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Interlacing families I: Bipartite Ramanujan graphs of all degrees

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Abstract

We prove that there exist infinite families of regular bipartite Ramanujan graphs of every degree bigger than 2. We do this by proving a variant of a conjecture of Bilu and Linial about the existence of good 2-lifts of every graph. We also establish the existence of infinite families of "irregular Ramanujan" graphs, whose eigenvalues are bounded by the spectral radius of their universal cover. Such families were conjectured to exist by Linial and others. In particular, we prove the existence of infinite families of (c, d)-biregular bipartite graphs with all nontrivial eigenvalues bounded by $\sqrt{c-1} + \sqrt{d-1}$ for all $c, d \geq 3$. Our proof exploits a new technique for controlling the eigenvalues of certain random matrices, which we call the "method of interlacing polynomials."

1. Introduction

Ramanujan graphs have been the focus of substantial study in Theoretical Computer Science and Mathematics. They are graphs whose nontrivial adjacency matrix eigenvalues are as small as possible. Previous constructions of Ramanujan graphs have been sporadic, only producing infinite families of Ramanujan graphs of certain special degrees. In this paper, we prove a variant of a conjecture of Bilu and Linial [4] and use it to realize an approach they suggested for constructing bipartite Ramanujan graphs of every degree.

Our main technical contribution is a novel existence argument. The conjecture of Bilu and Linial requires us to prove that every graph has a signed adjacency matrix with all of its eigenvalues in a small range. We do this by proving that the roots of the *expected* characteristic polynomial of a randomly signed adjacency matrix lie in this range. In general, a statement like this is useless, as the roots of a sum of polynomials do not necessarily have anything to do with the roots of the polynomials in the sum. However, there seem to be many sums of combinatorial polynomials for which this intuition is wrong.

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With this in mind, we identify certain special collections of polynomials, which we call "interlacing families," and prove that such families always contain a polynomial whose largest root is at most the largest root of the sum. We show that the polynomials arising from the signings of a graph form such a family. To finish the proof, we then bound the largest root of the sum of the characteristic polynomials of the signed adjacency matrices of a graph by observing that this sum is the well-studied matching polynomial of the graph.

This paper is the first one in a series that develops the method of interlacing polynomials. In the second paper [31], we use the method to give a positive resolution to an open problem of Kadison and Singer.

2. **Technical introduction and preliminaries**

2.1. *Ramanujan graphs.* Ramanujan graphs are defined in terms of the eigenvalues of their adjacency matrices. If G is a d-regular graph and A is its adjacency matrix, then d is always an eigenvalue of A. The matrix A has an eigenvalue of $-d$ if and only if G is bipartite. The eigenvalues of d, and $-d$ when G is bipartite, are called the *trivial* eigenvalues of A. Following Lubotzky, Phillips, and Sarnak [28], we say that a d-regular graph is *Ramanujan* if all of its nontrivial eigenvalues lie between $-2\sqrt{d-1}$ and $2\sqrt{d-1}$. It is easy to construct Ramanujan graphs with a small number of vertices: d-regular complete graphs and complete bipartite graphs are Ramanujan. The challenge is to construct an infinite family of d-regular graphs that are all Ramanujan. One cannot construct infinite families of d-regular graphs whose eigenvalues lie in a smaller range: the Alon–Boppana bound (see [34]) tells us that for every constant $\varepsilon > 0$, every sufficiently large d-regular graph has a nontrivial eigenvalue with absolute value at least $2\sqrt{d-1} - \varepsilon$.

Lubotzky, Phillips, and Sarnak [28] and Margulis [32] were the first to construct infinite families of Ramanujan graphs of constant degree. They built both bipartite and nonbipartite Ramanujan graphs from Cayley graphs. All of their graphs are regular and have degrees $p+1$ where p is a prime. There have been very few other constructions of Ramanujan graphs [7], [23], [33], [36]. To the best of our knowledge, the only degrees for which infinite families of Ramanujan graphs were previously known to exist were those of the form $q+1$ where q is a prime power. Lubotzky [29, Prob. 10.7.3] asked whether there exist infinite families of Ramanujan graphs of every degree greater than 2. We resolve this conjecture in the affirmative in the bipartite case.

2.2. 2-*lifts.* Bilu and Linial's strategy for constructing infinite families of expanders was to find a way of doubling the number of vertices of a graph without changing its degree or introducing any large nontrivial eigenvalues.

Given a graph $G = (V, E)$, a 2-*lift* of G is a graph $\hat{G} = (\hat{V}, \hat{E})$ that has two vertices $\{v_0, v_1\} \subseteq \hat{V}$ for each vertex $v \in V$. This pair of vertices is called the *fibre* of the original vertex. Every edge in G corresponds to two edges in $\hat{G}$. If (u,v) is an edge in E, $\{u_0, u_1\}$ is the fibre of u, and $\{v_0, v_1\}$ is the fibre of v, then $\hat{E}$ can either contain the pair of edges

$$\{(u_0,v_0),(u_1,v_1)\}\,, \text{ or} \tag{1}$$

$$\{(u_0,v_1),(u_1,v_0)\}\,. \tag{2}$$

If only edge pairs of the first type appear, then the 2-lift is just two disjoint copies of the original graph. If only edge pairs of the second type appear, then we obtain the *double-cover* of G.

The definition of a 2-lift is equivalent to saying that there is a $2:1$ *covering map* from $\hat{G}$ to G. Recall that a covering map is a graph homomorphism $\pi : \hat{V} \to V$ that bijectively maps the star of every vertex $\hat{v} \in \hat{V}$ to the star of $\pi(v)$, where the star of a vertex is the set of edges incident to it. If there is a covering map (of any order) from $\hat{G}$ to G, we say that $\hat{G}$ is a *lift* of G and G is a *quotient* of $\hat{G}$.

To analyze the eigenvalues of a 2-lift, Bilu and Linial study *signings* $s : E \to \{\pm 1\}$ of the edges of G. They place signings in one-to-one correspondence with 2-lifts by setting $s(u,v) = 1$ if edges of type (1) appear in the 2-lift and $s(u,v) = -1$ if edges of type (2) appear. They then define the signed adjacency matrix A_s to be the same as the adjacency matrix of G, except that the entries corresponding to an edge (u,v) are $s(u,v)$. They prove [4, Lemma 3.1] that the eigenvalues of the 2-lift are the union, taken with multiplicity, of the eigenvalues of the adjacency matrix A and those of the signed adjacency matrix A_s. Following Friedman [13], they refer to the eigenvalues of A as the *old eigenvalues* and the eigenvalues of A_s as the *new eigenvalues*. The main result of their paper is that every graph of maximum degree d has a signing in which all of the new eigenvalues have absolute value at most $O(\sqrt{d \log^3 d})$. They then build arbitrarily large d-regular expander graphs by repeatedly taking 2-lifts of a complete graph on $d+1$ vertices.

Bilu and Linial conjectured that every d-regular graph has a signing in which all of the new eigenvalues have absolute value at most $2\sqrt{d-1}$. If their conjecture is true, one can obtain an infinite family of d-regular Ramanujan graphs by starting with the d-regular complete graph and then repeatedly forming the appropriate 2-lifts. We prove a weak version of Bilu and Linial's conjecture: every d-regular graph has a signing in which all of the new eigenvalues are at most $2\sqrt{d-1}$. The difference between our result and the original conjecture is that we do not control the smallest new eigenvalue. This is why we consider bipartite graphs. The eigenvalues of the adjacency matrices of bipartite graphs are symmetric about zero (see, for example, [16, Th. 2.4.2]). Thus any upper bound on the largest nontrivial eigenvalue implies a corresponding lower bound on the smallest one. Since the 2-lift of a bipartite graph is also bipartite, we can start with a d-regular complete bipartite graph and inductively form the appropriate 2-lifts to obtain an infinite sequence of d-regular bipartite Ramanujan graphs.

2.3. *Irregular Ramanujan graphs and universal covers.* We say that a bipartite graph is (c, d)-biregular if all vertices on one side of the bipartition have degree c and all vertices on the other side have degree d. (To avoid technicalities, we will require $c, d \geq 2$.) The adjacency matrix of a (c, d)-biregular graph always has eigenvalues $\pm\sqrt{cd}$; these are its trivial eigenvalues. Feng and Li [12] (see also [24]) prove a generalization of the Alon–Boppana bound that applies to (c, d)-biregular graphs: for all $\varepsilon > 0$, all sufficiently large (c, d)-biregular graphs have a nontrivial eigenvalue that is at least $\sqrt{c-1} + \sqrt{d-1} - \varepsilon$. Thus, we say that a (c, d)-biregular graph is Ramanujan if all of its nontrivial eigenvalues have absolute value at most $\sqrt{c-1} + \sqrt{d-1}$. We prove the existence of infinite families of (c, d)-biregular Ramanujan graphs for all $c, d \geq 3$.

The regular and biregular Ramanujan graphs discussed above are actually special cases of a more general phenomenon. To describe it, we use a topological construction known as the *universal cover*. The universal cover of a graph G is the unique infinite tree T such that every connected lift of G is a quotient of T (see, e.g., [22, §6]). It can also be defined concretely in terms of walks in G. Recall that a *walk* in a graph is a sequence of vertices $(v_0, v_1, \ldots, v_\ell)$ such that each consecutive pair (v_{i-1}, v_i) is an edge in G. A walk is called *simple* if all the vertices are distinct and *nonbacktracking* if $v_{i-1} \neq v_{i+1}$ for all i. We say that a walk w' is a *continuation* of another walk w if it is obtained by adding a single vertex to w, i.e., $w = (v_0, \ldots, v_\ell)$ and $w' = (v_0, \ldots, v_\ell, v_{\ell+1})$ for some $v_{\ell+1}$.

To construct the universal cover, fix a "root" vertex $v_0 \in G$, and then place one vertex in T for every nonbacktracking walk w in G starting at v_0 of every length $\ell \in \mathbb{N}$. Two vertices w, w' are adjacent in T whenever w' is a continuation of w (or vice versa). It is easy to check that the universal cover of a graph is unique up to isomorphism and is independent of the choice of v_0.

The adjacency matrix A_T of the universal cover T is an infinite symmetric matrix. We will be interested in the *spectral radius* $\rho(T)$ of T, which may be defined[1] as

$$\rho(T) := \sup_{\|x\|_2 = 1} \|A_T x\|_2, \tag{3}$$

where $\|x\|_2^2 := \sum_{i=1}^{\infty} x(i)^2$ whenever the series converges. Naturally, the spectral radius of a finite tree is defined to be the norm of its adjacency matrix.

[1] In functional analysis, the spectral radius of an infinite-dimensional operator A is traditionally defined to be the supremum of $|\lambda|$ over the $\lambda \in \mathbb{C}$ such that $A - \lambda I$ is not invertible. However, in the case of self-adjoint operators, this definition is equivalent to the one presented here (see, for example, Theorem VI.6 in [38]).

With these notions in hand, we can state the definition of an irregular Ramanujan graph. As before, the largest (and smallest, in the bipartite case) eigenvalues of finite adjacency matrices are considered trivial. Greenberg [19] (see also [9]) showed that for every $\varepsilon > 0$ and every infinite family of graphs that have the same universal cover T, all sufficiently large graphs in the family have a nontrivial eigenvalue that is at least $\rho(T) - \varepsilon$. Following Greenberg [19] (see also [22, Def. 6.7]), we therefore define an arbitrary graph to be Ramanujan if all of its nontrivial eigenvalues have absolute value at most the spectral radius of its universal cover.

The universal cover of every d-regular graph is the infinite d-ary tree, and the universal cover of every (c, d)-biregular graph is the infinite (c, d)-biregular tree in which the degrees alternate between c and d on every other level [24]. The former tree is known to have a spectral radius of $2\sqrt{d-1}$ and the latter a spectral radius of $\sqrt{c-1} + \sqrt{d-1}$ (see [18], [24]). Thus a definition based on universal covers generalizes both the regular and biregular definitions of Ramanujan graphs, and the bound of Greenberg generalizes both the Alon-Boppana and Feng-Li bounds.

In this general setting, we show that every graph G has a 2-lift in which all of the new eigenvalues are at most the spectral radius of its universal cover. Applying these 2-lifts inductively to any finite irregular bipartite Ramanujan graph yields an infinite family of irregular bipartite Ramanujan graphs whose degree distribution matches that of the initial graph (since taking a 2-lift simply doubles the number of vertices of each degree). In particular, applying them to the (c, d)-biregular complete bipartite graph yields an infinite family of (c, d)-biregular Ramanujan graphs. As far as we know, infinite families of irregular Ramanujan graphs were not known to exist prior to this work.

2.4. *Related work.* There have been numerous studies of random lifts of graphs. For some results on the spectra of random lifts, we point the reader to [1], [2], [3], [26], [25], [27]. Friedman [14] has proved that almost every d-regular graph almost meets the Ramanujan bound: he shows that for every $\varepsilon > 0$, the absolute value of all the nontrivial eigenvalues of almost every sufficiently large d-regular graph are at most $2\sqrt{d-1} + \varepsilon$. In the irregular case, Puder [37] has shown that with high probability a random high-order lift of a graph G has new eigenvalues that are bounded in absolute value by $\sqrt{3}\rho$, where ρ is the spectral radius of the universal cover of G.

We remark that constructing bipartite Ramanujan graphs is at least as easy as constructing nonbipartite ones: the double-cover of a d-regular nonbipartite Ramanujan graph is a d-regular bipartite Ramanujan graph. For a survey of applications of expander graphs, we refer the reader to [22].

3. 2-lifts and the matching polynomial

A *matching* in a graph is a subset of its edges, no two of which share a common vertex. For a graph G, let m_i denote the number of matchings in G consisting of i edges (with $m_0 = 1$). Heilmann and Lieb [21] defined the *matching polynomial* of G to be the polynomial

$$\mu_G(x) \stackrel{\text{def}}{=} \sum_{i\geq 0} x^{n-2i}(-1)^i m_i,$$

where n is the number of vertices in the graph. They proved two remarkable theorems about the matching polynomial that we will exploit in this paper. It is worth mentioning that the proofs of these theorems are elementary and short, relying only on simple recurrence formulas for the matching polynomial.

THEOREM 3.1 ([21, Th. 4.2]). *For every graph G, $\mu_G(x)$ has only real roots.*

THEOREM 3.2 ([21, Th. 4.3]). *For every graph G of maximum degree d, all of the roots of $\mu_G(x)$ have absolute value at most $2\sqrt{d-1}$.*

The preceding theorems will allow us to prove the existence of infinite families of d-regular bipartite Ramanujan graphs. To handle the irregular case, we will require a refinement of these results due to Godsil. This refinement uses the concept of a *path tree*, which was also introduced by Godsil (see [15] or [16, §6]).

Definition 3.3. Given a graph G and a vertex u, the path tree $P(G, u)$ contains one vertex for every simple walk in G beginning at u. Two vertices in $P(G, u)$ are adjacent if the simple walk corresponding to one is a continuation of the simple walk corresponding to the other (as defined in Section 2.3).

Remark 3.4. The use of the term "path tree" derives from the fact that simple walks (as we have defined them) are sometimes called *paths* in the literature. Since this is not standard, we chose to define everything in terms of walks, but we retain the name "path tree" since it signifies a specific construction of Godsil.

The path tree provides a natural relationship between the roots of the matching polynomial of a graph and the spectral radius of its universal cover:

THEOREM 3.5 (Godsil [15]). *Let $P(G, u)$ be a path tree of G. Then the matching polynomial of G divides the characteristic polynomial of the adjacency matrix of $P(G, u)$. In particular, all of the roots of $\mu_G(x)$ are real and have absolute value at most $\rho(P(G, u))$.*

LEMMA 3.6. *Let G be a graph, and let T be its universal cover. Then the roots of $\mu_G(x)$ are bounded in absolute value by $\rho(T)$.*

Proof. Let u be any vertex of G and let P be the path tree rooted at u. Since the simple walks that correspond to the vertices of P are (in particular) nonbacktracking walks, P is a finite induced subgraph of the universal cover T, and A_P is a finite submatrix of A_T. By Theorem 3.5, the roots of μ_G are bounded by

$$\begin{aligned}\|A_P\|_2 &= \sup_{\|x\|_2=1} \|A_P x\|_2 \\ &\leq \sup_{\|y\|_2=1, \mathrm{supp}(y)\subset P} \|A_T y\|_2 \\ &\leq \sup_{\|y\|_2=1} \|A_T y\|_2 = \rho(T),\end{aligned}$$

as desired. □

We remark that one can directly prove an upper bound of $2\sqrt{d-1}$ on the spectral radius of a path tree of a d-regular graph and an upper bound of $\sqrt{c-1}+\sqrt{d-1}$ on the spectral radius of a path tree of a (c,d)-regular bipartite graph without considering infinite trees. We point the reader to Section 5.6 of Godsil's book [16] for an elementary argument.

We now recall an identity of Godsil and Gutman that relates the expected characteristic polynomial over uniformly random signings of the adjacency matrix of a graph to its matching polynomial. To associate a signing of the edges of G with a vector in $\{\pm 1\}^m$, we choose an arbitrary ordering of the m edges of G, denote the edges by $e_1, \ldots, e_m$, and denote a signing of these edges by $s \in \{\pm 1\}^m$. We then let A_s denote the signed adjacency matrix corresponding to s and define $f_s(x) = \det(xI - A_s)$ to be the characteristic polynomial of A_s.

Theorem 3.7 (Corollary 2.2 of Godsil and Gutman [17]).

$$\mathbb{E}_{s\in\{\pm 1\}^m}\left[f_s(x)\right] = \mu_G(x).$$

For the convenience of the reader, we present a simple proof of this theorem in Appendix A.

To prove that a good lift exists, it suffices, by Theorems 3.2 and 3.7, to show that there is a signing s so that the largest root of $f_s(x)$ is at most the largest root of $\mathbb{E}_{s\in\{\pm 1\}^m}\left[f_s(x)\right]$. To do this, we prove that the polynomials $\{f_s(x)\}_{s\in\{\pm 1\}^m}$ form what we call an "interlacing family." We define interlacing families and examine their properties in the next section.

4. Interlacing families

Definition 4.1. We say that a polynomial $g(x) = \prod_{i=1}^{n-1}(x-\alpha_i)$ *interlaces* a polynomial $f(x) = \prod_{i=1}^{n}(x-\beta_i)$ if

$$\beta_1 \leq \alpha_1 \leq \beta_2 \leq \alpha_2 \leq \cdots \leq \alpha_{n-1} \leq \beta_n.$$

We say that polynomials $f_1, \ldots, f_k$ have a *common interlacing* if there is a single polynomial g that interlaces each of the f_i.

Let $\beta_{i,j}$ be the jth smallest root of f_i. An equivalent characterization of the polynomials $f_1, \ldots, f_k$ having a common interlacing is the existence of numbers $\alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_n$ so that $\beta_{i,j} \in [\alpha_{j-1}, \alpha_j]$ for all i and j. The numbers $\alpha_1, \ldots, \alpha_{n-1}$ can be taken to be the roots of the polynomial g, and α_0 (α_n) can be chosen to be any number that is smaller (larger) than all of the roots of all of the f_i.

LEMMA 4.2. *Let $f_1, \ldots, f_k$ be degree-n real-rooted polynomials with positive leading coefficients, and define*

$$f_\emptyset = \sum_{i=1}^{k} f_i.$$

If $f_1, \ldots, f_k$ have a common interlacing, then there exists an i for which the largest root of f_i is at most the largest root of $f_\emptyset$.

Proof. Let g be a polynomial that interlaces all of the f_i, and let α_{n-1} be the largest root of g. As each f_i has a positive leading coefficient, it is positive for sufficiently large x. As each f_i has exactly one root that is at least α_{n-1}, each f_i is nonpositive at α_{n-1}. Therefore $f_\emptyset$ is also nonpositive at α_{n-1} and eventually becomes positive. This tells us that $f_\emptyset$ has a root that is at least α_{n-1}, and so its largest root is at least α_{n-1}. Let β_n be this root.

As $f_\emptyset$ is the sum of the f_i, there must be some i for which $f_i(\beta_n) \geq 0$. As f_i has at most one root that is at least α_{n-1}, and $f_i(\alpha_{n-1}) \leq 0$, the largest root of f_i is it at least α_{n-1} and at most β_n. □

One can show that the assumptions of the lemma imply that $f_\emptyset$ is itself a real-rooted polynomial. The conclusion of the lemma also holds for the kth largest root by a similar argument. However, we will not require these facts here.

If the polynomials do not have a common interlacing, the sum may fail to be real-rooted: consider $(x+1)(x+2)+(x-1)(x-2)$. Even if the sum of two polynomials is real-rooted, the conclusion of Lemma 4.2 may fail to hold if the interval containing the largest roots of each polynomial overlaps the interval containing their second-largest roots. For example, consider the sum of the polynomials $(x+5)(x-9)(x-10)$ and $(x+6)(x-1)(x-8)$. It has roots at approximately -5.3, 6.4, and 7.4, so its largest root is smaller than the largest root of both polynomials of which it is the sum.

Definition 4.3. Let $S_1, \ldots, S_m$ be finite sets, and for every assignment $s_1, \ldots, s_m \in S_1 \times \cdots \times S_m$, let $f_{s_1,\ldots,s_m}(x)$ be a real-rooted degree n polynomial with positive leading coefficient. For a partial assignment $s_1, \ldots, s_k \in S_1 \times \cdots \times S_k$ with $k < m$, define

$$f_{s_1,\ldots,s_k} \stackrel{\text{def}}{=} \sum_{s_{k+1} \in S_{k+1}, \ldots, s_m \in S_m} f_{s_1,\ldots,s_k,s_{k+1},\ldots,s_m}$$

as well as

$$f_{\emptyset} \stackrel{\text{def}}{=} \sum_{s_1 \in S_1, \ldots, s_m \in S_m} f_{s_1,\ldots,s_m}.$$

We say that the polynomials $\{f_{s_1,\ldots,s_m}\}_{s_1,\ldots,s_m}$ form an *interlacing family* if for all $k = 0, \ldots, m-1$ and all $s_1, \ldots, s_k \in S_1 \times \cdots \times S_k$, the polynomials

$$\{f_{s_1,\ldots,s_k,t}\}_{t \in S_{k+1}}$$

have a common interlacing.

THEOREM 4.4. *Let $S_1, \ldots, S_m$ be finite sets, and let $\{f_{s_1,\ldots,s_m}\}$ be an interlacing family of polynomials. Then, there exists some $s_1, \ldots, s_m \in S_1 \times \cdots \times S_m$ so that the largest root of $f_{s_1,\ldots,s_m}$ is at most the largest root of $f_{\emptyset}$.*

Proof. From the definition of an interlacing family, we know that the polynomials $\{f_t\}$ for $t \in S_1$ have a common interlacing and that their sum is $f_{\emptyset}$. Lemma 4.2 then guarantees one of the polynomials (say f_{s_1}) has largest root at most the largest root of $f_{\emptyset}$. We now proceed inductively. For any $s_1, \ldots, s_k$, we know that the polynomials $\{f_{s_1,\ldots,s_k,t}\}$ for $t \in S_{k+1}$ have a common interlacing and that their sum is $f_{s_1,\ldots,s_k}$. So for some choice of t (say s_{k+1}), the largest root of the polynomial $f_{s_1,\ldots,s_{k+1}}$ is at most the largest root of $f_{s_1,\ldots,s_k}$. □

We will prove that the polynomials $\{f_s\}_{s \in \{\pm 1\}^m}$ defined in Section 3 form an interlacing family. According to Definition 4.3, this requires establishing the existence of certain common interlacings. There is a systematic way to do this based on the fact that common interlacings are equivalent to real-rootedness statements. In particular, the following result seems to have been discovered a number of times. It appears as Theorem 2.1 of Dedieu [10], (essentially) as Theorem $2'$ of Fell [11], and as (a special case of) Theorem 3.6 of Chudnovsky and Seymour [8].

LEMMA 4.5. *Let $f_1, \ldots, f_k$ be (univariate) polynomials of the same degree with positive leading coefficients. Then $f_1, \ldots, f_k$ have a common interlacing if and only if $\sum_{i=1}^{k} \lambda_i f_i$ is real-rooted for all nonnegative $\lambda_1, \ldots, \lambda_k$.*

5. The main result

Our proof that the polynomials $\{f_s\}_{s \in \{\pm 1\}^m}$ form an interlacing family relies on the following generalization of the fact that the matching polynomial is real-rooted. It says that if the sign of each edge is chosen independently, each with its own probability, then the resulting polynomial is real-rooted.

THEOREM 5.1. *The polynomial*

$$\sum_{s \in \{\pm 1\}^m} \left(\prod_{i: s_i = 1} p_i \right) \left(\prod_{i: s_i = -1} (1 - p_i) \right) f_s(x)$$

is real-rooted for all values of $p_1, \ldots, p_m \in [0, 1]$.

We will prove this theorem using machinery that we develop in Section 6. It immediately implies our main technical result as follows.

THEOREM 5.2. *The polynomials* $\{f_s\}_{s\in\{\pm 1\}^m}$ *form an interlacing family.*

Proof. We will show that for every $0 \leq k \leq m-1$, every partial assignment $s_1 \in \pm 1, \ldots, s_k \in \pm 1$, and every $\lambda \in [0,1]$, the polynomial

$$\lambda f_{s_1,\ldots,s_k,1}(x) + (1-\lambda) f_{s_1,\ldots,s_k,-1}(x)$$

is real-rooted. The theorem then follows from Lemma 4.5. However this follows directly from Theorem 5.1 with $p_{k+1} = \lambda$, $p_{k+2}, \ldots, p_m = 1/2$, and $p_i = (1+s_i)/2$ for $1 \leq i \leq k$. □

THEOREM 5.3. *Let G be a graph with adjacency matrix A and universal cover T. Then there exists a signing s of A so that all of the eigenvalues of A_s are at most $\rho(T)$. In particular, if G is d-regular, there is a signing s so that the eigenvalues of A_s are at most $2\sqrt{d-1}$.*

Proof. The first statement follows immediately from Theorems 3.7, 4.4, 5.2, and Lemma 3.6. The second statement follows by noting that the universal cover of a d-regular graph is the infinite d-regular tree, which has spectral radius at most $2\sqrt{d-1}$, or by directly appealing to Theorem 3.2. □

LEMMA 5.4. *Every nontrivial eigenvalue of a complete (c,d)-biregular graph is zero.*

Proof. The adjacency matrix of this graph has rank 2, so all its eigenvalues other than $\pm\sqrt{cd}$ must be zero. □

THEOREM 5.5. *For every $d \geq 3$, there exists an infinite sequence of d-regular bipartite Ramanujan graphs.*

Proof. We know from Lemma 5.4 that the complete bipartite graph of degree d is Ramanujan. By Lemma 3.1 of [4] and Theorem 5.3, for every d-regular bipartite Ramanujan graph G, there is a 2-lift in which every nontrivial eigenvalue is at most $2\sqrt{d-1}$. As the 2-lift of a bipartite graph is bipartite and the eigenvalues of a bipartite graph are symmetric about 0, this 2-lift is also a d-regular bipartite Ramanujan graph.

Thus for every d-regular bipartite Ramanujan graph G, there exists another d-regular bipartite Ramanujan graph with twice as many vertices. □

THEOREM 5.6. *There exists an infinite sequence of (c,d)-biregular bipartite Ramanujan graphs for all $c, d \geq 3$.*

Proof. We know from Lemma 5.4 that the complete (c,d)-biregular graph is Ramanujan. We will use this as a base for a construction of an infinite sequence of (c, d)-biregular bipartite Ramanujan graphs. Let G be any (c, d)-biregular bipartite Ramanujan graph. As mentioned in Section 2.3, the universal cover of G is the infinite (c, d)-biregular tree, which has spectral radius $\sqrt{c-1} + \sqrt{d-1}$. Thus, Theorem 5.3 tells us that there is a 2-lift of G with all new eigenvalues at most $\sqrt{c-1} + \sqrt{d-1}$. As this graph is bipartite, all of its nontrivial eigenvalues have absolute value at most $\sqrt{c-1} + \sqrt{d-1}$. The resulting 2-lift is therefore a larger (c, d)-biregular bipartite Ramanujan graph. □

To conclude the section, we remark that repeated application of Theorem 5.3 can be used to generate an infinite sequence of irregular Ramanujan graphs from *any* finite irregular bipartite Ramanujan graph, since all of the lifts produced will have the same universal cover. In other words, if an infinite tree has a single bipartite Ramanujan quotient, then it must have infinitely many. In contrast, Lubotzky and Nagnibeda [30] have shown that there exist infinite trees that cover infinitely many finite graphs but such that none of the finite graphs are Ramanujan.

6. **Real stable polynomials**

In this section we will establish the real-rootedness of a class of polynomials that includes the polynomials of Theorem 5.1. We will do this by considering a multivariate generalization of real-rootedness known as *real stability* (see, e.g., the surveys [35], [40]). In particular, we will show that the univariate polynomials we are interested in are the images, under a well-behaved linear transformation, of a multivariate real stable polynomial.

Definition 6.1. A multivariate polynomial $f \in \mathbb{R}[z_1, \ldots, z_n]$ is called *real stable* if it is the zero polynomial or if

$$f(z_1, \ldots, z_n) \neq 0$$

whenever the imaginary part of each z_i is strictly positive.

Note that a real stable polynomial has real coefficients but may be evaluated on complex inputs. We begin by considering certain determinantal polynomials whose real stability is guaranteed by the following lemma, which may be found in Borcea and Brändén [6, Prop. 2.4].

LEMMA 6.2. *Let $A_1, \ldots, A_m$ be positive semidefinite matrices. Then*

$$\det (z_1 A_1 + \cdots + z_m A_m)$$

is real stable.

Real stable polynomials enjoy a number of useful closure properties. In particular, it is easy to see that if $f(x_1, \ldots, x_k)$ and $g(y_1, \ldots y_j)$ are real stable, then $f(x_1, \ldots, x_k)g(y_1, \ldots, y_j)$ is real stable. A standard limiting argument based on Hurwitz's theorem shows that the real stability of $f(x_1, \ldots, x_k)$ implies the real stability of $f(x_1, \ldots, x_{k-1}, c)$ for every $c \in \mathbb{R}$ (see, e.g., Lemma 2.4 in [40]). For a variable x_i, we define Z_{x_i} to be the operator that acts on polynomials by setting the variable x_i to zero.

In [5], Borcea and Brändén characterize a class of differential operators that preserve real stability. To simplify notation, we will let ∂_{z_i} denote the operation of partial differentiation with respect to z_i. For $\alpha, \beta \in \mathbb{N}^n$, we use the notation

$$z^\alpha = \prod_{i=1}^{n} z_i^{\alpha_i} \quad \text{and} \quad \partial^\beta = \prod_{i=1}^{n} (\partial_{z_i})^{\beta_i}.$$

THEOREM 6.3 (Theorem 1.3 in [5]). *Let $T : \mathbb{R}[z_1, \ldots, z_n] \to \mathbb{R}[z_1, \ldots, z_n]$ be an operator of the form*

$$T = \sum_{\alpha, \beta \in \mathbb{N}^n} c_{\alpha,\beta} z^\alpha \partial^\beta,$$

where $c_{\alpha,\beta} \in \mathbb{R}$ and $c_{\alpha,\beta}$ is zero for all but finitely many terms. Define

$$F_T(z, w) := \sum_{\alpha,\beta} c_{\alpha,\beta} z^\alpha w^\beta.$$

Then T preserves real stability if and only if $F_T(z, -w)$ is real stable.

We will use a special case of this result.

COROLLARY 6.4. *For all real numbers $a, b \geq 0$ and variables x, y, the operator $T = 1 + a\partial_x + b\partial_y$ preserves real stability.*

Proof. If $a = b = 0$, then T is the identity operator and thus preserves real stability. If a or b is nonzero, we just need to show that the polynomial $r(x, y) = 1 - ax - by$ is real stable. To see this, consider x and y with positive imaginary parts. The imaginary part of $1 - ax - by$ will then be negative, and so r cannot be zero. □

We now show how operators of the preceding kind can be used to generate the expected characteristic polynomial that appears in Theorem 5.1.

LEMMA 6.5. *For an invertible matrix A, vectors u and v, and a number $p \in [0, 1]$,*

$$\begin{aligned} Z_x Z_y (1 + p\partial_x + (1-p)\partial_y) \det\left(A + xuu^T + yvv^T\right) \\ = p \det\left(A + uu^T\right) + (1-p) \det\left(A + vv^T\right). \end{aligned}$$

Proof. The matrix determinant lemma (see, e.g., [20]) states that for every nonsingular matrix A and every real number t,

$$\det\left(A + tuu^T\right) = \det(A)\,(1 + tu^T A^{-1} u).$$

One consequence of this is Jacobi's formula for the derivative of the determinant:

$$\partial_t \det\left(A + tuu^T\right) = \det(A)\,(u^T A^{-1} u).$$

This formula implies that

$$Z_x Z_y (1 + p\partial_x + (1-p)\partial_y) \det\left(A + xuu^T + yvv^T\right) \\ = \det(A)\left(1 + p(u^T A^{-1} u) + (1-p)(v^T A^{-1} v)\right).$$

By the matrix determinant lemma, this equals

$$p \det\left(A + uu^T\right) + (1-p)\det\left(A + vv^T\right). \qquad \square$$

Using these tools, we prove our main technical result on real-rootedness.

THEOREM 6.6. *Let $u_1, \ldots, u_m$, $v_1, \ldots, v_m$ be vectors in $\mathbb{R}^n$, let $p_1, \ldots, p_m$ be real numbers in $[0, 1]$, and let D be a positive semidefinite matrix. Then the* (*univariate*) *polynomial*

$$P(x) \stackrel{\text{def}}{=} \sum_{S \subseteq [m]} \left(\prod_{i \in S} p_i\right)\left(\prod_{i \notin S}(1 - p_i)\right) \det\left(xI + D + \sum_{i \in S} u_i u_i^T + \sum_{i \notin S} v_i v_i^T\right)$$

is real-rooted.

Proof. Let $y_1, \ldots, y_m$ and $z_1, \ldots, z_m$ be formal variables, and define

$$Q(x, y_1, \ldots, y_m, z_1, \ldots, z_m) = \det\left(xI + D + \sum_i y_i u_i u_i^T + \sum_i z_i v_i v_i^T\right).$$

Lemma 6.2 and the fact mentioned in the paragraph after Lemma 6.2 that specializing variables to real numbers preserves real stability imply that Q is real stable. We claim that we can rewrite P as

$$P(x) = \left(\prod_{i=1}^{m} Z_{y_i} Z_{z_i} T_i\right) Q(x, y_1, \ldots, y_m, z_1, \ldots, z_m),$$

where $T_i = 1 + p_i \partial_{y_i} + (1 - p_i)\partial_{z_i}$. To see this, we prove by induction on k that

$$\left(\prod_{i=1}^{k} Z_{y_i} Z_{z_i} T_i\right) Q(x, y_1, \ldots, y_m, z_1, \ldots, z_m)$$

equals

$$\sum_{S \subseteq [k]} \left(\prod_{i \in S} p_i\right)\left(\prod_{i \in [k]\setminus S}(1 - p_i)\right) \det\left(xI + D + \sum_{i \in S} u_i u_i^T \right.$$
$$\left. + \sum_{i \in [k]\setminus S} v_i v_i^T + \sum_{i > k} y_i u_i u_i^T + z_i v_i v_i^T\right).$$

The base case ($k = 0$) is trivially true, as it is the definition of Q. The inductive step follows from Lemma 6.5. The case $k = m$ is exactly the claimed identity.

Starting with Q (a real stable polynomial) we can then apply Corollary 6.4 and the closure of real stable polynomials under the restrictions of variables to real constants to see that each of the polynomials above, including $P(x)$, is also real stable. As $P(x)$ is real stable and has one variable, it is real-rooted. □

Alternatively, one can prove Theorem 6.6 by observing that P is a *mixed characteristic polynomial*, as defined in [31], and then applying Corollary 4.4 of that paper.

Proof of Theorem 5.1. For each vertex u, let d_u be its degree, and let $d = \max_u d_u$. We need to prove that the polynomial

$$\sum_{s\in\{\pm 1\}^m} \left(\prod_{i:s_i=1} p_i\right)\left(\prod_{i:s_i=-1}(1-p_i)\right)\det\left(xI - A_s\right)$$

is real-rooted. This is equivalent to proving that the the following polynomial is real-rooted:

$$\sum_{s\in\{\pm 1\}^m} \left(\prod_{i:s_i=1} p_i\right)\left(\prod_{i:s_i=-1}(1-p_i)\right)\det\left(xI + dI - A_s\right), \tag{4}$$

as their roots only differ by d.

We now observe that the matrix $dI - A_s$ is a signed Laplacian matrix of G plus a nonnegative diagonal matrix. For each edge (u, v), define the rank-1 matrices

$$L^{1}_{u,v} = (e_u - e_v)(e_u - e_v)^T, \quad \text{and}$$
$$L^{-1}_{u,v} = (e_u + e_v)(e_u + e_v)^T,$$

where e_u is the elementary unit vector in direction u. Consider a signing s, and let $s_{u,v}$ denote the sign it assigns to edge (u, v). Since the original graph had maximum degree d, we have

$$dI - A_s = \sum_{(u,v)\in E} L^{s_{u,v}}_{u,v} + D,$$

where D is the diagonal matrix whose uth diagonal entry equals $d - d_u$. As the diagonal entries of D are nonnegative, it is positive semidefinite. If we now set $a_{u,v} = (e_u - e_v)$ and $b_{u,v} = (e_u + e_v)$, we can express the polynomial in (4) as

$$\sum_{s\in\{\pm 1\}^m} \left(\prod_{i:s_i=1} p_i\right)\left(\prod_{i:s_i=-1}(1-p_i)\right)\det\left(xI + D + \sum_{s_{u,v}=1} a_{u,v}a_{u,v}^T + \sum_{s_{u,v}=-1} b_{u,v}b_{u,v}^T\right).$$

The fact that this polynomial is real-rooted now follows from Theorem 6.6. □

7. Conclusion

We conclude by drawing an analogy between our proof technique and the probabilistic method, which relies on the fact that for every random variable $X : \Omega \to \mathbb{R}$, there is an $\omega \in \Omega$ for which $X(\omega) \le \mathbb{E}[X]$. We have shown that for certain special polynomial-valued random variables $P : \Omega \to \mathbb{R}[x]$, there must be an ω with $\lambda_{\max}(P(\omega)) \le \lambda_{\max}(\mathbb{E}[P])$. In fact it is possible to define interlacing families in greater generality than we have done here, using probabilistic notation. In particular, we call a polynomial-valued random variable P *useful* if P is deterministic and real-rooted *or* if there exist disjoint nontrivial events $E_1, \ldots, E_k$ with $\sum_{i \le k} \Pr[E_i] = 1$ such that the polynomials $\{\mathbb{E}[P|E_i]\}_{i \le k}$ have a common interlacing and each polynomial $\mathbb{E}[P|E_i]$ is itself useful. The conclusion of Theorem 4.4 continues to hold for this definition, and we suspect it will be applicable in nonproduct settings. In the case of this paper, the events E_i are particularly simple: they correspond to setting one sign of a lift to be $+1$ or -1, and the resulting sequence of polynomials $f_\emptyset, f_{s_1}, \ldots, f_{s_1,\ldots,s_m}$ forms a martingale (a fact that we do not use, but may be interesting in its own right).

Like many applications of the probabilistic method, our proof does not yield a polynomial-time algorithm. In the particular case of random lifts, the polynomial $f_\emptyset$ is itself a matching polynomial, and its last coefficient is the number of perfect matchings in the graph. Valiant [39] proved that it is $\#P$-hard to compute the number of matchings in a graph; so, we do not expect to find an efficient algorithm for computing the matching polynomial. It would be very interesting to find a computationally efficient analogue of our method.

It has been pointed out that the proofs in this paper carry over to the case of multigraphs (ones allowing loops and multiple edges) without much effort. For the sake of simplicity, we chose not to pursue that direction in detail. We mention it, however, as it might provide an interesting set of "seed" graphs apart from the default choice of the complete bipartite graph.

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Appendix A. Proof of Theorem 3.7

Let $\mathrm{sym}(S)$ denote the set of permutations of a set S, and let $|\pi|$ denote the number of inversions of a permutation π. We recall that $(-1)^{|\pi|}$ is the *signature* of the permutation π. Expanding the determinant as a sum over permutations $\sigma \in \mathrm{sym}([n])$, we have

$$\begin{aligned}\mathbb{E}_s\left[\det(xI - A_s)\right] &= \mathbb{E}_s\left[\sum_{\sigma\in\mathrm{sym}([n])}(-1)^{|\sigma|}\prod_{i=1}^{n}(xI-A_s)_{i,\sigma(i)}\right]\\ &= \sum_{k=0}^{n} x^{n-k}(-1)^k \sum_{S\subset[n],|S|=k}\ \sum_{\pi\in\mathrm{sym}(S)} \mathbb{E}_s\left[(-1)^{|\pi|}\prod_{i\in S}(A_s)_{i,\pi(i)}\right]\\ &= \sum_{k=0}^{n} x^{n-k}(-1)^k \sum_{S\subset[n],|S|=k}\ \sum_{\pi\in\mathrm{sym}(S)} \mathbb{E}_s\left[(-1)^{|\pi|}\prod_{i\in S}s_{i,\pi(i)}\right].\end{aligned}$$

Since the s_{ij} are independent with $\mathbb{E}\left[s_{ij}\right] = 0$, only those products that contain even powers (0 or 2) of the s_{ij} survive. Thus, we may restrict our attention to the permutations π that contain only orbits of size two (disjoint transpositions). These are just the perfect matchings on S. There are no perfect matchings when $|S|$ is odd; when $|S|$ is even, each matching consists of $|S|/2$ disjoint transpositions. Since $\mathbb{E}_s\left[s_{ij}^2\right] = 1$, we are left with

$$\begin{aligned}\mathbb{E}_s\left[\det(xI - A_s)\right] &= \sum_{\substack{k=0\\ k \text{ even}}}^{n} x^{n-k} \sum_{|S|=k}\ \sum_{\text{matchings } \pi \text{ on } S} (-1)^{|S|/2}\cdot 1\\ &= \mu_G(x),\end{aligned}$$

as desired.

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