Annals of Mathematics **183** (2016), 1–71
http://dx.doi.org/10.4007/annals.2016.183.1.1

Hasse principles for higher-dimensional fields

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dedicated to Jürgen Neukirch

Abstract

For rather general excellent schemes X, K. Kato defined complexes of Gersten-Bloch-Ogus type involving the Galois cohomology groups of all residue fields of X. For arithmetically interesting schemes, he developed a fascinating web of conjectures on some of these complexes, which generalize the classical Hasse principle for Brauer groups over global fields, and proved these conjectures for low dimensions. We prove Kato's conjecture over number fields in any dimension. This gives a cohomological Hasse principle for function fields F over a number field K, involving the corresponding function fields F_v over the completions K_v of K. For global function fields K we prove the part on injectivity for coefficients invertible in K. Assuming resolution of singularities, we prove a similar conjecture of Kato over finite fields, and a generalization to arbitrary finitely generated fields.

Contents

0. Introduction

In this paper we prove some conjectures of K. Kato [Kat86] which were formulated to generalize the classical exact sequence of Brauer groups for a global field K,

$$0 \longrightarrow \mathrm{Br}(K) \longrightarrow \bigoplus_v \mathrm{Br}(K_v) \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0, \tag{0.1}$$

to function fields F over K and varieties X over K. In the above sequence, which is also called the Hasse-Brauer-Noether sequence, the sum runs over all places v of K, and K_v is the completion of K with respect to v. The injectivity of the restriction map into the sum of local Brauer groups is called the Hasse principle.

Kato's generalization does not concern Brauer groups but rather the following cohomology groups. Let L be any field, and let $n > 0$ be an integer. Define the following Galois cohomology groups for $i, j \in \mathbb{Z}$:

(0.2)
$$H^i(L, \mathbb{Z}/n\mathbb{Z}(j)) := \begin{cases} H^i(L, \mu_n^{\otimes j}), & \mathrm{char}(L) = 0, \\ H^i(L, \mu_m^{\otimes j}) \oplus H^{i-j}(L, W_r\Omega^j_{L,\log}), & \mathrm{char}(L) = p > 0, n = mp^r, p \nmid m, \end{cases}$$

where μ_m is the Galois module of m-th roots of unity (in the separable closure L^{sep} of L) and $W_r\Omega^j_{L,\log}$ is the logarithmic part of the de Rham-Witt sheaf $W_r\Omega^j_L$ [Ill79, I 5.7] (an étale sheaf, regarded as a Galois module). It is a fact that $\mathrm{Br}(L)[n] = H^2(L, \mathbb{Z}/n\mathbb{Z}(1))$, where $A[n] = \{ x \in A \mid nx = 0\}$ denotes the n-torsion in an abelian group A, so the n-torsion of the sequence (0.1) can be identified with an exact sequence

$$0 \longrightarrow H^2(K, \mathbb{Z}/n\mathbb{Z}(1)) \longrightarrow \bigoplus_v H^2(K_v, \mathbb{Z}/n\mathbb{Z}(1)) \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0. \tag{0.3}$$

In fact, this sequence is often used for the Galois cohomology of number fields, independently of Brauer groups; it is closely related to class field theory and Tate-Poitou duality.

For the generalization, let F be a function field in d variables over a global field K and assume F/K is primary, i.e., that K is separably closed in F. For each place v of K, let F_v be the corresponding function field over K_v: If $F = K(V)$, the function field of a geometrically integral variety V over K, then $F_v = K_v(V \times_K K_v)$. Then Kato [Kat86] conjectured

Conjecture 1. *The following restriction map is injective*:
$$\alpha_n : H^{d+2}(F, \mathbb{Z}/n\mathbb{Z}(d+1)) \longrightarrow \bigoplus_v H^{d+2}(F_v, \mathbb{Z}/n\mathbb{Z}(d+1)).$$

Note that this generalizes the injectivity in (0.3), which is the case $d = 0$ and $F = K$. On the other hand it is known that the corresponding restriction map for Brauer groups is not, in general, injective for $d \geq 1$: If X is a smooth projective curve over a number field which has a K-rational point, then for $F = K(X)$, the kernel of $\mathrm{Br}(F) \to \prod_v \mathrm{Br}(F_v)$ is isomorphic to the Tate-Shafarevich group of the Jacobian $\mathrm{Jac}(X)$. Kato [Kat86] proved Conjecture 1 for $d = 1$. Here we prove

Theorem 0.4. *Conjecture* 1 *is true if* n *is invertible in* K.

The proof uses three main ingredients. First we prove the analogue for infinite coefficients. For a field L, a prime ℓ, and integers i and j, we let

$$H^i(L, \mathbb{Q}_\ell/\mathbb{Z}_\ell(j)) = \varinjlim H^i(L, \mathbb{Z}/\ell^n\mathbb{Z}(j)), \tag{0.5}$$

where the inductive limit is taken via the obvious monomorphisms $\mathbb{Z}/\ell^n\mathbb{Z}(j) \hookrightarrow \mathbb{Z}/\ell^{n+1}(j)$. Then we prove (see Theorem 2.10)

THEOREM 0.6. *Let K be a global field, let ℓ be a prime invertible in K, and let F be a function field in d variables over K such that F/K is primary. Then the restriction map*

$$\alpha_{\ell^\infty} : \; H^{d+2}(F, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d+1)) \longrightarrow \bigoplus_v H^{d+2}(F_v, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d+1))$$

is injective.

For number fields and $d = 2$, this result was already proved in [Jan92]. Concerning the case of finite coefficients, i.e., the original Conjecture 1, we use the following. For any field L, any prime ℓ, and any integer $t \geq 0$, there is a symbol map

$$h^t_{L,\ell} : K^M_t(L)/\ell \longrightarrow H^t(L, \mathbb{Z}/\ell\mathbb{Z}(t)),$$

where $K^M_t(L)$ denotes the t-th Milnor K-group of L ([Mil70] and [BK86, §2]). Extending an earlier conjecture of Milnor [Mil70] for $\ell = 2 \neq \mathrm{char}(L)$, Bloch and Kato stated the following conjecture:

BK(L, t, ℓ): *The map $h^t_{L,\ell}$ is an isomorphism.*

This conjecture was proved in recent years. In fact, for $\ell = \mathrm{char}(L)$ it was proved by Bloch, Gabber, and Kato [BK86], and for $\ell \neq \mathrm{char}(L)$ it is classical for $t = 1$ (Kummer theory), was proved for $t = 2$ by Merkurjev and Suslin [MS83]), for $\ell = 2$ by Voevodsky [Voe03], and for arbitrary t and ℓ by work of Rost and Voevodsky (see [Ros02], [SJ06], [Voe11], [Voe10], [HW09]).

Property BK$(F, d+1, \ell)$ for all ℓ dividing n allows us to deduce Theorem 0.4 from Theorem 0.6 for all ℓ dividing n as follows. One has the exact cohomology sequence

$$\begin{aligned} H^{d+1}(F, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d+1)) \xrightarrow{\ell^m} H^{d+1}(F, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d+1)) \to H^{d+2}(F, \mathbb{Z}/\ell^m(d+1)) \\ \xrightarrow{i} H^{d+2}(F, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d+1)), \end{aligned}$$

and it follows from BK(F, t, ℓ) that $H^{d+1}(F, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d+1))$ is divisible. Therefore i is injective, and this shows that the injectivity of α_{ℓ^∞} in Theorem 0.4 implies the injectivity of α_{ℓ^m} in Conjecture 1. It should be noted that Kato did in fact use BK$(K, 2, \ell)$, i.e., the Merkurjev-Suslin theorem, in his proof of Conjecture 1 for $d = 1$.

Finally the proof of Theorem 0.6 uses weights, i.e., Deligne's proof of the Weil conjectures, and some results on resolution of singularities, to control the weights. Over number fields the required resolution of singularities holds by work of Hironaka. For ℓ invertible in K, we observe that a weaker form of resolution suffices. More precisely we use alterations, as introduced by de Jong, but in a refined version established by Gabber; see [ILO14] or Theorem 2.11 below.

As in the classical case $d = 0$ and the case of $d = 1$ (see the appendix to [Kat86]) and the case of $d = 2$ in [Jan92], Theorem 0.4 has applications to quadratic forms over F (see [CTJ91]).

Corollary 0.7. *If F is a finitely generated field of characteristic zero, then the Pythagoras number of F is finite. More precisely, if F is of transcendence degree d over $\mathbb{Q}$, then any sum of squares in F is a sum of 2^{d+1} squares, provided $d \geq 2$.*

The proof uses the following instance of Theorem 0.4, which only needs the proof of the Milnor conjecture, i.e., the theorem of Voevodsky in [Voe03].

Corollary 0.8. *The restriction map*

$$H^{d+2}(F, \mathbb{Z}/2\mathbb{Z}) \longrightarrow \bigoplus_v H^{d+2}(F_v, \mathbb{Z}/2\mathbb{Z}).$$

is injective.

It should be mentioned that the finiteness of the Pythagoras number, with the weaker bound 2^{d+2}, can be obtained by some more elementary means, still using the Milnor conjecture [Pfi00].

Kato also stated a conjecture on the cokernel of the above restriction maps, in the following way. Let L be a global or local field, let X be any variety over L, and let n be an integer. Then in [Kat86] Kato defined a certain homological complex $C^{2,1}(X, \mathbb{Z}/n\mathbb{Z})$ of Galois cohomology groups:

$$\begin{aligned} \cdots \longrightarrow \bigoplus_{x \in X_a} H^{a+2}(k(x), \mathbb{Z}/n\mathbb{Z}(a+1)) \bigoplus_{x \in X_{a-1}} H^{a+1}(k(x), \mathbb{Z}/n\mathbb{Z}(a)) \\ \longrightarrow \cdots \longrightarrow \bigoplus_{x \in X_1} H^3(k(x), \mathbb{Z}/n\mathbb{Z}(2)) \longrightarrow \bigoplus_{x \in X_0} H^2(k(x), \mathbb{Z}/n\mathbb{Z}(1)). \end{aligned}$$

Here X_a denotes the set of points $x \in X$ of dimension a, the term involving X_a is placed in degree a, and $k(x)$ denotes the residue field of x. A complex of the same shape can also be defined via the method of Bloch and Ogus, and it is shown in [JSS14] that these two definitions agree up to (well-defined) signs. (Also see Section 4 for a discussion of more general complexes $C^{a,b}(X, \mathbb{Z}/n\mathbb{Z})$.)

Now let K be a global field, and let X be a variety over K. Then there are obvious maps of complexes $C^{2,1}(X, \mathbb{Z}/n\mathbb{Z}) \to C^{2,1}(X_v, \mathbb{Z}/n\mathbb{Z})$ for each place v of K, where $X_v = X \times_K K_v$, and these induce a map of complexes

$$\alpha_{X,n} : C^{2,1}(X, \mathbb{Z}/n\mathbb{Z}) \longrightarrow \bigoplus_v C^{2,1}(X_v, \mathbb{Z}/n\mathbb{Z}).$$

Then Kato [Kat86] conjectured the following.

CONJECTURE 2. *Let K be a global field, let $n > 0$ be an integer, and let X be a connected, smooth proper variety over K. Then the above map induces isomorphisms*

$$H_a(C^{2,1}(X, \mathbb{Z}/n\mathbb{Z})) \xrightarrow{\sim} \bigoplus_v H_a(C^{2,1}(X_v, \mathbb{Z}/n\mathbb{Z}))$$

for $a > 0$, and an exact sequence

$$0 \longrightarrow H_0(C^{2,1}(X, \mathbb{Z}/n\mathbb{Z})) \longrightarrow \bigoplus_v H_0(C^{2,1}(X_v, \mathbb{Z}/n\mathbb{Z})) \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0.$$

Note that we obtain the sequence (0.3) for $X = \mathrm{Spec}(K)$, where the complexes are concentrated in degree zero. Kato [Kat86] proved Conjecture 2 for $d = 1$. Here we prove (see Theorems 4.8 and 4.19).

THEOREM 0.9. *Conjecture 2 is true if K is a number field or if n is invertible in K and resolution of singularities* (*see Definition* 4.18) *holds over K. More precisely, in this case there is an exact sequence of complexes*

$$0 \to C^{2,1}(X, \mathbb{Z}/n\mathbb{Z}) \longrightarrow \bigoplus_v C^{2,1}(X_v, \mathbb{Z}/n\mathbb{Z}) \longrightarrow C'(X, \mathbb{Z}/n\mathbb{Z}) \to 0$$

with $H_0(C'(X, \mathbb{Z}/n\mathbb{Z})) = \mathbb{Z}/n\mathbb{Z}$, and $H_a(C'(X, \mathbb{Z}/n\mathbb{Z})) = 0$ for $a > 0$.

Again this version is deduced from a version with infinite coefficients by using the property $\mathrm{BK}(L, d+1, \ell)$ (for all residue fields of X and all ℓ dividing n), and the version with infinite coefficients is proved using weight arguments and resolution of singularities.

For global fields K of positive characteristic, Kerz and Saito [KS12] proved the same result unconditionally, by using Theorem 0.4, and the weaker result on resolution of singularities proved by Gabber, quoted above. An alternative proof, still using Gabber's result, can be found in [Jan09].

Our techniques also allow us to get results on another conjecture of Kato, over finite fields. For any variety over a finite field k and any natural number n, Kato considered a complex $C^{1,0}(X, \mathbb{Z}/n\mathbb{Z})$ which is of the form

$$\cdots \longrightarrow \bigoplus_{x \in X_a} H^{a+1}(k(x), \mathbb{Z}/n\mathbb{Z}(a)) \longrightarrow \bigoplus_{x \in X_{a-1}} H^a(k(x), \mathbb{Z}/n\mathbb{Z}(a-1))$$
$$\longrightarrow \cdots \longrightarrow \bigoplus_{x \in X_1} H^2(k(x), \mathbb{Z}/n\mathbb{Z}(1)) \longrightarrow \bigoplus_{x \in X_0} H^1(k(x), \mathbb{Z}/n\mathbb{Z})$$

with the term involving X_a placed in (homological) degree a. (This is another special case of the general complexes $C^{a,b}(X, \mathbb{Z}/n\mathbb{Z})$.) Kato conjectured the following (where the case $a = 0$ is easy):

Conjecture 3. *If X is connected, smooth, and proper over a finite field k, then one has*

$$H_a(C^{1,0}(X,\mathbb{Z}/n\mathbb{Z})) = \begin{cases} 0, & a \neq 0, \\ \mathbb{Z}/n\mathbb{Z}, & a = 0. \end{cases}$$

For $\dim(X) = 1$, this conjecture amounts to the exact sequence (0.3) with $K = k(X)$, and for $\dim(X) = 2$, the conjecture follows from [CTSS83] for n invertible in k and from [Gro85b] and [Kat86] if n is a power of $\mathrm{char}(k)$. S. Saito [Sai93] proved that $H_3(C^{2,1}(X,\mathbb{Q}_\ell/\mathbb{Z}_\ell)) = 0$ for $\dim(X) = 3$ and $\ell \neq \mathrm{char}(k)$. For X of any dimension, Colliot-Thélène [CT93] (for $\ell \neq \mathrm{char}(k)$) and Suwa [Suw95] (for $\ell = \mathrm{char}(k)$) proved that $H_a(C^{1,0}(X,\mathbb{Q}_\ell/\mathbb{Z}_\ell)) = 0$ for $0 < a \leq 3$. Here we prove the following (see Theorem 4.19 and Lemma 4.20).

Theorem 0.10. *Conjecture* 3 *holds if resolution of singularities holds over k.*

This result also follows from the technique in [JS09]. These techniques show unconditionally that $H_a(C^{1,0}(X,\mathbb{Z}/n\mathbb{Z})) = 0$ for X smooth projective of any dimension, any n, and $0 < a \leq 4$. Moreover, Kerz and Saito [KS12] proved Conjecture 3 for coefficients invertible in k, by using Gabber's weak resolution of singularities quoted above. Another proof can be found in [Jan09]. Finally, Kato also formulated an arithmetic analogue of Conjecture 3, for regular flat proper schemes over $\mathrm{Spec}(\mathbb{Z})$, and in [JS03] some results on this are obtained using Theorem 0.6.

Our method of proof is the same for Theorems 0.9 and 0.10. In fact, under certain conditions, which are always fulfilled in our cases, Kato defined more general complexes $C^{r,s}(X,\mathbb{Z}/n\mathbb{Z})$ of the form

$$\begin{aligned} \cdots \longrightarrow & \bigoplus_{x\in X_a} H^{r+a}(k(x),\mathbb{Z}/n\mathbb{Z}(s+a)) \\ \longrightarrow & \bigoplus_{x\in X_{a-1}} H^{r+a-1}(k(x),\mathbb{Z}/n\mathbb{Z}(s+a-1)) \\ \longrightarrow \cdots \longrightarrow & \bigoplus_{x\in X_1} H^{r+1}(k(x),\mathbb{Z}/n\mathbb{Z}(s+1)) \longrightarrow \bigoplus_{x\in X_0} H^{r}(k(x),\mathbb{Z}/n\mathbb{Z}(s)). \end{aligned}$$

For n invertible in K, we construct a canonical quasi-isomorphism between the complex $C'(X,\mathbb{Z}/n\mathbb{Z})$ in Theorem 0.9 and the complex $C^{0,0}(\overline{X},\mathbb{Z}/n\mathbb{Z})_{G_K}$ obtained from the Kato complex $C^{0,0}(\overline{X},\mathbb{Z}/n\mathbb{Z})$ by taking coinvariants under the absolute Galois group G_K, where $\overline{X} = X \times_K \overline{K}$. On the other hand, for a finite field k, one has a canonical isomorphism $C^{0,0}(\overline{X},\mathbb{Z}/n\mathbb{Z})_{G_k} \cong C^{1,0}(X,\mathbb{Z}/n\mathbb{Z})$ for a variety X over a finite field k. Therefore Theorems 0.9 and 0.10 follow from the following more general result (see Theorem 4.19).

THEOREM 0.11. *Let K be a finitely generated field with algebraic closure $\overline{K}$, let X be a smooth proper variety over K, and let n be natural number. Then*

$$H_a(C^{0,0}(X \times_K \overline{K}, \mathbb{Z}/n\mathbb{Z})_{G_K}) = \begin{cases} \mathbb{Z}/n\mathbb{Z}, & a = 0, \\ 0, & a \neq 0 \end{cases}$$

if resolution of singularities holds over K.

This paper had a rather long evolution time. Theorem 0.6 for number fields was obtained in 1990, rather shortly after the proofs of Theorem 0.6 and Corollary 0.4 for number fields and $d = 2$ in [Jan92]. In 1996, right after the appearance of [GS96], it became clear to me how to obtain Theorem 0.9 (for number fields and infinite coefficients), but a first account was only written in 2004. Meanwhile I had also noticed that these methods allow a proof of Theorem 0.11, i.e., a proof of Kato's conjecture over finite fields, with infinite coefficients, assuming resolution of singularities. Part of the delay was caused by the long time to complete the comparison of Kato's original complexes with the complexes of Gersten-Bloch-Ogus type used here, which was recently accomplished [JSS14].

I dedicate this paper to my teacher and friend Jürgen Neukirch, who helped and inspired me in so many ways by his support and enthusiasm. I also thank Jean-Louis Colliot-Thélène for his long lasting interest in this work, for the discussions on the rigidity Theorems 2.12 and 4.11, and for the proof of Theorem 2.13. Moreover, I thank Wayne Raskind, Florian Pop, Tamás Szamuely and Thomas Geisser for their interest and useful hints and discussions. In establishing the strategy for proving Theorems 0.11 and 0.9, I profited from an incomplete preprint by Michael Spieß. My contact with Shuji Saito started with the subject of this paper, and I thank him for all these years of a wonderful collaboration and the countless inspirations I got from our discussions.

1. **First reductions and a Hasse principle for global fields**

Let K be a global field, and let F be a function field of transcendence degree d over K. We assume that K is separably closed in F. For every place v of K, let K_v be the completion of K at v, and let F_v be the corresponding function field over K_v: there exists a geometrically irreducible variety V of dimension d over K, such that $F = K(V)$, and then $F_v = K_v(V_v)$, where $V_v = V \times_K K_v$. (This is integral, since F/K is primary and K_v/K is separable; see [Gro65, (4.3.2) and (4.3.5)].) This definition does not depend on the choice of V.

Fix a prime $\ell \neq \mathrm{char}(K)$. We want to study the map

$$\mathrm{res}\colon\ H^{d+2}(F, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d+1)) \to \prod_v H^{d+2}(F_v, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d+1))$$

induced by the restrictions from F to F_v. For this it will be useful to first replace the completions K_v by the Henselizations. For each place v of K, denote by $K_{(v)}$ the Henselization of K at v. It can be regarded as a subfield of a fixed separable closure $\overline{K}$ of K, equal to the fixed field of a decomposition group G_v at v. For V as above, let $F_{(v)} = K_{(v)}(V \times_K K_{(v)})$ be the corresponding function field over $K_{(v)}$. Since $K_{(v)}$ is separably algebraic over K and linearly disjoint from F, $F_{(v)}$ is equal to the composite $FK_{(v)}$ in a fixed separable closure $\overline{F}$ of F. We obtain a diagram of fields

(1.1)
$$\begin{tikzcd}[arrows=dash]
 & \overline{F} \\
\overline{K} \arrow[r] & F\overline{K} \arrow[u] \\
K_{(v)} \arrow[r] \arrow[u] & F_{(v)} \arrow[u] \\
K \arrow[r] \arrow[u] & F \arrow[u]
\end{tikzcd}$$

which identifies $G_K = \mathrm{Gal}(\overline{K}/K)$ with $\mathrm{Gal}(F\overline{K}/F)$ and $G_{K_{(v)}} = \mathrm{Gal}(\overline{K}/K_{(v)})$ with $\mathrm{Gal}(F\overline{K}/FK_{(v)})$.

Proposition 1.2. *Let M be a discrete ℓ-primary torsion G_F-module. The restriction map*

$$H^{d+2}(F, M) \to \prod_v H^{d+2}(F_{(v)}, M)$$

has image in the direct sum $\bigoplus_v H^{d+2}(F_{(v)}, M)$. There is a commutative diagram

$$\begin{array}{cccc}
f: & H^{d+2}(F, M) & \longrightarrow & \bigoplus_v H^{d+2}(F_{(v)}, M) \\
 & \downarrow & & \downarrow \\
g: & H^2(K, H^d(F\overline{K}, M)) & \longrightarrow & \bigoplus_v H^2(K_{(v)}, H^d(F\overline{K}, M)),
\end{array}$$

in which the horizontal maps are induced by the restrictions and the vertical maps by the Hochschild-Serre spectral sequences. This diagram is functorial in M and induces canonical isomorphisms

$$\ker(f) \xrightarrow{\sim} \ker(g) \qquad \textit{and} \qquad \mathrm{coker}(f) \xrightarrow{\sim} \mathrm{coker}(g) \cong H^d(F\overline{K}, M)(-1)_{G_K}.$$

Here $N(n)$ denotes the n-fold Tate twist of a ℓ-primary discrete torsion G_K-module N, and N_{G_K} denotes its cofixed module, i.e., the maximal quotient on which G_K acts trivially.

Proof. Diagram (1.1) gives Hochschild-Serre spectral sequences

$$E_2^{p,q}(K) = H^p(K, H^q(F\overline{K}, M)) \Rightarrow H^{p+q}(F, M),$$
$$E_2^{p,q}(K_{(v)}) = H^p(K_{(v)}, H^q(F\overline{K}, M)) \Rightarrow H^{p+q}(F_{(v)}, M).$$

Moreover, for each v we obtain a natural map $E(K) \to E(K_{(v)})$ between the above spectral sequences which gives the restriction maps for $K \subset K_{(v)}$ on the E_2-terms and the restriction maps for $F \subset F_{(v)}$ on the abutment, respectively. On the other hand, the field $F\overline{K}$ has cohomological dimension d, so that $E_2^{p,q}(K) = 0 = E_2^{p,q}(K_{(v)})$ for $q > d$. This gives a commutative diagram

$$\begin{array}{ccc} H^{d+2}(F, M) & \longrightarrow & H^{d+2}(F_{(v)}, M) \\ \downarrow & & \downarrow \\ H^2(K, H^d(F\overline{K}, M)) & \longrightarrow & H^2(K_{(v)}, H^d(F\overline{K}, M)), \end{array}$$

where the vertical maps are edge morphisms of the spectral sequences. If v is not a real archimedean place, or if $\ell \neq 2$, we have $cd_\ell(K_{(v)}) \leq 2$ and, hence, $E_2^{p,q}(K_{(v)}) = 0$ for $p > 2$, and the right vertical edge morphism is an isomorphism. This already shows the first claim of the proposition, since the restriction map

$$H^2(K, N) \to \prod_v H^2(K_{(v)}, N)$$

is known to have image in the direct sum $\bigoplus_v$ for any torsion G_K-module N. If K has no real archimedean valuations (or if $\ell \neq 2$), then $cd_\ell(K) = 2$, the left-hand edge morphism is an isomorphism as well, and the second claim follows. If this is not the case, we use the following lemma.

Lemma 1.3. *If K is a number field, then the above maps between the spectral sequences induce*

(a) *surjections for all $r \geq 2$ and all $p + q = d + 1$,*

$$E_r^{p,q}(K) \twoheadrightarrow \bigoplus_{v|\infty} E_r^{p,q}(K_{(v)});$$

(b) *surjections for all $r \geq 2$,*

$$E_r^{2,d}(K) \twoheadrightarrow \bigoplus_{v|\infty} E_r^{2,d}(K_{(v)});$$

(c) *isomorphisms between the kernels and between the cokernels of the maps*

$$E_r^{2,d}(K) \to \bigoplus_v E_r^{2,d}(K_{(v)}) \quad \text{and} \quad E_{r+1}^{2,d}(K) \to \bigoplus_v E_{r+1}^{2,d}(K_{(v)})$$

for all $r \geq 2$;

(d) *isomorphisms*

$$E_r^{p,q}(K) \xrightarrow{\sim} \bigoplus_{v|\infty} E_r^{p,q}(K_{(v)})$$

for all $r \geq 2$ and all $(p, q) \neq (2, d)$ with $p + q \geq d + 2$.

Proof. By induction on r. Recall that $E_r^{p,q}(K) = 0 = E_r^{p,q}(K_{(v)})$ for all $q > d$ and all $r \geq 2$. Hence, for $r = 2$, the claims (a), (b), and (d) follow from the following well-known facts of global Galois cohomology: the maps

$$H^1(K, N) \to \bigoplus_{v|\infty} H^1(K_{(v)}, N),$$
$$H^2(K, N) \to \bigoplus_{v|S} H^2(K_{(v)}, N)$$

are surjective for any torsion G_K-module N and any finite set S of places, and the maps

$$H^i(K, N) \xrightarrow{\sim} \bigoplus_{v|\infty} H^i(K_{(v)}, N)$$

are isomorphisms for such N and all $i \geq 3$. Note that here we could replace $K_{(v)}$ by the more common completion K_v, since $G_{K_{(v)}} \cong G_v \cong G_{K_v}$.

Now let $r \geq 2$. For (a) look at the commutative diagram

$$\begin{tikzcd}
E_r^{p-r,q+r-1}(K) \arrow[r, "d_r"] \arrow[d, "\alpha"'] & E_r^{p,q}(K) \arrow[r, "d_r"] \arrow[d, "\beta"'] & E_r^{p+r,q-r+1}(K) \arrow[d, "\gamma"'] \\
\bigoplus\limits_{v|\infty} E_r^{p-r,q+r-1}(K_{(v)}) \arrow[r, "d_r"] & \bigoplus\limits_{v|\infty} E_r^{p,q}(K_{(v)}) \arrow[r, "d_r"] & \bigoplus\limits_{v|\infty} E_r^{p+r,q-r+1}(K_{(v)})
\end{tikzcd}$$

coming from the map of spectral sequences. We may assume $p \geq 1$(since $E_r^{0,d+1} = 0$), and hence $(p+r, q-r+1) \neq (2, d)$. Then β is surjective and γ is an isomorphism, by induction assumption (for (a) and (d)). By taking homology of both rows, we obtain a surjection $E_{r+1}^{p,q}(K) \twoheadrightarrow \bigoplus\limits_{v|\infty} E_{r+1}^{p,q}(K_{(v)})$ as wanted for (a).

For (d), we look at the same diagram where now we may assume that $p \geq 2, (p,q) \neq (2,d) \neq (p+r, q-r+1)$, that β and γ are bijective, and that α is surjective (by induction assumption for (a), (b) and (d)). Hence we get the isomorphism

$$E_{r+1}^{p,q}(K) \xrightarrow{\sim} \bigoplus_{v|\infty} E_{r+1}^{p,q}(K_{(v)}).$$

For (b) and (c), consider the *exact* commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & E_{r+1}^{2,d}(K) \arrow[r] \arrow[d, "\beta'"] & E_r^{2,d}(K) \arrow[r, "d_r"] \arrow[d, "\beta"] & E_r^{2+r,d-r+1}(K) \arrow[d, "\gamma", "\wr"'] \\
0 \arrow[r] & \bigoplus\limits_{v\in S'} E_{r+1}^{2,d}(K_{(v)}) \arrow[r] & \bigoplus\limits_{v\in S'} E_r^{2,d}(K_{(v)}) \arrow[r, "\partial"] & \bigoplus\limits_{v|\infty} E_r^{2+r,d-r+1}(K_{(v)})
\end{tikzcd}$$

for any set of places $S' \supset \{v \mid \infty\}$ in which $\partial = \bigoplus\limits_{v\in S'} d_r(K_{(v)})$. (Note that $E_r^{p,q}(K_{(v)}) = 0$ for $p > 2$ and $v \nmid \infty$.) The map γ is an isomorphism by induction assumption (for (d).) Hence for $S' = \{v \mid \infty\}$, the surjectivity of β implies the one for β'; i.e., we get (b) for $r+1$. If S' is the set of all places, we see that clearly $\ker(\beta') = \ker(\beta)$ and that $\mathrm{coker}(\beta') = \mathrm{coker}(\beta)$, since $\mathrm{im}(\partial \circ \beta) = \mathrm{im}(\partial)$ by induction assumption for (b). Thus we get (c) for r from (b) and (d) for r. □

We use Lemma 1.3 to complete the proof of Proposition 1.2. From what we have shown, we have $E_\infty^{0,d+2} = E_\infty^{0,d+1} = 0$ for K and all $K_{(v)}$, and isomorphisms

$$E_\infty^{p,q}(K) \xrightarrow{\sim} \bigoplus_v E_\infty^{p,q}(K_{(v)})$$

for all (p,q) with $p+q = d+2, p \geq 3$. (Note that $E_2^{p,q}(K_{(v)}) = 0$ for $p \geq 3$ and $v \nmid \infty$.) Hence kernel and cokernel of

$$H^{d+2}(F, M) \to \bigoplus_v H^{d+2}(F_{(v)}, M)$$

can be identified with kernel and cokernel of

$$E_\infty^{2,d}(K) \to \bigoplus_v E_\infty^{2,d}(K_{(v)}),$$

respectively. But these coincide with kernel and cokernel of

$$E_2^{2,d}(K) = H^2(K, H^d(F\overline{K}, M)) \to \bigoplus_v H^2(K_{(v)}, H^d(F\overline{K}, M)) = \bigoplus_v E_2^{2,d}(K_{(v)})$$

respectively, by (c) of the lemma. Finally, for any finite ℓ-primary G_K-module N, Poitou-Tate duality gives an exact sequence

$$H^2(K, N) \longrightarrow \bigoplus_v H^2(K_{(v)}, N) \longrightarrow H^0(K, N^*)^\vee \longrightarrow 0,$$

where N^* denotes the finite G_K-module $\mathrm{Hom}(N, \mu)$ where μ is the group of roots of unity in $\overline{K}$ and $M^\vee$ is the Pontrjagin dual of a finite G_K-module. But then we have canonical identifications

$$\begin{aligned} H^0(K, \mathrm{Hom}(N, \mu))^\vee &= \mathrm{Hom}_{G_K}(N, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1))^\vee \\ &\cong \mathrm{Hom}_{G_K}(N(-1), \mathbb{Q}_\ell/\mathbb{Z}_\ell)^\vee = \mathrm{Hom}(N(-1)_{G_K}, \mathbb{Q}_\ell/\mathbb{Z}_\ell)^\vee \\ &= (N(-1)_{G_K})^{\vee\vee} \cong N(-1)_{G_K}. \end{aligned}$$

This shows the last isomorphism of Proposition 1.2. □

Remarks 1.4. (a) Proposition 1.2 extends to the case where F is a function field over K, but K is not necessarily separably closed in F, by replacing $F_{(v)}$ with $F \otimes_K K_{(v)}$ and $F\overline{K}$ with $F \otimes_K \overline{K}$. The cohomology groups of these rings have to be interpreted as the étale cohomology groups of the associated affine schemes; with this the proof carries over verbatim. In more down-to-earth (but more tedious) terms, we may note that $(F \otimes_K \overline{K})_{\mathrm{red}} \cong \prod_\sigma (F \otimes_{\tilde{K},\sigma} \overline{K})$, where $\tilde{K}$ is the separable closure of K in F and σ runs over the K-embeddings of $\tilde{K}$ into $\overline{K}$. Similarly, $F \otimes_K K_{(v)} \cong \prod_\sigma (\prod_w F \otimes_{\tilde{K}} \sigma(\tilde{K})_{(\sigma w)})$, where w runs over the places of $\tilde{K}$ above v, σw is the corresponding place of $\sigma(\tilde{K})$ above v, and $\sigma(\tilde{K})_{(\sigma w)}$ is the Henselization of $\sigma(\tilde{K})$ at σw. The étale cohomology groups referred to above can thus be identified with sums of Galois cohomology groups of the fields introduced above, and the claim also follows by applying Proposition 1.2 to $F/\tilde{K}$.

(b) A consequence of Proposition 1.2 is that the restriction map

$$f' : H^{d+2}(F, M) \to \prod_v H^{d+2}(F_v, M)$$

has image in the direct sum $\bigoplus_v \subset \prod_v$ as well, since it factors through the map f in 1.2. Moreover, as we shall see in Section 2, the maps $H^{d+2}(F_{(v)}, M) \to H^{d+2}(F_v, M)$ are injective, so that $\ker(f') = \ker(f)$. For $d > 0$, however, $H^{d+2}(F_v, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d+1))$ is much bigger than $H^{d+2}(F_{(v)}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d+1))$, and Proposition 1.2 does not extend to the completions. In particular, everywhere in [Jan92] the completions K_v should be replaced by the Henselizations $K_{(v)}$. (In loc. cit., Proof of Th. $1'$ and later, the notation FK_v and $F\overline{K}_v$ are problematic; they should be interpreted as F_v and $F_v\overline{K_v}$. Even then $\mathrm{Gal}(\overline{FK_v}/F\overline{K_v}) = \mathrm{Gal}(\overline{F_v}/F_v\overline{K_v})$ is not isomorphic to $\mathrm{Gal}(\overline{F}/F\overline{K})$, but much bigger, as was kindly pointed out to me by J.-L. Colliot-Thélène and J.-P. Serre.) The comparison of $\mathrm{coker}(f')$ and $\mathrm{coker}(f)$ is more subtle; see Section 4.

By Proposition 1.2, the restriction map

$$H^{d+2}(F, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d+1)) \to \bigoplus_v H^{d+2}(F_{(v)}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d+1))$$

has the same kernel and cokernel as the restriction map

$$\beta_N : H^2(K, N) \to \bigoplus_v H^2(K_{(v)}, N) \cong \bigoplus_v H^2(K_v, N)$$

for the G_K-module $N = H^d(F\overline{K}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d+1))$. Here we have used the isomorphism $G_{K_v} \xrightarrow{\sim} G_{K_{(v)}}$ to rewrite the latter map in terms of the more familiar completions K_v. Recall that $F = K(V)$, the function field of a geometrically irreducible variety V of dimension d over K. From this we obtain

$$H^d(F\overline{K}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d+1)) = \varinjlim_{U\subset V} H^d_{\text{ét}}(U \times_K \overline{K},\, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d+1)),$$

where the limit is over all affine open subvarieties U of V. In fact étale cohomology commutes with this limit ([Mil80, III 1.16]), so that the right-hand side is the étale cohomology group $H^d_{\text{ét}}(\mathrm{Spec}(\overline{K}(V)), \mathbb{Q}_\ell/\mathbb{Z}_\ell(d+1))$, which can be identified with the Galois cohomology group on the left-hand side. Since

$$H^2(K, \varinjlim N_i) = \varinjlim H^2(K, N_i)$$

for a direct limit of G_K-modules N_i, and since the same holds for $\bigoplus_v H^2(K_v, -)$, it thus suffices to study the maps

$$\beta_B : H^2(K, B) \to \bigoplus_v H^2(K_v, B)$$

for $B = H^d(U \times_K \overline{K}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d+1))$, where $U \subseteq V$ runs through all open subvarieties of V or through a cofinal set of them. For this we shall use the following Hasse principle, which generalizes [Jan88, Th. 3].

THEOREM 1.5. *Let K be a global field, and let $\ell \neq \mathrm{char}(K)$ be a prime number.*

(a) *Let A be a discrete G_K-module which is isomorphic to $(\mathbb{Q}_\ell/\mathbb{Z}_\ell)^m$ for some m as an abelian group and mixed of weights $\neq -2$ as a Galois module. Then the restriction map induces isomorphisms*

$$\beta_A : H^2(K, A) \xrightarrow{\sim} \bigoplus_v H^2(K_v, A) = \bigoplus_{v \in S \ \text{or}\ v|\ell} H^2(K_v, A),$$

where S is a finite set of bad places for A.

(b) *Let T be a finitely generated free $\mathbb{Z}_\ell$-module with continuous action of G_K making T mixed of weights $\neq 0$. Then for any finite set S' of places of K, the restriction map in continuous cohomology*

$$\alpha_T : H^1(K, T) \to \prod_{v \notin S'} H^1(K_v, T)$$

is injective.

Before we prove this, let us explain the notion of a mixed G_K-representation and a bad place v for it. *A priori*, this is defined for a $\mathbb{Q}_\ell$-representation V of G_K (i.e., a finite-dimensional $\mathbb{Q}_\ell$-vector space with a continuous action of G_K); see [Del80, (1.2) and (3.4.10)] and Definition 1.6 below. We extend it to a module like A above or, more generally, to a discrete ℓ-primary torsion G_K-module of cofinite type (resp. to a finitely generated $\mathbb{Z}_\ell$-module T with continuous action of G_K), by calling A (resp. T) pure of weight w or mixed, if this holds for the $\mathbb{Q}_\ell$-representation $T_\ell A \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ (resp. $T \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$), where $T_\ell A = \varprojlim_n A[\ell^n]$ is the Tate module of A. In the same way we define the bad places for A (resp. T) to be those of the associated $\mathbb{Q}_\ell$-representations. It remains to recall

Definition 1.6. (a) A $\mathbb{Q}_\ell$-representation V of G_K is pure of weight $w \in \mathbb{Z}$ if there is a finite set $S \supset \{v \mid \infty\}$ of places of K such that

(i) V is unramified outside $S \cup \{v \mid \ell\}$, i.e., for $v \notin S, v \nmid \ell$, the inertia group I_v at v acts trivially on V;

(ii) for every place $v \notin S, v \nmid \ell$, the eigenvalues α of the geometric Frobenius Fr_v at v acting on V are pure of weight w, i.e., algebraic numbers with

$$|\iota\alpha| = (Nv)^{\frac{w}{2}}$$

for every embedding $\iota : \mathbb{Q}(\alpha) \hookrightarrow \mathbb{C}$, where Nv is the cardinality of the finite residue field of v.

Every such set S will be called a set of bad places for V; the places not in S are called good.

(b) V is called mixed if it has a filtration $0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$ by subrepresentations such that every quotient V_i/V_{i-1} is pure of some weight w_i. The weights and bad places of V are those present in some nontrivial quotient V_i/V_{i-1}.

Remarks and Examples 1.7. For a field L, denote by $\overline{L}$ its separable closure, and let $G_L = \mathrm{Gal}(\overline{L}/L)$ be its absolute Galois group.

(a) If v is a place of K, then any extension w of v to $\overline{K}$ determines a decomposition group $G_w \subset G_K$ and an inertia group $I_w \subset G_w$. The arithmetic Frobenius φ_w is a well-defined element in G_w/I_w; under the canonical isomorphism $G_w/I_w \xrightarrow{\sim} \mathrm{Gal}(k(w)/k(v))$, it corresponds to the automorphism $x \mapsto x^{Nv}$ of $k(w)$. The geometric Frobenius Fr_w is the inverse of φ_w. If I_w acts trivially on V, then the action of Fr_w on V is well defined. If we do not fix a choice of w, everything is well defined up to conjugacy in G_K, and we use the notation G_v, I_v, and Fr_v. Thus "I_v acts trivially" means that one and hence any I_w for $w \mid v$, acts trivially, and then the eigenvalues of Fr_v are well defined, since they depend only on the conjugacy class.

(b) If V is pure of weight w, then the same holds for every $\mathbb{Q}_\ell$-G_K-subquotient. If V' is pure of weight w', then $V \otimes_{\mathbb{Q}_\ell} V'$ is pure of weight $w + w'$.

(c) The representation $\mathbb{Q}_\ell(1)$ is unramified outside $S = \{v \mid \infty \cdot \ell\}$, and for $v \notin S, \varphi_v$ acts on $\mathbb{Q}_\ell(1)$ by multiplication with Nv. Therefore $\mathbb{Q}_\ell(1)$ is pure of weight -2, and $\mathbb{Q}_\ell(i)$ is pure of weight $-2i$.

(d) Let A or T or V be G_K-representations as in Definition 1.6, which are mixed of weights $\neq 0$. Then $V^{G_K} = 0 = V_{G_K}$, $T^{G_K} = 0$, and $A_{G_K} = 0$. The first statement is easily reduced to the pure case, where it follows from the fact that the eigenvalues of Fr_v as in 1.6(ii) are different from 1. The other claims follow from the injection $T \hookrightarrow T \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ and the surjection $T_\ell A \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \twoheadrightarrow A$.

(e) If X is a smooth and proper variety over K, then the i-th étale cohomology group $H^i_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell)$ of $\overline{X} = X \times_K \overline{K}$ is pure of weight i by the smooth and proper base change theorems and by Deligne's proof of the Weil conjectures over finite fields (cf., e.g., [Jan89, proof of Lemma 3]). The set S can be taken to be the set of places where X has bad reduction, i.e., such that for $v \notin S, X$ has good reduction at v, viz., a smooth proper model $\mathcal{X}_v$ over $\mathcal{O}_v$, the ring of integers in K_v, with $\mathcal{X}_v \times_{\mathcal{O}_v} K_v = X_v$.

(f) For later purposes, we note that the whole theory above has a generalization to an arbitrary finitely generated field K (see [Del80, (3.4.10)]). A $\mathbb{Q}_\ell$-representation V of G_K (for $\ell \neq \mathrm{char}(K)$) is called pure of weight w if there is a normal scheme T of finite type over $\mathbb{Z}$ with fraction field K such that V comes from a $\mathbb{Q}_\ell$-representation of the algebraic fundamental group $\pi(T, \mathrm{Spec}(\overline{K}))$ via the natural epimorphism $G_K \to \pi(T, \mathrm{Spec}(\overline{K}))$ (i.e., from a smooth $\mathbb{Q}_\ell$-sheaf on T) such that for any closed point $t \in T$ with residue field $k(t)$ of characteristic $\neq \ell$, the eigenvalues of the geometric Frobenius Fr_t are pure of weight w in the sense of 1.6(i). (Replace Nv by Nt, the cardinality of the residue field $k(t)$ of t, which is finite.) The geometric Frobenius Fr_t is the image of the geometric Frobenius under the homomorphism $G_{k(t)} = \pi(\mathrm{Spec}(k(t)), \overline{k(t)}) \to \pi(T, \mathrm{Spec}(\overline{K}))$ which is well defined up to conjugation. The other notions (mixed representations, the notions for T and A) extend literally, as well as the properties (b) to (e) above. In (e) one takes T such that X/K extends to a smooth proper model $\pi : \mathcal{X} \to T$, and one uses the base change isomorphism

$$H^i(\overline{X}, \mathbb{Q}_\ell) \cong H^i(\mathcal{X}_t \times_{k(t)} \overline{k(t)}, \mathbb{Q}_\ell),$$

where $\mathcal{X}_t = \mathcal{X} \times_T k(t)$ is the fiber of π over $t \in T$.

(g) Moreover, we note that there is even an analogue for a finitely generated field K and $\ell = p = \mathrm{char}(K) > 0$. First we note that the notions of pure and mixed representations still make sense, and that properties (a), (b) and (d) also hold in this situation, while (c) does not have any counterpart. On the other hand, one has the following analogue of (e). For a scheme Z of finite type over a perfect field L and $m \in \mathbb{N}$, let

$$H^i(Z, \mathbb{Z}/p^m\mathbb{Z}(j)) := H^{i-j}(Z, W_m\Omega^j_{X,\log})$$

be the étale cohomology of the logarithmic part $W_m\Omega^j_{X,\log}$ of the de Rham-Witt sheaf $W_m\Omega^j_X$. (See [Ill79, I 5.7] and compare (0.2).) Moreover, let

$$H^i(Z, \mathbb{Q}_p(j)) = H^i(Z, \mathbb{Z}_p(j)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p,$$

where

$$H^i(Z, \mathbb{Z}_p(j)) = \varprojlim_{m} H^i(Z, \mathbb{Z}/p^m(j)),$$

with the inverse limit taken with respect to the natural epimorphisms

$$W_{m+1}\Omega^j_{X,\log} \to W_m\Omega^j_{X,\log}.$$

Then for X smooth and proper over a finite field k of characteristic p, the $\mathbb{Q}_p$-G_k-representation $H^i(\overline{X}, \mathbb{Q}_p(j))$ is finite-dimensional, and it follows from the work of Deligne [Del74], Katz-Messing [KM74], and Milne [Mil86] that it is pure of weight $i - 2j$; cf. [Jan10, §3]. If X is smooth and proper over a finitely generated field K of characteristic p and $\pi : \mathcal{X} \to T$ is a smooth proper model as in (f) (so that T is of finite type over $\mathbb{F}_p$), then Gros and Suwa ([GS88, Th. 2.1]) established base change isomorphisms

$$H^i(X \times_K \overline{K}, \mathbb{Q}_p(j)) \cong H^i(\mathcal{X}_t \times_{k(t)} \overline{k(t)}, \mathbb{Q}_p(j))$$

for all closed points t in a nonempty open $U \subset T$, where $\overline{K}$ now stands for an algebraic closure of K. These isomorphisms are compatible with the actions of the absolute Galois groups G_K (on the left) and $G_{k(t)}$ (on the right), so that the representation $H^i(\overline{X}, \mathbb{Q}_p(j)) = H^i(X \times_K \overline{K}, \mathbb{Q}_p(j))$ is pure of weight $i - 2j$

in exactly the same sense as for the ℓ-adic case in (f). Here we regard G_K as the Galois group $\mathrm{Gal}(\overline{K}/K^{\mathrm{in}})$, where K^{in} is the maximal inseparable extension of K in $\overline{K}$.

Proof of Theorem 1.5. Part (a) is implied by (b). In fact, A is mixed of weights $\neq -2$ if and only if its Kummer dual $T = \mathrm{Hom}(A, \mu)$ (where μ is the Galois module of roots of unity in $\overline{K}$) is mixed of weights $\neq 0$, and the kernels of β_A and α_T for $S' = \emptyset$ are dual to each other by the theorem of Tate-Poitou (and passing to the limits over the finite modules $A[\ell^n]$ and $T/\ell^n T = \mathrm{Hom}(A[\ell^n], \mu)$, respectively). Moreover, by Tate-Poitou, the cokernel of α_A is isomorphic to $H^0(K,T)^\vee \cong A(-1)_{G_k}$, and this is zero by the hypothesis on the weights. Finally, by local Tate duality, $H^2(K_v, A)$ is dual to $H^0(K_v, T)$, and for good places $v \nmid \ell$, this is zero if T is mixed of weights $\neq 0$.

Part (b) generalizes [Jan88, Th. 3(a)], which covers the case of a pure T. The generalization follows by induction: Let

$$0 \to T' \to T \to T'' \to 0$$

be an exact sequence of $\mathbb{Z}_\ell$-G_K-modules as in (b), and let S' be a finite set of primes. Then there is a commutative diagram with exact rows

$$\begin{tikzcd}
\prod\limits_{v\notin S'} H^0(K_v,T'') \arrow[r] & \prod\limits_{v\notin S'} H^1(K_v,T') \arrow[r] & \prod\limits_{v\notin S'} H^1(K_v,T) \arrow[r] & \prod\limits_{v\notin S'} H^1(K_v,T'') \\
H^0(K,T'') \arrow[r] \arrow[u] & H^1(K,T') \arrow[r] \arrow[u, "\beta_{T'}"'] & H^1(K,T) \arrow[r] \arrow[u, "\beta_T"'] & H^1(K,T''). \arrow[u, "\beta_{T''}"']
\end{tikzcd}$$

If $\beta_{T''}$ is injective and $H^0(K_v, T'') = 0$ for all $v \notin S'$ (which is the case for T'' pure of weight $\neq 0$ and S' containing all bad places for T'' and all $v \mid \ell$, by loc. cit.), then β_T is injective if and only if $\beta_{T'}$ is. Since we may always enlarge the set S', the proof proceeds by induction on the length of a filtration with pure quotients, which exists on $T \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$, by definition, and hence on T by pullback. □

2. Injectivity of the global-local map for coefficients invertible in K

Let K be a global field, let $\ell \neq \mathrm{char}(K)$ be a prime, and let U be a smooth, quasi-projective, geometrically irreducible variety of dimension d over K. Following the strategy of Section 1, we study the G_K-module $H^d(\overline{U}, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$. Assume the following condition, which holds for number fields by Hironaka's resolution of singularities in characteristic zero [Hir64a], [Hir64b].

> RS2(U): There is a *good compactification* for U; i.e., a smooth projective variety X over K containing U as an open subvariety such that $Y = X \smallsetminus U$, with its reduced closed subscheme structure, is a divisor with simple normal crossings.

Recall that Y is said to have simple normal crossings if its irreducible components $Y_1, \ldots, Y_N$ are smooth projective subvarieties $Y_i \subset X$ such that for all $1 \le i_1 < \cdots < i_\nu \le N$, the ν-fold intersection $Y_{i_1,\ldots,i_\nu} := Y_{i_1} \cap \cdots \cap Y_{i_\nu}$ is empty or smooth projective of pure dimension $d-\nu$, so the same is true for the disjoint union

$$Y^{[\nu]} := \coprod_{1\le i_1<\cdots<i_\nu\le N} Y_{i_1,\ldots,i_\nu} \qquad (1 \le \nu \le d)$$

and for $Y^{[0]} := Y_\emptyset := X$.

This geometric situation gives rise to a spectral sequence

$$E_2^{p,q} = H^p(\overline{Y^{[q]}}, \mathbb{Q}_\ell(-q)) \Rightarrow H^{p+q}(\overline{U}, \mathbb{Q}_\ell); \tag{2.1}$$

see, e.g., [Jan90, 3.20]. It is called the weight spectral sequence because it induces the weight filtration on the ℓ-adic representation $H^n(\overline{U}, \mathbb{Q}_\ell)$. In fact, $E_2^{p,q}$ is pure of weight $p+2q$. Therefore the same is true for the $E_\infty^{p,q}$-terms, and if $\tilde{W}_q$ denotes the canonical ascending filtration on the limit term $H^n(\overline{U}, \mathbb{Q}_\ell)$ for which $\tilde{W}_q/\tilde{W}_{q-1} = E_\infty^{n-q,q}$, then its n-fold shift $W_. := \tilde{W}_.[-n]$ (i.e., $W_i = \tilde{W}_{i-n}$) is the unique weight filtration, i.e., has the property that the quotient $W_i/W_{i-1} \cong E_\infty^{2n-i,i-n}$ is pure of weight i. Moreover, for $r > 3$, the differentials

$$d_r^{p,q} : E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}$$

are morphisms between Galois $\mathbb{Q}_\ell$-representation of different weights (viz., $p+2q$ and $p+2q-r+2$) and hence vanish, so that $E_\infty^{p,q} = E_3^{p,q}$.

Note that $E_2^{p,q} = 0$ for $p < 0$ or $q < 0$. Hence the weights occurring in $H^n(\overline{U}, \mathbb{Q}_\ell)$ lie in $\{n, \ldots, 2n\}$, W_{2n-1} is mixed of weights $w \le 2n-1$, and

$$\begin{aligned} H^n(\overline{U}, \mathbb{Q}_\ell)/W_{2n-1} &= W_{2n}/W_{2n-1} = E_3^{0,n} \\ &= \ker(H^0(\overline{Y^{[n]}}, \mathbb{Q}_\ell(-n)) \xrightarrow{d_2^{0,n}} H^2(\overline{Y^{[n-1]}},\ \mathbb{Q}_\ell(-n+1)). \end{aligned}$$

In particular, the Galois action on $(W_{2n}/W_{2n-1})(n)$ factors through a finite quotient, since this is the case for $H^0(\overline{Y^{[n]}}, \mathbb{Q}_\ell)$.

We want to say something similar for $H^n(\overline{U}, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$, at least for $n = d$ ($= \dim U$). If U is affine, then we have an exact sequence

$$\cdots \to H^d(\overline{U}, \mathbb{Z}_\ell) \to H^d(\overline{U}, \mathbb{Q}_\ell) \to H^d(\overline{U}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \to 0,$$

since $H^{d+1}(\overline{U}, \mathbb{Z}_\ell) = 0$ by weak Lefschetz [Mil80, VI 7.2]. From this we conclude that $B_1 = H^d(\overline{U}, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ is divisible and that there is an exact sequence

$$0 \longrightarrow A_1 \longrightarrow B_1 \longrightarrow C_1 \longrightarrow 0$$

in which $A_1 = \mathrm{im}(W_{2d-1}H^d(\overline{U}, \mathbb{Q}_\ell) \longrightarrow H^d(\overline{U}, \mathbb{Q}_\ell/\mathbb{Z}_\ell))$ is divisible and of weights $w \in \{d, \ldots, 2d-1\}$, and in which C_1 is a quotient of $H^d(\overline{U}, \mathbb{Q}_\ell)/W_{2d-1}$, divisible and pure of weight $2d$. We need to know C_1 precisely, not just up to isogeny, and this requires more arguments — note that in the $\mathbb{Q}_\ell/\mathbb{Z}_\ell$-analogue of (2.1), the differentials $d_1^{p,q}$ will not in general vanish for $r \geq 3$.

For better control of this spectral sequence, we replace U by a smaller variety, as follows. By the Bertini theorem, there is a hyperplane H in the ambient projective space whose intersection with X and all $Y_{i_1,\ldots,i_\nu}$ is transversal, i.e., gives smooth divisors in these. (In particular, the intersection with $Y_{i_1,\ldots,i_d}$ is empty for all d-tuples $(i_1,\ldots,i_d)$.) This means that $\tilde{Y} = \bigcup\limits_{i=1}^{N+1} Y_i$, with $Y_{N+1} := H \cap X$, is again a divisor with strict normal crossings on X. As explained in Section 1, it is possible for our purposes to replace U by the open subscheme $U^0 = X \smallsetminus \tilde{Y} = U \smallsetminus (H \cap U)$, because such subschemes form a cofinal subset in the set of all opens $U \subseteq V$, $F = K(V)$. Now we have the following description for $B_0 := H^d(\overline{U^0}, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$.

Proposition 2.2. *There is an exact sequence*

$$0 \to A_0 \to H^d(\overline{U^0}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \to C_0 \to 0$$

in which A_0 is divisible and mixed of weights in $\{d, \ldots, 2d-1\}$, and

$$C_0 = I \otimes_{\mathbb{Z}} \mathbb{Q}_\ell/\mathbb{Z}_\ell(-d)$$

for a finitely generated free $\mathbb{Z}$-module I with discrete action of G_K. Moreover, there is an exact sequence

$$0 \to I' \to I \to I'' \to 0$$

of G_K-modules with

$$I'' = \mathbb{Z}[\pi_0(\overline{Y^{[d]}})],$$

$$I' = \ker(\mathbb{Z}[\pi_0(\overline{Y^{[d-1]} \cap H})] \overset{\beta}{\twoheadrightarrow} \mathbb{Z}[\pi_0(\overline{Y^{[d-1]}})]),$$

where $Y^{[d-1]} \cap H := \coprod\limits_{1 \leq i_1 < \cdots < i_{d-1} \leq N} Y_{i_1,\ldots,i_{d-1}} \cap H$ and where β is induced by the inclusions $Y_{i_1,\ldots,i_\nu} \cap H \hookrightarrow Y_{i_1,\ldots,i_\nu}$.

Proof. For $1 \leq i_1 < \cdots < i_\nu \leq N$, define

$$Y^0_{i_1,\ldots,i_\nu} := Y_{i_1,\ldots,i_\nu} \smallsetminus (Y_{i_1,\ldots,i_\nu} \cap H)$$

by removing the smooth hyperplane section with H, and let $Y^{0[\nu]} \subseteq Y^{[\nu]}$ be the disjoint union of these open subvarieties for fixed ν (with $Y^{0[0]} := X^0 := X \smallsetminus (X \cap H)$). Then $Y^0 = \bigcup\limits_{i=1}^{N} Y_i^0$ is a divisor with (strict) normal crossing on X^0 with $U^0 = X^0 \smallsetminus Y^0$, and hence there is a spectral sequence

$$E_2^{p,q} = H^p(\overline{Y^{0[q]}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(-q)) \Rightarrow H^{p+q}(\overline{U^0}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \tag{2.3}$$

by the same arguments as for (2.1) (the properness is not needed in the proof). □

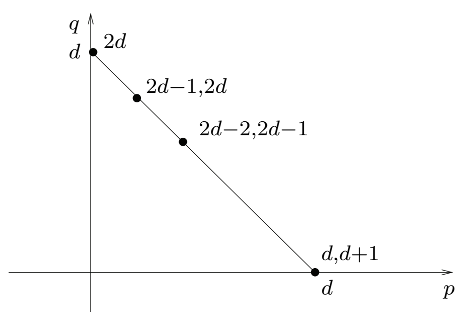
But now the $Y^0_{i_1,\dots,i_q}$ are affine varieties, as complements of hyperplane sections, and of dimension $d-q$, so that

$$H^p(\overline{Y^{0[q]}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(-q)) = \begin{cases} 0 & \text{for } p > d-q, \\ \text{divisible} & \text{for } p = d-q \end{cases}$$

by weak Lefschetz. Moreover, by the Gysin sequences

$$\cdots \to H^p(\overline{Y_{\underline{i}}}, \mathbb{Q}_\ell) \to H^p(\overline{Y^0_{\underline{i}}}, \mathbb{Q}_\ell) \to H^{p-1}(\overline{Y_{\underline{i}} \cap H}, \mathbb{Q}_\ell(-1)) \to \cdots,$$

$H^p(\overline{Y^{0[q]}}, \mathbb{Q}_\ell)$ is mixed with weights p and $p+1$, since $H^p(\overline{Y_{\underline{i}}}, \mathbb{Q}_\ell)$ is pure of weight p and $H^{p-1}(\overline{Y_{\underline{i}} \cap H}, \mathbb{Q}_\ell(-1))$ is pure of weight $p+1$. Hence the spectral sequence (2.3) is much simpler than (2.1) and has the following E_2-layer:



The terms vanish for $p+q > d$, and on the line $p+q = d$, the $E_2^{p,q}$-terms — and hence also the $E_\infty^{p,q}$-terms which are quotients — are divisible and mixed of the indicated weights. Note that $H^0(\overline{Y^{0[d]}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(-d)) = H^0(\overline{Y^{[d]}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(-d)))$ is pure of weight $2d$.

Let $F^\cdot$ be the descending filtration on $B_0 = H^d(\overline{U^0}, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ for which $F^\nu/F^{\nu-1} = E_\infty^{\nu,d-\nu}$. Then we see that F^2 is divisible and mixed of weights $\leq 2d-1$. Next,

$$F^1/F^2 \cong E_2^{1,d-1} = H^1(\overline{Y^{0[d-1]}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(-d+1))$$

is the cohomology of a (usually nonconnected) smooth affine curve, and by the Gysin sequence

$$\begin{aligned} 0 &\to H^1(\overline{Y^{[d-1]}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \to H^1(\overline{Y^{0[d-1]}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \\ &\to H^0(\overline{Y^{[d-1]} \cap H}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(-1)) \to H^2(\overline{Y^{[d-1]}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \to 0 \end{aligned}$$

there is an exact sequence

$$0 \to A' \to F^1/F^2 \to C' \to 0,$$

where A' is divisible of weight $2d-1$ and where $C' = I' \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell$, with I' defined by the exact sequence

$$0 \to I' \to \mathbb{Z}[\pi_0(\overline{Y^{[d-1]} \cap H})] \to \mathbb{Z}[\pi_0(\overline{Y^{[d-1]}})] \to 0.$$

Finally,

$$C'' := F^0/F^1 \cong H^0(\overline{Y^{0[d]}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(-d))$$

is the cohomology of $Y^{0[d]} = Y^{[d]}$ which is a union of points, and $C'' = I'' \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell$ for

$$I'' = \mathbb{Z}[\pi_0(\overline{Y^{[d]}})].$$

Let A_0 be the preimage of A' in F^1, and let $C_0 = B_0/A_0$. Then we have exact sequences

$$\begin{array}{ccccccccc} 0 & \to & A_0 & \to & B_0 & \to & C_0 & \to & 0, \\ 0 & \to & F^2 & \to & A_0 & \to & A' & \to & 0, \\ 0 & \to & C' & \to & C_0 & \to & C'' & \to & 0. \end{array}$$

Hence A_0 is divisible and mixed of weights $\leq 2d-1$, and C_0 is divisible of weight $2d$. This determines A_0 and C_0 uniquely (there is no nontrivial G_K-morphism between such modules), and so the spectral sequence (2.1) for $U^0 = X \smallsetminus \tilde{Y}$ instead of $U = X \smallsetminus Y$ shows that C_0 is a quotient of

$$\ker\left(H^0(\overline{\tilde{Y}^{[d]}}, \mathbb{Q}_\ell(-d)) \to H^2(\overline{\tilde{Y}^{[d-1]}}, \mathbb{Q}_\ell(-d+1))\right).$$

Hence the action of G_K on $C_0(d)$ factors through a finite quotient G. This in turn shows that the extension

$$0 \to C' \to C_0 \to C'' \to 0$$

comes from an extension

$$0 \to I' \to I \to I'' \to 0$$

of G-modules by tensoring with $\mathbb{Q}_\ell/\mathbb{Z}_\ell(d)$. In fact, applying a Tate twist is an exact functor on $\mathbb{Z}_\ell$-G_k-modules, and one has isomorphisms (where the tensor products are over $\mathbb{Z}$)

$$\begin{array}{rcl} \mathrm{Ext}^1_G(I'', I') \otimes_{\mathbb{Z}} \mathbb{Z}_\ell & \xrightarrow{\sim} & \mathrm{Ext}^1_{\mathbb{Z}_\ell[G]}(I'' \otimes \mathbb{Z}_\ell, I' \otimes \mathbb{Z}_\ell) \\ & \xrightarrow{\sim} & \mathrm{Ext}^1_G(I'' \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell, I' \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell), \end{array}$$

since $\mathbb{Z}_\ell$ is flat over $\mathbb{Z}$, and since the functor $T \mapsto T \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell/\mathbb{Z}_\ell$ is an equivalence between $\mathbb{Z}_\ell$-lattices and divisible ℓ-torsion modules of cofinite type (with action of G) preserving exact sequences. Finally,

$$\mathrm{Ext}^1_G(I'', I') \to \mathrm{Ext}^1_G(I'', I') \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_\ell$$

is surjective for a finite group G.

We are now ready to prove

Theorem 2.4. *The restriction map*

$$\beta_B : H^2(K, B) \to \bigoplus_v H^2(K_v, B)$$

is injective for $B = H^d(\overline{U^0}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d+1))$.

Proof. We follow the method of [Jan92]. By applying the $(d+1)$-fold Tate twist to the sequence $0 \to A_0 \to B_0 \to C_0 \to 0$ of Proposition 2.2, we get an exact sequence

$$0 \to A \to B \to C \to 0.$$

It induces a commutative diagram with exact rows

$$\begin{tikzcd}
\cdots \arrow[r] & \bigoplus_{v\in S} H^1(K_v,C) \arrow[r] \arrow[r, phantom, shift right=6ex, "(*)"] & \bigoplus_{v} H^2(K_v,A) \arrow[r] & \bigoplus_{v} H^2(K_v,B) \\
\cdots \arrow[r] & H^1(K,C) \arrow[r] \arrow[u, "\alpha_{C,S}"'] & H^2(K,A) \arrow[r] \arrow[u, "\beta_A"', "\wr"] & H^2(K,B) \arrow[u, "\beta_B"']
\end{tikzcd}$$

$$\begin{tikzcd}
\arrow[r] & \bigoplus_{v} H^2(K_v,C) \arrow[r] & \bigoplus_{v\infty} H^3(K_v,A) \arrow[r] & \cdots \\
\arrow[r] & H^2(K,C) \arrow[r] \arrow[u, "\beta_C"'] & H^3(K,A) \arrow[r] \arrow[u, "\gamma_A"', "\wr"] & \cdots
\end{tikzcd}$$

for a suitable finite set S of places of K. In fact, if S_{bad} is a set of bad places for A, then for any $S \supset S_{\mathrm{bad}} \cup \{v \mid \ell\}$, $H^2(K_v,A) = 0$ for $v \notin S$, and thus $(*)$ is commutative. By Theorem 1.5, β_A is an isomorphism, since A is divisible and mixed of weights ≤ -3, and by Tate duality, γ_A is an isomorphism (for all torsion modules A). To show the injectivity of β_B by the 5-lemma, it therefore suffices to show that C satisfies

(H) (i) $\alpha_{C,S} : H^1(K,C) \;\to\; \bigoplus_{v\in S} H^1(K_v,C)$ is surjective for all finite S;
(ii) $\beta_C : H^2(K,C) \;\to\; \bigoplus_{v} H^2(K_v,C)$ is injective.

Let I, I' and I'' be as in Proposition 2.2, so that $C = I \otimes_{\mathbb{Z}} \mathbb{Q}_\ell/\mathbb{Z}_\ell(1)$. We have exact sequences

$$\begin{array}{ccccccccc} 0 & \to & I' & \to & I & \to & I'' & \to & 0, \\ 0 & \to & I' & \to & I_2 & \to & I_3 & \to & 0, \end{array}$$

in which I'', I_2 and I_3 are permutation modules, i.e., of the form $\mathbb{Z}[M]$ for a G_K-set M. Thus (H) holds for C by repeated application (first to I'', I_2, and I_3, then to I', and finally to I) of the following result. $\square$

THEOREM 2.5. *Let I_1, I_2, and I_3 be finitely generated free $\mathbb{Z}$-modules with discrete G_K-action, and let $C_i = I_i \otimes_{\mathbb{Z}} \mathbb{Q}_\ell/\mathbb{Z}_\ell(1)$ for $i = 1,2,3$. Assume that I_3 is a permutation module.*

(a) *Property* (H) *holds for C_3.*
(b) *If $0 \to I_1 \to I_2 \to I_3 \to 0$ is an exact sequence, then* (H) *holds for C_1 if and only if it holds for C_2.*

The following observation will help to prove part (b).

Lemma 2.6. *Let I be a finitely generated free $\mathbb{Z}$-module with discrete G_K-action, and let T be the torus over K with cocharacter module $X_*(T) = I$. Then property* (H)(i) (*resp.* (H)(ii)) *holds for $C = I \otimes_{\mathbb{Z}} \mathbb{Q}_\ell/\mathbb{Z}_\ell(1)$ if and only if T satisfies*

$$\begin{aligned} &(\mathrm{H}'_\ell)(\mathrm{i}){:}\quad \alpha_{T,S,\ell} : H^1(K,T)\{\ell\} \to \bigoplus_{v\in S} H^1(K_v,T)\{\ell\} \text{ is surjective for all finite } S,\\ &(\textit{resp. } (\mathrm{H}'_\ell)(\mathrm{ii}){:}\quad \beta_{T,\ell} : H^2(K,T)\{\ell\} \to \bigoplus_{v} H^2(K_v,T)\{\ell\} \text{ is injective}). \end{aligned}$$

Proof. Recall that $T(\overline{K}) = I \otimes_{\mathbb{Z}} \overline{K^\times}$ and $H^i(K,T) = H^i(K,T(\overline{K}))$ by definition. Since $\ell \neq \mathrm{char}(K)$, $T(\overline{K})$ is ℓ-divisible, and the Kummer sequences

$$0 \to I \otimes_{\mathbb{Z}} \mu_{\ell^n} \to T(\overline{K}) \xrightarrow{\ell^n} T(\overline{K}) \to 0 \tag{2.7}$$

identify C with $T(\overline{K})\{\ell\}$, the ℓ-primary torsion subgroup of $T(\overline{K})$. Similar results hold for the fields K_v, and the cohomology sequences associated to (2.7) for all n give rise to a commutative diagram with exact rows

$$\begin{array}{ccccccc} \bigoplus_{v\in S} T(K_v)\otimes\mathbb{Q}_\ell/\mathbb{Z}_\ell & \longrightarrow & \bigoplus_{v\in S} H^1(K_v,C) & \longrightarrow & \bigoplus_{v\in S} H^1(K_v,T)\{\ell\} & \longrightarrow & 0\\ \uparrow{\scriptstyle \omega_{T,S}} & & \uparrow{\scriptstyle \alpha_{C,S}} & & \uparrow{\scriptstyle \alpha_{T,S,\ell}} & & \\ T(K)\otimes\mathbb{Q}_\ell/\mathbb{Z}_\ell & \longrightarrow & H^1(K,C) & \longrightarrow & H^1(K,T)\{\ell\} & \longrightarrow & 0 \end{array}$$

and to a commutative diagram with horizontal isomorphisms

$$\begin{array}{ccc} \bigoplus_{v} H^2(K_v,C) & \xrightarrow{\sim} & \bigoplus_{v} H^2(K_v,T)\{\ell\}.\\ \uparrow{\scriptstyle \beta_C} & & \uparrow{\scriptstyle \beta_{T,\ell}}\\ H^2(K,C) & \xrightarrow{\sim} & H^2(K,T)\{\ell\} \end{array}$$

Here the vertical maps are induced by the various restriction maps, and we used that $T(K) = H^0(K,T(\overline{K}))$ and $T(K_v) = H^0(K_v,T(\overline{K}_v))$ for a separable closure $\overline{K_v}$ of K_v. Note that $H^i(K,T)$ and $H^i(K_v,T)$ are torsion groups for $i \geq 1$.

Now the map $\omega_{T,S}$ is surjective for any torus T and any finite set of places S ([Jan92, Lemma 2]). This proves the lemma. □

Proof of Theorem 2.5. Let I_3 be a permutation module. Then I_3 is a direct sum of modules of the form $I_0 = \mathrm{Ind}_{K'}^{K}(\mathbb{Z}) = \mathbb{Z}[G_K/G_{K'}]$ for some finite separable extension K' of K. Let T_0 be the torus with cocharacter module I_0. Then

$$H^i(K,T_0) \cong H^i(K',\mathbb{G}_m) = H^i(K',\overline{K}^\times) \tag{2.8}$$

by Shapiro's lemma, and similarly

$$H^i(K_v, T_0) \cong \bigoplus_{w|v} H^i(K'_\omega, \mathbb{G}_m), \tag{2.9}$$

where w runs through the places of K' above v. Thus

$$H^1(K, T_0) = 0 = H^1(K_v, T_0)$$

by Hilbert's Theorem 90, and $\beta_{T_0} : H^2(K, T_0) \bigoplus_v H^2(K_v, T_0)$ is injective by the classical theorem of Brauer-Hasse-Noether for K'. This shows property (H'_ℓ) for the torus T_3 with cocharacter module I_3, for all primes ℓ, and hence part (a) of Theorem 2.5.

For part (b), let T_i be the torus with cocharacter module I_i ($i = 1, 2, 3$). Then we have an exact sequence

$$0 \to T_1 \to T_2 \to T_3 \to 0$$

with $H^1(K, T_3) = 0 = H^1(K_v, T_3)$ by assumption and the above. This gives exact commutative diagrams

$$\begin{tikzcd}
\bigoplus_{v\in S} T_3(K_v) \arrow[r, "\delta"] & \bigoplus_{v\in S} H^1(K_v, T_1) \arrow[r] & \bigoplus_{v\in S} H^1(K_v, T_2) \arrow[r] & 0 \\
T_3(K) \arrow[r] \arrow[u, "\omega"] & H^1(K, T_1) \arrow[r] \arrow[u, "\alpha_{T_1,S}"] & H^1(K, T_2) \arrow[r] \arrow[u, "\alpha_{T_2,S}"] & 0
\end{tikzcd}$$

and

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus_{v} H^2(K_v, T_1) \arrow[r] & \bigoplus_{v} H^2(K_v, T_2) \arrow[r] & \bigoplus_{v} H^2(K_v, T_3). \\
0 \arrow[r] & H^2(K, T_1) \arrow[r] \arrow[u, "\beta_{T_1}"] & H^2(K_v, T_2) \arrow[r] \arrow[u, "\beta_{T_2}"] & H^2(K_v, T_3) \arrow[u, "\beta_{T_3}"]
\end{tikzcd}$$

Since β_{T_3} is injective by assumption, one has an isomorphism $\ker\beta_{T_1} \xrightarrow{\sim} \ker\beta_{T_2}$. On the other hand, the groups $H^1(K_v, T_1)$ have finite exponent n. (By Hilbert's Theorem 90 we can take $n = [K' : K]$, if K'/K is a finite Galois extension splitting T_1.) Hence δ factors through $\bigoplus_{v\in S} T_3(K_v)/n$. But $\omega \otimes \mathbb{Z}/n\mathbb{Z}$ is surjective for every n: Indeed,

$$K^\times/(K^\times)^n \to \bigoplus_{v\in S} K_v^\times/(K_v^\times)^n$$

is surjective for all n by weak approximation for K, and the same for all finite extensions K' of K gives the result for T_3 (cf. (2.8) and (2.9) for $i = 0$). This gives an isomorphism coker $\alpha_{T_1,S} \xrightarrow{\sim}$ coker $\alpha_{T_2,S}$ and hence (b). $\square$

This completes the proof of Theorem 2.4, and we can now show the following theorem, which is a variant of Theorem 0.6, in which the fields F_v are replaced by the fields $F_{(v)}$.

THEOREM 2.10. *Let F be a function field in d variables over K, such that K is separably closed in F, and let ℓ be a prime invertible in K. Then the restriction map*

$$H^{d+2}(F, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d+1)) \longrightarrow \bigoplus_v H^{d+2}(F_{(v)}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d+1))$$

is injective.

Proof. Let $F = K(V)$ for a geometrically integral variety V of dimension d over K. By Proposition 1.2 it is equivalent to show the injectivity of

$$\alpha:\ H^2(K, H^d(F\overline{K}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d+1))) \longrightarrow \bigoplus_v H^2(K_{(v)}, H^d(F\overline{K}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d+1))).$$

Let x be an element in the kernel of α. By the limit property recalled above Theorem 1.5, there is an open affine $U \subset V$ such that x is the image of an element y lying in the kernel of

$$H^2(K, H^d(\overline{U}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d+1))) \longrightarrow \bigoplus_v H^2(K_{(v)}, H^d(\overline{U}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d+1))).$$

If K is a number field, then we may assume that U is smooth over K, and there is a good compactification $U \subset X$ as in property RS2(U) at the beginning of this section. Thus the claim follows immediately by restricting to the subset U^0 constructed before Proposition 2.2 and applying Theorem 2.4. $\square$

If K has positive characteristic, we use the following result of Gabber, which refines de Jong's theorem on alterations.

THEOREM 2.11 (Gabber; see [ILO14]). *If X is separated and integral of finite type over a field L and ℓ is a prime which is invertible in L, and $Y \subset X$ is a proper closed subscheme, then there exists a finite extension L'/L of degree prime to ℓ and a connected, smooth quasi-projective variety X' over L' together with a proper surjective L-morphism $\pi : X' \to X$ such that the extension of function fields $L'(X')/L(X)$ is finite of degree prime to ℓ, and such that $Y' = \pi^{-1}(Y)$, with the reduced subscheme structure, is a divisor with strict normal crossings on X'.*

We apply this to a compactification $U \subset X$ for our affine variety with a proper integral variety X over K and the closed subset $Y = X - U$. Let $\pi : X' \to X$ and $Y' = \pi^{-1}(Y)$ be as in (G), so that X' is smooth projective, without loss of generality geometrically irreducible, and Y' is a simple normal crossings divisor. Let $U' = X' - Y'$, and let $(U')^0 \subset U'$ be constructed as the complement of a well-chosen hyperplane section like before Proposition 2.2.

Then the image y' of y under the restriction map for $U' \to U$ lies in the kernel of

$$H^2(K', H^d(\overline{(U')^0}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d+1))) \longrightarrow \bigoplus_w H^2(K'_{(w)}, H^d(\overline{(U')^0}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d+1))),$$

where w runs over all places of K'. Thus $y' = 0$ by Theorem 2.4. By restricting to $F'\overline{K'}$ and once more applying Proposition 1.2, this implies that the image of x under

$$H^{d+2}(F, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d+1)) \longrightarrow H^{d+2}(F', \mathbb{Q}_\ell/\mathbb{Z}_\ell(d+1))$$

is zero. It remains to remark that this restriction map is injective, because the degree $[F' : F]$ is prime to ℓ. In fact, we can decompose the extension F'/F as $F'/F_i/F$, where F_i is the maximal inseparable extension inside F'/F. Then the restriction from F to F_i is an isomorphism, and the restriction Res from F_i to F' is injective, since, for the corestriction Cor from F' to F_i, we have Cor Res = multiplication by $[F' : F_i]$, which is prime to ℓ.

To have the same result with F_v in place of $F_{(v)}$, and thus obtain Theorem 0.6, it suffices to show

Theorem 2.12. *For any $n \in \mathbb{N}$ and all $i, j \in \mathbb{Z}$, the restriction map*

$$H^i(F_{(v)}, \mathbb{Z}/n\mathbb{Z}(j)) \to H^i(F_v, \mathbb{Z}/n\mathbb{Z}(j))$$

is injective.

This is related to a more precise rigidity result (for n invertible in K) on the Kato complexes recalled in Theorem 4.11, which we shall also need in the following sections. However, as was pointed out to me by J.-L. Colliot-Thélène, the injectivity above follows by a simple argument and in the following general version.

Theorem 2.13. *Let K/k be a field extension satisfying the following property*:

(SD) *If a variety Y over k has a K-rational point, then it also has a k-rational point.*

Let F be a set-valued contravariant functor on the category of all k-schemes such that

(FP) *For any inductive system (A_i) of k-algebras and $A = \varinjlim_i A_i$, the natural map $\varinjlim_i F(A_i) \cong F(A)$ is an isomorphism.* (*Here we write $F(B) := F(\operatorname{Spec} B)$ for a k-algebra B.*)

Let V be a geometrically integral variety over k, and write $k(V)$ (resp. $K(V)$) for the function field of V (resp. $V \times_k K$). Then the map

$$F(k(V)) \to F(K(V))$$

is injective.

Proof. The field K can be written as the union of its subfields K_i which are finitely generated (as fields) over k. Every K_i can of course be written as the fraction field of a finitely generated k-algebra A_i.

Now let $\alpha \in F(k(V))$ and assume that α vanishes in $F(K(V))$. By (FP), there is an i such that α already vanishes in $F(K_i(V))$. Moreover, there is a nonempty affine open $V' \subseteq V$ and a $\beta \in F(V')$ mapping to α in $F(k(V))$. Finally there is a nonempty affine open $U \subseteq Z_i \times_k V'$, where $Z_i = \text{Spec } A_i$, such that β vanishes under the composite map $F(V') \to F(Z_i \times_k V') \to F(U)$.

Now it follows from Chevalley's theorem that the image of U under the projection $p : Z_i \times_k V' \to Z_i$ contains a nonempty affine open U'. (p maps constructible set to constructible sets, and is dominant.) Now U' has a K-point $\text{Spec}(K) \to \text{Spec}(K_i) \hookrightarrow U'$. Hence, by property (SD), U' has a k-rational point Q. Then $W = p^{-1}(Q) \cap U$ is open and nonempty in $p^{-1}(Q) = Q \times_k V' \cong V'$. By functoriality, β vanishes in $F(W)$ and thus α in $F(k(V))$. $\square$

Proof of Theorem 2.12. We may apply Theorem 2.13 to the extension $K_v/K_{(v)}$ and the functor $F(X) = H^i_{\text{ét}}(X, M)$ for any fixed discrete $G_{K_{(v)}}$-module M (regarded as étale sheaf by pullback) to get the injectivity of

$$H^i(F_{(v)}, M) \to H^i(F_v, M).$$

In fact, property (SD) (for "strongly dense") is known to hold in this case (cf. [Gre66, Th. 1]), and the commuting with limits as in (FP) (for "finitely presented") is a standard property of étale cohomology (cf. [Mil80, III 1.16]). $\square$

3. A crucial exact sequence, and a Hasse principle for unramified cohomology

To investigate the cokernel of β_B (notation as in Section 2), we could follow the method of [Jan92] and show that it is isomorphic to $\text{coker}(\beta_C)$. By describing the edge morphisms in the spectral sequence (2.3) we could prove the crucial Theorem 3.1 below for global fields. Instead, we prefer to argue more directly, which allows us to treat arbitrary finitely generated fields and use 3.1 also for the remaining sections.

We shall make repeated use of the following. Let $i : Y \hookrightarrow X$ be a closed immersion of smooth varieties over a field L, of pure codimension c. Then, for every integer n invertible in L and every integer r, one has a long exact Gysin sequence

$$\begin{aligned} \cdots \to H^{\nu-1}(U, \mathbb{Z}/n\mathbb{Z}(r)) &\xrightarrow{\delta} H^{\nu-2c}(Y, \mathbb{Z}/n\mathbb{Z}(r-c)) \xrightarrow{i_*} H^{\nu}(X, \mathbb{Z}/n\mathbb{Z}(r)) \\ &\xrightarrow{j^*} H^{\nu}(U, \mathbb{Z}/n\mathbb{Z}(r)) \to \cdots, \end{aligned}$$

where $U = X \smallsetminus Y$ is the open complement of Y and $j : U \hookrightarrow X$ is the open immersion. We call i_* and δ the Gysin map and the residue map for $i : Y \hookrightarrow X$, respectively. If $i' : Y' \hookrightarrow Y$ is another closed immersion, with Y' smooth and of pure codimension c' in Y, then the diagram of Gysin sequences

$$\begin{tikzcd}
H^{\nu-1}(U,\mathbb{Z}/n\mathbb{Z}(r)) \arrow[r, "\delta"] & H^{\nu-2c}(Y,\mathbb{Z}/n\mathbb{Z}(r-c)) \arrow[r, "i_*"] & H^{\nu}(X,\mathbb{Z}/n\mathbb{Z}(r)) \arrow[r, "j^*"] & {} \\
H^{\nu-1}(U',\mathbb{Z}/n\mathbb{Z}(r)) \arrow[u, "j'^*"] \arrow[r, "\delta"] & H^{\nu-2(c+c')}(Y',\mathbb{Z}/n\mathbb{Z}(r-c-c')) \arrow[u, "i'_*"] \arrow[r, "(i\circ i')_*"] & H^{\nu}(X,\mathbb{Z}/n\mathbb{Z}(r)) \arrow[u, equal] \arrow[r, "(j\circ j')^*"] & {}
\end{tikzcd}$$

is commutative, where $j' : U \hookrightarrow U'$ is the open immersion. In fact, the first sequence comes from the long exact relative sequence involving $H^*_Y(X,\mathbb{Z}/n\mathbb{Z}(r)))$, together with canonical Gysin isomorphisms

$$H^{\nu-2c}(Y,\mathbb{Z}/n\mathbb{Z}(r-c)) \overset{\sim}{\to} H^{\nu}_Y(X,\mathbb{Z}/n\mathbb{Z}(r))).$$

If L is a perfect field of characteristic $p>0$ and $n=p^m$, then the one has still Gysin morphisms i_* with the transitivity property, by work of Gros [Gro85a], but the remaining properties are not in general true anymore, except for the following special case. If X is smooth of pure dimension d, then one has canonical Gysin isomorphisms

$$H^{\nu-2c}(Y,\mathbb{Z}/p^m\mathbb{Z}(d-c)) \overset{\sim}{\to} H^{\nu}_Y(X,\mathbb{Z}/p^m\mathbb{Z}(d)))$$

(see [Suw95, Cor. 2.6.]) and gets an exact Gysin sequence as above for $r=d$.

With these preparations we can now prove a crucial exact sequence for a specialization map which is not only used for Theorem 3.8 below, giving a Hasse principle for unramified cohomology, but is also essential in the proofs of Theorems 0.9, 0.10, and 0.11.

Theorem 3.1. *Let K be a finitely generated field with algebraic closure $\overline{K}$, and let X be a smooth, proper, irreducible variety of dimension d over K. Let $Y=\bigcup_{i=1}^r Y_i$, with $r\geq 1$, be a union of smooth irreducible divisors on X intersecting transversally such that $X \smallsetminus Y_1$ is affine (this holds, e.g., if X is projective and Y_1 is a smooth hyperplane section), and let $U = X \smallsetminus Y$. Then, for any prime ℓ, and with the notation of the beginning of Section 2, the sequence*

$$0 \to H^d(\overline{U},\mathbb{Q}_\ell/\mathbb{Z}_\ell(d))_{G_K} \overset{e}{\to} H^0(\overline{Y^{[d]}},\mathbb{Q}_\ell/\mathbb{Z}_\ell)_{G_K} \overset{d_2}{\to} H^2(\overline{Y^{[d-1]}},\mathbb{Q}_\ell/\mathbb{Z}_\ell(1))_{G_K}$$

is exact, where we write $\overline{X} = X \times_K \overline{K}$, and similarly for the other varieties, and where we regard G_K as $\mathrm{Gal}(\overline{K}/K^{\mathrm{per}})$ for the perfect hull K^{per} of K in $\overline{K}$, which is the maximal inseparable extension of K inside $\overline{K}$ and is a perfect field. Moreover, e and d_2 are defined as follows. The specialization map e is *induced by the compositions*

$$(3.2)\qquad H^d(\overline{U},\mathbb{Q}_\ell/\mathbb{Z}_\ell(d)) \xrightarrow{\delta} H^{d-1}(\overline{Y_{i_d} \smallsetminus (\bigcup_{i\neq i_d} Y_i)},\mathbb{Q}_\ell/\mathbb{Z}_\ell(d-1))$$
$$\xrightarrow{\delta} H^{d-2}(\overline{Y_{i_{d-1},i_d} \smallsetminus (\bigcup_{i\neq i_{d-1},i_d} Y_i)},\mathbb{Q}_\ell/\mathbb{Z}_\ell(d-2))$$
$$\longrightarrow \cdots \xrightarrow{\delta} H^1(\overline{Y_{i_2,\dots,i_d} \smallsetminus (\bigcup_{i\neq i_2,\dots,i_d} Y_i)},\mathbb{Q}_\ell/\mathbb{Z}_\ell(1)) \xrightarrow{\delta} H^0(\overline{Y_{i_1,\dots,i_d}},\mathbb{Q}_\ell/\mathbb{Z}_\ell),$$

where each δ is the connecting morphism in the obvious Gysin sequence. On the other hand, $d_2 = \sum_{\mu=1}^{d} (-1)^\mu\, \delta_\mu$, *where δ_μ is induced by the Gysin map associated to the inclusions*

$$Y_{i_1,\dots,i_d} \hookrightarrow Y_{i_1,\dots,\hat{i}_\nu,\dots,i_d}$$

(*and $\hat{i}_\nu$ means omission of i_ν, as usual*).

Proof. We note that here the absolute Galois group G_K of K can be regarded as the Galois group $\mathrm{Gal}(\overline{K}/K^{\mathrm{per}})$, where $K^{\mathrm{per}} \subset \overline{K}$ is the perfect hull of K (the maximal inseparable extension of K in $\overline{K}$). For ℓ invertible in K, we could replace the algebraic closure of K by its separable closure and, by a standard property of étale cohomology, we get isomorphic groups above, which are the ones used in Section 2. For $\ell = \mathrm{char}(K)$ however, we need $\overline{K}$ to be the algebraic closure.

Write $H^i(\overline{Z}, j)$ instead of $H^i(\overline{Z},\mathbb{Q}_\ell/\mathbb{Z}_\ell(j))$, for short, and note that U is affine, because $X \setminus Y_1$ is affine and $U \hookrightarrow X$ is an affine morphism, because Y is defined by a locally principal ideal. Hence $H^d(\overline{U}, d)$ is divisible, since $H^{d+1}(\overline{U},\mathbb{Z}/\ell\mathbb{Z}(d)) = 0$ by weak Lefschetz, which also holds for $\ell = \mathrm{char}(K)$; see [Suw95, Lemma 2.1]. We now proceed by induction on r, the number of components of Y. If $r = 1$, then the Gysin sequence

$$\cdots \to H^d(\overline{X}, d) \to H^d(\overline{U}, d) \to H^{d-1}(\overline{Y}, d-1) \to \cdots$$

shows that $H^d(\overline{U}, d)$ is mixed with weights $-d$ and $-d+1$; see 1.7(e)–(f). Hence, using 1.7(d), we can only have $H^d(\overline{U}, d)_{G_K} \neq 0$ and $Y^{[d]} \neq \emptyset$ for $d = 1$. In this case we have an exact sequence

$$0 \to H^1(\overline{X}, 1) \to H^1(\overline{U}, 1) \xrightarrow{\delta} H^0(\overline{Y}, 0) \to H^2(\overline{X}, 1) \to 0.$$

Without loss of generality, we may assume that X is geometrically irreducible over K. (Otherwise, this is the case over a finite extension K' of K, and everything reduces to this situation, since we have induced modules.) Letting $C = \mathrm{im}(\delta)$, we have

$$H^1(\overline{U}, 1)_{G_K} \xrightarrow{\sim} C_{G_K}$$

since $H^1(\overline{X},1)_{G_K} = 0$ ($H^1(\overline{X},1)$ is divisible and of weight -1), and there is an exact sequence

$$0 \to C \to \mathrm{Ind}^K_{K(x)}(\mathbb{Q}_\ell/\mathbb{Z}_\ell) \to \mathbb{Q}_\ell/\mathbb{Z}_\ell \to 0,$$

where $K(x)$ is the residue field of the unique point $x \in Y$, which is a separable extension of K, by assumption. But this sequence stays exact after taking cofixed modules: the action of G_K factors through a finite quotient G, and $H_1(G, \mathbb{Q}_\ell/\mathbb{Z}_\ell) = 0$. (This group is dual to $H^1(G,\mathbb{Z}_\ell) = 0$.) Putting things together, we have an exact sequence

$$0 \to H^1(\overline{U},1)_{G_K} \xrightarrow{e} H^0(\overline{Y},0)_{G_K} \xrightarrow{d_2} H^2(\overline{X},1)_{G_K} \to 0.$$

Now let $r > 1$. Then $Z = \bigcup_{i=1}^{r-1} Y_i$ is a divisor with normal crossings on X which fulfills all the assumptions of the theorem, and the same is true for $Z_r = Y_r \cap Z = \bigcup_{i=1}^{r-1}(Y_r \cap Y_i)$ on Y_r. □

We claim that we obtain a commutative diagram

(3.3)

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & H^d(\overline{X \smallsetminus Z}, d)_{G_K} \arrow[r] \arrow[d, "e"] \arrow[dr, phantom, "(1)"] & H^d(\overline{U}, d)_{G_K} \arrow[r, "\delta"] \arrow[d, "e"] \arrow[dr, phantom, "(2)"] & H^{d-1}(Y_r \smallsetminus (Y_r \cap Z), d-1)_{G_K} \arrow[r] \arrow[d, "e"] & 0 \\
0 \arrow[r] & H^0(\overline{Z^{[d]}}, 0)_{G_K} \arrow[r] \arrow[d, "d_2"] \arrow[dr, phantom, "(3)"] & H^0(\overline{Y^{[d]}}, 0)_{G_K} \arrow[r] \arrow[d, "d_2"] \arrow[dr, phantom, "(4)"] & H^0(\overline{(Y_r \cap Z)^{[d-1]}}, 0)_{G_K} \arrow[r] \arrow[d, "d_2"] & 0 \\
0 \arrow[r] & H^2(\overline{Z^{[d-1]}}, 1)_{G_K} \arrow[r] & H^2(\overline{Y^{[d-1]}}, 1)_{G_K} \arrow[r] & H^2(\overline{(Y_r \cap Z)^{[d-2]}}, 1)_{G_K} \arrow[r] & 0
\end{tikzcd}$$

with exact rows. The first row comes from the Gysin sequence for $(X \smallsetminus Z, Y_r \smallsetminus Z_r)$,

$$\cdots \to H^d(\overline{X \smallsetminus Z}, d) \to H^d(\overline{U}, d) \xrightarrow{\delta} H^{d-1}(\overline{Y_r \smallsetminus Z_r}, d-1) \to 0,$$

in which $H^{d+1}(\overline{X \smallsetminus Z}, d) = 0$ by weak Lefschetz. Next note that

$$\begin{aligned}
Y^{[\nu]} &= \coprod_{1 \le i_1 < \cdots < i_\nu \le r} Y_{i_1,\ldots,i_\nu}, \\
Z^{[\nu]} &= \coprod_{1 \le i_1 < \cdots < i_\nu \le r-1} Y_{i_1,\ldots,i_\nu}, \\
Y_r \cap Z^{[\nu-1]} = (Y_r \cap Z)^{[\nu-1]} &= \coprod_{1 \le i_1 < \cdots < i_{\nu-1} \le r-1} Y_r \cap Y_{i_1,\ldots,i_{\nu-1}}, \\
&= \coprod_{1 \le i_1 < \ldots < i_\nu = r} Y_{i_1,\ldots,i_\nu}
\end{aligned}$$

so that $Y^{[\nu]} = Z^{[\nu]} \coprod (Y_r \cap Z)^{[\nu-1]}$. Hence one has commutative diagrams (3.4)

$$\begin{tikzcd}
0 \arrow[r] & H^i(\overline{Z^{[d]}}, j) \arrow[r] \arrow[d, "d_2"] \arrow[dr, phantom, "(3')"] & H^i(\overline{Y^{[d]}}, j) \arrow[r] \arrow[d, "d_2"] \arrow[dr, phantom, "(4')"] & H^i(\overline{(Y_r \cap Z)^{[d-1]}}, j) \arrow[r] \arrow[d, "d_2"] & 0 \\
0 \arrow[r] & H^{i+2}(\overline{Z^{[d-1]}}, j+1) \arrow[r] & H^{i+2}(\overline{Y^{[d-1]}}, j+1) \arrow[r] & H^{i+2}(\overline{(Y_r \cap Z)^{[d-2]}}, j+1) \arrow[r] & 0
\end{tikzcd}$$

with canonically split exact rows, where both left maps are $d_2 = \sum\limits_{\mu=1}^{\nu} (-1)^\mu \delta_\mu$, with δ_μ being induced by the inclusions

$$Z_{i_1,\ldots,i_\nu} \hookrightarrow Z_{i_1,\ldots,\widehat{i_\mu},\ldots,i_\nu} \quad \text{and} \quad Y_{i_1,\ldots,i_\nu} \hookrightarrow Y_{i_1,\ldots,\widehat{i_\mu},\ldots,i_\nu}$$

respectively, and the right-hand d_2 is defined as $d_2 = \sum\limits_{\mu=1}^{\nu-1} (-1)^\mu \delta_\mu$, with δ_μ being induced by the inclusions

$$Y_{i_1,\ldots,i_{\nu-1}} \cap Y_r \hookrightarrow Y_{i_1,\ldots,\widehat{i_\mu},\ldots,i_{\nu-1}} \cap Y_r.$$

In fact, the commuting of $(3')$ is trivial, and the square $(4')$ commutes since it commutes with $\delta_\mu, 1 \le \mu \le \nu - 1$ in place of d_2, whereas δ_ν vanishes after projection onto $(Y_r \cap Z)^{(\nu-1)}$ (the last component of $(i_1 \ldots, \hat{i}_\nu)$ cannot be r). This implies the commutativity of (3) and (4), and the exactness of the two involved rows.

The commutativity of (2) is clear: For $1 \le i_1 < \cdots < i_d = r$, the specialization map (3.2) is the composition

$$H^d(\overline{U}, d) \xrightarrow{\delta} H^{d-1}(\overline{Y_r \smallsetminus Z}, d-1) \xrightarrow{\delta} H^{d-2}(\overline{Y_{i_{\nu-1}} \cap Y_r \smallsetminus (\bigcup_{i \ne i_{\nu-1,r}} (Y_i \cap Y_r))}, d-2)$$
$$\cdots \xrightarrow{\delta} H^1(\overline{Y_{i_2}, \ldots, r \smallsetminus (\bigcup_{i \ne i_2,\ldots,r} (Y_i \cap Y_r)}, 1) \xrightarrow{\delta} H^0(\overline{Y_{i_1,\ldots,r}}, 0).$$

The commutativity of (1) is implied by the commutativity of

$$\begin{tikzcd}
e: & H^d(\overline{U}, d) \arrow[r, "\delta"] & H^{d-1}(\overline{Y_{i_d} \smallsetminus \bigcup_{i \ne i_d} Y_i}, d-1) \arrow[r] & \cdots & \cdots \arrow[r] & H^0(\overline{Y_{i_1,\ldots,i_d}}, 0) \\
e: & H^d(\overline{X - Z}, d) \arrow[r, "\delta"] \arrow[u] & H^{d-1}(\overline{Y_{i_d} \smallsetminus \bigcup_{i \ne i_{d,r}} Y_i}, d-1) \arrow[r] \arrow[u] & \cdots & \cdots \arrow[r] \arrow[u] & H^0(\overline{Y_{i_1,\ldots,i_d}}, 0) \arrow[u, equal]
\end{tikzcd}$$

for $1 \le i_1 < \cdots < i_d < r$, where the vertical maps are the restriction maps for the open immersions obtained by deleting Y_r everywhere. (Note that $Y_{i_1,\ldots,i_d} \cap Y_r = \emptyset$ for $i_d \ne r$.) This commutativity follows from the compatibility of the corresponding Gysin sequences with restriction to open subschemes.

Given the diagram (3.3), we can carry out the induction step. It is easy to check that the middle column is a complex, and by induction the left and right column are exact. Hence the middle column is exact, by a straightforward diagram chase.

We give a first application to function fields. Recall the following definition [CT95, 2.1.8 and 4.1.1].

Definition 3.5. Let k be a field, and let F be a function field over k. For an integer n invertible in k, the unramified cohomology $H^i_{\mathrm{nr}}(F/k, \mathbb{Z}/n\mathbb{Z}(j)) \subseteq H^i(F, \mathbb{Z}/n\mathbb{Z}(j))$ is defined as the subset of elements lying in the image of

$$H^i_{\text{ét}}(\operatorname{Spec} A, \mathbb{Z}/n\mathbb{Z}(j)) \to H^i(F, \mathbb{Z}/n\mathbb{Z}(j))$$

for all discrete valuation rings $A \subseteq F$ containing k.

If λ is a discrete valuation of F which is trivial on k, and if A_λ and $k(\lambda)$ are the associated valuation ring and residue field, respectively, then one has an exact Gysin sequence

$$\begin{aligned} \cdots H^i_{\text{ét}}(\operatorname{Spec} A_\lambda, \mathbb{Z}/n\mathbb{Z}(j)) &\to H^i(F, \mathbb{Z}/n\mathbb{Z}(j)) \\ &\xrightarrow{\delta_\lambda} H^{i-1}(k(\lambda), \mathbb{Z}/n\mathbb{Z}(j-1)) \to \cdots, \end{aligned}$$

since purity is known to hold in this situation. We call the map δ_λ the residue map for λ. This shows

LEMMA 3.6. *One has*

$$H^i_{\mathrm{nr}}(F/k, \mathbb{Z}/n\mathbb{Z}(j)) = \ker\Big(H^i(F, \mathbb{Z}/n\mathbb{Z}(j)) \to \prod_\lambda H^{i-1}(k(\lambda), \mathbb{Z}/n\mathbb{Z}(j-1))\Big),$$

where the sum is over all discrete valuations λ of F/k, and the components of the map are the residue maps δ_λ.

We will need the following fact (cf. [CT95, 2.1.8 and 4.1.1]).

PROPOSITION 3.7. *Let X be a smooth proper variety over k, and let $F = k(X)$ be its function field. Then*

$$H^i_{\mathrm{nr}}(F/k, \mathbb{Z}/n\mathbb{Z}(j)) = \ker\Big(H^i(F, \mathbb{Z}/n\mathbb{Z}(j)) \xrightarrow{\delta_X} \bigoplus_{x \in X^1} H^{i-1}(k(x), \mathbb{Z}/n\mathbb{Z}(j-1))\Big),$$

where $X^i = \{x \in X \mid \dim \mathcal{O}_{X,x} = i\}$ for $i \geq 0$, $k(x)$ is the residue field of $x \in X$, and δ is the map from the Bloch-Ogus complexes for étale cohomology [BO74]. *In particular,*

$$H^i_{\mathrm{nr}}(F/k, \mathbb{Z}/n\mathbb{Z}(j)) \cong H^0_{\mathrm{Zar}}(X, \mathcal{H}^i_n(j)),$$

where $\mathcal{H}^i_n(j)$ is the Zariski sheaf on X associated to the presheaf

$$U \mapsto H^i_{\text{ét}}(U, \mathbb{Z}/n\mathbb{Z}(j)).$$

Proof. Since we need a variant below, we recall the beautiful argument. First note that, by definition of the Bloch-Ogus sequence, the components of δ_X are the residue maps $\delta_{X,x} := \delta_{\lambda(x)}$, where $\lambda(x)$ is the discrete valuation associated to x (so that $A_{\lambda(x)} = \mathcal{O}_{X,x}$ and $k(\lambda(x)) = k(x)$). This shows that the kernel of 3.6 is contained in the kernel of 3.7. Conversely, let $A \subset F$ be a discrete valuation ring. Then by properness of X we have a factorization $\mathrm{Spec}(F) \to \mathrm{Spec}(A) \to X$, and hence a factorization

$$\mathrm{Spec}(F) \to \mathrm{Spec}(A) \to \mathrm{Spec}(\mathcal{O}_{X,x}) \to X,$$

where $x \in X$ is the image of the closed point of $\mathrm{Spec}(A)$. By the results of Bloch and Ogus [BO74], since X is smooth, the sequence

$$\begin{aligned} H^i(\mathrm{Spec}(\mathcal{O}_{X,x}), \mathbb{Z}/n\mathbb{Z}(j)) \xrightarrow{j^*} H^i(\mathrm{Spec}(F), \mathbb{Z}/n\mathbb{Z}(j)) & \\ \to \bigoplus_x H^{i-1}(k(x), \mathbb{Z}/n\mathbb{Z}(j-1)) & \end{aligned}$$

is exact, where x runs over the codimension 1 points of $\mathrm{Spec}(\mathcal{O}_{X,x})$. Therefore any element in the kernel of 3.7 lies in the image of j^*, and hence in the image of $H^i(\mathrm{Spec}\, A, \mathbb{Z}/n\mathbb{Z}(j)) \to H^i(F, \mathbb{Z}/n\mathbb{Z}(j))$, by the above factorization. Since A was arbitrary, the element lies in the unramified cohomology. $\square$

The second main result of the present section is now

THEOREM 3.8. *Let K be a global field, let $n \in \mathbb{N}$ be invertible in K, and let F be a function field in d variables over K, $d > 0$, such that K is separably closed in F. For every place v of K, let $K_{(v)}$ be the Henselization of K at v, and let $F_{(v)} = FK_{(v)}$ be the corresponding function field over $K_{(v)}$. Then the restriction maps induce an isomorphism*

$$H^{d+2}_{\mathrm{nr}}(F/K, \mathbb{Z}/n\mathbb{Z}(d+1)) \xrightarrow{\ \sim\ } \bigoplus_v H^{d+2}_{\mathrm{nr}}(F_{(v)}/K_{(v)}, \mathbb{Z}/n\mathbb{Z}(d+1)).$$

Proof. It suffices to consider the case $n = \ell^m$, where ℓ is a prime invertible in K. Moreover, it suffices to show that the map

$$H^{d+2}_{\mathrm{nr}}(F/K, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d+1)) \longrightarrow \bigoplus_v H^{d+2}_{\mathrm{nr}}(F_{(v)}/K_{(v)}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d+1)) \tag{3.9}$$

is an isomorphism. In fact, if this holds, the bijectivity for $n = \ell^m$ follows from the commutative diagram with exact columns

$$\begin{array}{ccc} H^{d+2}_{\mathrm{nr}}(F/K, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d+1)) & \longrightarrow & \bigoplus_v H^{d+2}_{\mathrm{nr}}(F_{(v)}/K_{(v)}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d+1)). \\ \uparrow{\scriptstyle \ell^m} & & \uparrow{\scriptstyle \ell^m} \\ H^{d+2}_{\mathrm{nr}}(F/K, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d+1)) & \longrightarrow & \bigoplus_v H^{d+2}_{\mathrm{nr}}(F_{(v)}/K_{(v)}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d+1)) \\ \uparrow & & \uparrow \\ H^{d+2}_{\mathrm{nr}}(F/K, \mathbb{Z}/\ell^m\mathbb{Z}(d+1)) & \longrightarrow & \bigoplus_v H^{d+2}_{\mathrm{nr}}(F_{(v)}/K_{(v)}, \mathbb{Z}/\ell^m\mathbb{Z}(d+1)) \\ \uparrow & & \uparrow \\ 0 & & 0 \end{array}$$

The exactness of the columns follows from Lemma 3.6 and the exactness of

$$0 \to H^{i+1}(L, \mathbb{Z}/\ell^m\mathbb{Z}(i)) \longrightarrow H^{i+1}(L, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i)) \xrightarrow{\ell^m} H^{i+1}(L, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i))$$

for any field L and any natural number i, which in turn follows from the theorem of Rost and Voevodsky, i.e., the proof of the Bloch-Kato conjecture $BK(L, i, \ell)$; see the introduction.

We know already from Theorem 2.10 that (3.9) is injective; therefore it suffices to show the surjectivity in (3.9).

Case 3.8.A. First assume that there is a smooth projective variety X over K with function field $K(X) = F$. This is certainly the case if K is a number field. In fact, there is a geometrically irreducible variety U over K with $K(U) = F$, and after possibly shrinking U we may assume that U is smooth. Then, by resolution of singularities (more precisely by property RS2(U) from the beginning of Section 2) it can be embedded in a smooth projective variety X over K as an open subvariety. Then, abbreviating $H^i(?, j)$ for $H^i(?, \mathbb{Q}_\ell/\mathbb{Z}_\ell(j))$, Proposition 3.7 gives a commutative diagram with exact rows and injections j, j_ν, (3.10)

$$\begin{tikzcd}
\bigoplus_v H^{d+2}_{\mathrm{nr}}(F_{(v)}/K_{(v)}, d+1) \arrow[r, hook, "\oplus_v j_v"] & \bigoplus_v H^{d+2}(F_{(v)}, d) \arrow[r, "\oplus_v \delta_{X_{(v)}}"] & \bigoplus_v \bigoplus_{y \in X^1_{(v)}} H^{d+1}(K_{(v)}(y), d) \\
H^{d+2}_{\mathrm{nr}}(F/K, d+1) \arrow[r, hook, "j"] \arrow[u, "\beta'"] & H^{d+2}(F, d+1) \arrow[r, "\delta_X"] \arrow[u, "\beta(F/K)"] & \bigoplus_{x \in X^1} H^{d+2}(K(x), d), \arrow[u, "\beta''"]
\end{tikzcd}$$

where $X_{(v)} = X \times_K K_{(v)}$ and in which $\beta(F/K)$ is the restriction map, β'' is induced by the restrictions for the field extensions $K_{(v)}/K(x)$ for y lying above x, and β' is the induced map. Note that $F_{(v)} \cong F \otimes_K K_{(v)}$ is the function field of $X_{(v)}$ over $K_{(v)}$. The commutativity of the right square is easily checked (contravariance of Gysin sequences for pro-étale maps). Now, for $x \in X^1$ and a place v of K, every $y \in X_{(v)}$ lying above x is again of codimension 1, since $X_{(v)} \to X$ is integral. Hence,

$$\coprod_{\substack{y \in X^1_{(v)} \\ y|x}} \mathrm{Spec}(K_{(v)}(y)) = \coprod_{\substack{y \in X_{(v)} \\ y|x}} \mathrm{Spec}(K_{(v)}(y)) = X_{(v)} \times_X K(x), \tag{3.11}$$

the fibre of the pro-étale morphism $X_{(v)} \to X$ over x. This is again isomorphic to

$$\begin{aligned}
(X \times_K K_{(v)}) \times_X K(x) &\cong \mathrm{Spec}(K(x) \otimes_K K_{(v)}) \\
&\cong \mathrm{Spec}(K(x) \otimes_{K\{x\}} (K\{x\} \otimes_K K_{(v)})) \\
&\cong \coprod_{w|v} \mathrm{Spec}(K(x) \otimes_{K\{x\}} K\{x\}_{(w)}),
\end{aligned} \tag{3.12}$$

where $K\{x\}$ is the separable closure of K in $K(x)$ (which is a finite extension of K) and where w runs over the places w of $K\{x\}$ above v. This shows that β'' can be identified with the map

$$\bigoplus_{x\in X^1}\beta(K(x)/K\{x\}): \bigoplus_{x\in X^1} H^{d+1}(K(x),d) \longrightarrow \bigoplus_{x\in X^1}\bigoplus_{w\in P(K\{x\})} H^{d+1}(K(x)_{(w)},d), \tag{3.13}$$

where $P(K\{x\})$ is the set of places of the global field $K\{x\}$. Hence β'' is injective as well as $\beta(F/K)$, by Theorem 2.10. (Note that $K(x)$, for $x\in X^1$, is a function field in $d-1$ variables over $K\{x\}$.) By diagram (3.10) it now suffices to show that the following map is injective:

$$\text{coker } \beta(F/K) \longrightarrow \text{coker } \beta'' = \bigoplus_{x\in X^1} \text{coker } \beta(K(x)/K\{x\}). \tag{3.14}$$

Lemma 3.15. *The map* (3.14) *can be identified with the map of cofixed modules*

$$H^d(F\overline{K},d)_{G_K} \longrightarrow \bigoplus_{x\in X^1} H^{d-1}(K(x)\otimes_K \overline{K},d-1)_{G_K} \cong \Big(\bigoplus_{y\in\overline{X}^1} H^{d-1}(\overline{K}(y),d-1)\Big)_{G_K}$$

induced by the residue map $\delta_{\overline{X}}$ for $\overline{X}=X\times_K\overline{K}$.

Proof. By (3.11) and (3.12), the map β'' can also be identified with the map

$$\bigoplus_{x\in X^1}\Big[\beta(K(x)/K):\ H^{d+1}(K(x),d)\longrightarrow \bigoplus_{v\in P(K)} H^{d+1}(K(x)\otimes_K K_{(v)},d)\Big].$$

Therefore the map (3.14) can be identified with the map coker $\beta_1\to$ coker β_2 induced by the commutative diagram

$$\begin{tikzcd}
\bigoplus_v H^2(K_v,H^d(F\overline{K},d+1)) \arrow[r] & \bigoplus_{x\in X^1}\bigoplus_v H^2(K_v,H^{d-1}(K(x)\otimes_K\overline{K},d)), \\
H^2(K,H^d(F\overline{K},d+1)) \arrow[r] \arrow[u, "\beta_1"] & \bigoplus_{x\in X^1} H^2(K,H^{d-1}(K(x)\otimes_K\overline{K},d)) \arrow[u, "\beta_2"]
\end{tikzcd}$$

where the vertical maps are the obvious restriction maps, and the horizontal maps are induced by the residue maps

$$H^d(F\overline{K},d+1)\longrightarrow \bigoplus_{x\in X^1} H^{d-1}(K(x)\otimes_K\overline{K},d)\cong \bigoplus_{y\in\overline{X}^1} H^{d-1}(\overline{K}(y),d)$$

for $\overline{X}$. This follows from Proposition 1.2, Remark 1.4(a), and the fact that the Hochschild-Serre spectral sequence is compatible with the connecting morphisms for Gysin sequences. The latter statement follows from the fact that the Hochschild-Serre spectral sequence for étale (hyper)cohomology of complexes is functorial with respect to morphisms in the derived category and that the Gysin isomorphisms are compatible with pro-étale base change.

Finally, for all discrete torsion $\mathbb{Z}_\ell$-G_K-modules M, there are canonical isomorphisms

$$\operatorname{coker}\Big[\beta_M : H^2(K,M) \to \bigoplus_v H^2(K_v,M)\Big] \xrightarrow{\sim} M(-1)_{G_K}, \tag{3.16}$$

which are functorial in M; see (the proof of) Proposition 1.2. This proves Lemma 3.15. □

We are now ready to prove Theorem 3.8 for Case 3.8.A, which assumes the existence of X with $F = K(X)$. By Lemma 3.15 it suffices to show the following more general theorem, which will also be used in the later sections.

THEOREM 3.17. *Let K be a finitely generated field with perfect hull L and algebraic closure $\overline{K}$, let X be a smooth proper irreducible variety of dimension d over K, and let ℓ be a prime. Assume that ℓ is invertible in K or that condition* RS1(U) *(see the beginning of Section* 2*) holds for any open $U \subset X \times_K L$. Then the map*

$$\Big(\bigoplus_{y\in\overline{X}^0} H^d(\overline{K}(y),\mathbb{Q}_\ell/\mathbb{Z}_\ell(d))\Big)_{G_K} \longrightarrow \Big(\bigoplus_{x\in\overline{X}^1} H^{d-1}(\overline{K}(x),\mathbb{Q}_\ell/\mathbb{Z}_\ell(d-1))\Big)_{G_K}$$

induced by the Bloch-Ogus complex for $\overline{X} = X \times_K \overline{K}$ (via taking coinvariants under G_K) is injective.

Proof. We note that here we regard the absolute Galois group G_K of K as $\mathrm{Gal}(\overline{K}/L)$, and we may replace K by L and call this K again. Moreover, the above map can be identified with a map

$$H^d(K(X)\otimes_K \overline{K},\mathbb{Q}_\ell/\mathbb{Z}_\ell(d))_{G_K} \longrightarrow \bigoplus_{x\in X^1} H^{d-1}(K(x)\otimes_K \overline{K},\mathbb{Q}_\ell/\mathbb{Z}_\ell(d-1))_{G_K}.$$

Let a be an element in the kernel of the above map. Then there is an open $U \subset X$ such that a is the image of an element $a_U \in H^d(\overline{U},\mathbb{Q}_\ell/\mathbb{Z}_\ell(d))_{G_K}$, where $\overline{U} = U \times_K \overline{K}$. We distinguish the following three cases.

Case 3.17.A. First assume that RS1(U) holds and that K is infinite. Then there exists an open embedding $U \subset X'$ into a smooth projective variety X' such that the complement $Y = X' \smallsetminus U$ is a divisor with simple normal crossings, say with smooth components Y_i $(i = 1,\dots,r)$. By possibly applying Bertini's theorem as in Section 2 (before Proposition 2.2) and removing a suitable hyperplane section (which does not matter for our purposes), we may assume that $X' \smallsetminus Y_1$ is affine, i.e., that $U \subseteq X' \supseteq Y$ satisfies the assumptions of Theorem 3.1.

Next we note that the kernel of the map in 3.17 only depends on $F = K(X)$ and not on the smooth projective model X of F. In the case of a global field K and ℓ invertible in K, this is clear from Lemma 3.15, the diagram (3.10), and Proposition 3.7 for F and the $F_{(v)}$. In general, the argument is the same as in the proof of Proposition 3.7, noting the following two facts. By properness, for any $x' \in (X')^1$, the discrete valuation ring $\mathcal{O}_{X',x'}$ dominates a local ring $\mathcal{O}_{X,y}$ of X. Moreover, for this regular ring and any finite Galois extension M/K, the Bloch-Ogus sequence of $\mathrm{Gal}(M/K)$-modules

$$\begin{aligned} H^d(\mathrm{Spec}(\mathcal{O}_{X,y} \otimes_K M), \Lambda(d)) &\xrightarrow{j^*} H^d(K(X) \otimes_K M), \Lambda(d)) \\ &\to \bigoplus_{x \in X^1} H^{d-1}(K(x) \otimes_K M, \Lambda(d-1)) \end{aligned}$$

is exact for $\Lambda = \mathbb{Q}_\ell/\mathbb{Z}_\ell$, by purity for the semi-local ring $\mathcal{O}_{X,y} \otimes_K M$. In addition, it stays exact after taking coinvariants under $\mathrm{Gal}(M/K)$, because the Gersten resolution is universally exact (see [CTHK97, Cor. 6.2.4 together with Ex. 7.3(1)] (for $\ell \neq \mathrm{char}(K)$) and loc. cit. Ex. 7.4(3) (for $\ell = \mathrm{char}(K)$ and the Tate twist d). By passing to the inductive limit we get the corresponding result for $\overline{K}$ and G_K in place of M and $\mathrm{Gal}(M/K)$. The same holds for the discrete valuation ring $\mathcal{O}_{X',x'}$. As in the proof of Proposition 3.7 we get that the kernel of 3.17 for X lies in the kernel of 3.17 for X'. Interchanging the roles of X and X' we get the wanted equality.

Therefore we may replace X above by X' and call it X again. Now we claim that a_U lies in the kernel of the map

$$e : H^d(\overline{U}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d))_{G_K} \to H^0(\overline{Y^{[d]}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(0))_{G_K}$$

introduced in Theorem 3.1. Since the assumptions of 3.1 are fulfilled for U, we then conclude that a_U is zero and hence a is zero as wanted.

With the notation of (3.2), the claimed vanishing of $e(a_U)$ follows from the following commutative diagram for each $(i_1, \dots, i_d)$ and each $y \in Y_{i_1,\dots,i_d}$:
(3.18)

$$\begin{array}{ccccc} H^d(\overline{U}, d)_{G_K} & \longrightarrow & H^{d-1}(\overline{Y_{i_d} \smallsetminus (\bigcup_{i \neq i_d} Y_i)}, d\text{-}1)_{G_K} & \longrightarrow & H^{d-2}(\overline{Y_{i_{d-1},i_d} \smallsetminus (\bigcup_{i \neq i_{d-1},i_d} Y_i)}, d\text{-}2)_{G_K} \\ \downarrow & & \downarrow & & \downarrow \\ H^d(F\overline{K}, d)_{G_K} & \longrightarrow & H^{d-1}(K(y_{i_d}) \otimes_K \overline{K}, d\text{-}1)_{G_K} & \longrightarrow & H^{d-2}(K(y_{i_{d-1},i_d}) \otimes_K \overline{K}, d\text{-}2)_{G_K} \end{array}$$

$$\begin{array}{ccccc} \dots & \longrightarrow & H^1(\overline{Y_{i_2,\dots,i_d} \smallsetminus Y_{i_1}}, 1)_{G_K} & \longrightarrow & H^0(\overline{\{y\}}, 0)_{G_K} \\ & & \downarrow & & \downarrow \\ \dots & \longrightarrow & H^1(K(y_{i_2,\dots,i_d}) \otimes_K \overline{K}, 1)_{G_K} & \longrightarrow & H^0(K(y_{i_1,\dots,i_d}) \otimes_K \overline{K}, 0)_{G_K} \end{array}$$

in which $y_{\underline{i}}$ is the generic point of the component $Y_{\underline{i}}^y$ of $Y_{\underline{i}}$ in which y lies, for any $\underline{i} = (i_r, \dots, i_d)$, so that $K(y_{\underline{i}})$ is the function field of $Y_{\underline{i}}^y$. In fact, the maps in the bottom line are all induced by the residue maps, by definition, and the image of $\overline{a_U}$ under the left vertical map is $\overline{a}$. As we have noted, the image of $\overline{a}$ in $H^{d-1}(K(y_{i_d}) \otimes_K \overline{K}, d-1)_{G_K}$ vanishes for every choice of $(i_1, \ldots, i_d)$, $1 \leq i_1 < \cdots < i_d \leq r$. (Note that $y_{i_d} \in \tilde{X}^1$.) Therefore the image of $\overline{a_U}$ in $H^0(\overline{\{y\}}, 0)_{G_K}$ vanishes for every $y \in \overline{Y^{[d]}}$ as claimed. This finishes the proof of Case 3.17.A. □

Case 3.17.B. For the case of a finite field K, we note the following. First of all we have canonical functorial isomorphisms $M_{G_K} \cong H^1(K, M)$ for all discrete G_K-modules M. Therefore the map in 3.17 can be identified with the map

$$H^{d+1}(K(X), \mathbb{Q}_\ell/\mathbb{Z}_\ell(d)) \to \bigoplus_{x \in X^1} H^d(k(x), \mathbb{Q}_\ell/\mathbb{Z}_\ell(d-1)),$$

and it follows directly from Proposition 3.7 that the kernel of this map is independent of X and, in fact, equal to the unramified cohomology

$$H^{d+1}_{\mathrm{nr}}(K(X)/K, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d)).$$

If a, U, and a_U are as above and we have a good compactification $U \subset X \supset Y$ as above, we may not have a suitable hyperplane section defined over K, but we get one after taking a base extension to a field extension K'/K of degree prime to ℓ. Then we conclude that a maps to zero under the restriction Res : $H^{d+1}(K(X), \mathbb{Q}_\ell/\mathbb{Z}_\ell(d)) \to H^{d+1}(K'(X), \mathbb{Q}_\ell/\mathbb{Z}_\ell(d))$. But this map is injective, by the existence of the corestriction Cor in the other direction with Cor Res = multiplication with $[K' : K]$ which is prime to ℓ. This finishes the proof of Case 3.17.B. □

Case 3.17.*C.* Finally consider the case that $\mathrm{char}(K) = p > 0$ and $\ell \neq p$, and that we have no good compactification of U, where a, U, and a_U are as above. By the weaker resolution of singularities due to Gabber (see Theorem 2.11), we get a diagram

$$\begin{array}{ccccc} U' & \subset & X' & \supset & Y' \\ \downarrow & & \downarrow \pi & & \downarrow \\ U & \subset & X & \supset & Y, \end{array} \tag{3.19}$$

where X' is a geometrically irreducible, smooth, and projective variety over a finite extension K' of K with ℓ not dividing $[K' : K]$, π is a proper surjective morphism which is generically finite of degree prime to ℓ, $U' = \pi^{-1}(U)$, and $Y' = \pi^{-1}(Y)$ is a divisor with strict normal crossings on X'. Since $\ell \neq p$, the $\mathbb{Q}_\ell/\mathbb{Z}_\ell$-cohomology does not change under radicial maps, and we may pass to the perfect hull of K in K' and thus assume that K'/K is separable. Then X and X' are smooth projective over K.

For any smooth variety V over K, let $\overline{\mathcal{H}}^i(d)_{G_K}$ be the Zariski sheaf on V associated to the presheaf $U \mapsto H^i(\overline{U}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d))_{G_K}$ for $U \subseteq V$ open, where

$\overline{U} = U \times_K \overline{K}$. Then the Bloch-Ogus theory and the universal exactness used above show that the kernel of the map in Theorem 3.17 is canonically isomorphic to $H^0(X, \overline{\mathcal{H}}^d(d)_{G_K})$, and the pull-back maps for étale cohomology induce a natural pull-back map

$$\pi^* : H^0(X, \overline{\mathcal{H}}^d(d)_{G_K}) \to H^0(X', \overline{\mathcal{H}}^d(d)_{G_K}).$$

We claim that this map is injective. In fact, it embeds into the restriction map for the function fields, which can be factored as

$$(3.20)\quad H^d(K(X)\otimes_K \overline{K})_{G_K} \to H^d(K'(X)\otimes_K \overline{K})_{G_K} \to H^d(K'(X')\otimes_{K'} \overline{K})_{G_{K'}},$$

where we have omitted the coefficients $\mathbb{Q}_\ell/\mathbb{Z}_\ell(d)$. Both restriction maps are injective, because the degrees $[K'(X) : K(X)]$ and $[K'(X') : K'(X)]$ are prime to ℓ. (See the corestriction argument at the end of the proof of Theorem 2.10, which also works for the modules of coinvariants.) Thus the restriction map is injective and π^* is injective as well.

Now for an element $a \in H^0(X, \overline{\mathcal{H}}^i(d)_{G_K})$, its image $a' \in H^0(X', \overline{\mathcal{H}}^i(d)_{G_K})$ is represented by the image $a_{U'}$ of a_U under the restriction map

$$H^d(\overline{U}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d))_{G_K} \to H^d(\overline{U'}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d))_{G'_K}.$$

By the choice of U', and the Cases 3.17.A and 3.17.B, we get that $a_{U'} = 0$, hence $a' = 0$, and so $a = 0$ by the injectivity of (3.20). This finishes the proof of Case 3.17.C and thus Theorem 3.17. □

Case 3.8.B. With similar arguments we can now also complete the proof of Theorem 3.8, in the case where the function field F over K does not have any smooth projective model, but the prime ℓ is invertible in K. Let U be an affine integral geometrically irreducible variety of dimension d over K with function field $K(U) = F$. Then we have an open embedding $U \subset X$ into a projective integral variety, and we get again a diagram as in (3.19). (The smoothness of U or X was not needed.) Let $F' = K'(X')$. It follows from the definition of unramified cohomology that the morphism $F/K \to F'/K'$, i.e., the commutative diagram

$$\begin{array}{ccc} F & \to & F', \\ \uparrow & & \uparrow \\ K & \to & K' \end{array}$$

induces a restriction map $H^{d+1}_{\mathrm{nr}}(F/K, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d)) \longrightarrow H^{d+1}_{\mathrm{nr}}(F'/K', \mathbb{Q}_\ell/\mathbb{Z}_\ell(d))$. (Any discrete valuation of F' over K' induces by restriction a discrete valuation of F over K.) The same holds for the morphism $F_{(v)}/K_{(v)} \to F'_{(w)}/K'_{(w)}$ for a place v of K and a place w of K' above v. Moreover, the commutative diagrams

$$\begin{array}{ccc} F_{(v)}/K_{(v)} & \to & F'_{(w)}/K'_{(w)} \\ \uparrow & & \uparrow \\ F/K & \to & F'/K' \end{array}$$

induce a commutative diagram

$$\begin{array}{ccc} \bigoplus_v \ H^{d+1}_{\mathrm{nr}}(F_{(v)}/K_{(v)}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d)) & \longrightarrow & \bigoplus_w \ H^{d+1}_{\mathrm{nr}}(F'_{(w)}/K'_{(w)}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d)). \\ \uparrow{\scriptstyle \beta'(F/K)} & & \uparrow{\scriptstyle \beta'(F'/K')} \\ H^{d+1}_{\mathrm{nr}}(F/K, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d)) & \longrightarrow & H^{d+1}_{\mathrm{nr}}(F'/K', \mathbb{Q}_\ell/\mathbb{Z}_\ell(d)) \end{array}$$

By the first part of the proof (having the existence of the smooth projective model X' of F'), the cokernel of $\beta'(F'/K')$ is zero. Now we claim that the map coker $\beta'(F/K) \to$ coker $\beta'(F'/K')$ induced by the diagram is injective; then we have coker $\beta'(F/K) = 0$ as wanted. First of all, the above diagram is obtained from the following commutative diagram of restriction maps:

$$\begin{array}{ccc} \bigoplus_v \ H^{d+1}(F_{(v)}/K_{(v)}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d)) & \longrightarrow & \bigoplus_w \ H^{d+1}(F'_{(w)}/K'_{(w)}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d)) \\ \uparrow{\scriptstyle \beta(F/K)} & & \uparrow{\scriptstyle \beta(F'/K')} \\ H^{d+1}(F/K, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d)) & \longrightarrow & H^{d+1}(F'/K', \mathbb{Q}_\ell/\mathbb{Z}_\ell(d)) \end{array}$$

by passing to the unramified subgroups, so that we have a commutative diagram

$$\begin{array}{ccc} \text{coker } \beta(F/K) & \xrightarrow{r} & \text{coker } \beta(F'/K'). \\ \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle i'} \\ \text{coker } \beta'(F/K) & \xrightarrow{r'} & \text{coker } \beta'(F'/K') \end{array}$$

We claim that the maps i and r are injective; then we obtain the injectivity of r'.

The injectivity of i follows from the commutative diagram with exact rows

$$\begin{array}{ccccc} \bigoplus_v H^{d+2}_{nr}(F_{(v)}/K_{(v)}, d+1) & \xhookrightarrow{\oplus_v j_v} & \bigoplus_v H^{d+2}(F_{(v)}, d+1) & \xrightarrow{\oplus_v \delta_v} & \bigoplus_v \bigoplus_{\mu \in P_v} H^{d+1}(K_{(v)}(y), d). \\ {\scriptstyle \beta'(F/K)}\uparrow & & {\scriptstyle \beta(F/K)}\uparrow & & {\scriptstyle \beta''}\uparrow \\ H^{d+2}_{nr}(F/K, d+1) & \xhookrightarrow{j} & H^{d+2}(F, d+1) & \xrightarrow{\delta} & \oplus_{\lambda \in P} H^{d+1}(K(\lambda), d) \end{array}$$

Here we have omitted the coefficients $\mathbb{Q}_\ell/\mathbb{Z}_\ell$, P (resp. P_v) is the set of discrete valuations of F/K (resp. $F_{(v)}/K_{(v)}$), and the components of δ (resp. δ_v) are the residue maps for the valuations $\lambda \in P$ (resp. $\mu \in P_v$). The restriction map β'' is defined as follows. If the valuation μ of $F_{v)}/K_{(v)}$ lies over the valuation λ of F/K, then the corresponding component is induced by the inclusion of the corresponding valuation rings; otherwise the component is zero. The map β'' is injective, by similar arguments as in the beginning of the proof of 3.8:

If A_λ is the valuation ring of λ and M/K is a finite separable field extension, then $A_\lambda \otimes_K M$ is a regular semi-local ring of dimension 1, and hence the integral closure of A_λ in $F \otimes_K M$. Hence the extensions of the valuation λ to $F \otimes_K M$ correspond to the fiber above the closed point of A_λ, i.e., to the points of $\mathrm{Spec}(K(\lambda) \otimes_K M)$. Therefore the extensions of λ to $F_{(v)}$ correspond to the points of $\mathrm{Spec}(K(\lambda) \otimes_K K_{(v)}) \cong \oplus_w \mathrm{Spec}(K\{\lambda\}_{(w)})$, where $K\{\lambda\}$ is the separable closure of K in $K(\lambda)$ (which is a finite extension), and w runs over all places of $K\{\lambda\}$ lying above v. Thus the restriction of β'' to the component for λ can be identified with the map $\beta(K(\lambda)/K\{\lambda\})$, which is injective by Theorem 2.10. (Note that $K(\lambda)$ is a geometrically irreducible function field in $d-1$ variables over $K\{\lambda\}$.) The injectivity of β'' now implies by a diagram chase that $i : \mathrm{coker}\ \beta'(F/K) \to \mathrm{coker}\ \beta(F/K)$ is injective.

Now we consider the injectivity of r. Since the ℓ-adic cohomology does not change under radicial/inseparable extensions, we may assume that F'/F and K'/K are separable. Then we get a commutative diagram with exact rows

$$\begin{tikzcd}
H^{d+2}(F', d+1) \arrow[r] & \bigoplus_v H^{d+2}(F' \otimes_K K_{(v)}, d+1) \arrow[r, two heads] & \mathrm{coker}\ \beta(F'/K') \\
H^{d+2}(F, d+1) \arrow[r] \arrow[u, "\mathrm{Res}"] & \bigoplus_v H^{d+2}(F \otimes_K K_{(v)}, d+1) \arrow[r, two heads] \arrow[u, "\mathrm{Res}"] & \mathrm{coker}\ \beta(F/K) \arrow[u, "r"']
\end{tikzcd}$$

induced by the restriction for F'/F. The cokernel in the upper row can indeed be identified with coker $\beta(F'/K')$ (compare Remark 1.4(a)), and then the right-hand map can be identified with r as indicated. Now the finite étale map $\pi : \mathrm{Spec}(F') \to \mathrm{Spec}(F)$ also induces compatible downward maps π_* in the left square, such that π_* is the usual corestriction Cor on the left and such that $\pi_*\mathrm{Res}$ is the multiplication with $[F' : F]$ in both cases. Since F'/F and the extensions $K_{(v)}/K$ are separable, these properties follow from obvious calculations in Galois cohomology, which are left to the readers. We obtain an induced map π_* on the right with $\pi_* r$ being the multiplication with $[F' : F]$. Since this degree is prime to ℓ, and we consider $\mathbb{Q}_\ell/\mathbb{Z}_\ell$-coefficients, we obtain the injectivity of r as claimed.

Remark 3.21. In the considerations of this section we have preferred to work with the explicit descriptions (via Gysin and specialization maps) of the maps e and d_2 in Theorem 3.1, but we note that they coincide with the corresponding edge morphisms and differentials of the weight spectral sequence (2.3) for $U \subset X \supset Y$, up to signs.

4. **A Hasse principle for Bloch-Ogus-Kato complexes**

Let X be an excellent scheme, let $n \geq 1$ be an integer, and let $r, s \in \mathbb{Z}$. Under some conditions on X, n, and (r, s), there are homological complexes of

Gersten-Bloch-Ogus-Kato type

$$\begin{aligned} C^{r,s}(X,\mathbb{Z}/n\mathbb{Z}) : \cdots \to & \bigoplus_{x\in X_i} H^{r+i}(k(x),\mathbb{Z}/n\mathbb{Z}(s+i)) \\ & \xrightarrow{\partial} \bigoplus_{x\in X_{i-1}} H^{r+i-1}(k(x),\mathbb{Z}/n\mathbb{Z}(s+i-1)) \\ \to \cdots \to & \bigoplus_{x\in X_i} H^{r+i} \cdots \to \bigoplus_{x\in X_0} H^{r}(k(x),\mathbb{Z}/n\mathbb{Z}(s)), \end{aligned}$$

where the term for $X_i = \{x \in X \mid \dim(x) = i\}$ is placed in degree i. If X is separated of finite type over a field L, n is invertible in L, and (r,s) are arbitrary, these complexes where defined by Bloch and Ogus [BO74], by using the étale homology for such schemes, defined by

$$H_a(X,\mathbb{Z}/n\mathbb{Z}(b)) := H^{-a}(X, Rf^!\mathbb{Z}/n\mathbb{Z}(-b)), \tag{4.1}$$

where $f : X \to \operatorname{Spec} L$ is the structural morphism and $Rf^!$ is the extraordinary inverse image functor on constructible étale $\mathbb{Z}/n\mathbb{Z}$-sheaves defined in [SGA, XVIII]. In fact, Bloch and Ogus constructed a niveau spectral sequence

$$E^1_{p,q}(X,\mathbb{Z}/n\mathbb{Z}(b)) = \bigoplus_{x\in X_p} H_{p+q}(k(x),\mathbb{Z}/n\mathbb{Z}(b)) \Rightarrow H_{p+q}(X,\mathbb{Z}/n\mathbb{Z}(b)),$$

where, by definition, $H_a(k(x),\mathbb{Z}/n\mathbb{Z}(b)) = \varinjlim H_a(U,\mathbb{Z}/n\mathbb{Z}(b))$ for $x \in X$, where the limit is over all open subschemes $U \subseteq \overline{\{x\}}$ of the Zariski closure of x. By purity, there is an isomorphism $H_a(U,\mathbb{Z}/n\mathbb{Z}(b)) \cong H^{2p-a}(U,\mathbb{Z}/n\mathbb{Z}(p-b))$ for U irreducible and smooth of dimension p over L. Thus one has a canonical isomorphism

$$E^1_{p,q}(X,\mathbb{Z}/n\mathbb{Z}(b)) \cong \bigoplus_{x\in X_p} H^{p-q}(k(x),\mathbb{Z}/n\mathbb{Z}(p-b)). \tag{4.2}$$

This is clear for a perfect field L, because then $\overline{\{x\}}$ is generically smooth. So the limit can be carried out over the smooth $U \subseteq \overline{\{x\}}$, and

$$\varinjlim_{U\subseteq\overline{\{x\}}} H^{2p-a}(U,\mathbb{Z}/n\mathbb{Z}(p-b)) = H^{2p-a}(k(x),\mathbb{Z}/n\mathbb{Z}(p-b)),$$

since $\varprojlim U = \operatorname{Spec} k(x)$, and since étale cohomology commutes with this limit. For a general field L, we may pass to the separable hull, because of invariance of étale cohomology with respect to base change with radical morphisms.

Using the identification (4.2), one may define

$$C^{r,s}(X,\mathbb{Z}/n\mathbb{Z}) = E^1_{*,-r}(X,\mathbb{Z}/n\mathbb{Z}(-s)).$$

With this definition, one obtains the following description of the differential:

$$\partial : \bigoplus_{x\in X_i} H^{r+i}(k(x),\mathbb{Z}/n\mathbb{Z}(s+i)) \to \bigoplus_{x\in X_{i-1}} H^{r+i-1}(k(x),\mathbb{Z}/n\mathbb{Z}(s+i-1)).$$

We may assume that L is perfect. For $y \in X_i$ and $x \in X_{i-1}$, let $\partial_{y,x} = \partial^X_{y,x}$ be the (y,x)-component of ∂. If $x \notin \overline{\{y\}}$, then $\partial_{y,x} = 0$. If x is a smooth point of $\overline{\{y\}}$, then there is an open smooth neighbourhood $x \in U \subseteq \overline{\{y\}}$. Moreover, any $\alpha \in H^{r+i}(k(x), \mathbb{Z}/n\mathbb{Z}(s+i))$ lies in the image of $H^{r+i}(V, \mathbb{Z}/n\mathbb{Z}(s+i)) \to H^{r+i}(k(x), \mathbb{Z}/n\mathbb{Z}(s+i))$ for some open $V \subseteq U$. Moreover, by making U (and V) smaller we may assume that $Z = U \setminus V$ is irreducible and smooth as well and that x is the generic point of Z. Then one has a commutative diagram

$$\begin{array}{ccc} H^{r+i}(k(y), \mathbb{Z}/n\mathbb{Z}(s+i) & \xrightarrow{\partial_{y,x}} & H^{r+i-1}(k(x), \mathbb{Z}/n\mathbb{Z}(s+i-1)), \\ \uparrow & & \uparrow \\ H^{r+i}(V, \mathbb{Z}/n\mathbb{Z}(s+i) & \xrightarrow{\partial} & H^{r+i-1}(Z, \mathbb{Z}/n\mathbb{Z}(s+i-1)) \end{array} \tag{4.3}$$

where the vertical maps come from passing to the generic points, and ∂ is the connecting morphism for the Gysin sequence for (U, Z). This determines $\partial_{y,x}(\alpha)$. If $x \in \overline{\{y\}}$, but is not necessarily a smooth point of $Y = \overline{\{y\}}$, let $\tilde{Y} \to Y$ be the normalization of Y. Any point $x' \in \tilde{Y}$ above x has codimension 1 and thus is a regular point in $\tilde{Y}$. Since the niveau spectral sequence is covariant with respect to proper morphisms, there is a commutative diagram (4.4)

$$\begin{array}{ccc} H^{r+i}(k(y), \mathbb{Z}/n\mathbb{Z}(s+i)) & \xrightarrow{\bigoplus_{x'|x} \partial^{\tilde{Y}}_{y,x'}} & \bigoplus_{x'|x} H^{r+i-1}(k(x'), \mathbb{Z}/n\mathbb{Z}(s+i-1)) \\ \| & & \downarrow \pi_* \\ H^{r+i}(k(y), \mathbb{Z}/n\mathbb{Z}(s+i)) & \xrightarrow{\partial_{y,x}} & H^{r+i-1}(k(x), \mathbb{Z}/n\mathbb{Z}(s+i-1)), \end{array}$$

where π_* is induced by $\pi : \tilde{Y} \to Y \hookrightarrow X$. One can check that $\pi_*((\alpha_{x'})) = \sum_{x'|x} \mathrm{Cor}_{x'|x}(\alpha_{x'})$, where $\mathrm{Cor}_{x'|x} : H^\mu(k(x'), \mathbb{Z}/n\mathbb{Z}(\nu)) \to H^\mu(k(x), \mathbb{Z}/n\mathbb{Z}(\nu))$ is the corestriction for the finite extension $k(x')/k(x)$. (This also makes sense if this extension has some inseparable part.) Since $\partial^{\tilde{Y}}_{y,x'}$ can be treated as before, this determines $\partial_{y,x}$.

For a function field L of transcendence degree d over a perfect field k of characteristic $p > 0$, a separated scheme X of finite type over L and n a power of p, it was shown in [JS03] and [JSS14, 3.11.3] that the theory of Bloch and Ogus can be literally extended to this situation for the case $b = -d$, where the cohomology groups $H^i(X, \mathbb{Z}/n\mathbb{Z}(j))$ and $H^i(k(x), \mathbb{Z}/n\mathbb{Z}(j))$ are defined as in (0.2).

For any excellent scheme X and arbitrary n, the complexes $C^{r,s}(X, \mathbb{Z}/n\mathbb{Z})$ were defined by Kato (and named $C^{r,s}_n(X)$; cf. [Kat86]), in a more direct way, by using the Galois cohomology of discrete valuation fields, assuming the following condition:

($*$) If $r = s+1$ and p is a prime dividing n, then for any $x \in X_0$ with $\mathrm{char}(k(x)) = p$, one has $[k(x) : k(x)^p] \le s$.

It is shown in [JSS14] that both definitions agree (up to well-defined signs) for varieties over fields in the cases discussed above.

Now let K be a global field, and let X be a variety over K. For every place v of K, let $X_v = X \times_K K_v$. Then condition ($*$) holds for X and the X_v for $(r,s) = (2,1)$ and arbitrary n. Moreover, one has natural restriction maps $\alpha_v : C^{r,s}(X, \mathbb{Z}/n\mathbb{Z}) \to C^{r,s}(X_v, \mathbb{Z}/n\mathbb{Z})$, and Kato stated the following conjecture (see Conjecture 2).

Conjecture 4.5. *Let X be connected, smooth, and proper. Then the α_v induce isomorphisms*

$$H_a(C^{2,1}(X, \mathbb{Z}/n\mathbb{Z})) \xrightarrow{\sim} \bigoplus_v H_a(C^{2,1}(X_v, \mathbb{Z}/n\mathbb{Z})) \text{ for all } a \neq 0,$$

and an exact sequence

$$0 \to H_0(C^{2,1}(X, \mathbb{Z}/n\mathbb{Z})) \to \bigoplus_v H_0(C^{2,1}(X_v, \mathbb{Z}/n\mathbb{Z})) \to \mathbb{Z}/n\mathbb{Z} \to 0.$$

Remark 4.6. (a) For $X = \mathrm{Spec}\ (L)$, L any finite extension of K, the cohomology groups vanish for $a \neq 0$, and the sequence for $a = 0$ becomes the exact sequence

$$0 \to H^2(L, \mathbb{Z}/n\mathbb{Z}(1)) \to \bigoplus_{w \in P(L)} H^2(L_w, \mathbb{Z}/n\mathbb{Z}(1)) \to \mathbb{Z}/n\mathbb{Z} \to 0,$$

which is the n-torsion of the classical exact sequence

$$0 \to \mathrm{Br}(L) \to \bigoplus_{w \in p(L)} Br(L_w) \xrightarrow{\sum_w \mathrm{inv}_w} \mathbb{Q}/\mathbb{Z} \to 0$$

for the Brauer groups (where $\mathrm{inv}_w : \mathrm{Br}(L_w) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}$ is the 'invariant' map). Thus Kato's conjecture is a generalization of this famous sequence to higher dimensional varieties.

(b) As we will see below, the α_v induce a map

$$\alpha_{X,n} : C^{2,1}(X, \mathbb{Z}/n\mathbb{Z}) \longrightarrow \bigoplus_v C^{2,1}(X_v, \mathbb{Z}/n\mathbb{Z}). \tag{4.7}$$

Let $C'(X, \mathbb{Z}/n\mathbb{Z})$ be its cokernel. Then Conjecture 4.5 is implied by the following two statements:

(i) $\alpha_{X,n}$ is injective;
(ii) $H_0(C'(X, \mathbb{Z}/n\mathbb{Z})) = \mathbb{Z}/n\mathbb{Z}$, and $H_a(C'(X, \mathbb{Z}/n\mathbb{Z})) = 0$ for $a > 0$.

Conversely, Conjecture 4.5 implies (i) and (ii) by the known case (a) and induction on dimension, provided the occurring function fields have smooth and proper models over the perfect hull of K (which holds over number fields).

We prove the following on Conjecture 4.5; compare with Remark 4.6(b).

THEOREM 4.8. *Let K be a global field, let $n \in \mathbb{N}$ be invertible in K, and let X be a connected, smooth proper variety over K.*

(a) *The map $\alpha_{X,n} : C^{2,1}(X, \mathbb{Z}/n\mathbb{Z}) \longrightarrow \bigoplus_v C^{2,1}(X_v, \mathbb{Z}/n\mathbb{Z})$ is well defined and injective.*
(b) *Let $C'(X, \mathbb{Z}/n\mathbb{Z})$ be the cokernel of $\alpha_{X,n}$. If K is a number field or if resolution of singularities holds over K (see Definition 4.18), then*

$$H_a(C'(X, \mathbb{Z}/n\mathbb{Z})) = \begin{cases} 0, & a \neq 0, \\ \mathbb{Z}/n\mathbb{Z}, & a = 0. \end{cases}$$

Proof of Theorem 4.8(a). First note that the restriction map α_v factors as

$$\alpha_v : C^{2,1}(X, \mathbb{Z}/n\mathbb{Z}) \xrightarrow{\beta_v} C^{2,1}(X_{(v)}, \mathbb{Z}/n\mathbb{Z}) \to C^{2,1}(X_v, \mathbb{Z}/n\mathbb{Z}),$$

where $X_{(v)} = X \times_K K_{(v)}$. These maps of complexes have components

$$\bigoplus_{x \in X_i} H^{i+2}(k(x), \mathbb{Z}/n\mathbb{Z}(i+1)) \to \bigoplus_{x \in (X_{(v)})_i} H^{i+2}(k(x), \mathbb{Z}/n\mathbb{Z}(i+1))$$
$$\to \bigoplus_{x \in (X_v)_i} H^{i+2}(k(x), \mathbb{Z}/n\mathbb{Z}(i)),$$

which in turn can be written as the sum, over all $x \in X_i$, of maps

$$H^{i+2}(k(x), \mathbb{Z}/n\mathbb{Z}(i+1)) \to \bigoplus_{\substack{x' \in (X_{(v)})_i \\ x' | x}} H^{i+2}(k(x'), \mathbb{Z}/n\mathbb{Z}(i+1))$$
$$\to \bigoplus_{\substack{x'' \in (X_v)_i \\ x'' | x}} H^{i+2}(k(x''), \mathbb{Z}/n\mathbb{Z}(i+1)).$$

By the same reasoning as in the proof of Theorem 3.8, the first map can be identified with

$$H^{i+2}(k(x), \mathbb{Z}/n\mathbb{Z}(i+1)) \to \bigoplus_{w|v} H^{i+2}(k(x)K\{x\}_{(w)}, \mathbb{Z}/n\mathbb{Z}(i+1)),$$

where $K\{w\}$ is the separable closure of K in $k(x)$, which is a finite extension of K, and where w runs over all places of $K\{x\}$ above v. Note that $k(x)$ is a function field of transcendence degree i over $K\{x\}$. Therefore the restriction maps above induce an injective map into the direct sum

$$H^{i+2}(k(x), \mathbb{Z}/n\mathbb{Z}(i+1)) \to \bigoplus_{w \in P(K\{x\})} H^{i+2}(k(x)_{(w)}, \mathbb{Z}/n\mathbb{Z}(i+1))$$

by Proposition 1.2 and Theorem 2.10 (since the latter implies Theorem 0.4; see the introduction). This shows that we get maps

$$\text{(4.9)} \quad \alpha_{X,n} : C^{2,1}(X, \mathbb{Z}/n\mathbb{Z}) \xrightarrow{\beta_{X,n}} \bigoplus_v C^{2,1}(X_{(v)}, \mathbb{Z}/n\mathbb{Z}) \to \bigoplus_v C^{2,1}(X_v, \mathbb{Z}/n\mathbb{Z})$$

of which the first one is injective. The claim of Theorem 4.8(a) therefore follows from the next claim.

PROPOSITION 4.10. *Let K be a global field, and let v be a place of K. For every variety V over $K_{(v)}$, every integer n invertible in K and all $r, s \in \mathbb{Z}$, the natural map*

$$C^{r,s}(V, \mathbb{Z}/n\mathbb{Z}) \to C^{r,s}(V \times_{K_{(v)}} K_v, \mathbb{Z}/n\mathbb{Z})$$

is injective.

Proof. In degree i, this map is the sum over all $x \in V_i$ of restriction maps

$$H^{r+i}(k(x), \mathbb{Z}/n\mathbb{Z}(s+i)) \to \bigoplus_{\substack{x' \in \tilde{V}_i \\ x' | x}} H^{r+i}(k(x'), \mathbb{Z}/n\mathbb{Z}(s+i)),$$

where $\tilde{V} = V \times_{K_{(v)}} K_v$. For $x \in V_i$, $k(x)$ is the function field of the integral subscheme (of dimension i) $Z = \overline{\{x\}} \subseteq V$. Because $K_v/K_{(v)}$ is separable, and $K_{(v)}$ is algebraically closed in K_v, $\tilde{Z} = Z \times_{K_{(v)}} K_v \hookrightarrow \tilde{V}$ is a closed integral subscheme whose generic point $\tilde{x}$ is in $\tilde{V}_i$ and lies above x. Let L be the algebraic closure of $K_{(v)}$ in $k(x)$. Then $\tilde{L} = L \otimes_{K_{(v)}} K_v$ is a field, Z is geometrically integral over L with function field $L(Z) = k(x)$, and $\tilde{Z} = Z \times_L \tilde{L}$ with function field $\tilde{L}(\tilde{Z}) = k(\tilde{x})$. Moreover, L is henselian, with completion $\tilde{L}$. Thus it follows from Theorem 2.12 that the natural map

$$H^{r+i}(k(x), \mathbb{Z}/n\mathbb{Z}(s+i)) \to H^{r+i}(k(\tilde{x}), \mathbb{Z}/n\mathbb{Z}(s+i))$$

is injective for all $r, s, i \in \mathbb{Z}$ and all $n \in \mathbb{N}$ invertible in $K_{(v)}$. This proves Proposition 4.10 and thus Theorem 4.8(a). $\square$

Now we start the proof of Theorem 4.8(b). The following rigidity result is shown in [Jan15].

THEOREM 4.11. *Let K be a global field, and let v be a place of K. For every variety V over $K_{(v)}$, every integer n invertible in K, and all $r, s \in \mathbb{Z}$, the natural morphism of complexes*

$$C^{r,s}(V, \mathbb{Z}/n\mathbb{Z}) \to C^{r,s}(V \times_{K_{(v)}} K_v, \mathbb{Z}/n\mathbb{Z})$$

is a quasi-isomorphism, i.e., induces isomorphisms in the homology.

In view of this result and of the factorization (4.9), it suffices to prove Theorem 4.8(b) after replacing X_v by $X_{(v)}$ for each v. In fact, by Theorem 4.11 we have a canonical quasi-isomorphism

$$\overline{C}(X,\mathbb{Q}_\ell/\mathbb{Z}) \xrightarrow{\text{quis}} C'(X,\mathbb{Z}/n\mathbb{Z}), \tag{4.12}$$

where the complex $\overline{C}(X,\mathbb{Q}_\ell/\mathbb{Z})$ is defined by the exact sequence

$$0 \to C^{2,1}(X,\mathbb{Z}/n\mathbb{Z}) \overset{\beta_{X,n}}{\to} \bigoplus_v C^{2,1}(X_{(v)},\mathbb{Z}/n\mathbb{Z}) \to \overline{C}(X,\mathbb{Z}/n\mathbb{Z}) \to 0. \tag{4.13}$$

So our task is to show $H_0(\overline{C}(X,\mathbb{Z}/n\mathbb{Z})) = \mathbb{Z}/n\mathbb{Z}$, and $H_a(\overline{C}(X,\mathbb{Z}/n\mathbb{Z})) = 0$ for $a \neq 0$, if X is connected, smooth, and proper over K. Note that all complexes in (4.12) are concentrated in degrees $0,\dots,d := \dim(X)$.

Next we note the following.

LEMMA 4.14. *For $n \in \mathbb{N}$ invertible in K, the complex $\overline{C}(X,\mathbb{Z}/n\mathbb{Z})$ can be canonically identified with the complex $C^{0,0}(\overline{X},\mathbb{Z}/n\mathbb{Z})_{G_K}$:*

$$\cdots \to \bigoplus_{x\in X_r} H^r(K(x)\otimes_K \overline{K},\mathbb{Z}/n\mathbb{Z}(r))_{G_K} \to \cdots \to \bigoplus_{x\in X_0} H^0(K(x)\otimes_K \overline{K},\mathbb{Z}/n\mathbb{Z}(0))_{G_K},$$

obtained from the Kato complex $C^{0,0}(\overline{X},\mathbb{Z}/n\mathbb{Z})$ by taking coinvariants.

Proof. This follows easily by means of the arguments used in the proof of Lemma 3.15, together with the explicit description of the differentials in this complex in (4.4) and the covariance of the Hochschild-Serre spectral sequence for corestrictions. □

With these tools at hand, we can reduce the proof of Theorem 4.8(b) to a $\mathbb{Q}_\ell/\mathbb{Z}_\ell$-version. Note that it suffices to prove Theorem 4.8(b) for $n = \ell^m$, for any prime ℓ invertible in K and any natural number m. For any prime ℓ and any integers r,s and any scheme Z where it is defined, define the Kato complex $C^{r,s}(Z,\mathbb{Q}_\ell/\mathbb{Z}_\ell)$ as the direct limit of the complexes $C^{r,s}(Z,\mathbb{Z}/\ell^n\mathbb{Z})$ via the transition maps induced by the canonical injections $\mathbb{Z}/\ell^n\mathbb{Z} \to \mathbb{Z}/\ell^{n+1}$. Then we have, in fact,

LEMMA 4.15. *Let K be a global field, let X be a connected, smooth proper variety over K, and let ℓ be any prime. Define the map*

$$\beta_{\ell^\infty} : C^{2,1}(X,\mathbb{Q}_\ell/\mathbb{Z}_\ell) \longrightarrow \bigoplus_v C^{2,1}(X_{(v)},\mathbb{Q}_\ell/\mathbb{Z}_\ell)$$

as the inductive limit of the maps β_{X,ℓ^m} for all $m \in \mathbb{N}$, and let $\overline{C}(X,\mathbb{Q}_\ell/\mathbb{Z}_\ell)$ be its cokernel.

(a) *The injectivity of β_{X,ℓ^∞} is equivalent to the injectivity of the β_{ℓ^m} for all m.*

(b) *To have* $H_0(\overline{C}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell) = \mathbb{Q}_\ell/\mathbb{Z}_\ell$ *and* $H_a(\overline{C}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell)) = 0$ *for* $a \neq 0$ *is equivalent to having* $H_0(\overline{C}(X, \mathbb{Z}/\ell^m\mathbb{Z})) = \mathbb{Z}/\ell^m\mathbb{Z}$ *and* $H_a(\overline{C}(X, \mathbb{Z}/\ell^m\mathbb{Z})) = 0$ *for all* $a \neq 0$ *for all* $m \in \mathbb{N}$.

Proof. Let F be a function field in d variables over K. Then the theorem of Rost-Voevodsky, more precisely, the validity of the condition $\mathrm{BK}(F, d+1, \ell)$ recalled in the introduction, implies that the sequence

$$0 \to H^{d+2}(F, \mathbb{Z}/\ell^n\mathbb{Z}(d+1)) \xrightarrow{i} H^{d+2}(F, \mathbb{Q}/\mathbb{Z}_\ell(d+1)) \xrightarrow{\ell^n} H^{d+2}(F, \mathbb{Q}/\mathbb{Z}_\ell(d+1))$$

is exact (see the introduction), and the same holds for the fields F_v, for all places v of K. By applying this to all residue fields of X and X_v, for all v, we get a commutative diagram with exact rows

$$\begin{tikzcd}
\bigoplus_v C^{2,1}(X_{(v)}, \mathbb{Z}/\ell^n\mathbb{Z}) \arrow[r, hook, "i'"] & \bigoplus_v C^{2,1}(X_{(v)}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \arrow[r, "\ell^n"] & \bigoplus_v C^{2,1}(X_{(v)}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \\
C^{2,1}(X, \mathbb{Z}/\ell^n\mathbb{Z}) \arrow[u, "\beta_{X,\ell^n}"] \arrow[r, hook, "i"] & C^{2,1}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \arrow[u, "\beta_{X,\ell^\infty}"] \arrow[r, "\ell^n"] & C^{2,1}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \arrow[u, "\beta_{X,\ell^\infty}"]
\end{tikzcd} \tag{4.16}$$

with injections i and i', and we deduce the claim in (a).

Now we consider the cokernels of the vertical maps. First assume that K is a global function field. Then we claim that we even have an exact sequence

$$\begin{aligned} 0 \to H^{d+2}(F, \mathbb{Z}/\ell^n\mathbb{Z}(d+1)) &\xrightarrow{i} H^{d+2}(F, \mathbb{Q}/\mathbb{Z}_\ell(d+1)) \\ &\xrightarrow{\ell^n} H^{d+2}(F, \mathbb{Q}/\mathbb{Z}_\ell(d+1)) \to 0, \end{aligned}$$

and similarly for all $F_{(v)}v$. In fact, we have $H^{d+3}(F, \mathbb{Z}/\ell^n\mathbb{Z}(d+1)) = 0$: If $\ell \neq \mathrm{char}(F)$, then F has ℓ-cohomological dimension $d+2$, and if $\ell = p = \mathrm{char}(F)$, then we have $H^{d+3}(F, \mathbb{Z}/\ell^n\mathbb{Z}(d+1)) = H^2(F, W_n\Omega_{F,\log})$, but F has p-cohomological dimension 1. Exactly the same reasoning works for $F_{(v)}$. Writing, for n a positive integer or $n = \infty$,

$$C_n := \mathrm{coker}[H^{d+2}(F, \mathbb{Z}/\ell^n\mathbb{Z}(d+1)) \longrightarrow \bigoplus_v \; H^{d+2}(F_{(v)}, \mathbb{Z}/\ell^n\mathbb{Z}(d+1))],$$

where we set $\mathbb{Z}/\ell^\infty\mathbb{Z} := \mathbb{Q}_\ell/\mathbb{Z}_\ell$, we obtain an exact sequence

$$0 \to C_n \to C_\infty \to C_\infty \to 0.$$

Applied to the points of X and the X_v and the morphisms β_{X,ℓ^m} for $n \in \mathbb{N} \cup \{\infty\}$, we now get an exact sequence

$$0 \to \overline{C}(X, \mathbb{Z}/\ell^n\mathbb{Z}) \to \overline{C}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \xrightarrow{\ell^n} \overline{C}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \to 0,$$

and the claim of 4.15(b) follows in this case.

Now let K be a number field. If $\ell \neq 2$ or if K has no real places, then F has ℓ-cohomological dimension $d+2$, and we can argue in the same way. In general, we can argue in the following way. In any case, a function field F of transcendence degree d over an algebraically closed field has ℓ-cohomological dimension d for ℓ invertible in F. It follows that for any variety X over K the sequence

$$C^{0,0}(\overline{X}, \mathbb{Z}/\ell^n\mathbb{Z}) \to C^{0,0}(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \xrightarrow{\ell^n} C^{0,0}(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \to 0 \tag{4.17}$$

is exact, where $\overline{X} = X \times_K \overline{K}$ for an algebraic closure $\overline{K}$ of K. Obviously this complex stays exact if we pass to the co-invariants under G_K, the absolute Galois group of K.

Now β_{X,ℓ^∞} is injective by Theorem 2.10. (Note that ℓ is invertible in K.) By (4.17) and Lemma 4.14, we get the following commmutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & & & 0 & \\
0 \arrow[r] & C^{2,1}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \arrow[r, "\beta_{X,\ell^\infty}"] & \bigoplus_v C^{2,1}(X_{(v)}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \arrow[r] & \overline{C}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & C^{2,1}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \arrow[r, "\beta_{X,\ell^\infty}"] \arrow[u, "\ell^n"'] & \bigoplus_v C^{2,1}(X_{(v)}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \arrow[r] \arrow[u, "\ell^n"'] & \overline{C}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \arrow[r] \arrow[u, "\ell^n"'] & 0 \\
 & C^{2,1}(X, \mathbb{Z}/\ell^n\mathbb{Z}) \arrow[r, "\beta_{X,\ell^n}"] \arrow[u] & \bigoplus_v C^{2,1}(X_{(v)}, \mathbb{Z}/\ell^n\mathbb{Z}) \arrow[r] \arrow[u] & \overline{C}(X, \mathbb{Z}/\ell^n\mathbb{Z}) \arrow[r] \arrow[u, "i"'] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & &
\end{tikzcd}$$

A simple diagram chase now shows that β_{X,ℓ^n} and i are injective, which gives an exact sequence

$$0 \longrightarrow \overline{C}(X, \mathbb{Z}/\ell^n\mathbb{Z}) \longrightarrow \overline{C}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \longrightarrow \overline{C}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \longrightarrow 0.$$

This implies Lemma 4.15(b). □

Definition 4.18. Let L be a perfect field. We say that resolution of singularities holds over L, or that $(\mathrm{RS})_L$ holds, if the following two conditions hold:

$(\mathrm{RS1})_L$: For any integral and proper variety X over L, there exists a proper birational morphism $\pi : \tilde{X} \to X$ such that $\tilde{X}$ is smooth over L.

$(\mathrm{RS2})_L$: For any smooth affine variety U over L, there is an open immersion $U \hookrightarrow X$ such that X is projective smooth over L and $Y = X - U$ (with the reduced subscheme structure) is a simple normal crossing divisor on X.

By Hironaka's fundamental results [Hir64a], [Hir64b], resolution of singularities holds over fields L of characteristic zero.

Using the quasi-isomorphism (4.12) as well as Lemmas 4.14 and 4.15, Theorem 4.8(b) is obviously implied by the following result.

THEOREM 4.19. *Let K be a finitely generated field with algebraic closure $\overline{K}$ and perfect hull K^{per}, let ℓ be a prime, and let X be an irreducible smooth* *proper variety over K. Assume that resolution of singularities holds over $L = K^{\mathrm{per}}$. Then for the Kato complex $\overline{C}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell) := C^{0,0}(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell)_{G_K}$, one has*

$$H_a(\overline{C}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell)) = \begin{cases} \mathbb{Q}_\ell/\mathbb{Z}_\ell, & a = 0, \\ 0, & a \neq 0. \end{cases}$$

Here G_K, the absolute Galois group of K, is regarded as $\mathrm{Gal}(\overline{K}/K^{\mathrm{per}})$.

Note that Theorem 4.19 implies Theorem 0.11. By the following lemma, it also implies Theorem 0.10, concerning Kato's conjecture over finite fields.

LEMMA 4.20. *Let k be a finite field, and let X be any variety over k.*

(a) *One has a canonical isomorphism of complexes*

$$C^{1,0}(X, \mathbb{Z}/\ell^n\mathbb{Z}) \cong C^{0,0}(X \times_k \overline{k}, \mathbb{Z}/\ell^n\mathbb{Z})_{G_k}.$$

(b) *The canonical sequence*

$$0 \to C^{1,0}(X, \mathbb{Z}/\ell^n\mathbb{Z}) \to C^{1,0}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \xrightarrow{\ell^n} C^{1,0}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \to 0$$

is exact.

Proof. Let F be a function field of transcendence degree m over k. Then one has canonical isomorphisms

$$H^{m+1}(F, \mathbb{Z}/\ell^n\mathbb{Z}(m)) \cong H^1(k, H^m(F\overline{k}, \mathbb{Z}/\ell^n\mathbb{Z}(m))) \cong H^m(F\overline{k}, \mathbb{Z}/\ell^n\mathbb{Z}(m))_{G_k},$$

where $F\overline{k}$ is the function field over $\overline{k}$ deduced from F; i.e., $F\overline{k} = F \otimes_{\{k\}} \overline{k}$, where $\{K\}$ is the algebraic closure of k in F. In fact, the first isomorphism follows from the Hochschild-Serre spectral sequence, because $F\overline{k}$ has ℓ-cohomological dimension m, and the second isomorphism comes from the canonical identification $H^1(k, M) = M_{G_k}$ for any G_k-module M if k is a finite field. By applying this to all fields $k(x)$ for $x \in X$, we obtain (a).

(b) follows from the exact sequence

$$0 \to H^{m+1}(F, \mathbb{Z}/\ell^n\mathbb{Z}(m)) \to H^{m+1}(F, \mathbb{Q}_\ell/\mathbb{Z}_\ell(m)) \xrightarrow{\ell^n} H^{m+1}(F, \mathbb{Q}_\ell/\mathbb{Z}_\ell(m)) \to 0,$$

in which the exactness on the left follows from the results of Bloch-Kato-Gabber (for $\ell = p = \mathrm{char}(k)$) and Merkurjev-Suslin-Rost-Voevodsky (for ℓ invertible in k) — see the introduction — and the exactness on the right follows from the cohomological dimension of F; compare the proof of Theorem 4.15. □

The proof of Theorem 4.19 will be given in the next section. The idea is to 'localize' the question; but for this we will have to leave the realm of smooth proper varieties. First recall that the complexes $C^{r,s}(X, \mathbb{Z}/n\mathbb{Z})$ exist for arbitrary varieties X over a field L, under the conditions on X, n, and (r, s) stated at the beginning of this section. If K is a global field, the restriction map $C^{2,1}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \to \Pi_v C^{2,1}(X_{(v)}, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ still has image in the direct sum and is injective (by Proposition 1.2 and the same argument as for 4.8 (a)), and we may define $\overline{C}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ for arbitrary varieties X over K by exactness of the sequence

$$0 \to C^{2,1}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \to \bigoplus_v C^{2,1}(X_{(v)}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \to \overline{C}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \to 0. \tag{4.21}$$

At the same time, by the same arguments as in Lemma 4.14, we have a canonical isomorphism

$$\overline{C}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \cong C^{0,0}(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell)_{G_K}$$

for ℓ invertible in K.

Definition 4.22. Let L be a field, and let $\mathcal{C}$ be a category of schemes of finite type over L such that for each scheme X in $\mathcal{C}$, also every closed immersion $i : Y \hookrightarrow X$ and every open immersion $j : U \hookrightarrow X$ is in $\mathcal{C}$.

(a) Let $\mathcal{C}_*$ be the category with the same objects as $\mathcal{C}$, but where morphisms are just the proper maps in $\mathcal{C}$. A homology theory on $\mathcal{C}$ is a sequence of covariant functors

$$H_a(-) : \mathcal{C}_* \to \text{(abelian groups)} \qquad (a \in \mathbb{Z})$$

satisfying the following conditions:

  (i) For each open immersion $j : V \hookrightarrow X$ in $\mathcal{C}$, there is a map $j^* : H_a(X) \to H_a(V)$, associated to j in a functorial way.

  (ii) If $i : Y \hookrightarrow X$ is a closed immersion in $\mathcal{C}$, with open complement $j : V \hookrightarrow X$, there is a long exact sequence (called localization sequence)

$$\cdots \xrightarrow{\delta} H_a(Y) \xrightarrow{i_*} H_a(X) \xrightarrow{j^*} H_a(V) \xrightarrow{\delta} H_{a-1}(Y) \longrightarrow \cdots .$$

  (The maps δ are called the connecting morphisms.) This sequence is functorial with respect to proper maps or open immersions, in an obvious way.

(b) A morphism between homology theories H and H' is a morphism $\phi : H \to H'$ of functors on $\mathcal{C}_*$, which is compatible with the long exact sequences from (ii).

Lemma 4.23.

(a) *Let L be a field, and let r, s, and $n \geq 1$ be fixed integers with n invertible in L, or $r \neq s+1$, or $p = \mathrm{char}(L) \mid n$ and $r = s+1$ and $[L : L^p] \leq p^s$. There is a natural way to extend the assignments*

$$X \quad \leadsto \quad H_a^{r,s}(X, \mathbb{Z}/n\mathbb{Z}) := H_a(C^{r,s}(X, \mathbb{Z}/n\mathbb{Z})) \qquad (a \in \mathbb{Z})$$

to a homology theory on the category of all varieties over L.

(b) *The same holds for the assignment*

$$X \quad \leadsto \quad \overline{H}^{r,s}_a(X,\mathbb{Z}/n\mathbb{Z}) := H_a(\overline{C}^{r,s}(X,\mathbb{Z}/n\mathbb{Z})) \qquad (a \in \mathbb{Z}),$$

where $\overline{C}^{r,s}(X,\mathbb{Z}/n\mathbb{Z}) := C^{r,s}(\overline{X},\mathbb{Z}/n\mathbb{Z})_{G_L}$, *with* $\overline{X} = X \times_L \overline{L}$ *for a separable closure of* L.

Proof. (a) The Bloch-Ogus-Kato complexes are covariant with respect to proper morphisms and contravariant with respect to open immersions. The localization sequence for a closed immersion $i : Y \hookrightarrow X$ with open complement $j : U = X \smallsetminus Y \hookrightarrow X$ is obtained by the short exact sequence of complexes

$$0 \to C^{r,s}(Y,\mathbb{Z}/n\mathbb{Z}) \xrightarrow{i_*} C^{r,s}(X,\mathbb{Z}/n\mathbb{Z}) \xrightarrow{j^*} C^{r,s}(U,\mathbb{Z}/n\mathbb{Z}) \to 0$$

which are componentwise canonically split (cf. also [JS03, Cor. 2.10]).

(b) This follows from (a), because the mentioned splitting is equvariant, so that the sequences stay exact after taking coinvariants. □

The mentioned localization is now obtained by the following observation.

LEMMA 4.24. *Let* L *be a perfect field, let* $\mathcal{C}$ *be a category of schemes of finite type over* L *as in* 4.20, *and let* $\varphi : H \to \widetilde{H}$ *be a morphism of homology theories on the category* $\mathcal{C}_*$ *of all schemes in* $\mathcal{C}$ *with proper morphisms. For every integral variety* Z *over* L, *let* $L(Z)$ *be its function field. Define*

$$H_a(L(Z)) := \varinjlim H_a(U),$$

where the limit is over all nonempty open subvarieties U *of* Z, *and define* $\widetilde{H}_a(L(Z))$ *similarly. Suppose the following holds for every integral variety* Z *of dimension* d *over* L:

(i) $H_a(L(Z)) = 0$ *for* $a \neq d$,
(ii) $\widetilde{H}_a(L(Z)) = 0$ *for* $a \neq d$, *and*
(iii) *the map* $\varphi : H_d(L(Z)) \to \widetilde{H}_d(L(Z))$ *induced by* φ *is an isomorphism.*

Then φ *is an isomorphism of homology theories.*

Before we give a proof for this, we note the following.

Remark 4.25. The homology theories of 4.23(a) clearly satisfy condition 4.24(i). In fact, setting

$$C^{r,s}(L(X),\mathbb{Z}/n\mathbb{Z}) = \varinjlim C^{r,s}(U,\mathbb{Z}/n\mathbb{Z})$$

for an integral X, where the limit is over all nonempty open subvarieties U of X, we trivially have $C^{r,s}_a(L(X),\mathbb{Z}/n\mathbb{Z}) = 0$ for $a \neq \dim X$, because for any $x \in X$ different from the generic point, there is a nontrivial open $U \subset X$ not containing x. Hence 4.24(i) also holds for the homology theories from 4.23(b). The proof of Theorem 4.19 will then be achieved as follows. In the next section we will define a homology theory $H^W_*(-,\mathbb{Q}_\ell/\mathbb{Z}_\ell)$, over any field

K of characteristic 0, or over a perfect field of positive characteristic assuming resolution of singularities, which *a priori* satisfies

$$H_a^W(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell) = \begin{cases} \mathbb{Q}_\ell/\mathbb{Z}_\ell, & a = 0, \\ 0, & a \neq 0 \end{cases} \tag{4.26}$$

if X is smooth, proper and irreducible. Moreover, we will show that 4.22(ii) holds for H^W. Still under the same assumptions we will construct a morphism

$$\varphi : \overline{H}_*(-, \mathbb{Q}_\ell/\mathbb{Z}_\ell) := \overline{H}_*^{0,0}(-, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \to H^W(-, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$$

of homology theories which satisfies 4.24(iii) if K is finitely generated. Thus, by Lemma 4.24, φ is an isomorphism, and hence (4.26) also holds for $\overline{H}_*(-, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$, which proves 4.19.

Proof of Lemma 4.24. For every homology theory H over L, there is a strongly converging niveau spectral sequence

$$E^1_{p,q}(X) = \bigoplus_{x \in X_p} H_{p+q}(k(x)) \Rightarrow H_{p+q}(X)$$

for every X; cf. [BO74] and [JS03]. If $\widetilde{E}^1_{p,q} \Rightarrow \widetilde{H}_{p+q}$ is associated to another homology theory $\widetilde{H}$, then every morphism $\varphi : H \to \widetilde{H}$ induces a morphism $E \to \widetilde{E}$ of these spectral sequences, compatible with φ on the E^1-terms and limit terms. In the situation of 4.24, conditions (i), (ii), and (iii) imply that φ induces isomorphisms on the E^1-terms, and hence φ also gives an isomorphism between the limit terms, i.e., between H and $\widetilde{H}$. □

5. **Weight complexes and weight cohomology**

Let k be a field. Let X be a smooth, proper variety of dimension d over k, and let $Y = \bigcup_{i=1}^{r} Y_i$ be a divisor with simple normal crossings in X — with a fixed ordering of the smooth components as indicated.

Definition 5.1. Let F be a covariant functor on the category $\mathcal{SP}_k$ of smooth projective varieties with values in an abelian category $\mathcal{A}$ which is additive in the sense that the natural arrow

$$F(X_1) \bigoplus F(X_2) \to F(X_1 \coprod X_2)$$

is an isomorphism in $\mathcal{A}$, where $X_1 \coprod X_2$ is the sum (disjoint union) of two varieties X_1, X_2 in $\mathcal{SP}_k$. Then define $L^iF(X, Y)$ as the i-th homology of the complex

$$C.F(X, Y) : \; 0 \to F(Y^{[d]}) \to F(Y^{[d-1]}) \to \cdots \to F(Y^{[1]}) \to F(X) \to 0.$$

Here $F(Y^{[j]})$ is placed in degree j, and the differential $\partial : F(Y^{[j]}) \to F(Y^{[j-1]})$ is $\sum\limits_{\nu=1}^{j}(-1)^\nu \delta_\nu$, where δ_ν is induced by the inclusions

$$Y_{i_1,\ldots,i_j} \hookrightarrow Y_{i_1,\ldots,\hat{i}_\nu,\ldots,i_j}$$

for $1 \le i_1 < \cdots < i_j \le r$ (and where $Y^{[0]} = X$, as usual).

Remark 5.2. There is the dual notion of an additive, contravariant functor G from $\mathcal{SP}_k$ to $\mathcal{A}$, and here we define $R^iG(X,Y)$ to be the i-th cohomology of the complex

$$C^{\cdot}G(X,Y): \; G(X) \to G(Y^{[1]}) \to \cdots \to G(Y^{[d-1]}) \to G(Y^{[d]}),$$

with $G(Y^{[j]})$ placed in degree j.

We may apply this to the following functors. Let *Ab* be the category of abelian groups.

Definition 5.3. For any abelian group A, define the covariant functor $H_0(-,A) : \mathcal{SP}_k \to Ab$ and the covariant functor $H^0(-,A) : \mathcal{SP}_k \to Ab$ by

$$H_0(X,A) = \bigoplus_{\alpha\in\pi_0(X)} A = A \otimes_{\mathbb{Z}} \mathbb{Z}[\pi_0(X)],$$
$$H^0(X,A) = A^{\pi_0(X)} = \mathrm{Map}(\pi_0(X), A),$$

where $\mathbb{Z}[M]$ is the free abelian group on a set M and $\mathrm{Map}(M,N)$ is the set of maps between two sets M and N. (Hence if A happens to be a ring, then $H_0(X,A)$ is the free A-module on $\pi_0(X)$, and $H_0(X,A) = \mathrm{Hom}_A(H_0(X,A),A)$ is its A-dual.) We write $C^W_{\cdot}(X,Y;A)$ for $C_{\cdot}H_0(-,A)(X,Y)$ and call

$$H_i^W(X,Y;A) := L^iH_0(-,A)(X,Y) = H_i(C_{\cdot}(X,Y;A))$$

the weight homology of (X,Y). Similarly define

$$C_W^{\cdot}(X,Y;A) = C^{\cdot}H_0(-,A)(X,Y)$$

and call $H^i_W(X,Y;A) = H^i(C^{\cdot}_W(X,Y;A))$ the weight cohomology of (X,Y).

Proposition 5.4. *Let Y_{r+1} be a smooth divisor on X such that the intersections with the connected components of $Y^{[j]}$ are transversal for all j and connected for all $j \le d-2$.* (*Note*: *If k is infinite, then by the Bertini theorems such a Y_{r+1} exists by taking a suitable hyperplane section, since* $\dim Y^{[j]} = d-j \ge 2$ *for $j \le d-2$.*) *Let $Z = \bigcup\limits_{i=1}^{r+1} Y_i$* (*which, by the assumption, is again a divisor with normal crossings on X*). *Then, for any abelian group A,*

$$H_i^W(X,Z;A) = 0 = H^i_W(X,Z;A) \; \textit{for } i \le d-1.$$

Proof. Fix A, and omit it in the notation. We get a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & H_0(Y^{[d-1]} \cap Y_{r+1}) \arrow[r, "\delta_{d-1}"] \arrow[d, two heads, "\psi_{d-1}"] & H_0(Y^{[d-2]} \cap Y_{r+1}) \arrow[r] \arrow[d, "\psi_{d-2}", "\wr"'] & \cdots \\
0 \arrow[r] & H_0(Y^{[d]}) \arrow[r, "\partial_d"] & H_0(Y^{[d-1]}) \arrow[r, "\partial_{d-1}"] & H_0(Y^{[d-2]}) \arrow[r] & \cdots
\end{tikzcd}$$

$$\begin{tikzcd}
\cdots \arrow[r] & H_0(Y^{[1]} \cap Y_{r+1}) \arrow[r] \arrow[d, "\psi_1", "\wr"'] & H_0(Y_{r+1}) \arrow[r] \arrow[d, "\psi_0", "\wr"'] & 0 \\
\cdots \arrow[r] & H_0(Y^{[1]}) \arrow[r] & H_0(X) \arrow[r] & 0,
\end{tikzcd}$$

where the bottom line is the complex $C_\cdot^W(X,Y)$ and the top line is $C_\cdot^W(Y_{r+1}, Y \cap Y_{r+1})$ (note that $Y \cap Y_{r+1} = \bigcup_{i=1}^{r} (Y_i \cap Y_{r+1})$ is a divisor with strict normal crossings on the smooth, projective variety Y_{r+1}), and where ψ_ν is induced by the inclusion $Y^{[\nu]} \cap Y_{r+1} \hookrightarrow Y^{[\nu]}$.

By the assumption, ψ_ν is an isomorphism for $\nu \leq d-2$ and a (noncanonically) split surjection for $\nu = d-1$. Hence we have isomorphisms

$$H_i^W(Y_{r+1}, Y \cap Y_{r+1}) \xrightarrow{\sim} H_i^W(X,Y) \text{ for } i \leq d-2$$

and a surjection

$$H_{d-1}^W(Y_{r+1}, Y \cap Y_{r+1}) \twoheadrightarrow H_{d-1}^W(X,Y).$$

Moreover, let $C_{\cdot\cdot}$ be the associated double complex, with $H_0(X)$ placed in degree $(0,0)$ and ψ_ν being replaced by $(-1)^\nu \psi_\nu$. Then the associated total complex $s(C_{\cdot\cdot})$ is just the complex $C_\cdot^W(X,Z)$. Hence the result follows, and we have exact sequences

$$0 \to H_d^W(X,Y) \to H_d^W(X,Z) \to H_{d-1}^W(Y_{r+1}, Y \cap Y_{r+1}) \to H_{d-1}^W(X,Y) \to 0,$$

$$0 \to \ker(\psi)_{d-1} \to H_d^W(X,Z) \to H_0^W(Y^{[d]}) \to 0.$$

The proof for $H_W^i(X,Z)$ is dual. (Note, however, that in general, $H_\cdot^W(X,Y)$ and $H_W^\cdot(X,Y)$ are related by a coefficient theorem in an nontrivial way.) $\square$

Corollary 5.5. *$H_d^W(X,Z;\mathbb{Z})$ is a finitely generated free $\mathbb{Z}$-module, and we have an isomorphism $H_d^W(X,Z;A) \cong H_d(X,Z;\mathbb{Z}) \otimes_{\mathbb{Z}} A$. The same holds for $H_W^d(X,Z)$.*

Proof. The first statement follows since $\ker(\psi)_{d-1}$ and $H^0(Y^{[d]})$ have this property for $A = \mathbb{Z}$, and the second claim follows from 5.4 and the universal coefficient theorem. Similarly for $H_W^d(X,Z)$. $\square$

Corollary 5.6. *$H_d^W(X,Z;\mathbb{Q}_\ell/\mathbb{Z}_\ell)$ is divisible.*

Now let $U = X \smallsetminus Y$.

PROPOSITION 5.7. *Let $\overline{U} = U \times_k \overline{k}$, where $\overline{k}$ is the algebraic closure of k. Then there are canonical homomorphisms*

$$H^d_{\acute{e}t}(\overline{U}, \mathbb{Z}/n\mathbb{Z}(d)) \xrightarrow{e} H^W_d(X, Y; \mathbb{Z}/n\mathbb{Z})$$

for all $n \in \mathbb{N}$. If k is finitely generated and if $X \smallsetminus Y_\nu$ is affine for one $\nu \in \{1, \dots, r\}$, then these induce isomorphisms

$$H^d_{\acute{e}t}(\overline{U}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d))_{G_k} \xrightarrow{\sim} H^W_d(X, Y; \mathbb{Q}_\ell/\mathbb{Z}_\ell)$$

for all primes ℓ.

This is just a reformulation of Theorem 3.1, in which the construction of e does not depend on the assumption that some $X \smallsetminus Y_\nu$ is affine.

Remark 5.8. In particular, with the notation and assumptions of 5.4, this applies to (X, Z) and $U = X \smallsetminus Z$.

We want to have these results in a more functorial setting. This is possible if resolution of singularities holds in a suitable form.

THEOREM 5.9. *Let k be a field, with perfect hull $L = k^{\mathrm{per}}$, let A be an abelian group, and assume that condition* $(\mathrm{RS1})_L$ *from* 4.18 *holds. Then there exists a homology theory* (*in the sense of Definition* 4.22) $(H^W_a(-, A), a \in \mathbb{Z})$ *on the category $(\mathcal{V}_k)_*$ of all varieties over k with proper morphisms such that the following holds*:

(i) *For any smooth, proper, and connected variety X over k, one has*

$$H^W_a(X, A) = \begin{cases} 0, & a \neq 0, \\ A, & a = 0. \end{cases}$$

(ii) *If X is smooth and proper over k and Y is a divisor with simple normal crossings on X, then one has a canonical isomorphism for $U = X \setminus Y$,*

$$H^W_a(U, A) \cong H^W_a(X, Y; A) = H_a\Big(\bigoplus_{\pi_0(Y^{[e]})} A \to \cdots \to \bigoplus_{\pi_0(Y^{[1]})} A \to \bigoplus_{\pi_0(X)} A\Big),$$

where the right-hand side is defined in Definition 5.3. *We call $H^W_a(-, A)$ the weight homology with coefficients in A.*

Proof. First assume that k is perfect. We want to show that the covariant functor (cf. 5.3)

$$F: \; X \quad \leadsto \quad H_0(X, A) = \bigoplus_{\pi_0(X)} A$$

on the category $\mathcal{SP}_k$ of all smooth proper varieties over k extends to a homology theory on all of $\mathcal{V}_k$ and fulfills (ii). By the method of Gillet and Soulé

([GS96, proof of 3.1.1]) this holds if $(\mathrm{RS1})_k$ holds and if F extends to a contravariant functor on Chow motives over k, i.e., admits an action of algebraic correspondences modulo rational equivalence. But the latter is clear — in fact, one has $F(X) = \mathrm{Hom}(\mathrm{CH}^0(X), A)$, where $\mathrm{CH}^j(X)$ is the Chow group of algebraic cycles of codimension j on X, modulo rational equivalence. $\square$

If k is general, we just define

$$H_a^W(Z, A) := H_a^W(Z \times_k k^{\mathrm{per}}),$$

where the theory on the right is the one existing over k^{per} by our assumptions and the case of a perfect field. Note that $Z \times_k k^{\mathrm{per}}$ is again connected for connected Z.

Theorem 5.10. *Let k be a field, and assume that resolution of singularities holds over the perfect hull L of k (see Definition 4.18). Then the homology theory $H_*^W(-, A)$ of Theorem 5.9 has the property 4.24(ii).*

Proof. By construction we may assume k is perfect. For every integral variety Z of dimension d over k, we have to show

$$H_a^W(k(Z), A) := \varinjlim H_a^W(V, A) = 0 \text{ for } a \neq d, \tag{5.11}$$

where the inductive limit is over all nonempty open subvarieties $V \subset Z$.

Now assume property $(\mathrm{RS2})_k$ from 4.18 holds. Then, by perfectness of k, for every nonempty open subvariety $V \subset Z$, there is a nonempty smooth open subvariety $U \subset V$, and by $(\mathrm{RS2})_k$, there is an open embedding $U \hookrightarrow X$ into a smooth projective variety X such that the complement $Y = X \setminus U$ is a divisor with strict normal crossings. If k is infinite then, by Bertini's theorem, there exists a smooth hyperplane section H of X whose intersection with all connected components of $Y^{[i]}$ is smooth, and connected for $i \leq d-2$. Writing $Z = Y \cup H$ (which is a divisor with strict normal crossings on X) and $U^0 = X \setminus Z \subset U \subset V$, we get $H_a^W(U^0, A) = H_a^W(X, Z; A) = 0$ for $a \neq d$ by Property 5.9(ii) and Proposition 5.4. Since V was arbitrary, we get property (5.11).

If k is finite, we use a suitable norm argument. By what has been shown, for each prime p, we find such a hyperplane section after base change to an extension k'/k of degree $[k' : k] = p^r$, a power of p (the maximal pro-p-extension of k is an infinite field). Then the map

$$H_a^W(V_{k'}, A) \to H_a^W(k'(Z_{k'}), A)$$

is zero.

Now we note that there is a homology theory $H^W(-, A; k')$ on all varieties over k, defined by $H_a^W(Z, A; k') = H_a^W(Z_{k'}, A)$ and the induced structure maps. This is also the homology theory which is obtained by the method of Theorem 5.9, by extending the covariant functor

$$F' : \mathcal{SP}_k \longrightarrow Ab\ ,\ \ X \leadsto \bigoplus_{\pi_0(X_{k'})} A$$

to a homology theory on all varieties. There is an obvious morphism of functors $\mathrm{Tr} : F' \to F$ (trace, or norm), induced by the natural maps $\pi_0(X_{k'}) \to \pi_0(X)$. On the other hand there is also a morphism of functors $\mathrm{Res} : F \to F'$ (restriction) such that $\mathrm{Tr}\ \mathrm{Res} = [k' : k]$.

This is best seen by noting that for any smooth proper variety X/k one has a canonical isomorphism

$$F(X) = \Big(\bigoplus_{\pi_0(X\times_k\overline{k})} A\Big)_{G_k} \xrightarrow{\sim} \bigoplus_{\pi_0(X)} A,$$

where $\overline{k}$ is an algebraic closure of k. Similarly, $F'(X) = B_{G_{k'}}$, where

$$B = \bigoplus_{\pi_0((X\times_k k')\times_{k'}\overline{k})} A = \bigoplus_{\pi_0(X\times_k\overline{k})} A.$$

In these terms, Tr is induced by the natural map $B_{G_{k'}} \to B_{G_k}$. Conversely, for any profinite group G, any open subgroup U, and any discrete G-module C, we have a well-defined functorial map

$$\mathrm{Cor}^\vee(C) : C_G \longrightarrow C_H, \quad \text{class of} \quad a \quad \mapsto \quad \text{class of} \quad \sum_{\sigma\in G/H} \sigma a,$$

and the composition $\mathrm{Cor}^\vee \circ \pi$ is the multiplication by $(G : H)$. Applied to $(G, H, C) = (G_k, G_{k'}, B)$, we get the claim.

By the construction of Gillet and Soulé (or by Theorem 5.13 and Remark 5.15 below), Tr extends to morphism of homology theories

$$\mathrm{Tr} : H^W(-, A; k') \to H^W(-, A),$$

and one checks that the induced maps $H_a^W(Z_{k'}, A) \to H_a^W(Z, A)$ are just the maps obtained from functoriality for proper morphisms. Similarly, Res extends to a morphism of homology theories $\mathrm{Res} : H(-, A) \to H^W(-, A; k')$, and one has $\mathrm{Tr}\ \mathrm{Res} = [k' : k] = p^r$, because this holds for the restriction to the functors F and F'. The outcome is that the kernel of Res is killed by p^r. From the commutative diagram

$$\begin{array}{ccc} H_a^W(V_{k'}, A) & \to & H_a^W(k'(Z_{k'}), A), \\ \uparrow & & \uparrow \\ H_a^W(V, A) & \to & H_a^W(k(Z), A) \end{array}$$

we then get that the image of $H_a^W(V, A)$ in $H_a^W(k(Z), A)$ is killed by p^r for $a \neq e$, because its image in $H_a^W(k'(Z_{k'}), A)$ is zero. Since this holds for all V (with varying powers of p), we conclude that every element in the group $H_a^W(k(Z), A)$ is killed by a power of p. Since this also holds for any second prime $q \neq p$, we conclude the vanishing of $H_a^W(k(Z), A)$. $\square$

THEOREM 5.12. *Let ℓ be a prime, and let K be a finitely generated field such that resolution of singularities holds over the perfect hull $L = K^{\mathrm{per}}$ of K. Let $\overline{H}_*(-, \mathbb{Q}_\ell/\mathbb{Z}_\ell) = H_*(\overline{C}^{0,0}(-, \mathbb{Q}_\ell/\mathbb{Z}_\ell))$ and $H_*^W(-, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ be the homology theories on $\mathcal{V}_K$ defined in* 4.23(b) *and* 5.9, *respectively. There exists a morphism*

$$\varphi : \overline{H}_*(-, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \longrightarrow H_*^W(-, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$$

of homology theories such that properties (i)–(iii) *of Lemma* 4.24 *are fulfilled for $\overline{H}$, H^W, and φ. Consequently, φ is an isomorphism of homology theories.*

Evidently this theorem implies Theorem 4.19, in view of 5.9(i). Since $H_*^W(-, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ is defined via the method of Gillet and Soulé in [GS96, 3.1.1], we need to analyze the constructions in [GS96] more closely. There functors on Chow motives with values in abelian categories are extended to homology theories on all varieties. We give a more general version for complexes in the following form.

THEOREM 5.13. *Let k be a perfect field, let $\mathcal{C}_{\geq 0}(\mathcal{A})$ be the category of non-negative homological complexes $\cdots \to C_2 \to C_1 \to C_0$ in an abelian category $\mathcal{A}$, and let*

$$C : \mathcal{SP}_k \to \mathcal{C}_{\geq 0}(\mathcal{A})$$

be a covariant functor on the category $\mathcal{SP}_k$ of all smooth proper varieties over k. Assume that the associated functors

$$H_a^C : \mathcal{SP}_k \to \mathcal{A},\ X \leadsto H_a(C(X))$$

extend to contravariant functors on the category $\mathcal{CHM}^{\mathrm{eff}}(k,d)$ of effective Chow motives (*i.e.*, *$\mathbb{Z}$-linear motives modulo rational equivalence*) *generated by all smooth proper varieties over k. Assume further that resolution of singularities holds over k. Then the above functors H_a^C extend in a natural way to a homology theory $\hat{H}_a^C$ on the category $\mathcal{V}_k$ of all varieties over k.*

Proof (*cf. the reasoning in* [GS96, 5.3]). For the proof it is better to consider the category $(\mathcal{CHM}^{\mathrm{eff}}(k,d))^{\mathrm{op}}$ of covariant effective Chow motives. By (RS1) any proper variety Z over k has a smooth proper hyperenvelope, i.e., a hyperenvelope (cf. [GS96, 1.4.1 and Lemma 2]) $h : \tilde{Z}_. \to Z$ where the components $\tilde{Z}_r$ of the simplicial scheme $\tilde{Z}_.$ are smooth and proper, and every morphism $f : Y \to X$ of proper varieties has a smooth proper hyperenvelope, i.e., there is a commutative square

$$\begin{array}{ccc} Y_. & \xrightarrow{f_.} & X_. \\ {\scriptstyle h_Y}\big\downarrow & & \big\downarrow{\scriptstyle h_X} \\ Y & \xrightarrow{f} & X, \end{array}$$

in which h_Y and h_X are smooth proper hyperenvelopes. Fixing such a diagram for every morphism f of proper varieties, we can proceed as follows. If V is an arbitrary reduced variety over k, fix an open embedding $j : V \subset Z$ into a proper variety, and let $Y = Z \setminus V$, so that $i : Y \hookrightarrow Z$ is a closed immersion. Then take the corresponding morphism of hypercoverings $i_. : Y_. \to Z_.$, and the associated morphism of complexes of effective (covariant) Chow motives

$$(i_.)_* : M(Y_.) \longrightarrow M(Z_.),$$

where the differentials are the alternating sums of the simplicial morphisms d_j. Finally take the cone $\mathrm{Cone}(M(Y_.) \xrightarrow{i_.^*} M(Z_.))$, which gives a complex of motives

$$M(V). : \quad \cdots \to M(Z_n) \oplus M(Y_{n-1}) \to M(Z_{n-1}) \oplus M(Y_{n-2}) \to \cdots ,$$

which is called the weight complex of V. It is known that this complex does not depend on the choices, up to homotopy of complexes, and can be represented by a finite complex. We can now define the functor C on all varieties by letting $C(V)$ be the associated total complex of the double complex $C(M(V).)$, which is well defined since, by assumption, $C(X)$ only depends on the motive of X for a smooth proper variety X. Then we have a convergent spectral sequence

$$E^1_{p,q}(V_.) = H_p(H_q(C(V_.))) \Rightarrow H_{p+q}(tC(V_.)). \tag{5.14}$$

To avoid confusion: $E^1_{p,q}$ is the p-th homology of the complex

$$H_q(V_.) : \qquad \cdots \to H_q(V_r) \to H_q(V_{r-1}) \to \cdots \to H_q(V_0).$$

For every morphism $g : W_. \to Z_.$ of smooth proper simplicial schemes, we have a morphism $E(W_.) \to E(Z_.)$ of spectral sequences. If g is a hyperenvelope of simplicial schemes (see [GS96, 1.4.1]), then the fundamental result [GS96, Prop. 2] asserts that the induced morphism

$$g^* : M(Z_.) \longrightarrow M(W_.)$$

is a homotopy equivalence. Hence g induces an isomorphism on the E^2-terms of the above spectral sequences, and thus an isomorphism $g_* : H_a(W_.) \to H_a(Z_.)$ for all a.

Using this, we get the functoriality of our homology theory on $\mathcal{V}_k$ following the reasoning in [GS96, 2.3]: If $f : V_1 \to V_2$ is a proper morphism of varieties and $i_\nu = i_{V_\nu} : Z_\nu \setminus V_\nu \hookrightarrow Z_\nu$ is as chosen above ($\nu = 1, 2$), then there is a canonical diagram in the category $\mathrm{Ar}(\mathcal{P}_k)$ of morphisms in the category $\mathcal{P}_k$ of proper varieties over k

$$\begin{array}{ccc} i_f & \xrightarrow{\pi_2} & i_2 \\ {\scriptstyle \pi_1}\downarrow & & \\ i_1 & & \end{array}$$

in which π_1 is Gersten acyclic (loc. cit.) and hence induces a quasi-isomorphism $C(\tilde{i}_f) \to C(\tilde{i}_1)$. (For this one reasons via the spectral sequences (5.14).) This gives

$$f_* := H_a(f) := H_a((\pi_1)_*)^{-1} H_a((\pi_2)_*) : H_a(Z_1) \to H_a(Z_2).$$

This is functorial with the same argument as in [GS96, 2.3].

Moreover, we remark that, for another choice of compactifications j_Z and hence maps i_Z, say j'_Z and i'_Z, the morphisms $H_a(id_Z)$ give canonical isomorphisms between the different constructions of H_a. Also if we take another choice for the hyperenvelopes $\tilde{f}$, then the reasoning in [GS96, 2.2] shows that the resulting homology theory is canonically isomorphic to the one for the first choice. In this sense, the homology theory $\hat{H}^C$ with $\hat{H}^C(V)_a = H_a(C(V)$ is canonical.

To obtain properties 4.22(a)(i) and (ii) for our homology theory, i.e., contravariance for open immersions and the exact localization sequences, we proceed as in [GS96, 2.4]: For a variety Z and a closed subvariety $Z' \subset Z$ with open complement $U = Z \setminus Z'$, choose a compactification $Z \hookrightarrow \overline{Z}$, and let $Y = \overline{Z} \setminus Z$, and $Y' = \overline{Z} \setminus U$, so that $Z' = Y' \setminus Y$. Then we choose smooth projective hyperenvelopes $\tilde{\overline{Z}} \to \overline{Z}$, $\tilde{Y} \to Y$ and $\tilde{Y}' \to Y'$ such that one has morphisms

$$\tilde{i} : \tilde{Y} \xrightarrow{\tilde{k}} \tilde{Y}' \xrightarrow{\tilde{i}'} \tilde{\overline{Z}}$$

lifting the closed immersions $i : Y \overset{k}{\hookrightarrow} Y' \overset{i'}{\hookrightarrow} \overline{Z}$. We obtain a triangle of mapping cones

$$C(\tilde{k}) \longrightarrow C(\tilde{i}) \longrightarrow C(\tilde{i}') \longrightarrow C(\tilde{k})[-1],$$

which represents the desired triangle

$$C(Z') \longrightarrow C(Z) \longrightarrow C(U) \longrightarrow C(Z')[-1]$$

in the derived category which, in turn, gives rise to the exact localization sequence

$$\cdots \to H_a(Z') \to H_a(Z) \to H_a(U) \to H_{a-1}(Z') \to \cdots$$

and thereby also to the pullback j^* for the open immersion $j : U \to Z$. The functorial properties are easily checked. □

Remark 5.15. Assume that $C' : \mathcal{SP}_k \to \mathcal{C}_{\geq 0}(\mathcal{A})$ is another functor with the properties required in Theorem 5.13 and $\varphi : C \to C'$ is a morphism of functors. Then it is clear from the construction that there is a canonical induced morphism $\varphi : \hat{H}^C \to \hat{H}^{C'}$ of the associated homology theories on $\mathcal{V}_k$.

PROPOSITION 5.16. *Let $(\mathcal{V}_k)_*$ be the category of all varieties over k with all proper morphisms between them. Let*

$$C : (\mathcal{V}_k)_* \longrightarrow \mathcal{C}_{\geq 0}(\mathcal{A})$$

be a covariant functor which is equipped with the following additional data:

(i) *For every open immersion* $j : U \hookrightarrow Z$ *in* $(\mathcal{V}_k)_*$, *there is a morphism* $j^* : C(Z) \to C(U)$, *associated to* j *in a functorial way.*

(ii) *If* $i : Y \hookrightarrow Z$ *is a closed immersion in* $(\mathcal{V}_k)_*$, *with open complement* $j : U \hookrightarrow Z$, *then there is a short exact sequence of complexes*

$$0 \to C(Y) \xrightarrow{i_*} C(X) \xrightarrow{j^*} C(U) \to 0.$$

This sequence is functorial with respect to proper morphisms and open immersions, *in an obvious way.*

Let H *be the obvious homology theory on* $(\mathcal{V}_k)_*$ *deduced from* C, *with* $H_a(Z) = H_a(C(Z))$, *and let* C_{SP} *be the restriction of* C *to* $\mathcal{SP}_k$. *Assume that the associated functors*

$$H_a : \mathcal{SP}_k \to \mathcal{A}\ ,\ X \leadsto H_a(C(X))$$

extend to contravariant functors on the category $\mathcal{CHM}^{\mathrm{eff}}(k)$ *of effective Chow motives*, *and let* $\hat{H}$ *be the homology theory on* $\mathcal{V}_k$ *derived from* C_{SP} *via Theorem* 5.12. *Then* $\hat{H}$ *and* H *are canonically isomorphic.*

Proof. This follows from the following descent lemma (which we only need for the case that X is proper and $Z_.$ is a smooth proper hyperenvelope). □

Lemma 5.17. *If* $Z_. \to X$ *is a hyperenvelope of a variety* X *over* k, *then the canonical morphism*

$$tC'(Z_.) \longrightarrow C'(X)$$

(*induced by the morphism* $\tilde{Z}_0 \to X$) *is a quasi-isomorphism. Here* $C'(Z_.)$ *and* $tC'(Z_.)$ *are defined as in the proof of Theorem* 5.13.

Proof. This follows in a similar way as in the descent theorem [Gil84, Th. 4.1]. Let me very briefly recall the three steps.

(I) If $Z_. = \mathrm{cosk}_0^X(Z)$ for an envelope $Z \to X$, and $Z \to X$ has a section, then $Z_.$ is homotopy equivalent to the constant simplicial variety X, and the claim follows via the convergent spectral sequence

$$E^1_{p,q}(Z_.) = H_p(H_q(C'(Z_.))) \Rightarrow H_{p+q}(tC'(Z_.)), \tag{5.18}$$

whose existence follows with the same argument as for 5.13. (It is the spectral sequence for the filtration with respect to the 'simplicial' degree of the bicomplex $C'(Z_.)$.)

(II) If still $Z_. = \mathrm{cosk}_0^X(Z)$ for a morphism, then by localization in X, i.e., by the exact sequence 5.16(ii) and the induced one for Z, and by noetherian induction, we may assume that $Z \to X$ has a section.

(III) Then, to extend this to the general case it suffices to show that the morphism of simplicial schemes

$$f : Z_.[n+1] = \mathrm{cosk}^X_{n+1} sk_{n+1}(Z_.) \longrightarrow \mathrm{cosk}^X_n sk_n(Z_.) = Z_.[n] \tag{5.19}$$

induces a quasi-isomorphism $tC'(Z_.[n+1]) \to C'(Z_.[n])$ for all $n \geq 0$, because $Z_j[n] = Z_j$ for $j \leq n$ and hence $H_j(tC'(Z_.)) \xrightarrow{\sim} H_j(Z_.[n])$ for $j < n$ by the spectral sequence (5.18). To show that the morphism (5.19), abbreviated $f : X'_. \to X_.$, induces a quasi-isomorphism, one then follows the proof of [SGA, (3.3.3.2)]. In fact, as noted in [Gil84], the reasoning of loc. cit. (3.3.3.3) shows that all morphisms $F_i : X'_i = Z_i[n+1] \to X_i = Z_i[n]$ are envelopes. Note that the diagram in loc. cit. 3.3.3.3 should read

$$\begin{array}{ccccc} K'_\iota & \longrightarrow & \prod X'_r & \overset{\mathrm{pr}_i}{\underset{X'(\iota)\mathrm{pr}_j}{\rightrightarrows}} & X'_i \\ \downarrow & & \downarrow & & \downarrow \\ K_\iota & \longrightarrow & \prod X_r & \overset{\mathrm{pr}_i}{\underset{X(\iota)\mathrm{pr}_j}{\rightrightarrows}} & X_i \end{array}$$

for the morphism $\iota : i \to j$ in the category $\underset{n+1[p]}{\Delta^+}$ of monomorphisms $[q] \to [m]$ with $q \leq n+1$ in the category of simplicial sets, and where the product is over all objects r in $\underset{n+1[p]}{\Delta^+}$. Moreover, the object $\cap K_\iota$ should rather read $\prod K_\iota$, which is $\times_X K_\iota$ here.

By looking at the bi-simplical scheme $[X'_./X_.]$ with components $[X'_./X_.]_p = X'_. \times_{X_.} \cdots \times_{X_.} X'_.$ ($(p+1)$ times) and its base change with $X'_. \to X_.$, one sees that it suffices to replace $X'_. \to X_.$ by its base change with $[X'_./X_.]_p$ for all $p \geq 0$. This is again of the form (5.18) and has a section s. So $fs = \mathrm{id}$, and because sf is the identity on $sk_n(Z_.)$, it is homotopic to the identity ([SGA, 4, Vbis, (3.0.2.4)]. Therefore f in this situation is a homotopy equivalence, and hence induces a quasi-isomorphism by (5.18). □

Lemma 5.20.

(a) *For a smooth proper variety X over a field k and for integers $n, r, s, a \in \mathbb{Z}$, let*

$$H_a^{r,s}(X, \mathbb{Z}/n\mathbb{Z}) := H_a(C^{r,s}(X, \mathbb{Z}/n\mathbb{Z}))$$

be the a-th homology of the Bloch-Ogus-Kato complex $C^{r,s}(X, \mathbb{Z}/n\mathbb{Z})$ when it is defined. (*See the beginning of Section* 4.) *If n is invertible in k, then each of the functors*

$$H_a^{r,s}(-, \mathbb{Z}/n\mathbb{Z}) : \mathcal{SP}_k \longrightarrow Ab \ , \ X \leadsto H_a^{r,s}(X, \mathbb{Z}/n\mathbb{Z})$$

extends to a contravariant functor on the category $\mathcal{CHM}^{\mathrm{eff}}(k)$ of effective Chow motives over k. The same holds if k is a perfect field of positive characteristic p and n is a power of p if $(r, s) = (0, 0)$.

(b) *The same holds for the category $\mathcal{CHM}^{\mathrm{eff}}(k)$ and the functors*

$$X \quad \leadsto \quad \overline{H}_a^{r,s}(X, \mathbb{Z}/n\mathbb{Z}) := H_a(\overline{C}^{r,s}(X, \mathbb{Z}/n\mathbb{Z})) \qquad (a \in \mathbb{Z}),$$

where $\overline{C}^{r,s}(X,\mathbb{Z}/n\mathbb{Z}) := C^{r,s}(\overline{X},\mathbb{Z}/n\mathbb{Z})_{G_L}$, *with* $\overline{X} = X \times_L \overline{L}$ *for a separable closure of* L.

Proof. (a) By work of Bloch and Ogus [BO74], one has canonical isomorphisms for n invertible in k and X irreducible smooth proper of dimension d,

$$H_a^{r,s}(X,\mathbb{Z}/n\mathbb{Z}) := H_a(C^{r,s}(X,\mathbb{Z}/n\mathbb{Z})) \cong H^{d-a}(X_{\mathrm{Zar}},\mathcal{H}_n^{r+d}(s+d)), \tag{5.21}$$

where $\mathcal{H}_n^i(j)$ is the Zariski sheaf associated to the presheaf $U \mapsto H^i_{\text{ét}}(U,\mathbb{Z}/n\mathbb{Z}(j))$. The same holds in the second case of (a) by work of Gros and Suwa [GS88] for $(r,s)=(0,0)$. On the other hand, by Barbieri-Viale ([BV97, 5.5]) one has products (omitting the subscript n)

$$H^i(X,\mathcal{H}(a)) \times H^j(X,\mathcal{H}(b)) \longrightarrow H^{i+j}(X,\mathcal{H}(a+b))$$

and a formalism of pullbacks $f^* : H^i(Y,\mathcal{H}^m(j)) \to H^i(X,\mathcal{H}^m(j))$ for arbitrary morphisms $f : X \to Y$, and pushforwards for proper morphisms of irreducible varieties $f : X \to Y$,

$$f_* : H^i(X,\mathcal{H}^j(k)) \to H^{i-r}(Y,\mathcal{H}^{j-r}(k-r)), \tag{5.22}$$

where $r = \dim(X) - \dim(Y)$, such that the pair (f^*, f_*) satisfies the projection formula. With this one can get the usual formalism of correspondences (contravariant version): Letting X and Y be smooth and proper of dimensions d and e, respectively, the group of correspondences from X to Y is defined as $H^e(Y\times X,\mathcal{H}(e))$, and the correspondences α induce homomorphisms $\alpha_* : H^i(Y,\mathcal{H}^j(k)) \to H^i(X,\mathcal{H}^j(k))$ via the formula $\alpha_*(\beta) = p_{2*}(\alpha \cdot p_1{}^*(\beta))$ in the diagram

$$\begin{array}{ccc}
H^e(Y\times X,\mathcal{H}^e(e)) \times H^i(Y\times X,\mathcal{H}^j(k)) & \longrightarrow & H^{e+i}(Y\times X,\mathcal{H}^{e+j}(e+k)) \\
\big\downarrow{=} \qquad\qquad \big\uparrow{(p_1)^*} & & \big\downarrow{(p_2)_*} \\
H^e(Y\times X,\mathcal{H}^e(e)) \times \quad H^i(Y,\mathcal{H}^j(k)) & \longrightarrow & H^i(X,\mathcal{H}^j(k)),
\end{array}$$

and this action is compatible with the composition of correspondences. On the other hand, from (5.21) we get a canonical isomorphism

$$H^e(Y\times X,\mathcal{H}_n^e(e)) \cong \mathrm{CH}^e(Y\times X)/n$$

for any smooth projective varieties Y and X, and these morphisms are compatible with pullbacks, pushforwards, and with products after some suitable sign modifications [Gil87]. So we obtain the wanted action of Chow correspondences, at first in a covariant way. But there is a canonical equivalence between the category of Chow motives and its dual (see [KMP07, Lemma 1.2]), so that we obtain the wanted contravariant version as well.

As remarked above, for $n = p^m$ and k perfect of characteristic $p > 0$ and $(r,s) = (0,0)$, property (5.21) also holds, moreover, purity holds in this setting, and there are canonical pushforward maps like in (5.22). Thus one can apply the theory of [BV97] and get the same results as above.

(b) By the universal exactness of the Gersten complexes in the situations above, we get a similar formula

$$\overline{H}_a^{r,s}(X,\mathbb{Z}/n\mathbb{Z}) := H_a(\overline{C}^{r,s}(X,\mathbb{Z}/n\mathbb{Z})) \cong H^{d-a}(X_{\mathrm{Zar}}, \overline{\mathcal{H}}_n^{r+d}(s+d)), \tag{5.23}$$

where $\overline{\mathcal{H}}_n^{r+d}(m)$ is the Zariski sheaf associated to the presheaf

$$U \mapsto H^{r+d}(\overline{U}, \mathbb{Z}/n\mathbb{Z}(m))_{G_k}.$$

Then we obtain a similar theory as above by replacing the groups $H^i(Z,\mathcal{H}_n^j(k))$ above by the groups $H^i(Z,\overline{\mathcal{H}}_n^j(k))$. So via the obvious morphisms

$$H^i(Z,\mathcal{H}_n^j(k)) \to H^i(Z,\overline{\mathcal{H}}_n^j(k))$$

which are compatible with all structures (pullbacks, pushforwards, products and correspondences) used in (a) we get an extension to a functor on the Chow correspondences over k as before. □

Proof of Theorem 5.12. Let K be a finitely generated field, and let ℓ be a prime. Consider the Kato complex figuring in 5.12 (and 4.19)

$$\overline{C}(X,\mathbb{Q}_\ell/\mathbb{Z}_\ell) = C^{0,0}(\overline{X},\mathbb{Q}_\ell/\mathbb{Z}_\ell)_{G_K},$$

where $\overline{X} = X \times_K \overline{K}$ for the algebraic closure $\overline{K}$ of K and M_{G_K} is the module of coinvariants of a G_K-module. By Lemma 5.20 and Theorem 5.13, the covariant functors

$$\overline{H}_a : \mathcal{SP}_K \longrightarrow C_{\geq 0}(Ab), \quad \overline{H}_a(X) = H_a(\overline{C}(X,\mathbb{Q}_\ell/\mathbb{Z}_\ell))$$

extend to a homology theory $\hat{\overline{H}}$ on the category $\mathcal{V}_K$ of all varieties over K.

Next, one has a direct sum decomposition

$$\overline{C}(X_1 \textstyle\coprod X_2,\mathbb{Q}_\ell/\mathbb{Z}_\ell) \cong \overline{C}(X_1,\mathbb{Q}_\ell/\mathbb{Z}_\ell) \oplus \overline{C}(X_2,\mathbb{Q}_\ell/\mathbb{Z}_\ell)$$

for varieties X_1, X_2, and a morphism $\overline{C}(X,\mathbb{Q}_\ell/\mathbb{Z}_\ell) \to \overline{C}(K,\mathbb{Q}_\ell/\mathbb{Z}_\ell) = \mathbb{Q}_\ell/\mathbb{Z}_\ell$ induced by the structural morphism $X \to \mathrm{Spec}(K)$ for any variety. Applying this to the connected components of each smooth proper variety, we get a functorial map

$$\overline{C}(X,\mathbb{Q}_\ell/\mathbb{Z}_\ell) \longrightarrow \bigoplus_{\pi_0(X)} \mathbb{Q}_\ell/\mathbb{Z}_\ell = C^W(X,\mathbb{Q}_\ell/\mathbb{Z}_\ell)$$

for all smooth proper varieties over K and hence (cf. Remark 5.15) a morphism of homology theories

$$\phi : H_*(-,\mathbb{Q}_\ell/\mathbb{Z}_\ell) \longrightarrow H_*^W(-,\mathbb{Q}_\ell/\mathbb{Z}_\ell) \tag{5.24}$$

as wanted. In fact, we first get it for the homology $\hat{H}$ constructed above, which however coincides with H by Lemma 5.17.

It remains to show property 4.24(iii) for the morphism (5.24). Let Z be an integral variety of dimension d over K, and let $V \subset Z$ be any nonempty smooth subvariety. Let $U \subset V$ and $U \subset X$, $Y \subset X$ be as in property (RS2) (which holds because K has characteristic zero). By possibly removing a further suitable smooth hyperplane section we may assume that $X \setminus Y_1$ is affine.

As noted above, the complex $\overline{C}(W, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ exists for any variety W over K. If W is irreducible of dimension d, then $\overline{H}_d(W, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ can be identified with

$$\ker(H^d(K(W) \otimes_K \overline{K}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d))_{G_K} \to \bigoplus_{x \in W^1} H^{d-1}(K(x) \otimes_K \overline{K}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d-1))_{G_K}),$$

and if W is irreducible and smooth of dimension d, then the Bloch-Ogus spectral sequence ([BO74, 3.9] gives a canonical edge morphism

$$\gamma_W : H^d(\overline{W}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d))_{G_K} \longrightarrow H^0(W, \overline{\mathcal{H}}^d(d)) = \overline{H}_d(W, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$$

(compare (5.23)), which is just induced by the restriction map

$$H^d(\overline{W}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d)) \longrightarrow H^d(K(W) \otimes_K \overline{K}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d))_{G_K}.$$

Now we have the following result.

LEMMA 5.25. *There is a commutative diagram*

(5.26)
$$\begin{tikzcd}
0 \arrow[r] & H^d(\overline{U}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d))_{G_K} \arrow[r, "e"] \arrow[d, "\gamma_U"] & H^0(\overline{Y^{[d]}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(0))_{G_K} \arrow[r, "d_2"] \arrow[d, "\gamma_{Y^{[d]}}"] & H^2(\overline{Y^{[d-1]}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1)) \arrow[d, "\gamma'"] \\
0 \arrow[r] & H_d(U, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \arrow[r, "e"] \arrow[d, "\varphi_U"] & H_0(Y^{[d]}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \arrow[r, "d_2"] \arrow[d, "\varphi_{Y^{[d]}}"] & H_0(Y^{[d-1]}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \arrow[d, "\varphi_{Y^{[d-1]}}"] \\
0 \arrow[r] & \tilde{H}_d(U, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \arrow[r, "e"] & \tilde{H}_0(Y^{[d]}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \arrow[r, "d_2"] & \tilde{H}_0(Y^{[d-1]}, \mathbb{Q}_\ell/\mathbb{Z}_\ell).
\end{tikzcd}$$

Here the maps e and d_2 in the first row are those occuring in Theorem 3.1. *The maps in the second and third row are the homological analogues*: *e is the composition of the morphisms*

$$H_d(U, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \xrightarrow{\delta} H_{d-1}(Y_{i_d} \smallsetminus (\bigcup_{i \neq i_d} Y_i), \mathbb{Q}_\ell/\mathbb{Z}_\ell)$$
$$\xrightarrow{\delta} \cdots \xrightarrow{\delta} H^1(Y_{i_2,\dots,i_d} \smallsetminus (\bigcup_{i \neq i_2,\dots,i_d} Y_i), \mathbb{Q}_\ell/\mathbb{Z}_\ell) \xrightarrow{\delta} H^0(Y_{i_1,\dots,i_d}, \mathbb{Q}_\ell/\mathbb{Z}_\ell),$$

where each δ is the connecting morphism for the obvious localization sequence for the homology theory, and $d_2 = \sum_{\mu=1}^d (-1)^\mu \delta_\mu$, where δ_μ is induced by the push-forward morphisms for the inclusions $Y_{i_1,\dots,i_d} \hookrightarrow Y_{i_1,\dots,\hat{i_\mu},\dots,i_d}$. Finally, $\delta' = \gamma_{Y^{[d-1]}} \circ \mathrm{tr}'$, where

$$\mathrm{tr}' : H^2(\overline{Y^{[d-1]}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1))_{G_K} \xrightarrow{\sim} H^0(\overline{Y^{[d-1]}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(0))_{G_K}$$

is the map induced by the trace map.

Proof. For any smooth irreducible variety W of dimension d over K and any smooth irreducible divisor $i : W' \hookrightarrow W$, we have a commutative diagram

$$\begin{array}{ccc} H^d(\overline{W \smallsetminus W'}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d)) & \stackrel{\delta}{\to} & H^{d-1}(\overline{W'}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d-1)) \\ \downarrow & & \downarrow \\ H^d(K(W) \otimes_K \overline{K}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d)) & \stackrel{\delta}{\to} & H^{d-1}(K(W') \otimes_K \overline{K}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d-1)), \end{array}$$

where the δ in the top line is the connecting morphism for the Gysin sequence for $W' \subseteq W \supseteq W \smallsetminus W'$, and the δ in the bottom line is the residue map for the point in W^1 corresponding to W'. The latter induces the connecting morphism

$$H_d(W \smallsetminus W', \mathbb{Q}_\ell/\mathbb{Z}_\ell) \stackrel{\delta}{\to} H_{d-1}(W', \mathbb{Q}_\ell/\mathbb{Z}_\ell)$$

for the localization sequence for $(W', W, W \smallsetminus W')$. This shows the commutativity of the top left square in (5.26), by definition of the maps e.

On the other hand, for a smooth projective curve C over $\overline{K}$ and a closed point $P : \mathrm{Spec}(\overline{K}) \to C$, the composition

$$H^0(\overline{K}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(0)) \xrightarrow{P_*} H^2(C, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1)) \xrightarrow{tr} H^0(\overline{K}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(0))$$

is the identity. This shows the commutativity of the top right square in (5.26). The two bottom squares commute because φ is a morphism of homology theories. □

We proceed with the proof of property 4.24(iii) for the morphism (5.24). The compositions of the vertical maps in the middle column and the right column of (5.26) are isomorphisms, and the top row is exact by Theorem 3.1 (and our assumption on U). But the bottom line is exact as well: This follows in a similar (but simpler) way as in the proof of Theorem 3.1 by noting that $\tilde{H}_a(T, \mathbb{Q}_\ell/\mathbb{Z}_\ell) = 0$ for $a \neq 0$ if T is smooth and projective of positive dimension, by definition. It can also be deduced from the fact that $\tilde{H}_a(U, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ is computed as the a-th homology of the complex

$$\bigoplus_{\pi_0(Y^{[d]})} \mathbb{Q}_\ell/\mathbb{Z}_\ell \to \bigoplus_{\pi_0(Y^{[d-1]})} \mathbb{Q}_\ell/\mathbb{Z}_\ell \to \cdots \to \bigoplus_{\pi_0(Y^{[0]})} \mathbb{Q}_\ell/\mathbb{Z}_\ell$$

as noted before.

This shows that the composition

$$H^d(\overline{U}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d))_{G_K} \xrightarrow{\gamma_U} H_d(U, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \xrightarrow{\varphi_U} \tilde{H}_d(U, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$$

figuring in the left column of (5.26) is an isomorphism. Because the subvarieties U as constructed above form a cofinal family in the set of open subvarieties of Z, by passing to the limit we get an isomorphism

$$H^d(K(Z) \otimes_K \overline{K}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d))_{G_K} \xrightarrow{\gamma} H_d(K(Z), \mathbb{Q}_\ell/\mathbb{Z}_\ell) \xrightarrow{\varphi} \tilde{H}_d(K(Z), \mathbb{Q}_\ell/\mathbb{Z}_\ell)$$

in which the first map γ is an isomorphism by definition. Therefore φ is an isomorphism as wanted, and Theorem 5.12 is proved.

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(Received: October 14, 2009)
(Revised: April 22, 2015)

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