

Isolation, equidistribution, and orbit closures for the $\mathrm{SL}(2, \mathbb{R})$ action on moduli space

By ALEX ESKIN, MARYAM MIRZAKHANI, and AMIR MOHAMMADI

Abstract

We prove results about orbit closures and equidistribution for the $\mathrm{SL}(2, \mathbb{R})$ action on the moduli space of compact Riemann surfaces, which are analogous to the theory of unipotent flows. The proofs of the main theorems rely on the measure classification theorem of the first two authors and a certain isolation property of closed $\mathrm{SL}(2, \mathbb{R})$ invariant manifolds developed in this paper.

1. Introduction

Suppose $g \geq 1$, let $\alpha = (\alpha_1, \dots, \alpha_n)$ be a partition of $2g - 2$, and let $\mathcal{H}(\alpha)$ be a stratum of Abelian differentials, i.e., the space of pairs (M, ω) where M is a Riemann surface and ω is a holomorphic 1-form on M whose zeroes have multiplicities $\alpha_1 \dots \alpha_n$. The form ω defines a canonical flat metric on M with conical singularities at the zeros of ω . Thus we refer to points of $\mathcal{H}(\alpha)$ as *flat surfaces* or *translation surfaces*. For an introduction to this subject, see the survey [Zor06].

The space $\mathcal{H}(\alpha)$ admits an action of the group $\mathrm{SL}(2, \mathbb{R})$ which generalizes the action of $\mathrm{SL}(2, \mathbb{R})$ on the space $\mathrm{GL}(2, \mathbb{R})/\mathrm{SL}(2, \mathbb{Z})$ of flat tori.

Affine measures and manifolds. The area of a translation surface is given by

$$a(M, \omega) = \frac{i}{2} \int_M \omega \wedge \bar{\omega}.$$

Research of the first author is partially supported by NSF grants DMS-0604251, DMS-0905912 and DMS-1201422.

Research of the second author is partially supported by the Clay foundation and by NSF grant DMS-0804136.

Research of the third author is partially supported by NSF grant DMS-1200388 and Alfred P. Sloan Research Fellowship.

© 2015 Department of Mathematics, Princeton University.

A “unit hyperboloid” $\mathcal{H}_1(\alpha)$ is defined as a subset of translation surfaces in $\mathcal{H}(\alpha)$ of area one. For a subset $\mathcal{N}_1 \subset \mathcal{H}_1(\alpha)$, we write

$$\mathbb{R}\mathcal{N}_1 = \{(M, t\omega) \mid (M, \omega) \in \mathcal{N}_1, \quad t \in \mathbb{R}\} \subset \mathcal{H}(\alpha).$$

Definition 1.1. An ergodic $\mathrm{SL}(2, \mathbb{R})$ -invariant probability measure ν_1 on $\mathcal{H}_1(\alpha)$ is called *affine* if the following hold:

- (i) The support \mathcal{M}_1 of ν_1 is an *immersed submanifold* of $\mathcal{H}_1(\alpha)$; i.e., there exist a manifold \mathcal{N} and a proper continuous map $f : \mathcal{N} \rightarrow \mathcal{H}_1(\alpha)$ so that $\mathcal{M}_1 = f(\mathcal{N})$. The self-intersection set of \mathcal{M}_1 , i.e., the set of points of \mathcal{M}_1 which do not have a unique preimage under f , is a closed subset of \mathcal{M}_1 of ν_1 -measure 0. Furthermore, each point in \mathcal{N} has a neighborhood U such that locally $\mathbb{R}f(U)$ is given by a complex linear subspace defined over \mathbb{R} in the period coordinates.
- (ii) Let ν be the measure supported on $\mathcal{M} = \mathbb{R}\mathcal{M}_1$ so that $d\nu = d\nu_1 da$. Then each point in \mathcal{N} has a neighborhood U such that the restriction of ν to $\mathbb{R}f(U)$ is an affine linear measure in the period coordinates on $\mathbb{R}f(U)$; i.e., it is (up to normalization) the restriction of the Lebesgue measure λ to the subspace $\mathbb{R}f(U)$.

Definition 1.2. We say that any suborbifold \mathcal{M}_1 for which there exists a measure ν_1 such that the pair (\mathcal{M}_1, ν_1) satisfies (i) and (ii) an *affine invariant submanifold*.

Note that, in particular, any affine invariant submanifold is a closed subset of $\mathcal{H}_1(\alpha)$ which is invariant under the $\mathrm{SL}(2, \mathbb{R})$ action, and which in period coordinates looks like an affine subspace. We also consider the entire stratum $\mathcal{H}_1(\alpha)$ to be an (improper) affine invariant submanifold. It follows from Theorem 2.2 below that the self-intersection set of an affine invariant manifold is itself a finite union of affine invariant manifolds of lower dimension.

Notational conventions. In case there is no confusion, we will often drop the subscript 1 and denote an affine manifold by \mathcal{N} . Also we will always denote the affine probability measure supported on \mathcal{N} by $\nu_{\mathcal{N}}$. (This measure is unique since it is ergodic for the $\mathrm{SL}(2, \mathbb{R})$ action on \mathcal{N} .)

Let $P \subset \mathrm{SL}(2, \mathbb{R})$ denote the subgroup $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$. In this paper we prove statements about the action of P and $\mathrm{SL}(2, \mathbb{R})$ on $\mathcal{H}_1(\alpha)$ which are analogous to the statements proved in the theory of unipotent flows on homogeneous spaces. For some additional results in this direction, see also [CE13].

The following theorem is the main result of [EM13]:

THEOREM 1.3. *Let ν be any P -invariant probability measure on $\mathcal{H}_1(\alpha)$. Then ν is $\mathrm{SL}(2, \mathbb{R})$ -invariant and affine.*

Theorem 1.3 is a partial analogue of Ratner's celebrated measure classification theorem in the theory of unipotent flows; see [Rat91a].

2. The main theorems

2.1. Orbit closures.

THEOREM 2.1. *Suppose $x \in \mathcal{H}_1(\alpha)$. Then, the orbit closure $\overline{Px} = \overline{\mathrm{SL}(2, \mathbb{R})x}$ is an affine invariant submanifold of $\mathcal{H}_1(\alpha)$.*

The analogue of Theorem 2.1 in the theory of unipotent flows is due in full generality to M. Ratner [Rat91b]. See also the discussion in Section 2.8 below.

THEOREM 2.2. *Any closed P -invariant subset of $\mathcal{H}_1(\alpha)$ is a finite union of affine invariant submanifolds.*

2.2. The space of ergodic P -invariant measures.

THEOREM 2.3. *Let \mathcal{N}_n be a sequence of affine manifolds, and suppose $\nu_{\mathcal{N}_n} \rightarrow \nu$. Then ν is a probability measure. Furthermore, ν is the affine measure $\nu_{\mathcal{N}}$, where \mathcal{N} is the smallest submanifold with the following property: there exists some $n_0 \in \mathbb{N}$ such that $\mathcal{N}_n \subset \mathcal{N}$ for all $n > n_0$.*

In particular, the space of ergodic P -invariant probability measures on $\mathcal{H}_1(\alpha)$ is compact in the weak- topology.*

Remark 2.4. In the setting of unipotent flows, the analogue of Theorem 2.3 is due to Mozes and Shah [MS95].

We state a direct corollary of Theorem 2.3:

COROLLARY 2.5. *Let \mathcal{M} be an affine invariant submanifold, and let \mathcal{N}_n be a sequence of affine invariant submanifolds of \mathcal{M} such that no infinite subsequence is contained in any proper affine invariant submanifold of \mathcal{M} . Then the sequence of affine measures $\nu_{\mathcal{N}_n}$ converges to $\nu_{\mathcal{M}}$.*

2.3. Equidistribution for sectors.

Let $a_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$, $r_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$.

THEOREM 2.6. *Suppose $x \in \mathcal{H}_1(\alpha)$, and let \mathcal{M} be an affine invariant submanifold of minimum dimension which contains x . Then for any $\varphi \in C_c(\mathcal{H}_1(\alpha))$ and any interval $I \subset [0, 2\pi)$,*

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \frac{1}{|I|} \int_I \varphi(a_t r_\theta x) d\theta dt = \int_{\mathcal{M}} \varphi d\nu_{\mathcal{M}}.$$

Remark. It follows from Theorem 2.6 that for any $x \in \mathcal{H}_1(\alpha)$, there exists a unique affine invariant manifold of minimal dimension which contains x .

We also have the following uniform version (cf. [DM93, Th. 3]):

THEOREM 2.7. *Let \mathcal{M} be an affine invariant submanifold. Then for any $\varphi \in C_c(\mathcal{H}_1(\alpha))$ and any $\varepsilon > 0$, there are affine invariant submanifolds $\mathcal{N}_1, \dots, \mathcal{N}_\ell$ properly contained in \mathcal{M} such that for any compact subset $F \subset \mathcal{M} \setminus (\cup_{j=1}^\ell \mathcal{N}_j)$, there exists T_0 so that for all $T > T_0$ and any $x \in F$,*

$$\left| \frac{1}{T} \int_0^T \frac{1}{|I|} \int_I \varphi(a_t r_\theta x) d\theta dt - \int_{\mathcal{M}} \varphi d\nu_{\mathcal{M}} \right| < \varepsilon.$$

We remark that the analogue of Theorem 2.7 for unipotent flows, due to Dani and Margulis [DM93], plays a key role in the applications of the theory.

2.4. Equidistribution for random walks. Let μ be a probability measure on $\mathrm{SL}(2, \mathbb{R})$ which is compactly supported and is absolutely continuous with respect to the Haar measure. Even though it is not necessary, for clarity of presentation, we will also assume that μ is $\mathrm{SO}(2)$ -bi-invariant. Let $\mu^{(k)}$ denote the k -fold convolution of μ with itself.

We now state “random walk” analogues of Theorems 2.6 and 2.7.

THEOREM 2.8. *Suppose $x \in \mathcal{H}_1(\alpha)$, and let \mathcal{M} be the affine invariant submanifold of minimum dimension which contains x . Then for any $\varphi \in C_c(\mathcal{H}_1(\alpha))$,*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \int_{\mathrm{SL}(2, \mathbb{R})} \varphi(gx) d\mu^{(k)}(g) = \int_{\mathcal{M}} \varphi d\nu_{\mathcal{M}}.$$

We also have the following uniform version, similar in spirit to [DM93, Th. 3]:

THEOREM 2.9. *Let \mathcal{M} be an affine invariant submanifold. Then for any $\varphi \in C_c(\mathcal{H}_1(\alpha))$ and any $\varepsilon > 0$, there are affine invariant submanifolds $\mathcal{N}_1, \dots, \mathcal{N}_\ell$ properly contained in \mathcal{M} such that for any compact subset $F \subset \mathcal{M} \setminus (\cup_{j=1}^\ell \mathcal{N}_j)$, there exists n_0 so that for all $n > n_0$ and any $x \in F$,*

$$\left| \frac{1}{n} \sum_{k=1}^n \int_{\mathrm{SL}(2, \mathbb{R})} \varphi(gx) d\mu^{(k)}(g) - \int_{\mathcal{M}} \varphi d\nu_{\mathcal{M}} \right| < \varepsilon.$$

2.5. Equidistribution for some Følner sets. Let $u_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$.

THEOREM 2.10. *Suppose $x \in \mathcal{H}_1(\alpha)$, and let \mathcal{M} be the affine invariant submanifold of minimum dimension which contains x . Then for any $\varphi \in C_c(\mathcal{H}_1(\alpha))$ and any $r > 0$,*

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \frac{1}{r} \int_0^r \varphi(a_t u_s x) ds dt = \int_{\mathcal{M}} \varphi d\nu_{\mathcal{M}}.$$

We also have the following uniform version (cf. [DM93, Th. 3]):

THEOREM 2.11. *Let \mathcal{M} be an affine invariant submanifold. Then for any $\varphi \in C_c(\mathcal{H}_1(\alpha))$ and any $\varepsilon > 0$, there are affine invariant submanifolds*

$\mathcal{N}_1, \dots, \mathcal{N}_\ell$ properly contained in \mathcal{M} such that for any compact subset $F \subset \mathcal{M} \setminus (\cup_{j=1}^\ell \mathcal{N}_j)$, there exists T_0 so that for all $T > T_0$, for all $r > 0$, and for any $x \in F$,

$$\left| \frac{1}{T} \int_0^T \frac{1}{r} \int_0^r \varphi(a_t u_s x) ds dt - \int_{\mathcal{M}} \varphi d\nu_{\mathcal{M}} \right| < \varepsilon.$$

2.6. *Counting periodic trajectories in rational billiards.* Let Q be a rational polygon, and let $N(Q, T)$ denote the number of cylinders of periodic trajectories of length at most T for the billiard flow on Q . By a theorem of H. Masur [Mas90] [Mas88], there exist $c_1 > 0$ and $c_2 > 0$ depending on Q such that for all $t > 1$,

$$c_1 e^{2t} \leq N(Q, e^t) \leq c_2 e^{2t}.$$

As a consequence of Theorem 2.7, we get the following “weak asymptotic formula” (cf. [AEZ12]):

THEOREM 2.12. *For any rational polygon Q , there exists a constant $c = c(Q) > 0$ such that*

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t N(Q, e^s) e^{-2s} ds = c.$$

The constant c in Theorem 2.12 is the Siegel-Veech constant (see [Vee98], [EMZ03]) associated to the affine invariant submanifold $\mathcal{M} = \overline{\mathrm{SL}(2, \mathbb{R})S}$, where S is the flat surface obtained by unfolding Q .

It is natural to conjecture that the extra averaging on Theorem 2.12 is not necessary, and one has $\lim_{t \rightarrow \infty} N(Q, e^t) e^{-2t} = c$. This can be shown if one obtains a classification of the measures invariant under the subgroup $U = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ of $\mathrm{SL}(2, \mathbb{R})$. Such a result is in general beyond the reach of the current methods. However, it is known in a few very special cases; see [EMS03], [EMM06], [CW10] and [Bai10].

2.7. *The main proposition and countability.* For a function $f : \mathcal{H}_1(\alpha) \rightarrow \mathbb{R}$, let

$$(A_t f)(x) = \frac{1}{2\pi} \int_0^{2\pi} f(a_t r_\theta x).$$

Following the general idea of Margulis introduced in [EMM98], the strategy of the proof is to define a function which will satisfy a certain inequality involving A_t . In fact, the main technical result of this paper is the following:

PROPOSITION 2.13. *Let $\mathcal{M} \subset \mathcal{H}_1(\alpha)$ be an affine invariant submanifold. (In this proposition $\mathcal{M} = \emptyset$ is allowed). Then there exists an $\mathrm{SO}(2)$ -invariant function $f_{\mathcal{M}} : \mathcal{H}_1(\alpha) \rightarrow [1, \infty]$ with the following properties:*

- (a) $f_{\mathcal{M}}(x) = \infty$ if and only if $x \in \mathcal{M}$, and $f_{\mathcal{M}}$ is bounded on compact subsets of $\mathcal{H}_1(\alpha) \setminus \mathcal{M}$. For any $\ell > 0$, the set $\overline{\{x : f_{\mathcal{M}}(x) \leq \ell\}}$ is a compact subset of $\mathcal{H}_1(\alpha) \setminus \mathcal{M}$.

- (b) *There exists $b > 0$ (depending on \mathcal{M}), and for every $0 < c < 1$, there exists $t_0 > 0$ (depending on \mathcal{M} and c) such that for all $x \in \mathcal{H}_1(\alpha) \setminus \mathcal{M}$ and all $t > t_0$,*

$$(A_t f_{\mathcal{M}})(x) \leq c f_{\mathcal{M}}(x) + b.$$

- (c) *There exists $\sigma > 1$ such that for all $g \in \mathrm{SL}(2, \mathbb{R})$ in some neighborhood of the identity and all $x \in \mathcal{H}_1(\alpha)$,*

$$\sigma^{-1} f_{\mathcal{M}}(x) \leq f_{\mathcal{M}}(gx) \leq \sigma f_{\mathcal{M}}(x).$$

The proof of Proposition 2.13 consists of Sections 4–10. It is based on the recurrence properties of the $\mathrm{SL}(2, \mathbb{R})$ -action proved by Athreya in [Ath06] and also on the fundamental result of Forni on the uniform hyperbolicity in compact sets of the Teichmüller geodesic flow [For02, Cor. 2.1].

Remark 2.14. In the case \mathcal{M} is empty, a function satisfying the conditions of Proposition 2.13 has been constructed in [EM01] and used in [Ath06].

Remark 2.15. In fact, we show that the constant b in Proposition 2.13(b) depends only on the “complexity” of \mathcal{M} (defined in Section 8). This is used in Section 11 for the proof of the following:

PROPOSITION 2.16. *There are at most countably many affine invariant submanifolds in each stratum.*

Another proof of Proposition 2.16 is given in [Wri14], where it is shown that any affine invariant submanifold is defined over a number field.

2.8. Analogy with unipotent flows and historical remarks. In the context of unipotent flows, i.e., the left-multiplication action of a unipotent subgroup U of a Lie group G on the space G/Γ where Γ is a lattice in G , the analogue of Theorem 2.1 was conjectured by Raghunathan. In the literature the conjecture was first stated in the paper [Dan81] and in a more general form in [Mar89a] (when the subgroup U is not necessarily unipotent but generated by unipotent elements). Raghunathan’s conjecture was eventually proved in full generality by M. Ratner (see [Rat90b], [Rat90a], [Rat91a] and [Rat91b]). Earlier it was known in the following cases: (a) G is reductive and U is horospherical (see [Dan81]); (b) $G = \mathrm{SL}(3, \mathbb{R})$ and $U = \{u(t)\}$ is a one-parameter unipotent subgroup of G such that $u(t) - I$ has rank 2 for all $t \neq 0$, where I is the identity matrix (see [DM90]); (c) G is solvable (see [Sta87] and [Sta89]). We remark that the proof given in [Dan81] is restricted to horospherical U and the proof given in [Sta87] and [Sta89] cannot be applied for nonsolvable G .

However, the proof in [DM90] together with the methods developed in [Mar89b], [Mar90], [Mar91], [Mar04] and [DM89] suggest an approach for proving the Raghunathan conjecture in general by studying the minimal invariant

sets and the limits of orbits of sequences of points tending to a minimal invariant set. This program was being actively pursued at the time Ratner's results were announced (cf. [Sha94]).

3. Proofs of the main theorems

In this section we derive all the results of Sections 2.1–2.6 from Theorem 1.3 and Propositions 2.13 and 2.16.

The proofs are much simpler than the proofs of the analogous results in the theory of unipotent flows. This is related to Proposition 2.16. In the setting of unipotent flows there may be continuous families of invariant manifolds (which involve the centralizer and normalizer of the acting group).

3.1. Random walks. Many of the arguments work most naturally in the random walk setting. But first we need to convert Theorem 1.3 and Proposition 2.13 to the random walk setup.

Stationary measures. Recall that μ is a compactly supported probability measure on $\mathrm{SL}(2, \mathbb{R})$ which is $\mathrm{SO}(2)$ -bi-invariant and is absolutely continuous with respect to Haar measure. A measure ν on $\mathcal{H}_1(\alpha)$ is called μ -stationary if $\mu * \nu = \nu$, where

$$\mu * \nu = \int_{\mathrm{SL}(2, \mathbb{R})} (g_* \nu) d\mu(g).$$

Recall that by a theorem of Furstenberg [Fur63b], [Fur63a], restated as [NZ99, Th. 1.4], μ -stationary measures are in one-to-one correspondence with P -invariant measures. Therefore, Theorem 1.3 can be reformulated as the following:

THEOREM 3.1. *Any μ -stationary measure on $\mathcal{H}_1(\alpha)$ is $\mathrm{SL}(2, \mathbb{R})$ invariant and affine.*

The operator \mathbb{A}_μ . Let $\mathbb{A}_\mu : C_c(\mathcal{H}_1(\alpha)) \rightarrow C_c(\mathcal{H}_1(\alpha))$ denote the linear operator

$$(\mathbb{A}_\mu f)(x) = \int_{\mathrm{SL}(2, \mathbb{R})} f(gx) d\mu(g).$$

LEMMA 3.2. *Let $f_{\mathcal{M}}$ be as in Proposition 2.13. Then there exists $b > 0$, and for any $c > 0$, there exists $n_0 > 0$ such that for $n > n_0$ and any $x \in \mathcal{H}_1(\alpha)$,*

$$(\mathbb{A}_\mu^n f_{\mathcal{M}})(x) \leq cf_{\mathcal{M}}(x) + b.$$

Proof. Recall the KAK decomposition:

$$g = k_1 a_t k_2, \quad g \in \mathrm{SL}(2, \mathbb{R}), \quad k_1, k_2 \in \mathrm{SO}(2), \quad t \in \mathbb{R}^+.$$

We may think of k_1, t, k_2 as coordinates on $\mathrm{SL}(2, \mathbb{R})$. Since $\mu^{(n)}$ is $\mathrm{SO}(2)$ -bi-invariant and absolutely continuous with respect to the Haar measure on

$\mathrm{SL}(2, \mathbb{R})$, we have

$$d\mu^{(n)}(g) = K_n(t) dm(k_1) dm(k_2) dt,$$

where m is the Haar measure on $\mathrm{SO}(2)$, and $K_n : \mathbb{R}^+ \rightarrow \mathbb{R}$ is a compactly supported function satisfying $K_n(t) \geq 0$, $\int_0^\infty K_n(t) dt = 1$. Also, since the top Lyapunov exponent of the random walk on $\mathrm{SL}(2, \mathbb{R})$ given by μ is positive, for any $t_0 > 0$ and any $\varepsilon > 0$, there exists n_0 such that for $n > n_0$,

$$(1) \quad \int_0^{t_0} K_n(t) dt < \varepsilon.$$

We have, since $f_{\mathcal{M}}$ is $\mathrm{SO}(2)$ -invariant,

$$(2) \quad (\mathbb{A}_\mu^n f_{\mathcal{M}})(x) = \int_0^\infty K_n(t) (A_t f_{\mathcal{M}})(x) dt.$$

Now let t_0 be as in Proposition 2.13(b) for $c/2$ instead of c . By Proposition 2.13(c), there exists $R > 0$ such that

$$(3) \quad f_{\mathcal{M}}(a_t r_\theta x) < R f_{\mathcal{M}}(x) \quad \text{when } t < t_0.$$

Then let n_0 be such that (1) holds with $\varepsilon = c/(2R)$. Then, for $n > n_0$,

$$\begin{aligned} (\mathbb{A}_\mu^n f_{\mathcal{M}})(x) &= \int_0^{t_0} K_n(t) (A_t f_{\mathcal{M}})(x) dt \\ &\quad + \int_{t_0}^\infty K_n(t) (A_t f_{\mathcal{M}})(x) dt && \text{by (2)} \\ &\leq \int_0^{t_0} K_n(t) (R f_{\mathcal{M}}(x)) dt \\ &\quad + \int_{t_0}^\infty ((c/2) f_{\mathcal{M}}(x) + b) dt && \text{by (3) and 2.13(b)} \\ &\leq (c/2R) R f_{\mathcal{M}}(x) + ((c/2) f_{\mathcal{M}}(x) + b) && \text{by (1)} \\ &= c f_{\mathcal{M}}(x) + b. \end{aligned}$$

□

Notational conventions. Let

$$\bar{\mu}^{(n)} = \frac{1}{n} \sum_{k=1}^n \mu^{(k)}.$$

For $x \in \mathcal{H}_1(\alpha)$, let δ_x denote the Dirac measure at x , and let $*$ denote convolution of measures.

We have the following:

PROPOSITION 3.3. *Let \mathcal{N} be a (possibly empty) proper affine invariant submanifold. Then for any $\varepsilon > 0$, there exists an open set $\Omega_{\mathcal{N}, \varepsilon}$ containing \mathcal{N} with $(\Omega_{\mathcal{N}, \varepsilon})^c$ compact such that for any compact $F \subset \mathcal{H}_1(\alpha) \setminus \mathcal{N}$, there exists $n_0 \in \mathbb{N}$ so that for all $n > n_0$ and all $x \in F$, we have*

$$(\bar{\mu}^{(n)} * \delta_x)(\Omega_{\mathcal{N}, \varepsilon}) < \varepsilon.$$

Proof. Let $f_{\mathcal{N}}$ be the function of Proposition 2.13. Let $b > 0$ be as in Lemma 3.2, and let

$$\Omega_{\mathcal{N},\varepsilon} = \{p : f_{\mathcal{N}}(p) > (b+1)/\varepsilon\}^0,$$

where E^0 denotes the interior of E .

Suppose F is a compact subset of $\mathcal{H}_1(\alpha) \setminus \mathcal{N}$. Let $m_F = \sup\{f_{\mathcal{N}}(x) : x \in F\}$. Let $n_0 \in \mathbb{N}$ be as in Lemma 3.2 for $c = 0.5/m_F$. Then, by Lemma 3.2,

$$(\mathbb{A}_{\mu}^n f_{\mathcal{N}})(x) < \frac{0.5}{m_F} f_{\mathcal{N}}(x) + b \leq 0.5 + b \quad \text{for all } n > n_0 \text{ and all } x \in F.$$

It follows that for n_0 sufficiently large, for all $x \in F$ and all $n > n_0$,

$$(\bar{\mu}^{(n)} * \delta_x)(f_{\mathcal{N}}) \leq 1 + b.$$

Thus for any $x \in F$ and $L > 0$, we have

$$(4) \quad (\bar{\mu}^{(n)} * \delta_x)(\{p : f_{\mathcal{N}}(p) > L\}) < \frac{b+1}{L}.$$

Then (4) implies that $(\bar{\mu}^{(n)} * \delta_x)(\Omega_{\mathcal{N},\varepsilon}) < \varepsilon$. Also, Proposition 2.13(a) implies that $\Omega_{\mathcal{N},\varepsilon}$ is a neighborhood of \mathcal{N} and

$$(\Omega_{\mathcal{N},\varepsilon})^c = \overline{\{p : f_{\mathcal{N}}(p) \leq (b+1)/\varepsilon\}}$$

is compact. \square

Proof of Theorem 2.8. Let \mathcal{M} be an affine manifold containing x of minimal dimension. (At this point we do not yet know that \mathcal{M} is unique.) Suppose the assertion of the theorem does not hold. Then there exist a $\varphi \in C_c(\mathcal{H}_1(\alpha))$, $\varepsilon > 0$, $x \in \mathcal{M}$ and a sequence $n_k \rightarrow \infty$ such that

$$|(\bar{\mu}^{(n_k)} * \delta_x)(\varphi) - \nu_{\mathcal{M}}(\varphi)| \geq \varepsilon.$$

Recall that the space of measures on $\mathcal{H}_1(\alpha)$ of total mass at most 1 is compact in the weak star topology. Therefore, after passing to a subsequence if necessary, we may and will assume that $\bar{\mu}^{(n_k)} * \delta_x \rightarrow \nu$, where ν is some measure on $\mathcal{H}_1(\alpha)$ (which could *a priori* be the zero measure). Below, we will show that in fact ν is the probability measure $\nu_{\mathcal{M}}$, which leads to a contradiction.

First note that it follows from the definition that ν is an μ -stationary measure. Therefore, by Theorem 3.1, ν is $\text{SL}(2, \mathbb{R})$ -invariant. Also since \mathcal{M} is $\text{SL}(2, \mathbb{R})$ -invariant, we get $\text{supp}(\nu) \subset \mathcal{M}$. The measure ν need not be ergodic, but by Theorem 1.3, all of its ergodic components are affine measures supported on affine invariant submanifolds of \mathcal{M} . By Proposition 2.16 there are only countably many affine invariant submanifolds of \mathcal{M} . Therefore, we have the ergodic decomposition

$$(5) \quad \nu = \sum_{\mathcal{N} \subseteq \mathcal{M}} a_{\mathcal{N}} \nu_{\mathcal{N}},$$

where the sum is over all the affine invariant submanifolds $\mathcal{N} \subset \mathcal{M}$ and $a_{\mathcal{N}} \in [0, 1]$. To finish the proof we will show that ν is a probability measure and that $a_{\mathcal{N}} = 0$ for all $\mathcal{N} \subsetneq \mathcal{M}$.

Suppose $\mathcal{N} \subsetneq \mathcal{M}$. (Here we allow $\mathcal{N} = \emptyset$.) Note that $x \notin \mathcal{N}$ (since $\dim \mathcal{N} < \dim \mathcal{M}$ and \mathcal{M} is assumed to be an affine manifold containing x of minimal dimension). We now apply Proposition 3.3 with \mathcal{N} and the compact set $F = \{x\}$. We get that for any $\varepsilon > 0$, there exists some n_0 so that if $n > n_0$, then $(\bar{\mu}^{(n)} * \delta_x)((\Omega_{\mathcal{N}, \varepsilon})^c) \geq 1 - \varepsilon$. Therefore, passing to the limit, we get

$$\nu((\Omega_{\mathcal{N}, \varepsilon})^c) \geq 1 - \varepsilon.$$

Note that $\varepsilon > 0$ is arbitrary. From the case $\mathcal{N} = \emptyset$, we get that ν is a probability measure. Also, for any $\mathcal{N} \subsetneq \mathcal{M}$, this implies that $\nu(\mathcal{N}) = 0$. Hence $a_{\mathcal{N}} \leq \nu(\mathcal{N}) = 0$. \square

Proof of Theorem 2.3. Since the space of measures of mass at most 1 on $\mathcal{H}_1(\alpha)$ is compact in the weak-* topology, the statement about weak-* compactness in Theorem 2.3 follows from the others.

Suppose that $\nu_{\mathcal{N}_n} \rightarrow \nu$. We first prove that ν is a probability measure. Let $\Omega_{\emptyset, \varepsilon}$ be as in Proposition 3.3 with $\mathcal{M} = \emptyset$. By the random ergodic theorem [Fur02, Th. 3.1], for almost every $x_n \in \mathcal{N}_n$,

$$(6) \quad \lim_{m \rightarrow \infty} (\bar{\mu}^{(m)} * \delta_{x_n})((\Omega_{\emptyset, \varepsilon})^c) = \nu_{\mathcal{N}_n}((\Omega_{\emptyset, \varepsilon})^c).$$

Choose x_n such that (6) holds. By Proposition 3.3, for all m large enough (depending on x_n),

$$(\bar{\mu}^{(m)} * \delta_{x_n})((\Omega_{\emptyset, \varepsilon})^c) \geq 1 - \varepsilon.$$

Passing to the limit as $n \rightarrow \infty$, we get

$$\nu((\Omega_{\emptyset, \varepsilon})^c) \geq 1 - \varepsilon.$$

Since ε is arbitrary, this shows that ν is a probability measure.

In view of the fact that the ν_n are invariant under $\mathrm{SL}(2, \mathbb{R})$, the same is true of ν . As in (5), let

$$\nu = \sum_{\mathcal{N} \subseteq \mathcal{H}_1(\alpha)} a_{\mathcal{N}} \nu_{\mathcal{N}}$$

be the ergodic decomposition of ν , where $a_{\mathcal{N}} \in [0, 1]$. By Theorem 1.3, all the measures $\nu_{\mathcal{N}}$ are affine, and by Proposition 2.16, the number of terms in the ergodic decomposition is countable.

For any affine invariant submanifold \mathcal{N} , let

$$X(\mathcal{N}) = \bigcup \{ \mathcal{N}' \subsetneq \mathcal{N} : \mathcal{N}' \text{ is an affine invariant submanifold} \}.$$

Let $\mathcal{N} \subseteq \mathcal{H}_1(\alpha)$ be a submanifold such that $\nu(X(\mathcal{N})) = 0$ and $\nu(\mathcal{N}) > 0$. This implies $a_{\mathcal{N}} = \nu(\mathcal{N})$.

Let K be a large compact set, such that $\nu(K) > (1 - a_{\mathcal{N}}/4)$. Then, $\nu(K \cap \mathcal{N}) > (3/4)a_{\mathcal{N}}$. Let $\varepsilon = a_{\mathcal{N}}/4$, and let $\Omega_{\mathcal{N},\varepsilon}$ be as in Proposition 3.3. Since $K \cap \mathcal{N}$ and $(\Omega_{\mathcal{N},\varepsilon})^c$ are both compact sets, we can choose a continuous compactly supported function φ such that $0 \leq \varphi \leq 1$, $\varphi = 1$ on $K \cap \mathcal{N}$ and $\varphi = 0$ on $(\Omega_{\mathcal{N},\varepsilon})^c$. Then,

$$\nu(\varphi) \geq \nu(K \cap \mathcal{N}) > (3/4)a_{\mathcal{N}}.$$

Since $\nu_{\mathcal{N}_n}(\varphi) \rightarrow \nu(\varphi)$, there exists $n_0 \in \mathbb{N}$ such that for $n > n_0$,

$$\nu_{\mathcal{N}_n}(\varphi) > a_{\mathcal{N}}/2.$$

For each n , let $x_n \in \mathcal{N}_n$ be a generic point for $\nu_{\mathcal{N}_n}$ for the random ergodic theorem [Fur02, Th. 3.1]; i.e.,

$$(7) \quad \lim_{m \rightarrow \infty} (\bar{\mu}^{(m)} * \delta_{x_n})(\varphi) = \nu_{\mathcal{N}_n}(\varphi) \text{ for all } \varphi \in C_c(\mathcal{H}_1(\alpha)).$$

Suppose $n > n_0$. Then, by (7), we get

$$\text{if } m \text{ is large enough, then } (\bar{\mu}^{(m)} * \delta_{x_n})(\varphi) > a_{\mathcal{N}}/4.$$

Therefore, since $0 \leq \varphi \leq 1$ and $\varphi = 0$ outside of $\Omega_{\mathcal{N},\varepsilon}$, we get

$$\text{if } m \text{ is large enough, then } (\bar{\mu}^{(m)} * \delta_{x_n})(\Omega_{\mathcal{N},\varepsilon}) > a_{\mathcal{N}}/4.$$

Proposition 3.3, applied with $\varepsilon = a_{\mathcal{N}}/4$ now implies that $x_n \in \mathcal{N}$ which, in view of the genericity of x_n , implies that $\mathcal{N}_n \subset \mathcal{N}$ for all $n > n_0$. This implies $\nu(\mathcal{N}) = 1$, and since $\nu(X(\mathcal{N})) = 0$, we get $\nu = \nu_{\mathcal{N}}$. Also, since $\nu(X(\mathcal{N})) = 0$, \mathcal{N} is the minimal affine invariant manifold which eventually contains the \mathcal{N}_n . \square

LEMMA 3.4. *Given any $\varphi \in C_c(\mathcal{H}_1(\alpha))$, any affine invariant submanifold \mathcal{M} , and any $\varepsilon > 0$, there exists a finite collection \mathcal{C} of proper affine invariant submanifolds of \mathcal{M} with the following property: if $\mathcal{N}' \subset \mathcal{M}$ is such that $|\nu_{\mathcal{N}'}(\varphi) - \nu_{\mathcal{M}}(\varphi)| \geq \varepsilon$, then there exists some $\mathcal{N} \in \mathcal{C}$ such that $\mathcal{N}' \subset \mathcal{N}$.*

Proof. Let φ and $\varepsilon > 0$ be given. We will prove this by inductively choosing \mathcal{N}_j 's as follows. Suppose $k > 0$, and put

$$\mathcal{A}_k = \{\mathcal{N} \subseteq \mathcal{M} : \mathcal{N} \text{ has codimension } k \text{ in } \mathcal{M} \text{ and } |\nu_{\mathcal{N}}(\varphi) - \nu_{\mathcal{M}}(\varphi)| \geq \varepsilon\}.$$

Let $\mathcal{B}_1 = \mathcal{A}_1$, and define

$$\mathcal{B}_k = \{\mathcal{N} \in \mathcal{A}_k : \text{such that } \mathcal{N} \text{ is not contained in any } \mathcal{N}' \in \mathcal{A}_\ell \text{ with } \ell < k\}.$$

CLAIM. \mathcal{B}_k is a finite set for each k .

We will show this inductively. Note that by Corollary 2.5, we have \mathcal{A}_1 , and hence \mathcal{B}_1 , is a finite set. Suppose we have shown $\{\mathcal{B}_j : 1 \leq j \leq k-1\}$ is a finite set. Let $\{\mathcal{N}_j\}$ be an infinite collection of elements in \mathcal{B}_k . By Theorem 2.3, we may pass to subsequence (which we continue to denote by \mathcal{N}_j) such that

$\nu_{\mathcal{N}_j} \rightarrow \nu$. Theorem 2.3 also implies that $\nu = \nu_{\mathcal{N}}$ for some affine invariant submanifold \mathcal{N} and that there exists some j_0 such that $\mathcal{N}_j \subset \mathcal{N}$ for all $j > j_0$. Note that \mathcal{N} has codimension $\ell \leq k - 1$.

Since $\nu_{\mathcal{N}_j} \rightarrow \nu_{\mathcal{N}}$ and $\mathcal{N}_j \in \mathcal{B}_k \subset \mathcal{A}_k$, we have $|\nu_{\mathcal{N}}(\varphi) - \nu_{\mathcal{M}}(\varphi)| \geq \varepsilon$. Therefore, $\mathcal{N} \in \mathcal{A}_\ell$. But this is a contradiction to the definition of \mathcal{B}_k since $\mathcal{N}_j \subset \mathcal{N}$ and $\mathcal{N}_j \in \mathcal{B}_k$. This completes the proof of the claim.

Now let

$$\mathcal{C} = \{\mathcal{N} : \mathcal{N} \in \mathcal{B}_k, \text{ for } 0 < k \leq \dim \mathcal{M}\}.$$

This is a finite set that satisfies the conclusion of the lemma. \square

Proof of Theorem 2.9. Let φ and $\varepsilon > 0$ be given, and let \mathcal{C} be given by Lemma 3.4. Write $\mathcal{C} = \{\mathcal{N}_1, \dots, \mathcal{N}_\ell\}$. We will show that the theorem holds with this choice of the \mathcal{N}_j .

Suppose not. Then there exists a compact subset $F \subset \mathcal{M} \setminus \bigcup_{j=1}^\ell \mathcal{N}_j$ such that for all $m_0 \geq 0$,

$$\{x \in F : |(\bar{\mu}^{(m)} * \delta_x)(\varphi) - \nu_{\mathcal{M}}(\varphi)| > \varepsilon \text{ for some } m > m_0\} \neq \emptyset.$$

Let $m_n \rightarrow \infty$ and $\{x_n\} \subset F$ be a sequence such that $|(\bar{\mu}^{(m_n)} * \delta_{x_n})(\varphi) - \nu_{\mathcal{M}}(\varphi)| > \varepsilon$.

Since the space of measures on $\mathcal{H}_1(\alpha)$ of total mass at most 1 is compact in the weak star topology, after passing to a subsequence if necessary, we may and will assume that $\bar{\mu}^{(m_n)} * \delta_{x_n} \rightarrow \nu$, where ν is some measure on \mathcal{M} (which could *a priori* be the zero measure). We will also assume that $x_n \rightarrow x$ for some $x \in F$.

Note that ν is $\mathrm{SL}(2, \mathbb{R})$ -invariant. Let

$$\nu = \sum_{\mathcal{N} \in \mathcal{C}} a_{\mathcal{N}} \nu_{\mathcal{N}}$$

be the ergodic decomposition of ν , as in (5).

We claim that ν is a probability measure and $\nu(\mathcal{N}) = 0$ for all $\mathcal{N} \in \mathcal{C}$. To see this, suppose $\mathcal{N} \in \mathcal{C}$ or $\mathcal{N} = \emptyset$ and apply Proposition 3.3 with \mathcal{N} and F . We get that for any $\varepsilon' > 0$, there exists some n_0 so that if $n > n_0$, then $(\bar{\mu}^{(n)} * \delta_y)((\Omega_{\mathcal{N}, \varepsilon'})^c) \geq 1 - \varepsilon'$ for all $y \in F$. Therefore, passing to the limit, we get

$$\nu((\Omega_{\mathcal{N}, \varepsilon})^c) \geq 1 - \varepsilon'.$$

Since $\varepsilon' > 0$ is arbitrary, this implies that ν is a probability measure and $\nu(\mathcal{N}) = 0$. The claim now follows since \mathcal{C} is a finite family.

The claim and Lemma 3.4 imply that $|\nu(\varphi) - \nu_{\mathcal{M}}(\varphi)| < \varepsilon$. This and the definition of ν imply that $|(\bar{\mu}^{(m_n)} * \delta_{x_n})(\varphi) - \nu_{\mathcal{M}}(\varphi)| < \varepsilon$ for all large enough n . This contradicts the choice of x_n and m_n and completes the proof. \square

The only properties of the measures $\bar{\mu}^{(n)}$ which were used in this subsection were Proposition 3.3 and the fact that any limit of the measures $\bar{\mu}^{(n)} * \delta_x$ is $\mathrm{SL}(2, \mathbb{R})$ invariant. In fact, we proved the following theorem, which we will record for future use:

THEOREM 3.5. *Suppose $\{\eta_t : t \in \mathbb{R}\}$ is a family of probability measures on $\mathrm{SL}(2, \mathbb{R})$ with the following properties:*

- (a) *Proposition 3.3 holds for η_t instead of $\bar{\mu}^{(n)}$ (and t instead of n).*
- (b) *Any weak-* limit of measures of the form $\eta_{t_i} * \delta_{x_i}$ as $t_i \rightarrow \infty$ is $\mathrm{SL}(2, \mathbb{R})$ -invariant.*

Then,

- (i) (cf. Theorem 2.8). *Suppose $x \in \mathcal{H}_1(\alpha)$, and let \mathcal{M} be the smallest affine invariant submanifold containing x . Then for any $\varphi \in C_c(\mathcal{H}_1(\alpha))$,*

$$\lim_{t \rightarrow \infty} (\eta_t * \delta_x)(\varphi) = \nu_{\mathcal{M}}(\varphi).$$

- (ii) (cf. Theorem 2.9). *Let \mathcal{M} be an affine invariant submanifold. Then for any $\varphi \in C_c(\mathcal{H}_1(\alpha))$ and any $\varepsilon > 0$, there are affine invariant submanifolds $\mathcal{N}_1, \dots, \mathcal{N}_\ell$ properly contained in \mathcal{M} such that for any compact subset $F \subset \mathcal{M} \setminus (\cup_{j=1}^\ell \mathcal{N}_j)$, there exists T_0 so that for all $T > T_0$ and any $x \in F$,*

$$|(\eta_t * \delta_x)(\varphi) - \nu_{\mathcal{M}}(\varphi)| < \varepsilon.$$

3.2. Equidistribution for sectors. We define a sequence of probability measures ϑ_t on $\mathrm{SL}(2, \mathbb{R})$ by

$$\vartheta_t(\varphi) = \frac{1}{t} \int_0^t \frac{1}{2\pi} \int_0^{2\pi} \varphi(a_s r_\theta) d\theta ds.$$

More generally, if $I \subset [0, 2\pi]$ is an interval, then we define

$$\vartheta_{t,I}(\varphi) = \frac{1}{t} \int_0^t \frac{1}{|I|} \int_I \varphi(a_s r_\theta) d\theta ds.$$

We have the following:

PROPOSITION 3.6. *Let \mathcal{N} be a (possibly empty) proper affine invariant submanifold. Then for any $\varepsilon > 0$, there exists an open set $\Omega_{\mathcal{N},\varepsilon}$ containing \mathcal{N} with $(\Omega_{\mathcal{N},\varepsilon})^c$ compact such that for any compact $F \subset \mathcal{H}_1(\alpha) \setminus \mathcal{N}$, there exists $t_0 \in \mathbb{R}$ so that for all $t > t_0$ and all $x \in F$, we have*

$$(\vartheta_{t,I} * \delta_x)(\Omega_{\mathcal{N},\varepsilon}) < \varepsilon.$$

Proof. This proof is virtually identical to the proof of Proposition 3.3. It is enough to prove the statement for the case $I = [0, 2\pi]$. Let $f_{\mathcal{N}}$ be the function of Proposition 2.13. Let $b > 0$ be as in Proposition 2.13(b), and let

$$\Omega_{\mathcal{N},\varepsilon} = \{p : f_{\mathcal{N}}(p) > (b+1)/\varepsilon\}^0,$$

where E^0 denotes the interior of E .

Suppose F is a compact subset of $\mathcal{H}_1(\alpha) \setminus \mathcal{N}$. Let $m_F = \sup\{f_{\mathcal{N}}(x) : x \in F\}$. By Proposition 2.13(b) with $c = \frac{1}{2m_F}$, there exists $t_1 > 0$ such that

$$(A_t f_{\mathcal{N}})(x) < \frac{1}{m_F} f_{\mathcal{N}}(x) + b \leq 1 + b \quad \text{for all } t > t_1 \text{ and all } x \in F.$$

By Proposition 2.13(a) there exists $R > 0$ such that $f_{\mathcal{N}}(a_t x) \leq R f_{\mathcal{N}}(x)$ for $0 \leq t \leq t_1$. Now choose t_0 so that $t_1 R/t_0 < m_F/2$. Then, for $t > t_0$,

$$\begin{aligned} (\vartheta_t * \delta_x)(f_{\mathcal{N}}) &= \frac{1}{t} \int_0^t (A_s f_{\mathcal{N}})(x) ds = \frac{1}{t} \int_0^{t_1} (A_s f_{\mathcal{N}})(x) ds + \frac{1}{t} \int_{t_1}^t (A_s f_{\mathcal{N}})(x) ds \\ &\leq \frac{t_1 R}{t} f_{\mathcal{N}}(x) + \left(\frac{m_F}{2} f_{\mathcal{N}}(x) + b \right) \leq m_F f_{\mathcal{N}}(x) + b \leq 1 + b. \end{aligned}$$

Thus for any $x \in F$, $t > t_0$ and $L > 0$, we have

$$(8) \quad (\vartheta_t * \delta_x)(\{p : f_{\mathcal{N}}(p) > L\}) < (b+1)/L.$$

Then (8) implies that $(\vartheta_t * \delta_x)(\Omega_{\mathcal{N}, \varepsilon}) < \varepsilon$. Also, Proposition 2.13(a) implies that $\Omega_{\mathcal{N}, \varepsilon}$ is a neighborhood of \mathcal{N} and

$$(\Omega_{\mathcal{N}, \varepsilon})^c = \overline{\{p : f_{\mathcal{N}}(p) \leq (b+1)/\varepsilon\}}$$

is compact. \square

LEMMA 3.7. *Suppose $t_i \rightarrow \infty$, $x_i \in \mathcal{H}_1(\alpha)$, and $\vartheta_{t_i, I} * \delta_{x_i} \rightarrow \nu$. Then ν is invariant under P (and then by Theorem 1.3 also invariant under $\mathrm{SL}(2, \mathbb{R})$).*

Proof. Let A denote the diagonal subgroup of $\mathrm{SL}(2, \mathbb{R})$, and let $U = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$. From the definition it is clear that ν is A -invariant. We will show it is also U -invariant; indeed, it suffices to show this for $u_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$ with $0 \leq s \leq 1$.

First note that for any $0 < \theta < \pi/2$, we have

$$(9) \quad r_{\theta} = g_{\theta} u_{\tan \theta}, \text{ where } g_{\theta} = \begin{pmatrix} \cos \theta & 0 \\ \sin \theta & 1/\cos \theta \end{pmatrix}.$$

Therefore, for all $\tau > 0$, we have $a_{\tau} g_{\theta} a_{\tau}^{-1} = \begin{pmatrix} \cos \theta & 0 \\ e^{-2\tau} \sin \theta & 1/\cos \theta \end{pmatrix}$. We have

$$(10) \quad a_{\tau} r_{\theta} = a_{\tau} g_{\theta} u_{\tan \theta} = a_{\tau} g_{\theta} a_{\tau}^{-1} u_{e^{2\tau} \tan \theta} a_{\tau}.$$

Fix some $0 < s < 1$, and define s_{τ} by $e^{2\tau} \tan s_{\tau} = s$. Then, (10) becomes

$$(11) \quad a_{\tau} r_{s_{\tau}} = (a_{\tau} g_{s_{\tau}} a_{\tau}^{-1}) u_s a_{\tau}.$$

For any $\varphi \in C_c(\mathcal{H}_1(\alpha))$ and all x , we have

$$(12) \quad \varphi(u_s a_{\tau} r_{\theta} x) - \varphi(a_{\tau} r_{\theta} x) = (\varphi(u_s a_{\tau} r_{\theta} x) - \varphi(a_{\tau} r_{\theta + s_{\tau}} x)) + (\varphi(a_{\tau} r_{\theta + s_{\tau}} x) - \varphi(a_{\tau} r_{\theta} x)).$$

We compute the contribution from the two parentheses separately. Note that terms in the first parenthesis are close to each other thanks to (11) and the definition of s_{τ} . The contribution from the second is controlled as the integral over I and a “small” translate of I are close to each other.

We carry out the computation here. First note that $s_\tau \rightarrow 0$ as $\tau \rightarrow \infty$. Furthermore, this and (9) imply that $a_\tau g_{s_\tau} a_\tau^{-1}$ tends to the identity matrix as $\tau \rightarrow \infty$. Therefore, given $\varepsilon > 0$, thanks to (11) and the uniform continuity of φ , we have

$$|\varphi(u_s a_\tau r_\theta x) - \varphi(a_\tau r_{\theta+s_\tau} x)| \leq \varepsilon$$

for all large enough τ and all $x \in \mathcal{H}_1(\alpha)$. Thus, for large enough n (depending on ε and φ), we get

$$(13) \quad \frac{1}{t_n} \int_0^{t_n} \frac{1}{|I|} \int_I |\varphi(u_s a_\tau r_\theta x_n) - \varphi(a_\tau r_{\theta+s_\tau} x_n)| d\theta d\tau \leq 2\varepsilon.$$

As for the second parentheses on the right side of (12), we have

$$\begin{aligned} & \left| \frac{1}{t_n} \int_0^{t_n} \frac{1}{|I|} \int_I (\varphi(a_\tau r_{\theta+s_\tau} x_n) - \varphi(a_\tau r_\theta x_n)) d\theta d\tau \right| \\ & \leq \frac{1}{t_n} \int_0^{t_n} \left| \frac{1}{|I|} \int_{I+s_\tau} \varphi(a_\tau r_\theta x_n) d\theta - \frac{1}{|I|} \int_I \varphi(a_\tau r_\theta x_n) d\theta \right| d\tau \leq \frac{C_\varphi}{t_n} \int_0^{t_n} s_\tau d\tau \\ & \leq \frac{C'_\varphi}{t_n}, \quad \text{since } s_\tau = O(e^{-2\tau}), \text{ and thus the integral converges.} \end{aligned}$$

This, together with (13) and (12), implies $|\nu(u_s \varphi) - \nu(\varphi)| \leq 2\varepsilon$; the lemma follows. \square

Now in view of Proposition 3.6 and Lemma 3.7, Theorems 2.6 and 2.7 hold by Theorem 3.5.

3.3. Equidistribution for some Følner sets. In this subsection, we prove Theorems 2.10 and 2.11. These theorems can be easily derived from Theorems 2.6 and 2.7, but we choose to derive them directly from Theorem 3.5.

Fix $r > 0$, and define a family of probability measures $\lambda_{t,r}$ on $\text{SL}(2, \mathbb{R})$ by

$$\lambda_{t,r}(\varphi) = \frac{1}{rt} \int_0^t \int_0^r \varphi(a_\tau u_s) ds d\tau.$$

The supports of the measures $\lambda_{t,r}$ form a Følner family as $t \rightarrow \infty$ (and r is fixed). Thus, any limit measure of the measures $\lambda_{t_i,r} * \delta_{x_i}$ is P -invariant (and thus $\text{SL}(2, \mathbb{R})$ -invariant by Theorem 1.3). Therefore, it remains to prove

PROPOSITION 3.8. *Let \mathcal{N} be a (possibly empty) proper affine invariant submanifold. Then for any $\varepsilon > 0$, there exists an open set $\Omega_{\mathcal{N},\varepsilon}$ containing \mathcal{N} with $(\Omega_{\mathcal{N},\varepsilon})^c$ compact such that for any compact $F \subset \mathcal{H}_1(\alpha) \setminus \mathcal{N}$, there exists $t_0 \in \mathbb{R}$ so that for all $t > t_0$ and all $x \in F$, we have*

$$(\lambda_{t,r} * \delta_x)(\Omega_{\mathcal{N},\varepsilon}) < \varepsilon.$$

Proof. It is enough to prove the statements for $r = \tan 0.01$. As in the proof of Lemma 3.7, we may write

$$r_\theta = g_\theta u_{\tan \theta}$$

and thus

$$a_t u_{\tan \theta} = a_t g_\theta^{-1} r_\theta = (a_t g_\theta^{-1} a_t^{-1}) a_t r_\theta.$$

Let $I = (0, 0.01)$. Note that $a_t g_\theta^{-1} a_t^{-1}$ remains bounded for $\theta \in I$ as $t \rightarrow \infty$. Also, the derivative of $\tan \theta$ is bounded between two nonzero constants for $\theta \in I$. Therefore, by Proposition 2.13(c), for all t and x ,

$$(\lambda_{t,r} * \delta_x)(f_{\mathcal{N}}) \leq C(\vartheta_{t,I} * \delta_x)(f_{\mathcal{N}}),$$

where C depends only on the constant σ in Proposition 2.13(c). Therefore, for all t and x ,

$$(\lambda_{t,r} * \delta_x)(f_{\mathcal{N}}) \leq C'(\vartheta_t * \delta_x)(f_{\mathcal{N}}),$$

where $C' = C/|I|$. Now let

$$\Omega_{\mathcal{N},\varepsilon} = \{p : f_{\mathcal{N}}(p) > C(b+1)/\varepsilon\}^0.$$

The rest of the proof is exactly as in Proposition 3.6. \square

Now Theorems 2.10 and 2.11 follow from Theorem 3.5.

3.4. Proofs of Theorems 2.1, 2.2 and 2.12.

Proof of Theorem 2.1. This is an immediate consequence of Theorem 2.10. \square

Proof of Theorem 2.2. Suppose $\mathcal{A} \subset \mathcal{H}_1(\alpha)$ is a closed P -invariant subset. Let Y denote the set of affine invariant manifolds contained in \mathcal{A} , and let Z consist of the set of maximal elements of Y (i.e., elements of Y which are not properly contained in another element of Y). By Theorem 2.1,

$$\mathcal{A} = \bigcup_{\mathcal{N} \in Y} \mathcal{N} = \bigcup_{\mathcal{N} \in Z} \mathcal{N}.$$

We now claim that Z is finite. Suppose not. Then there exists an infinite sequence \mathcal{N}_n of distinct submanifolds in Z . Then by Theorem 2.3, there exists a subsequence \mathcal{N}_{n_j} such that $\nu_{\mathcal{N}_{n_j}} \rightarrow \nu_{\mathcal{N}}$, where \mathcal{N} is another affine invariant manifold which contains all but finitely many \mathcal{N}_{n_j} . Without loss of generality, we may assume that $\mathcal{N}_{n_j} \subset \mathcal{N}$ for all j .

Since $\nu_{\mathcal{N}_{n_j}} \rightarrow \nu_{\mathcal{N}}$, the union of the \mathcal{N}_{n_j} is dense in \mathcal{N} . Since $\mathcal{N}_{n_j} \subset \mathcal{A}$ and \mathcal{A} is closed, $\mathcal{N} \subset \mathcal{A}$. Therefore, $\mathcal{N} \in Y$. But $\mathcal{N}_{n_j} \subset \mathcal{N}$, and therefore $\mathcal{N}_{n_j} \notin Z$. This is a contradiction. \square

Proof of Theorem 2.12. This is a consequence of Theorem 2.6; see [EM01, §§3–5] for the details. See also [EMM06, §8] for an axiomatic formulation and an outline of the argument.

We note that we do not have a convergence theorem for averages of the form

$$\lim_{t \rightarrow \infty} \frac{1}{2\pi} \int_0^{2\pi} \varphi(a_t r_\theta x) d\theta$$

and therefore we do not know that, e.g., assumption (C) of [EMM06, Th. 8.2] is satisfied. But by Theorem 2.6, we do have convergence for the averages

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \frac{1}{2\pi} \int_0^{2\pi} \varphi(a_s r_\theta x) d\theta ds.$$

Since we also have an extra average on the right-hand side of Theorem 2.12, the proof goes through virtually without modifications. \square

4. Recurrence properties

Recall that for a function $f : \mathcal{H}_1(\alpha) \rightarrow \mathbb{R}$,

$$(A_t f)(x) = \frac{1}{2\pi} \int_0^{2\pi} f(a_t r_\theta x) d\theta.$$

THEOREM 4.1 ([EM01] and [Ath06]). *There exists a continuous, proper, $\mathrm{SO}(2)$ -invariant function $u : \mathcal{H}_1(\alpha) \rightarrow [2, \infty)$ such that*

(i) *there exists $m \in \mathbb{R}$ such that for all $x \in \mathcal{H}_1(\alpha)$ and all $t > 0$,*

$$(14) \quad e^{-mt} u(x) \leq u(a_t x) \leq e^{mt} u(x);$$

(ii) *there exist constants $t_0 > 0$, $\tilde{\eta} > 0$ and $\tilde{b} > 0$ such that for all $t \geq t_0$ and all $x \in \mathcal{H}_1(\alpha)$, we have*

$$(15) \quad A_t u(x) \leq \tilde{c} u(x) + \tilde{b}, \quad \text{with } \tilde{c} = e^{-\tilde{\eta}t}.$$

We state some consequences of Theorem 4.1, mostly from [Ath06]:

THEOREM 4.2. *For any $\rho > 0$, there exists a compact $K_\rho \subset \mathcal{H}_1(\alpha)$ such that for any $\mathrm{SL}(2, \mathbb{R})$ -invariant probability measure ν ,*

$$\nu(K_\rho) > 1 - \rho.$$

Proof. The fact that this follows from Theorem 4.1 is well known and can be extracted, e.g., from the proof of [EM04, Lemma 2.2]. For a self-contained argument, one may use Lemma 11.1 in the present paper with $\sigma = e^{-m}$, $c = c_0(\sigma)$ and t_0 sufficiently large so that $e^{-\tilde{\eta}t_0} < c$, to obtain the estimate

$$\int_{\mathcal{H}_1(\alpha)} u(x) d\nu(x) < B,$$

where B depends only on the constants of Theorem 4.1. This implies that

$$\nu(\{x : u(x) > B/\rho\}) < \rho,$$

as required. \square

THEOREM 4.3. *Let K_ρ be as in Theorem 4.2. Then, if $\rho > 0$ is sufficiently small, there exists a constant $m'' > 0$ such that for all $x \in \mathcal{H}_1(\alpha)$, there exist $\theta \in [0, 2\pi]$ and $\tau \leq m'' \log u(x)$ such that $x' \equiv a_\tau r_\theta x \in K_\rho$.*

Proof. This follows from [Ath06, Th. 2.2], with $\delta = 1/2$. \square

THEOREM 4.4. *For $x \in \mathcal{H}_1(\alpha)$ and a compact set $K_* \subset \mathcal{H}_1(\alpha)$, define*

$$\mathcal{I}_1(t) = \{\theta \in [0, 2\pi] : |\{\tau \in [0, t] : a_\tau r_\theta x \in K_*\}| > t/2\}$$

and

$$\mathcal{I}_2(t) = [0, 2\pi] \setminus \mathcal{I}_1(t).$$

Then, there exist some $\eta_1 > 0$, a compact subset K_ , and constants $L_0 > 0$ and $\eta_0 > 0$ such that for any $t > 0$,*

$$(16) \quad \text{if } \log u(x) < L_0 + \eta_0 t, \quad \text{then } |\mathcal{I}_2(t)| < e^{-\eta_1 t}.$$

Theorem 4.4 is not formally stated in [Ath06], but it is a combination of [Ath06, Th. 2.2] and [Ath06, Th. 2.3]. (In the proof of [Ath06, Th. 2.3], one should use [Ath06, Th. 2.2] to control the distribution of τ_0 .)

5. Period coordinates and the Kontsevich-Zorich cocycle

Let $\Sigma \subset M$ denote the set of zeroes of ω . Let $\{\gamma_1, \dots, \gamma_k\}$ denote a \mathbb{Z} -basis for the relative homology group $H_1(M, \Sigma, \mathbb{Z})$. (It is convenient to assume that the basis is obtained by extending a symplectic basis for the absolute homology group $H_1(M, \mathbb{Z})$.) We can define a map $\Phi : \mathcal{H}(\alpha) \rightarrow \mathbb{C}^k$ by

$$\Phi(M, \omega) = \left(\int_{\gamma_1} \omega, \dots, \int_{\gamma_k} \omega \right).$$

The map Φ (which depends on a choice of the basis $\{\gamma_1, \dots, \gamma_n\}$) is a local coordinate system on (M, ω) . Alternatively, we may think of the cohomology class $[\omega] \in H^1(M, \Sigma, \mathbb{C})$ as a local coordinate on the stratum $\mathcal{H}(\alpha)$. We will call these coordinates *period coordinates*.

The $\mathrm{SL}(2, \mathbb{R})$ -action and the Kontsevich-Zorich cocycle. We write $\Phi(M, \omega)$ as a $2 \times n$ matrix x . The action of $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R})$ in these coordinates is linear. Let $\mathrm{Mod}(M, \Sigma)$ be the mapping class group of M fixing each zero of ω . We choose some fundamental domain for the action of $\mathrm{Mod}(M, \Sigma)$, and we

think of the dynamics on the fundamental domain. Then, the $\mathrm{SL}(2, \mathbb{R})$ action becomes

$$x = \begin{pmatrix} x_1 & \dots & x_n \\ y_1 & \dots & y_n \end{pmatrix} \rightarrow gx = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 & \dots & x_n \\ y_1 & \dots & y_n \end{pmatrix} A(g, x),$$

where $A(g, x) \in \mathrm{Sp}(2g, \mathbb{Z}) \times \mathbb{R}^{n-1}$ is the *Kontsevich-Zorich cocycle*. Thus, $A(g, x)$ is the change of basis one needs to perform to return the point gx to the fundamental domain. It can be interpreted as the monodromy of the Gauss-Manin connection (restricted to the orbit of $\mathrm{SL}(2, \mathbb{R})$).

6. The Hodge norm

Let M be a Riemann surface. By definition, M has a complex structure. Let \mathcal{H}_M denote the set of holomorphic 1-forms on M . One can define *Hodge inner product* on \mathcal{H}_M by

$$\langle \omega, \eta \rangle = \frac{i}{2} \int_M \omega \wedge \bar{\eta}.$$

We have a natural map $r : H^1(M, \mathbb{R}) \rightarrow \mathcal{H}_M$ which sends a cohomology class $\lambda \in H^1(M, \mathbb{R})$ to the holomorphic 1-form $r(\lambda) \in \mathcal{H}_M$ such that the real part of $r(\lambda)$ (which is a harmonic 1-form) represents λ . We can thus define the Hodge inner product on $H^1(M, \mathbb{R})$ by $\langle \lambda_1, \lambda_2 \rangle = \langle r(\lambda_1), r(\lambda_2) \rangle$. We have

$$\langle \lambda_1, \lambda_2 \rangle = \int_M \lambda_1 \wedge * \lambda_2,$$

where $*$ denotes the Hodge star operator, and we choose harmonic representatives of λ_1 and $*\lambda_2$ to evaluate the integral. We denote the associated norm by $\|\cdot\|_M$. This is the *Hodge norm*; see [FK80].

If $x = (M, \omega) \in \mathcal{H}_1(\alpha)$, we will often write $\|\cdot\|_x$ to denote the Hodge norm $\|\cdot\|_M$ on $H^1(M, \mathbb{R})$. Since $\|\cdot\|_x$ depends only on M , we have $\|\lambda\|_{kx} = \|\lambda\|_x$ for all $\lambda \in H^1(M, \mathbb{R})$ and all $k \in \mathrm{SO}(2)$.

Let $E(x) = \mathrm{span}\{\Re(\omega), \Im(\omega)\}$. (Many authors refer to $E(x)$ as the “standard space”.) We let $p : H^1(M, \Sigma, \mathbb{R}) \rightarrow H^1(M, \mathbb{R})$ denote the natural projection; using this map, $p(E(x)) \subset H^1(M, \mathbb{R})$. For any $v \in E(x)$ and any point y in the $\mathrm{SL}(2, \mathbb{R})$ orbit of x , the Hodge norm $\|v\|_y$ of v at y can be explicitly computed. In fact, the following elementary lemma holds:

LEMMA 6.1. Suppose $x \in \mathcal{H}_1(\alpha)$, $g = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R})$,

$$v = v_1 p(\Re(\omega)) + v_2 p(\Im(\omega)) \in p(E(x)).$$

Let

$$(17) \quad \begin{pmatrix} u_1 & u_2 \end{pmatrix} = \begin{pmatrix} v_1 & v_2 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^{-1}.$$

Then,

$$(18) \quad \|v\|_{gx} = \|u_1^2 + u_2^2\|^{1/2}.$$

Proof. Let

$$(19) \quad c_1 = a_{11}p(\Re(\omega)) + a_{12}p(\Im(\omega)), \quad c_2 = a_{21}p(\Re(\omega)) + a_{22}p(\Im(\omega)).$$

By the definition of the $\mathrm{SL}(2, \mathbb{R})$ action, $c_1 + ic_2$ is holomorphic on gx . Therefore, by the definition of the Hodge star operator, at gx ,

$$*c_1 = c_2, \quad *c_2 = -c_1.$$

Therefore,

$$\|c_1\|_{gx}^2 = c_1 \wedge *c_1 = c_1 \wedge c_2 = (\det g)p(\Re(\omega)) \wedge p(\Im(\omega)) = 1,$$

where for the last equality we used the fact that $x \in \mathcal{H}_1(\alpha)$. Similarly, we get

$$(20) \quad \|c_1\|_{gx} = 1, \quad \|c_2\|_{gx} = 1, \quad \langle c_1, c_2 \rangle_{gx} = 0.$$

Write

$$v = v_1p(\Re(\omega)) + v_2p(\Im(\omega)) = u_1c_1 + u_2c_2.$$

Then, in view of (19), u_1 and u_2 are given by (17). Equation (18) follows from (20). \square

On the complementary subspace to $p(E(x))$, there is no explicit formula comparable to Lemma 6.1. However, we have the following fundamental result due to Forni [For02, Cor. 2.1] (see also [FMZ14, Cor. 2.1]):

LEMMA 6.2. *There exists a continuous function $\Lambda : \mathcal{H}_1(\alpha) \rightarrow (0, 1)$ such that for any $c \in H^1(M, \mathbb{R})$ with $c \wedge p(E(x)) = 0$, any $x \in \mathcal{H}_1(\alpha)$ and any $t > 0$, we have*

$$\|c\|_x e^{-\beta_t(x)} \leq \|c\|_{a_t x} \leq \|c\|_x e^{\beta_t(x)},$$

where $\beta_t(x) = \int_0^t \Lambda(a_\tau x) d\tau$.

Let $\mathcal{I}_1(t)$ and $\mathcal{I}_2(t)$ be as in Theorem 4.4. Now compactness of K_* and Lemma 6.2 imply that

$$(21) \quad \text{there exists } \eta_2 > 0 \text{ such that for all } x \in \mathcal{H}_1(\alpha), \\ \text{if } t > t_0 \text{ and } \theta \in \mathcal{I}_1(t), \text{ then } \beta_t(r_\theta x) < (1 - \eta_2)t.$$

7. Expansion on average of the Hodge norm

Recall that $p : H^1(M, \Sigma, \mathbb{R}) \rightarrow H^1(M, \mathbb{R})$ denotes the natural projection. Let \mathcal{M}_1 be an affine invariant suborbifold of $\mathcal{H}_1(\alpha)$, and let $\mathcal{M} = \mathbb{R}\mathcal{M}_1$ be as above. Then \mathcal{M} is given by complex linear equations in period coordinates and is $\mathrm{GL}(2, \mathbb{R})$ -invariant. We let L denote this subspace in $H^1(M, \Sigma, \mathbb{R})$.

Recall that $H^1(M, \mathbb{R})$ is endowed with a natural symplectic structure given by the wedge product on de Rham cohomology and also the Hodge inner product. It is shown in [AEM15] that the wedge product restricted to $p(L)$ is nondegenerate. Therefore, there exists an $\mathrm{SL}(2, \mathbb{R})$ -invariant complement for $p(L)$ in $H^1(M, \mathbb{R})$, which we denote by $p(L)^\perp$.

We will use the following elementary lemma with $d = 2, 3$:

LEMMA 7.1. *Let V be a d -dimensional vector space on which $\mathrm{SL}(2, \mathbb{R})$ acts irreducibly, and let $\|\cdot\|$ be any $\mathrm{SO}(2)$ -invariant norm on V . Then there exists $\delta_0(d) > 0$ (depending on d) such that for any $\delta < \delta_0(d)$, any $t > 0$ and any $v \in V$,*

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{\|a_t r_\theta v\|^\delta} \leq \frac{e^{-k_d t}}{\|v\|^\delta},$$

where $k_d = k_d(\delta) > 0$.

Proof. This is essentially the case $G = \mathrm{SL}(2, \mathbb{R})$ of [EMM98, Lemma 5.1]. The exponential estimate in the right-hand side is not stated in [EMM98, Lemma 5.1] but follows easily from the proof of the lemma. \square

The space $H'(x)$ and the function ψ_x . For $x = (M, \omega)$, let

$$H'(x) = \{v \in H^1(M, \mathbb{C}) : v \wedge \overline{p(\omega)} + p(\omega) \wedge \overline{v} = 0\}.$$

We have, for any $x = (M, \omega)$,

$$H^1(M, \mathbb{C}) = \mathbb{R} p(\omega) \oplus H'(x).$$

(Here and below, we are considering $H^1(M, \mathbb{C})$ as a real vector space.) For $v \in H^1(M, \mathbb{C})$, let

$$\psi_x(v) = \frac{\|v\|_x}{\|v'\|_x}, \quad \text{where } v = \lambda p(\omega) + v', \lambda \in \mathbb{R}, v' \in H'(x).$$

Then $\psi_x(v) \geq 1$, and $\psi_x(v)$ is bounded if v is bounded away from $\mathbb{R} p(\omega)$.

7.1. Absolute cohomology. Fix some $\delta \leq 0.1 \min(\eta_1, \eta_2, \delta_0(2), \delta_0(3))$. For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $v \in H^1(M, \mathbb{C})$, we write

$$(22) \quad gv = a \Re(v) + b \Im(v) + i(c \Re(v) + d \Im(v)).$$

LEMMA 7.2. *There exists $C_0 > 1$ such that for all $x = (M, \omega) \in \mathcal{H}_1(\alpha)$, all $t > 0$ and all $v \in H^1(M, \mathbb{C})$, we have*

$$(23) \quad \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{(\|a_t r_\theta v\|_{a_t r_\theta x})^{\delta/2}} \leq \min \left(\frac{C_0}{\|v\|_x^{\delta/2}}, \frac{\psi_x(v)^{\delta/2} \kappa(x, t)}{\|v\|_x^{\delta/2}} \right),$$

where

(a) $\kappa(x, t) \leq C_0$ for all x and all t ; and

(b) *there exists $\eta > 0$ such that*

$$\kappa(x, t) \leq C_0 e^{-\eta t}, \quad \text{provided } \log u(x) < L_0 + \eta_0 t,$$

where the constants L_0 and η_0 are as in Theorem 4.4.

Proof. For $x = (M, \omega) \in \mathcal{H}_1(\alpha)$, we have an $\mathrm{SL}(2, \mathbb{R})$ -invariant and Hodge-orthogonal decomposition

$$H^1(M, \mathbb{R}) = p(E(x)) \oplus H^1(M, \mathbb{R})^\perp,$$

where $E(x) = \mathrm{span}\{\Re(\omega), \Im(\omega)\}$ and

$$H^1(M, \mathbb{R})^\perp(x) = \{c \in H^1(M, \mathbb{R}) : c \wedge p(E(x)) = 0\}.$$

For a subspace $V \subset H^1(M, \mathbb{R})$, let $V_{\mathbb{C}} \subset H^1(M, \mathbb{C})$ denote its complexification. Then, we have

$$(24) \quad H^1(M, \mathbb{C}) = p(E(x))_{\mathbb{C}} \oplus H^1(M, \mathbb{R})_{\mathbb{C}}^\perp(x).$$

Note that $H^1(M, \mathbb{R})_{\mathbb{C}}^\perp(x) \subset H'(x)$. We can write

$$v = \lambda \omega + u + w,$$

where $\lambda \in \mathbb{R}$, $u \in p(E(x))_{\mathbb{C}} \cap H'(x)$, $w \in H^1(M, \mathbb{R})_{\mathbb{C}}^\perp(x)$. Since $u \in p(E(x))_{\mathbb{C}}$, we may write

$$u = u_{11} p(\Re(\omega)) + u_{12} p(\Im(\omega)) + i(u_{21} p(\Re(\omega)) + u_{22} p(\Im(\omega))).$$

Since $u \in H'(x)$,

$$(25) \quad u_{11} + u_{22} = 0.$$

Recall that the Hilbert-Schmidt norm $\|\cdot\|_{\mathrm{HS}}$ of a matrix is the square root of the sum of the squares of the entries. Then,

(26)

$$\begin{aligned} & (\|a_t r_\theta(p(\omega) + u)\|_{a_t r_\theta x})^2 \\ &= \left\| (a_t r_\theta) \begin{pmatrix} \lambda + u_{11} & u_{12} \\ u_{21} & \lambda + u_{22} \end{pmatrix} (a_t r_\theta)^{-1} \right\|_{\mathrm{HS}}^2 \quad \text{by Lemma 6.1 and (22)} \\ &= \lambda^2 + \left\| (a_t r_\theta) \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} (a_t r_\theta)^{-1} \right\|_{\mathrm{HS}}^2 \quad \text{by (25).} \end{aligned}$$

Since the decomposition (24) is Hodge orthogonal, it follows that for all t and all θ ,

$$(27) \quad (\|a_t r_\theta v\|_{a_t r_\theta x})^2 = \lambda^2 + (\|a_t r_\theta u\|_{a_t r_\theta x})^2 + (\|a_t r_\theta w\|_{a_t r_\theta x})^2.$$

By (26), (25) and Lemma 7.1,

$$(28) \quad \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{(\|a_t r_\theta u\|_{a_t r_\theta x})^{\delta/2}} \leq e^{-k_3 t} \|u\|_x^{\delta/2},$$

where $k_3 > 0$. We now claim that

$$(29) \quad \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{\|a_t r_\theta w\|^{\delta/2}} \leq \frac{\kappa_2(x, t)}{\|w\|^{\delta/2}},$$

where for some absolute constant $C > 0$ and $\eta > 0$, and for L_0 and η_0 as in Theorem 4.4, we have

$$(30) \quad \begin{cases} \kappa_2(x, t) \leq C & \text{for all } x \in \mathcal{H}_1(\alpha), t \geq 0, \\ \kappa_2(x, t) \leq C e^{-\eta t} & \text{provided } \log u(x) < L_0 + \eta_0 t. \end{cases}$$

Assuming (29) and (30), we have

$$\begin{aligned} & \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{(\|a_t r_\theta v\|_{a_t r_\theta x})^{\delta/2}} \\ & \leq \frac{1}{2\pi} \int_0^{2\pi} \min \left(\frac{1}{\lambda^{\delta/2}}, \frac{1}{(\|a_t r_\theta u\|_{a_t r_\theta x})^{\delta/2}}, \frac{1}{(\|a_t r_\theta w\|_{a_t r_\theta x})^{\delta/2}} \right) d\theta \quad \text{by (27)} \\ & \leq \min \left(\frac{1}{\lambda^{\delta/2}}, \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{(\|a_t r_\theta u\|_{a_t r_\theta x})^{\delta/2}}, \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{(\|a_t r_\theta w\|_{a_t r_\theta x})^{\delta/2}} \right) \\ & \leq \min \left(\frac{1}{\lambda^{\delta/2}}, \frac{e^{-k_3 t}}{\|u\|_x^{\delta/2}}, \frac{\kappa_2(x, t)}{\|w\|_x^{\delta/2}} \right) \quad \text{by (28) and (29).} \end{aligned}$$

Since we must have either $\lambda > \|v\|_x/3$, or $\|u\|_x > \|v\|_x/3$ or $\|w\|_x > \|v\|_x/3$, we have for all x, t ,

$$\min \left(\frac{1}{\lambda^{\delta/2}}, \frac{e^{-k_3 t}}{\|u\|_x^{\delta/2}}, \frac{\kappa_2(x, t)}{\|w\|_x^{\delta/2}} \right) \leq \frac{3^{\delta/2} \max(1, e^{-k_3 t}, \kappa_2(x, t))}{\|v\|_x^{\delta/2}} \leq \frac{C_0}{\|v\|_x^{\delta/2}},$$

where for the last estimate, we used the fact that both k_3 and κ_2 are bounded functions. Also, we have $\|u + w\|_x = \psi_x(v)^{-1} \|v\|_x$, hence either $\|u\|_x \geq \psi_x(v)^{-1} \|v\|_x/2$ or $\|w\|_x \geq \psi_x(v)^{-1} \|v\|_x/2$, and therefore, for all x, t ,

$$\begin{aligned} \min \left(\frac{1}{\lambda^{\delta/2}}, \frac{e^{-k_3 t}}{\|u\|_x^{\delta/2}}, \frac{\kappa_2(x, t)}{\|w\|_x^{\delta/2}} \right) & \leq \frac{\psi_x(v)^{\delta/2} \max(e^{-k_3 t}, \kappa_2(x, t))}{\|v\|_x^{\delta/2}} \\ & \equiv \frac{\psi_x(v)^{\delta/2} \kappa(x, t)}{\|v\|_x^{\delta/2}}. \end{aligned}$$

Therefore, (23) holds. This completes the proof of the lemma, assuming (29) and (30).

It remains to prove (29) and (30). Let L_0 and η_0 be as in Theorem 4.4, and suppose $\log u(x) < L_0 + \eta_0 t$. Recall that $\mathcal{I}_1(t)$ and $\mathcal{I}_2(t)$ are defined relative to the compact set K_* in Theorem 4.4. We have

$$\int_0^{2\pi} \frac{d\theta}{(\|a_t r_\theta w\|_{a_t r_\theta x})^{\delta/2}} = \int_{\mathcal{I}_1(t)} \frac{d\theta}{(\|a_t r_\theta w\|_{a_t r_\theta x})^{\delta/2}} + \int_{\mathcal{I}_2(t)} \frac{d\theta}{(\|a_t r_\theta w\|_{a_t r_\theta x})^{\delta/2}}.$$

Using (16) and Lemma 6.2, we get

$$\int_{\mathcal{I}_2(t)} \frac{d\theta}{(\|a_t r_\theta w\|_{a_t r_\theta x})^{\delta/2}} \leq \frac{e^{-\eta_1 t} e^{\delta t/2}}{\|v\|^{\delta/2}}.$$

Also,

$$\begin{aligned} & \int_{\mathcal{I}_1(t)} \frac{d\theta}{(\|a_t r_\theta w\|_{a_t r_\theta x})^{\delta/2}} \\ & \leq \int_{\mathcal{I}_1(t)} \frac{d\theta}{(\|\Re(a_t r_\theta w)\|_{a_t r_\theta x})^{\delta/2}} && \text{since } \|z\| \geq \|\Re(z)\| \\ & = \int_{\mathcal{I}_1(t)} \frac{d\theta}{(\|e^t \Re(r_\theta w)\|_{a_t r_\theta x})^{\delta/2}} && \text{by (22)} \\ & \leq \int_{\mathcal{I}_1(t)} \frac{e^{-(1-\beta_t(r_\theta x))\delta t/2}}{\|\Re(r_\theta w)\|_x^{\delta/2}} && \text{by Lemma 6.2} \\ & \leq e^{-\eta_2 \delta t/2} \int_0^{2\pi} \frac{d\theta}{\|\Re(r_\theta w)\|_x^{\delta/2}} && \text{by (21)} \\ & = e^{-\eta_2 \delta t/2} \int_0^{2\pi} \frac{d\theta}{\|\cos \theta \Re(w) + \sin \theta \Im(w)\|_x^{\delta/2}} \\ & \leq \frac{C_2 e^{-\eta_2 \delta t/2}}{\|w\|^{\delta/2}}. && \text{since the integral converges.} \end{aligned}$$

These estimates imply (29) and (30) for the case when $\log u(x) < L_0 + \eta_0 t$. If x is arbitrary, we need to show (29) holds with $\kappa_2(x, t) \leq C$. Note that

$$\begin{aligned} \|a_t r_\theta w\|_{a_t r_\theta x} & \geq \|\Re(a_t r_\theta w)\|_{a_t r_\theta x} && \text{since } \|z\| \geq \|\Re(z)\| \\ & = \|e^t (\cos \theta \Re(w) + \sin \theta \Im(w))\|_{a_t r_\theta x} && \text{by (22)} \\ & = e^t \|\cos \theta \Re(w) + \sin \theta \Im(w)\|_{a_t r_\theta x} \\ & \geq \|\cos \theta \Re(w) + \sin \theta \Im(w)\|_x && \text{by Lemma 6.2.} \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{(\|a_t r_\theta w\|_{a_t r_\theta x})^{\delta/2}} & \leq \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{\|\cos \theta \Re(w) + \sin \theta \Im(w)\|_x^{\delta/2}} \\ & \leq \frac{C_2}{\|w\|_x^{\delta/2}} \text{ since } \delta \leq 0.1 \text{ and the integral converges.} \end{aligned}$$

This completes the proof of (29) and (30) for arbitrary x . \square

7.2. The modified Hodge norm. For the application in Section 7.3, we will need to consider a modification of the Hodge norm in the thin part of moduli space.

The classes $c_\alpha, *c_\alpha$. Let α be a homology class in $H_1(M, \mathbb{R})$. We can define the cohomology class $*c_\alpha \in H^1(M, \mathbb{R})$ so that for all $\omega \in H^1(M, \mathbb{R})$,

$$\int_\alpha \omega = \int_M \omega \wedge *c_\alpha.$$

Then,

$$\int_M *c_\alpha \wedge *c_\beta = I(\alpha, \beta),$$

where $I(\cdot, \cdot)$ denotes algebraic intersection number. Let $*$ denote the Hodge star operator, and let

$$c_\alpha = *^{-1}(*c_\alpha).$$

Then, for any $\omega \in H^1(M, \mathbb{R})$, we have

$$\langle \omega, c_\alpha \rangle = \int_M \omega \wedge *c_\alpha = \int_\alpha \omega,$$

where $\langle \cdot, \cdot \rangle$ is the Hodge inner product. We note that $*c_\alpha$ is a purely topological construction which depends only on α , but c_α depends also on the complex structure of M .

Fix $\varepsilon_* > 0$ (the *Margulis constant*) so that any two curves of hyperbolic length less than ε_* must be disjoint.

Let σ denote the hyperbolic metric in the conformal class of M . For a closed curve α on M , $\ell_\alpha(\sigma)$ denotes the length of the geodesic representative of α in the metric σ .

We recall the following:

THEOREM 7.3 ([ABEM12, Th. 3.1]). *For any constant $D > 1$, there exists a constant $c > 1$ such that for any simple closed curve α with $\ell_\alpha(\sigma) < D$,*

$$(31) \quad \frac{1}{c} \ell_\alpha(\sigma)^{1/2} \leq \|c_\alpha\| < c \ell_\alpha(\sigma)^{1/2}.$$

Furthermore, if $\ell_\alpha(\sigma) < \varepsilon_*$ and β is the shortest simple closed curve crossing α , then

$$\frac{1}{c} \ell_\alpha(\sigma)^{-1/2} \leq \|c_\beta\| < c \ell_\alpha(\sigma)^{-1/2}.$$

Short bases. Suppose $(M, \omega) \in \mathcal{H}_1(\alpha)$. Fix $\varepsilon_1 < \varepsilon_*$, and let $\alpha_1, \dots, \alpha_k$ be the curves with hyperbolic length less than ε_1 on M . For $1 \leq i \leq k$, let β_i be the shortest curve in the flat metric defined by ω with $i(\alpha_i, \beta_i) = 1$. We can pick simple closed curves γ_r , $1 \leq r \leq 2g - 2k$ on M so that the hyperbolic length of each γ_r is bounded by a constant L depending only on the genus, and so that the α_j , β_j and γ_j are a symplectic basis \mathcal{S} for $H_1(M, \mathbb{R})$. We will call such a basis *short*. A short basis is not unique, and in the following we fix some measurable choice of a short basis at each point of $\mathcal{H}_1(\alpha)$.

We now define a modification of the Hodge norm, which is similar to the one used in [ABEM12]. The modified norm is defined on the tangent

space to the space of pairs (M, ω) , where M is a Riemann surface and ω is a holomorphic 1-form on M . Unlike the Hodge norm, the modified Hodge norm will depend not only on the complex structure on M but also on the choice of a holomorphic 1-form ω on M . Let $\{\alpha_i, \beta_i, \gamma_r\}_{1 \leq i \leq k, 1 \leq r \leq 2g-2k}$ be a short basis for $x = (M, \omega)$.

We can write any $\theta \in H^1(M, \mathbb{R})$ as

$$(32) \quad \theta = \sum_{i=1}^k a_i (*c_{\alpha_i}) + \sum_{i=1}^k b_i \ell_{\alpha_i}(\sigma)^{1/2} (*c_{\beta_i}) + \sum_{r=1}^{2g-2k} u_r (*c_{\gamma_r}).$$

We then define

$$(33) \quad \|\theta\|_x'' = \|\theta\|_x + \left(\sum_{i=1}^k |a_i| + \sum_{i=1}^k |b_i| + \sum_{r=1}^{2g-2k} |u_r| \right).$$

We note that $\|\cdot\|''$ depends on the choice of short basis; however, switching to a different short basis can change $\|\cdot\|''$ by at most a fixed multiplicative constant depending only on the genus. To manage this, we use the notation $A \approx B$ to denote the fact that A/B is bounded from above and below by constants depending on the genus.

From (33), for $1 \leq i \leq k$, we have

$$(34) \quad \|*c_{\alpha_i}\|_x'' \approx 1.$$

Similarly, we have

$$(35) \quad \|*c_{\beta_i}\|_x'' \approx \|*c_{\beta_i}\|_x \approx \frac{1}{\ell_{\alpha_i}(\sigma)^{1/2}}.$$

In addition, in view of Theorem 7.3, if γ is any other moderate length curve on M , then $\|*c_{\gamma}\|_x'' \approx \|*c_{\gamma}\|_x = O(1)$. Thus, if \mathcal{B} is a short basis at $x = (M, \omega)$, then for any $\gamma \in \mathcal{B}$,

$$(36) \quad \text{Ext}_{\gamma}(x)^{1/2} \approx \|*c_{\gamma}\| \leq \|*c_{\gamma}\|''.$$

(By $\text{Ext}_{\gamma}(x)$ we mean the extremal length of γ in M , where $x = (M, \omega)$.)

Remark. From the construction, we see that the modified Hodge norm is greater than the Hodge norm. Also, if the flat length of shortest curve in the flat metric defined by ω is greater than ε_1 , then for any cohomology class λ , for some C depending on ε_1 and the genus,

$$(37) \quad \|\lambda\|'' \leq C\|\lambda\|;$$

i.e., the modified Hodge norm is within a multiplicative constant of the Hodge norm.

From the definition, we have the following:

LEMMA 7.4. *There exists a constant $C > 1$ depending only on the genus such that for any $t > 0$, any $x \in \mathcal{H}_1(\alpha)$ and any $\lambda \in H^1(M, \mathbb{R})$,*

$$C^{-1}e^{-2t}\|\lambda\|_x'' \leq \|\lambda\|_{a_tx}'' \leq Ce^{2t}\|\lambda\|_x''.$$

Proof. From the definition of $\|\cdot\|''$, for any $x \in \mathcal{H}_1(\alpha)$,

$$(38) \quad C_1^{-1}\|\lambda\|_x \leq \|\lambda\|_x'' \leq C_1\ell_{\text{hyp}}(x)^{-1/2}\|\lambda\|_x,$$

where C depends only on the genus and $\ell_{\text{hyp}}(x)$ is the hyperbolic length of the shortest closed curve on x . It is well known that for very short curves, the hyperbolic length is comparable to the extremal length; see, e.g., [Mas85]. It follows immediately from Kerckhoff's formula for the Teichmüller distance that

$$e^{-2t} \text{Ext}_\gamma(x) \leq \text{Ext}_\gamma(a_tx) \leq e^{2t} \text{Ext}_\gamma(x).$$

Therefore,

$$(39) \quad C_2e^{-2t}\ell_{\text{hyp}}(x) \leq \ell_{\text{hyp}}(a_tx) \leq C_2e^{2t}\ell_{\text{hyp}}(x),$$

where C_2 depends only on the genus. Now the lemma follows immediately from (38), (39) and Lemma 6.2. \square

One annoying feature of our definition is that for a fixed absolute cohomology class λ , $\|\lambda\|_x''$ is not a continuous function of x , as x varies in a Teichmüller disk, due to the dependence on the choice of short basis. To remedy this, we pick a positive continuous $\text{SO}(2)$ -bi-invariant function ϕ on $\text{SL}(2, \mathbb{R})$ supported on a neighborhood of the identity e such that $\int_{\text{SL}(2, \mathbb{R})} \phi(g) dg = 1$, and we define

$$\|\lambda\|_x' = \|\lambda\|_x + \int_{\text{SL}(2, \mathbb{R})} \|\lambda\|_{gx}'' \phi(g) dg.$$

Then, it follows from Lemma 7.4 that for a fixed λ , $\log \|\lambda\|_x'$ is uniformly continuous as x varies in a Teichmüller disk. In fact, there is a constant m_0 such that for all $x \in \mathcal{H}_1(\alpha)$, all $\lambda \in H^1(M, \mathbb{R})$ and all $t > 0$,

$$(40) \quad e^{-m_0t}\|\lambda\|_x' \leq \|\lambda\|_{a_tx}' \leq e^{m_0t}\|\lambda\|_x'.$$

Remark. Even though $\|\cdot\|_x'$ is uniformly continuous as long as x varies in a Teichmüller disk, it may be only measurable in general (because of the choice of short basis). In the end, this causes our function $f_{\mathcal{M}}$ of Proposition 2.13 to be discontinuous.

7.3. Relative cohomology. For $c \in H^1(M, \Sigma, \mathbb{R})$ and $x = (M, \omega) \in \mathcal{H}_1(\alpha)$, let $\mathbf{p}_x(c)$ denote the harmonic representative of $p(c)$, where $p : H^1(M, \Sigma, \mathbb{R}) \rightarrow H^1(M, \mathbb{R})$ is the natural map. We view $\mathbf{p}_x(c)$ as an element of $H^1(M, \Sigma, \mathbb{R})$.

Then, (similarly to [EMR12]) we define the Hodge norm on $H^1(M, \Sigma, \mathbb{R})$ as

$$\|c\|'_x = \|p(c)\|'_x + \sum_{(z, z') \in \Sigma \times \Sigma} \left| \int_{\gamma_{z, z'}} (c - \mathbf{p}_x(c)) \right|,$$

where $\gamma_{z, z'}$ is any path connecting the zeroes z and z' of ω . Since $c - \mathbf{p}_x(c)$ represents the zero class in absolute cohomology, the integral does not depend on the choice of $\gamma_{z, z'}$. Note that the $\|\cdot\|'$ norm on $H^1(M, \Sigma, \mathbb{R})$ is invariant under the action of $\mathrm{SO}(2)$.

As above, we pick a positive continuous $\mathrm{SO}(2)$ -bi-invariant function ϕ on $\mathrm{SL}(2, \mathbb{R})$ supported on a neighborhood of the identity e such that $\int_{\mathrm{SL}(2, \mathbb{R})} \phi(g) dg = 1$, and we define

$$(41) \quad \|\lambda\|_x = \int_{\mathrm{SL}(2, \mathbb{R})} \|\lambda\|'_{gx} \phi(g) dg.$$

Then, the $\|\cdot\|$ norm on $H^1(M, \Sigma, \mathbb{R})$ is also invariant under the action of $\mathrm{SO}(2)$.

Notational warning. If λ is an absolute cohomology class, then $\|\lambda\|_x$ denotes the Hodge norm of λ at x defined in Section 6. If, however, λ is a relative cohomology class, then $\|\lambda\|_x$ is defined in (41). We hope the meaning will be clear from the context.

We will use the following crude version of Lemma 6.2. (Much more accurate versions are possible, especially in compact sets; see, e.g., [EMR12].)

LEMMA 7.5. *There exists a constant $m' > m_0 > 0$ such that for any $x \in \mathcal{H}_1(\alpha)$, any $\lambda \in H^1(M, \Sigma, \mathbb{R})$ and any $t > 0$,*

$$e^{-m't} \|\lambda\|_x \leq \|\lambda\|_{a_t x} \leq e^{m't} \|\lambda\|_x.$$

Proof. We remark that this proof fails if we use the standard Hodge norm on absolute homology. It is enough to prove the statement assuming $0 \leq t \leq 1$, since the statement for arbitrary t then follows by iteration. It is also enough to check this for the case when $p(\lambda) = *c_\gamma$, where γ is an element of a short basis.

Let $\alpha_1, \dots, \alpha_n$ be the curves with hyperbolic length less than ε_1 . For $1 \leq k \leq n$, let β_k be the shortest curve with $i(\alpha_k, \beta_k) = 1$, where $i(\cdot, \cdot)$ denotes the geometric intersection number. Let γ_r , $1 \leq r \leq 2g - 2k$ be moderate length curves on M so that the α_j , β_j and γ_j are a symplectic basis \mathcal{S} for $H_1(M, \mathbb{R})$. Then \mathcal{S} is a short basis for $x = (M, \omega)$.

We now claim that for any curve $\gamma \in \mathcal{S}$ and any i, j ,

$$(42) \quad \left| \int_{\zeta_{ij}} *c_\gamma \right| \leq C \|*c_\gamma\|''_x,$$

where C is a universal constant and ζ_{ij} is the path connecting the zeroes z_i and z_j of ω and minimizing the hyperbolic distance. (Of course since $*\gamma$ is

harmonic, only the homotopy class of ζ_{ij} matters in the integral on the left-hand side of (42)).

It is enough to prove (42) for the α_k and the β_k . (The estimate for other $\gamma \in \mathcal{S}$ follows from a compactness argument.)

We can find a collar region around α_k as follows: take two annuli

$$\{z_k : 1 > |z_k| > |t_k|^{1/2}\} \text{ and } \{w_k : 1 > w_k > |t_k|^{1/2}\},$$

and identify the inner boundaries via the map $w_k = t_k/z_k$. (This coordinate system on the neighborhood of a boundary point in the Deligne-Mumford compactification of the moduli space of curves is used in, e.g., [Mas76], [Wol03, §3], also [Fay73, Chap. 3], [For02], and elsewhere. For a self-contained modern treatment, see [HK14, §8].) The hyperbolic metric σ in the collar region is approximately $|dz|/(|z| \log |z|)$. Then $\ell_{\alpha_k}(\sigma) \approx 1/|\log t_k|$ where, as above, $A \approx B$ means that A/B is bounded above and below by a constant depending only on the genus. (In fact, we choose the parameters t_k so that $\ell_{\alpha_k}(\sigma) = 1/|\log t_k|$.)

By [Fay73, Chap. 3], any holomorphic 1-form ω can be written in the collar region as

$$\left(a_0(z_k + t_k/z_k, t_k) + \frac{a_1(z_k + t_k/z_k, t_k)}{z_k} \right) dz_k,$$

where a_0 and a_1 are holomorphic in both variables. (We assume here that the limit surface on the boundary of Teichmüller space is fixed; this is justified by the fact that the Deligne-Mumford compactification is indeed compact, and if we normalize ω by fixing its periods along g disjoint curves, then in this coordinate system, the dependence of ω on the limit surface in the boundary is continuous.) This implies that as $t_k \rightarrow 0$,

$$\omega = \left(\frac{a}{z_k} + h(z_k) + O(t_k/z_k^2) \right) dz_k,$$

where h is a holomorphic function which remains bounded as $t_k \rightarrow 0$, and the implied constant is bounded as $t_k \rightarrow 0$. (Note that when $|z_k| \geq |t_k|^{1/2}$, $|t_k/z_k^2| \leq 1$.) Now from the condition $\int_{\alpha_k} *c_{\beta_k} = 1$, we see that on the collar of α_j ,

$$(43) \quad c_{\beta_k} + i *c_{\beta_k} = \left(\frac{\delta_{kj}}{(2\pi)z_j} + h_{kj}(z_j) + O(t_j/z_j^2) \right) dz_j,$$

where the h_{kj} are holomorphic and bounded as $t_j \rightarrow 0$. (We use the notation $\delta_{kj} = 1$ if $k = j$ and zero otherwise.) Also from the condition $\int_{\beta_k} *c_{\alpha_k} = 1$, we have

$$(44) \quad c_{\alpha_k} + i *c_{\alpha_k} = \frac{i}{|\log t_j|} \left(\frac{\delta_{kj}}{z_j} + s_{kj}(z_j) + O(t_j/z_j^2) \right) dz_j,$$

where s_{kj} also remains holomorphic and is bounded as $t_j \rightarrow 0$.

Then, on the collar of α_j ,

$$*c_{\alpha_k} = \frac{\delta_{jk}}{|\log t_j|} d \log |z_j|^2 + \text{bounded 1-form}$$

and thus,

$$\left| \int_{\zeta_{ij}} *c_{\alpha_k} \right| = O(1).$$

Also, on the collar of α_j ,

$$*c_{\beta_k} = \frac{\delta_{jk}}{2\pi} d \arg |z_j| + \text{bounded 1-form}$$

and so

$$\left| \int_{\zeta_{ij}} *c_{\beta_k} \right| = O(1).$$

By Theorem 7.3,

$$\| *c_{\alpha_k} \|'' \approx O(1) \quad \text{and} \quad \| *c_{\beta_k} \|'' \approx \| *c_{\beta_k} \| \approx \ell_{\alpha_k}(\sigma)^{1/2} \gg 1.$$

Thus, (42) holds for $*c_{\beta_k}$ and $*c_{\alpha_k}$, and therefore for any $\gamma \in \mathcal{S}$. By the definition of $\| \cdot \|''$, (42) holds for any $\lambda \in H^1(M, \Sigma, \mathbb{R})$. For $0 \leq t \leq 1$, let θ_t denote the harmonic representative of $p(\lambda)$ on $g_t x$. Then, for $0 \leq t < 1$,

$$\begin{aligned} \|\lambda\|'_{g_t x} &= \|p(\lambda)\|'_{g_t x} + \sum_{i,j} \left| \int_{z_i}^{z_j} (\lambda - \theta_t) \right| \\ &\leq C \|p(\lambda)\|'_x + \sum_{i,j} \left| \int_{z_i}^{z_j} (\lambda - \theta_0) \right| + \sum_{i,j} \left| \int_{z_i}^{z_j} (\theta_t - \theta_0) \right| \quad \text{by (40)} \\ &\leq C \|\lambda\|'_x + \sum_{i,j} \left| \int_{\gamma_{ij}} (\theta_t - \theta_0) \right| \\ &\leq C \|\lambda\|'_x + C' \sum_{i,j} (\|p(\lambda)\|'_{g_t x} + \|p(\lambda)\|'_x) \quad \text{by (42)} \\ &\leq C'' \|\lambda\|'_x \quad \text{by (40).} \end{aligned}$$

In the above computation we have identified the basis at time 0 and time $t \in [0, 1]$. This is justified since the change of basis from x to $a_t x$ for $t \in [0, 1]$ only involve Dehn twists along the short curves α_j 's. This only effects the above computations for $*c_{\beta_j}$. However, the number of twists needed is at most $\ell_{\alpha_j}(\sigma)^{-1/2}$, which in view of Theorem 7.3 is controlled by the Hodge norm of $*c_{\beta_j}$. Therefore, there exists m' such that for $0 \leq t \leq 1$ and any $\lambda \in H^1(M, \Sigma, \mathbb{R})$,

$$\|\lambda\|_{g_t x} \leq e^{m' t} \|\lambda\|_x.$$

This implies the lemma for all t . □

In the sequel we will need to have a control of the matrix coefficients of the cocycle. Let $x \in \mathcal{H}_1(\alpha)$ and $t \in \mathbb{R}$. We let $A(x, t) \equiv A(x, a_t)$ denote the cocycle. Using the map p above, we may write

$$(45) \quad A(x, t) = \begin{pmatrix} I & U(x, t) \\ 0 & S(x, t) \end{pmatrix}.$$

(Note that since we are labelling the zeroes of ω , the action of the cocycle on $\ker p$ is trivial.)

The following is an immediate corollary of Lemma 7.5:

LEMMA 7.6. *There is some $m' \in \mathbb{N}$ such that for all $x \in \mathcal{H}_1(\alpha)$ and all $t \in \mathbb{R}$, we have*

$$\|U(x, t)\| \leq e^{m'|t|},$$

where

$$(46) \quad \|U(x, t)\| \equiv \sup_{c \in H^1(M, \Sigma, \mathbb{R})} \frac{\|\mathfrak{p}_x(c) - \mathfrak{p}_{a_t x}(c)\|}{\|p(c)\|'_x}.$$

Note that since $\mathfrak{p}_x(c) - \mathfrak{p}_{a_t x}(c) \in \ker p$, $\|\mathfrak{p}_x(c) - \mathfrak{p}_{a_t x}(c)\|_y$ is independent of y .

Suppose $L \subset H^1(M, \Sigma, \mathbb{R})$ is a subspace such that $p(L) \subset H^1(M, \mathbb{R})$ is symplectic (in the sense that the intersection form restricted to $p(L)$ is nondegenerate). Let $p(L)^\perp$ denote the symplectic complement of $p(L)$ in $H^1(M, \mathbb{R})$. Suppose $x \in \mathcal{H}_1(\alpha)$. For any $c \in H^1(M, \Sigma, \mathbb{R})$, we may write

$$c = h + c' + v,$$

where h is harmonic with $p(h) \in p(L)^\perp$, $v \in L$ and $c' \in \ker p$. This decomposition is not unique since for $u \in L \cap \ker p$, we can replace c' by $c' + u$ and v by $v - u$. We denote the c' with smallest possible $\|\cdot\|_x$ norm by c'_L . Thus, we have the decomposition

$$(47) \quad c = \mathfrak{p}_{x,L}(c) + c'_L + v,$$

where $\mathfrak{p}_{x,L}(c)$ is the harmonic representative at x of $p_L(c) \equiv \pi_{L^\perp}(p(c))$, $c'_L \in \ker p$, $v \in L$, and c'_L has minimal norm.

Define $\nu_{x,L} : H^1(M, \Sigma, \mathbb{R}) \rightarrow \mathbb{R}$ by

$$\nu_{x,L}(c) = \begin{cases} \max\{\|c'_L\|_x, (\|p_L(c)\|'_x)^{1/2}\} & \text{if } \max\{\|c'_L\|_x, \|p_L(c)\|'_x\} \leq 1, \\ 1 & \text{otherwise.} \end{cases}$$

We record (without proof) some simple properties of $\nu_{x,L}$.

LEMMA 7.7. *We have*

- (a) $\nu_{x,L}(c) = 0$ if and only if $c \in L$;
- (b) for $v \in L$, $\nu_{x,L}(c + v) = \nu_{x,L}(c)$;
- (c) for $v' \in \ker p$, $\nu_{x,L}(c) - \|v'\|_x \leq \nu_{x,L}(c + v') \leq \nu_{x,L}(c) + \|v'\|_x$.

In view of Lemma 7.7, for an affine subspace $\mathcal{L} = v_0 + L$ of $H^1(M, \Sigma, \mathbb{R})$, we can define $\nu_{x, \mathcal{L}}(c)$ to be $\nu_{x, L}(c - v_0)$.

Extend $\nu_{x, \mathcal{L}}$ to $H^1(M, \Sigma, \mathbb{C})$ by

$$\nu_{x, \mathcal{L}}(c_1 + ic_2) = \max\{\nu_{x, \mathcal{L}}(c_1), \nu_{x, \mathcal{L}}(c_2)\}.$$

For an affine subspace $\mathcal{L} \subset H^1(M, \Sigma, \mathbb{R})$, let $\mathcal{L}_{\mathbb{C}} \subset H^1(M, \Sigma, \mathbb{C})$ denote the complexification $\mathbb{C} \otimes \mathcal{L}$. We use the notation (here we are working in period coordinates)

$$d'(x, \mathcal{L}) = \nu_{x, \mathcal{L}}(x - v),$$

where v is any vector in $\mathcal{L}_{\mathbb{C}}$. (The choice of v does not matter by Lemma 7.7(b).) Note that $d'(\cdot, \mathcal{L})$ is defined only if $\mathcal{L} = v_0 + L$ where $p(L)$ is symplectic. We think of $d'(x, \mathcal{L})$ as measuring the distance between x and $\mathcal{L}_{\mathbb{C}} \subset H^1(M, \mathbb{C})$. In view of Lemma 7.5, for all $t > 0$, we have

$$(48) \quad e^{-m't} d'(x, \mathcal{L}) \leq d'(a_t x, a_t \mathcal{L}) \leq e^{m't} d'(x, \mathcal{L}).$$

Recall that $\delta > 0$ is defined in the beginning of Section 7.1.

LEMMA 7.8. *Let the notation be as above. Then, there exist constants $C_0 > 0$, $L_0 > 0$, $\eta'_0 > 0$, $\eta_3 > 0$, $t'_0 > 0$ and continuous functions $\kappa_1 : \mathcal{H}_1(\alpha) \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ and $b : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that*

- $\kappa_1(x, t) \leq C_0 e^{m'\delta t}$ for all $x \in \mathcal{H}_1(\alpha)$ and all $t > 0$;
- $\kappa_1(x, t) \leq e^{-\eta_3 t}$ for all $x \in \mathcal{H}_1(\alpha)$ and $t > t'_0$ with $\log u(x) < L_0 + \eta'_0 t$,

so that for any affine subspace $\mathcal{L} \subset H^1(M, \Sigma, \mathbb{R})$ such that the projection of the linear part of \mathcal{L} to $H^1(M, \mathbb{R})$ is symplectic, we have

$$(49) \quad \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{d'(a_t r_\theta x, a_t r_\theta \mathcal{L})^\delta} \leq \frac{\kappa_1(x, t)}{d'(x, \mathcal{L})^\delta} + b(t).$$

Proof. Suppose $d'(x, \mathcal{L}) \geq 1$, or $d'(a_t r_\theta x, a_t r_\theta \mathcal{L}) \geq 1$ for some $\theta \in [0, 2\pi]$. Then, (49) with $b(t) = e^{2m'\delta t}$ follows immediately from (48). Therefore, we may assume that

$$(50) \quad d'(x, \mathcal{L}) < 1 \quad \text{and} \quad d'(a_t r_\theta x, a_t r_\theta \mathcal{L}) < 1 \text{ for all } \theta.$$

Therefore, in particular,

$$(51) \quad d'(x, \mathcal{L}) = \nu_{x, \mathcal{L}}(v) = \max(\|v'\|_x, (\|p(v)\|'_x)^{1/2}),$$

where

$$(52) \quad v = \mathfrak{p}_x(v) + v', \quad p(v) \in p(L_{\mathbb{C}})^\perp, \quad L \text{ is the linear part of } \mathcal{L}, \text{ and } v' \in \ker p.$$

We remark that the main difficulty of the proof of this lemma is to control the interaction between absolute and pure relative cohomology. The strategy is roughly as follows: we quickly reduce to the case where v is extremely small. Then, if the size of the absolute part $\|p(v)\|'_x$ is comparable to the size of

pure relative part $\|v'\|$, then the quantities $d(x, \mathcal{L})$ and $d(a_t r_\theta x, a_t r_\theta \mathcal{L})$ are all controlled by the absolute part (because of the square root in (51)). In fact, the only situation in which the pure relative part v' has an effect is when $\|p(v)\|'_x$ is essentially smaller than $\|v'\|^2$ (so it is tiny). In this regime, the influence of the absolute part on the relative part is very small, in view of Lemma 7.6. This allows us to separate the contribution of absolute and pure relative cohomology in all cases; for a precise statement, see (56) below. We now give the detailed implementation of this strategy.

Suppose $d'(x, \mathcal{L}) = \nu_{x, \mathcal{L}}(v) \geq \frac{1}{2}e^{-3m't}$. Then, using (48), we have the crude estimate

$$d'(a_t r_\theta x, a_t r_\theta \mathcal{L})^{-\delta} \leq d'(a_t r_\theta x, a_t r_\theta \mathcal{L})^{-1} \leq 2e^{5m't},$$

and thus (49) holds with $b(t) = 2e^{5m't}$. Hence, we may assume that $\nu_{x, \mathcal{L}}(v) < \frac{1}{2}e^{-3m't}$. Then,

$$\begin{aligned} (53) \quad e^{2m't}(\|p(a_t r_\theta v)\|'_{a_t r_\theta x})^{1/2} &\leq e^{(2m'+0.5)t}(\|p(v)\|'_{a_t r_\theta x})^{1/2} && \text{by (22)} \\ &\leq e^{(2m'+0.5+0.5m_0)t}(\|p(v)\|'_x)^{1/2} && \text{by (40)} \\ &\leq e^{3m't}\nu_{x, \mathcal{L}}(v) && \text{since } m' > m_0 > 1 \\ &\leq \frac{1}{2}. \end{aligned}$$

Let us introduce the notation, for $u \in \ker p$,

$$\|u\|_{\mathcal{L}} = \inf\{\|u - w\| : w \in \mathcal{L} \cap \ker p\}.$$

Then, by (50),

$$(54) \quad d'(a_t r_\theta x, a_t r_\theta \mathcal{L}) = \max((\|p(a_t r_\theta v)\|'_{a_t r_\theta x})^{1/2}, \|a_t r_\theta v - \mathfrak{p}_{a_t r_\theta x}(a_t r_\theta v)\|_{a_t r_\theta \mathcal{L}}).$$

But,

$$\begin{aligned} (55) \quad &\|a_t r_\theta v - \mathfrak{p}_{a_t r_\theta x}(a_t r_\theta v)\|_{a_t r_\theta \mathcal{L}} \\ &= \|a_t r_\theta(v' + \mathfrak{p}_x(v)) - \mathfrak{p}_{a_t r_\theta x}(a_t r_\theta v)\|_{a_t r_\theta \mathcal{L}} && \text{by (52)} \\ &= \|a_t r_\theta v' + \mathfrak{p}_x(a_t r_\theta v) - \mathfrak{p}_{a_t r_\theta x}(a_t r_\theta v)\|_{a_t r_\theta \mathcal{L}} \\ &\geq \|a_t r_\theta v'\|_{a_t r_\theta \mathcal{L}} \\ &\quad - \|\mathfrak{p}_x(a_t r_\theta v) - \mathfrak{p}_{a_t r_\theta x}(a_t r_\theta v)\| && \text{by the reverse triangle inequality} \\ &\geq \|a_t r_\theta v'\|_{a_t r_\theta \mathcal{L}} - \|U(r_\theta x, t)\| \|p(a_t r_\theta v)\|'_x && \text{by (46)} \\ &\geq \|a_t r_\theta v'\|_{a_t r_\theta \mathcal{L}} - e^{2m't} \|p(a_t r_\theta v)\|'_{a_t r_\theta x} && \text{by Lemma 7.6.} \end{aligned}$$

Therefore,

(56)

$$\begin{aligned}
& d'(a_t r_\theta x, a_t r_\theta \mathcal{L}) \\
&= \max \left((\|p(a_t r_\theta v)\|'_{a_t r_\theta x})^{1/2}, \|a_t r_\theta v - \mathbf{p}_{a_t r_\theta x}(a_t r_\theta v)\|_{a_t r_\theta \mathcal{L}} \right) \quad \text{by (54)} \\
&\geq \frac{1}{2} \left(\|a_t r_\theta v - \mathbf{p}_{a_t r_\theta x}(a_t r_\theta v)\|_{a_t r_\theta \mathcal{L}} + (\|p(a_t r_\theta v)\|'_{a_t r_\theta x})^{1/2} \right) \\
&\geq \frac{1}{2} \left(\|a_t r_\theta v'\|_{a_t r_\theta \mathcal{L}} - e^{2m't} \|p(a_t r_\theta v)\|'_{a_t r_\theta x} + (\|p(a_t r_\theta v)\|'_{a_t r_\theta x})^{1/2} \right) \quad \text{by (55)} \\
&= \frac{1}{2} \left(\|a_t r_\theta v'\|_{a_t r_\theta \mathcal{L}} + (\|p(a_t r_\theta v)\|'_{a_t r_\theta x})^{1/2} (1 - e^{2m't} (\|p(a_t r_\theta v)\|'_{a_t r_\theta x})^{1/2}) \right) \\
&\geq \frac{1}{2} \left(\|a_t r_\theta v'\|_{a_t r_\theta \mathcal{L}} + \frac{1}{2} (\|p(a_t r_\theta v)\|'_{a_t r_\theta x})^{1/2} \right) \quad \text{by (53)} \\
&\geq \frac{1}{2} \left(\|a_t r_\theta v'\|_{a_t r_\theta \mathcal{L}} + \frac{1}{2} (\|p(a_t r_\theta v)\|_{a_t r_\theta x})^{1/2} \right) \quad \text{since } \|\cdot\|' \geq \|\cdot\|.
\end{aligned}$$

However, since the action of the cocycle on $\ker p$ is trivial, $v' \in \ker p$ and \mathcal{L} is invariant,

$$\|a_t r_\theta v'\|_{a_t r_\theta \mathcal{L}} = \|a_t r_\theta v'\|.$$

Then, (with v and v' as in (51) and (52)),

$$\begin{aligned}
& \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{d'(a_t r_\theta x, a_t r_\theta \mathcal{L})^\delta} \\
&\leq \frac{1}{2\pi} \int_0^{2\pi} 4 \min \left(\frac{1}{\|a_t r_\theta v'\|^\delta}, \frac{1}{\|a_t r_\theta p(v)\|_{\delta/2}} \right) d\theta \\
&\leq 4 \min \left(\frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{\|a_t r_\theta v'\|^\delta}, \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{\|a_t r_\theta p(v)\|_{\delta/2}} \right) d\theta \\
&\leq 4 \min \left(\frac{e^{-k_2(\delta)t}}{\|v'\|_x^\delta}, \frac{\min(C_0, \psi_x(p(v))^{\delta/2} \kappa(x, t))}{\|p(v)\|_x^{\delta/2}} \right) \quad \text{by Lemmas 7.1 and 7.2.}
\end{aligned}$$

Let $\eta'_0 > 0$ be a constant to be chosen later. Suppose $\log u(x) < L_0 + \eta'_0 t$. By Theorem 4.3 there exist $\theta \in [0, 2\pi]$ and $\tau \leq m'' \log u(x)$ such that $x' \equiv a_\tau r_\theta x \in K_\rho$. Then,

$$\tau \leq m'' L_0 + m'' \eta'_0 t.$$

Then, for any v ,

$$\|p(v)\|'_x \leq e^{m_0 \tau} \|p(v)\|'_{x'} \leq C_0 e^{m_0 \tau} \|p(v)\|_{x'} \leq C_0 e^{(m_0+2)\tau} \|p(v)\|_x.$$

Therefore, by Lemma 7.2(b),

$$\begin{aligned}
\frac{\kappa(x, t)}{\|p(v)\|_x^{\delta/2}} &\leq e^{-\eta t} C_0 e^{(\delta/2)(m_0+2)(m'' L_0 + m'' \eta'_0 t)} (\|p(v)\|'_x)^{-\delta/2} \\
&\leq e^{-(\eta/2)t} (\|p(v)\|'_x)^{-\delta/2},
\end{aligned}$$

provided $(\delta/2)m''\eta'_0 < \eta/2$ and t'_0 is sufficiently large.

Let v be as defined in (51). Note that $x + v \in \mathcal{L}_{\mathbb{C}}$ (in period coordinates), and $p(v)$ is (symplectically) orthogonal to $p(\mathcal{L}_{\mathbb{C}})$. Let $w = a_{\tau}r_{\theta}v$. Then, since \mathcal{L} is invariant, $p(w)$ is symplectically orthogonal to $p(\mathcal{L}_{\mathbb{C}})$. Therefore, $\psi_{x'+w}(p(w)) = 1$. Also, by definition, the subspace $E(x')$ varies continuously with x' , hence for any $y \in \mathcal{L}_{\mathbb{C}}$,

$$\lim_{x' \rightarrow y} \psi_{x'}(p(w)) = 1.$$

Since we are assuming that $d'(x', \mathcal{L})$ is small (in fact, $d'(x, \mathcal{L}) \leq \frac{1}{2}e^{-m't}$ and $\tau \ll t$), we conclude that $\psi_{x'}(p(w))$ is uniformly bounded. Therefore,

$$\psi_x(p(v))^{\delta/2} \leq e^{C\eta'_0(\delta/2)2\tau} \leq e^{(\eta/4)t}$$

provided η'_0 is small enough. Thus, we get, for $t > t'_0$ and $x \in \mathcal{H}_1(\alpha)$ so that $\log u(x) < L_0 + \eta'_0 t$,

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{d'(a_t r_{\theta} x, a_t r_{\theta} \mathcal{L})^{\delta}} \leq 4 \min \left(\frac{e^{-k_2(\delta)t}}{\|v'\|_x^{\delta}}, \frac{e^{-(\eta/4)t}}{(\|p(v)\|'_x)^{\delta/2}} \right).$$

The estimate (49) now follows. \square

8. The sets $J_{k,\mathcal{M}}$

Let $\tilde{\mathcal{H}}_1(\alpha)$ denote the space of markings of translation surfaces in $\mathcal{H}_1(\alpha)$ with the zeroes labeled. Then $\tilde{\mathcal{H}}_1(\alpha)$ is a bundle over (a finite cover of) the Teichmüller space of Riemann surfaces, or alternatively a stratum of the Teichmüller space of holomorphic 1-forms.

Fix $0 < \rho < 1/2$ so that Theorems 4.2 and 4.3 hold. Let K_{ρ} be as in Theorem 4.2, and let $K' := \{x : d(x, K_{0.01}) \leq 1\}$ where d denotes the Teichmüller distance. Then, K' is a compact subset of $\mathcal{H}_1(\alpha)$. We lift K' to a compact subset of the Teichmüller space $\tilde{\mathcal{H}}_1(\alpha)$, which we also denote by K' .

Definition 8.1 (Complexity). For an affine invariant submanifold $\mathcal{M} \subset \mathcal{H}_1(\alpha)$, let $n(\mathcal{M})$ denote the smallest integer such that $\mathcal{M} \cap K'$ is contained in a union of at most $n(\mathcal{M})$ affine subspaces. We call $n(\mathcal{M})$ the “complexity” of \mathcal{M} .

Since \mathcal{M} is closed and K' is compact, $n(\mathcal{M})$ is always finite. Clearly $n(\mathcal{M})$ depends also on the choice of K' , but since K' is fixed once and for all, we drop this dependence from the notation.

LEMMA 8.2. *Let \mathcal{M} be an affine manifold, and let $\tilde{\mathcal{M}}$ be a lift of \mathcal{M} to the Teichmüller space $\tilde{\mathcal{H}}_1(\alpha)$. For $x \in \mathcal{H}_1(\alpha)$, let*

$$J_{k,\mathcal{M}}(x) = \{\mathcal{L} : d'(\mathcal{L}, x) \leq u(x)^{-k}, \text{ } \mathcal{L} \text{ is an affine subspace tangent to } \tilde{\mathcal{M}} \}.$$

Then, there exists $k > 0$, depending only on α such that for any affine manifold $\mathcal{M} \subset \mathcal{H}_1(\alpha)$,

$$|J_{k,\mathcal{M}}(x)| \leq n(\mathcal{M}),$$

where $|J_{k,\mathcal{M}}(x)|$ denotes the cardinality of $J_{k,\mathcal{M}}(x)$ and $n(\mathcal{M})$ is as in Definition 8.1.

Proof. We lift x to the Teichmüller space $\tilde{\mathcal{H}}_1(\alpha)$. Working in period coordinates, let

$$B'(x, r) = \{x + h + v : h \text{ harmonic}, v \in \ker p, \max(\|h\|_x^{1/2}, \|v\|_x) \leq r\}.$$

For every $x \in \mathcal{H}_1(\alpha)$, there exists $r(x) > 0$ such that $B'(x, r(x))$ is embedded (in the sense that the projection from the Teichmüller space $\tilde{\mathcal{H}}_1(\alpha)$ to the moduli space $\mathcal{H}_1(\alpha)$, restricted to $B'(x, r(x))$, is injective). Furthermore, we may choose $r(x) > 0$ small enough so that the periods on $B'(x, r(x))$ are a coordinate system (both on the Teichmüller space $\tilde{\mathcal{H}}_1(\alpha)$ and on the moduli space $\mathcal{H}_1(\alpha)$). Let $r_0 = \inf_{x \in K_\rho} r(x)$. By compactness of K_ρ , $r_0 > 0$. Then, choose k_0 so that

$$(57) \quad 2^{m''m'-k_0} < r_0,$$

where m'' is as in Theorem 4.3 and m' is as in (48).

We now claim that for any $k > k_0$ and any $x \in \mathcal{H}_1(\alpha)$, $B'(x, u(x)^{-k_0})$ is embedded. Suppose not. Then there exist $x \in \mathcal{H}_1(\alpha)$ and $x_1, x_2 \in B'(x, u(x)^{-k_0})$ such that $x_2 = \gamma x_1$ for some γ in the mapping class group. Write

$$x_i = x + h_i + v_i, \quad h_i \text{ harmonic}, v_i \in \ker p, \max(\|h_i\|_x^{1/2}, \|v_i\|_x) \leq u(x)^{-k_0}.$$

By Theorem 4.3, there exist $\theta \in [0, 2\pi]$ and $\tau \leq m'' \log u(x)$ such that $x' \equiv a_\tau r_\theta x \in K_\rho$.

Let $x'_i = a_\tau t_\theta x_i$. Then, by Lemma 7.5, we have

$$\max(\|h_i\|_{x'}^{1/2}, \|v_i\|_{x'}) \leq e^{-m'\tau} u(x)^{-k_0} \leq u(x)^{m'm''-k_0} \leq 2^{mm'-k_0} \leq r_0,$$

where for the last estimate we used (57) and the fact that $u(x) \geq 2$. Thus, both x'_1 and x'_2 belong to $B'(x', r_0)$, which is embedded by construction, contradicting the fact that $x'_2 = \gamma x'_1$. Thus, $B'(x, u(x)^{-k})$ is embedded.

Now suppose $\mathcal{L} \in J_{k,\mathcal{M}}(x)$, so that

$$d'(x, \mathcal{L}) \leq u(x)^{-k}.$$

Write $\mathcal{L}' = a_\tau r_\theta \mathcal{L}$. Then, by (48),

$$d'(x', \mathcal{L}') \leq e^{m'\tau} u(x)^{-k} \leq u(x)^{m'm''} u(x)^{-k} < r_0.$$

Hence, \mathcal{L}' intersects $B'(x', r_0)$. Furthermore, since $B'(x', r_0)$ and $B'(x, u(x)^{-k})$ are embedded, there is a one-to-one map between subspaces contained in $J_{k,\mathcal{M}}(x)$ and subspaces intersecting $B'(x', r_0)$.

Since $x' \in K_\rho$, and $r_0 < 1$, $B'(x', r_0) \subset K'$. Hence, there are at most $n(\mathcal{M})$ possibilities for \mathcal{L}' , and hence at most $n(\mathcal{M})$ possibilities for \mathcal{L} . \square

9. Standard recurrence lemmas

LEMMA 9.1. *For every $\sigma > 1$, there exists a constant $c_0 = c_0(\sigma) > 0$ such that the following holds: Suppose X is a space on which $\mathrm{SL}(2, \mathbb{R})$ acts, and suppose $f : X \rightarrow [2, \infty]$ is an $\mathrm{SO}(2)$ -invariant function with the following properties:*

(a) *for all $0 \leq t \leq 1$ and all $x \in X$,*

$$(58) \quad \sigma^{-1}f(x) \leq f(a_t x) \leq \sigma f(x);$$

(b) *there exist $\tau > 0$ and $b_0 > 0$ such that for all $x \in X$,*

$$A_\tau f(x) \leq c_0 f(x) + b_0.$$

Then,

(i) *for all $c < 1$, there exist $t_0 > 0$ (depending on σ , and c) and $b > 0$ (depending only on b_0 , c_0 and σ) such that for all $t > t_0$ and all $x \in X$,*

$$(A_t f)(x) \leq c f(x) + b;$$

(ii) *there exists $B > 0$ (depending only on c_0 , b_0 and σ) such that for all $x \in X$, there exists $T_0 = T_0(x, c_0, b_0, \sigma)$ such that for all $t > T_0$,*

$$(A_t f)(x) \leq B.$$

For completeness, we include the proof of this lemma. It is essentially taken from [EMM98, §5.3], specialized to the case $G = \mathrm{SL}(2, \mathbb{R})$. The basic observation is the following standard fact from hyperbolic geometry:

LEMMA 9.2. *There exist absolute constants $0 < \delta' < 1$ and $\delta > 0$ such that for any $t > 0$, any $s > 0$ and any $z \in \mathbb{H}$, for at least δ' -fraction of $\phi \in [0, 2\pi]$,*

$$(59) \quad t + s - \delta \leq d(a_t r_\phi a_s z, z) \leq t + s,$$

where $d(\cdot, \cdot)$ is the hyperbolic distance in \mathbb{H} , normalized so that $d(a_t r_\theta z, z) = t$.

COROLLARY 9.3. *Suppose $f : X \rightarrow [1, \infty]$ satisfies (58). Then, there exists $\sigma' > 1$ depending only on σ such that for any $t > 0$, $s > 0$ and any $x \in X$,*

$$(60) \quad (A_{t+s} f)(x) \leq \sigma' (A_t A_s f)(x).$$

Outline of proof. Fix $x \in \mathcal{H}_1(\alpha)$. For $g \in \mathrm{SL}(2, \mathbb{R})$, let $f_x(g) = f(gx)$, and let

$$\tilde{f}_x(g) = \int_0^{2\pi} f(gr_\theta x) d\theta.$$

Then, $\tilde{f}_x : \mathbb{H} \rightarrow [2, \infty]$ is a spherically symmetric function, i.e., $\tilde{f}_x(g)$ depends only on $d(g \cdot o, o)$, where o is the point fixed by $\mathrm{SO}(2)$.

We have

$$(61) \quad (A_t A_s f)(x) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{2\pi} \int_0^{2\pi} f(a_t r_\phi a_s r_\theta x) d\phi d\theta = \frac{1}{2\pi} \int_0^{2\pi} \tilde{f}_x(a_t r_\phi a_s).$$

By Lemma 9.2, for at least δ' -fraction of $\phi \in [0, 2\pi]$, (59) holds. Then, by (58), for at least δ' -fraction of $\phi \in [0, 2\pi]$,

$$\tilde{f}_x(a_t r_\phi a_s) \geq \sigma_1^{-1} \tilde{f}_x(a_{t+s}),$$

where $\sigma_1 = \sigma_1(\sigma, \delta) > 1$. Plugging into (61), we get

$$(A_t A_s f)(x) \geq (\delta' \sigma_1^{-1}) \tilde{f}_x(a_{t+s}) = (\delta' \sigma_1^{-1}) (A_{t+s} f)(x),$$

as required. \square

Proof of Lemma 9.1. Let $c_0(\sigma)$ be such that $\kappa \equiv c_0 \sigma' < 1$, where σ' is as in Corollary 9.3. Then, for any $s \in \mathbb{R}$ and for all x ,

$$\begin{aligned} (A_{s+\tau} f)(x) &\leq \sigma' A_s(A_\tau f)(x) && \text{by (60)} \\ &\leq \sigma' A_s(c_0 f(x) + b_0) && \text{by condition (b)} \\ &= \kappa(A_s f)(x) + \sigma' b_0 && \text{since } \sigma' c_0 = \kappa. \end{aligned}$$

Iterating this, for $n \in \mathbb{N}$, we get

$$(A_{n\tau} f)(x) \leq \kappa^n f(x) + \sigma' b_0 + \kappa \sigma' b_0 + \cdots + \kappa^{n-1} \sigma' b_0 \leq \kappa^n f(x) + B,$$

where $B = \frac{\sigma' b_0}{1-\kappa}$. Since $\kappa < 1$, $\kappa^n f(x) \rightarrow 0$ as $n \rightarrow \infty$. Therefore both (i) and (ii) follow for $t \in \tau\mathbb{N}$. The general case of both (i) and (ii) then follows by applying again condition (a). \square

10. Construction of the function

Note that by Jensen's inequality, for $0 < \varepsilon < 1$,

$$(62) \quad A_t(f^\varepsilon) \leq (A_t f)^\varepsilon.$$

Also, we will repeatedly use the inequality

$$(63) \quad (a+b)^\varepsilon \leq a^\varepsilon + b^\varepsilon,$$

valid for $\varepsilon < 1$, $a \geq 0$, $b \geq 0$.

Fix an affine invariant submanifold \mathcal{M} , and let k be as in Lemma 8.2. For $\varepsilon > 0$, let

$$s_{\mathcal{M}, \varepsilon}(x) = \begin{cases} \sum_{\mathcal{L} \in J_{k, \mathcal{M}}(x)} d'(x, \mathcal{L})^{-\varepsilon \delta} & \text{if } J_{k, \mathcal{M}}(x) \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

where $\delta > 0$ is as in Lemma 7.8.

PROPOSITION 10.1. *Suppose $\mathcal{M} \subset \mathcal{H}_1(\alpha)$ is an affine manifold and $0 < c < 1$. For $\varepsilon > 0$ and $\lambda > 0$, let*

$$f_{\mathcal{M}}(x) = s_{\mathcal{M},\varepsilon}(x)u(x)^{1/2} + \lambda u(x).$$

Then, $f_{\mathcal{M}}$ is $\mathrm{SO}(2)$ -invariant, and $f(x) = +\infty$ if and only if $x \in \mathcal{M}$. Also, if ε is sufficiently small (depending on α) and λ is sufficiently large (depending on α , c and $n(\mathcal{M})$), there exists $t_1 > 0$ (depending on $n(\mathcal{M})$ and c) such that for all $t \geq t_1$, we have

$$(64) \quad A_t f_{\mathcal{M}}(x) < c f_{\mathcal{M}}(x) + b,$$

where $b = b(\alpha, n(\mathcal{M}))$.

For the proof of Proposition 10.1, will use Lemma 9.1. Thus, in order to prove Proposition 10.1, it is enough to show that $f_{\mathcal{M}}$ satisfies conditions (a) and (b) of Lemma 9.1. We start with the following:

CLAIM 10.2. *For $\varepsilon > 0$ sufficiently small and $\lambda > 0$ sufficiently large, $f_{\mathcal{M}}$ satisfies condition (a) of Lemma 9.1, with $\sigma = \sigma(k, m, m')$.*

Proof of Claim 10.2. We will choose $\varepsilon < 1/(2k\delta)$. Suppose $x \in \mathcal{H}_1(\alpha)$ and $0 \leq t < 1$. We consider three sets of subspaces:

$$\begin{aligned} \Delta_1 &= \{\mathcal{L} \in J_{k,\mathcal{M}}(x) : a_t \mathcal{L} \in J_{k,\mathcal{M}}(a_t x)\}, \\ \Delta_2 &= \{\mathcal{L} \in J_{k,\mathcal{M}}(x) : a_t \mathcal{L} \notin J_{k,\mathcal{M}}(a_t x)\}, \\ \Delta_3 &= \{\mathcal{L} \notin J_{k,\mathcal{M}}(x) : a_t \mathcal{L} \in J_{k,\mathcal{M}}(a_t x)\}. \end{aligned}$$

We remark that the rest of the proof is a routine verification. Note that the cardinality of all Δ_i is bounded by $n(\mathcal{M})$, which is fixed. For any $0 \leq t \leq 1$, in view of (48), the contribution of each \mathcal{L} in Δ_1 at $a_t x$ is within a fixed multiplicative factor of the contribution at x . Furthermore, if $\mathcal{L} \in \Delta_2 \cup \Delta_3$, then in view of (48), $d'(x, \mathcal{L})$ is bounded from below by a negative power of $u(x)$, and then (with the proper choice of parameters), its contribution to both $f_{\mathcal{M}}(x)$ and $f_{\mathcal{M}}(a_t x)$ is negligible. We now give the details.

Let

$$S_i(x) = \sum_{\mathcal{L} \in \Delta_i} d'(x, \mathcal{L})^{-\varepsilon\delta}.$$

Then,

$$s_{\mathcal{M},\varepsilon}(x) = S_1(x) + S_2(x), \quad s_{\mathcal{M},\varepsilon}(a_t x) = S_1(a_t x) + S_3(a_t x).$$

For $\mathcal{L} \in \Delta_1$, by (48) with $0 \leq t \leq 1$,

$$e^{-m'\varepsilon\delta} d'(x, \mathcal{L})^{-\varepsilon\delta} \leq d'(a_t x, a_t \mathcal{L})^{-\varepsilon\delta} \leq e^{m'\varepsilon\delta} d'(x, \mathcal{L})^{-\varepsilon\delta},$$

and thus

$$e^{-m'\varepsilon\delta} S_1(x) \leq S_1(a_t x) \leq e^{m'\varepsilon\delta} S_1(x).$$

Then, using (14),

$$e^{-m'\varepsilon\delta-m/2}S_1(x)u(x)^{1/2} \leq S_1(a_tx)u(a_tx)^{1/2} \leq e^{m'\varepsilon\delta+m/2}S_1(x)u(x)^{1/2}.$$

Suppose $\mathcal{L} \in \Delta_2 \cup \Delta_3$. Then, by (14) and (48),

$$d'(x, \mathcal{L}) \geq Cu(x)^{-k},$$

where $C = O(1)$ (depending only on k, m and m'), and thus, for $i = 2, 3$, and using Lemma 8.2,

$$S_i(a_tx) \leq Cn(\mathcal{M})u(x)^{-\varepsilon\delta k} \quad \text{and} \quad S_i(x) \leq Cn(\mathcal{M})u(a_tx)^{-\varepsilon\delta k}, \quad i = 2, 3.$$

Now choose $\varepsilon > 0$ so that $k\varepsilon\delta < 1/2$ and $\lambda > 0$ so that $\lambda > 10Ce^mn(\mathcal{M})$. Then,

$$S_i(a_tx)u(a_tx)^{1/2} \leq (0.1)\lambda u(x) \quad \text{and} \quad S_i(x)u(x)^{1/2} \leq (0.1)\lambda u(a_tx), \quad i = 2, 3.$$

Then,

$$\begin{aligned} f_{\mathcal{M}}(a_tx) &= S_1(a_tx)u(a_tx)^{1/2} + S_3(a_tx)u(a_tx)^{1/2} + \lambda u(a_tx) \\ &\leq e^{m'\varepsilon\delta+m/2}S_1(x)u(x)^{1/2} + (0.1)\lambda u(x) + e^m\lambda u(x) \quad \text{by (14) and (48)} \\ &\leq (e^{m'\varepsilon\delta+m/2} + (0.1) + e^m)(S_1(x)u(x)^{1/2} + \lambda u(x)) \\ &\leq (e^{m'\varepsilon+\delta m/2} + (0.1) + e^m)f_{\mathcal{M}}(x). \end{aligned}$$

In the same way,

$$\begin{aligned} f_{\mathcal{M}}(x) &= S_1(x)u(x)^{1/2} + S_2(x)u(x)^{1/2} + \lambda u(x) \\ &\leq e^{m'\varepsilon\delta+m/2}S_1(a_tx)u(a_tx)^{1/2} \\ &\quad + (0.1)\lambda u(a_tx) + e^m\lambda u(a_tx) \quad \text{by (14) and (48)} \\ &\leq (e^{m'\varepsilon\delta+m/2} + (0.1) + e^m)(S_1(a_tx)u(a_tx)^{1/2} + \lambda u(a_tx)) \\ &\leq (e^{m'\varepsilon+\delta m/2} + (0.1) + e^m)f_{\mathcal{M}}(a_tx). \end{aligned} \quad \square$$

We now begin the verification of condition (b) of Lemma 9.1. The first step is the following:

CLAIM 10.3. *Suppose ε is sufficiently small (depending on k, δ). Then there exist $t_2 > 0$ and $\tilde{b} > 0$ such that for all $x \in \mathcal{H}_1(\alpha)$ and all $t > t_2$,*

$$\begin{aligned} (65) \quad A_t(s_{\mathcal{M},\varepsilon}u^{1/2})(x) &\leq \kappa_1(x, t)^{\varepsilon}\tilde{c}^{1/2}s_{\mathcal{M},\varepsilon}(x)u(x)^{1/2} \\ &\quad + \kappa_1(x, t)^{\varepsilon}\tilde{b}^{1/2}s_{\mathcal{M},\varepsilon}(x) + b_3(t)n(\mathcal{M})u(x), \end{aligned}$$

where $\tilde{c} = e^{-\tilde{\eta}t}$ and $\kappa_1(x, t)$ is as in Lemma 7.8.

Remark. The proof of Claim 10.3 is a straightforward verification, where we again have to show that contribution of the subspaces which contribute at x but not at $a_t r_\theta x$ (or vice versa) is negligible (or, more precisely, can be absorbed into the right-hand side of (65)). The main feature of (65) is the appearance of the “cross term” $\kappa_1(x, t)^\varepsilon \tilde{b}^{1/2} s_{\mathcal{M}, \varepsilon}(x)$. In order to proceed further, we will need to show (for a properly chosen t), that for all $x \in \mathcal{H}_1(\alpha)$, $\kappa_1(x, t)^\varepsilon \tilde{b}^{1/2} \leq (0.1)c_0 u(x)^{1/2}$, where c_0 is in Lemma 9.1(b). This will be done, on a case by case basis, in the proof of Proposition 10.1 below.

Proof of Claim 10.3. In this proof, the $b_i(t)$ denote constants depending on t . Choose $\varepsilon > 0$ so that $2k\varepsilon\delta \leq 1$. Suppose $t > 0$ is fixed. Let $J'(x) \subset J_{k, \mathcal{M}}(x)$ be the subset

$$J'(x) = \{\mathcal{L} : a_t r_\theta \mathcal{L} \in J_{k, \mathcal{M}}(a_t r_\theta x) \text{ for all } 0 \leq \theta \leq 2\pi\}.$$

Suppose $\mathcal{L} \subset J'(x)$. For $0 \leq \tau \leq t$ and $0 \leq \theta \leq 2\pi$, let

$$\ell_{\mathcal{L}}(a_\tau r_\theta x) = d'(a_\tau r_\theta \mathcal{L}, a_\tau r_\theta x)^{-\delta}.$$

Then,

$$\begin{aligned} (66) \quad A_t(\ell_{\mathcal{L}}^{2\varepsilon})(x) &\leq (A_t \ell_{\mathcal{L}})^{2\varepsilon}(x) && \text{by (62)} \\ &\leq (\kappa_1(x, t) \ell_{\mathcal{L}}(x) + b(t))^{2\varepsilon} && \text{by Lemma 7.8} \\ &\leq \kappa_1(x, t)^{2\varepsilon} \ell_{\mathcal{L}}(x)^{2\varepsilon} + b(t)^{2\varepsilon} && \text{by (63).} \end{aligned}$$

Recall that

$$(67) \quad u(x) \geq 2 \quad \text{for all } x.$$

We have, at the point x ,

$$\begin{aligned} (68) \quad A_t(\ell_{\mathcal{L}}^\varepsilon u^{1/2}) &\leq (A_t \ell_{\mathcal{L}}^{2\varepsilon})^{1/2} (A_t u)^{1/2} \quad \text{by Cauchy-Schwartz} \\ &\leq [(\kappa_1(x, t)^{2\varepsilon} \ell_{\mathcal{L}}(x)^{2\varepsilon} + b_1(t)u(x))]^{1/2} (\tilde{c}u(x) + \tilde{b})^{1/2} \quad \text{by (66), (15), (67)} \\ &\leq [\kappa_1(x, t)^\varepsilon \ell_{\mathcal{L}}(x)^\varepsilon + b_1(t)^{1/2} u(x)^{1/2}] (\tilde{c}^{1/2} u(x)^{1/2} + \tilde{b}^{1/2}) \quad \text{by (63)} \\ &= \kappa_1(x, t)^\varepsilon \ell_{\mathcal{L}}(x)^\varepsilon (\tilde{c}^{1/2} u(x)^{1/2} + \tilde{b}^{1/2}) \\ &\quad + b_1(t)^{1/2} \tilde{c}^{1/2} u(x) + b_1(t)^{1/2} \tilde{b}^{1/2} u(x)^{1/2} \\ &\leq \kappa_1(x, t)^\varepsilon \ell_{\mathcal{L}}(x)^\varepsilon (\tilde{c}^{1/2} u(x)^{1/2} + \tilde{b}^{1/2}) \\ &\quad + b_1(t)^{1/2} (\tilde{c}^{1/2} + \tilde{b}^{1/2}) u(x) \quad \text{since } u(x) \geq 1 \\ &= \kappa_1(x, t)^\varepsilon \tilde{c}^{1/2} \ell_{\mathcal{L}}(x)^\varepsilon u(x)^{1/2} + \kappa_1(x, t)^\varepsilon \tilde{b}^{1/2} \ell_{\mathcal{L}}(x)^\varepsilon \\ &\quad + b_2(t) u(x). \end{aligned}$$

For $0 \leq \tau \leq t$ and $0 \leq \theta \leq 2\pi$, let

$$h(a_\tau r_\theta x) = \sum_{\mathcal{L} \in J'(x)} d'(a_\tau r_\theta \mathcal{L}, a_\tau r_\theta x)^{-\varepsilon\delta} = \sum_{\mathcal{L} \in J'(x)} \ell_{\mathcal{L}}(a_\tau r_\theta x)^\varepsilon.$$

Then, $h(a_\tau r_\theta x) \leq s_{\mathcal{M},\varepsilon}(a_\tau r_\theta x)$. Summing (68) over $\mathcal{L} \in J'(x)$ and using Lemma 8.2, we get

(69)

$$A_t(hu^{1/2})(x) \leq \kappa_1(x, t)^\varepsilon \tilde{c}^{1/2} h(x) u(x)^{1/2} + \kappa_1(x, t)^\varepsilon \tilde{b}^{1/2} h(x) + b_2(t) n(\mathcal{M}) u(x).$$

We now need to estimate the contribution of subspaces not in $J'(x)$. Suppose $0 \leq \theta \leq 2\pi$, and suppose

$$a_t r_\theta \mathcal{L} \in J_{k,\mathcal{M}}(a_t r_\theta x), \quad \text{but } \mathcal{L} \notin J'(x).$$

Then, either $\mathcal{L} \notin J_{k,\mathcal{M}}(x)$ or for some $0 \leq \theta' \leq 2\pi$, $a_t r_{\theta'} \mathcal{L} \notin J_{k,\mathcal{M}}(a_t r_{\theta'} x)$. Then in either case, for some $\tau' \in \{0, t\}$ and some $0 \leq \theta' \leq 2\pi$, $a_{\tau'} r_{\theta'} \mathcal{L} \notin J_{k,\mathcal{M}}(a_{\tau'} r_{\theta'} x)$. Hence

$$d'(a_{\tau'} r_{\theta'} x, a_{\tau'} r_{\theta'} \mathcal{L}) \geq u(a_{\tau'} r_{\theta'} x)^{-k}.$$

Then, by (48) and (14),

$$d'(x, \mathcal{L}) \geq b_0(\tau')^{-1} u(x)^{-k} \geq b_0(t)^{-1} u(x)^{-k}$$

and thus, for all $\theta \in [0, 2\pi]$, by (14) and (48),

$$d'(a_t r_\theta x, a_t r_\theta \mathcal{L}) \geq b_0(t)^{-2} u(x)^{-k}.$$

Hence, using (14) again,

$$(70) \quad d'(a_t r_\theta x, a_t r_\theta \mathcal{L})^{-\varepsilon\delta} u(a_t r_\theta x)^{1/2} \leq b_1(t) u(x)^{k\varepsilon\delta+1/2} \leq b_1(t) u(x),$$

where for the last estimate we used $k\varepsilon\delta \leq 1/2$. Thus, for all $0 \leq \theta \leq 2\pi$,

$$\begin{aligned} s_{\mathcal{M},\varepsilon}(a_t r_\theta x) u(a_t r_\theta x)^{1/2} &\leq h(a_t r_\theta x) u(a_t r_\theta x)^{1/2} \\ &\quad + |J(a_t r_\theta x)| b_1(t) u(x) \quad \text{using (70)} \\ &\leq h(a_t r_\theta x) u(a_t r_\theta x)^{1/2} \\ &\quad + b_1(t) n(\mathcal{M}) u(x) \quad \text{using Lemma 8.2.} \end{aligned}$$

Hence,

$$\begin{aligned} A_t(s_{\mathcal{M},\varepsilon} u^{1/2})(x) &\leq A_t(hu^{1/2})(x) + b_1(t) n(\mathcal{M}) u(x) \\ &\leq \kappa_1(x, t)^\varepsilon \tilde{c}^{1/2} h(x) u(x)^{1/2} \\ &\quad + \kappa_1(x, t)^\varepsilon \tilde{b}^{1/2} h(x) + b_3(t) n(\mathcal{M}) u(x) \quad \text{using (69)} \\ &\leq \kappa_1(x, t)^\varepsilon \tilde{c}^{1/2} s_{\mathcal{M},\varepsilon}(x) u(x)^{1/2} \\ &\quad + \kappa_1(x, t)^\varepsilon \tilde{b}^{1/2} s_{\mathcal{M},\varepsilon}(x) + b_3(t) n(\mathcal{M}) u(x) \quad \text{since } h \leq s_{\mathcal{M},\varepsilon}. \end{aligned}$$

□

Proof of Proposition 10.1. Let σ be as in Claim 10.2, and let $c_0 = c_0(\sigma)$ be as in Lemma 9.1. Let $L_0, \eta'_0, \eta_3, m', \delta$ be as in Lemma 7.8. Suppose $\varepsilon > 0$ is small enough so that

$$(71) \quad \varepsilon m' \delta < \frac{1}{2} \tilde{\eta},$$

where $\tilde{\eta}$ is as in Theorem 4.1. We also assume that $\varepsilon > 0$ is small enough so that

$$(72) \quad \varepsilon m' \delta < \frac{1}{2} \min(\eta_3, \eta'_0),$$

where η_3 is as in Lemma 7.8. Choose $t_0 > 0$ so that Theorem 4.1 holds for $t > t_0$ and so that $e^{-\tilde{\eta}t_0} < (0.1)c_0$. Since $\kappa_1(x, t) < e^{m'\delta t}$, we can also, in view of (71), make sure that for $t > t_0$,

$$(73) \quad \kappa_1(x, t)^\varepsilon e^{-\tilde{\eta}t/2} \leq (0.1)c_0.$$

Let $t_2 > 0$ be such that Claim 10.3 holds. By (72), there exists $t_3 > 0$ so that for $t > t_3$,

$$(74) \quad \kappa_1(x, t)^\varepsilon \tilde{b}^{1/2} \leq e^{m'\delta\varepsilon t} \tilde{b}^{1/2} \leq (0.1)c_0 e^{\eta'_0 t/2}.$$

By Lemma 7.8 there exists $\tau > \max(t_0, t_2, t_3)$ such that for all x with $\log u(x) < L_0 + \eta'_0 \tau$,

$$\kappa_1(x, \tau)^\varepsilon \tilde{b}^{1/2} \leq (0.1)c_0 \leq (0.1)c_0 u(x)^{1/2}.$$

If $\log u(x) \geq L_0 + \eta'_0 \tau$, then $u(x)^{1/2} \geq e^{(\eta'_0/2)\tau}$, and therefore, since $\tau > t_3$, by (74),

$$\kappa_1(x, \tau)^\varepsilon \tilde{b}^{1/2} \leq e^{m'\delta\varepsilon\tau} \tilde{b}^{1/2} \leq (0.1)c_0 e^{\eta'_0 \tau/2} \leq (0.1)c_0 u(x)^{1/2}.$$

Thus, for all $x \in \mathcal{H}_1(\alpha)$,

$$(75) \quad \kappa_1(x, \tau)^\varepsilon \tilde{b}^{1/2} \leq (0.1)c_0 u(x)^{1/2}.$$

Thus, substituting (73) and (75) into (65), for all $x \in \mathcal{H}_1(\alpha)$, we get

$$(76) \quad A_\tau(s_{\mathcal{M}, \varepsilon} u^{1/2})(x) \leq (0.2)c_0 s_{\mathcal{M}, \varepsilon}(x) u(x)^{1/2} + b_3(\tau) n(\mathcal{M}) u(x).$$

Choose

$$\lambda > 10b_3(\tau) n(\mathcal{M}) / c_0.$$

Then, in view of (76), we have

$$(77) \quad A_\tau(s_{\mathcal{M}, \varepsilon} u^{1/2})(x) \leq (0.2)c_0 s_{\mathcal{M}, \varepsilon}(x) u^{1/2} + (0.1)c_0 \lambda u(x).$$

Finally, since $\tilde{c} \leq (0.1)c_0$, we have

$$\begin{aligned} A_\tau(f_{\mathcal{M}})(x) &= A_\tau(s_{\mathcal{M}, \varepsilon} u^{1/2})(x) + A_\tau(\lambda u)(x) \\ &\leq [(0.2)c_0 s_{\mathcal{M}, \varepsilon}(x) u^{1/2} \\ &\quad + (0.1)c_0 \lambda u(x)] + (0.1)c_0 \lambda u(x) + \lambda \tilde{b} \quad \text{by (77) and (15)} \\ &\leq (0.2)c_0 f_{\mathcal{M}}(x) + b_{\mathcal{M}} \quad \text{where } b_{\mathcal{M}} = \lambda \tilde{b}. \end{aligned}$$

Thus, condition (b) of Lemma 9.1 holds for $f_{\mathcal{M}}$. In view of Lemma 9.1 this completes the proof of Proposition 10.1. \square

11. Countability

The following lemma is standard:

LEMMA 11.1. *Suppose $\mathrm{SL}(2, \mathbb{R})$ acts on a space X , and suppose there exists a proper function $f : X \rightarrow [1, \infty]$ such that for some $\sigma > 1$ all $0 \leq t \leq 1$ and all $x \in X$,*

$$\sigma^{-1}f(x) \leq f(a_t x) \leq \sigma f(x),$$

and also there exist $0 < c < c_0(\sigma)$ (where $c_0(\sigma)$ is as in Lemma 9.1), $t_0 > 0$ and $b > 0$ such that for all $t > t_0$ and all $x \in X$,

$$A_t f(x) \leq c f(x) + b.$$

Suppose that ν is an ergodic $\mathrm{SL}(2, \mathbb{R})$ -invariant measure on X , such that $\nu(\{f < \infty\}) > 0$. Then,

$$(78) \quad \int_X f d\nu \leq B,$$

where B depends only on b , c and σ .

Proof. For $n \in \mathbb{N}$, let $f_n = \min(f, n)$. By the Moore ergodicity theorem, the action of $A \equiv \{a_t : t \in \mathbb{R}\}$ on X is ergodic. Then, by the Birkhoff ergodic theorem, there exists a point $x_0 \in X$ such that for almost all $\theta \in [0, 2\pi]$ and all $n \in \mathbb{N}$,

$$(79) \quad \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f_n(a_t r_\theta x_0) dt = \int_X f_n d\nu.$$

Therefore, for each n , there exists a subset $E_n \subset [0, 2\pi]$ of measure at least π such that the convergence in (79) is uniform over $\theta \in E_n$. Then there exists $T_n > 0$ such that for all $T > T_n$,

$$(80) \quad \frac{1}{T} \int_0^T f_n(a_t r_\theta x_0) dt \geq \frac{1}{2} \int_X f_n d\nu \quad \text{for } \theta \in E_n.$$

We integrate (80) over $\theta \in [0, 2\pi]$. Then for all $T > T_n$,

$$(81) \quad \frac{1}{T} \int_0^T \left(\int_0^{2\pi} f_n(a_t r_\theta x_0) d\theta \right) dt \geq \frac{1}{4} \int_X f_n d\nu.$$

But, by Lemma 9.1(ii), for sufficiently large T , the integral in parenthesis on the left-hand side of (81) is bounded above by $B' = B'(c, b, \sigma)$. Therefore, for all n ,

$$\int_X f_n d\nu \leq 4B'.$$

Taking the limit as $n \rightarrow \infty$, we get that $f \in L^1(X, \nu)$ and (78) holds. \square

Proof of Proposition 2.16. Let $X_d(\alpha)$ denote the set of affine manifolds of dimension d . It enough to show that each $X_d(\alpha)$ is countable.

For an affine subspace $\mathcal{L} \subset H^1(M, \Sigma, \mathbb{R})$ whose linear part is L , let $H_{\mathcal{L}} : p(L) \rightarrow \ker p / (L \cap \ker p)$ denote the linear map such that for $v \in p(L)$, $v + H_{\mathcal{L}}(v) \in L \bmod L \cap \ker p$. For an affine manifold \mathcal{M} , let

$$H(\mathcal{M}) = \sup_{x \in \mathcal{M} \cap K'} \|H_{\mathcal{M}_x}\|_x,$$

where we use the notation \mathcal{M}_x for the affine subspace tangent to \mathcal{M} at x .

For an integer $R > 0$, let

$$X_{d,R}(\alpha) = \{\mathcal{M} \in X_d(\alpha) : n(\mathcal{M}) \leq R \text{ and } H(\mathcal{M}) \leq R\}.$$

Since $X_d(\alpha) = \bigcup_{R=1}^{\infty} X_{d,R}(\alpha)$, it is enough to show that each $X_{d,R}(\alpha)$ is finite.

Let K' be as in Definition 8.1 of $n(\cdot)$, and let $L_R(K')$ denote the set of (unordered) $\leq R$ -tuples of d dimensional affine subspaces intersecting K' . Then $L_R(K')$ is compact, and we have the map $\phi : X_{d,R} \rightarrow L_R(K')$ which takes the affine manifold \mathcal{M} to the (minimal) set of affine subspaces containing $\mathcal{M} \cap K'$.

Suppose $\mathcal{M}_j \in X_{d,R}(\alpha)$ is an infinite sequence, with $\mathcal{M}_j \neq \mathcal{M}_k$ for $j \neq k$. Then, $\mathcal{M}_j \cap K' \neq \mathcal{M}_k \cap K'$ for $j \neq k$. (If $\mathcal{M}_j \cap K' = \mathcal{M}_k \cap K'$, then by the ergodicity of the $\text{SL}(2, \mathbb{R})$ action, $\mathcal{M}_j = \mathcal{M}_k$.)

Since $L_R(K')$ is compact, after passing to a subsequence, we may assume that $\phi(\mathcal{M}_j)$ converges. Therefore,

$$(82) \quad \text{hd}(\mathcal{M}_j \cap K', \mathcal{M}_{j+1} \cap K') \rightarrow 0 \quad \text{as } j \rightarrow \infty,$$

where $\text{hd}(\cdot, \cdot)$ denotes the Hausdorff distance. (We use any metric on $\mathcal{H}_1(\alpha)$ for which the period coordinates are continuous.) Then, because of (82) and the bound on $H(\mathcal{M})$, for all $x \in \mathcal{M}_{j+1} \cap K'$, $d'(x, \mathcal{M}_j) \rightarrow 0$. From the definition of $f_{\mathcal{M}}$, we have $f_{\mathcal{M}}(x) \rightarrow \infty$ as $d'(x, \mathcal{M}) \rightarrow 0$. Therefore, there exists a sequence $T_j \rightarrow \infty$ such that we have

$$(83) \quad f_{\mathcal{M}_{j+1}}(x) \geq T_j \quad \text{for all } x \in \mathcal{M}_j \cap K'.$$

Let ν_j be the affine $\text{SL}(2, \mathbb{R})$ -invariant probability measure whose support is \mathcal{M}_j . Then, by Proposition 10.1 and Lemma 11.1, we have for all j ,

$$\int_{\mathcal{H}_1(\alpha)} f_{\mathcal{M}_{j+1}} d\nu_j \leq B,$$

where B is independent of j . But, by the definition of K' and Theorem 4.2,

$$\nu_j(\mathcal{M}_j \cap K') \geq 1 - \rho \geq 1/2.$$

This is a contradiction to (83). Therefore, $X_{d,R}(\alpha)$ is finite. \square

References

- [AEZ12] J. ATHREYA, A. ESKIN, and A. ZORICH, Rectangular billiards and volumes of spaces of quadratic differentials on \mathbb{CP}^1 (with an appendix by Jon Chaika), 2012. arXiv 1212.1660.
- [Ath06] J. ATHREYA, Quantitative recurrence and large deviations for Teichmüller geodesic flow, *Geom. Dedicata* **119** (2006), 121–140. MR 2247652. Zbl 1108.32007. <http://dx.doi.org/10.1007/s10711-006-9058-z>.
- [ABEM12] J. ATHREYA, A. BUFETOV, A. ESKIN, and M. MIRZAKHANI, Lattice point asymptotics and volume growth on Teichmüller space, *Duke Math. J.* **161** (2012), 1055–1111. MR 2913101. Zbl 1246.37009. <http://dx.doi.org/10.1215/00127094-1548443>.
- [AEM15] A. AVILA, A. ESKIN, and M. MOELLER, Symplectic and isometric $\mathrm{SL}(2, \mathbb{R})$ invariant subbundles of the Hodge bundle, *J. Reine Angew. Math.*, published online April 14, 2015. <http://dx.doi.org/10.1515/crelle-2014-0142>.
- [Bai10] M. BAINBRIDGE, Billiards in L-shaped tables with barriers, *Geom. Funct. Anal.* **20** (2010), 299–356. MR 2671280. Zbl 1213.37062. <http://dx.doi.org/10.1007/s00039-010-0065-8>.
- [CW10] K. CALTA and K. WORTMAN, On unipotent flows in $\mathcal{H}(1, 1)$, *Ergodic Theory Dynam. Systems* **30** (2010), 379–398. MR 2599885. Zbl 1201.37036. <http://dx.doi.org/10.1017/S0143385709000108>.
- [CE13] J. CHAIKA and A. ESKIN, Every flat surface is Birkhoff and Osceledets generic in almost every direction, 2013. arXiv 1305.1104.
- [Dan81] S. G. DANI, Invariant measures and minimal sets of horospherical flows, *Invent. Math.* **64** (1981), 357–385. MR 0629475. Zbl 0498.58013. <http://dx.doi.org/10.1007/BF01389173>.
- [DM89] S. G. DANI and G. A. MARGULIS, Values of quadratic forms at primitive integral points, *Invent. Math.* **98** (1989), 405–424. MR 1016271. Zbl 0682.22008. <http://dx.doi.org/10.1007/BF01388860>.
- [DM90] S. G. DANI and G. A. MARGULIS, Orbit closures of generic unipotent flows on homogeneous spaces of $\mathrm{SL}(3, \mathbb{R})$, *Math. Ann.* **286** (1990), 101–128. MR 1032925. Zbl 0679.22007. <http://dx.doi.org/10.1007/BF01453567>.
- [DM93] S. G. DANI and G. A. MARGULIS, Limit distributions of orbits of unipotent flows and values of quadratic forms, in *I. M. Gel'fand Seminar, Adv. Soviet Math.* **16**, Amer. Math. Soc., Providence, RI, 1993, pp. 91–137. MR 1237827. Zbl 0814.22003.
- [EM04] A. ESKIN and G. MARGULIS, Recurrence properties of random walks on finite volume homogeneous manifolds, in *Random Walks and Geometry*, Walter de Gruyter GmbH & Co. KG, Berlin, 2004, pp. 431–444. MR 2087794. Zbl 1064.60092.
- [EMM98] A. ESKIN, G. MARGULIS, and S. MOZES, Upper bounds and asymptotics in a quantitative version of the Oppenheim conjecture, *Ann. of Math.*

- 147** (1998), 93–141. MR 1609447. Zbl 0906.11035. <http://dx.doi.org/10.2307/120984>.
- [EMM06] A. ESKIN, J. MARKLOF, and D. W. MORRIS, Unipotent flows on the space of branched covers of Veech surfaces, *Ergodic Theory Dynam. Systems* **26** (2006), 129–162. MR 2201941. Zbl 1085.37021. <http://dx.doi.org/10.1017/S0143385705000234>.
- [EM01] A. ESKIN and H. MASUR, Asymptotic formulas on flat surfaces, *Ergodic Theory Dynam. Systems* **21** (2001), 443–478. MR 1827113. Zbl 1096.37501. <http://dx.doi.org/10.1017/S0143385701001225>.
- [EMS03] A. ESKIN, H. MASUR, and M. SCHMOLL, Billiards in rectangles with barriers, *Duke Math. J.* **118** (2003), 427–463. MR 1983037. Zbl 1038.37031. <http://dx.doi.org/10.1215/S0012-7094-03-11832-3>.
- [EMZ03] A. ESKIN, H. MASUR, and A. ZORICH, Moduli spaces of Abelian differentials: the principal boundary, counting problems, and the Siegel-Veech constants, *Publ. Math. Inst. Hautes Études Sci.* **97** (2003), 61–179. MR 2010740. Zbl 1037.32013. <http://dx.doi.org/10.1007/s10240-003-0015-1>.
- [EM13] A. ESKIN and M. MIRZAKHANI, Invariant and stationary measures for the $SL(2, \mathbb{R})$ action on moduli space, 2013. arXiv 1302.3320.
- [EMR12] A. ESKIN, M. MIRZAKHANI, and K. RAFL, Counting closed geodesics in strata, 2012. arXiv 1206.5574.
- [FK80] H. M. FARKAS and I. KRA, *Riemann Surfaces*, *Grad. Texts in Math.* **71**, Springer-Verlag, New York, 1980. MR 0583745. Zbl 0475.30001.
- [Fay73] J. D. FAY, *Theta Functions on Riemann Surfaces*, *Lecture Notes in Math.* **352**, Springer-Verlag, New York, 1973. MR 0335789. Zbl 0281.30013.
- [For02] G. FORNI, Deviation of ergodic averages for area-preserving flows on surfaces of higher genus, *Ann. of Math.* **155** (2002), 1–103. MR 1888794. Zbl 1034.37003. <http://dx.doi.org/10.2307/3062150>.
- [FMZ14] G. FORNI, C. MATHEUS, and A. ZORICH, Lyapunov spectrum of invariant subbundles of the Hodge bundle, *Ergodic Theory Dynam. Systems* **34** (2014), 353–408. MR 3233697. Zbl 1290.37002. <http://dx.doi.org/10.1017/etds.2012.148>.
- [Fur02] A. FURMAN, Random walks on groups and random transformations, in *Handbook of Dynamical Systems*, Vol. 1A, North-Holland, Amsterdam, 2002, pp. 931–1014. MR 1928529. Zbl 1053.60045. [http://dx.doi.org/10.1016/S1874-575X\(02\)80014-5](http://dx.doi.org/10.1016/S1874-575X(02)80014-5).
- [Fur63a] H. FURSTENBERG, Noncommuting random products, *Trans. Amer. Math. Soc.* **108** (1963), 377–428. MR 0163345. Zbl 0203.19102. <http://dx.doi.org/10.2307/1993589>.
- [Fur63b] H. FURSTENBERG, A Poisson formula for semi-simple Lie groups, *Ann. of Math.* **77** (1963), 335–386. MR 0146298. Zbl 0192.12704. <http://dx.doi.org/10.2307/1970220>.
- [HK14] J. H. HUBBARD and S. KOCH, An analytic construction of the Deligne-Mumford compactification of the moduli space of curves, *J. Differential*

- Geom.* **98** (2014), 261–313. MR 3263519. Zbl 06322533. Available at <http://projecteuclid.org/euclid.jdg/1406552251>.
- [Mar89a] G. A. MARGULIS, Discrete subgroups and ergodic theory, in *Number Theory, Trace Formulas and Discrete Groups* (Oslo, 1987), Academic Press, Boston, 1989, pp. 377–398. MR 0993328. Zbl 0675.10010.
- [Mar89b] G. A. MARGULIS, Indefinite quadratic forms and unipotent flows on homogeneous spaces, in *Dynamical Systems and Ergodic Theory* (Warsaw, 1986), *Banach Center Publ.* **23**, PWN, Warsaw, 1989, pp. 399–409. MR 1102736. Zbl 0689.10026.
- [Mar90] G. A. MARGULIS, Orbits of group actions and values of quadratic forms at integral points, in *Festschrift in Honor of I. I. Piatetski-Shapiro on the Occasion of his Sixtieth Birthday, Part II* (Ramat Aviv, 1989), *Israel Math. Conf. Proc.* **3**, Weizmann, Jerusalem, 1990, pp. 127–150. MR 1159112. Zbl 0718.11027.
- [Mar91] G. A. MARGULIS, Dynamical and ergodic properties of subgroup actions on homogeneous spaces with applications to number theory, in *Proceedings of the International Congress of Mathematicians, Vol. I, II* (Kyoto, 1990), Math. Soc. Japan, Tokyo, 1991, pp. 193–215. MR 1159213. Zbl 0747.58017.
- [Mar04] G. A. MARGULIS, Random walks on the space of lattices and the finiteness of covolumes of arithmetic subgroups, in *Algebraic Groups and Arithmetic*, Tata Inst. Fund. Res., Mumbai, 2004, pp. 409–425. MR 2094118. Zbl 1110.22005.
- [Mas85] B. MASKIT, Comparison of hyperbolic and extremal lengths, *Ann. Acad. Sci. Fenn. Ser. A I Math.* **10** (1985), 381–386. MR 0802500. Zbl 0587.30043. <http://dx.doi.org/10.5186/aasfm.1985.1042>.
- [Mas76] H. MASUR, The extension of the Weil-Petersson metric to the boundary of Teichmüller space, *Duke Math. J.* **43** (1976), 623–635. MR 0417456. Zbl 0358.32017. <http://dx.doi.org/10.1215/S0012-7094-76-04350-7>.
- [Mas88] H. MASUR, Lower bounds for the number of saddle connections and closed trajectories of a quadratic differential, in *Holomorphic Functions and Moduli, Vol. I* (Berkeley, CA, 1986), *Math. Sci. Res. Inst. Publ.* **10**, Springer-Verlag, New York, 1988, pp. 215–228. MR 0955824. Zbl 0661.30034. http://dx.doi.org/10.1007/978-1-4613-9602-4_20.
- [Mas90] H. MASUR, The growth rate of trajectories of a quadratic differential, *Ergodic Theory Dynam. Systems* **10** (1990), 151–176. MR 1053805. Zbl 0706.30035. <http://dx.doi.org/10.1017/S0143385700005459>.
- [MS95] S. MOZES and N. SHAH, On the space of ergodic invariant measures of unipotent flows, *Ergodic Theory Dynam. Systems* **15** (1995), 149–159. MR 1314973. Zbl 0818.58028. <http://dx.doi.org/10.1017/S0143385700008282>.
- [NZ99] A. NEVO and R. J. ZIMMER, Homogeneous projective factors for actions of semi-simple Lie groups, *Invent. Math.* **138** (1999), 229–252. MR 1720183. Zbl 0936.22007. <http://dx.doi.org/10.1007/s002220050377>.

- [Rat90a] M. RATNER, On measure rigidity of unipotent subgroups of semisimple groups, *Acta Math.* **165** (1990), 229–309. MR 1075042. Zbl 0745.28010. <http://dx.doi.org/10.1007/BF02391906>.
- [Rat90b] M. RATNER, Strict measure rigidity for unipotent subgroups of solvable groups, *Invent. Math.* **101** (1990), 449–482. MR 1062971. Zbl 0745.28009. <http://dx.doi.org/10.1007/BF01231511>.
- [Rat91a] M. RATNER, On Raghunathan’s measure conjecture, *Ann. of Math.* **134** (1991), 545–607. MR 1135878. Zbl 0763.28012. <http://dx.doi.org/10.2307/2944357>.
- [Rat91b] M. RATNER, Raghunathan’s topological conjecture and distributions of unipotent flows, *Duke Math. J.* **63** (1991), 235–280. MR 1106945. Zbl 0733.22007. <http://dx.doi.org/10.1215/S0012-7094-91-06311-8>.
- [Sha94] N. A. SHAH, Unipotent flows on homogeneous spaces, Ph.D. thesis, 1994, Tata Institute for Fundamental Research.
- [Sta87] A. N. STARKOV, Solvable homogeneous flows, *Mat. Sb.* **176** (1987), 242–259. MR 0922418. Zbl 0667.58058. <http://dx.doi.org/10.1070/SM1989v062n01ABEH003238>.
- [Sta89] A. N. STARKOV, Ergodic decomposition of flows on homogeneous spaces of finite volume, *Mat. Sb.* **180** (1989), 1614–1633. MR 1038220. Zbl 0787.22009. <http://dx.doi.org/10.1070/SM1991v068n02ABEH001200>.
- [Vee98] W. A. VEECH, Siegel measures, *Ann. of Math.* **148** (1998), 895–944. MR 1670061. Zbl 0922.22003. <http://dx.doi.org/10.2307/121033>.
- [Wol03] S. A. WOLPERT, Geometry of the Weil-Petersson completion of Teichmüller space, in *Surveys in Differential Geometry, Vol. VIII* (Boston, MA, 2002), Int. Press, Somerville, MA, 2003, pp. 357–393. MR 2039996. Zbl 1049.32020. <http://dx.doi.org/10.4310/SDG.2003.v8.n1.a13>.
- [Wri14] A. WRIGHT, The field of definition of affine invariant submanifolds of the moduli space of abelian differentials, *Geom. Topol.* **18** (2014), 1323–1341. MR 3254934. Zbl 06323273. <http://dx.doi.org/10.2140/gt.2014.18.1323>.
- [Zor06] A. ZORICH, Flat surfaces, in *Frontiers in Number Theory, Physics, and Geometry. I*, Springer-Verlag, New York, 2006, pp. 437–583. MR 2261104. Zbl 1129.32012.

(Received: February 23, 2014)

(Revised: February 21, 2015)

UNIVERSITY OF CHICAGO, CHICAGO, IL
E-mail: eskin@math.uchicago.edu

STANFORD UNIVERSITY, STANFORD, CA
E-mail: mmirzakh@math.stanford.edu

UNIVERSITY OF TEXAS, AUSTIN, TX
E-mail: amir@math.utexas.edu