule, it then follows that a satisfies $a(p) = 1$, $\partial_{z^i} a(p) = \partial_{z^i}\partial_{z^j} a(p) = 0$.

Now cover D by $\cup_{q\in D} B_{\hat{g}}(q, \varepsilon(q)/2)$. By compactness of D the conclusion follows.

(ii) Denote by $\{w^i\}_{i=1}^n$ the coordinates obtained in (i). Following [29, p. 108], let $\tilde{z}^k := w^k - w^k(p) + \frac{1}{2} b^k_{st}(w^s - w^s(p))(w^t - w^t(p))$, with $b^k_{st} = b^k_{ts}$, define a new coordinate system. Then, $\omega_0(\frac{\partial}{\partial w^i}, \frac{\partial}{\overline{\partial w^j}}) = \omega_0(\frac{\partial}{\partial \tilde{z}^i}, \frac{\partial}{\overline{\partial \tilde{z}^j}}) + \hat{g}_{t\bar{j}} b^t_{ip} w^p + \hat{g}_{i\bar{t}}\overline{b^t_{jp} w^p} + O(\sum_{i=1}^n |w^i - w^i(p)|^2)$, and

$$d_{i\bar{j}k} := \frac{\partial}{\partial w^k}\omega_0(\frac{\partial}{\partial w^i}, \frac{\partial}{\overline{\partial w^j}})|_p = \frac{\partial}{\partial \tilde{z}^k}\omega_0(\frac{\partial}{\partial \tilde{z}^i}, \frac{\partial}{\overline{\partial \tilde{z}^j}})|_p + \hat{g}_{t\bar{j}}(p) b^t_{ik} =: e_{i\bar{j}k} + \hat{g}_{t\bar{j}}(p) b^t_{ik}.$$

Let $\hat{g}'_{r\bar{s}} := \hat{g}_{r\bar{s}}$ for each $r, s > 1$, and denote the inverse of the $(n-1)\times(n-1)$ matrix $[\hat{g}'_{r\bar{s}}]$ by $[\hat{g}'^{r\bar{s}}]$. Let $b^1_{ik} = 0$. Then, for each $j > 1$, the equations can be rewritten as $d_{i\bar{j}k} - \sum_{t>1} \hat{g}'_{t\bar{j}}(p) b^t_{ik} = e_{i\bar{j}k}$. Hence, $\sum_{j>1} \hat{g}'^{s\bar{j}} e_{i\bar{j}k} = \sum_{j>1} \hat{g}'^{s\bar{j}} d_{i\bar{j}k} -$

b^s_{ik}, $s > 1$. For each $s > 1$, define b^s_{ik} so that the right-hand side vanishes. Multiplying the equations by $[\hat{g}'_{s\bar{t}}]$, we obtain $e_{i\bar{t}k} = 0$ for each $t > 1$. Finally, set $z^i := \tilde{z}^i + w^i(p)$, $i = 1, \ldots, n$. Since $b^1_{ik} = 0$, we have $z^1 = w^1$, and therefore these coordinates satisfy both properties (i) and (ii) of the statement, as desired. $\square$

Let $H := a^\beta$, then $|s|^{2\beta}_h = |z_1 e|^{2\beta}_h = H|z_1|^{2\beta}$. Note that both a and H are locally defined smooth positive functions. Let $\omega = \frac{\sqrt{-1}}{2} g_{i\bar{j}} dz^i \wedge \overline{dz^j}$, $\omega_0 = \frac{\sqrt{-1}}{2} \hat{g}_{i\bar{j}} dz^i \wedge \overline{dz^j}$, and write $z \equiv z_1$ and $\rho := |z|$. Using the symmetry for subindices, we can calculate in a straightforward manner:

$$
\begin{aligned}
g_{i\bar{j}} &= \hat{g}_{i\bar{j}} + H_{i\bar{j}}|z|^{2\beta} + \beta H_i \delta_{1\bar{j}} |z|^{2\beta-2} z + \beta H_{\bar{j}} \delta_{1i} |z|^{2\beta-2} \bar{z} + \beta^2 H |z|^{2\beta-2} \delta_{1i} \delta_{1\bar{j}}, \\
g_{i\bar{j},k} &= \hat{g}_{i\bar{j},k} + H_{i\bar{j}k}|z|^{2\beta} + \beta H_{ik} \delta_{1\bar{j}} |z|^{2\beta-2} z + \beta (H_{k\bar{j}} \delta_{1i} + H_{i\bar{j}} \delta_{1k}) |z|^{2\beta-2} \bar{z} \\
&\quad + \beta^2 (H_i \delta_{1\bar{j}} \delta_{1k} + H_k \delta_{1i} \delta_{1\bar{j}} + H_{\bar{j}} \delta_{1i} \delta_{1k}) |z|^{2\beta-2} \\
&\quad + \beta^2 (\beta - 1) H |z|^{2\beta-4} \bar{z} \delta_{1i} \delta_{1\bar{j}} \delta_{1k},
\end{aligned}
$$

$$
\begin{aligned}
g_{i\bar{j},k\bar{\ell}} &= \hat{g}_{i\bar{j},k\bar{\ell}} + H_{i\bar{j}k\bar{\ell}} |z|^{2\beta} \\
&\quad + \beta \left[(H_{ik\bar{\ell}} \delta_{1\bar{j}} + H_{ik\bar{j}} \delta_{1\bar{\ell}}) |z|^{2\beta-2} z + (H_{\bar{j}\bar{\ell}i} \delta_{1k} + H_{\bar{j}\bar{\ell}k} \delta_{1i}) |z|^{2\beta-2} \bar{z} \right] \\
&\quad + \beta^2 (H_{k\bar{j}} \delta_{1i} \delta_{1\bar{\ell}} + H_{i\bar{j}} \delta_{1k} \delta_{1\bar{\ell}} + H_{k\bar{\ell}} \delta_{1i} \delta_{1\bar{j}} + H_{i\bar{\ell}} \delta_{1\bar{j}} \delta_{1k}) |z|^{2\beta-2} \\
&\quad + \beta(\beta - 1) \left[H_{ik} \delta_{1\bar{j}} \delta_{1\bar{\ell}} |z|^{2\beta-4} z^2 + H_{\bar{j}\bar{\ell}} \delta_{1i} \delta_{1k} |z|^{2\beta-4} \bar{z}^2 \right] \\
&\quad + \beta^2 (\beta - 1) \left[(H_i \delta_{1k} + H_k \delta_{1i}) \delta_{1\bar{j}} \delta_{1\bar{\ell}} |z|^{2\beta-4} z \right. \\
&\qquad\qquad \left. + (H_{\bar{j}} \delta_{1\bar{\ell}} + H_{\bar{\ell}} \delta_{1\bar{j}}) \delta_{1i} \delta_{1k} |z|^{2\beta-4} \bar{z} \right] \\
&\quad + \beta^2 (\beta - 1)^2 H |z|^{2\beta-4} \delta_{1i} \delta_{1\bar{j}} \delta_{1k} \delta_{1\bar{\ell}}.
\end{aligned}
$$

Let $p \in M \setminus D$ satisfy $\operatorname{dist}_{\hat{g}}(p, D) \le \varepsilon_0$. The lemma implies, in particular, $H(p) = 1$, $H_i(p) = H_{ij}(p) = 0$, and the expressions above simplify to

$$
\begin{aligned}
g_{i\bar{j}}(p) &= \hat{g}_{i\bar{j}} + H_{i\bar{j}} |z|^{2\beta} + \beta^2 |z|^{2\beta-2} \delta_{i1} \delta_{1\bar{j}}, \\
g_{i\bar{j},k}(p) &= \hat{g}_{i\bar{j},k} + H_{i\bar{j}k} |z|^{2\beta} + \beta (\delta_{i1} H_{k\bar{j}} + \delta_{k1} H_{i\bar{j}}) |z|^{2\beta-2} \bar{z} \\
&\quad + \beta^2 (\beta - 1) \delta_{i1} \delta_{\bar{j}1} \delta_{k1} |z|^{2\beta-4} \bar{z}, \\
g_{i\bar{j},k\bar{\ell}}(p) &= \hat{g}_{i\bar{j},k\bar{\ell}} + H_{i\bar{j}k\bar{\ell}} |z|^{2\beta} + \beta (\delta_{i1} H_{\bar{j}k\bar{\ell}} + \delta_{k1} H_{i\bar{j}\bar{\ell}}) |z|^{2\beta-2} \bar{z} \\
&\quad + \beta (\delta_{\bar{j}1} H_{ik\bar{\ell}} + \delta_{\bar{\ell}1} H_{i\bar{j}k}) |z|^{2\beta-2} z + \beta^2 (\delta_{i1} \delta_{\bar{j}1} H_{k\bar{\ell}} + \delta_{i1} \delta_{\bar{\ell}1} H_{k\bar{j}} \\
&\quad + \delta_{k1} \delta_{\bar{j}1} H_{i\bar{\ell}} + \delta_{k1} \delta_{\bar{\ell}1} H_{i\bar{j}}) |z|^{2\beta-2} + \beta^2 (\beta - 1)^2 \delta_{i1} \delta_{\bar{j}1} \delta_{k1} \delta_{\bar{\ell}1} |z|^{2\beta-4}.
\end{aligned}
$$

It follows that

$$
g^{r\bar{s}}(p) = O(1), \quad g^{1\bar{s}}(p) = O(\rho^{2-2\beta}) \quad \text{for } r, s > 1 \tag{97}
$$

and

$$
g^{1\bar{1}}(p) = \beta^{-2} \rho^{2-2\beta} \frac{1}{1 + b(p) \rho^{2-2\beta}} + O(\rho^2), \tag{98}
$$

where $O(\rho^2) < C_3\rho^2$ and $b(p) := \beta^{-2}\det[\hat{g}_{i\bar{j}}]/\det[\hat{g}_{r\bar{s}}]_{r,\bar{s}>1}|_p$ with $0 < C_1 < b(p) < C_2$, and C_1, C_2, C_3 independent of $p \in M\backslash D$.

Take two unit vectors $\eta = \eta^i\frac{\partial}{\partial z^i}, \nu = \nu^i\frac{\partial}{\partial z^i} \in T^{1,0}_p M$, so that $g(\eta,\eta)|_p = g(\nu,\nu)|_p = 1$. Then from the expression of $g_{i\bar{j}}$, we have

$$\eta^1, \nu^1 = O(\rho^{1-\beta}) \quad \eta^r, \nu^r = O(1) \text{ for } r > 1. \tag{99}$$

Set

$$\mathrm{Bisec}_\omega(\eta,\nu) = R(\eta,\bar{\eta},\nu,\bar{\nu}) = R_{i\bar{j}k\bar{\ell}}\eta^i\overline{\eta^j}\nu^k\overline{\nu^\ell} = \sum_{i,j,k,l}\Lambda_{i\bar{j}k\bar{\ell}} + \Pi_{i\bar{j}k\bar{\ell}},$$

with $\Lambda_{i\bar{j}k\bar{\ell}} := -g_{i\bar{j},k\bar{\ell}}\eta^i\overline{\eta^j}\nu^k\overline{\nu^\ell}$ and $\Pi_{i\bar{j}k\bar{\ell}} := g^{s\bar{t}}\,g_{i\bar{t},k}\,g_{s\bar{j},\bar{\ell}}\eta^i\overline{\eta^j}\nu^k\overline{\nu^\ell}$ (no summations). By (97)–(99), we have $|\Lambda_{i\bar{j}k\bar{\ell}}| \le C$ except for

$$\Lambda_{1\bar{1}1\bar{1}} = -\beta^2(\beta-1)^2|z|^{2\beta-4}|\eta^1|^2|\nu^1|^2,$$

hence

$$\sum_{i,j,k,l}\Lambda_{i\bar{j}k\bar{\ell}}(p) = O(1) + \Lambda_{1\bar{1}1\bar{1}}(p) = O(1) - \beta^2(\beta-1)^2|z|^{2\beta-4}|\eta^1|^2|\nu^1|^2. \tag{100}$$

The proposition follows immediately by combining (100) and the following estimate.

LEMMA A.3. *There exists a uniform constant $C > 0$ such that for every $p \in M\backslash D$,*

$$\sum_{i,j,k,l}\Pi_{i\bar{j}k\bar{\ell}}(p) \le C + \beta^2(\beta-1)^2|z|^{2\beta-4}|\eta^1|^2|\nu^1|^2.$$

Proof. Define a bilinear Hermitian form of two tensors $a = [a_{i\bar{j}k}]$, $b = [b_{p\bar{q}r}] \in (\mathbb{C}^n)^3$ satisfying $a_{i\bar{j}k} = a_{k\bar{j}i}$ and $b_{p\bar{q}r} = b_{r\bar{q}p}$ by setting

$$\langle[a_{i\bar{j}k}],[b_{p\bar{q}r}]\rangle := \sum_{i,j,k,p,q,r} g^{q\bar{j}}(\eta^i a_{i\bar{j}k}\nu^k)(\overline{\eta^p b_{p\bar{q}r}\nu^r}).$$

It is easy to see that this is a nonnegative bilinear form. We denote by $\|\cdot\|$ the associated norm. Then $\sum_{i,j,k,l}\Pi_{i\bar{j}k\bar{\ell}} = \|[g_{i\bar{j},k}]\|^2$. Write

$$g_{i\bar{j},k} = A_{i\bar{j}k} + B_{i\bar{j}k} + D_{i\bar{j}k} + E_{i\bar{j}k},$$

with $A_{i\bar{j}k} := \hat{g}_{i\bar{j},k},\ B_{i\bar{j}k} := H_{i\bar{j}k}|z|^{2\beta},\ D_{i\bar{j}k} := \beta(\delta_{i1}H_{k\bar{j}} + \delta_{k1}H_{i\bar{j}})|z|^{2\beta-2}\bar{z}$ and $E_{i\bar{j}k} := \beta^2(\beta-1)\delta_{i1}\delta_{\bar{j}1}\delta_{k1}|z|^{2\beta-4}\bar{z}$. Denote $A = [A_{i\bar{j}k}]$ and similarly B, D, E. Using (97),

$$\langle D, E\rangle \le C\left|\sum_j g^{1\bar{j}}|\eta^1|^2\overline{\nu^1}\rho^{2\beta-1}\rho^{2\beta-3}\right| \le C\rho^{1-\beta},$$

and similarly we conclude that $\|[g_{i\bar{j},k}]\|^2 \le C + \|A+E\|^2$. Now, since $\|\frac{1}{\sqrt{\varepsilon}}A - \sqrt{\varepsilon}E\|^2 \ge 0$, we obtain $\|A+E\|^2 \le (1+\frac{1}{\varepsilon})\|A\|^2 + (1+\varepsilon)\|E\|^2$. Note now that by (98),

$$\|E\|^2 = g^{1\bar{1}}|E_{1\bar{1}1}|^2|\eta^1|^2|\nu^1|^2 \leq C + \frac{\beta^2(1-\beta)^2}{1+b(p)\rho^{2-2\beta}}\rho^{2\beta-4}|\eta^1|^2|\nu^1|^2.$$

Thus, letting $\varepsilon = \varepsilon(p) = b(p)\rho^{2-2\beta}$, we will have proved the lemma provided we can bound $(1+\rho^{2\beta-2})\|A\|^2$. Now, by (98) and Lemma A.2(ii),

$$\rho^{2\beta-2}\|A\|^2 = \sum_{i,k,p,r} \rho^{2\beta-2}\hat{g}_{i\bar{1},k}\hat{g}_{1\bar{j},\bar{\ell}}g^{1\bar{1}}\eta^i\overline{\eta^j}\nu^k\overline{\nu^\ell} \leq C.$$

This concludes the proof of Lemma A.3. □

Appendix B. A local third derivative estimate (after Tian)

A general result due to Tian [57], proved in his M.Sc. thesis, gives a local *a priori* estimate in $W^{3,2}$ for solutions of both real and complex Monge–Ampère equations under the assumption that the solution has bounded real or complex Hessian and the right-hand side is at least Hölder. By the classical integral characterization of Hölder spaces this implies a uniform Hölder estimate on the Laplacian. This result can be seen as an alternative to the Evans–Krylov theorem (and, in fact, appeared independently around the same time).

We present a very special case of this here that applies, in particular, to $\varphi(s)$ along the Ricci continuity path (30). Unlike Calabi's estimates, this local estimate does not require curvature bounds on the reference geometry (which works only when $\beta < 1/2$ [15]). The argument here is an immediate adaptation of [57] to the complex edge setting and is based entirely on the presentation in [57] and Tian's unpublished notes [64]. He understood the applicability of this method in the edge setting for some time and had described this in various courses and lectures over the years.

Theorem B.1 (Tian [57], [64]). *Let $\varphi(s) \in \mathcal{D}^{0,0}_w \cap \mathcal{C}^4(M\setminus D)\cap \mathrm{PSH}(M,\omega)$ be a solution to* (30), *with $s > S$ and $0<\beta<1$. For any $\gamma \in (0,\beta^{-1}-1)\cap(0,1)$, there are constants $r_0 \in (0,1)$ and $C>0$ such that for any $x\in M$ and $0<a<r_0$,*

$$\int_{B_a(x)} |\nabla\omega_\varphi|^2\,\omega^n \;\leq\; C\,a^{2n-2+2\gamma}, \tag{101}$$

where $B_a(x)$ denotes the geodesic ball with center x and radius a, ∇ the covariant derivative and $|\cdot|$ the norm, all taken with respect to ω_β (3). *The constant C depend only on $\gamma,\beta,\omega,n,||\Delta_\omega\varphi||_{L^\infty(M)}$ and $||\varphi||_{L^\infty(M)}$.*

For the proof, we may assume that $x \in D$ and fix some neighborhood U of x in M. We will also always assume $1/2<\beta<1$ purely for simplicity of notation. Setting $t=1$ in (87),

$$\log\det[u_{i\bar{j}}] = f_\omega - s\varphi + \log\det[\psi_{i\bar{j}}] =: \log h, \tag{102}$$

and differentiating twice, multiplying by h and using that $(hu^{i\bar{j}})_i = 0$ (this, in turn, uses that $h = \det u_{i\bar{j}}$) yields

$$-\,hu^{i\bar{s}}u^{t\bar{j}}u_{t\bar{s},\bar{l}}u_{i\bar{j},k} + (hu^{i\bar{j}}u_{k\bar{l},i})_{\bar{j}} = h_{k\bar{l}} - h_k h_{\bar{l}}/h, \tag{103}$$

Combining Lemmas 7.2 and 8.2(iv) yields a uniform bound for the right-hand side of (103). (In fact, even with weaker bounds on h one could replace terms of order a^{2n} that appear later by terms of order $a^{2n-\delta}$ and still run the argument with $a^{-\delta} + |\nabla\omega|$ instead of $1 + |\nabla\omega|$.) Here all the derivatives are with respect to $\zeta, z_2, \ldots, z_n$, equivalently, covariant derivatives with respect to ω_β defined in (3).

Define $\mathbf{B}_\beta(R) \subset U$ to be the domain in $\mathbb{C} \times \mathbb{C}^{n-1}$ consisting of all (ζ, Z), where $Z = (z_2, \cdots, z_n)$, satisfying $|\zeta|^2 + |Z|^2 \le R^2$; recall $\zeta = re^{\sqrt{-1}\beta\theta}, r \in [0, R], \theta \in [0, 2\pi]$. We often identify $\mathbf{B}_\beta(R) \subset U$ with the standard ball B_R in $\mathbb{C}^n$.

Lemma B.2.

(i) *Let h be a harmonic function on $\mathbf{B}_\beta(1)$ such that*

$$\begin{aligned} h(re^{\sqrt{-1}2\pi\beta}, Z) &= e^{\sqrt{-1}2\pi(1-\beta)}h(r, Z), \\ \partial_{z_1}h(re^{\sqrt{-1}2\pi\beta}, Z) &= e^{\sqrt{-1}2\pi(1-\beta)}\partial_{z_1}h(r, Z), \ ||dh||_{L^2(\mathbf{B}_\beta(1),\omega_\beta)} < \infty. \end{aligned} \tag{104}$$

Then for any $a < 1$, there is a constant $C = C(\beta, n)$,

$$||dh||^2_{L^2(\mathbf{B}_\beta(a),\omega_\beta)} \le Ca^{2n-4+2\beta^{-1}}||dh||^2_{L^2(\mathbf{B}_\beta(1),\omega_\beta)}. \tag{105}$$

(ii) *Let f be a smooth function on $\mathbf{B}_\beta(1)$ satisfying* (104). *Then for some $C = C(\beta, n)$,*

$$||f||^2_{L^2(\mathbf{B}_\beta(1),\omega_\beta)} \le C||df|^2_{L^{\frac{2n}{n+1}}(\mathbf{B}_\beta(1),\omega_\beta)}. \tag{106}$$

This lemma can be proved by standard methods (e.g., Sobolev embedding $L^2 \subset W^{1,\frac{2n}{n+1}}$, separation of variables and consideration of the indicial roots associated to the harmonic functions $z_1^{1-\beta+k} = \zeta^{\frac{k+1}{\beta}-1}$, with $k = 0, 1, \ldots$; the exponent $2n - 4 + 2\beta^{-1}$ is sharp, corresponding to the first indicial root of the problem, i.e., the harmonic function $\zeta^{\frac{1-\beta}{\beta}}$). The boundary condition (104) corresponds to the $d\zeta$-coefficient of a smooth 1-forms written in the $\zeta, z_2, \ldots, z_n$ coordinates. To be more specific, in our application, we will consider a smooth 1-form defined on a neighborhood in M of a point $p \in D$ and write this 1-form with respect to the aforementioned coordinates. The $d\zeta$-coefficient of this 1-form is then multivalued. Choosing any branch, the coefficient is a function on the wedge $\mathbf{B}_\beta(R)$ that satisfies (104).

The lemma above is the only place where we need to modify [57]. The rest of the proof below uses arguments identical to those of [57, §2]. Since the proof was originally written for real Monge–Ampère equations, we write out details here for the complex Monge–Ampère equation. (This involves purely a change in notation.)

LEMMA B.3 ([57, Lemma 2.2]). *Let $\lambda_{i\bar{i}}, i = 1, \ldots, n$ be positive numbers. Then,*

$$\left| \sum_k u_{k\bar{k}} \Pi_{i\neq k} \lambda_{i\bar{i}} - \det[u_{p\bar{q}}] - (n-1)\Pi_i \lambda_{i\bar{i}} \right| \leq C \sum_{i,j} |u_{i\bar{j}} - \delta_{ij}\lambda_{i\bar{j}}|^2, \tag{107}$$

where C is a constant depending only on $\lambda_{i\bar{i}}, u_{i\bar{j}}, i, j = 1, \ldots, n$.

Proof. First, by using the homogeneity and positivity of $[u_{i\bar{j}}]$, it suffices to prove the case when $[u_{i\bar{j}}] = [\delta_{ij}] = I$. Next, if we denote the left side of (107) by $f(\lambda_{1\bar{1}}, \ldots, \lambda_{n\bar{n}})$, then by a direct computation, $f(I) = 0$, $\frac{\partial f}{\partial \lambda_{i\bar{i}}}(I) = 0$ for $i = 1, \cdots, n$. Then (107) follows from the Taylor expansion of f at I. $\square$

LEMMA B.4. [57, Lemma 2.3] *There are some uniform constants $q > 2$ and $C > 0$, depending only on $\beta, n, \omega, ||u_{i\bar{j}}||_{L^\infty}, ||h_{i\bar{j}}||_{L^\infty}$, $i, j = 1, \ldots, n$, such that for any $B_{2a}(y) \subset U$,*

$$||1 + |\nabla\omega|^2||_{L^{q/2}(B_a(y),\omega_\beta)} \leq Ca^{2n(-1+2/q)}||1 + |\nabla\omega|^2||_{L^1(B_{2a}(y),\omega_\beta)}. \tag{108}$$

Proof. First we assume $y = x$. Set

$$\lambda_{i\bar{j}} := a^{-2n} \int_{B_a(x)} u_{i\bar{j}}\, \omega_\beta^n, \qquad i, j = 1, \ldots, n. \tag{109}$$

By using unitary transformations if necessary, we may assume $\lambda_{i\bar{j}} = 0$ for any $i \neq j$ and $i, j \geq 2$. Let $C > 0$ be such that $C^{-1}I \leq \Lambda = [\lambda_{i\bar{j}}] \leq CI$. Choose a radial cut-off function $\eta : B_a(x) \to \mathbb{R}_+$ equal to 1 on $B_{3a/4}(x)$ and supported in $B_{4a/5}(x)$, such that $|\eta''| \leq C/a^2$. Multiplying (103) by η and integrating by parts gives

$$\begin{aligned} c\int_{B_a(x)} &\eta|\nabla\omega_\varphi|^2\omega_\beta^n - Ca^{2n} \\ &\leq \int_{B_a(x)} hu^{i\bar{j}} \left(\sum_{k=1}^n u_{k\bar{k}}\Pi_{i\neq k}\lambda_{i\bar{i}} - h - (n-1)\Pi\lambda_{i\bar{i}} \right) \eta_{i\bar{j}}\, \omega_\beta^n. \end{aligned}$$

Thus,

$$\int_{B_{\frac{3a}{4}}(x)} |\nabla\omega_\varphi|^2\, \omega_\beta^n \leq C \left(a^{2n} + \int_{B_a(x)} \sum_{i,j=1}^n |u_{i\bar{j}} - \lambda_{i\bar{i}}\delta_{ij}|^2\, \omega_\beta^n \right),$$

by Lemma B.3 applied to the matrix $\mathrm{diag}(\lambda_{1\bar{1}}, \ldots, \lambda_{n\bar{n}})$.

Applying Lemma B.2(ii) to the terms $u_{1\bar{j}}$ and the usual Sobolev inequality to the term $u_{1\bar{1}} - \lambda_{1\bar{1}}$ and the terms $u_{i\bar{j}} - \lambda_{i\bar{i}}\delta_{ij}, i, j \geq 2$, it follows that $||1 + |\nabla\omega_\varphi|^2||_{L^1(B_{\frac{3a}{4}}(x),\omega_\beta)} \leq Ca^{-2}||1 + |\nabla\omega_\varphi|^2||_{L^{\frac{n}{n+1}}(B_a(x),\omega_\beta)}$. This inequality still holds if we replace $B_a(x)$ by any $B_a(y)$ that is disjoint from the singular set $\{z_1 = 0\}$. This can be proved by using the same arguments (without Lemma B.2(ii)). From this and a covering argument, then

$$||1 + |\nabla\omega_\varphi|^2||_{L^1(B_a(y),\omega_\beta)} \le Ca^{-2}||1 + |\nabla\omega_\varphi|^2||_{L^{\frac{n}{n+1}}(B_{2a}(y),\omega_\beta)}$$

for any ball $B_{2a}(y) \subset U$. Then (108) follows from Gehring's inverse Hölder inequality [27, Lemma 3]. □

LEMMA B.5 ([57, Lemma 2.4]). *For any $B_{4a}(y) \subset U$ and $\sigma < a$, we have*

$$\begin{aligned}||\nabla\omega_\varphi||^2_{L^2(B_\sigma(y),\omega_\beta)} &- Ca^{2n}\\ &\le C\left[\left(\frac{\sigma}{a}\right)^{2n-4+\frac{2}{\beta}} + a^{(2n-2)(\frac{2}{q}-1)}||\nabla\omega_\varphi||^{\frac{2(q-2)}{q}}_{L^2(B_a(y),\omega_\beta)}\right]||\nabla\omega_\varphi||^2_{L^2(B_{2a}(y),\omega_\beta)}.\end{aligned}$$

The dependence of C is as in the previous lemma.

Proof. Let v be the unique (1,1)-form on $B_a(y)$ solving

$$\sum_{k=1}^{n} \Pi_{i\neq k}\lambda_{i\bar{i}}v_{k\bar{k}} = 0 \text{ on } B_a(y),\ v = \omega_\varphi \text{ on } \partial B_a(y). \tag{110}$$

We emphasize that here $v_{k\bar{k}} = \nabla_k\nabla_{\bar{k}}v$ denotes covariant derivatives with respect to ω_β of the full $(1,1)$-form v. Set $\hat{\omega} := \omega_\varphi - v$. Then,

$$||\nabla\omega_\varphi||^2_{L^2(B_\sigma(y),\omega_\beta)} \le 2||\nabla\hat{\omega}||^2_{L^2(B_\sigma(y),\omega_\beta)} + 2||\nabla v||^2_{L^2(B_\sigma(y),\omega_\beta)}\ .$$

Note that v is harmonic with respect to a constant coefficient metric equivalent to ω_β. Thus, $||\nabla v||_{L^2(B_a(y),\omega_\beta)} \le C||\nabla\omega_\varphi||_{L^2(B_a(y),\omega_\beta)}$, and applying Lemma B.2(i) to v (or more precisely to each of the components $v_{1\bar{j}}, v_{i\bar{1}}, i, j \ge 2$) gives

$$\begin{aligned}||\nabla\omega_\varphi||^2_{L^2(B_\sigma(y),\omega_\beta)} &\le 2||\nabla\hat{\omega}||^2_{L^2(B_\sigma(y),\omega_\beta)} + 2C\,(\sigma/a)^{2n-4+2\beta^{-1}}\,||\nabla v||^2_{L^2(B_a(y),\omega_\beta)}\\ &\le 2||\nabla\hat{\omega}||^2_{L^2(B_\sigma(y),\omega_\beta)}\\ &\quad + 2C'\,(\sigma/a)^{2n-4+2\beta^{-1}}\,||\nabla\omega_\varphi||^2_{L^2(B_a(y),\omega_\beta)}.\end{aligned} \tag{111}$$

It remains to estimate the first term on the right-hand side. Similarly to before, multiplying (102) by $\hat{\omega}\wedge\omega_\beta^{n-1}$ and integrating by parts,

$$\int_{B_r(y)} |\nabla\hat{\omega}|^2\omega_\beta^n \le C\Big(r^{2n} + \int_{B_r(y)} (|\hat{\omega}| + \sum_{i,j} |u_{i\bar{j}} - \lambda_i\delta_{ij}|^2)(1 + |\nabla\omega_\varphi|^2)\omega_\beta^n\Big). \tag{112}$$

By using Lemma B.4 and the Poincaré inequality (the usual one with matching boundary data $f(re^{\sqrt{-1}2\pi\beta}, Z) = f(r, Z)$ as well as Lemma B.2(ii)),

$$\begin{aligned}
&|| \, |u_{i\bar{j}} - \lambda_i \delta_{ij}|^2 (1 + |\nabla\omega_\varphi|^2) ||_{L^1(B_a(y),\omega_\beta)} \\
&\quad \le || 1 + |\nabla\omega_\varphi|^2 \, ||_{L^{q/2}(B_a(y),\omega_\beta)} || \, |u_{i\bar{j}} - \lambda_i\delta_{ij}| \, ||^2_{L^{\frac{2q}{q-2}}(B_a(y),\omega_\beta)} \\
&\quad \le C a^{\frac{q-2}{q}(2-2n)} ||1 + |\nabla\omega_\varphi|^2||_{L^1(B_{2a}(y),\omega_\beta)} ||u_{i\bar{j}} - \lambda_i\delta_{ij}||^{\frac{2(q-2)}{q}}_{L^2(B_a(y),\omega_\beta)} \\
&\quad \le C a^{\frac{q-2}{q}(2-2n)} ||1 + |\nabla\omega_\varphi|^2||_{L^1(B_{2a}(y),\omega_\beta)} ||1 + |\nabla\omega_\varphi|^2||^{\frac{q-2}{q}}_{L^1(B_a(y),\omega_\beta)}.
\end{aligned}$$

Without loss of generality, we may assume that $q \ge 2(q-2)$. Since $\hat{\omega}$ vanishes on $\partial B_r(y)$, its L^2-norm is controlled by the $L^2(B_a(y), \omega_\beta)$-norm of $|\nabla\hat{\omega}|$ and, consequently, of $|\nabla\omega_\varphi|$ (as $||\nabla\hat{\omega}||^2_{L^2} \le 2||\nabla\omega_\varphi||^2_{L^2} + 2||\nabla v||^2_{L^2} \le 2(C+1)||\nabla\omega_\varphi||^2_{L^2}$). Also, $\hat{\omega}$ is uniformly bounded in L^∞ as both ω_φ (by the Laplacian estimate) and v (by the maximum principle) are; thus its L^2 norm is equivalent to its $L^{\frac{q}{q-2}}$ norm. Then,

$$\begin{aligned}
&|| \, |\hat{\omega}| (1 + |\nabla\omega_\varphi|)^2 ||_{L^1(B_a(y),\omega_\beta)} \\
&\quad \le ||1 + |\nabla\omega_\varphi|^2||_{L^{q/2}(B_a(y),\omega_\beta)} ||\hat{\omega}||_{L^{\frac{q}{q-2}}(B_a(y),\omega_\beta)} \\
&\quad \le C a^{2n(-1+2/q)} ||1 + |\nabla\omega_\varphi|^2 \, ||_{L^1(B_{2a}(y),\omega_\beta)} ||\hat{\omega}||^{\frac{2(q-2)}{q}}_{L^2(B_a(y),\omega_\beta)} \\
&\quad \le C a^{2n(-1+2/q)} ||1 + |\nabla\omega_\varphi|^2 \, ||_{L^1(B_{2a}(y),\omega_\beta)} a^{\frac{2(q-2)}{q}} ||\nabla\hat{\omega}||^{\frac{2(q-2)}{q}}_{L^2(B_a(y),\omega_\beta)} \\
&\quad \le C a^{(2n-2)(-1+2/q)} ||1 + |\nabla\omega_\varphi|^2 \, ||_{L^1(B_{2a}(y),\omega_\beta)} ||\nabla\omega_\varphi||^{\frac{2(q-2)}{q}}_{L^2(B_a(y),\omega_\beta)}.
\end{aligned}$$

Combining all the estimates above concludes the proof. □

The next lemma gives an estimate on the smallness of the coefficient in the right-hand side of the previous lemma.

LEMMA B.6. [57, Lemma 2.5] *For any $\varepsilon_0 > 0$, there is an ℓ depending only on ε_0, $||\Delta u||_{L^\infty}$ and $||h_{i\bar{j}}||_{L^\infty}$ satisfying: For any $a > 0$ with $B_a(y) \subset U$, there is $\sigma \in [2^{-\ell}a, 2^{-1}a]$ such that*

$$||\nabla\omega_\varphi||^2_{L^2(B_\sigma(y),\omega_\beta)} \le \varepsilon_0 \sigma^{2n-2}. \tag{113}$$

Proof. From (102),

$$\Delta_{\omega_\varphi} \Delta u = \sum_k u^{i\bar{j}} u^{p\bar{q}} u_{i\bar{q}k} u_{\bar{j}p\bar{k}} + \Delta \log h, \tag{114}$$

where Δ denotes, as before, the Laplacian of ω_β. Let η be a nonnegative radial function on $B_a(y)$ equal to 1 on $B_{a/2}(y)$, supported in $B_{3a/4}(y)$, and such that

$a^2|\eta''| \le C$. Let $M_a := \sup_{B_a(y)} \Delta u$. Then,

$$\int_{B_a(y)} \eta |\nabla \omega_\varphi|^2 \, \omega_\beta^n \le C a^{2n} - \int_{B_a(y)} \Delta_{\omega_\varphi} \eta \, (M_a - \Delta u) \, \omega^n$$
$$\le C \, a^{2n} + C a^{-2} \int_{B_a(y)} (M_a - \Delta u) \, \omega_\beta^n.$$

From (114), there exists $c > 0$, such that $Z := M_a - \Delta u - c a^2$ satisfies $\Delta_{\omega_\varphi} Z \le 0$. Thus,

$$\frac{1}{C} a^{-2n} \int_{B_a(y)} Z \, \omega^n \le \inf_{B_{a/2}(y)} Z + a^2$$

by [28, Th. 8.18]. Thus, $a^{2-2n} ||\nabla \omega_\varphi||^2_{L^2(B_{a/2}(y), \omega_\beta)} \le C(M_a - M_{a/2} + a^2)$. Hence, if (113) does not hold for $\sigma = 2^{-1}a, \cdots, 2^{-k}a$, then $(k-1)\varepsilon_0 \le C \, (M_a - M_{2^{-k}a} + 2a^2)$. This is impossible if k is sufficiently large. $\square$

We now complete the proof of Theorem B.1. Using the previous two lemmas, there exist uniform $\chi, \lambda \in (0,1)$ such that

$$(\lambda a)^{\frac{2}{\beta}-2} + (\lambda a)^{2-2n} ||\nabla \omega_\varphi||^2_{L^2(B_{\lambda a}(y), \omega_\beta)} \le \chi \Big[a^{\frac{2}{\beta}-2} + a^{2-2n} ||\nabla \omega_\varphi||^2_{L^2(B_a(y), \omega_\beta)} \Big].$$

Thus, as in Section 8, from [28, Lemma 8.23] it follows that there exists some $\gamma \in (0, \frac{1}{\beta} - 1)$ for which (101) holds, which is sufficient for the purposes of this article. (It is easy to see that in fact χ, λ can be chosen to give (101) even for all $\gamma \in (0, \frac{1}{\beta} - 1)$.) This concludes the proof.

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