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# Regularity of Einstein manifolds and the codimension 4 conjecture

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## Abstract

In this paper, we are concerned with the regularity of noncollapsed Riemannian manifolds $(M^n, g)$ with bounded Ricci curvature, as well as their Gromov-Hausdorff limit spaces $(M^n_j, d_j) \xrightarrow{d_{\mathrm{GH}}} (X, d)$, where $d_j$ denotes the Riemannian distance. Our main result is a solution to the codimension 4 conjecture, namely that $X$ is smooth away from a closed subset of codimension 4. We combine this result with the ideas of quantitative stratification to prove *a priori* $L^q$ estimates on the full curvature $|\mathrm{Rm}|$ for all $q < 2$. In the case of Einstein manifolds, we improve this to estimates on the regularity scale. We apply this to prove a conjecture of Anderson that the collection of 4-manifolds $(M^4, g)$ with $|\mathrm{Ric}_{M^4}| \leq 3$, $\mathrm{Vol}(M) > \mathrm{v} > 0$, and $\mathrm{diam}(M) \leq D$ contains at most a finite number of diffeomorphism classes. A local version is used to show that noncollapsed 4-manifolds with bounded Ricci curvature have *a priori* $L^2$ Riemannian curvature estimates.

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## 1. **Introduction**

In this paper, we consider pointed Riemannian manifolds $(M^n, g, p)$ with bounded Ricci curvature

$$|\mathrm{Ric}_{M^n}| \leq n-1, \tag{1.1}$$

which satisfy the noncollapsing assumption

$$\mathrm{Vol}(B_1(p)) > \mathrm{v} > 0. \tag{1.2}$$

We will be particularly concerned with pointed Gromov-Hausdorff limits

$$(M^n_j, d_j, p_j) \xrightarrow{d_{\mathrm{GH}}} (X, d, p) \tag{1.3}$$

of sequences of such manifolds, where $d_j$ always denotes the Riemannian distance. Our main result is that $X$ is smooth away from a closed subset of codimension 4.[1] We will combine this with the previous work of the authors on quantitative stratification to show that $X$ satisfies *a priori* $L^q$ estimates on the curvature $|\mathrm{Rm}|$ for all $q < 2$; see Theorems 1.4 and 1.8. Finally, we will apply the results in the dimension 4 setting in which there are various improvements, including a finiteness theorem up to diffeomorphism and an *a priori* $L^2$ curvature bound, for noncollapsed manifolds with bounded Ricci curvature; see Theorems 1.12 and 1.13.

[1] In the Kähler case, this was shown in [Che03], and independently by Tian, by means of an $\varepsilon$-regularity theorem that exploits the first Chern form and its relation to Ricci curvature.

The first major results on limit spaces satisfying (1.1)–(1.3) were proved in Einstein case; see [And89], [BKN89], and [Tia90]. A basic assumption is that the $L^{n/2}$ norm of the of the curvature tensor is bounded. From this, together with an appropriate $\varepsilon$-regularity theorem, it was shown that any limit space as above is smooth away from at most a definite number of points at which the singularities are of orbifold type.

In dimension 4, given (1.1), it follows directly from the Chern-Gauss-Bonnet formula that the $L^2$-norm of the curvature is bounded in terms of the Euler characteristic. In [And90], it is shown that the collection of noncollapsed 4-manifolds with definite bounds on Ricci curvature, diameter, and Euler characteristic contains only finitely many diffeomorphism types. Assuming an $L^{n/2}$ bound on curvature (in place of the Euler characteristic bound) the finiteness theorem was extended to arbitrary dimensions in [AC91], a precursor of which was [Ban90].

It was conjectured in [And95] that for the finiteness theorem in dimension 4, the Euler characteristic bound is an unnecessary assumption. In Theorem 1.12, we prove this conjecture.

The first step toward the study of such Gromov-Hausdorff limits as in relations (1.1)–(1.3), without the need for assumptions implying integral curvature bounds, was taken in [CC96]. There, a stratification theory for noncollapsed limits with only lower Ricci curvature bounds was developed. By combining this with the $\varepsilon$-regularity results of [And90], it was proved that a noncollapsed Gromov-Hausdorff limit of manifolds with bounded Ricci curvature is smooth outside a closed subset of codimension 2. More recently, it was shown in [CN13] that one can then prove *a priori* $L^q$-bounds on the curvature for all $q < 1$.

Based on knowledge of the 4-dimensional case, early workers conjectured that the singular set of a noncollapsed limit space satisfying (1.1)–(1.3) should have codimension 4. This was shown in [CCT02] under the additional assumption of an $L^q$ curvature bound for all $q < 2$. The following is the main result of this paper.

THEOREM 1.4. *Let* $(M_j^n, d_j, p_j) \xrightarrow{d_{\mathrm{GH}}} (X, d, p)$ *be a Gromov-Hausdorff limit of manifolds with* $|\mathrm{Ric}_{M_j^n}| \le n-1$ *and* $\mathrm{Vol}(B_1(p_j)) > \mathrm{v} > 0$. *Then the singular set* $\mathcal{S}$ *satisfies*

$$\dim \mathcal{S} \le n-4. \tag{1.5}$$

*The dimension can be taken to be the Hausdorff or Minkowski dimension.*

We will outline the proof of Theorem 1.4 in Section 1.1. First we will discuss various applications. Our first applications are to the regularity theory of Einstein manifolds. To make this precise, let us begin with the following definition (see also [CN13]).

*Definition* 1.6. For $x \in X$, we define the regularity scale $r_x$ by

$$r_x \equiv \max_{0<r\leq 1} \left\{ \sup_{B_r(x)} |\mathrm{Rm}| \leq r^{-2} \right\}. \tag{1.7}$$

If $x \in \mathcal{S}$ is in the singular set of $X$, then $r_x \equiv 0$.

Let $T_r(S) = \{x \in M : d(x, S) < r\}$ denote the $r$-tube around the set $S$. By combining Theorem 1.4 with the quantitative stratification ideas of [CN13], we show the following.

THEOREM 1.8. *There exists $C = C(n, \mathrm{v}, q)$ such that if $M^n$ satisfies $|\mathrm{Ric}_{M^n}| \leq n - 1$ and $\mathrm{Vol}(B_1(p)) > \mathrm{v} > 0$, then for each $q < 2$,*

$$\fint_{B_1(p)} |\mathrm{Rm}|^q \leq C. \tag{1.9}$$

*If, in addition, $M^n$ is assumed to be Einstein, then for every $q < 2$, we have that*

$$\mathrm{Vol}(T_r(\{x \in B_1(p) : r_x \leq r\})) \leq C\, r^{2q}. \tag{1.10}$$

*Remark* 1.11. If we replace the assumption that $M^n$ is Einstein with just a bound on $|\nabla \mathrm{Ric}_{M^n}|$, we obtain the same conclusion. In fact, if we only assume a bound on the Ricci curvature $|\mathrm{Ric}_{M^n}|$, then (1.10) holds with the regularity scale $r_x$ replaced by the harmonic radius $r_h$; see Definition 2.9. Note that estimates on the regularity scale are *much* stronger than corresponding $L^q$ estimates for the curvature given in (1.9).

The final theorems of the paper concern the 4-dimensional case in which we can make some marked improvements on the results in the general case. Let us begin with the following, which is a conjecture of Anderson [And95].

THEOREM 1.12. *There exists $C = C(\mathrm{v}, D)$ such that if $M^4$ satisfies $|\mathrm{Ric}_{M^4}| \leq 3$, $\mathrm{Vol}(B_1(p)) > \mathrm{v} > 0$, and $\mathrm{diam}(M^n) \leq D$, then $M^4$ can have one of at most $C$ diffeomorphism types.*

By proving a more local version of the above theorem, we can improve Theorem 1.8 in the 4-dimensional case and show that the $L^q$-bounds on the curvature for $q < 2$ may be pushed all the way to an *a priori* $L^2$-bound in dimension 4. We conjecture in Section 9 that this holds in all dimensions.

THEOREM 1.13. *There exists $C = C(\mathrm{v})$ such that if $M^4$ satisfies $|\mathrm{Ric}_{M^4}| \leq 3$ and $\mathrm{Vol}(B_1(p)) > \mathrm{v} > 0$, then*

$$\fint_{B_1(p)} |\mathrm{Rm}|^2 \leq C. \tag{1.14}$$

*Furthermore, we have the sharp weak type $L^2$-estimate on the harmonic radius,*

$$\mathrm{Vol}(T_r(\{x \in B_1(p) : r_h \leq r\})) \leq Cr^4. \tag{1.15}$$

*If we assume in addition that $M^4$ is Einstein, then the same result holds with the harmonic radius $r_h$ replaced by the regularity scale $r_x$.*

*Remark* 1.16. If the assumption that $M^4$ is Einstein is weakened to assuming a bound on $|\nabla \mathrm{Ric}_{M^n}|$, then (1.15) still holds with the harmonic radius $r_h$ replaced by the stronger regularity scale $r_x$.

Next, we will give a brief outline of the paper. We begin in Section 1.1 by outlining the proof of Theorem 1.4. This includes statements and explanations of some of the main technical theorems of the paper.

In Section 2 we go over some basic background and preliminary material. This includes the basics of stratifications for limit spaces, the standard $\varepsilon$-regularity theorem for spaces with bounded Ricci curvature, and some motivating examples.

Sections 3 and 4 are the the most crucial sections of the paper. There, we prove Theorems 1.32 and Theorem 1.23, the Transformation and Slicing Theorems which, roughly speaking, allow us to blow up along a collection of points that is large enough to see into the singular set; see Section 1.1 for more on this.

Section 5 is dedicated to proving the main result of the paper, Theorem 1.4. The argument is a blowup argument that exploits the Slicing Theorem of Section 4. In Section 6, based on Theorem 1.4, we give a new $\varepsilon$-regularity theorem. Theorem 6.1 states that if a ball in a space with bounded Ricci curvature is close enough in the Gromov-Hausdorff sense to a ball in a metric cone, $\mathrm{R}^{n-3} \times C(Z)$, then the concentric ball of half the radius must be smooth.

In Section 7, the $\varepsilon$-regularity theorem of Section 6 is combined with the ideas of quantitative stratification to give effective improvements on all the results of the paper. We show that the singular set has codimension 4 in the Minkowski sense, and we give effective estimates for tubes around the regions of curvature concentration. This culminates in the proof of Theorem 1.8. In Section 7.2, we use the effective estimates of Theorem 1.8 to prove new estimates for harmonic functions on spaces with bounded Ricci curvature. These estimates, which can fail on manifolds with only lower Ricci curvature bounds, give the first taste of how analysis on manifolds with bounded Ricci curvature improves over that on manifolds with only lower Ricci curvature bounds.

Finally, in Section 8, we discuss the 4-dimensional case and prove the finiteness up to diffeomorphism theorem, Theorem 1.12. We also prove the $L^2$ curvature estimates of Theorem 1.13.

1.1. *Outline of the proof Theorem* 1.4, *the codimension* 4 *conjecture.* Let $S^1_\beta$ denote the circle of circumference $\beta < 2\pi$. It has been understood since [CC96] that to prove Theorem 1.4, the key step is to show that the cone $\mathbb{R}^{n-2} \times C(S^1_\beta)$ does not occur as the (pointed) Gromov-Hausdorff limit of some sequence $M^n_j$ with $|\mathrm{Ric}_{M^n_j}| \to 0$. This was shown in [CCT02] assuming just a lower bound $\mathrm{Ric}_{M^n_i} \geq -(n-1)$, but with the additional assumption that the $L^1$-norm of the curvature is sufficiently small. In [Che03], it was proved for the Kähler-Einstein case, which was also done by Tian. A common feature of the proofs is an argument by contradiction, implemented by the use of harmonic almost splitting maps $u : B_2(p) \to \mathbb{R}^{n-2}$; see Lemma 1.21. In each case, it is shown that for most points $s \in \mathbb{R}^{n-2}$ in the range, the slice $u^{-1}(s)$ has a certain good property which, when combined with the assumed curvature bounds, enables one to deduce a contradiction. In particular, in [CCT02] it is shown that most slices $u^{-1}(s)$ have integral bounds on the second fundamental form, which when combined with the assumed integral curvature bounds, enables one to apply the Gauss-Bonnet formula for 2-dimensional manifolds with boundary to derive a contradiction.

However, prior to the present paper it was not known how, in the general case, to implement a version of the above strategy, which would rule out the cones $\mathbb{R}^{n-2} \times C(S^1_\beta)$, without assuming the integral curvature bounds. In the remainder of this subsection, we will state the main results that are used in the present implementation, which enables us to prove Theorem 1.4.

Thus, we consider a sequence of Riemannian manifolds $(M^n_j, d_j, p_j)$, with $|\mathrm{Ric}_{M^n_j}| \to 0$ and $\mathrm{Vol}(B_1(p_j) > \mathrm{v} > 0$, such that

$$(M^n_j, d_j, p_j) \xrightarrow{d_{\mathrm{GH}}} \mathbb{R}^{n-2} \times C(S^1_\beta). \tag{1.17}$$

As above, we have harmonic almost splitting maps

$$u_j : B_2(p_j) \to \mathbb{R}^{n-2}; \tag{1.18}$$

see Lemma 1.21 below. The key ingredient will be Theorem 1.23 (the Slicing Theorem), which states that there exist $s_j \in \mathbb{R}^{n-2}$ such that for all $x \in u_j^{-1}(s_j)$ and for all $r < 1$, the ball $B_r(x)$ is $\varepsilon_j r$-close in the Gromov-Hausdorff sense to a ball in an isometric product $\mathbb{R}^{n-2} \times S_{j,x,r}$, where $\varepsilon_j \to 0$ as $j \to \infty$.

Granted this, we can apply a blowup argument in the spirit of [And90] to obtain a contradiction. Namely, it is easy to see that since $\beta < 2\pi$, then the minimum of the harmonic radius $r_h$ at points of the slice $u_j^{-1}(s_j)$ is obtained at some $x_j \in u_j^{-1}(s_j)$ and is going to zero as $j \to \infty$. We rescale the metric by the inverse of the harmonic radius $r_j = r_h(x_j)$ and find a subsequence converging in the pointed Gromov-Hausdorff sense a smooth noncompact Ricci flat manifold,

$$(M^n_j, r_j^{-1} d_j, x_j) \to (X, d_X, x), \tag{1.19}$$

such that $X = \mathbb{R}^{n-2} \times S$ splits off $\mathbb{R}^{n-2}$ isometrically, with $S$ a smooth 2-dimensional surface. It follows that $S$ is Ricci flat, and hence flat. From the noncollapsing assumption, it follows that $X$ has Euclidean volume growth. Thus, $X = \mathbb{R}^n$ is Euclidean space. However, the 2-sided Ricci bound implies that the harmonic radius behaves continuously in the limit. Hence, the harmonic radius at $x$ is $r_h(x_\infty) = 1$; a contradiction. See Section 5.1 for more details on the blowup argument.

Clearly then, the key issue is to show the existence of the points $s_j \in \mathbb{R}^{n-2}$, such that at *all* points $x \in u_j^{-1}(s_j)$, we have the above mentioned splitting property on $B_r(x)$ for *all* $r < 1$. To indicate the proof, we now recall some known connections between isometric splittings, the Gromov-Haudorff distance and harmonic maps to Euclidean spaces $\mathbb{R}^k$. We begin with a definition.

*Definition* 1.20. A $\varepsilon$-splitting map $u = (u^1, \dots, u^k) : B_r(p) \to \mathbb{R}^k$ is a harmonic map such that

(1) $|\nabla u| \leq 1 + \varepsilon$;
(2) $\fint_{B_r(p)} |\langle \nabla u^\alpha, \nabla u^\beta \rangle - \delta^{\alpha\beta}|^2 < \varepsilon^2$;
(3) $r^2 \fint_{B_r(p)} |\nabla^2 u^\alpha|^2 < \varepsilon^2$.

Note that the condition that $u$ is harmonic is equivalent to the harmonicity of the individual component functions $u^1, \dots, u^k$.

The following lemma summarizes the basic facts about splitting maps.[2]

LEMMA 1.21 ([CC96]). *For every $\varepsilon, R > 0$, there exists $\delta = \delta(n, \varepsilon, R) > 0$ such that if $\mathrm{Ric}_{M^n} \geq -(n-1)\delta$, then*

(1) *If $u : B_{2R}(p) \to \mathbb{R}^k$ is a $\delta$-splitting map, then there exists a map $f : B_R(p) \to u^{-1}(0)$ such that*

$$(u, f) : B_R(p) \to \mathbb{R}^k \times u^{-1}(0)$$

*is an $\varepsilon$-Gromov Hausdorff map, where $u^{-1}(0)$ is given the induced metric.*

(2) *If*

$$d_{\mathrm{GH}}(B_{\delta^{-1}}(p), B_{\delta^{-1}}(0)) < \delta, \tag{1.22}$$

*where $0 \in \mathbb{R}^k \times Y$, then there exists an $\varepsilon$-splitting map $u : B_R(p) \to \mathbb{R}^k$.*

Let us return to the consideration of the maps $u_j$ from (1.18), which in our situation arise from (2) of Lemma 1.21. We can thus assume that the $u_j$ are $\delta_j$-splitting maps, with $\delta_j \to 0$. We wish to find slices $u_j^{-1}(s_j)$ such that $B_r(x)$ continues to almost split for all $x \in u_j^{-1}(s_j)$ and all $r \leq 1$. One might

[2] In [CC96], only a uniform bound $|\nabla u| < C(n)$ is proved. This would actually suffice for our present purposes. The improved bound, $|\nabla u| < 1 + \varepsilon$, in (1) above, is derived in (3.42)–(3.46) in a context that passes over almost verbatim to the present one.

hope that there always exist $s_j$ such that by restricting the map $u_j$ to each such ball $B_r(x)$, one obtains an $\varepsilon_j$-splitting map. However, it turns out that there are counterexamples to this statement; see Example 2.14.

The essential realization is that for our purposes, it actually suffices to show the existence of $s_j$ such that for all $x \in u_j^{-1}(s_j)$ and all $0 < r \leq 1$, there exists a matrix $A = A(x,r) \in \mathrm{GL}(n-2)$ such that the harmonic map $A \circ u_j : B_r(x) \to \mathbb{R}^{n-2}$ is our desired $\varepsilon_j$-splitting map. Thus, while $u_j$ might not itself be an $\varepsilon_j$-splitting map on $B_r(x)$, it might only differ from one by a linear transformation of the image. This turns out to hold. In fact, we will show that $A$ can be chosen to be lower triangular with positive diagonal entries. Since this condition plays a role in the proof of Theorem 1.32 below, we will incorporate it from now on.

THEOREM 1.23 (The Slicing Theorem). *For each $\varepsilon > 0$, there exists $\delta(n,\varepsilon) > 0$ such that if $M^n$ satisfies $\mathrm{Ric}_{M^n} \geq -(n-1)\delta$ and if $u : B_2(p) \to \mathbb{R}^{n-2}$ is a harmonic $\delta$-splitting map, then there exists a subset $G_\varepsilon \subseteq B_1(0^{n-2})$ that satisfies the following*:

(1) $\mathrm{Vol}(G_\varepsilon) > \mathrm{Vol}(B_1(0^{n-2})) - \varepsilon$;
(2) *if $s \in G_\varepsilon$, then $u^{-1}(s)$ is nonempty*;
(3) *for each $x \in u^{-1}(G_\varepsilon)$ and $r \leq 1$, there exists a lower triangular matrix $A \in \mathrm{GL}(n-2)$ with positive diagonal entries such that $A \circ u : B_r(x) \to \mathbb{R}^{n-2}$ is an $\varepsilon$-splitting map.*

The proof of the Slicing Theorem is completed in Section 4. In the next subsection, we give the proof modulo the key technical results on which it depends. These will be indicated in the remainder of the present subsection.

Given a harmonic function with values in $\mathrm{R}^k$, for $1 \leq \ell \leq k$, we put

$$\omega^\ell =: du^1 \wedge \cdots \wedge du^\ell. \tag{1.24}$$

The forms $\omega^\ell$ and, in particular, the Laplacians $\Delta|\omega^\ell|$ of their norms, play a key role in the sequel. We point out that in general, $\Delta|\omega^\ell|$ is a distribution, not just a function. Put

$$\begin{aligned} Z_{|\nabla u^a|} &=: \{x \,:\, |\nabla u^a|(x) = 0\}, \\ Z_{|\omega^\ell|} &=: \{x \,:\, |\omega^\ell|(x) = 0\}. \end{aligned} \tag{1.25}$$

Then the functions $|\nabla u^a|$, $|\omega^\ell|$, are Lipschitz on $B_2(p)$ and are smooth away from $Z_{|\nabla u^a|}$, $Z_{|\omega^\ell|}$, respectively. An important structural point, which is contained in the next theorem, is that $\Delta|\nabla u^a|$ is in fact a function and $\Delta|\omega^\ell|$ is at least a Borel measure. As usual, $|\Delta|\omega^\ell||$ denotes the absolute value of the measure $\Delta|\omega^\ell|$. Thus, $\int_U |\Delta|\omega^\ell||$ denotes the mass of the restriction of $\Delta|\omega^\ell|$ to $U$ and $\fint_U |\Delta|\omega^\ell||$ denotes this mass divided by $\mathrm{Vol}(U)$.

THEOREM 1.26 (Higher order estimates). *For every $\varepsilon > 0$, there exists $\delta(n,\varepsilon) > 0$ such that if $\mathrm{Ric}_{M^n} \geq -(n-1)\delta$ and $u : B_2(p) \to \mathbb{R}^k$ is a $\delta$-splitting map, then the following hold*:

(1) *There exists $\alpha(n) > 0$ such that for each $1 \leq a \leq k$,*

$$\fint_{B_{3/2}(p)} \frac{|\nabla^2 u^a|^2}{|\nabla u^a|^{1+\alpha}} < \varepsilon. \tag{1.27}$$

(2) *Let $\omega^\ell \equiv du^1 \wedge \cdots \wedge du^\ell$, $1 \leq \ell \leq k$. The Laplacians $\Delta|\omega^\ell|$ taken in the distributional sense are Borel measures with singular part a nonnegative locally finite Borel measure supported on $\partial Z_{|\omega^\ell|}$. For $\ell = 1$, the singular part vanishes. The normalized mass of $\Delta|\omega^\ell|$ satisfies*

$$\fint_{B_{3/2}(p)} |\Delta|\omega^\ell|| < \varepsilon. \tag{1.28}$$

*Remark* 1.29. In actuality, we will need only the case $\alpha = 0$ of (1.27).

As will be clear from Theorem 1.32 below (the Transformation Theorem), the following definition is key.

*Definition* 1.30. Let $u : B_2(p) \to \mathbb{R}^k$ be a harmonic function. For $x \in B_1(p)$ and $\delta > 0$, define the *singular scale* $s_x^\delta \geq 0$ to be the infimum of all radii $s$ such that for all $r$ with $s \leq r < \frac{1}{2}$ and all $1 \leq \ell \leq k$ we have

$$r^2 \fint_{B_r(x)} |\Delta|\omega^\ell|| \leq \delta \fint_{B_r(x)} |\omega^\ell|. \tag{1.31}$$

Note that there is an invariance property for (1.31). Namely, if (1.31) holds for $u$, then it holds for $A \circ u$ for any lower triangular matrix $A \in \mathrm{GL}(k)$. That is, the singular scale of $u$ and the singular scale of $A \circ u$ are equal. In view of (1.28), this means essentially that (1.31) is a necessary condition for the existence of $A$ as in the Slicing Theorem. Our next result, which is by far the most technically difficult of the paper, provides a sort of converse. We will not attempt to summarize the proof except to say that it involves a contradiction argument, as well as an induction on $\ell$. It is proved in Section 3.

THEOREM 1.32 (The Transformation Theorem). *For every $\varepsilon > 0$, there exists $\delta = \delta(n,\varepsilon) > 0$ such that if $\mathrm{Ric}_{M^n} \geq -(n-1)\delta$ and $u : B_2(p) \to \mathbb{R}^k$ is a $\delta$-splitting map, then for each $x \in B_1(p)$ and $1/2 \geq r \geq s_x^\delta$, there exists a lower triangular matrix $A = A(x,r)$ with positive diagonal entries such that $A \circ u : B_r(x) \to \mathbb{R}^k$ is an $\varepsilon$-splitting map.*

1.2. *Proof the the Slicing Theorem modulo technical results.* Granted Theorems 1.26 and 1.32, the Transformation Theorem, we now give the proof of the Slicing Theorem, modulo two additional technical results, (1.37) and (1.38).

These will be seen in Section 4 to be easy consequences of the Transformation Theorem.

Fix $\varepsilon > 0$ as in the Slicing Theorem. We must show that there exists $\delta = \delta(n, \varepsilon)$ such that if $u : B_2(p) \to \mathbb{R}^{n-2}$ denotes a $\delta$-splitting map, then the conclusions of the the Slicing Theorem hold.

Let us write $\delta_3 = \delta_3(n, \varepsilon)$ for what was denoted by $\delta(n, \varepsilon)$ in the Transformation theorem. Put

$$\mathcal{B}_{\delta_3} =: \bigcup_{x \in B_1(p) \,|\, s_x^{\delta_3} > 0} B_{s_x^{\delta_3}}(x). \tag{1.33}$$

We can assume that $\delta$ of the Slicing Theorem is small enough so that $s_x^{\delta_3} \leq 1/32$ (which will be used in (1.40)).

Let $|u(V)|$ denote the $(n-2)$-dimensional measure of $V \subset \mathbb{R}^{n-2}$. According to Theorem 2.37 of [CCT02], there exists $\delta_1 = \delta_1(n, \varepsilon/2)$ such that if $\delta(n, \varepsilon) \leq \delta_1(n, \varepsilon/2)$, then

$$|B_1(0^{n-2})) \setminus u(B_1(p))| < \varepsilon/2. \tag{1.34}$$

It follows from the Transformation Theorem and (1.34) that if we choose $\delta$ to satisfy in addition $\delta \leq \delta_1$, then to conclude the proof of the Slicing Theorem, it suffices to show that $\delta$ can be chosen so that we also have

$$|u(\mathcal{B}_{\delta_3})| \leq \varepsilon/2. \tag{1.35}$$

To this end, we record two perhaps nonobvious, but easily verified, consequences of Theorem 1.32.

Denote by $\mu$ the measure such that for all open sets $U$,

$$\mu(U) = \left( \int_{B_{3/2}(p)} |\omega| \right)^{-1} \cdot \int_U |\omega|. \tag{1.36}$$

The first consequence (see Lemma 4.1) is that for each $x \in B_1(p)$ and $1/4 \geq r \geq s_x^{\delta_3}$, we have the doubling condition

$$\mu(B_{2r}(x)) \leq C(n) \cdot \mu(B_r(x)). \tag{1.37}$$

The second consequence (see Lemma 4.10) is that if $x \in B_1(p)$ and $1/2 \geq r \geq s_x^{\delta_3}$, then we have the volume estimate

$$|u(B_r(x))| \leq C(n) \cdot r^{-2} \mu(B_r(x)). \tag{1.38}$$

The proof of these results exploits the fact that $A \circ u : B_r(x) \to \mathbb{R}^{n-2}$ is an $\varepsilon$-splitting map for some lower triangular matrix $A$ with positive diagonal entries.

By a standard covering lemma, there exists a collection of mutually disjoint balls, $\{B_{s_j}(x_j)\}$ with $s_j = s_{x_j}^{\delta_3}$, such that

$$\mathcal{B}_{\delta_3} \subset \bigcup_j B_{6s_j}(x_j). \tag{1.39}$$

Since the balls $B_{s_j}(x_j)$ are mutually disjoint, we can apply Theorem 1.26 together with (1.38) and the (three times iterated) doubling property (1.37) of $\mu$ to obtain

(1.40)

$$\begin{aligned}|u(\mathcal{B}_{\delta_3})| &\le \sum_j |u(B_{6s_j}(x_j))| \le \sum_j (6s_j)^{-2}\mu(B_{6s_j}(x_j))\\ &\le C(n)\sum_j s_j^{-2}\mu(B_{s_j}(x_j)) \le C\delta_3^{-1}\sum_j \left(\int_{B_{3/2}(p)}|\omega|\right)^{-1}\cdot\int_{B_{s_j}(x_j)}|\omega|\\ &\le C\delta_3^{-1}\cdot \fint_{B_{3/2}(p)}|\omega|.\end{aligned}$$

(For the iterated doubling property of $\mu$, we used $s_x^{\delta_3} \le 1/32$.)

Write $\delta_2(n,\,\cdot\,)$ for what was denoted by $\delta(n,\,\cdot\,)$ in Theorem 1.26. If, in addition, we choose $\delta \le \delta_2(n, \frac{1}{2}C^{-1}\delta_3\varepsilon)$, where $C = C(n)$ is the the constant on the last line in (1.40), then by Theorem 1.26, the right-hand side of (1.40) is $\le \varepsilon/2$; i.e. (1.35) holds. As we have noted, this suffices to complete the proof of the Slicing Theorem.

## 2. **Background and preliminaries**

In this section we review some standard constructions and techniques, which will be used throughout the paper.

2.1. *Stratification of limit spaces.* In this subsection we recall some basic properties of pointed Gromov-Hausdorff limit spaces

$$(M_j^n, d_j, p_j) \xrightarrow{d_{\mathrm{GH}}} (X, d, p), \tag{2.1}$$

where the $\mathrm{Ric}_{M_j^n} \ge -(n-1)$ and the noncollapsing assumption $\mathrm{Vol}(B_1(p_j)) \ge \mathrm{v} > 0$ holds. In particular, we recall the stratification of a noncollapsed limit space, which was first introduced in [CC96] and which will play an important role in the proof of Theorem 1.4. The effective version, called the *quantitative stratification*, which was first introduced in [CN13], will be recalled in Section 7. It will play an important role in the estimates of Theorem 1.8.

Given $x \in X$, we call a metric space $X_x$ a *tangent cone* at $x$ if there exists a sequence $r_i \to 0$ such that

$$(X, r_i^{-1}d, x) \xrightarrow{d_{\mathrm{GH}}} X_x. \tag{2.2}$$

That tangent cones exist at every point is a consequence of Gromov's compactness theorem; see, for instance, the book [Pet98]. A point is called *regular* if every tangent cone is isometric to $\mathbb{R}^n$ and otherwise *singular*. The set of singular points is denoted by $\mathcal{S}$. As explained below, for noncollapsed limit spaces with a uniform lower Ricci bound, the singular set has codimension

$\geq 2$. At singular points, tangent cones may be highly nonunique, to the extent that neither the dimension of the singular set nor the homeomorphism type is uniquely defined; see, for instance, [CN13]. Easy examples show that the singular set need not be closed if one just assumes a uniform a lower bound $\mathrm{Ric}_{M_j^n} \geq -(n-1)$. However, under the assumption of a 2-sided bound $|\mathrm{Ric}_{M_j^n}| \leq (n-1)$, the singular set is indeed closed; see [And90] and [CC97].

For noncollapsed limit spaces, as shown in [CC96], every tangent cone is isometric to a *metric cone*; i.e.,

$$X_x = C(Z) \tag{2.3}$$

for some compact metric space $Z$, with $\mathrm{diam}(Z) \leq \pi$. With this as our starting point, we introduce the following notion of symmetry.

*Definition* 2.4. A metric space $Y$ is called *$k$-symmetric* if $Y$ is isometric to $\mathbb{R}^k \times C(Z)$ for some compact metric space $Z$. We define the *closed $k^{\mathrm{th}}$-stratum* by

$$\mathcal{S}^k(X) =: \{x \in X : \text{ no tangent cone at } x \text{ is } (k+1)\text{-symmetric}\} \tag{2.5}$$

Thus, in the noncollapsed case, every tangent cone is 0-symmetric.

The key result of [CC96] is the following:

$$\dim \mathcal{S}^k \leq k, \tag{2.6}$$

where dimension is taken in the Hausdorff sense. Thus, away from a set of Hausdorff dimension $k$, every point has some tangent cone with $(k+1)$ degrees of symmetry. For an effective refinement of this theorem, see [CN13] and Section 7.

2.2. *The $\varepsilon$-regularity theorems.* A central result of this paper is the $\varepsilon$-regularity theorem, Theorem 6.1. The original $\varepsilon$-regularity theorems for Einstein manifolds were given in [BKN89] and [And89], [Tia90]. They state that if $M^n$ is an Einstein manifold with $\mathrm{Ric}_{M^n} = \lambda g$, $|\lambda| \leq n-1$, $\mathrm{Vol}(B_1(p)) \geq \mathrm{v}$, and

$$\int_{B_2(p)} |\mathrm{Rm}|^{n/2} < \varepsilon(n, \mathrm{v}), \tag{2.7}$$

then $\sup_{B_1(p)} |\mathrm{Rm}| \leq 1$.

In [CCT02], [Che03], and [CD13], $\varepsilon$-regularity theorems were proved under the assumption of $L^q$ curvature bounds, $1 \leq q < n/2$, provided $B_2(p)$ is assumed sufficiently close to a ball in a cone that splits off an isometric factor $\mathbb{R}^{n-2q}$.

On the other hand, the regularity theory of [CN13] for Einstein manifolds, depends on $\varepsilon$-regularity theorems that do not assume $L^q$ curvature bounds. In

particular, it follows from the work of [And90] that there exists $\varepsilon(n) > 0$ such that if $|\mathrm{Ric}_{M^n}| \leq \varepsilon(n)$ and if

$$d_{\mathrm{GH}}(B_2(p), B_2(0^n)) < \varepsilon(n), \tag{2.8}$$

where $B_2(0^n) \subseteq \mathbb{R}^n$, then $|\mathrm{Rm}| \leq 1$ on $B_1(p)$.

This result can be extended in several directions. In order to state the extension in full generality, we first recall the notion of the harmonic radius.

*Definition* 2.9. For $x \in X$, we define the harmonic radius $r_h(x)$ so that $r_h(x) = 0$ if no neighborhood of $x$ is a Riemannian manifold. Otherwise, we define $r_h(x)$ to be the largest $r > 0$ such that there exists a mapping $\Phi : B_r(0^n) \to X$ such that

(1) $\Phi(0) = x$ with $\Phi$ is a diffeomorphism onto its image;
(2) $\Delta_g x^\ell = 0$, where $x^\ell$ are the coordinate functions and $\Delta_g$ is the Laplace Beltrami operator;
(3) if $g_{ij} = \Phi^* g$ is the pullback metric, then

$$||g_{ij} - \delta_{ij}||_{C^0(B_r(0^n))} + r||\partial_k g_{ij}||_{C^0(B_r(0^n))} \leq 10^{-3}. \tag{2.10}$$

We call a mapping $\Phi : B_r(0^n) \to X$ as above a harmonic coordinate system. Harmonic coordinates have an abundance of good properties when it comes to regularity issues; see the book [Pet98] for a nice introduction. In particular, if the Ricci curvature is uniformly bounded, then in harmonic coordinates, the metric, $g_{ij}$ has *a priori* $C^{1,\alpha} \cap W^{2,q}$-bounds for all $\alpha < 1$ and $q < \infty$. If, in addition, there is a bound on $|\nabla \mathrm{Ric}_{M^n}|$, then in harmonic coordinates, $g_{ij}$ has $C^{2,\alpha}$-bounds for all $\alpha < 1$.

The primary theorem we wish to review in this subsection is the following

Theorem 2.11 ([And90], [CC96]). *There exists $\varepsilon(n, \mathrm{v}) > 0$ such that if $M^n$ satisfies $|\mathrm{Ric}_{M^n}| \leq \varepsilon$, $\mathrm{Vol}(B_1(p)) > \mathrm{v} > 0$, and*

$$d_{\mathrm{GH}}(B_2(p), B_2(0)) < \varepsilon(n), \tag{2.12}$$

*where $0 \in \mathbb{R}^{n-1} \times C(Z)$, then the harmonic radius $r_h(p)$ satisfies*

$$r_h(p) \geq 1. \tag{2.13}$$

*If $M^n$ is further assumed to be Einstein, then the regularity scale $r_p$ satisfies $r_p \geq 1$.*

By the results of the previous subsection, it is possible to find balls satisfying the above constraint off a subset of Hausdorff codimension 2. Moreover, when combined with the quantitative stratification of [CN13], see also Section 7, this $\varepsilon$-regularity theorem leads to *a priori* $L^p$-bounds on the curvature. The primary result of the present paper can be viewed as Theorem 6.1, which states that the conclusions of Theorem 2.11 continue to hold if $\mathbb{R}^{n-1}$ is replaced by $0 \in \mathbb{R}^{n-3} \times C(Z)$.

2.3. *Examples.* In this subsection, we indicate some simple examples that play an important role in guiding the results of this paper.

*Example* 2.14 (The Cone Space $\mathbb{R}^{n-2} \times C(S^1_\beta)$). The main result of this paper, Theorem 1.4, states that $\mathbb{R}^{n-2} \times C(S^1_\beta)$, with $\beta < 2\pi$, is not the noncollapsed Gromov-Hausdorff limit of a sequence of manifolds with bounded Ricci curvature. However, it is clear that this space is the Gromov-Hausdorff limit of a sequence of noncollapsed manifolds with a uniform lower Ricci curvature bound. Indeed, by rounding off $C(S^1_\beta)$, we see that $\mathbb{R}^{n-2} \times C(S^1_\beta)$ can appear as a noncollapsed limit of manifolds with nonnegative sectional curvature.

In this example, let us just consider the 2-dimensional cone $C(S^1_\beta)$ with $\beta < 2\pi$. Regard $S^1_\beta$ as $0 \le \theta \le 2\pi$, with the end points identified. Then the Laplacian on $S^1_\beta$ is $(\frac{2\pi}{\beta})^2 \cdot \frac{\partial^2}{\partial\theta^2}$. The eigenfunctions are of the form $e^{ik\theta}$, where $k$ is an integer. Written in polar coordinates, a basis for the bounded harmonic functions on $C(S^1_\beta)$ is $\{r^{\frac{2\pi}{\beta}|k|} \cdot e^{ik\theta}\}$. In particular, we see from this that if $\beta < 2\pi$, then $|\nabla(r^{\frac{2\pi}{\beta}|k|} \cdot e^{ik\theta})| \to 0$ as $r \to 0$. As a consequence, *every* bounded harmonic function has vanishing gradient at the vertex, which is a set of positive $(n-2)$-dimensional Hausdorff measure. By considering examples with more vertices, we can construct limit spaces where bounded harmonic functions $h$ must have vanishing gradient on bounded subsets sets of arbitrarily large, or even infinite, $(n-2)$-dimensional Hausdorff measure. This set can even be taken to be dense.

*Example* 2.15 (The Eguchi-Hanson manifold). The Eguchi-Hanson metric $g$ is a complete Ricci flat metric on the cotangent bundle of $S^2$, which at infinity becomes rapidly asymptotic to the metric cone on $\mathbb{RP}(3)$ or equivalently to $\mathbb{R}^4/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts on $\mathbb{R}^4$ by $x \to -x$. When the metric $g$ is scaled down by $g \to r^2 g$, with $r \to 0$, one obtains a family of Ricci flat manifolds whose Gromov-Hausdorff limit is $C(\mathbb{RP}(3)) = \mathbb{R}^4/\mathbb{Z}_2$. This is the simplest example that shows that even under the assumption of Ricci flatness and noncollapsing, Gromov-Hausdorff limit spaces can contain codimension 4 singularities.

*Example* 2.16 (Infinitely many topological types in dimension 4). Let $T^3$ denote a flat 3-torus. According to Anderson [And92], there is a collapsing sequence of manifolds $(M^4_j, d_j) \xrightarrow{d_{\mathrm{GH}}} T^3$ satisfying

$$\begin{aligned} &\operatorname{diam}(M^4_j) \le 1, \\ &|\operatorname{Ric}_{M^n_j}| \le \varepsilon_j \to 0, \\ &\operatorname{Vol}(M^4_j) \to 0, \\ &b_2(M^4_j) \to \infty, \end{aligned} \tag{2.17}$$

where $b_2(M_j^4)$ denotes the second Betti number of $M_j^4$. In particular, Theorem 1.12, the finiteness theorem in dimension 4, does not extend to the case in which the lower volume bound is dropped.

## 3. **Proof of the Transformation Theorem**

In this section we prove the Transformation Theorem (Theorem 1.32), which is the main technical tool in the proof of the Slicing Theorem (Theorem 1.23). As motivation, let us mention the following. Given $\varepsilon, \eta > 0$ and a $\delta(\varepsilon, \eta)$-splitting map $u : B_2(p) \to \mathbb{R}^k$, one can use a weighted maximal function estimate for $|\nabla^2 u|$ to conclude there exists a set $B$ with small $(n-2+\eta)$-content, such that for each $x \notin B$ and every $0 < r < 1$, the restriction $u : B_r(x) \to \mathbb{R}^k$ is an $\varepsilon$-splitting map. However, as we have observed in Example 2.14, we cannot take $\eta = 0$, since $|\nabla u|$ can vanish on a set of large $(n-2)$-content. For purposes of proving the Slicing Theorem, this set is too large.

Suppose instead that we consider the collection of balls $B_r(x)$ such that for no lower triangular matrix $A \in \mathrm{GL}(n-2)$ with positive diagonal entries is $A \circ u$ an $\varepsilon$-splitting map on $B_r(x)$. Though we cannot show that this set has small $(n-2)$-content, we will prove that its image under $u$ has small $(n-2)$-dimensional measure. This will be what is required for the Slicing Theorem.

For the case of a single function, $k = 1$, the basic idea can be explained as follows. In order to obtain an $\varepsilon$-splitting function on $B_r(x)$, it is not necessary that the Hessian of $u$ is small and the gradient is close to 1. Rather, we need only that the Hessian of $u$ is small relative to the gradient. That is, for $\varepsilon > 0$, $0 < r \le 1$, consider the condition

$$r \fint_{B_{2r}(x)} |\nabla^2 u| \le \delta(\varepsilon) \cdot \fint_{B_{2r}(x)} |\nabla u|. \tag{3.1}$$

Now if $\fint_{B_{2r}(x)} |\nabla u|$ is very small, then the restricted map $u : B_r(x) \to \mathbb{R}$ will not define a splitting map. However, if (3.1) holds, we may simply rescale $u$ so that $\fint_{B_{2r}(x)} |\nabla u| = 1$, in which case arguments as in the proof of Lemma 1.21 tell us that after such a rescaling, $u : B_r(x) \to \mathbb{R}$ becomes an $\varepsilon$-splitting map.

To control the collection of balls that do not satisfy the inequality (3.1), we start with what is essentially (1.27):

$$\fint_{B_{3/2}(p)} \frac{|\nabla^2 u|^2}{|\nabla u|} < \delta^2.$$

By arguing as in Section 1.2, this enables us to control the set of balls $B_{2r}(x)$ that do not satisfy

$$r^2 \fint_{B_{2r}(x)} \frac{|\nabla^2 u|^2}{|\nabla u|} < \delta \fint_{B_{2r}(x)} |\nabla u| \tag{3.2}$$

and, in particular, to show that the image under $u$ of this collection of balls has small $(n-2)$-dimensional measure. On the other hand, (3.2) implies (3.1), since

(3.3)
$$r \fint_{B_{2r}(x)} |\nabla^2 u| \leq \Big(r^2 \fint_{B_{2r}(x)} \frac{|\nabla^2 u|^2}{|\nabla u|}\Big)^{1/2} \cdot \Big(\fint_{B_{2r}(x)} |\nabla u|\Big)^{1/2} \leq \delta^{1/2} \fint_{B_{2r}(x)} |\nabla u|.$$

For the case $k > 1$, serious new issues arise. For one thing, even if on some ball $B_r(x)$ the individual gradients, $\nabla u^1, \dots, \nabla u^{n-2}$, satisfy (3.1) and we then normalize them to have $L^2$ norm 1, it still might be the case that in the $L^2$ sense, this normalized collection looks close to being linearly dependent. Then $u$ would still be far from defining an $\varepsilon$-splitting map. This issue is related to the fact that for $\ell > 1$, the distributional Laplacian $\Delta|\omega^\ell|$ may have a singular part. Additionally, for $k > 1$, we are unable to obtain a precise analog of (1.27), which was the tool for handling the case $k = 1$. Instead, we have to proceed on the basis of (1.28), the bound on the normalized mass of the distributional Laplacian $\Delta|\omega^\ell|$. These points make the proof of the Transformation Theorem in the general case substantially more difficult.

3.1. *Higher order estimates.* We begin by recalling the existence of a good cutoff function. According to [CC96], if $\mathrm{Ric}_{M^n} \geq -\delta$, then for any $B_r(x) \subset M^n$ with $0 < r \leq 1$, there exists a cutoff function, with $0 \leq \varphi \leq 1$, such that

$$\begin{aligned} &\varphi(x) \equiv 1 \text{ if } x \in B_{9r/5}(x), \\ &\operatorname{supp} \varphi \subset B_{2r}(x), \end{aligned} \tag{3.4}$$

and such that

$$\begin{aligned} r|\nabla \varphi| &\leq C(n), \\ r^2|\Delta \varphi| &\leq C(n). \end{aligned} \tag{3.5}$$

In preparation for proving part (2) of Theorem 1.26, we state a general lemma on distributional Laplacians.

Let $w$ be a smooth section of a Riemannian vector bundle with orthogonal connection over $B_2(p)$. Let $\Delta w$ denote the rough Laplacian of $w$. Note that $|w|$ is a Lipschitz function that is smooth off of the set $Z_{|w|} =: \{x \,|\, |w|(x) = 0\}$. We put

$$U_r = \{x \,:\, |w|(x) \leq r\}. \tag{3.6}$$

LEMMA 3.7. *The distributional Laplacian $\Delta|w|$ is a locally finite Borel measure $\mu = \mu_{\mathrm{ac}} + \mu_{\mathrm{sing}}$. The measure $\mu$ is absolutely continuous on $B_2(p) \backslash Z_{|w|}$, with density*

$$\mu_{\mathrm{ac}} = \frac{\langle \Delta w, w\rangle}{|w|} + \frac{|\nabla w|^2 - |\nabla |w||^2}{|w|}. \tag{3.8}$$

*The singular part* $\mu_{\text{sing}}$ *is a nonnegative locally finite Borel measure supported on* $B_2(p)\cap\partial Z_{|w|}$. *There exists* $r_j \to 0$ *such that for any nonnegative continuous function* $\varphi$, *with* $\operatorname{supp}\varphi \subset B_2(p)$, *we have*

$$\mu_{\text{sing}}(\varphi) = \lim_{r_i\to 0}\int_{B_2(p)\cap\partial U_{r_i}} \varphi\cdot|\nabla|w||\geq 0. \tag{3.9}$$

*If, in addition,* $\operatorname{Ric}_{M^n}\geq -(n-1)\kappa$, *then on each ball* $B_{2-s}(p)$, *the distributional Laplacian* $\Delta|w|$ *satisfies the normalized mass bound*

$$\fint_{B_{2-s}(p)}|\Delta|w|\,|\leq C(n,\kappa,s)\cdot\inf_c \fint_{B_{2-s/2}(p)}||w|-c|-2\fint_{B_{2-s/2}(p)}\frac{\langle\Delta w,w\rangle_-}{|w|}, \tag{3.10}$$

*where* $\frac{\langle\Delta w,w\rangle_-}{|w|} =: \min\bigl(0,\frac{\langle\Delta w,w\rangle}{|w|}\bigr)$.

*Proof.* The computation of the absolutely continuous part (3.8) on $B_2(p)\setminus Z_{|w|}$ is standard. Before continuing, let us mention the following technical point. Fix $2>s>0$. Since on $B_{2-s}(p)$, $|w|$ is Lipschtiz and

$$\nabla|w| = \frac{\langle\nabla w,w\rangle}{|w|}\qquad(\text{on } B_{2-s}(p)\setminus Z_{|w|}) \tag{3.11}$$

has uniformly bounded norm $|\nabla|w||\leq|\nabla w|$, we have by the coarea formula that as $r\to 0$,

$$\begin{aligned} o(r) &= \int_{B_{2-s}(p)\cap(U_r\setminus U_{r/2})}|\nabla|w|\,| \\ &= \int_{r/2}^{r}\mathcal{H}^{n-1}(B_{2-s}\cap\partial U_t)\,dt, \end{aligned} \tag{3.12}$$

where $\mathcal{H}^{n-1}$ denotes $(n-1)$-dimensional Hausdorff measure. By combining this with Sard's theorem it follows, in particular, that there exist decreasing sequences $r_i\searrow 0$ such that for any $2>s>0$, we have that $B_{2-s}(p)\cap\partial U_{r_i}$ is smooth and

$$\lim_{r_i\to 0} r_i\cdot\mathcal{H}^{n-1}(B_{2-s}(p)\cap\partial U_{r_i}) = 0. \tag{3.13}$$

Let $\varphi \geq 0$ denote a smooth function with $\operatorname{supp}\varphi \subset B_2(p)$, and let $r_i \searrow 0$ be as in (3.13). Then for any constant $c$, we have

$$\begin{aligned}(3.14)\quad \int_{B_2(p)} \Delta\varphi \cdot \big(|w| - c\big) &= \lim_{r_i\to 0} \int_{B_2(p)\setminus U_{r_i}} \Delta\varphi \cdot |w| \\ &= \lim_{r_i\to 0} \int_{B_2(p)\setminus U_{r_i}} \varphi \cdot \Delta|w| \\ &\quad + \lim_{r_i\to 0} \int_{B_2(p)\cap \partial U_{r_i}} \varphi \cdot N(|w|) - \lim_{r_i\to 0} \int_{B_2(p)\cap \partial U_{r_i}} N(\varphi)\cdot r_i \\ &= \int_{B_2(p)\setminus Z_{|w|}} \varphi \cdot \Delta|w| + \lim_{r_i\to 0} \int_{B_R(p)\cap \partial U_{r_i}} \varphi \cdot N(|w|) \\ &= \int_{B_2(p)\setminus Z_{|w|}} \varphi \cdot \left\langle \Delta w, \frac{w}{|w|} \right\rangle \\ &\quad + \lim_{r_i\to 0} \int_{B_2(p)\setminus U_{r_i}} \varphi \cdot \frac{|\nabla w|^2 - |\nabla|w||^2}{|w|} + \lim_{r_i\to 0} \int_{B_2(p)\cap \partial U_{r_i}} \varphi \cdot |\nabla|w||,\end{aligned}$$

where the third term on the right-hand side of the second line of (3.14) vanishes because of (3.13). Note that since $w$ is smooth and the second and third integrands on the last line are nonnegative, it follows that all three limits on the last line exist.

For fixed $i$, each term on the last line above defines a Borel measure. To see that the weak limits of these measures define Borel measures that satisfy the mass bound in (3.10), we assume $\operatorname{Ric}_{M^n} \geq -(n-1)\kappa$ and choose $\varphi$ in (3.14) to be a cutoff function as in [CC96] with $\varphi \equiv 1$ on $B_{2-s}(p)$, $\operatorname{supp}\varphi \subset B_{2-s/2}(p)$, and $s|\nabla\varphi|$, $s^2|\Delta\varphi| \leq c(n,\kappa,s)$. From the elementary fact that $a-b \geq 0$ implies $|a| \leq a + 2b_-$, where $b_- =: \min(0,b)$, we get the mass bound

$$\begin{aligned}(3.15)\qquad \min_c \int_{B_{2-s/2}(p)} |\Delta\varphi| \cdot |(|w| - c)| - 2\int_{B_{2-s/2}(p)} \varphi \cdot \frac{\langle \Delta w, w\rangle_-}{|w|} \\ \geq \int_{B_{2-s}(p)\setminus Z_{|w|}} |\Delta|w|| + \lim_{r_i\to 0} \int_{B_{2-s}(p)\cap \partial U_{r_i}} |\nabla|w||,\end{aligned}$$

which suffices to complete the proof. □

*Remark* 3.16. Note that for the proof of the mass bound in Lemma 3.7, on which the mass bound in Theorem 1.26 is based, it is crucial that the singular term has the correct sign: $\int_{B_2(p)\cap\partial U_{r_i}} \varphi \cdot |\nabla|\omega|| \geq 0$, where $\varphi \geq 0$.

Now we can finish the proof of Theorem 1.26. First, we recall the following statement.

*For every $\varepsilon > 0$, there exists $\delta(n,\varepsilon) > 0$ such that if $\operatorname{Ric}_{M^n} \geq -\delta$, with $u : B_2(p) \to \mathbb{R}^k$ a $\delta$-splitting map, then the following hold*:

(1) *There exists* $\alpha(n) > 0$ *such that for each* $1 \leq a \leq k$,

$$\fint_{B_{3/2}(p)} \frac{|\nabla^2 u^a|^2}{|\nabla u^a|^{1+\alpha}} < \varepsilon. \tag{3.17}$$

(2) *Let* $\omega^\ell \equiv du^1 \wedge \cdots \wedge du^\ell$, $1 \leq \ell \leq k$. *The Laplacians* $\Delta|\omega^\ell|$, *taken in the distributional sense, are Borel measures with singular part a locally finite nonnegative Borel measure supported on* $\partial Z_{|\omega^\ell|}$. *For* $\ell = 1$, *the singular part vanishes. The normalized mass of* $\Delta|\omega^\ell|$ *satisfies*

$$\fint_{B_{3/2}(p)} \left|\Delta|\omega^\ell|\right| < \varepsilon. \tag{3.18}$$

*Proof of Theorem* 1.26. We begin by proving (1).[3] The main observation is that for all $0 \leq \alpha < 1$, we have that the distributional Laplacian $\Delta|\nabla u|^{1-\alpha}$ satisfies

$$\Delta|\nabla u|^{1-\alpha} = (1-\alpha)\frac{\Big(|\nabla^2 u|^2 - (1+\alpha)|\nabla|\nabla u||^2 + \mathrm{Ric}(\nabla u, \nabla u)\Big)}{|\nabla u|^{1+\alpha}}. \tag{3.19}$$

In particular, unlike for $\omega^\ell$ with $\ell > 1$, there is no possibility of a singular contribution. The proof of this is similar to arguments in [Don92], but for the sake of convenience, we will outline it here. There are two key facts that play a role in the vanishing of the singular part of $\Delta|\nabla u|$:

(a) The critical set $Z_{|\nabla u|}$ has Hausdorff dimension $\leq n-2$.

(b) $u$ vanishes to finite order at each point of $x \in Z_{|\nabla u|}$. That is, $u$ has a leading order Taylor expansion at $x$ of degree $k_x \geq 1$, with $k_x$ uniformly bounded on compact subsets.

The previous two properties are standard. They follow by working in a sufficiently smooth coordinate chart and using the monotonicity of the frequency; see [HL], [CNV15], and [NV14]. Note that the frequency is not monotone until one restricts to a sufficiently regular coordinate chart. Since in our situation there is no *a priori* estimate on the size of such a coordinate chart, although there is finite vanishing order at each point, there is no *a priori* estimate on the size of the vanishing order.

Now let us finish outlining the proof of (3.19). Let $\varphi$ be a smooth function with support contained in $B_2(p)$. Put $S_r(\,\cdot\,) = \partial T_r(\,\cdot\,)$. Now $Z_{|\nabla u|}$ is a closed set that satisfies the Hausdorff dimension estimate of (a). While this

[3]We remind the reader that in our subsequent applications, we encounter only the case $\alpha = 0$. Moreover, in view of Lemma 3.7, our subsequent arguments would go through even without knowing that the singular part of $\Delta|\nabla u|$ is absent. However, for the sake of completeness, we start by considering all $0 \leq \alpha < 1$ and then specialize to the case $0 \leq \alpha < \frac{1}{n-1}$, in which we can give an effective estimate.

is sufficient, to simplify the argument we use [CNV15] and [NV14] to see that the following Minkowski estimate holds:

$$\mathrm{Vol}(S_r(Z_{|\nabla u|} \cap \operatorname{supp}\varphi)) < Cr. \tag{3.20}$$

Then we compute

(3.21)

$$\begin{aligned}
\int_{B_2(p)} \Delta\varphi \cdot |\nabla u|^{1-\alpha} &= \lim_{r\to 0} \int_{B_2(p)\setminus T_r(Z_{|\nabla u|})} \Delta\varphi \cdot |\nabla u|^{1-\alpha} \\
&= -\lim_{r\to 0} \int_{B_2(p)\setminus T_r(Z_{|\nabla u|})} \langle \nabla\varphi, \nabla |\nabla u|^{1-\alpha}\rangle + \lim_{r\to 0} \int_{S_r(Z_{|\nabla u|})} N(\varphi)\cdot |\nabla u|^{1-\alpha}, \\
&= -\lim_{r\to 0} \int_{B_2(p)\setminus T_r(Z_{|\nabla u|})} \langle \nabla\varphi, \nabla |\nabla u|^{1-\alpha}\rangle, \\
&= \lim_{r\to 0} \int_{B_2(p)\setminus T_r(Z_{|\nabla u|})} \varphi \cdot \Delta |\nabla u|^{1-\alpha} \\
&\quad - \frac{(1-\alpha)}{2} \lim_{r\to 0} \int_{B_2(p)\cap S_r(Z_{|\nabla u|})} \varphi \cdot \frac{N(|\nabla u|^2)}{|\nabla u|^{1+\alpha}}, \\
&= (1-\alpha) \int_{B_2(p)\setminus Z_{|\nabla u|}} \varphi \cdot \left( \frac{\left( |\nabla^2 u|^2 - (1+\alpha)|\nabla |\nabla u||^2 + \mathrm{Ric}(\nabla u, \nabla u)\right)}{|\nabla u|^{1+\alpha}} \right),
\end{aligned}$$

where in dropping the last boundary term, we have used

(3.22)

$$\begin{aligned}
\lim_{r\to 0} \int_{B_2(p)\cap S_r(Z_{|\nabla u|})} \varphi \cdot \frac{N(|\nabla u|^2)}{|\nabla u|^{1+\alpha}} &= 2 \lim_{r\to 0} \int_{B_2(p)\cap S_r(Z_{|\nabla u|})} \varphi \cdot \frac{|\nabla |\nabla u||}{|\nabla u|^{\alpha}} \\
&\le C \lim_{r\to 0} r^1 r_i^{-1+(1-\alpha)} = C \lim_{r\to 0} r^{1-\alpha} \to 0.
\end{aligned}$$

For a related argument, see [Don92].

To finish the proof, observe that since $\mathrm{trace}(\nabla^2 u) = \Delta u = 0$, it follows if $\lambda_1, \dots, \lambda_n$ are the eigenvalues of $\nabla^2 u$, then $\sum \lambda_i = 0$. In particular, if $\lambda_n$ is the largest eigenvalue, then by the Schwarz inequality,

$$\lambda_1^2 + \cdots + \lambda_n^2 \ge \frac{1}{n-1}(\lambda_1 + \cdots + \lambda_{n-1})^2 + \lambda_n^2 \ge \frac{n}{n-1}\lambda_n^2. \tag{3.23}$$

This leads to the improved Kato inequality

$$|\nabla^2 u(\mathrm{v})|^2 \le \left(1 - \frac{1}{n}\right) |\nabla^2 u|^2, \tag{3.24}$$

where v is any vector with $|\mathrm{v}| = 1$.

Thus, if rewrite

$$|\nabla|\nabla u|| = \left|\nabla^2 u\left(\frac{\nabla u}{|\nabla u|}\right)\right| \tag{3.25}$$

and apply the improved Kato inequality and $\mathrm{Ric}_{M^n} \geq -\delta$, we get

$$\Delta|\nabla u|^{1-\alpha} \geq \frac{1-(n-1)\alpha}{n}\frac{|\nabla^2 u|^2}{|\nabla u|^{1+\alpha}} - (1-\alpha)\delta|\nabla u|^{1-\alpha}, \tag{3.26}$$

which gives nontrivial information for any $\alpha < \frac{1}{n-1}$, which we now assume. Namely, we get the distributional inequality

$$\frac{|\nabla^2 u|^2}{|\nabla u|^{1+\alpha}} \leq C(n,\alpha)\Big(\Delta|\nabla u|^{1-\alpha} + \delta|\nabla u|^{1-\alpha}\Big). \tag{3.27}$$

Finally let $\varphi \geq 0$ be a smooth function as in (3.4), (3.5), with $\operatorname{supp}\varphi \subset B_2(p)$, $|\nabla\varphi|, |\Delta\varphi| \leq C(n)$.

By multiplying both sides of (3.27) by $\varphi$ and integrating we obtain

$$\begin{aligned}\int_{B_2(p)} \varphi\frac{|\nabla^2 u|^2}{|\nabla u|^{1+\alpha}} &\leq C(n,\alpha)\int_{B_2(p)}\Big(\varphi\Delta|\nabla u|^{1-\alpha} + \varphi\delta|\nabla u|^{1-\alpha}\Big),\\ &\leq C(n,\alpha)\int_{B_2(p)}\Delta\varphi\Big(|\nabla u|^{1-\alpha} - \fint_{B_2(p)}|\nabla u|^{1-\alpha}\Big)\\ &\quad + C(n,\alpha)\delta\int_{B_2(p)}|\nabla u|^{1-\alpha},\\ &\leq C(n,\alpha)\int_{B_2(p)}\left||\nabla u|^{1-\alpha} - \fint_{B_2(p)}|\nabla u|^{1-\alpha}\right|\\ &\quad + C(n,\alpha)\delta\int_{B_2(p)}|\nabla u|^{1-\alpha}.\end{aligned} \tag{3.28}$$

Now we use that if $u$ is a harmonic $\delta$-splitting map, then $|\nabla u|^{1-\alpha}$ is bounded and $\fint_{B_2(p)}\left||\nabla u|^{1-\alpha} - \fint_{B_2(p)}|\nabla u|^{1-\alpha}\right|$ is small. In particular, for $\delta$ sufficiently small, we have

$$\begin{aligned}\fint_{B_{3/2}(p)}\frac{|\nabla^2 u|^2}{|\nabla u|^{1+\alpha}} &\leq C(n)\fint_{B_2(p)}\varphi\frac{|\nabla^2 u|^2}{|\nabla u|^{1+\alpha}}\\ &\leq C(n,\alpha)\fint_{B_2(p)}\left||\nabla u|^{1-\alpha} - \fint_{B_2(p)}|\nabla u|^{1-\alpha}\right|\\ &\quad + C(n,\alpha)\delta\fint_{B_2(p)}|\nabla u|^{1-\alpha} \leq \varepsilon,\end{aligned} \tag{3.29}$$

which proves (3.17).

*Remark* 3.30. For $0 \leq \alpha < \frac{1}{n-1}$, there is another way of seeing that (3.19) holds in the distributional sense, which uses only the fact that $Z_{|\nabla u|} \cap \operatorname{supp}\varphi$ has

Hausdorff dimension $\leq n - \frac{n}{n-1}$ and the improved Kato inequality. From the Hausdorff dimension bound, it follows that there is a nondecreasing sequence of cutoff functions $\psi_i$ converging pointwise to 1 on $B_2(p)\setminus(Z_{|\nabla u|}\cap \operatorname{supp}\varphi)$, each of which vanishes in a neighborhood of $Z_{|\nabla u|}$, and such that $|\nabla\psi_i|_{L^q}\to 0$ for all $q<\frac{n}{n-1}$. For the case $\alpha=0$, the claim follows by applying the divergence theorem to the vector fields $\psi_i\nabla|\nabla u|$, noting that $|\nabla|\nabla u||\in L^\infty\subset L^{q'}$ and using Hölder's inequality. For $0<\alpha<\frac{1}{n-1}$, one uses an iterative version of the above argument. For additional details on this instance of the divergence theorem see, e.g., Section 2 of [Che03].

Next we prove (2). The vanishing of the singular part for $\ell=1$ is contained in part (1). By invoking Lemma 3.7, all that remains is to bound from below the term $\langle\Delta\omega^\ell,\frac{\omega^\ell}{|\omega^\ell|}\rangle_-$ in (3.18). On $B_2(p)\setminus Z_{|\omega^\ell|}$, by Bochner's formula, we have

$$\begin{aligned}\Delta\omega^\ell &= \sum_a du^1\wedge\cdots\operatorname{Ric}(du^a)\wedge\cdots\wedge du^\ell\\ &\quad + 2\sum_{a\neq b,j} du^1\wedge\nabla_j(du^a)\wedge\cdots\wedge\nabla_j(du^b)\wedge\cdots\wedge du^\ell,\end{aligned}\tag{3.31}$$

from which it follows that

$$\langle\Delta\omega^\ell,\frac{\omega^\ell}{|\omega^\ell|}\rangle_-\geq -C(n)\left(\delta+\sum_a|\nabla u^j|^2\right)|\omega^\ell|.\tag{3.32}$$

Since $u$ is a $\delta$-splitting map, by using (3.10) of Lemma 3.7, this suffices to complete the proof. □

*Remark* 3.33. A simple example of a harmonic map $u:\mathrm{R}^n\to\mathrm{R}^k$, for which the distributional Laplacian $\Delta|\omega|$ has a singular part with positive mass, is furnished by the 2-form $dx\wedge d(x^2-y^2)=-2y\cdot dx\wedge dy$ (which can be thought of as depending on $(n-2)$ additional variables). We do not know whether an $\varepsilon$-splitting map with small $\varepsilon$ can furnish such an example, though this seems within reason.

3.2. *Proof of the Transformation Theorem.* In this subsection we prove the Transformation Theorem (Theorem 1.32), which constitutes the technical heart of the Slicing Theorem (Theorem 1.23). We will assume for notational simplicity that $M^n$ is complete, but it is an easy exercise to show that this may be weakened to the local assumption that $B_4(p)$ has compact closure in $M^n$. First we recall the definition of the singular scale:

*Let $u:B_2(p)\to\mathbb{R}^k$ be a harmonic function. For $\delta>0$, let us define for $x\in B_1(p)$ the singular scale $s_x^\delta\geq 0$ as the infimum of all radii $s$ such that for*

*all* $s < r < \frac{1}{2}$ *and all* $1 \leq \ell \leq k$, *we have the estimate*

$$r^2 \fint_{B_r(x)} |\Delta|\omega^\ell|| \leq \delta \fint_{B_r(x)} |\omega^\ell|,$$

*where* $\omega^\ell = du^1 \wedge \cdots \wedge du^\ell$.

Next recall that Theorem 1.32 states:

*For every* $\varepsilon > 0$, *there exists* $\delta = \delta(n, \varepsilon) > 0$ *such that if* $\mathrm{Ric}_{M^n} \geq -\delta$ *and* $u : B_2(p) \to \mathbb{R}^k$ *is a harmonic* $\delta$*-splitting map, then for each* $x \in B_1(p)$ *and* $r \geq s^\delta_x$, *there exists a lower triangular matrix* $A = A(x, r)$ *with positive diagonal entries such that* $A \circ u : B_r(x) \to \mathbb{R}^k$ *is a harmonic* $\varepsilon$*-splitting map.*

*Proof of Theorem* 1.32. The strategy will be a proof by induction. Thus, we will begin with the simplest case of $k = 1$. The following is a slightly more general form of the statement we wish to prove

LEMMA 3.34. *Let* $u : B_{2r}(x) \to \mathbb{R}$ *be a harmonic function with* $r \leq 1$. *Then for every* $\varepsilon > 0$, *there exists* $\delta(n, \varepsilon) > 0$ *such that if* $\mathrm{Ric}_{M^n} \geq -(n-1)\delta$ *and*

$$r^2 \fint_{B_{2r}(x)} |\Delta|\nabla u|| \leq \delta \fint_{B_{2r}(x)} |\nabla u|, \tag{3.35}$$

*then for* $A = \left( \fint_{B_r(x)} |\nabla u| \right)^{-1} > 0$, *we have that* $A \circ u : B_r(x) \to \mathbb{R}$ *is an* $\varepsilon$*-splitting map.*

As in the proof of Theorem 1.26, the fact that $u$ is harmonic leads to the improved Kato inequality, $|\nabla|\nabla u^a|\,|^2 \leq \frac{n-1}{n}|\nabla^2 u^a|^2$, from which we can compute

$$\Delta|\nabla u| \geq \frac{1}{n}\frac{|\nabla^2 u|^2}{|\nabla u|} - (n-1)\delta|\nabla u|. \tag{3.36}$$

In particular, the estimate (3.35) gives rise to the estimate

$$r^2 \fint_{B_{2r}(x)} \frac{|\nabla^2 u|^2}{|\nabla u|} \leq C(n)\delta \fint_{B_{2r}(x)} |\nabla u| \tag{3.37}$$

from which, as previously noted (see (3.2), (3.3)), we get

$$\begin{aligned} r \fint_{B_{2r}(x)} |\nabla^2 u| &\leq \Big( r^2 \fint_{B_{2r}(x)} \frac{|\nabla^2 u|^2}{|\nabla u|} \Big)^{1/2} \cdot \Big( \fint_{B_{2r}(x)} |\nabla u| \Big)^{1/2} \\ &\leq C\delta^{1/2} \fint_{B_{2r}(x)} |\nabla u|. \end{aligned} \tag{3.38}$$

Let us put $v = \left( \fint_{B_{2r}(x)} |\nabla u| \right)^{-1} u$, so that $\fint_{B_{2r}(x)} |\nabla v| = 1$. The lower Ricci bound implies that a Poincaré inequality holds. When combined with

the last inequality this gives

$$\fint_{B_{2r}(x)} \Big||\nabla v| - 1\Big| \leq C(n)\delta^{1/2}. \tag{3.39}$$

By using the doubling property, after possible increasing $C(n)$, we have that for every $y \in B_{3r/2}(x)$,

$$\fint_{B_{r/2}(y)} \Big||\nabla v| - 1\Big| \leq C\delta^{1/2}. \tag{3.40}$$

In particular,

$$1 - C\delta^{1/2} \leq \frac{\fint_{B_{2r}(x)} |\nabla v|}{\fint_{B_r(x)} |\nabla v|} \leq 1 + C\delta^{1/2}. \tag{3.41}$$

Let us observe that we may also use the Ricci lower bound, the Poincaré inequality, and the Harnack inequality to conclude the weak gradient estimate $\sup_{B_{\frac{7}{4}r}(x)} |\nabla v| \leq C(n)$. Now, if we can show that for $\delta$ sufficiently small, the map $v : B_r(x) \to \mathbb{R}$ is an $\varepsilon/2$-splitting, then for $k = 1$, the proof will be complete

Now as in [CC96], let $\varphi \geq 0$ be a cutoff function satisfying $\varphi(y) = 1$ if $y \in B_{5r/3}(x)$ with $\varphi(y) \equiv 0$ if $y \notin B_{2r}(x)$, and such that $r|\nabla\varphi|, r^2|\Delta\varphi| \leq C(n)$. Let $\rho_t(y, dz)$ be the heat kernel on $M^n$. Consider for $y \in B_{3r/2}(x)$ the one parameter family

$$\int \big(|\nabla v| - 1\big)\varphi\, \rho_t(y, dz). \tag{3.42}$$

Note that for all $y \in A(0, r)$, $z \in A(3r/2, 2r)$ and $t \in [0, r^2]$, we have $|\rho_t(y, dz)|$ $< C(n)\mathrm{Vol}(B_{\sqrt{t}}(x))^{-1}$; see (3.68). It follows that for $t \in [0, r^2]$, we have

$$\begin{aligned}
\frac{d}{dt} &\int_{B_{2r}(x)} \big(|\nabla v| - 1\big)\varphi\, \rho_t(y, dz) \\
&\geq \int_{B_{2r}(x)} \bigg( \bigg( \frac{|\nabla^2 v|^2 - |\nabla|\nabla v||^2}{|\nabla v|} - (n-1)\delta^2 |\nabla v| \bigg) \varphi \\
&\qquad\qquad + 2\langle \nabla|\nabla v|, \nabla\varphi\rangle + (|\nabla v| - 1)\Delta\varphi \bigg) \rho_t(y, dz), \\
&\geq -C(n)\delta^2 - C(n) \int_{A(3r/2,2r)} \Big( r^{-1}|\nabla^2 v| + r^{-2}\Big||\nabla v| - 1\Big| \Big) \rho_t(y, dz), \\
&\geq -C\delta^{1/2} r^{-2}.
\end{aligned} \tag{3.43}$$

Integrating this yields

$$(|\nabla v|(y)-1) \le C\delta^{1/2} + \int \big(|\nabla v|-1\big)\varphi\, \rho_{r^2}(y,dz) \tag{3.44}$$
$$\le C\delta^{1/2} + C \fint_{B_{2r}(x)} \Big||\nabla v|-1\Big| \le C\delta^{1/2}.$$

In particular, we have

$$\sup_{B_{3r/2}(x)} |\nabla v| \le 1 + C\delta^{1/2}. \tag{3.45}$$

Combining this with the integral estimate (3.40) we get

$$\fint_{B_{3r/2}(x)} \Big||\nabla v|^2-1\Big| \le C\delta^{1/2}. \tag{3.46}$$

Now using the Bochner formula

$$\Delta|\nabla v|^2 = 2|\nabla^2 v|^2 + 2\mathrm{Ric}(\nabla v,\nabla v) \ge 2|\nabla^2 v|^2 - C\delta^2|\nabla v|^2, \tag{3.47}$$

we can estimate

$$\fint_{B_r(x)} |\nabla^2 v|^2 \le C(n) \fint_{B_{3r/2}(x)} \varphi|\nabla^2 v|^2 \tag{3.48}$$
$$\le C \fint_{B_{3r/2}(x)} \varphi\Big(\Delta(|\nabla v|^2-1) + \delta|\nabla v|^2\Big)$$
$$\le C \fint_{B_{3r/2}(x)} |\Delta\varphi|\Big||\nabla v|^2-1\Big| + C\delta \fint_{B_{3r/2}(x)} |\nabla v|^2,$$
$$\le Cr^{-2}\delta^{1/2}.$$

Hence, for $\delta(n,\varepsilon)$ sufficiently small, we have that $v$ is an $\varepsilon/2$-splitting, which as previously remarked, proves the theorem for the case $k=1$.

We now turn to the proof of Theorem 1.32, which will proceed by induction. Assume the theorem has been proved for some $k-1 \ge 1$. We will prove the result for $k$ by arguing by contradiction.

Thus, we can suppose that for some $\varepsilon > 0$, the result is false. There is no harm in assuming $0 < \varepsilon \le \varepsilon(n)$ is sufficiently small, which we will do from time to time. Then, for some $\delta_j \to 0$, we can find a sequence of spaces $(M_j^n, g_j, p_j)$ with $\mathrm{Ric}_{M_j^n} \ge -\delta_j$, and mappings $u_j : B_2(p_j) \to \mathbb{R}^k$, which are $\delta_j$-splitting mappings, for which there exists $x_j \in B_1(p_j)$ and radii $r_j \ge s^{\delta_j}(x_j)$, such that there is no lower triangular matrix $A$ with positive diagonal entries, such that $A \circ u : B_{r_j}(x_j) \to \mathbb{R}^k$ is an $\varepsilon$-splitting map. Without loss of generality, we can assume $r_j$ is the supremum of those radii for which there is no such matrix. In particular, there exists such a matrix $A_j$ corresponding to the radius $2r_j$. Observe that $r_j \to 0$. Indeed, we can see this just by using the identity map $A = I$, since $\delta_j \to 0$ and $u : B_2(p) \to \mathbb{R}^2$ is a $\delta_j$-splitting map.

Now, set $v_j = A_j \circ \big(u_j - u_j(x_j)\big)$ and consider the rescaled spaces $(M_j^n, g_j', x_j)$ with $g_j' \equiv r_j^{-2} g$. Thus, $v_j : B_{2r_j^{-1}}(x_j) \to \mathbb{R}^k$ is a harmonic function on this space. We have normalized so that $v(x_j) = 0$. As before, for all $2 \le r \le 2r_j^{-1}$, there is a lower triangular matrix $A_r$ with positive entries on the diagonal, such that $A_r \circ v_j : B_r(x_j) \to R^k$ is an $\varepsilon$-splitting map and with our current normalization, $A_2 = I$, the identity map.

*Note.* Throughout the remainder of the argument, when there is no danger of confusion, for ease of notation, we will sometimes omit the subscript $j$ from various quantities including $v$ and $A$, which in actuality depend on $j$. For example, we omit the subcript $j$ from the matrices $A_r$, $A_{2r}$ in Claim 1 below.

We will now break the proof into a series of claims.

*Claim* 1. For each $2 \le r \le 2r_j^{-1}$, we have

$$(1 - C(n)\varepsilon) A_{2r} \le A_r \le (1 + C(n)\varepsilon) A_{2r}. \tag{3.49}$$

Since $A_{2r} \circ v : B_{2r}(x_j) \to \mathbb{R}^k$ is an $\varepsilon$-splitting map, we have

$$\fint_{B_{2r}(x_j)} \Big| \langle \nabla (A_{2r} \circ v)^a, \nabla (A_{2r} \circ v)^b \rangle - \delta_{ab} \Big| < \varepsilon, \tag{3.50}$$

and thus, by doubling of the volume measure, we have

$$\fint_{B_r(x_j)} \Big| \langle \nabla (A_{2r} \circ v)^a, \nabla (A_{2r} \circ v)^b \rangle - \delta^{ab} \Big| < C(n)\varepsilon. \tag{3.51}$$

However, in addition, we also have

$$\fint_{B_r(x_j)} \Big| \langle \nabla (A_r \circ v)^a, \nabla (A_r \circ v)^b \rangle - \delta^{ab} \Big| < \varepsilon. \tag{3.52}$$

By using the Gram-Schmidt process, it follows that there exist lower triangular matrices, $T_1$ and $T_2$ with $|T_1 - I| < C(n)\varepsilon$, $|T_2 - I| < C(n)\varepsilon$, such that

$$\fint_{B_r(x_j)} \langle (T_2 A_{2r} \circ \nabla v)^a, (T_2 A_{2r} \circ \nabla v)^b \rangle = \delta^{ab},$$

$$\fint_{B_r(x_j)} \langle (T_1 A_r \circ \nabla v)^a, (T_1 A_r \circ \nabla v)^b \rangle = \delta^{ab}.$$

We can assume that $\varepsilon$ has been chosen small enough that $T_1$ and $T_2$ have positive diagonal entries, which implies that the lower triangular matrices $T_1 A_r$ and $T_2 A_{2r}$ do as well. Define $H$ by

$$(H)_{s,t} = \fint_{B_r(x_j)} \langle \nabla v_s, \nabla v_t \rangle.$$

It follows from the above that we have two so-called Cholesky decompositions of the positive definite symmetric matrix $H$ [GVL96]. Namely,

$$((T_1A_r)^{-1}))^*(T_1A_r)^{-1} = ((T_2A_{2r})^{-1}))^*(T_2A_{2r})^{-1} = H.$$

Since for lower triangular matrices with positive diagonal entries and $H$ positive definite, the Cholesky decomposition is unique, it follows that $(T_1A_r)^{-1} = (T_2A_{2r})^{-1}$. Therefore, $T_1A_r = T_2A_{2r}$, which suffices to prove the claim. □

Now let us record some very important consequences of Claim 1. First, since by our normalization $A_2 \equiv I$, we have for $r \geq 2$ the sublinear growth estimate

$$|A_r|,\ |A_r^{-1}| \leq r^{C(n)\varepsilon}. \tag{3.53}$$

In particular, since $A_r \circ v : B_r(x_j) \to \mathbb{R}^k$ is an $\varepsilon$-splitting, and hence

$$\sup_{B_r(x_j)} |\nabla(A_r \circ v)| \leq 1 + \varepsilon,$$

we have for any $2 \leq r \leq r_j^{-1}$ the *sublinear growth* conditions

$$\begin{aligned} \sup_{B_r(x_j)} |\nabla v_j^a| &\leq (1 + C\varepsilon)r^{C\varepsilon}, \\ \sup_{B_r(x_j)} |\omega_j| &\leq (1 + C\varepsilon)r^{C\varepsilon}, \\ r^2 \fint_{B_r(x_j)} |\nabla^2 v_j^a|^2 &\leq C\varepsilon r^{C\varepsilon}, \end{aligned} \tag{3.54}$$

where $\omega_j \equiv dv_j^1 \wedge \cdots \wedge dv_j^k$ is the pullback $k$-form.

*Remark* 3.55. The sublinearity of the growth estimates in (3.54) will play a fundamental role in the proof; see, in particular, Claims 3–5.

Our first application of these estimates is the following, which uses the induction statement to conclude that $v_j^1, \ldots, v_j^{k-1}$ are improving in their splitting behavior as $j \to \infty$.

*Claim* 2. There exists a lower triangular matrix $A$ such that $A \circ v : B_2(x_j) \to \mathbb{R}^k$ is a $C(n)\varepsilon$-splitting while for each $R > 0$ the restricted map $A \circ v : B_R(x_j) \to \mathbb{R}^{k-1}$, obtained by dropping the last function, is an $\varepsilon_j(R)$-splitting map, where $\varepsilon_j(R) \to 0$ if $j \to \infty$ and $R$ is fixed.

To prove the claim let us first denote by $\tilde{v} : B_{2r_j^{-1}}(x_j) \to \mathbb{R}^{k-1}$ the map obtained by dropping the last function $v^k$. By our induction hypothesis, there exists for every $r \geq 2$ a lower triangular matrix $\tilde{A}_r \in \mathrm{GL}(k-1)$ with positive diagonal entries, such that $\tilde{A}_r \circ \tilde{v} : B_r(x_j) \to \mathbb{R}^{k-1}$ is an $\varepsilon_j$-splitting map with $\varepsilon_j \to 0$. Since both $\tilde{v}$ and $\tilde{A}_2 \circ \tilde{v}$ are, in particular, $\varepsilon$-splittings on $B_2(x_j)$

with $\tilde{A}_2$ lower triangular, then arguments similar to those in Claim 1 give $|\tilde{A}_2 - I| < C(n)\varepsilon$ and the growth estimates

$$\sup_{B_r(x_j)} |\nabla(\tilde{A}_2 \circ \tilde{v})| \leq (1 + C\varepsilon_j) r^{C\varepsilon_j}, \tag{3.56}$$

$$r^2 \fint_{B_r(x_j)} |\nabla^2(\tilde{A}_2 \circ \tilde{v})|^2 \leq C\varepsilon_j r^{C\varepsilon_j}.$$

In particular, we can use the Hessian estimate and a Poincaré inequality to conclude

$$\begin{aligned} &\Big| \fint_{B_2(x)} \Big| \langle \nabla(\tilde{A}_2 \circ \tilde{v})^a, \nabla(\tilde{A}_2 \circ \tilde{v})^b \rangle - \delta^{ab} \Big| \\ &\qquad - \fint_{B_R(x)} \Big| \langle \nabla(\tilde{A}_2 \circ \tilde{v})^a, \nabla(\tilde{A}_2 \circ \tilde{v})^b \rangle - \delta^{ab} \Big| \Big| \\ &\leq \fint_{B_2(x)} \Big| \Big| \langle \nabla(\tilde{A}_2 \circ \tilde{v})^a, \nabla(\tilde{A}_2 \circ \tilde{v})^b \rangle - \delta^{ab} \Big| \\ &\qquad - \fint_{B_R(x)} \Big| \langle \nabla(\tilde{A}_2 \circ \tilde{v})^a, \nabla(\tilde{A}_2 \circ \tilde{v})^b \rangle - \delta^{ab} \Big| \Big| \\ &\leq C(n,R) \fint_{B_R(x)} \Big| \Big| \langle \nabla(\tilde{A}_2 \circ \tilde{v})^a, \nabla(\tilde{A}_2 \circ \tilde{v})^b \rangle - \delta^{ab} \Big| \\ &\qquad - \fint_{B_R(x)} \Big| \langle \nabla(\tilde{A}_2 \circ \tilde{v})^a, \nabla(\tilde{A}_2 \circ \tilde{v})^b \rangle - \delta^{ab} \Big| \Big| \\ &\leq C(n,R) \fint_{B_R(x)} \Big| \nabla \langle \nabla(\tilde{A}_2 \circ \tilde{v})^a, \nabla(\tilde{A}_2 \circ \tilde{v})^b \rangle \Big| \leq \varepsilon_j(R) \to 0. \end{aligned} \tag{3.57}$$

Thus, for each $R > 0$ fixed, we have that $\tilde{A}_2 \circ \tilde{v} : B_R(x_j) \to \mathbb{R}^{k-1}$ is an $\varepsilon_j(R)$-splitting, where $\varepsilon_j(R) \to 0$ when $j \to \infty$ with $R$. Finally, if we let $A = \tilde{A}_2 \oplus 1$ act on $\mathbb{R}^k$ by fixing the last component, then we have proved the claim. □

*Note.* As a point of notation, we mention that as above, from now on, the symbol $\varepsilon_j(R)$ will always denote a quantity, regardless of origin, satisfying $\varepsilon_j(R) \to 0$, when $j \to \infty$ with $R$ fixed.

*Note.* In the course of the proof, on more than one occasion, we will replace $v$ by $A \circ v$, where $A$ is a lower triangular matrix with positive diagonal entries and with $|A - I| < C(n)\varepsilon$. In particular, from this point on in the proof, we will assume $v^a$ has been normalized as in Claim 2. Thus $v^a : B_2(x_j) \to \mathbb{R}^k$ will be taken to be a $C\varepsilon$-splitting map, while $v^a : B_R(x_j) \to \mathbb{R}^{k-1}$ is an $\varepsilon_j(R)$-splitting map.

A useful consequence is that for each $R > 0$ and $1 \leq \ell \leq k-1$, we have

$$\fint_{B_R(x_j)} |\nabla^2 v^a|^2 \leq \varepsilon_j(R) \to 0. \tag{3.58}$$

*Remark* 3.59. By way of orientation, we mention at this point that our long term goal is to show

$$\fint_{B_R(x_j)} |\nabla^2 v_j^k|^2 \leq \varepsilon_j(R) \to 0,$$

which is the content of Claim 6. Once this has been achieved, the proof will be virtually complete.

Our next goal is to study in more detail the properties of $\omega_j = \omega_j^k = dv_j^1 \wedge \cdots \wedge dv_j^k$. First, since

$$\nabla \omega_j = \nabla(dv_j^1) \wedge \cdots \wedge dv_j^k + \cdots + dv_j^1 \wedge \cdots \wedge \nabla(dv_j^k), \tag{3.60}$$

we can use (3.54) to obtain

$$r^2 \fint_{B_r(x_j)} |\nabla \omega_j|^2 \leq C\varepsilon\, r^{C\varepsilon} \tag{3.61}$$

for $2 \leq r \leq r_j^{-1}$.

Recall that our underlying assumptions are that for every $r \geq 1$, we have

$$r^2 \fint_{B_r(x_j)} |\Delta |\omega_j|\,| \leq \delta_j \fint_{B_r(x_j)} |\omega_j|. \tag{3.62}$$

By combining this with (3.54), we get that for every $2 \leq r \leq r_j^{-1}$,

$$r^2 \fint_{B_r(x_j)} |\Delta |\omega_j|\,| \leq C\delta_j r^{C\varepsilon}. \tag{3.63}$$

Now we are ready to make our third claim.

*Claim* 3. For each fixed $R \geq 1$, we have $\fint_{B_R(x_j)} \left| |\omega_j|^2 - \fint_{B_R(x_j)} |\omega_j|^2 \right| \to 0$.

The proof of Claim 3 will rely on the sublinear growth estimates (3.54), (3.61), (3.63), standard heat kernel estimates for almost nonnegative Ricci curvature, (3.68)–(3.70), and the Bakry-Emery gradient estimate for the heat kernel (3.77). In particular, the sublinear growth condition in (3.61) enters crucially in (3.76) and its consequence (3.79).

Fix $R \geq 1$, and consider the maximal function

$$M^R(x) \equiv \sup_{r \leq R} \fint_{B_r(x)} |\Delta |\omega_j|| \tag{3.64}$$

for $x \in B_R(x_j)$. Since by the Bishop-Gromov inequality, the Riemannian measure is doubling, we can combine the usual maximal function arguments with (3.63) and conclude that there exists a subset $U_j \subseteq B_R(x_j)$ such that

$$\begin{aligned} &\frac{\mathrm{Vol}(B_R(x_j) \setminus U_j)}{\mathrm{Vol}(B_R(x_j))} \leq \varepsilon_j(R) \to 0, \\ &M^R(x) \leq \varepsilon_j(R) \to 0 \end{aligned} \tag{3.65}$$

for all $x \in U_j$. Relation (3.65) will be used in (3.72).

As with the symbol $\varepsilon_j(R)$, the symbol $\varepsilon_j(S)$ will always denote a quantity, regardless of origin, satisfying $\varepsilon_j(S) \to 0$ when $j \to \infty$ with $S$ fixed.

Now let $\varphi \geq 0$ be a smooth cutoff function as in [CC96], such that $\varphi \equiv 1$ on $B_{r_j^{-1}/2}(p)$, $\mathrm{supp}\,\varphi \subset B_{r_j^{-1}}(p)$, and such that $r_j|\nabla\varphi|, r_j^2|\Delta\varphi| \leq C(n)$. For $x \in B_R(x_j)$, let us consider the function

$$\int_{M_j^n} |\omega_j|\varphi\rho_t(x, dy), \tag{3.66}$$

where $\rho_t$ is the heat kernel centered at $x$. Then we have the equality

$$\frac{d}{dt}\int_{M_j^n} |\omega_j|\varphi\rho_t(x, dy) = \int \Big(\Delta|\omega_j|\varphi + 2\langle\nabla|\omega_j|, \nabla\varphi\rangle + |\omega_j|\Delta\varphi\Big)\rho_t(x, dy). \tag{3.67}$$

As a consequence of our assumption that $\mathrm{Ric}_{M_j^n} \geq -\delta_j r_j^2$, we have the usual heat kernel estimates ([SY94])

$$\rho_t(x, y) \leq C(n)\mathrm{Vol}(B_{\sqrt{t}}(x))^{-1/2}\mathrm{Vol}(B_{\sqrt{t}}(y))^{-1/2}e^{-\frac{d^2(x,y)}{4t} + C(n)\delta_j r_j^2 t}, \tag{3.68}$$

which implies that for $y \in B_{r_j^{-1}}(x)$ and $t \leq r_j^{-2}$, we have

$$\rho_t(x, y) \leq C(n)\mathrm{Vol}(B_{\sqrt{t}}(x))^{-1/2}\mathrm{Vol}(B_{\sqrt{t}}(y))^{-1/2}e^{-\frac{d^2(x,y)}{4t}}. \tag{3.69}$$

We can use the volume doubling and monotonicity properties to observe the following useful inequality. If $y \in B_r(x)$, then

$$\begin{aligned} \rho_t(x, y) &\leq C(n)\Big(\frac{\mathrm{Vol}(B_r(x))}{\mathrm{Vol}(B_{\sqrt{t}}(x))^{1/2}\mathrm{Vol}(B_{\sqrt{t}}(y))^{1/2}}\Big)\mathrm{Vol}(B_r(x))^{-1}e^{-\frac{d^2(x,y)}{4t}} \\ &\leq C(n)\Big(\frac{r}{t^{1/2}}\Big)^n\mathrm{Vol}(B_r(x))^{-1}e^{-\frac{d^2(x,y)}{4t}}. \end{aligned} \tag{3.70}$$

Let us fix $S >> R \geq 2$ and consider times $0 < t \leq S^2$. By combining the heat kernel estimate, (3.70), with the growth estimates for all $x \in B_R(x_j)$ and

$0 < t \leq S^2$, we can bound the second two terms of (3.67) by

$$
\begin{aligned}
(3.71)\quad & \int_{M_j^n} \Big|\langle \nabla|\omega_j|, \nabla\varphi\rangle + |\omega_j|\,|\Delta\varphi|\Big|\,\rho_t(x,dy) \\
&= \int_{A_{r_j^{-1}/2, r_j^{-1}}(x_j)} \Big|\langle \nabla|\omega_j|, \nabla\varphi\rangle + |\omega_j|\,|\Delta\varphi|\Big|\rho_t(x,dy) \\
&\leq C r_j r_j^{1-C\varepsilon} \mathrm{Vol}(B_{r_j^{-1}}(x_j))\mathrm{Vol}(B_{\sqrt{t}}(x))^{-1/2}\mathrm{Vol}(B_{\sqrt{t}}(y))^{-1/2}\, e^{-\frac{1}{4t}r_j^{-2}} \\
&\leq C r_j^{2-C\varepsilon}\Big(\frac{1}{r_j t^{1/2}}\Big)^n e^{-\frac{1}{4t}r_j^{-2}} \leq \varepsilon_j(S) \to 0.
\end{aligned}
$$

To estimate the first term of (3.67) is more involved. To this end, we begin with an estimate in which we must restrict attention to points $x \in U_j \subseteq B_R(x_j)$; see (3.65). Below, we write $t = r^2$, and so we consider $0 < r < S$. We also put $r_\alpha = 2^\alpha r$. Suppose first that $\sqrt{t} = r \leq R$. Then we have

$$
\begin{aligned}
(3.72)\quad & \int_{M_j^n} \Big|\Delta|\omega_j|\Big|\varphi\rho_{r^2}(x,dy) \\
&= \int_{B_r(x)} \Big|\Delta|\omega_j|\Big|\varphi\rho_{r^2}(x,dy) + \sum_\alpha \int_{A_{r_\alpha, r_{\alpha+1}}(x)} \Big|\Delta|\omega_j|\Big|\varphi\rho_{r^2}(x,dy) \\
&\leq C(n) \fint_{B_r(x)} \Big|\Delta|\omega_j|\,\Big| + C(n)\sum_\alpha \Big(\frac{r_\alpha}{r}\Big)^n e^{-\left(r^{-1}r_\alpha\right)^2} \fint_{B_{2^\alpha r}(x)} \Big|\Delta|\omega_j|\,\Big| \\
&\leq C(n) \fint_{B_r(x)} \Big|\Delta|\omega_j|\,\Big| + C(n)\sum_\alpha 2^{n\alpha} e^{-2^{2\alpha}} \fint_{B_{2^\alpha r}(x)} \Big|\Delta|\omega_j|\,\Big| \\
&= C(n) \fint_{B_r(x)} \Big|\Delta|\omega_j|\Big| + C(n)\sum_{r^\alpha \leq R} 2^{n\alpha} e^{-2^{2\alpha}} \fint_{B_{2^\alpha r}(x)} \Big|\Delta|\omega_j|\,\Big| \\
&\quad + C(n)\sum_{r_\alpha > R} 2^{n\alpha} e^{-2^{2\alpha}} \fint_{B_{2^\alpha r}(x)} \Big|\Delta|\omega_j|\,\Big| \\
&\leq C\varepsilon_j(R) + C\sum_{r_\alpha \leq R} 2^{n\alpha} e^{-2^{2\alpha}} \varepsilon_j(R) + CR^{-2}\sum_{r^\alpha > R} 2^{n\alpha} e^{-2^{2\alpha}} \delta_j \to 0.
\end{aligned}
$$

Note that in estimating the first two terms in the last line of (3.72) we use the maximal function estimate (3.65), which is the reason for restricting attention to $x \in U_j$. For the third term in the last line, we use (3.63).

Similarly, $\sqrt{t} = r > R$, the first two terms on the last line of (3.72) are absent, and we just get

$$
(3.73)\qquad \int_{M_j^n} \Big|\Delta|\omega_j|\Big|\varphi\rho_{r^2}(x,dy) \leq CR^{-2}\sum_\alpha 2^{n\alpha} e^{-2^{2\alpha}} \delta_j \to 0.
$$

By combining (3.67), (3.71), (3.72), and (3.73), for $x \in U_j$ and $0 < t \leq S^2$, we get

$$\left| \frac{d}{dt} \int_{M_j^n} |\omega_j| \varphi \rho_t(x, dy) \right| \leq \varepsilon_j(S) \to 0, \tag{3.74}$$

uniformly in $U_j$.

At this point, by using (3.74) and integrating with respect to $t$ from 0 to $S^2$, we have for any $x \in U_j \subseteq B_R(x_j)$,

$$\left| |\omega_j|(x) - \int_{M_j^n} |\omega_j| \varphi \rho_{S^2}(x, dy) \right| \leq \varepsilon_j(S) \cdot S^2 \to 0, \tag{3.75}$$

uniformly in $U_j$.

By arguing in a manner similar to the above (but without the need for a maximal function estimate) we can use (3.61) to see that for all $x \in B_{2R}(x_j)$,

$$\begin{aligned} \int_{M_j^n} \left| \nabla(|\omega_j|\varphi) \right|^2 \rho_{S^2}(x, dy) &\leq 2 \int_{M_j^n} |\nabla \omega_j|^2 + |\omega_j|^2 |\nabla \varphi|^2 \rho_{S^2}(x, dy) \\ &\leq C \sum 2^{n\alpha} e^{-2^{2\alpha}} \fint_{B_{2^\alpha S}(x)} |\nabla \omega_j|^2 \\ &\quad + C r_j^{2-C\varepsilon} \Big( \frac{1}{S r_j} \Big)^n e^{-\frac{1}{S^2} r_j^{-2}} \\ &\leq C\, S^{-2+C\varepsilon} + \varepsilon_j(S), \end{aligned} \tag{3.76}$$

where without loss of generality, we can assume that our original $\varepsilon$ has been chosen so that $-2 + C\varepsilon < 0$. As previously mentioned, it is at just this point that the sublinearity in (3.61) has entered crucially, giving rise to the *negative* power of $S$ in (3.76), which comes to fruition in (3.79).

We have that $H_t\big(|\omega_j|\varphi\big) = \int_{M_j^n} |\omega_j| \varphi \rho_t(x, dy)$ solves the heat equation. So using the Bakry-Emery gradient estimate, [BÉ85], we have for any $x \in B_{2R}(x_j)$,

$$|\nabla H_t\big(|\omega_j|\varphi\big)|^2(x) \leq e^{\delta_j r_j^2 t} H_t |\nabla(|\omega_j|\varphi)|^2(x). \tag{3.77}$$

In particular, using (3.76) we have

$$\sup_{B_{2R}(x_j)} \left| \nabla_x \int_{M_j^n} |\omega_j| \varphi \rho_{S^2}(x, dy) \right| \leq \frac{C}{S^{1-C\varepsilon/2}} + \varepsilon_j(S). \tag{3.78}$$

Combining this with (3.75), for any pair of points $x, y \in U_j$, we get

$$\begin{aligned}\Big|\,|\omega_j|(x) - |\omega_j|(y)\Big| &\le \Big|\omega_j(x) - \int_{M_j^n} |\omega_j|\varphi\rho_{S^2}(x, dz)\Big| \\ &\quad + \Big|\omega_j(y) - \int_{M_j^n} |\omega_j|\varphi\rho_{S^2}(y, dz)\Big| \\ &\quad + \Big|\int_{M_j^n} |\omega_j|\varphi\rho_{S^2}(x, dy) - \int_{M_j^n} |\omega_j|\varphi\rho_{S^2}(y, dz)\Big| \\ &\le \varepsilon_j(S) + \frac{CR}{S^{1-C\varepsilon/2}}.\end{aligned} \tag{3.79}$$

By letting $S$ tend to infinity sufficiently slowly, we get for $x, y \in U_j$ that

$$\Big|\,|\omega_j|(x) - |\omega_j|(y)\Big| \le \varepsilon_j(R) \to 0. \tag{3.80}$$

Finally, to finish the proof, we use the supremum bound (3.61) on $|\omega|$ to note that for $x \in U_j$, we have

(3.81)

$$\begin{aligned}&\Big|\fint_{B_R(x_j)} |\omega_j|^2 - |\omega_j|^2(x)\Big| \\ &\quad\le \fint_{B_R(x_j)} \Big||\omega_j|^2 - |\omega_j|^2(x)\Big| \\ &\quad\le \fint_{B_R(x_j)} \Big||\omega_j| - |\omega_j|(x)\Big| \cdot \Big||\omega_j| + |\omega_j|(x)\Big| \\ &\quad\le C(n,R) \fint_{B_R(x_j)} \Big||\omega_j| - |\omega_j|(x)\Big| \\ &\quad\le C(n,R) \fint_{U_j} \Big||\omega_j| - |\omega_j|(x)\Big| + C(n,R) \fint_{B_R(x_j)\setminus U_j} \Big||\omega_j| - |\omega_j|(x)\Big| \\ &\quad\le C(n,R)\varepsilon_j(R) + C(n,R) \cdot \frac{\mathrm{Vol}(B_R(x_j) \setminus U_j)}{\mathrm{Vol}(B_R(x_j))} \to 0.\end{aligned}$$

Hence, we have

$$\begin{aligned}\fint_{B_R(x_j)} \Big|\,|\omega_j|^2 - \fint_{B_R(x_j)} |\omega_j|^2\Big| &\le \Big|\fint_{B_R(x_j)} |\omega_j|^2 - |\omega_j|^2(x)\Big| \\ &\quad + \fint_{B_R(x_j)} \Big||\omega_j|^2 - |\omega_j|^2(x)\Big| \to 0,\end{aligned} \tag{3.82}$$

which proves the claim. □

We know from (3.54), (3.61) that $|\nabla\omega_j|$ has $L^2$-bounds. It is crucial to improve these to bounds that are small compared to $\varepsilon$. This is the content of the next claim.

*Claim* 4. For fixed $R$, we have

$$\fint_{B_R(x_j)} |\nabla \omega_j|^2 \leq \varepsilon_j(R) \to 0. \tag{3.83}$$

To see this fix $R$ and as in [CC96], let $\varphi : B_{2R}(x_j) \to \mathbb{R}^+$ be a cutoff function with $\varphi \equiv 1$ on $B_R(x_j)$ and $R|\nabla\varphi|$, $R^2|\Delta\varphi| \leq C(n)$. We use the Bochner formula

$$\begin{aligned} \Delta|\omega_j|^2 &= 2|\nabla\omega_j|^2 + 2\Big\langle \sum_b dv_j^1 \wedge \cdots \mathrm{Ric}(dv_j^b) \wedge \cdots \wedge dv_j^k, \omega \Big\rangle \\ &\qquad + \Big\langle \sum_{a\neq b} dv_j^1 \wedge \nabla^c(dv_j^a) \wedge \cdots \wedge \nabla_c(dv_j^b) \wedge \cdots \wedge dv_j^k, \omega_j \Big\rangle \\ &\geq 2|\nabla\omega_j|^2 - C(n)\delta_j^2 r_j^2 |\omega_j|^2 - C(n)|\nabla(dv)|^2|\omega_j|^2 \\ &\qquad + \Big\langle \sum_{a\neq b} dv_j^1 \wedge \nabla^c(dv_j^a) \wedge \cdots \wedge \nabla_c(dv_j^b) \wedge \cdots \wedge dv_j^k, \omega_j \Big\rangle, \end{aligned} \tag{3.84}$$

which together with the growth estimates (3.54) allows us to compute

$$\begin{aligned} \fint_{B_R(x_j)} |\nabla\omega_j|^2 &\leq C(n) \fint_{B_{2R}(x_j)} \varphi \Delta|\omega_j|^2 \\ &\quad + C(n,R) \sum_{a\neq b} \fint_{B_{2R}(x_j)} |\nabla^2 v_j^a|\,|\nabla^2 v^b| + C(n,R)\delta_j r_j^2 \\ &\leq C \fint_{B_{2R}(x_j)} \Delta\varphi \left( |\omega_j|^2 - \fint_{B_{2R}(x_j)} |\omega_j|^2 \right) \\ &\quad + C(n,R) \sum_{a\neq b} \Big( \fint_{B_{2R}(x_j)} |\nabla^2 v^a|^2 \Big)^{1/2} \Big( \fint_{B_{2R}(x_j)} |\nabla^2 v^b|^2 \Big)^{1/2} + \varepsilon_j(R) \\ &\leq C \fint_{B_{2R}(x_j)} \Big| |\omega_j|^2 - \fint_{B_{2R}(x)} |\omega_j|^2 \Big| + \varepsilon_j(R) \leq \varepsilon_j(R) \to 0, \end{aligned} \tag{3.85}$$

where we have used Claim 3 and (3.58). Note that it is important that we have $a \neq b$ in the summation, so that at least one of the Hessian terms in each factor is going to zero as $j \to \infty$. This proves the claim. □

As mentioned in Remark 3.59, to complete the proof we must show that $\fint_{B_R(x_j)} |\nabla^2 v_j^k|^2 \to 0$ as $j \to \infty$. To prove this we will first pass to limits and obtain information on the limiting space. That is, we have been considering a sequence $(M_j^n, d_j, x_j)$ with $\mathrm{Ric}_{M_j^n} \geq -\delta_j r_j^2 \to 0$. After passing to a subsequence if necessary, we can take a measured pointed Gromov-Hausdorff limit

$$(M_j^n, d_j', x_j) \xrightarrow{d_{\mathrm{GH}}} (X, d, x) \tag{3.86}$$

to obtain an $\mathrm{RCD}(n,0)$ space $X$; see [AGS14a] and [AGS14b]. The fact that $X$ is an $\mathrm{RCD}(n,0)$ space is used below in applying the mean value estimate (3.93), which is known to hold for such spaces.

In addition, we can assume that the functions $v_j^\ell$ converge to harmonic functions

$$v_j^\ell \to v^\ell : X \to \mathbb{R}. \tag{3.87}$$

Indeed, for any ball $B_R(x_j)$, we can characterize $v_j^\ell$ as minimizers of the Dirichlet energy with fixed Dirichlet boundary values. Our assertion then follows from the lower semicontinuity of the Dirichlet energy [AGS14b] combined with the Mosco convergence of the Dirichlet form [GMS13] to see that the limit also minimizes the Dirichlet energy on any ball.

Observe first that by using Claim 2 and Lemma 1.21, we have

$$X = \mathbb{R}^{k-1} \times Y, \tag{3.88}$$

where $v^1, \dots, v^{k-1} : X \to \mathbb{R}$ are linear functions that induce the $\mathbb{R}^{k-1}$ factor and we can identify $Y = (v^1, \dots, v^{k-1})^{-1}(0^{k-1})$. We are left with understanding the behavior of $v^k$. We will see in Claim 6 that it too is linear, and in the process we prove our Hessian estimate. We first show the following

*Claim* 5. There exists $a_1, \dots, a_{k-1} \in \mathbb{R}$ with $|a_\ell| < C(n)\varepsilon$ such that $v^k - a_1 v^1 - \cdots - a_{k-1} v^{k-1} : X \to \mathbb{R}$ is a function of only the $Y$ variable.

To prove the claim let us fix any vector $V \in \mathbb{R}^{k-1}$ and consider the map $Dv^k : X \to \mathbb{R}$ defined by

$$Dv^k(y) = v^k(y + V) - v^k(y) \tag{3.89}$$

where, of course, the translation $x \to x+V$ is well defined, since $X \equiv \mathbb{R}^{k-1} \times Y$. The function $v^k(y)$ is harmonic, and the translation map $x \to x + V$ is a measure preserving isometry. Thus, $v^k(x+V)$ is a harmonic function as well. Since $X$ is an RCD space, and hence the Laplacian $\Delta$ on $X$ is linear, it follows that $Dv^k$ is harmonic. Using the estimates (3.54) we have the growth condition

$$\sup_{B_r(x)} |Dv^k| \le C|V|^{1+C\varepsilon} \cdot r^{C\varepsilon}. \tag{3.90}$$

This is to say that $Dv^k$ is a harmonic function with sublinear growth. It follows that $Dv^k$ must be a constant. Indeed, let $\varphi$ be a cutoff on $B_{2S}(x)$ with $\varphi \equiv 1$ on $B_S(x)$ and $|\nabla\varphi| \le 10S^{-1}$. Then on the one hand, since $Dv^k$ is harmonic and the Dirichlet form is bilinear, we have that

$$\begin{aligned} 0 &= \fint_{B_{2S}(x)} \langle \nabla Dv^k, \nabla(\varphi^2 Dv^k)\rangle \\ &= \fint_{B_{2S}(x)} \varphi^2 |\nabla Dv^k|^2 + 2\fint_{B_{2S}(x)} \varphi\, Dv^k\, \langle \nabla Dv^k, \nabla\varphi\rangle. \end{aligned} \tag{3.91}$$

By rearranging terms, we obtain

$$\begin{aligned}(3.92)\qquad \fint_{B_S(x)} |\nabla Dv^k|^2 &\leq \fint_{B_{2S}(x)} \varphi^2 |\nabla Dv^k|^2 \\ &\leq \frac{1}{2}\fint_{B_{2S}(x)} \varphi^2 |\nabla Dv^k|^2 + 8\fint_{B_{2S}(x)} |Dv^k|^2 |\nabla\varphi|^2 \\ &\leq CS^{-2+C\varepsilon},\end{aligned}$$

where without loss of generality, we can assume that $\varepsilon$ is so small that $-2 + C\varepsilon < 0$.

On the other hand, $\mathrm{Ric}_{M_j^n} \geq -(n-1)\delta_j r_j^2 \to 0$ and so $X$ is an $\mathrm{RCD}(n,0)$ space. On such spaces, there is a mean value inequalilty for the norm squared of the gradient of a harmonic function; see, for instance, [MN14]. When applied to the harmonic function $Dv^k$ it gives for $r > 0$ fixed and $S \to \infty$,

$$(3.93)\qquad \sup_{B_r(x)} |\nabla Dv^k|^2 \leq C \fint_{B_S(x)} |\nabla Dv^k|^2 \leq CS^{-2+C\varepsilon} \to 0.$$

Note that once again, we have exploited the sublinearity of the growth estimates. In particular, it now follows that $Dv^k$ is a constant. Since this holds for any $V \in \mathbb{R}^{k-1}$, we have that $v^k$ is linear in the $\mathbb{R}^{k-1}$ variable. More precisely, since the $\mathbb{R}^{k-1}$ factor is spanned by $v^1, \ldots, v^{k-1}$, we have

$$(3.94)\qquad v^k = v^k_Y + a_1 v^1 + \cdots + a_{k-1} v^{k-1},$$

where $v^k_Y : Y \to \mathbb{R}$. Since $v_j \to v : X \to \mathbb{R}^k$ are $C\varepsilon$-splittings on $B_2(x_j)$, we automatically have the bounds $|a_\ell| \leq C(n)\varepsilon$. This finishes the claim. □

To complete the proof, we want to see that the Hessians of $v_j^k$ are tending to zero as $j \to \infty$. This is the content of Claim 6 below. However, prior to stating this claim, we will make some additional normalizations.

To begin with, we can use Claim 5 to further normalize the mappings $v_j$ by composing with another lower triangular matrix with positive diagonal entries. Indeed, as a corollary of Claim 5, we can choose a lower triangular matrix $A$ with $|A - I| < C(n)\varepsilon$, whose restriction to the first $(k-1)\times(k-1)$ terms is the identity, such that $Av_j : B_2(x_j) \to \mathbb{R}^k$ is still a $C(n)\varepsilon$-splitting, while $A \circ v_j^k \to A \circ v^k : \mathbb{R}^{k-1} \times Y \to \mathbb{R}$ is independent of the $\mathbb{R}^{k-1}$ factor. Further, let us consider the induced form $A \circ \omega_j = d(A \circ v_j^1) \wedge \cdots \wedge d(A \circ v_j^k) = dv_j^1 \wedge \cdots \wedge d(A \circ v_j^k)$. Then after multiplying the $k^{th}$ row of $A$ by a constant $c$ with $|c - 1| \leq C(n)\varepsilon$, we may further assume that

$$(3.95)\qquad \fint_{B_2(x_j)} |A \circ \omega_j|^2 = 1.$$

From this point forward in the proof, for ease of notation, we will write $v_j$ for what was denoted above by $A \circ v_j$. In particular, this $v_j$ differs from the

original mapping $u_j$ only by composition with a lower triangular matrix with positive diagonal entries. We will eventually see that $v_j : B_1(x_j) \to \mathbb{R}^k$ is an $\varepsilon_j$-splitting, which will give the desired contradiction and finish the proof.

*Claim* 6. For each $R > 0$, we have $\fint_{B_R(x_j)} |\nabla^2 v_j^k|^2 \leq \varepsilon_j(R) \to 0$.

The fact that $v_j : B_R(x_j) \to \mathbb{R}^{k-1}$ is an $\varepsilon_j(R)$-splitting,

$$\fint_{B_2(x_j)} |\omega_j|^2 = 1,$$

together with

$$\fint_{B_R(x_j)} |\nabla \omega_j|^2 \leq \varepsilon_j(R) \to 0,$$

implies

$$\fint_{B_R(x_j)} \Big||\omega_j^\ell| - 1\Big| \leq \varepsilon_j(R) \qquad (\text{for all } 1 \leq \ell \leq k). \tag{3.96}$$

Now we will show that

$$\fint_{B_R(x_j)} \Big||\nabla v_j^k|^2 - 1\Big| \leq \varepsilon_j(R) \to 0. \tag{3.97}$$

Once this is accomplished, as we have done repeatedly, we can argue with Bochner's formula to obtain the Hessian estimate in the claim.

Define the 1-form

$$V_j \equiv \langle \omega_j^{k-1}, \omega_j \rangle. \tag{3.98}$$

Note that $\omega_j^{k-1} \wedge V_j$ is proportional to $\omega_j = \omega_j^{k-1} \wedge dv_j^k = \omega_j^{k-1} \wedge \big(dv_j^k - \pi_{k-1} dv_j^k\big)$. More generally, we have that $V_j \in \text{span}\{\nabla v_j^1, \dots, \nabla v_j^k\}$ is perpendicular to $\text{span}\{\nabla v_j^1, \dots, \nabla v_j^{k-1}\}$. From the above, we get

$$\fint_{B_R(x_j)} |V_j - \big(dv_j^k - \pi_{k-1} dv_j^k\big)| \leq \varepsilon_j(R). \tag{3.99}$$

On the other hand, by (3.85) we have

$$\fint_{B_R(x_j)} |\nabla V_j|^2 \leq \varepsilon_j(R) \to 0, \tag{3.100}$$

and thus using (3.96) we have

$$\fint_{B_R(x_j)} \Big||V_j| - 1\Big| \leq \varepsilon_j(R). \tag{3.101}$$

Therefore, from (3.99) we get

$$\fint_{B_R(x_j)} \Big||dv_j^k - \pi_{k-1} dv_j^k| - 1\Big| \leq \varepsilon_j(R) \to 0. \tag{3.102}$$

It follows that our main concern is to show $|\pi_{k-1}(dv_j^k)| \to 0$ in $L^1$ as $j \to \infty$. Because we have the estimate

$$\int_{B_R(x_j)} |\langle \nabla v_j^a, \nabla v_j^b \rangle - \delta^{ab}| \leq \varepsilon_j(R) \to 0 \tag{3.103}$$

for $a, b < k$, this is equivalent to showing that

$$\int_{B_R(x_j)} |\langle \nabla v_j^\ell, \nabla v_j^k \rangle| \leq \varepsilon_j(R) \to 0 \tag{3.104}$$

for all $\ell < k$, which will be our primary goal now.

To accomplish this let us fix some $\ell < k$ and recall that $M_j^n \to X \equiv \mathbb{R}^{k-1} \times Y$, where the $v_j^\ell \to v^\ell$ converge to the linear splitting factors and $v_j^k \to v^k$ converges to a function of the $Y$ variable. In particular, notice in the limit that $|\langle \nabla v^\ell, \nabla v^k \rangle| = 0$. One could therefore prove the result by showing that the energies of a sequence of harmonic functions actually converge in $L^1_{\rm loc}$ to the energies of the limiting harmonic functions. We will proceed by essentially proving a more effective version of this statement.

Thus, for each $(s, y) \in \mathbb{R}^{k-1} \times Y \cap B_R(x_j)$ and $0 < \varepsilon_j \ll r_2 \ll r_1 \ll 1$, let us consider an open set $U(s, y, r_1, r_2)$ such that

(3.105)

$$\begin{aligned} &\big(B_{r_2}(s_1, \dots, s_{\ell-1}) \times (s_\ell - r_1, s_\ell + r_1) \times B_{r_2}(s_{\ell+1}, \dots, s_{k-1}) \times B_{r_2}(y)\big) \\ &\qquad \cap B_{R+r_1}(x_j) \subseteq U(s, y, r_1, r_2), \\ &U(s, y, r_1, r_2) \\ &\quad \subseteq \big(B_{2r_2}(s_1, \dots, s_{\ell-1}) \times (s_\ell - 2r_1, s_\ell + 2r_1) \times B_{2r_2}(s_{\ell+1}, \dots, s_{k-1})\big) \\ &\qquad \cap B_{R+2r_1}(x_j) \end{aligned}$$

with respect to the Gromov-Hausdorff map from $M_j^n$ to $\mathbb{R}^{k-1} \times Y$. Clearly, we have the volume estimate

$$C^{-1}(n, \mathrm{v}, R) \leq r_1^{-1} r_2^{-(n-1)} \mathrm{Vol}(U(s, y, r_1, r_2)) \leq C(n, \mathrm{v}, R), \tag{3.106}$$

where $\mathrm{v} > 0$ is the noncollapsing constant. Let us notice that as $r_2 \ll r_1 \to 0$, we have that $U(s, y, r_1, r_2)$ is approximately a product of balls with diameter tending to zero, and such that for each $z_1 \in U(s, y, r_1, r_2)$, we have the important estimates

$$\begin{aligned} &⨍_{U(s,y,r_1,r_2)} \frac{|v_j^a(z_1) - v_j^a(z_2)|}{d(z_1, z_2)} dv_g(z_2) < O\left(\frac{r_2}{r_1}\right) + \varepsilon_j(R) \text{ for } a \neq \ell, \\ &⨍_{U(s,y,r_1,r_2)} \sqrt{\left|\frac{|v_j^\ell(z_1) - v_j^\ell(z_2)|}{d(z_1, z_2)} - 1\right|} dv_g(z_2) < O\left(\frac{r_2}{r_1}\right) + \varepsilon_j(R). \end{aligned} \tag{3.107}$$

Note that the integrands above are bounded and converging to zero pointwise away from a set whose measure is going to zero relative to $U$ as $\frac{\varepsilon_j}{r_2}, \frac{r_2}{r_1} \to 0$. In

other words, the $v_j^a$ for $a \neq \ell$ are becoming approximately constant functions and $v_j^\ell$ is becoming a norm one linear function in the domains as $\frac{\varepsilon_j}{r_2}, \frac{r_2}{r_1} \to 0$.

Now let $z_1, z_2 \in U(s, y, r_1, r_2)$, with $\gamma_{z_1,z_2} : [0, d(z_1, z_2)] \to M$ a minimizing geodesic connecting them, and let $d \equiv d(z_1, z_2)$. Without loss of generality, let us assume that $v_j^\ell(z_2) \geq v_j^\ell(z_1)$. Otherwise, the argument below works with the reverse geodesic $\gamma_{z_2,z_1}$. We can estimate

(3.108)

$$\begin{aligned}
|\langle \nabla v_j^\ell, \nabla v_j^k\rangle|(z_1) &= \Big| \fint_0^d \langle \nabla v_j^\ell, \nabla v_j^k\rangle - \fint_0^d \int_0^t \nabla_{\dot\gamma}\langle \nabla v_j^\ell, \nabla v_j^k\rangle \Big| \\
&= \Big| \fint_0^d \langle \dot\gamma, \nabla v_j^k\rangle + \fint_0^d \langle \nabla v_j^\ell - \dot\gamma, \nabla v_j^k\rangle - \fint_0^d \int_0^t \nabla_{\dot\gamma}\langle \nabla v_j^\ell, \nabla v_j^k\rangle \Big|, \\
&\leq C\Big( \Big| \frac{v_j^k(z_2) - v_j^k(z_1)}{d} \Big| + \fint_{\gamma_{z_1,z_2}} |\nabla v_j^\ell - \dot\gamma| + \int_{\gamma_{z_1,z_2}} |\nabla^2 v_j^\ell| + \int_{\gamma_{z_1,z_2}} |\nabla^2 v_j^k| \Big).
\end{aligned}$$

To deal with the second term on the last line, let us observe that since $|\nabla v_j^\ell| \leq 1 + \varepsilon_j$, we have

$$\begin{aligned}
\Big( \fint_{\gamma_{z_1,z_2}} |\nabla v_j^\ell - \dot\gamma| \Big)^2 &\leq \fint_{\gamma_{z_1,z_2}} |\nabla v_j^\ell - \dot\gamma|^2 \\
&\leq \fint_{\gamma_{z_1,z_2}} 2\big(1 - \langle \nabla v_j^\ell, \dot\gamma\rangle\big) + \varepsilon_j \\
&\leq 2\Big(1 - \frac{v_j^\ell(z_2) - v^\ell(z_1)}{d}\Big) + \varepsilon_j \\
&\leq 2\Big|1 - \frac{|v_j^\ell(z_2) - v^\ell(z_1)|}{d}\Big| + \varepsilon_j,
\end{aligned} \tag{3.109}$$

where we have used our normalizing condition that $v_j^\ell(z_2) \geq v^\ell(z_1)$ in the last line. Plugging this into our estimate for $|\langle \nabla v_j^\ell, \nabla v_j^k\rangle|(z_1)$, we obtain

$$\begin{aligned}
|\langle \nabla v_j^\ell, \nabla v_j^k\rangle|(z_1) \leq C\Bigg( &\Big| \frac{v_j^k(z_2) - v_j^k(z_1)}{d} \Big| + \sqrt{\Big|1 - \frac{|v_j^\ell(z_2) - v^\ell(z_1)|}{d}\Big|} \\
&+ \int_{\gamma_{z_1,z_2}} |\nabla^2 v_j^\ell| + \int_{\gamma_{z_1,z_2}} |\nabla^2 v_j^k| \Bigg) + \varepsilon_j.
\end{aligned} \tag{3.110}$$

Since this holds for each $z_2 \in U(y, r, r_1, r_2)$, we can average both sides and use (3.107) to estimate

$$\begin{aligned}
|\langle \nabla v_j^\ell, \nabla v_j^k\rangle|(z_1) &\leq O\left(\frac{r_2}{r_1}\right) \\
&\quad + C \fint_{U(s,y,r_1,r_2)} \Big( \int_{\gamma_{z_1,z_2}} |\nabla^2 v_j^\ell| + \int_{\gamma_{z_1,z_2}} |\nabla^2 v_j^k| \Big) dv_g(z_2) + \varepsilon_j(R).
\end{aligned} \tag{3.111}$$

Integrating over $z_1$ then gives us the estimate

$$\begin{aligned}(3.112)\qquad &\fint_{U(s,y,r_1,r_2)} |\langle \nabla v_j^\ell, \nabla v_j^k\rangle| \\ &\leq O\left(\frac{r_2}{r_1}\right) + C \fint_{U\times U} \Big( \int_{\gamma_{z_1,z_2}} |\nabla^2 v_j^\ell| + \int_{\gamma_{z_1,z_2}} |\nabla^2 v_j^k|\Big) + \varepsilon_j, \\ &\leq O\left(\frac{r_2}{r_1}\right) + Cr_1 \fint_{U(s,y,10r_1,10r_2)} \Big(|\nabla^2 v_j^\ell| + |\nabla^2 v_j^k|\Big) + \varepsilon_j.\end{aligned}$$

In the last line we have used a sharpening of the standard segment inequality, which takes into account that all the minimizing geodesics beginning and ending in $U(s,y,r_1,r_2)$ are contained in $U(s,y,10r_1,10r_2)$. Given this the conclusion follows from the proof of the standard segment inequality. Rewriting the above gives us

$$\begin{aligned}(3.113)\qquad \int_{U(s,y,r_1,r_2)} |\langle \nabla v_j^\ell, \nabla v_j^k\rangle| &\leq \Big(O\left(\frac{r_2}{r_1}\right) + \varepsilon_j\Big)\mathrm{Vol}(U(s,y,r_1,r_2)) \\ &\quad + Cr_1 \int_{U(s,y,10r_1,10r_2)} \Big(|\nabla^2 v_j^\ell| + |\nabla^2 v_j^k|\Big).\end{aligned}$$

Now to complete the proof, let us choose for each $r_1, r_2$ fixed a covering

$$(3.114)\qquad B_R(x_j) \subseteq \bigcup U(s_i, y_i, r_1, r_2)$$

such that the sets $U(s_i,y_i,10r_1,10r_2)$ overlap at most $C(n)$ times. This is possible using the GH condition with $\varepsilon_j \ll r_2$. By applying (3.113) to each of these and summing, we obtain the estimate

$$\begin{aligned}(3.115)\qquad \int_{B_R(x_j)} |\langle \nabla v_j^\ell, \nabla v_j^k\rangle| &\leq \Big(O\Big(\frac{r_2}{r_1}\Big) + \varepsilon_j\Big)\mathrm{Vol}(B_{2R}(x_j)) \\ &\quad + Cr_1 \int_{B_{2R}(x_j)} \Big(|\nabla^2 v_j^\ell| + |\nabla^2 v_j^k|\Big),\end{aligned}$$

or that

$$\begin{aligned}(3.116)\qquad \fint_{B_R(x_j)} |\langle \nabla v_j^\ell, \nabla v_j^k\rangle| &\leq O\left(\frac{r_2}{r_1}\right) + Cr_1 \fint_{B_{2R}(x_j)} \Big(|\nabla^2 v_j^\ell| + |\nabla^2 v_j^k|\Big) + \varepsilon_j(R) \\ &\leq O\left(\frac{r_2}{r_1}\right) + Cr_1 + \varepsilon_j(R),\end{aligned}$$

where in the last line we have used that we have uniform $L^2$-estimates on the Hessians of $v_j^a$. The estimates above hold for all $0 < \varepsilon_j \ll r_2 \ll r_1 \ll 1$. To finish the proof let us now choose $r_{2,j}, r_{1,j} \to 0$ such that $\frac{r_{2,j}}{r_{1,j}}, \frac{\varepsilon_j}{r_{2,j}} \to 0$. This proves the estimate (3.104), and therefore by (3.102), we have that

$$(3.117)\qquad \fint_{B_R(x_j)} \Big| |\nabla v_j^k| - 1\Big| \leq \varepsilon_j(R) \to 0.$$

When combined with the $L^\infty$ estimate on $|\nabla v_j^k|$, this gives the $L^2$-estimate

$$\fint_{B_R(x_j)} \Big| |\nabla v_j^k|^2 - 1\Big| \leq \varepsilon_j(R) \to 0. \tag{3.118}$$

Finally, since $v_j^k$ is harmonic, we can now argue with Bochner's formula as in the proof of (1.27) to obtain the Hessian estimate, $\fint_{B_R(x_j)} |\nabla^2 v_j^k|^2 \leq \varepsilon_j(R) \to 0$. This completes the proof of the claim. □

Now we can finish the proof of the Transformation Theorem. Indeed, we will see that $v_j = A \circ u : B_1(x_j) \to \mathbb{R}^k$ is the desired $\varepsilon_j(R)$-splitting. Claim 6 gives

$$\fint_{B_R(x_j)} |\nabla^2 v_j^\ell|^2 \to 0 \tag{3.119}$$

for all $1 \leq \ell \leq k$, while (3.104), (3.103) and (3.118) imply

$$\fint_{B_R(x_j)} |\langle \nabla v_j^a, \nabla v_j^b\rangle - \delta^{ab}| \to 0. \tag{3.120}$$

To see that $v_j : B_1(x_j) \to \mathbb{R}^k$ is an $\varepsilon_j(R)$-splitting on $B_1(x_j)$, the last step is to show that $|\nabla v_j^k| \leq 1 + \varepsilon_j \to 1$. However this follows immediately from (3.119) and (3.120) by using precisely the same argument as in (3.42)–(3.46).

Thus, for $j$ sufficiently large, we see that $v_j : B_1(x_j) \to \mathbb{R}^k$ is an $\varepsilon$-splitting. This is a contradiction, so the proof is complete.

□

## 4. **Proof of Theorem 1.23, the Slicing Theorem**

The goal of this section is to prove the Slicing Theorem (Theorem 1.23). Recall the statement

*For each $\varepsilon > 0$, there exists $\delta(n,\varepsilon) > 0$ such that if $M^n$ satisfies $\mathrm{Ric}_{M^n} \geq -(n-1)\delta$ and if $u : B_2(p) \to \mathbb{R}^{n-2}$ is a harmonic $\delta$-splitting map, then there exists a subset $G_\varepsilon \subseteq B_1(0^{n-2})$ that satisfies the following*:

(1) $\mathrm{Vol}(G_\varepsilon) > \mathrm{Vol}(B_1(0^{n-2})) - \varepsilon$;
(2) *if $s \in G_\varepsilon$, then $u^{-1}(s)$ is nonempty*;
(3) *for each $x \in u^{-1}(G_\varepsilon)$ and $r \leq 1$, there exists a lower triangular matrix $A \in \mathrm{GL}(n-2)$ with positive diagonal entries such that $A \circ u : B_r(x) \to \mathbb{R}^{n-2}$ is an $\varepsilon$-splitting map.*

*Proof of Theorem* 1.23. Recall from Section 1.2 the measure $\mu$ defined in (1.36) and $\delta_3 = \delta_3(n,\varepsilon)$ in the Transformation Theorem; see the sentence prior to (1.33). It was shown Section 1.2 that in view of Theorem 1.26 and the transformation theorem, Theorem 1.32, to complete the proof of the Slicing Theorem, it suffices to verify that for $1/4 \geq r \geq s_x^{\delta_3}$, $\mu$ satisfies the doubling

condition $|u(B_r(x)| \leq C(n)\cdot r^{-2}\mu(B_r(x))$ and the volume estimate $|u(B_r(x))| \leq C(n)\cdot r^{-2}\mu(B_r(x))$ on the image of a ball; see (1.37), (1.38).

LEMMA 4.1. *For each $x$ and $1/4 \geq r \geq s_x^{\delta_3}$, we have the doubling condition*

$$\mu(B_{2r}(x)) \leq C(n)\mu(B_r(x)). \tag{4.2}$$

*Proof.* According to Theorem 1.32, there exists a lower triangular matrix $A \in \mathrm{GL}(n-2)$ with positive diagonal entries such that

$$u' = A \circ u : B_{2r}(x) \to \mathbb{R}^{n-2} \tag{4.3}$$

is an $\varepsilon$-splitting. Let $dv_g$ denote the Riemannian measure, and set $\omega' \equiv du'^1 \wedge \cdots \wedge du'^{n-2}$. Define the measure $\mu'$ by $\mu' = \Big(\int_{B_{3/2}(p)} |\omega|\Big)^{-1} |\omega'| dv_g$. Then

$$\mu' = \det(A)\mu. \tag{4.4}$$

In particular, this gives us

$$\frac{\mu'(B_{2r}(x))}{\mu'(B_r(x))} = \frac{\mu(B_{2r}(x))}{\mu(B_r(x))}, \tag{4.5}$$

and it is equivalent to show the ratio bound for $\mu'$. Since $u'$ is an $\varepsilon$-splitting, we have the estimate

$$\fint_{B_{2r}(x)} |\,|\omega'| - 1| \leq C(n)\varepsilon. \tag{4.6}$$

Hence, we also have the estimate

$$\fint_{B_r(x)} |\,|\omega'| - 1| \leq \frac{\mathrm{Vol}(B_{2r}(x))}{\mathrm{Vol}(B_r(x))} \fint_{B_{2r}(x)} |\,|\omega'| - 1| \leq C(n)\varepsilon, \tag{4.7}$$

which of course uses the doubling property for the Riemannian measure. By combining the previous two estimates we get

$$\begin{aligned} \big(1 - C\varepsilon\big)\mathrm{Vol}(B_r(x)) \leq \mu'(B_r(x)) \leq \big(1 + C\varepsilon\big)\mathrm{Vol}(B_r(x)), \\ \big(1 - C\varepsilon\big)\mathrm{Vol}(B_{2r}(x)) \leq \mu'(B_{2r}(x)) \leq \big(1 + C\varepsilon\big)\mathrm{Vol}(B_{2r}(x)). \end{aligned} \tag{4.8}$$

Finally, by using the definition of $\mu'$, we arrive at

$$\begin{aligned}\mu'(B_{2r}(x)) &= \Big(\int_{B_{3/2}(p)} |\omega|\, dv_g\Big)^{-1} \int_{B_{2r}(x)} |\omega'| \\ &\le (1+C(n)\varepsilon)\Big(\int_{B_{3/2})} |\omega|\, dv_g\Big)^{-1} \mathrm{Vol}(B_{2r}(x)) \\ &\le C(n)\Big(\int_{B_{3/2})} |\omega|\, dv_g\Big)^{-1} \mathrm{Vol}(B_r(x)) \\ &\le C(n)\Big(\int_{B_{3/2}p)} |\omega|\, dv_g\Big)^{-1} \int_{B_r(x)} |\omega'| \\ &= C(n)\mu'(B_r(x)),\end{aligned} \tag{4.9}$$

which by (4.5) completes the proof. □

Recall the collection $\mathcal{B}_{\delta_3}$ of bad balls, defined in (1.33). The proof of the Slicing Theorem (Theorem 1.23) requires that the image under $u$ of $\mathcal{B}_{\delta_3}$ has measure $< \varepsilon/2$; see (1.38), (1.40). If in (1.40) the measure $\mu$ were instead the usual riemannian measure, then since $u$ is Lipschitz, standard estimates could be used to show just that. On the face of it, however, the $\mu$-content estimate is much weaker, since for balls where the determinant $|\omega|$ of $u$ is small, then $\mu(B_r(x))/\mathrm{Vol}(B_r(x))$ is small as well.

On the other hand, in the spirit of Sard's theorem, we will see in the next lemma that at least for balls $B_r(x)$ with $1/4 \ge r \ge s_x^{\delta_3}$, we recover this loss because the volume of the image $u(B_r(x))$ is correspondingly small.

LEMMA 4.10. *If* $1/4 \ge r \ge s_x^{\eta}$, *then*

$$|u(B_r(x)| \le C(n)\cdot r^{-2}\mu(B_r(x)). \tag{4.11}$$

*Proof.* As in Lemma 4.1, choose a lower triangular matrix $A \in \mathrm{GL}(n-2)$ with positive diagonal entries, such that

$$u' = A\circ u : B_{2r}(x) \to \mathbb{R}^{n-2} \tag{4.12}$$

is an $\varepsilon$-splitting, and define the measure $\mu'$ as in Lemma 4.1. Then as in (4.3), $\mu' = \det(A)\mu$.

Since $u'$ is an $\varepsilon$-splitting, we have the estimates

$$\begin{aligned}\fint_{B_{2r}(x)} ||\omega'| - 1| &\le C(n)\varepsilon, \\ u'(B_r(x)) &\subseteq B_{2r}(u'(x)).\end{aligned} \tag{4.13}$$

By the first estimate above,

$$\begin{aligned}\mu'(B_r(x)) &= \Big(\int_{B_{3/2}(p)} |\omega|\Big)^{-1} \int_{B_r(x)} |\omega'|, \\ &\geq (1 - C(n)\varepsilon)\frac{\mathrm{Vol}(B_r(x))}{\mathrm{Vol}(B_{3/2}(p))} \fint_{B_r(x)} |\omega'| \\ &\geq (1 - C\varepsilon)\frac{\mathrm{Vol}(B_r(x))}{\mathrm{Vol}(B_{3/2}(x))} \geq C(n) r^n,\end{aligned} \tag{4.14}$$

where in the last step we have used volume monotonicity for the Riemannian measure. On the other hand, by the second estimate of (4.13),

$$|u'(B_r(x))| \leq C(n) r^{n-2}. \tag{4.15}$$

Combining these gives the estimate

$$|u'(B_r(x))| \leq C(n) r^{-2} \mu'(B_r(x)). \tag{4.16}$$

To relate these back to the original function $u$, we observe that

$$\begin{aligned}|u'(B_r(x))| &= \det(A)|u(B_r(x))|, \\ \mu'(B_r(x)) &= \det(A)|\mu(B_r(x))|,\end{aligned} \tag{4.17}$$

which immediately gives (4.11). This completes the proof. □

As previously noted, Lemmas 4.1 and 4.10 suffice to complete the proof of the Slicing Theorem. □

## 5. **Codimension** 4 **regularity of singular limits**

In this section we prove Theorem 1.4. Thus, we consider a Gromov-Hausdorff limit space,

$$(M_j^n, d_j, p_j) \xrightarrow{d_{\mathrm{GH}}} (X, d, p), \tag{5.1}$$

of a sequence of Riemannian manifolds $(M_j^n, g_j, p_j)$, satisfying $|\mathrm{Ric}_{M_j^n}| \leq n-1$ and $\mathrm{Vol}(B_1(p_j)) > \mathrm{v} > 0$. We will show that there exists a subset $\mathcal{S} \subseteq X$ of codimension 4 such that $X \setminus \mathcal{S}$ is a $C^{1,\alpha}$-Riemannian manifold. In this section, we will show that $\mathcal{S}$ has Hausdorff codimension 4. We will postpone the improvement to Minkowski codimension 4 until Section 7.

As mentioned in Section 1, it has been understood since [CC97] that the main technical challenge lies in showing that spaces of the form $\mathbb{R}^{n-2} \times C(S^1_\beta)$, where $S^1_\beta$ is the circle of circumference $\beta \leq 2\pi$, cannot arise as limit spaces unless $\beta = 2\pi$ and hence $\mathbb{R}^{n-2} \times C(S^1_\beta) = \mathbb{R}^n$. The Slicing Theorem (Theorem 1.23) was expressly designed to enable us to handle this point via a blowup argument. We will do this in Section 5.1.

In Section 5.2 we prove that more general spaces of the form $\mathbb{R}^{n-3} \times C(Y)$ cannot arise as limit spaces. The proof of this statement has a very different

feel than the proof ruling out the codimension 2 limits and essentially comes down to a bordism and curvature pinching argument for 3-manifolds.

Finally, in Section 5.3 we combine the tools developed in the previous subsections to prove the Hausdorff estimates of Theorem 1.4.

5.1. *Nonexistence of codimension* 2 *singularities.* In this subsection we use the tools of Section 4 in order to prove that spaces that are $(n-2)$-symmetric cannot arise as noncollapsed limits of manifolds with bounded Ricci curvature.

THEOREM 5.2 ($(n-2)$-Symmetric Limits). *Let $(M_j^n, g_j, p_j)$ be a sequence of Riemannian manifolds satisfying $|\mathrm{Ric}_{M_j^n}| \to 0$, $\mathrm{Vol}(B_1(p_j)) > \mathrm{v} > 0$ and such that*

$$(M_j^n, d_j, p_j) \xrightarrow{d_{\mathrm{GH}}} \mathbb{R}^{n-2} \times C(S^1_\beta). \tag{5.3}$$

*Then $\beta = 2\pi$ and $\mathbb{R}^{n-2} \times C(S^1_\beta) = \mathbb{R}^n$.*

*Proof of Theorem* 5.2. We will prove the result by contradiction. So let us assume it is false. Then there exists a sequence $(M_j^n, g_j, p_j)$ of Riemannian manifolds satisfying $|\mathrm{Ric}_{M_j^n}| \to 0$, $\mathrm{Vol}(B_1(p_j)) > \mathrm{v} > 0$ and such that

$$(M_j^n, d_j, p_j) \to \big(\mathbb{R}^{n-2} \times C(S^1_\beta), d, p\big), \tag{5.4}$$

with $\beta < 2\pi$ and $p$ a vertex.

Note first that by the noncollapsing assumption we have $\beta \geq \beta_0(n, v)$. Now by Lemma 1.21, there exist $\delta_j$-splitting maps $u_j : B_2(p_j) \to \mathbb{R}^{n-2}$ with $\delta_j \to 0$. Fix some sequence $\varepsilon_j \to 0$ that is tending to zero so slowly compared to $\delta_j$ that Theorem 1.23 holds for $u_j : B_2(0) \to \mathbb{R}^{n-2}$ with $\varepsilon_j$. Let $G_{\varepsilon_j} \subseteq B_1(0^{n-2})$ be the corresponding good values of $u_j$, and let $s_j \in G_{\varepsilon_j} \cap B_{10^{-1}}(0^{n-2})$ be fixed regular values.

Observe that $\mathbb{R}^{n-2} \times C(S^1_\beta)$ is smooth outside of the singular set $\mathcal{S} = \mathbb{R}^{n-2} \times \{0\} \subseteq \mathbb{R}^{n-2} \times C(S^1_\beta)$. In particular, on $\mathbb{R}^{n-2} \times C(S^1_\beta)$ we have $r_h(x) \approx 1/d(x, \mathcal{S})$, where $r_h$ is the harmonic radius as in Section 1 and $d$ denotes distance. By the standard $\varepsilon$-regularity theorem, it follows that the convergence of $M_j^n$ is in $C^{1,\alpha} \cap W^{2,q}$ away from $\mathcal{S}$ for every $\alpha < 1$ and $q < \infty$. Let $f_j : B_{\varepsilon_j^{-1}}(p) \to B_{\varepsilon_j^{-1}}(p_j)$ be the $\varepsilon_j$-Gromov Hausdorff maps, and let us denote $\mathcal{S}_j \equiv f_j(\mathcal{S}) \subseteq M_j^n$. Then by the previous statements, for every $\tau > 0$, all $j$ sufficiently large, and $x \in B_1(p_j) \setminus T_\tau(\mathcal{S}_j)$, we have $r_h(x) \geq \frac{\tau}{2}$.

Consider again the submanifold $u_j^{-1}(s_j) \cap B_1(p_j)$. Define the scale

$$r_j = \min\{r_h(x) : x \in u_j^{-1}(s_j) \cap B_1(p_j)\}. \tag{5.5}$$

By the discussion of the previous paragraph, this minimum is actually obtained at some $x_j \in u_j^{-1}(s_j) \cap B_1(p_j)$, with $x_j \to \mathcal{S}_j \cap B_{10^{-1}}(p_j)$. Moreover, since $S^1_\beta$, the cross-section of the cone factor, satisfies $0 < \beta < 2\pi$, it follows that $r_j \to 0$. According to Theorem 1.23, there exists a lower triangular matrix

$A_j \in \mathrm{GL}(n-2)$ with positive diagonal entries, such that $v_j \equiv A_j \circ \left(u_j - s_j\right) : B_{r_j}(x_j) \to \mathbb{R}^{n-2}$ is an $\varepsilon_j$-splitting map. Note that we have renormalized so that each of our regular values is the zero level set.

Now let us consider the sequence $(M_j^n, r_j^{-1} d_j, x_j)$. After passing to a subsequence if necessary, which we will continue to denote by $(M_j^n, r_j^{-1} d_j, x_j)$, we have

$$(M_j^n, r_j^{-1} d_j, x_j) \xrightarrow{d_{\mathrm{GH}}} (X, d_X, x), \tag{5.6}$$

in the pointed Gromov-Hausdorff sense, where $X$ splits off $\mathbb{R}^{n-2}$ isometrically.

By our noncollapsing assumption we have $\mathrm{Vol}(B_1(x_j)) > c(n)\mathrm{v} > 0$, and hence, in the rescaled spaces, we have $\mathrm{Vol}(B_r(x_j)) > c\mathrm{v} r^n$ for all $r \le R_j \to \infty$. In particular, $X$ has Euclidean volume growth at $\infty$; i.e., $\mathrm{Vol}(B_r(x')) > c\mathrm{v}\, r^n$ for all $r > 0$.

After possibly passing to another subsequence, we can limit the functions $v_j$ to a function $v : X \to \mathbb{R}^{n-2}$. Note that by our normalization, we have that $v_j : B_2(x_j) \to \mathbb{R}^{n-2}$ are $\varepsilon_j$-splittings and by Theorem 1.32, we have for each $R > 2$ that the maps $v_j : B_R(x_j) \to \mathbb{R}^{n-2}$ are $C(n,R)\varepsilon_j$-splittings. In particular, we can conclude that

$$X = \mathbb{R}^{n-2} \times S, \tag{5.7}$$

where $v : X \to \mathbb{R}^{n-2}$ is the projection map and $S = u^{-1}(0)$.

Now by construction, in the rescaled spaces we have for any $y \in u_j^{-1}(0)$ that $r_h(y) \ge 1$. Therefore, the limit $X$ is $C^{1,\alpha} \cap W^{2,q}$ in a neighborhood of $u^{-1}(0)$, and hence $S = u^{-1}(0)$ is a nonsingular surface. Thus, since $X = \mathbb{R}^{n-2} \times S$, it follows that $X$ is at least a $C^{1,\alpha} \cap W^{2,q}$ manifold with $r_h \ge 1$. Since the Ricci curvature is uniformly bounded, in fact tending to zero, we have by the standard $\varepsilon$-regularity theorem that the convergence $(M_j^n, r_j^{-1} d_j, x_j) \to (X, d_X, x)$ is in $C^{1,\alpha} \cap W^{2,q}$. Because the convergence is in $C^{1,\alpha} \cap W^{2,q}$, we have that $r_h$ behaves continuously in the limit [And90]. In particular, we have $r_h(x'_j) \to r_h(x')$ and so, $r_h(x') = 1$.

On the other hand, since $|\mathrm{Ric}_{M_j^n}| \to 0$ and $X$ is $C^{1,\alpha} \cap W^{2,q}$, it follows that $X$ is a smooth Ricci flat manifold. This is easiest to see by writing things out in harmonic coordinates on $X$; see [And90] for the argument. Now since $X = \mathbb{R}^{n-2} \times S$, we can conclude that $S$ is smooth and Ricci flat, hence flat. In particular, we have that $X$ is flat. Since we have already shown that $X$ has Euclidean volume growth, this implies that $X = \mathbb{R}^n$. However, we have also already concluded that $r_h(x') = 1$, which gives us our desired contradiction. □

We end this subsection with the following corollary, which states that a noncollapsed limit space is smooth away from a set of codimension 3. We will use this in the next subsection to show $(n-3)$-symmetric splittings cannot arise as limits.

COROLLARY 5.8. *Let $(M_j^n, g_j, p_j)$ denote a sequence of Riemannian manifolds satisfying $|\mathrm{Ric}_{M_j^n}| \le n-1$, $\mathrm{Vol}(B_1(p_j)) > \mathrm{v} > 0$ and such that*

$$(M_j^n, d_j, p_j) \to (X, d, p). \tag{5.9}$$

*Then there exists a subset, $\mathcal{S} \subseteq X$, with $\dim \mathcal{S} \le n-3$, such that for each $x \in X \setminus \mathcal{S}$, we have $r_h(x) > 0$. In particular, $x \in X \setminus \mathcal{S}$ is a $C^{1,\alpha}$ Riemannian manifold.*

*Proof.* Recall the standard stratification of $X$. In particular, if we consider the subset $\mathcal{S}^{n-3} \subset X$, we have that $\dim \mathcal{S}^{n-3} \le n-3$ and that for every point $x \notin \mathcal{S}^{n-3}$, there exists *some* tangent cone at $x$ that is isometric to $\mathbb{R}^{n-2} \times C(S^1_\beta)$. That is, there exists $r_a \to 0$ such that

$$(X, r_a^{-1} d, x) \to \mathbb{R}^{n-2} \times C(S^1_\beta). \tag{5.10}$$

However by Theorem 5.2, we then have $\beta = 2\pi$, which is to say that

$$(X, r_a^{-1} d, x) \to \mathbb{R}^n. \tag{5.11}$$

Thus, for $a \in \mathbb{N}$ sufficiently large, we can apply the standard $\varepsilon$-regularity theorem, Theorem 2.11, to see that a neighborhood of $x$ is a $C^{1,\alpha}$ Riemannian manifold, which proves the corollary. □

5.2. *Nonexistence of codimension 3 singularities.* In this subsection we use the tools of Sections 4 and 5.1 in order to prove that $(n-3)$-symmetric metric spaces cannot arise as limits of manifolds with bounded Ricci curvature. Specifically, we prove the following

THEOREM 5.12 ($(n-3)$-Symmetric Limits). *Let $(M_j^n, g_j, p_j)$ denote a sequence of Riemannian manifolds satisfying $|\mathrm{Ric}_{M_j^n}| \to 0$, $\mathrm{Vol}(B_1(p_j)) > \mathrm{v} > 0$ and such that*

$$(M_j^n, d_j, p_j) \to \mathbb{R}^{n-3} \times C(Y), \tag{5.13}$$

*in the pointed Gromov-Hausdorff sense, where $Y$ is some compact metric space. Then $Y$ is isometric to the unit 2-sphere, and hence $\mathbb{R}^{n-3} \times C(Y) = \mathbb{R}^n$.*

*Proof.* Let us assume that this is not the case and study such a limit space $\mathbb{R}^{n-3} \times C(Y)$. The first observation is that by Corollary 5.8, it follows that $Y$ is a smooth surface. Indeed, if there were a point $y \in Y$ such that $r_h(y) = 0$, then since $X = \mathbb{R}^{n-3} \times C(Y)$, it would follow that there is a set of codimension at least 2 such that $r_h \equiv 0$, which cannot happen by Corollary 5.8.

Since $Y$ is a $C^{1,\alpha} \cap W^{2,q}$ manifold and $|\mathrm{Ric}_{M_j^n}| \to 0$, it follows that $Y$ is a smooth Einstein manifold satisfying $\mathrm{Ric}_Y = g$. Because $Y$ is a surface, this means, in particular, that $Y$ has constant sectional curvature $\equiv 1$. Thus, either $Y = \mathbb{RP}^2$ or $Y = S^2$, the unit 2-sphere, and in the latter case we are done.

So let us study the case $Y = \mathbb{RP}^2$. For $\varepsilon > 0$ small, choose $u_j : B_2(p_j) \to \mathbb{R}^{n-3}$ to be an $\varepsilon$-splitting as in Lemma 1.21. Note that away from the singular set, $\mathcal{S} \equiv \mathbb{R}^{n-3} \times \{0\}$, we have that the $M_j^n$ converge to $\mathbb{R}^{n-3} \times C(Y)$ in $C^{1,\alpha}$. If $f_j : B_2(p) \to B_2(p_j)$ denote the Gromov-Hausdorff maps, we put $\mathcal{S}_j = f_j(\mathcal{S})$. Then for $\tau > 0$ small but fixed, we have for $j$ sufficiently large that on $B_1(p) \setminus T_\tau(\mathcal{S}_j)$, the estimates $|\nabla u_j| > \frac{1}{2}$ and $|\nabla^2 u_j| \leq 1$ hold.

Consider Poisson approximation $h_j$ to the square of distance function $d^2(x, p_j)$ on $B_2(p_j)$. That is, $\Delta h_j = 2n$ and $h_j = 1$ on $\partial B_2(p_j)$. We have (see, for instance, [CC96]) that $|h_j - d(\cdot, p_j)| \to 0$ uniformly in $B_2(p_j)$, and again, because the convergence is in $C^{1,\alpha}$, we have for $j$ sufficiently large that $|\nabla h| > \delta$ and $|\nabla^2 h| \leq 4n$ on $B_1(p_j) \setminus B_\tau(\mathcal{S}_j)$. Once again, appealing to the $C^{1,\alpha}$ convergence, for all $j$ sufficiently large and all $s \in B_1(0^{n-3})$, we have that $u^{-1}(s) \cap h^{-1}(1)$ is diffeomorphic to $\mathbb{RP}^2$. By Sard's theorem, there exists a regular value $s_j \in B_1(0^{n-3})$. Then for $j$ sufficiently large, $u_j^{-1}(s_j) \cap \{h \leq 1\}$ is a smooth 3-manifold, whose boundary is diffeomorphic to $\mathbb{RP}^2$. However, the second Stiefel-Whitney number of $\mathbb{RP}^2$ is nonzero and, in particular, $\mathbb{RP}^2$ does not bound a smooth 3-manifold. This contradicts $Y = \mathbb{RP}^2$. □

5.3. *Proof of Hausdorff estimates of Theorem* 1.4. With Theorem 5.12 in hand, the proof of Theorem 1.4 becomes standard and follows the same lines as the proof of Corollary 5.8. Thus, consider a sequence

$$(M_j^n, d_j, p_j) \to (X, d, p) \tag{5.14}$$

of Riemannian manifolds satisfying $|\mathrm{Ric}_{M_j^n}| \leq n - 1$ and $\mathrm{Vol}(B_1(p_j)) > \mathrm{v} > 0$, which Gromov-Hausdorff converges to some $X$. Recall again the standard stratification of $X$, which is reviewed in Section 2.1. More specifically let us consider the closed stratum $\mathcal{S}^{n-4}(X) \subseteq X$. On the one hand, we have from [CC96] that

$$\dim \mathcal{S}^{n-4} \leq n - 4. \tag{5.15}$$

On the other hand, we have that for every point $x \notin \mathcal{S}^{n-4}$, there exists *some* tangent cone at $x$ that is isometric to $\mathbb{R}^{n-3} \times C(Y)$. That is, for some sequence $r_a \to 0$, we have

$$(X, r_a^{-1} d, x) \to \mathbb{R}^{n-3} \times C(Y). \tag{5.16}$$

However, by Theorem 5.12, we have that $Y$ is isometric to the unit 2-sphere, and hence,

$$(X, r_a^{-1} d, x) \to \mathbb{R}^n. \tag{5.17}$$

Then for $a \in \mathbb{N}$ sufficiently large, we can apply the standard $\varepsilon$-regularity theorem, Theorem 2.11, to see that $r_h(x) > 0$, and thus, that a neighborhood of $x$ is a $C^{1,\alpha}$ Riemannian manifold. This proves the theorem.

## 6. The $\varepsilon$-regularity theorem

In Section 5, we showed that limit spaces satisfying our assumptions must be smooth away from a closed subset of codimension 4. However, the strongest applications come from a more effective version of this statement. In particular, the curvature estimates of Theorem 1.8 and the Minkowski estimates of Theorem 1.4 will require a more rigid statement. Namely, in this section, we will prove the following

THEOREM 6.1. *There exists $\varepsilon(n, \mathrm{v}) > 0$ such that if $M^n$ satisfies $|\mathrm{Ric}_{M^n}| \leq \varepsilon$, $\mathrm{Vol}(B_1(p)) > \mathrm{v} > 0$ and*

$$d_{\mathrm{GH}}\big(B_2(p), B_2(0)\big) < \varepsilon, \tag{6.2}$$

*where 0 is a vertex of the cone $\mathbb{R}^{n-3} \times C(Y)$ for some metric space $Y$, then we have*

$$r_h(p) \geq 1. \tag{6.3}$$

*Consequently, if $M^n$ is Einstein, we have the bound*

$$\sup_{B_1(p)} |\mathrm{Rm}| \leq 1. \tag{6.4}$$

*Proof.* Given $n$ and $\mathrm{v} > 0$, assume no such $\varepsilon$ exists. Then there exists a sequence of spaces $(M_j^n, g_j, p_j)$ such that $|\mathrm{Ric}_{M_i^n}| \leq \varepsilon_j \to 0$, $\mathrm{Vol}(B_1(p_j)) > \mathrm{v} > 0$ and

$$d_{\mathrm{GH}}\big(B_2(p_j), B_2(0_j)\big) < \varepsilon_j \to 0, \tag{6.5}$$

where $0_j \in \mathbb{R}^{n-3} \times C(Y_j)$ is a vertex but $r_h(p_j) < 1$. After possibly passing to a subsequence,we have

$$B_2(p_j) \to B_2(0), \tag{6.6}$$

where $0 \in \mathbb{R}^{n-3} \times C(Y) \equiv X$ is a vertex. But if $C(Y)$ has any point with $r_h(x) = 0$, then there is a set of Hausdorff codimension 3 in $X$ that is not smooth. By the Hausdorff estimate of Theorem 1.4 this is not possible, so it must be that $C(Y)$ is smooth. Thus, $Y$ is a smooth manifold and, in fact, $C(Y)$ is itself be smooth if and only if $Y$ is the unit 2-sphere. Thus,

$$B_2(p_j) \to B_2(0^n) \subseteq \mathbb{R}^n. \tag{6.7}$$

Now we can apply the standard $\varepsilon$-regularity theorem to conclude $r_h(p_j) \geq 1$, which is a contradiction. □

## 7. Quantitative stratification and effective estimates

Having shown in Sections 5 and 6 that noncollapsed limits of Einstein manifolds are smooth away from a closed codimension 4 subset, we will now give some applications. In particular, we will use the ideas of quantitative stratification first introduced in [CN13] in order to improve the codimension estimates on singular sets of limit spaces to curvature estimates on Einstein manifolds. More precisely, in this section, we will prove Theorem 1.8. We will also improve the Hausdorff dimension estimate of Theorem 1.4 to a Minkowski dimension estimate. One can view this as an easy corollary of Theorem 1.8.

We begin here by reviewing the quantitative stratification and the main results on it from [CN13]. These will play a crucial role in our estimates. In Section 7.1 we combine the main results concerning the quantitative stratification, stated in Theorem 7.4, with the $\varepsilon$-regularity of Theorem 6.1 in order to prove the main estimates on Einstein manifolds given in Theorem 1.8. In Section 7.2 we apply the regularity results of Theorem 1.8 in order to conclude stronger results about the behavior of harmonic functions on Einstein manifolds.

The idea of [CN13] was to make the notion of stratification more effective. The standard stratification, recalled in Section 2.1, is used to show that most points have a lot of symmetry infinitesimally. The quantitative stratification is used to show that most balls of a definite size have a lot of approximate symmetry. In particular, the quantitative stratification introduced in [CN13] exists and gives nontrivial information even on a smooth manifold, whereas, on a smooth space, the standard stratification is always trivial. This point is crucial to the proof of Theorem 1.8. To make this precise we begin by defining a more local version of approximate symmetry.

*Definition* 7.1. Given a metric space $Y$ with $y \in Y$, $r > 0$, and $\varepsilon > 0$, we say that $y$ is $(k, \varepsilon, r)$-symmetric if there exists a $k$-symmetric space $Y'$ such that $d_{\mathrm{GH}}(B_r(y), B_r(y')) < \varepsilon r$, where $y' \in Y'$ is a vertex.

Recall from Definition 2.4 that $Y'$ is $k$-symmetric if $Y' = \mathbb{R}^k \times C(Z')$. To state the definition in words, we say that $Y$ is $(k, \varepsilon, r)$-symmetric if the ball $B_r(x)$ looks very close to having $k$-symmetries. The quantitative stratification is then defined as follows:

*Definition* 7.2. For each $\varepsilon > 0$, $0 < r < 1$, and $k \in \mathbb{N}$, define the closed quantitative $k$-stratum, $\mathcal{S}^k_{\varepsilon,r}(X)$, by

(7.3)
$$\mathcal{S}^k_{\varepsilon,r}(X) \equiv \{x \in X : \text{ for no } r \le s \le 1 \text{ is } x \text{ a } (k+1, \varepsilon, s)\text{-symmetric point}\}.$$

Thus, the closed stratum $\mathcal{S}^k_{\varepsilon,r}(X)$ is the collection of points such that no ball of size at least $r$ is almost $(k+1)$-symmetric. The first main result of

[CN13] is to show that for manifolds that are noncollapsed and have lower Ricci curvature bounds, the set $\mathcal{S}^k_{\varepsilon,r}(X)$ is small in a very strong sense. To say this a little more carefully, if one pretends that the $k$-stratum is a well-behaved $k$-dimensional submanifold, then one would expect the volume of the $r$-tube around the set to behave like $Cr^{n-k}$. Although we do not know this to be the case, the following slightly weaker statement does hold.

THEOREM 7.4 (Quantitative Stratification [CN13]). *Let $M^n$ satisfy* $\mathrm{Ric} \geq -(n-1)$ *with* $\mathrm{Vol}(B_1(p)) > \mathrm{v} > 0$. *Then for every $\varepsilon, \eta > 0$, there exists $C = C(n, \mathrm{v}, \varepsilon, \eta)$ such that*

$$\mathrm{Vol}\left(T_r\left(\mathcal{S}^k_{\varepsilon,r}(M) \cap B_1(p)\right)\right) \leq Cr^{n-k-\eta}. \tag{7.5}$$

*Remark* 7.6. In [CN13], the theorem is stated with $\varepsilon \equiv \eta$, however it is easily seen to be equivalent to the statement above.

7.1. *Proof of Theorem* 1.8. In this subsection we combine Theorems 6.1 and 7.4 in order to prove Theorem 1.8.

*Proof of Theorem* 1.8. Let $(M^n, g, p)$ satisfy $|\mathrm{Ric}_{M^n}| \leq n-1$ and $\mathrm{Vol}(B_1(p)) > \mathrm{v} > 0$. We will first show that for every $q < 2$, there exists $C = C(n, \mathrm{v}, q) > 0$ such that

$$⨍_{B_1(p)} r_h^{-2q} \leq C. \tag{7.7}$$

Simultaneously, we will show that if $M^n$ is Einstein, then this can be improved to

$$⨍_{B_1(p)} r_x^{-2q} \leq C, \tag{7.8}$$

where $r_x$ denotes the regularity scale at $x$.

Let $q < 2$ be fixed, set $\eta = 4 - 2q$, and let us consider Theorem 7.4 with $\varepsilon = \varepsilon(n) > 0$ chosen from Theorem 6.1 and $\eta$ as above. Thus, there exists $C(n, \mathrm{v}, q)$ such that

$$\mathrm{Vol}(T_r(\{x \in \mathcal{S}^{n-4}_{\varepsilon,2r} \cap B_1(p)\})) < Cr^{4-\eta}. \tag{7.9}$$

Note that by rescaling, we may regard the $\varepsilon$-regularity theorem (Theorem 6.1) as stating that if $x$ is $(n-3, \varepsilon, 2r)$-symmetric, then $r_h > r$, and if $M^n$ is Einstein then $r_x > r$. In fact, we have that if $x$ is $(n-3, \varepsilon, s)$-symmetric for any $s \geq 2r$, then $r_h > r$. Thus, if $x \notin \mathcal{S}^{n-4}_{\varepsilon,2r}$, then $r_h > \frac{s}{2} > r$. The contrapositive gives the inclusion

$$\{x \in B_1(p) : r_h \leq r\} \subseteq \mathcal{S}^{n-4}_{\varepsilon,2r} \cap B_1(p), \tag{7.10}$$

which by (7.9) implies the desired estimate

$$\mathrm{Vol}(T_r(\{x \in B_1(p) : r_h \leq r\})) < Cr^{4-\eta} \leq Cr^{2q}. \tag{7.11}$$

If $M^n$ is Einstein, then Theorem 6.1 allows us to replace $r_h$ with $r_x$, as claimed.

Now, for $q < 2$, let us prove the $L^q$-bound on the curvature from Theorem 1.8. For this, note that if $r_h(x) > r$, then by definition there exists a harmonic coordinate system $\Phi : B_r(0^n) \to M$ with $\phi(0) = x$ and such that

$$||g_{ij} - \delta_{ij}||_{C^0(B_r(0))} + r||\partial_k g_{i,j}||_{C^0(B_r(0))} < 10^{-3}, \tag{7.12}$$

where $g_{ij} = \Phi^* g$ is the pullback metric. Since the Ricci curvature satisfies the bound $|\mathrm{Ric}_{M^n}| \leq n-1$, this implies that

$$|\Delta_x g_{ij}| < C(n) r^{-2}, \tag{7.13}$$

where $\Delta_x$ denotes the Laplacian written in coordinates. Then for every $\alpha < 1$ and $s < \infty$, we have the scale invariant estimates

$$\begin{aligned} r^{1+\alpha}||\partial_k g_{ij}||_{C^\alpha(B_{\frac{3r}{4}}(0))} &\leq C(n,\alpha), \\ r^2||g_{i,j}||_{W^{2,s}(B_{\frac{3r}{4}}(0))} &\leq C(n,s). \end{aligned} \tag{7.14}$$

In particular, applying this to $s = q$, we get

$$r^{2q} \fint_{B_{r/2}(x)} |\mathrm{Rm}|^q \leq C(n) r^{2q} \fint_{B_{3r/4}(0)} |\Phi^* \mathrm{Rm}|^q < C(n,q). \tag{7.15}$$

Put $\eta = 2 - q$. Then $q + \frac{\eta}{2} < 2$. Then we have already shown that

$$\mathrm{Vol}(T_r(\{x \in B_1(p) : r_h \leq r\})) < C r^{2q+\eta} \tag{7.16}$$

for $C(n, \mathrm{v}, q) > 0$. Consider the covering $\{B_{r_h(x)}(x)\}$ of $B_1(p)$, and a subcovering $\{B_{r_j}(x_j)\}$ by mutually disjoint balls, such that

(1) $B_1(p) \subseteq \bigcup B_{r_j}(x_j)$ with $r_j = \frac{1}{2} r_h(x)$;
(2) $\{B_{r_j/4}(x_j)\}$ are disjoint.

By using (7.16), we see that for each $\alpha \in \mathbb{N}$, we have

$$\sum_{2^{-\alpha-1} < r_j \leq 2^{-\alpha}} \mathrm{Vol}(B_{r_j}(x_j)) \leq C r_j^{2q+\eta} = C\, r_j^{2q}\, 2^{-\eta\alpha}. \tag{7.17}$$

By summing over $\alpha$, this gives

$$\sum r_j^{-2q} \mathrm{Vol}(B_{r_j}(x_j)) \leq C \sum 2^{-\eta\alpha} \leq C(n, \mathrm{v}, q). \tag{7.18}$$

Finally, combining this with (7.15) we get

$$\begin{aligned} \fint_{B_1(p)} |\mathrm{Rm}|^q &\leq \mathrm{v}^{-1} \sum \int_{B_{r_j}(x_j)} |\mathrm{Rm}|^q \\ &\leq C(n, \mathrm{v}, q) \sum r_j^{-2q} \mathrm{Vol}(B_{r_j}(x_j)) \leq C(n, \mathrm{v}, q), \end{aligned} \tag{7.19}$$

which finishes the proof of Theorem 1.8. □

7.2. *$L^q$ estimates for harmonic functions on Einstein manifolds.* In this subsection we give some applications of Theorem 1.8. In particular, we obtain Sobolev bounds for harmonic functions and solutions of more general equations on manifolds with bounded Ricci curvature. As we have used repeatedly, given a lower bound on Ricci curvature, there is a definite $L^2$-bound on the Hessian of a harmonic function; see (1.27). However, the example of a rounded off 2-dimensional cone shows that one does not have definite $L^q$-bounds for any $q > 2$; see Example 2.14. In this subsection, we will see that the situation is better for noncollapsed spaces with bounded Ricci curvature. Namely, one can obtain $L^q$-bounds on the Hessians of such harmonic functions for all $q < 4$. More generally, we show the following

THEOREM 7.20. *For every $q < 4$, there exists $C = C(n, \mathrm{v}, q)$ such that if $M^n$ satisfies $|\mathrm{Ric}_{M^n}| \le n - 1$ and $\mathrm{Vol}(B_1(p)) > \mathrm{v} > 0$ and $u : B_2(p) \to \mathbb{R}$ satisfies*

$$|u| \le 1, \qquad |\Delta u| \le 1,$$

*then for every $q < 4$,*

$$\fint_{B_1(p)} |\nabla^2 u|^q \le C. \tag{7.21}$$

*Remark* 7.22. It follows from the $L^2$ curvature estimate (1.14) of Theorem 1.13 and [Che03] that in dimension 4, we actually have a full $L^4$-bound on $|\nabla^2 u|$. If Conjecture 9.1 is correct, then this holds in all dimensions.

*Proof.* Let us first note that by a Green's function estimate, we have

$$\sup_{B_{3/2}(p)} |\nabla u| \le C(n, v). \tag{7.23}$$

Indeed, for $x \in B_2(p)$, we can write

(7.24)

$$u(x) = h(x) + \int_{B_2(p)} G(x, y)\Delta(u - h)\, dv_g(y) = h(x) + \int_{B_2(p)} G(x, y)\Delta u\, dv_g(y),$$

where $h$ is a harmonic function with $h \equiv u$ on $\partial B_2(p)$. Standard estimates, see [SY94], on the Green's function on spaces with lower Ricci bounds gives us $|\nabla_x G(x, y)| \le C(n, v)d(x, y)^{1-n}$ in our domain, and since $h$ is a bounded harmonic function, we have $|\nabla h| \le C(n)$ on $B_{3/2}(p)$. Combining these gives us (7.23). In fact, with a little more work one can drop the volume dependence in the estimate, though it makes no difference for our purposes since this is not true for the Hessian estimate.

Now using Theorem 1.8 we know that for each $\varepsilon > 0$,

$$\mathrm{Vol}(T_r(\{x \in B_1(p) : r_h(x) \le r\})) \le C_\varepsilon(n, \mathrm{v}, \varepsilon)r^{4-\varepsilon}. \tag{7.25}$$

In particular, let us consider the sets

$$\mathcal{C}_\alpha \equiv \{x \in B_1(p) : r_\alpha \le r_h(x) \le r_{\alpha-1}\}, \tag{7.26}$$

where $r_\alpha \equiv 2^{-\alpha}$. For the set $\mathcal{C}_\alpha$, we have the cover $\{B_{r_\alpha}(x)\}_{x\in\mathcal{C}_\alpha}$. We can choose a finite subcovering $\{B_{r_\alpha/2}(x_i)\}_1^{N_\alpha}$ such that the balls $B_{r_\alpha/8}(x_i)$ are mutually disjoint. Using (7.25) we have

$$N_\alpha \le C_\varepsilon r_\alpha^{4-n-\varepsilon}. \tag{7.27}$$

On each ball $B_{r_\alpha/2}(x_j)$ we can use standard elliptic estimates along with the gradient bound $|\nabla u| \le C(n)$ to get the scale-invariant estimate

$$r_\alpha^q \fint_{B_{r_\alpha/2}(x_j)} |\nabla^2 u|^q \le C(n, \mathrm{v}, q) \tag{7.28}$$

for any $q < \infty$. In particular, if we choose $q < 4$ and pick $\varepsilon = \frac{4-q}{2}$, then we have

$$\int_{B_{r_\alpha/2}(x_j)} |\nabla^2 u|^q \le C(n, \mathrm{v})\, r_\alpha^{n-4+2\varepsilon}. \tag{7.29}$$

Combining this with (7.25) gives

$$\int_{\mathcal{C}_\alpha} |\nabla^2 u|^q \le C(n, \mathrm{v})\, r_\alpha^{n-4+2\varepsilon} \cdot N_\alpha \le C(n, \mathrm{v}, q)\, r_\alpha^{\varepsilon}. \tag{7.30}$$

Finally, by summing over $\mathcal{C}_\alpha$ we get the estimate

$$\int_{B_1(p)} |\nabla^2 u|^q \le C(n, \mathrm{v}, p)\lambda^q \sum_\alpha r_\alpha^\varepsilon = C \sum 2^{-\varepsilon\alpha} = C(n, \mathrm{v}, q), \tag{7.31}$$

as claimed. □

## 8. **Improved estimates in dimension 4**

In this section we apply the codimension 4 estimates of Theorem 1.4 to prove the finite diffeomorphism and $L^2$ curvature bounds of Theorems 1.13 and 1.12. In Section 8.1, we recall some necessary preliminaries. In Section 8.2, we use the codimension 4 estimate of Theorem 1.4 to prove the existence of *good* annuli that have curvature and harmonic radius control.

In Section 8.3 we first use this to show that in the noncollapsed situation, at any point, away from a definite number of scales, *every* annulus is good. We combine this with a counting argument, which plays the role of an effective version of the fact any infinite collection of points has a limit point, in order to prove the harmonic radius estimates of Theorem 1.8.

In Section 8.4 we prove the finite diffeomorphism statement of Theorem 1.12. Morally, the argument is quite similar to the one in [AC92], though it is designed to be more effective in nature. In fact, the argument in Section 8.4 is quite general and works for any collection of uniformly noncollapsed smooth

manifolds with bounded Ricci curvature, such that all Gromov-Hausdorff limits and blowups have only isolated singularities.

In Section 8.5, we give a local version of the finite diffeomorphism theorem. Our main application of this is to prove *a priori* $L^2$-estimates on the curvature on a noncollapsed 4-manifold with bounded Ricci curvature.

8.1. *Diffeomorphisms and harmonic radius.* Roughly speaking, to control the diffeomorphism type of a manifold up to a finite number of possibilities, it suffices to know that there exists an atlas with a definite number of charts, for which nonempty intersections of chart domains have a definite size, and for which the change of coordinate maps have a definite bound on their norms in a suitably strong topology. This type of result has a long history, going back to [Che70] in the context of bounded sectional curvature. In particular, control on the harmonic radius enables one to implement such an argument.

In this subsection we recall two theorems that will be used later. We refer the reader to the book [Pet98] for proofs of these statements. The first theorem states that when two manifolds with harmonic radius bounded from below are sufficiently Gromov-Hausdorff close, then they must be diffeomorphic.

THEOREM 8.1. *For every $\varepsilon > 0$, there exists $\delta = \delta(n, \varepsilon)$ such that the following holds. If $M_1^n$ and $M_2^n$ are Riemannian manifolds and $U_j \subset M_j$ are subsets such that $r_h(x) > r > 0$ for each $x \in U_j$ and*

$$d_{\mathrm{GH}}(B_r(U_1), B_r(U_2)) < \delta r,$$

*then there exist open sets $B_{r/2}(U_j) \subseteq U_j' \subseteq B_r(U_j)$ and a $C^2$ diffeomorphism $\Phi : U_1' \to U_2'$ such that*

$$||g_1 - \Phi^* g_2||_{C^0} < \varepsilon. \tag{8.2}$$

*If we further assume $|\mathrm{Ric}_{M_j^n}| \leq n - 1$, $j = 1, 2$, then $\Phi$ is in $C^{2,\alpha} \cap W^{3,q}$ for all $\alpha < 1$ and $q < \infty$, and in harmonic coordinates on $U_1'$ we have*

$$||g_1 - \Phi^* g_2||_{C^0} + r^{1+\alpha}||\partial_i \Phi^* g_2||_{C^\alpha} + r^2||\partial_i \partial_j \Phi^* g_2||_{L^q} \leq C(n, \alpha, q)\varepsilon. \tag{8.3}$$

The idea of the proof of Theorem 8.1 is to cover the set $U_1$ by harmonic charts $B_{r/2}(x_j)$ of definite size, the intersection of whose domains also have a definite size or are empty and such that each chart domain intersects at most a definite number of distinct chart domains. By restricting the Gromov-Hausdorff map $f : U_1 \to U_2$ to $U_1$, and using that the image of each ball $f(B_r(x_j))$ lies in a harmonic coordinate chart of $U_2$, we can construct a suitable smooth approximation of $f$. Then using the estimates of the local charts one can see this smoothing of $f$ is the required diffeomorphism.

In a related direction, we can use the harmonic radius to bound the number of possibilities for the diffeomorphism type of a manifold. Precisely, we have the following

THEOREM 8.4. *There exists $C = C(n, D)$ with the following property. Let $(M^n, g)$ denote a Riemannian manifold and $U \subseteq M$ a subset such that $r_h(x) > r > 0$ for all $x \in U$ and such that* $\operatorname{diam}(U) \leq D \cdot r$. *Then there exists an open set $U'$ with $T_{r/2}(U) \subseteq U' \subseteq T_r(U)$ such that $U'$ has at most one of $C$ diffeomorphism types.*

The idea of the proof of the above is that $U$ may be covered by a controlled number of harmonic charts $B_{r/2}(x_j)$ with suitable control as above on the intersections of their domains. The bounds on the metric in the coordinate charts, together with the fact that the coordinate functions are harmonic, yields suitable control over the transition functions between these charts. It follows that (up to a small ambiguity, which does not effect the diffeomorphism type) there are only a definite number of ways that this collection of charts can be pasted together.

8.2. *Annulus estimates.* In this section, we use Theorem 1.4 to prove our basic annulus estimates on 4-manifolds with bounded Ricci curvature. These estimates are the key first steps towards proving the finite diffeomorphism statements and the corresponding curvature estimates of Theorem 1.13. To state our main result for this subsection let us recall the volume ratio

$$\mathcal{V}_r^\delta(x) := -\ln\left(\frac{\operatorname{Vol}(B_r(x))}{\operatorname{Vol}(B_r(0^4_{-\delta}))}\right), \tag{8.5}$$

where $0^4_{-\delta}$ is a base point in the 4-dimensional hyperbolic space of constant curvature $-\delta$; by the Bishop-Gromov theorem, this ratio is monotone *increasing* for a manifold with Ricci curvature bounded from below $\operatorname{Ric}_{M^n} \geq -3\delta$. It has been understood since [CC96] that almost constancy of $\mathcal{V}_r^\delta(x)$ over a range of scales leads to cone behavior of the underlying metric space. Our main result of this subsection states that in the context of bounded Ricci curvature and dimension 4, almost constancy of this volume ratio leads to much stronger control up to diffeomorphism and pointwise geometric control.

THEOREM 8.6. *For every $\varepsilon > 0$, there exists $\delta(\mathrm{v}, \varepsilon) > 0$ such that if $M^4$ satisfies $|\operatorname{Ric}_{M^4}| \leq 3\delta$, $\operatorname{Vol}(B_1(p)) > \mathrm{v} > 0$ and $|\mathcal{V}_4^\delta(p) - \mathcal{V}_{1/4}^\delta(p)| < \delta$, then there exists a discrete subgroup $\Gamma \subseteq \mathrm{O}(4)$, unique up to conjugacy, with $|\Gamma| \leq N(\mathrm{v})$ such that the following hold*:

(1) *for each $x \in A_{\varepsilon,2}(p)$, we have the harmonic radius lower bound $r_h(x) > r_0(\mathrm{v})\varepsilon$;*
(2) *there exists a subset $A_{\varepsilon,2}(p) \subseteq U \subseteq A_{\varepsilon/2,2+\varepsilon}(p)$ and a diffeomorphism $\Phi : A_{\varepsilon,2}(0) \to U$, with $0 \in \mathbb{R}^4/\Gamma$, such that if $g_{ij} = \Phi^* g$ is the pullback metric, then*

$$||g_{ij} - \delta_{ij}||_{C^0} + ||\partial_k g_{ij}||_{C^0} < \varepsilon. \tag{8.7}$$

*Proof.* The proof is by contradiction. So let us assume for some $\varepsilon > 0$ there is no such $\delta(\mathrm{v}, \varepsilon) > 0$. Thus, we have a sequence of spaces $(M^4_j, g_j, p_j)$ with $\mathrm{Vol}(B_1(p_j)) > \mathrm{v} > 0$, $|\mathrm{Ric}_{M^4_j}| \leq \delta_j \to 0$, and $|\mathcal{V}_4(p_j) - \mathcal{V}_{1/4}(p_j)| < \delta_j \to 0$, but the conclusions of the theorem fail. After passing to a subsequence we can take a limit

$$(M^4_j, d_j, p_j) \xrightarrow{d_{\mathrm{GH}}} (X, d, p). \tag{8.8}$$

Using the fact that almost volume cones are almost metric cones [CC96], we then have

$$B_4(p) = B_4\big(y_0\big), \tag{8.9}$$

where $y_0 \in C(Y)$ is the cone vertex and $Y$ some metric space of diameter $\leq \pi$.

Now using Theorem 1.4, we know that away from a set of codimension 4 in $C(Y)$, the harmonic radius $r_h > 0$ is bounded uniformly from below. Assume there is some point $y \in Y$ such that $r_h(y) = 0$, and consider the ray $\gamma_y$ in $C(Y)$ through the point $y$. In that case, it would follow that for every point of $\gamma_y$, the harmonic radius $r_h = 0$ vanishes. The ray $\gamma$ has Hausdorff dimension 1, and therefore its existence would contradict Theorem 1.4. Thus, we conclude that $r_h > 0$ and that $Y = (Y, g_Y)$ is a $C^{1,\alpha} \cap W^{2,q}$ manifold for every $\alpha < 1$ and $q < \infty$.

Now by writing the formula for the Ricci tensor in harmonic coordinates and using $|\mathrm{Ric}_{M^4_j}| \to 0$, it follows that $C(Y)$ is smooth and Ricci flat away from the vertex. In particular, since $C(Y)$ is a metric cone over $Y$, we have $\mathrm{Ric}_{Y^3} = 3g^Y$. Since in dimension 3, constant Ricci curvature implies constant sectional curvature, it follows that $Y = S^3/\Gamma$ has constant sectional curvature $\equiv 1$. Additionally, we know from the volume bound, $\mathrm{Vol}(B_1(p)) > \mathrm{v} > 0$, that the order $|\Gamma| < N(\mathrm{v})$ is uniformly bounded. In particular, we have that $C(Y) = \mathbb{R}^4/\Gamma$ is an orbifold with an isolated singularity.

It now follows that there exists $r_0(\mathrm{v}) > 0$ such that for $y \in \mathbb{R}^4/\Gamma$ with $|y| = 1$, we have

$$B_{2r_0}(y) = B_{2r_0}(0^4), \tag{8.10}$$

where $0^4 \in \mathbb{R}^4$. In particular, for all $j$ sufficiently large, we have from the standard $\varepsilon$-regularity theorem, Theorem 2.11, that for all $x \in A_{\varepsilon,2}(p_j)$, the harmonic radius $r_h(x) > r_0(\mathrm{v}, \varepsilon) = r_0(\mathrm{v})\varepsilon$ is bounded uniformly from below independent of $j$. Thus, if there exists $\varepsilon$ as above, for which there is no $\delta(\mathrm{v}, \varepsilon)$, it must be (2) that fails to hold.

However, by again using the diffeomorphism statement of Theorem 8.1, we have that for $j$ sufficiently large, there exist diffeomorphisms

$$\Phi_j : A_{\varepsilon,2}(0) \to M^4_j, \tag{8.11}$$

such that

$$\Phi_j^* g_j \overset{C^{1,\alpha}\cap W^{2,q}}{\longrightarrow} dr^2 + r^2 g_Y. \tag{8.12}$$

For $j$ sufficiently large, this implies that (2) holds; a contradiction. □

8.3. *Regularity scale estimates.* In this subsection we prove the harmonic and regularity scale estimates (1.15) of Theorem 1.13. We know already from Theorem 1.4 that if $M^4 \to X$ is a limit space, then the singular set of $X$ has dimension zero. The estimate (1.15) may be viewed as an effective version of this statement. Indeed, (1.15) not only gives a bound on the number of singularities that can appear, but it gives a bound on the number of balls with large curvature concentration. Motivated by Theorem 8.6 and the constructions of [CN13], we begin with the following definition, which will be useful in subsequent sections as well.

*Definition* 8.13. Consider the scales $r_\alpha = 2^{-\alpha}$. For each $x \in M$, we associate the infinite tuple $T(x) \in \mathbb{Z}_2^{\mathbb{N}}$ defined by

$$T_\alpha(x) \equiv \begin{cases} 1 \text{ if } |\mathcal{V}^\delta_{4r_\alpha}(x) - \mathcal{V}^\delta_{r_\alpha/4}(x)| \geq \delta, \\ 0 \text{ if } |\mathcal{V}^\delta_{4r_\alpha}(x) - \mathcal{V}^\delta_{r_\alpha/4}(x)| < \delta. \end{cases}$$

We denote by $|T|(x) = \sum T_\alpha(x)$ the number of *bad* scales at $x \in M^4$.

*Remark* 8.14. The definition of $T(x)$ relies on a choice of $\delta > 0$. When we want to stress this, we will write $T^\delta(x)$, but otherwise we will suppress this dependence.

We begin with the following; see also [CN13] for the same statement in a more general context.

LEMMA 8.15. *Let* $\mathrm{Ric}_{M^4} \geq -3\delta$ *and* $\mathrm{Vol}(B_1(p)) > \mathrm{v} > 0$ *with* $\delta \leq 1$. *Then for each* $\delta' > 0$ *and* $x \in B_2(p)$, *there exist at most* $N(\mathrm{v}, \delta')$ *scales* $\alpha \in \mathbb{N}$ *such that equation*

$$\left|\mathcal{V}^\delta_{r_{\alpha+1}}(x) - \mathcal{V}^\delta_{r_\alpha}(x)\right| > \delta'. \tag{8.16}$$

*Proof.* For $x \in B_2(p)$ fixed, we have

$$\mathrm{Vol}(B_1(x)) \geq C(n)^{-1}\mathrm{Vol}(B_3(x)) \geq C^{-1}\mathrm{Vol}(B_1(p)) \geq C^{-1}\mathrm{v} > 0, \tag{8.17}$$

and so

$$\mathcal{V}^\delta_1(x) \leq -\ln\left(C^{-1}\mathrm{v}\right) = C(n, \mathrm{v}). \tag{8.18}$$

The monotonicity of $\mathcal{V}_r^\delta(x)$ gives

$$C(n,\mathrm{v}) - 1 \geq \mathcal{V}_1^\delta(x) - \mathcal{V}_0^\delta(x) = \sum \left(\mathcal{V}_{r_\alpha}^\delta(x) - \mathcal{V}_{r_{\alpha+1}}^\delta(x)\right) = \sum \left|\mathcal{V}_{r_\alpha}^\delta(x) - \mathcal{V}_{r_{\alpha+1}}^\delta(x)\right|. \tag{8.19}$$

In particular, there are at most $N = C(n,\mathrm{v})(\delta')^{-1}$ elements $\alpha \in \mathbb{N}$ such that

$$\left|\mathcal{V}_{r_\alpha}(x) - \mathcal{V}_{r_{\alpha+1}}(x)\right| > \delta', \tag{8.20}$$

as claimed. □

Let us point out the following useful corollary.

COROLLARY 8.21. *Let $M^4$ satisfy $\mathrm{Ric}_{M^4} \geq -3\delta$ and $\mathrm{Vol}(B_1(p)) > \mathrm{v} > 0$. Then for each $x \in B_2(p)$, we have*

$$|T^\delta|(x) \leq N(\mathrm{v},\delta). \tag{8.22}$$

*Proof.* Put $\delta' = \delta/3$. Then for $x \in B_2(p)$, there are at most $N(\mathrm{v},\delta) = C(\mathrm{v})\delta^{-1}$ scales $\alpha$ for which

$$\left|\mathcal{V}_{r_\alpha}(x) - \mathcal{V}_{r_{\alpha+1}}(x)\right| > \frac{\delta}{3}. \tag{8.23}$$

Hence, there are at most $3N$ elements $\alpha \in \mathbb{N}$ such that

$$\left|\mathcal{V}_{r_\beta}(x) - \mathcal{V}_{r_{\beta+1}}(x)\right| > \frac{\delta}{3} \tag{8.24}$$

for some $\beta \in \{\alpha - 1, \alpha, \alpha + 1\}$. Therefore, for all other $\alpha$, we must have

$$\left|\mathcal{V}_{4r_\alpha}(x) - \mathcal{V}_{r_\alpha/4}(x)\right| < \delta, \tag{8.25}$$

which proves the corollary. □

*Remark* 8.26. In fact, both the lemma and the corollary work in all dimensions.

We end this subsection with a proof of the regularity scale estimate (1.15) from Theorem 1.13. One can view the proof as an effective version of the fact that an infinite collection of points must have a limit point.

*Proof of estimate* (1.15) *of Theorem* 1.13. Let $M^n$ satisfy $|\mathrm{Ric}_{M^4}| \leq 3$ and $\mathrm{Vol}(B_1(p)) > \mathrm{v} > 0$. We will prove the estimate for the harmonic radius $r_h$. The same argument works in the Einstein case to control the regularity scale.

So let $0 < \varepsilon \ll 1$ be fixed with $\delta(n,\varepsilon)$ chosen to satisfy Theorem 8.6. Consider the set

$$\{x \in B_1(p) : r_h(x) < r\}. \tag{8.27}$$

In view of the doubling condition implied by the Bishop-Gromov inequality, we have by a standard construction that there exists a covering $\{B_r(x_j)\}_1^N$ with

$$\{x \in B_1(p) : r_h(x) < r\} \subseteq \bigcup_j B_r(x_j) \tag{8.28}$$

but such that $\{B_{r/4}(x_j)\}$ are disjoint. Such coverings, which we will term "efficient," will be constructed several times below. Note that

$$T_r\big(\{x \in B_1(p) : r_h(x) < r\}\big) \subseteq \bigcup_j B_{2r}(x_j), \tag{8.29}$$

and thus

$$\mathrm{Vol}\Big(T_r\big(\{x \in B_1(p) : r_h(x) < r\}\big)\Big) \leq \sum_1^N \mathrm{Vol}(B_{2r}(x_j)) \leq C(n)N \cdot r^4. \tag{8.30}$$

Hence, our goal is to control the number of balls $N$ in the covering. Denote by $\mathcal{C} \equiv \{x_j\}_1^N$ the corresponding collection of centers.

Now note the following. If $x_j$ is one of our ball centers and $T_\alpha(x_j) = 0$, then by Theorem 8.6, we have for every $x \in A_{r_\alpha/2,2r_\alpha}(x_j)$ that $r_h(x) > \bar{r}(\mathrm{v}) \cdot r_\alpha$. In particular, if $r_\alpha > \bar{r}^{-1}r$, this implies that

$$x_k \notin A_{r_\alpha/2,2r_\alpha}(x_j), \tag{8.31}$$

for any other ball center $x_k$.

Now let us inductively build a sequence of decreasing subsets $\mathcal{C}^{k+1} \subseteq \mathcal{C}^k \subseteq \cdots \subseteq \mathcal{C}$ and associated radii $s_k = r_{\alpha_k} > 0$ with $\mathrm{diam}(\mathcal{C}^k) < 4s_k$. There are three key inductive properties that will be proved about these sets:

(1) there exists $C(n) > 0$ such that the cardinality of $\mathcal{C}^k$ satisfies

$$\Big|\#\mathcal{C}^k\Big| \geq C^{-k}\Big|\#\mathcal{C}\Big| = C^{-k}N; \tag{8.32}$$

(2) for every $x_j^k \in \mathcal{C}^k$, we have

$$\sum_{0 \leq \alpha \leq \alpha_k} T_\alpha^\delta(x_j^k) \geq k; \tag{8.33}$$

(3) if $\Big|\#\mathcal{C}^k\Big| > 1$ and $s_k > \bar{r}^{-1}r$, then $\mathcal{C}^{k+1} \neq \emptyset$.

Before constructing the sequence of sets, let us see that once the construction is complete, we will have proved our desired estimate on $N$. Indeed, let $k$ be the largest index such that $\mathcal{C}^k \neq \emptyset$. By the third property we must have either $|\#\mathcal{C}^k| = 1$ or $s_k \leq \bar{r}^{-1}r$, at which point we get by a covering argument that $|\#\mathcal{C}^k| < C(n)$. By Lemma 8.15 and the second property, we have that $k \leq k(n, \mathrm{v}, \delta) = k(n, \mathrm{v})$, and thus by the first property, we have

$$N \leq C(n)^{k(n,\mathrm{v})} \cdot |\#\mathcal{C}^k| \leq C(n, \mathrm{v}), \tag{8.34}$$

which proves the result.

Now let $\mathcal{C}^0 \equiv \mathcal{C}$ with $s_0 = 1$. Clearly, the inductive properties hold for $\mathcal{C}^0$. Assume we have built $\mathcal{C}^k \subseteq \mathcal{C}$ with $s_k > 0$ satisfying the inductive properties, and let us build $\mathcal{C}^{k+1}$. First note that if $|\#\mathcal{C}^k| = 1$ or $s_k \leq \bar{r}^{-1} r$, then we let $\mathcal{C}^{k+1} = \emptyset$. Our construction will otherwise give us a nonempty $\mathcal{C}^{k+1}$, so that the third inductive property will automatically be satisfied. So let us denote $s'_k = \mathrm{diam}(\mathcal{C}^k) \cdot 2^{-10}$. Choose an efficient covering $\{B_{s'_k}(x_j^k)\}$, where $x_j^k \in \mathcal{C}^k$, so that the balls in $\{B_{s'_k/4}(x_j^k)\}$ are disjoint. Note that because $\mathrm{diam}(\mathcal{C}^k) < 4s_k$, the usual doubling estimates imply that there are at most $C(n)$ balls in this covering. We choose the ball $B_{s'_k}(y)$ such that $\mathcal{C}^k \cap B_{s'_k}(y)$ has the largest cardinality of any ball from the covering. Then we define $\mathcal{C}^{k+1} = \mathcal{C}^k \cap B_{s'_k}(y)$.

By our choice of ball, $B_{s'_k}(y)$, we have

$$\left|\#\mathcal{C}^{k+1}\right| = \left|\#\mathcal{C}^k \cap B_{s'_k}(y)\right| \geq C(n)^{-1}\left|\#\mathcal{C}^k\right| \geq C^{-(k+1)}N, \tag{8.35}$$

so that $\mathcal{C}^{k+1}$ satisfies the first inductive property. To find $s_{k+1}$ and prove the second inductive property, let us define the following. For each $x_j^{k+1} \in \mathcal{C}^{k+1}$, if

$$T^{\delta}_{\alpha_k+7}(x_j^{k+1}) = 1, \tag{8.36}$$

then let us set $\beta_j = \alpha_k + 7$, and otherwise let $\beta_j \geq \alpha_k + 8$ be the largest integer such that $T^{\delta}_{\beta_j-1}(x_j^{k+1}) = 0$ but $T^{\delta}_{\beta_j}(x_j^{k+1}) = 1$. Note that $B_{s'_k}(y) \subseteq B_{r_{\alpha_k+7}}(x_j^k)$. Let $\alpha_{k+1} \equiv \max\{\beta_j, \lceil -\log_2\left(\bar{r}r^{-1}\right)\rceil\}$ with $x^{k+1} \in \mathcal{C}^{k+1}$ the associated element that attains the maximum, and note by (8.31) that

$$\mathcal{C}^{k+1} = \mathcal{C}^k \cap B_{s'_k}(y) = \mathcal{C}^k \cap B_{2^{-\alpha_{k+1}+1}}(x^{k+1}). \tag{8.37}$$

In particular, with $s_{k+1} = r_{\alpha_{k+1}}$, then $\mathrm{diam}(\mathcal{C}^{k+1}) < 4s_{k+1}$, and the second inductive property holds, which completes the induction step of the construction, and hence the proof. □

8.4. *Finite diffeomorphism type.* In this subsection we will prove Theorem 1.12 and give some refinements that will be useful for the $L^2$ curvature estimate of Theorem 1.13.

We begin by associating to a good scale the subgroup of O(4) occurring in Theorem 8.6.

*Definition* 8.38. Let $\varepsilon, \delta > 0$ be such that Theorem 8.6 holds. For $x \in B_1(p)$ and $\alpha \in \mathbb{N}$ such that $T^{\delta}_{\alpha}(x) = 0$, we denote by $[\Gamma_\alpha(x)] \subseteq O(4)$ the conjugacy class of the discrete subgroups arising from Theorem 8.6.

In the sequel, $\Gamma_\alpha$ will denote some arbitrary element of $[\Gamma_\alpha]$; only the isometry class of of $S^3/\Gamma_\alpha$, which is independent of the particular choice, is significant.

The following is the key *neck* lemma for our finite diffeomorphism of Theorem 1.12. In essence, the proof of Theorem 1.12 will come from decomposing

$M$ into a finite number of distinct pieces. What we are referring to informally as the *neck* regions will be diffeomorphic to cylinders $\mathbb{R} \times S^3/\Gamma$. They will connect the pieces that will be referred to as *body* regions.

LEMMA 8.39. *For every* $0 < \varepsilon \leq \varepsilon(\mathrm{v})$, *there exists* $\delta = \delta(\mathrm{v}, \varepsilon)$ *with the following properties. Let* $M^4$ *satisfy* $|\mathrm{Ric}_{M^4}| \leq 3\delta$ *and* $\mathrm{Vol}(B_1(p)) > \mathrm{v} > 0$. *Let* $x \in B_1(p)$, *and assume* $\alpha_1 \in \mathbb{N}$ *satisfies* $T^{\delta}_{\alpha_1}(x) = 0$ *with* $\Gamma_{\alpha_1}$ *the corresponding group. Then if* $\alpha_2 \in \mathbb{N}$ *is such that* $\mathcal{V}^{\delta}_{r_{\alpha_2}/4}(x) \geq \ln \left|\Gamma_{\alpha_1}\right| - \delta$, *there exists a subset* $A_{r_{\alpha_2}/2,2r_{\alpha_1}}(x) \subseteq U \subseteq A_{(1-\varepsilon)r_{\alpha_2}/2,2(1+\varepsilon)r_{\alpha_1}}(x)$ *and a diffeomorphism* $\Phi : A_{r_{\alpha_2}/2,2r_{\alpha_1}}(0) \to U$, *where* $0 \in \mathbb{R}^4/\Gamma_{\alpha_1}$, *such that if* $g_{ij} = \Phi^* g$ *is the pullback metric, we have*

$$||g_{ij} - \delta_{ij}||_{C^0(A_{r_\alpha/2,r_{2\alpha}})} + r_\alpha ||\partial_k g_{ij}||_{C^0(A_{r_\alpha/2,2r_\alpha})} < \varepsilon. \tag{8.40}$$

*Proof.* We will fix $\varepsilon(\mathrm{v}) > 0$ later. For the moment, let any $\varepsilon_1 > 0$ be arbitrary, with $\delta_1(\mathrm{v}, \varepsilon_1) > 0$ the corresponding number from Theorem 8.6. If $T^{\delta_1}_{\alpha_1}(x) = 0$, then there exists a diffeomorphism

$$\Phi_{\alpha_1} : A_{r_{\alpha_1}/2,2r_{\alpha_1}}(0) \to U_{\alpha_1}, \tag{8.41}$$

where $0 \in \mathbb{R}^4/\Gamma_{\alpha_1}$ and $A_{r_{\alpha_1}/2,2r_{\alpha_1}}(x) \subseteq U_\alpha \subseteq A_{(1-\varepsilon)r_{\alpha_1}/2,(1+\varepsilon)r_{\alpha_1}}(x)$, such that

$$||\Phi^*_{\alpha_1} g_{ij} - \delta_{ij}||_{C^0(A_{r_{\alpha_1}/2,2r_{\alpha_1}})} + r_{\alpha_1} ||\partial_k \Phi^*_{\alpha_1} g_{ij}||_{C^0(A_{r_{\alpha_1}/2,2r_{\alpha_1}})} < \varepsilon_1. \tag{8.42}$$

In particular, if $\varepsilon > 0$ is fixed and $2\delta(n, \varepsilon)$ is the corresponding number from Theorem 8.6, then we can choose $\varepsilon_1 = \varepsilon_1(\varepsilon, \mathrm{v})$ sufficiently small so that

$$\mathcal{V}^{\delta}_{r_{\alpha_1}}(x) < \ln |\Gamma_{\alpha_1}| + \delta. \tag{8.43}$$

Thus, if $\alpha_2$ is such that

$$\mathcal{V}^{\delta}_{r_{\alpha_2}/2}(x) \geq \ln |\Gamma_{\alpha_1}| - \delta, \tag{8.44}$$

then for all $\alpha_1 \leq \alpha \leq \alpha_2$, we have $T^{2\delta}_{\alpha}(x) = 0$.

By Theorem 8.6, there exists for each $\alpha_1 \leq \alpha \leq \alpha_2$ a diffeomorphism

$$\Phi_\alpha : A_{r_\alpha/2,2r_\alpha}(0) \to U_\alpha, \tag{8.45}$$

where $0 \in \mathbb{R}^4/\Gamma_\alpha$ and $A_{r_\alpha/2,2r_\alpha}(x) \subseteq U_\alpha \subseteq A_{(1-\varepsilon)r_\alpha/2,2(1+\varepsilon)r_\alpha}(x)$, such that

$$||\Phi^*_\alpha g_{ij} - \delta_{ij}||_{C^0(A_{r_\alpha/2,2r_\alpha})} + r_\alpha ||\partial_k \Phi^*_\alpha g_{ij}||_{C^0(A_{r_\alpha/2,2r_\alpha})} < \varepsilon. \tag{8.46}$$

In particular, this implies that $\Gamma_\alpha = \Gamma$ can be chosen independent of $\alpha$.

Next we focus on the inverse maps

$$\Phi^{-1}_\alpha : U_\alpha \to A_{r_\alpha/2,2r_\alpha}(0). \tag{8.47}$$

Observe that by (8.46), after possibly composing $\Phi_\alpha$ with a rotation of $\mathbb{R}^4/\Gamma$ we can assume for $x \in U_\alpha \cap U_\beta$ that

$$|\Phi^{-1}_\alpha(x) - \Phi^{-1}_\beta(x)| < \varepsilon r_\alpha. \tag{8.48}$$

Now let $\varepsilon < \varepsilon(\mathrm{v})$ be sufficiently small, so that if $x \in \mathbb{R}^4/\Gamma$, then $B_{\varepsilon|x|}(x) \subseteq \mathbb{R}^4/\Gamma$ is isometric to the standard Euclidean ball $B_{\varepsilon|x|}(0^4) \subseteq \mathbb{R}^4$. Note, in particular, that if $\{x_i\} \in B_{\varepsilon|x|}(x)$ is a collection of points, then any convex combination is well defined.

For each $\alpha$, let $\varphi'_\alpha : U_\alpha \to \mathbb{R}$ be a smooth cutoff function such that

$$\varphi'_\alpha(x) = \begin{cases} 1 \text{ if } x \in A_{3r_\alpha/8,15r_\alpha/8}(x), \\ 0 \text{ if } x \notin A_{r_\alpha/2,2r_\alpha}(x) \end{cases}$$

and such that $|\nabla\varphi'_\alpha| \leq 10r_\alpha^{-1}$. If we set $\varphi'(x) = \sum_\alpha \varphi'_\alpha(x)$, then $1 \leq \varphi'(x) \leq 4$. In particular,

$$\varphi_\alpha = \frac{\varphi'_\alpha(x)}{\varphi'(x)} : U_\alpha \to \mathbb{R} \tag{8.49}$$

satisfies $\sum \varphi_\alpha(x) = 1$ and so is a partition of unity, with $|\nabla\varphi_\alpha| \leq 40r_\alpha^{-1}$.

Define the map

$$\Phi^{-1} : U = \bigcup_\alpha U_\alpha \to A_{r_{\alpha_2}/2,2r_{\alpha_1}}(0) \tag{8.50}$$

given by

$$\Phi^{-1}(x) = \sum_\alpha \varphi_\alpha(x)\Phi_\alpha^{-1}(x). \tag{8.51}$$

(As previously noted, the convex combination is well defined since the $\Phi_\alpha^{-1}(x)$ all live in a ball that is isometric to a Euclidean ball.) On each domain, $U_\alpha$, we have by (8.42) and (8.48) that $\Phi^{-1}$ and $\Phi_\alpha^{-1}$ are $C^1$-close. Hence, $\Phi^{-1}$ is a diffeomorphism, and a quick computation using (8.42) and (8.48) verifies the desired estimates:

$$||\Phi^* g_{ij} - \delta_{ij}||_{C^0(A_{r_\alpha/2,2r_\alpha})} + r_\alpha||\partial_k\Phi_\alpha^* g_{ij}||_{C^0(A_{r_\alpha/2,2r_\alpha})} < C\varepsilon. \tag{8.52}$$

By choosing $\varepsilon$ appropriately small, we complete the proof. □

The following lemma could be termed a "gap lemma." It will be used to tell us that if we consider two distinct neck regions, then the complexity of the smaller neck region must be strictly less than that of the larger neck region.

LEMMA 8.53. *For each $\delta < \delta(\mathrm{v})$, there exists $\bar\alpha(\delta, \mathrm{v})$ with the following property. If $|\mathrm{Ric}_{M^4}| \leq 3\delta$, $\mathrm{Vol}(B_1(p)) > \mathrm{v} > 0$, and $\mathcal{V}^\delta_{r_1}(x) < \ln N - \delta$ for some $N \in \mathbb{N}$ and $x \in B_1(p)$, then we have*

$$\mathcal{V}^\delta_{r_{\bar\alpha}}(x) < \ln\Big(N - 1\Big) + \delta.$$

*Proof.* First note by Theorem 8.6 that if $\delta$ is fixed, then there exists $\delta'(\mathrm{v}, \delta)$ such that if $|\mathrm{Ric}| \leq 3\delta'$ and if $T_0^{\delta'} = 0$, then

$$\Big|\mathcal{V}_1^{\delta'}(x) - \ln|\Gamma_0|\Big| < \delta. \tag{8.54}$$

By rescaling this inequality, we see that in the context of this lemma, the following holds. If $x \in B_1(p)$, $\alpha > \bar{\alpha}(v,\delta)$ and

$$\left|\mathcal{V}^{\delta}_{4r_\alpha}(x) - \mathcal{V}^{\delta}_{r_\alpha/4}(x)\right| < \delta', \tag{8.55}$$

then we have

$$\left|\mathcal{V}^{\delta}_{r_\alpha}(x) - \ln|\Gamma_\alpha|\right| < \delta. \tag{8.56}$$

In particular, for $x \in B_1(p)$, we can apply Lemma 8.15 to see that there exists a scale $\alpha \le \bar{\alpha}(v,\delta)$ such that

$$\left|V^{\delta}_{r_\alpha}(x) - \mathcal{V}^{\delta}_{r_{\alpha+1}}(x)\right| < \delta', \tag{8.57}$$

and hence

$$\left|\mathcal{V}^{\delta}_{r_\alpha}(x) - \ln|\Gamma_\alpha|\right| < \delta. \tag{8.58}$$

However, if

$$\mathcal{V}^{\delta}_{r_\alpha}(x) < \ln N - \delta, \tag{8.59}$$

this implies $|\Gamma_\alpha| < N$, which completes the proof. □

In Lemma 8.39 we have built the required structure for constructing the *neck* regions of our decomposition. What is left is to build the *body* regions of the decomposition. The following lemma will be applied in the proof of Theorem 1.12 in order to construct the various body regions.

LEMMA 8.60. *For every $\delta > 0$, there exists $r_0(\mathrm{v},\delta), N(\mathrm{v},\delta) > 0$ with the following properties. Let $M^4$ satisfy $|\mathrm{Ric}_{M^4_j}| \le 3\delta$, $\mathrm{Vol}(B_1(p)) > \mathrm{v} > 0$. Then there exist points $\{x_j\}_1^N$ with $N \le N(\mathrm{v},\delta)$, and scales $\alpha_j \in \mathbb{N}$ with $r_j \equiv r_{\alpha_j} > r_0$, such that*

(1) $T^{\delta}_{\alpha_j}(x_j) = 0$;
(2) *if $x \in B_1(p) \setminus \bigcup_j B_{r_j}(x_j)$, then $r_h(x) > r_0$;*
(3) *if $\beta_j \in \mathbb{N}$ denotes the largest integer such that $\mathcal{V}^{\delta}_{r_{\beta_j}/4}(x_j) \ge \ln|\Gamma_j| - \delta$, then for every $x \in B_{2r_{\beta_j}}(x_j)$, we have*

$$\mathcal{V}^{\delta}_{r_{\beta_j}/8}(x) < \ln|\Gamma_j| - \delta. \tag{8.61}$$

*Proof.* Let $\delta > 0$ be chosen, with $\delta'(\mathrm{v},\delta)$ to be chosen later. Note that by Lemma 8.15, for each $x \in B_1(p)$, there exists $\alpha_x \le \bar{\alpha}(v,\delta')$ such that $T^{\delta'}_{\alpha_x}(x) = 0$. Consider the covering $\{B_{r_{\alpha_x}}(x)\}$ of $B_1(p)$, and choose an efficient subcovering $\{B_{r_j}(x'_j)\}_1^N$, where $r_j = r_{\alpha_{x'_j}}$ and the balls in $\{B_{r_j/4}(x'_j)\}$ are disjoint. The usual doubling arguments imply that $N \le N(v,\delta')$.

By Theorem 8.6, if we are given $\varepsilon > 0$, then we can choose $\delta'(\mathrm{v},\varepsilon,\delta)$ such that for each $x \in B_{\varepsilon r_j}(x'_j)$, we have $T^{\delta}_{\alpha_j}(x) = 0$ while for each $x \in A_{\varepsilon r_j, 2r_j}(x_j)$, we have $r_h(x) > \bar{r}(v,\varepsilon)r_j \ge r_0(v,\varepsilon,\delta')$. Let $\Gamma_j$ be the group associated to

$B_{r_j}(x'_j)$, and for each $x \in B_{\varepsilon r_j}(x'_j)$, let $\beta_j(x)$ be the largest integer such that $V^{\delta}_{r_{\beta_j}/4}(x) \geq \ln|\Gamma_j| - \delta$. Let $\beta_j = \max \beta_j(x)$ with $x_j$ the corresponding point. Note that for $\varepsilon(v,\delta)$ sufficiently small, we have $B_{2r_{\beta_j}}(x_j) \subseteq B_{\varepsilon r_j}(x'_j)$ and, in particular, for every $x \in B_{2r_{\beta_j}}(x_j)$,

$$V^{\delta}_{r_{\beta_j}/8}(x) < \ln|\Gamma_j| - \delta. \tag{8.62}$$

Consider the collection of balls $\{B_{r_j}(x_j)\}$. Clearly, by construction, conditions (1) and (3) are satisfied. If $x \in B_1(p)\setminus\{B_{r_j}(x_j)\}$, then since $\{B_{2r_j}(x_j)\}$ cover $B_1(p)$, we have that for some $x_j$, $x \in A_{r_j,2r_j}(x_j)$, which implies $r_h(x) \geq r_0(v,\delta)$, as claimed. □

By the previous lemma, the regions between necks, i.e., $B_1(p)\setminus\bigcup_j B_{r_{\alpha_j}}(x_j)$, can be written as the union of a definite number of balls of definite size, on which there is definite geometric control.

We are nearly in a position to prove Theorem 1.12. To do so we will in fact prove the following stronger result, which is the *bubble tree* decomposition of $M^4$.

THEOREM 8.63. *Let $M^4$ satisfy $|\mathrm{Ric}_{M^4}| \leq 3$, $\mathrm{Vol}(M) \geq \mathrm{v} > 0$ and $\mathrm{diam}(M) \leq D$. Then there exists a decomposition of $M^4$,*

$$M^4 = \mathcal{B}^1 \cup \bigcup_{j_2=1}^{N_2} \mathcal{N}^2_{j_2} \cup \bigcup_{j_2=1}^{N_2} \mathcal{B}^2_{j_2} \cup \cdots \cup \bigcup_{j_k=1}^{N_k} \mathcal{N}^k_{j_k} \cup \bigcup_{j_k=1}^{N_k} \mathcal{B}^k_{j_k}, \tag{8.64}$$

*into open sets that satisfy the following*:

(1) *if $x \in \mathcal{B}^\ell_j$, then $r_h(x) > r_0(n,\mathrm{v},D)\cdot \mathrm{diam}(\mathcal{B}^\ell_j)$;*
(2) *each neck $\mathcal{N}^\ell_j$ is diffeomorphic to $\mathbb{R}\times S^3/\Gamma^\ell_j$ for some $\Gamma^\ell_j \subset O(4)$;*
(3) *$\mathcal{N}^\ell_j \cap \mathcal{B}^\ell_j$ is diffeomorphic to $\mathbb{R}\times S^3/\Gamma^\ell_j$;*
(4) *$\mathcal{B}^{\ell-1}_{j'} \cap \mathcal{N}^\ell_j$ are either empty or diffeomorphic to $\mathbb{R}\times S^3/\Gamma^\ell_j$;*
(5) *$N_\ell \leq N(\mathrm{v},D)$ and $k \leq k(\mathrm{v},D)$.*

*Proof.* Let us remark first that if $p \in M^n$, then by volume ratio monotonicity, we have for every $r \leq 1$ that

$$\begin{aligned}\mathrm{Vol}(B_r(p)) &\geq \frac{\mathrm{Vol}_{-1}(B_r)}{\mathrm{Vol}_{-1}(B_D)}\mathrm{Vol}(B_D(p)) \\ &\geq C(n,D)^{-1}\mathrm{Vol}(M^4)r^n \geq C^{-1}\mathrm{v}r^n =: \mathrm{v}'r^n.\end{aligned} \tag{8.65}$$

Let $\varepsilon < \varepsilon(\mathrm{v}')$ from Lemma 8.39 with $\delta(\mathrm{v},D,\varepsilon)$ sufficiently small to satisfy Theorem 8.6 and Lemmas 8.39, 8.53, and 8.60. After rescaling, it is sufficient to consider a Riemannian manifold $(M^4,g)$ with $|\mathrm{Ric}_{M^4}| \leq 3\delta$, $\mathrm{diam}(M) \leq D\delta^{-2} = D'$, and $\mathrm{Vol}(B_1(p)) > \mathrm{v}' > 0$ for every $p \in M$.

Let us begin by efficiently covering $M^4$ by balls $\{B_1(x^0_j)\}$ such that the balls in $\{B_{1/4}(x^0_j)\}$ are disjoint. By the usual doubling argument, there are at

most $N(n, D, \mathrm{v})$ such balls. For each such ball, we apply Lemma 8.60 in order to produce a collection of balls $\{B_{r_j^1}(x_j^1)\}_1^{N_1}$ such that $r_j^1 = r_{\alpha_j^1} > \bar{r}(\mathrm{v}, D)$, $N_1 \leq N(\mathrm{v}', D')$, $T^{\delta}_{\alpha_j^1}(x_j^1) = 0$, and such that if $x \in M^4 \setminus \bigcup_j B_{r_j^1}(x_j^1)$, then $r_h(x) > \bar{r}$. Furthermore, if we denote by $\Gamma_j^2$ the group associated to $B_{r_j^1}(x_j^1)$, then if $\beta_j^1$ is the largest integer such that $V^{\delta}_{r_{\beta_j^1}/2}(x_j^1) \geq \ln|\Gamma_j^2| - \delta$, then for all $x \in B_{2r_{\beta_j^1}}(x_j^1)$, we have

$$\mathcal{V}^{\delta}_{r_{\beta_j^1}/4}(x_j^1) < \ln|\Gamma_j^2| - \delta. \tag{8.66}$$

Define

$$\mathcal{B}^1 =: M^4 \setminus \bigcup B_{r_j}(x_j) \tag{8.67}$$

as the first body region. Then we can write

$$M^4 = \mathcal{B}^1 \bigcup B_{2r_j^1}(x_j^1), \tag{8.68}$$

where by using Theorem 8.6, we have that $B_{2r_j^1}(x_j^1) \cap \mathcal{B}^1$ is diffeomorphic to $\mathbb{R} \times S^3/\Gamma_j^1$.

Now to prove the theorem, let us inductively build a decomposition of $M^4$,

$$M^4 = \mathcal{B}^1 \cup \bigcup_{j_2=1}^{N_2} \mathcal{N}^2_{j_2} \bigcup_{j_2=1}^{N_2} \mathcal{B}^2_{j_2} \cup \cdots \cup \bigcup_{j_k=1}^{N_k} \mathcal{N}^k_{j_k} \bigcup_{j_k=1}^{N_k} \mathcal{B}^k_{j_k} \cup \bigcup_{a=1}^{N_{k+1}} B_{2r_a^k}(x_a), \tag{8.69}$$

with the following properties:

(1) if $x \in \mathcal{B}_j^{\ell}$, then $r_h(x) > r_0(n, \mathrm{v}, D) \cdot \mathrm{diam}(\mathcal{B}_j^{\ell})$;
(2) each neck $\mathcal{N}_j^{\ell}$ is diffeomorphic to $\mathbb{R} \times S^3/\Gamma_j^{\ell}$ for some $\Gamma_j^{\ell} \subset O(4)$;
(3) each $\mathcal{N}_j^{\ell} \cap \mathcal{B}_j^{\ell}$ is diffeomorphic to $\mathbb{R} \times S^3/\Gamma_j^{\ell}$, and each $\mathcal{N}_j^{\ell} \cap \mathcal{B}_{j'}^{\ell-1}$ are either empty or diffeomorphic to $\mathbb{R} \times S^3/\Gamma_j^{\ell}$;
(4) $N_{\ell} \leq N(n, \mathrm{v}, D)$;
(5) if $\mathcal{N}_a^{\ell+1} \cap \mathcal{B}_j^{\ell} \neq \emptyset$, then $|\Gamma_a^{\ell}| \leq |\Gamma_j^{\ell}| - 1$;
(6) we have $r_a^k = r_{\alpha_a^k}$ with $T^{\delta}_{\alpha_a^k} = 0$, and $\mathcal{B}_j^k \cap B_{r_a^k}(x_a) \subseteq A_{r_a^k/2, r_a^k}(x_a)$;
(7) if $\beta_a^k$ is the largest integer such that $\mathcal{V}^{\delta}_{r_{\beta_a^k}/4}(x_a) \geq \ln|\Gamma_a^k| - \delta$, then for every $x \in B_{2r_{\beta_a^k}}(x_a)$, we have $\mathcal{V}^{\delta}_{r_{\beta_a^k}/8}(x) < \ln|\Gamma_a^k| - \delta$.

Before building the inductive decomposition, let us note that once we have it, we will have finished the proof. In fact, all we really need to see is that for some $k \leq k(n, \mathrm{v}, D)$, there are no balls $\{B_{r_a^k}(x_a)\}$ in the decomposition. To see this, observe that by the lower volume bound we have the upper order bound $|\Gamma_j^2| \leq C(\mathrm{v}, D)$. By condition (5) above we have by iteration that for each $j$,

there is some $j_2$ such that

$$0 \leq |\Gamma_j^k| \leq |\Gamma_{j_2}^2| - (k-2) \leq C(\mathrm{v}, D) - (k-2) \tag{8.70}$$

and, in particular, this immediately implies the upper bound

$$k \leq k(\mathrm{v}, D). \tag{8.71}$$

To prove the inductive decomposition, we begin by noting that (8.68) provides the basic case. So let us assume that the decomposition has been constructed for some $k$, and let us build the decomposition for $k+1$.

First, we use condition (7) and Lemma 8.39 to see that there exist an open set

$$A_{r_{\beta_a^k}/2,2r_{\alpha_a^k}}(x_a) \subseteq \mathcal{N}_a^{k+1} \subseteq A_{(1-\varepsilon)r_{\beta_a^k}/2,2(1+\varepsilon)r_{\alpha_a^k}}(x_a) \tag{8.72}$$

and a diffeomorphism $\Phi_a^{k+1} : \mathcal{N}_a^{k+1} \to A_{r_{\beta_a^k}/2,2r_{\alpha_a^k}}(0)$ with $0 \in \mathbb{R}^4/\Gamma_a^k$. By Lemma 8.53, there exists a radius $r_a = \bar{r}(\mathrm{v}, \delta) r_{\beta_a^k}$ such that

$$\mathcal{V}_{r_a}^{\delta}(x) < \ln\left(|\Gamma_a^{k+1}| - 1\right) + \delta \tag{8.73}$$

for every $x \in B_{2r_{\beta_a^k}}(x_a)$.

Pick some efficient covering $\{B_{r_a}(x_{aj}\}$ of $B_{2r_{\beta_a^k}}(x_a)$ such that the balls in $\{B_{r_a/4}(x_{aj}\}$ are disjoint. Now apply Lemma 8.60 to each ball $\{B_{r_a}(x_{aj}\}$ in order to construct a collection of balls $\{B_{r_{ab}^{k+1}}(x_{ab}^{k+1})\}$ with $r_{ab}^{k+1} = r_{\alpha_{ab}^{k+1}} > \bar{r}(\mathrm{v}, \delta) r_{\beta_a^k}$. Observe that since there are at most $N(\mathrm{v}, D)$ balls in the collection $\{B_{r_a/4}(x_{aj}\}$, and the application of Lemma 8.60 produces at most $N(\mathrm{v}, D)$ balls for each of these, we have at most $N(\mathrm{v}, D)$ such balls in total.

If we put

$$\mathcal{B}_a^{k+1} \equiv \; B_{2r_{\beta_a^k}}(x_a) \setminus \cup B_{r_{\alpha_{ab}^{k+1}}}(x_{ab}), \tag{8.74}$$

we see that $\mathcal{B}_a^{k+1}$ and the collection $\{B_{r_{ab}^{k+1}}(x_{ab}^{k+1})\}$ satisfy the inductive conditions. Specifically, what is left to check is condition (5). However, by construction, we have

$$\ln(|\Gamma_a^k| - 1) + \delta > \mathcal{V}_{r_{ab}^{k+1}}^{\delta}(x_{ab}) \geq \ln|\Gamma_{ab}^{k+1}| - \delta, \tag{8.75}$$

which for $\delta(\mathrm{v})$ sufficiently small implies $|\Gamma_{aj}^{k+1}| < |\Gamma_a^k|$. In particular, the decomposition

$$\begin{aligned} M^n \equiv \mathcal{B}^1 \cup \bigcup_{j_2=1}^{N_2} \mathcal{N}_{j_2}^2 \bigcup_{j_2=1}^{N_2} \mathcal{B}_{j_2}^2 \cup \cdots \cup \bigcup_{j_k=1}^{N_k} \mathcal{N}_{j_k}^k \bigcup_{j_k=1}^{N_k} \mathcal{B}_{j_k}^k \\ \cup \bigcup_{a=1}^{N_{k+1}} \mathcal{N}_a^{k+1} \bigcup \mathcal{B}_a^{k+1} \bigcup B_{2r_{\alpha_{ab}^{k+1}}}(x_{ab}) \end{aligned} \tag{8.76}$$

satisfies the inductive hypothesis as well, which completes the proof. □

Now that we have constructed the bubble tree in Theorem 8.63, let us finish the proof of Theorem 1.12.

*Proof of Theorem* 1.12. Let $M^4$ satisfy $|\mathrm{Ric}_{M^4}| \leq 3$, $\mathrm{Vol}(M) > \mathrm{v} > 0$ and $\mathrm{diam}(M^4) \leq D$. Then using Theorem 8.63, we can write

$$M^4 \equiv \mathcal{B}^1 \cup \bigcup_{j_2=1}^{N_2} \mathcal{N}^2_{j_2} \cup \bigcup_{j_2=1}^{N_2} \mathcal{B}^2_{j_2} \cup \cdots \cup \bigcup_{j_k=1}^{N_k} \mathcal{N}^k_{j_k} \cup \bigcup_{j_k=1}^{N_k} \mathcal{B}^k_{j_k}. \tag{8.77}$$

First we will analyze each body region $\mathcal{B}^k_j$. Indeed, by (1) and Theorem 8.4, it follows that there are at most $C(\mathrm{v}, D)$-diffeomorphism types for each $\mathcal{B}^k_j$. By (4) there are at most $C(\mathrm{v}, D)$ such body regions, each of which has at most $C(\mathrm{v}, D)$ boundary components. By (2), (3), and Lemma 8.39, we can suppose that $\varepsilon(\mathrm{v})$ is so small that for each neck, the induced attaching map between boundary components of the corresponding pair of bodies is sufficiently close to being an isometry of $S^3/\Gamma_\alpha$ that it is isotopic to such an isometry. Since the group of isometries of a compact manifold has finitely many components, it follows that for each neck, there are only finitely many possible isotopy classes of such attaching maps. As a consequence, there are at most $C(\mathrm{v}, D)$ diffeomorphism types that can arise by attaching together the body regions by the various necks. This proves the theorem. □

8.5. *$L^2$ curvature estimates.* We begin with the following, whose proof is essentially the same as that of Theorem 1.12 of the previous subsection.

THEOREM 8.78. *There exists $\delta(\mathrm{v}) > 0$ such that if $M^4$ satisfies $|\mathrm{Ric}_{M^4}| < 2\delta$, $\mathrm{Vol}(B_1(p)) > \mathrm{v} > 0$, and $T^\delta_0(p) = 0$, then there exists $B_1(p) \subseteq U \subseteq B_2(p)$ such that $U$ has at most $C(\mathrm{v})$ diffeomorphism types. Further, $U$ can be chosen so that its boundary $\partial U$ is diffeomorphic to $S^3/\Gamma$ and satisfies the second fundamental form estimate $|A| \leq C(v)$.*

*Proof.* The proof is the same as that of Theorem 1.12, except for the second fundamental form estimate on the boundary. To see this estimate, we use $T^\delta_0(p) = 0$ and Theorem 8.6 to find a diffeomorphism $\Phi : A_{1/2,2}(0) \to B_1(p)$ onto its image, such that if $g_{ij} = \Phi^* g$ is the pullback metric, then

$$||g_{ij} - \delta_{ij}||_{C^0} + ||\partial_k g_{ij}||_{C^0} < \varepsilon. \tag{8.79}$$

In particular, we can choose $U$ so that its boundary is $\partial U = \partial B_{3/2}(0)$ in these coordinates. The $C^1$ estimates on $g$ give rise to the appropriate second fundamental form estimates on $\partial U$. □

With this in hand we are in a position to finish the proof of Theorem 1.13.

*Proof of Theorem* 1.13. Let $M^4$ satisfy $|\mathrm{Ric}_{M^4}| \leq 3$ and $\mathrm{Vol}(B_1(p)) > \mathrm{v} > 0$. Using volume monotonicity, we have for every $x \in B_1(p)$ and $r \leq 1$,

$$\mathrm{Vol}(B_r(x)) \geq \frac{\mathrm{Vol}_{-1}(B_r)}{\mathrm{Vol}_{-1}(B_2)}\mathrm{Vol}(B_2(x)) \geq c(n)\mathrm{Vol}(B_1(p))\, r^4 \geq c\mathrm{v}\, r^4. \tag{8.80}$$

Let $\delta(\mathrm{v})$ be as in Theorem 8.78. By Lemma 8.15, we have that for each $x \in B_1(p)$, there exists a radius $r_{\alpha_x} = 2^{-\alpha_x} \in [C(\mathrm{v})\delta^3, \delta^2]$ such that $T^{\delta}_{\alpha_x}(x) = 0$. Let $\{B_{r_i}(x_i)\}$ be a subcovering such that the balls in $\{B_{r_i/4}(x_i)\}$ are disjoint, where $r_i = r_{\alpha_{x_i}}$. Since $r_i > \bar{r}(\mathrm{v})$, we have by the usual doubling estimates that there are at most $C(\mathrm{v})$ balls in this covering.

Note that for each ball $B_{r_i}(x_i)$, we can apply Theorem 8.78 in order to get a subset $U_i \supseteq B_{r_i}(x_i)$ with bounded diffeomorphism type and uniform boundary control. Now recall that in dimension 4 the Chern-Guass-Bonnet formula can be written as

$$\chi(U_i) = \frac{1}{32\pi^2}\int_{U_i} |\mathrm{Rm}|^2 - 4|\mathrm{Ric}|^2 + R^2 + \int_{\partial U_i} \Psi, \tag{8.81}$$

where $\Psi = \Psi(A)$ is a function of the second fundamental form. By reorganizing, we obtain the bound

$$\begin{aligned} \int_{U_i} |\mathrm{Rm}|^2 &\leq 32\pi^2|\chi(U_i)| + 4\int_{U_i} |\mathrm{Ric}|^2 + C\int_{U_i} |\Psi|, \\ &\leq C(\mathrm{v}), \end{aligned} \tag{8.82}$$

where we have used the bound on the diffeomorphism type, the Ricci bound, and the second fundamental form bound from Theorem 8.78. By summing over $i$, we get

$$⨍_{B_1(p)} |\mathrm{Rm}|^2 \leq C(\mathrm{v}) \sum \int_{U_i} |\mathrm{Rm}|^2 \leq C(\mathrm{v}), \tag{8.83}$$

as claimed. □

In view of [And89], [BKN89], [Tia90], and [And90], the $L^2$ curvature bound in Theorem 8.78 has the following consequence.

COROLLARY 8.84. *Let* $(M^4_j, d_j, p_j) \xrightarrow{d_{\mathrm{GH}}} (X, d, p)$ *be a Gromov-Hausdorff limit of manifolds with* $|\mathrm{Ric}_{M^4_j}| \leq n-1$ *and* $\mathrm{Vol}(B_1(p_j)) > \mathrm{v} > 0$. *Then* $X$ *is a Riemannian orbifold with at most* $c(v)$ *singular points.*

Similarly, we get

COROLLARY 8.85. *Let* $M^4$ *be a complete noncompact Ricci flat manifold with Euclidean volume growth. Then* $M^4$ *is an* ALE *space.*

## 9. **Open questions**

In this section, we briefly remark on some possible extensions of the results of this paper. To begin with, we recall that one of the main applications of this paper was to combine the codimension 4 estimates of Theorem 1.4 with the ideas of quantitative stratification in order to show for all $q < 2$ that $⨍_{B_1(p)} |\mathrm{Rm}|^q$ is uniformly bounded when $M^n$ is a noncollapsed manifold with bounded Ricci curvature. Furthermore, in dimension 4 we were able to improve this to show a bound on $⨍_{B_1(p)} |\mathrm{Rm}|^2$. We conjecture that this holds in any dimension.

*Conjecture* 9.1. There exists $C = C(n, \mathrm{v}) > 0$ such that if $M^n$ satisfies $|\mathrm{Ric}_{M^n}| \leq n - 1$ and $\mathrm{Vol}(B_1(p)) > \mathrm{v} > 0$, then

$$⨍_{B_1(p)} |\mathrm{Rm}|^2 \leq C. \tag{9.2}$$

In a different direction, another main result of the paper was to show that in dimension 4, noncollapsed manifolds with bounded diameter and Ricci curvature have finite diffeomorphism type. In higher dimensions, this is too much to hope for; see, for instance, [HN], where noncollapsed Calabi-Yau manifolds of real dimension $\geq 6$ are constructed with unbounded third Betti number. Nonetheless, it interesting to ask if under the assumption of bounded Ricci curvature, should one expect a bound on the second Betti number?

*Question* 9.3. Does there exist $C = C(n, \mathrm{v}, D)$ such that if $M^n$ satisfies $|\mathrm{Ric}_{M^n}| \leq n - 1$, $\mathrm{diam}(M^n) \leq D$, and $\mathrm{Vol}(B_1(p)) > \mathrm{v} > 0$, then $b_2(M^n) \leq C$?

Examples of Perelman show that if the 2-sided bound on the Ricci tensor is weakened to a lower bound, then the answer is negative.

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