Annals of Mathematics **181** (2015), 1139–1208
http://dx.doi.org/10.4007/annals.2015.181.3.6

A solution of an L^2 extension problem with an optimal estimate and applications

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Abstract

In this paper, we prove an L^2 extension theorem with an optimal estimate in a precise way, which implies optimal estimate versions of various well-known L^2 extension theorems. As applications, we give proofs of a conjecture of Suita on the equality condition in Suita's conjecture, the so-called L-conjecture, and the extended Suita conjecture. As other applications, we give affirmative answer to a question by Ohsawa about limiting case for the extension operators between the weighted Bergman spaces, and we present a relation of our result to Berndtsson's important result on log-plurisubharmonicity of the Bergman kernel.

1. Background and notation

The L^2 extension problem is stated as follows (for background, see Demailly [16]): for a suitable pair (M, S), where S is a closed complex subvariety of a complex manifold M, given a holomorphic function f (or a holomorphic section of a holomorphic vector bundle) on Y satisfying suitable L^2 conditions on S, find an L^2 holomorphic extension F on M together with a good or even optimal L^2 estimate for F on M.

The famous Ohsawa-Takegoshi L^2 extension theorem (Ohsawa wrote a series of papers on the L^2 extension theorem in more general settings) gives an answer to the first part of the problem — existence of the L^2 extension. There have been some new proofs and a lot of important applications of the theorem in complex geometry and several complex variables, thanks to the works of Y.-T. Siu, J. P. Demailly, Ohsawa, and Berndtsson et al. (see [49], [50], [37], [39], [16], [6], etc.). An unsolved problem is left — the second part of an optimal estimate in the L^2 extension problem. A first exception is Blocki's recent work on an optimal estimate of Ohsawa-Takegoshi's L^2 extension theorem for bounded pseudoconvex domains (see [11]) as a continuation of an earlier work

The authors were partially supported by NSFC.

towards the L^2 extension problem with optimal estimate [61] (see also [10]). Another exception is our recent work [26] based on [61] about an optimal estimate of Ohsawa's L^2 extension theorem with negligible weight for Stein manifolds [40].

In the present paper, we shall further discuss the L^2 extension problem with an optimal estimate and give a solution of the problem with its applications, by putting it into a vision with a wider scope.

The paper is organized as follows. In the rest of this section, we recall some notation used in the paper. In Section 2, we present our main theorems, solving the L^2 extension problem with optimal estimate. In Section 3, we introduce the main applications and corollaries of our main theorems, among others, including: we give proofs of a conjecture of Suita on the equality condition in Suita's conjecture, the so-called L-conjecture, and the extended Suita conjecture; we find a relation of our result to Berndtsson's theorem on log-plurisubharmonicity of the Bergman kernel; we give an affirmative answer to a question by Ohsawa in [43] about a limiting case for the extension operators between the weighted Bergman spaces; and we also obtain optimal estimate versions of various well-known L^2 extension theorems. In Section 4 we recall or prove some preliminary results used in the proofs of the main theorems and corollaries. In Section 5, we give the detailed proofs of the main theorems. In Section 6, we give the proofs of the main corollaries.

For basic knowledge and references on several complex variables, we refer to [12], [14], [21], [23], [24], [31], [32], [33], [52], [53], [58], [59], et al.

Now let us recall some notation in [41], [42]. Let M be a complex n-dimensional manifold, and let S be a closed complex subvariety of M. Let dV_M be a continuous volume form on M. We consider a class of the upper-semi-continuous function Ψ from M to the interval $[-\infty, A)$, where $A \in (-\infty, +\infty]$, such that

(1) $\Psi^{-1}(-\infty) \supset S$, and $\Psi^{-1}(-\infty)$ is a closed subset of M;

(2) if S is l-dimensional around a point $x \in S_{\mathrm{reg}}$ (S_{reg} is the regular part of S), there exists a local coordinate $(z_1, \ldots, z_n)$ on a neighborhood U of x such that $z_{l+1} = \cdots = z_n = 0$ on $S \cap U$ and

$$\sup_{U \setminus S} |\Psi(z) - (n-l) \log \sum_{l+1}^{n} |z_j|^2| < \infty.$$

The set of such polar functions Ψ will be denoted by $\#_A(S)$.

For each $\Psi \in \#_A(S)$, one can associate a positive measure $dV_M[\Psi]$ on S_{reg} as the minimum element of the partially ordered set of positive measures $d\mu$ satisfying

$$\int_{S_l} f d\mu \geq \limsup_{t \to \infty} \frac{2(n-l)}{\sigma_{2n-2l-1}} \int_M f e^{-\Psi} \mathbb{I}_{\{-1-t<\Psi<-t\}} dV_M$$

for any nonnegative continuous function f with $\operatorname{supp} f \subset\subset M$, we denote by $\mathbb{I}_{\{-1-t<\Psi<-t\}}$ the characteristic function of the set $\{-1-t < \Psi < -t\}$, S_l the l-dimensional component of S_{reg}, and σ_m the volume of the unit sphere in $\mathbb{R}^{m+1}$.

Let ω be a Kähler metric on $M \setminus (X \cup S)$, where X is a closed subset of M such that $S_{\mathrm{sing}} \subset X$. ($S_{\mathrm{sing}}$ is the singular part of S.)

We can define measure $dV_\omega[\Psi]$ on $S \setminus X$ as the minimum element of the partially ordered set of positive measures $d\mu'$ satisfying

$$\int_{S_l} f d\mu' \geq \limsup_{t\to\infty} \frac{2(n-l)}{\sigma_{2n-2l-1}} \int_{M\setminus(X\cup S)} f e^{-\Psi} \mathbb{I}_{\{-1-t<\Psi<-t\}} dV_\omega$$

for any nonnegative continuous function f with $\operatorname{supp}(f) \subset\subset M \setminus X$; as

$$\operatorname{Supp}(\mathbb{I}_{\{-1-t<\Psi<-t\}}) \cap \operatorname{Supp}(f) \subset\subset M \setminus (X \cup S),$$

the right-hand side of the above inequality is well defined.

Let u be a continuous section of $K_M \otimes E$, where E is a holomorphic vector bundle equipped with a continuous metric h on M. We define

$$|u|_h^2|_V := \frac{c_n h(e,e) v \wedge \bar{v}}{dV_M}$$

and

$$|u|_{h,\omega}^2|_V := \frac{c_n h(e,e) v \wedge \bar{v}}{dV_\omega},$$

where $u|_V = v \otimes e$ for an open set $V \subset M \setminus (X \cup S)$, v is a continuous section of $K_M|_V$ and e is a continuous section of $E|_V$; especially, we define

$$|u|^2|_V := \frac{c_n u \wedge \bar{u}}{dV_M},$$

when u is a continuous section of K_M. It is clear that $|u|_h^2$ is independent of the choice of V.

The following argument shows a relationship between $dV_\omega[\Psi]$ and $dV_M[\Psi]$ (resp. dV_ω and dV_M). Precisely,

$$\int_{M\setminus(X\cup S)} f|u|_{h,\omega}^2 dV_\omega[\Psi] = \int_{M\setminus(X\cup S)} f|u|_h^2 dV_M[\Psi], \tag{1.1}$$

$$\left(\text{resp. } \int_{M\setminus(X\cup S)} f|u|_{h,\omega}^2 dV_\omega = \int_{M\setminus(X\cup S)} f|u|_h^2 dV_M\right), \tag{1.2}$$

where f is a continuous function with compact support on $M \setminus X$.

For the neighborhood U, let $u|_U = v \otimes e$. Note that

$$\begin{aligned}
&\int_{M\setminus(X\cup S)} f\mathbb{I}_{\{-1-t<\Psi<-t\}}|u|^2_{h,\omega}e^{-\Psi}dV_\omega \\
&\quad = \int_{M\setminus(X\cup S)} f\mathbb{I}_{\{-1-t<\Psi<-t\}}h(e,e)c_n v\wedge\bar{v}e^{-\Psi} \\
&\quad = \int_{M\setminus(X\cup S)} f\mathbb{I}_{\{-1-t<\Psi<-t\}}|u|^2_h e^{-\Psi}dV_M
\end{aligned} \tag{1.3}$$

and

$$\begin{aligned}
&(\text{resp. } \int_{M\setminus(X\cup S)} f|u|^2_{h,\omega}e^{-\Psi}dV_\omega \\
&\quad = \int_{M\setminus(X\cup S)} fh(e,e)c_n v\wedge\bar{v}e^{-\Psi} \\
&\quad = \int_{M\setminus(X\cup S)} f|u|^2_h e^{-\Psi}dV_M),
\end{aligned} \tag{1.4}$$

where f is a continuous function with compact support on $M \setminus (X \cup S)$. As

$$\mathrm{Supp}(\mathbb{I}_{\{-1-t<\Psi<-t\}}) \cap \mathrm{Supp}(f) \subset\subset M \setminus (X \cup S),$$

equality (1.3) is well defined. Then we have equalities (1.1) and (1.2).

It is clear that $|u|^2_h$ is independent of the choice of U, while $|u|^2_h dV_M$ is independent of the choice of dV_M (resp. $|u|^2_h dV_M[\Psi]$ is independent of the choice of dV_M). Then the space of L^2 integrable holomorphic sections of K_M is denoted by $A^2(M, K_M \otimes E, dV_M^{-1}, dV_M)$ (resp. the space of holomorphic sections of $K_M|_S$ which is L^2 integrable with respect to the measure $dV_M[\Psi]$ is denoted by $A^2(S, K_M|_S \otimes E|_S, dV_M^{-1}, dV_M[\Psi])$).

Denote the norm of any continuous section u of $K_M \otimes E$ by

$$|u|^2_h dV_M := \{u, u\}_h.$$

Define

$$\{f, f\}_h := \langle e, e\rangle_h \sqrt{-1}^{\dim S^2} f_1 \wedge \bar{f}_1$$

for any continuous section f of $K_S \otimes E|_S$, where $f = f_1 \otimes e$ locally (see [16]). It is clear that $\{f, f\}_h$ is well defined.

Definition 1.1. Let M be a complex manifold with a continuous volume form dV_M, and let S be a closed complex subvariety of M. We say (M, S) satisfies condition (ab) if M and S satisfy the following conditions:

There exists a closed subset $X \subset M$ such that

(a) X is locally negligible with respect to L^2 holomorphic functions; i.e., for any local coordinate neighborhood $U \subset M$ and for any L^2 holomorphic function f on $U \setminus X$, there exists an L^2 holomorphic function $\tilde{f}$ on U such that $\tilde{f}|_{U\setminus X} = f$ with the same L^2 norm.

(b) $M \setminus X$ is a Stein manifold which intersects with every component of S, such that $S_{\text{sing}} \subset X$.

When S is smooth, condition (ab) is the same as condition (1) in Theorem 4 in [41], [42]. The data (M,S) with the condition (ab) includes all the following examples:

(1) M is a Stein manifold (including open Riemann surfaces), and S is any closed complex subvariety of M;
(2) M is a complex projective algebraic manifold (including compact Riemann surfaces), and S is any closed complex subvariety of M;
(3) M is a projective family (see [51]), and S is any closed complex subvariety of M.

The Hermitian metric h on E is said to be semipositive in the sense of Nakano if the curvature tensor Θ_h is semipositive definite as a hermitian form on $T_X \otimes E$, i.e. if for every $u \in T_X \otimes E$, we have $\sqrt{-1}\Theta_h(u,u) \geq 0$ (see [13]).

Let $\Delta_{A,h,\delta}(S)$ be the subset of functions Ψ in $\#_A(S)$ which satisfies that both $he^{-\Psi}$ and $he^{-(1+\delta)\Psi}$ are semipositive in the sense of Nakano on $M \setminus (X \cup S)$. Let $\Delta_A(S)$ be the subset of plurisubharmonic functions Ψ in $\#_A(S)$.

2. Main theorems

In the present section, we state an L^2 extension theorem with an optimal estimate, related to a kind of positive real function $c_A(t)$, which will be explained later on, solving the L^2 extension problem with an optimal estimate. The theorem is stated first in a general setting and then in a less general but sufficiently useful setting. By the way, the word "optimal" depends on the considered setting. If the setting becomes narrower, the estimate possibly could not be optimal again.

Given $\delta > 0$, let $c_A(t)$ be a positive function on $(-A, +\infty)$ $(A \in (-\infty, +\infty))$, which is in $C^\infty((-A, +\infty))$ and satisfies both $\int_{-A}^{\infty} c_A(t)e^{-t}dt < \infty$ and

(2.1)
$$\left(\frac{1}{\delta}c_A(-A)e^A + \int_{-A}^{t} c_A(t_1)e^{-t_1}dt_1\right)^2$$
$$> c_A(t)e^{-t}\left(\int_{-A}^{t}\left(\frac{1}{\delta}c_A(-A)e^A + \int_{-A}^{t_2} c_A(t_1)e^{-t_1}dt_1\right)dt_2 + \frac{1}{\delta^2}c_A(-A)e^A\right)$$

for any $t \in (-A, +\infty)$. If $c_A(t)e^{-t}$ is decreasing with respect to t, then inequality (2.1) holds.

We establish the following L^2 extension theorem with an optimal estimate as follows:

Theorem 2.1 (Main Theorem 1). *Let* (M, S) *satisfy condition* (ab), *and let* h *be a smooth metric on a holomorphic vector bundle* E *on* M *with rank* r. *Let* $\Psi \in \#_A(S) \cap C^\infty(M \setminus S)$, *which satisfies*

(1) $he^{-\Psi}$ *is semipositive in the sense of Nakano on* $M \setminus (S \cup X)$ (X *is as in the definition of condition* (ab));
(2) *there exists a continuous function* $a(t)$ *on* $(-A, +\infty]$, *such that* $0 < a(t) \leq s(t)$ *and* $a(-\Psi)\sqrt{-1}\Theta_{he^{-\Psi}} + \sqrt{-1}\partial\bar{\partial}\Psi$ *is semipositive in the sense of Nakano on* $M \setminus (S \cup X)$, *where*

$$s(t) = \frac{\int_{-A}^{t}(\frac{1}{\delta}c_A(-A)e^A + \int_{-A}^{t_2} c_A(t_1)e^{-t_1}dt_1)dt_2 + \frac{1}{\delta^2}c_A(-A)e^A}{\frac{1}{\delta}c_A(-A)e^A + \int_{-A}^{t} c_A(t_1)e^{-t_1}dt_1}.$$

Then there exists a uniform constant $\mathbf{C} = 1$, *which is optimal, such that, for any holomorphic section* f *of* $K_M \otimes E|_S$ *on* S *satisfying*

$$\sum_{k=1}^{n} \frac{\pi^k}{k!} \int_{S_{n-k}} |f|_h^2 dV_M[\Psi] < \infty, \tag{2.2}$$

there exists a holomorphic section F *of* $K_M \otimes E$ *on* M *satisfying* $F = f$ *on* S *and*

$$\begin{aligned} &\int_M c_A(-\Psi)|F|_h^2 dV_M \\ &\quad \leq \mathbf{C}\left(\frac{1}{\delta}c_A(-A)e^A + \int_{-A}^{\infty} c_A(t)e^{-t}dt\right) \sum_{k=1}^{n} \frac{\pi^k}{k!} \int_{S_{n-k}} |f|_h^2 dV_M[\Psi], \end{aligned} \tag{2.3}$$

where $c_A(t)$ *satisfies* $c_A(-A)e^A := \lim_{t \to -A^+} c_A(t)e^{-t} < \infty$ *and* $c_A(-A)e^A \neq 0$.

Using Remark 4.10 and Lemma 4.8, which will be discussed later on, we can replace smoothness of c_A in the above theorem by continuity.

Now we consider a useful and simpler class of functions as follows: Let $c_A(t)$ be a positive function in $C^\infty((-A, +\infty))$ ($A \in (-\infty, +\infty]$), satisfying $\int_{-A}^{\infty} c_A(t)e^{-t}dt < \infty$ and

$$\left(\int_{-A}^{t} c_A(t_1)e^{-t_1}dt_1\right)^2 > c_A(t)e^{-t}\int_{-A}^{t}\int_{-A}^{t_2} c_A(t_1)e^{-t_1}dt_1dt_2 \tag{2.4}$$

for any $t \in (-A, +\infty)$. When $c_A(t)e^{-t}$ is decreasing with respect to t and A is finite, inequality (2.4) holds.

For such a simpler and sufficiently useful class of functions, we establish the following L^2 extension theorem with an optimal estimate, whose simpler version was announced in [28]:

Theorem 2.2 (Main Theorem 2). *Let* (M, S) *satisfy condition* (ab), *and let* Ψ *be a plurisubharmonic function in* $\Delta_A(S) \cap C^\infty(M \setminus (S \cup X))$. ($X$ *is as in the definition of condition* (ab).) *Let* h *be a smooth metric on a holomorphic* *vector bundle E on M with rank r, such that $he^{-\Psi}$ is semipositive in the sense of Nakano on $M \setminus (S \cup X)$. (When E is a line bundle, h can be chosen as a semipositive singular metric.) Then there exists a uniform constant $\mathbf{C} = 1$, which is optimal, such that, for any holomorphic section f of $K_M \otimes E|_S$ on S satisfying condition (2.2), there exists a holomorphic section F of $K_M \otimes E$ on M satisfying $F = f$ on S and*

$$\int_M c_A(-\Psi)|F|_h^2 dV_M \leq \mathbf{C} \int_{-A}^{\infty} c_A(t)e^{-t}dt \sum_{k=1}^{n} \frac{\pi^k}{k!} \int_{S_{n-k}} |f|_h^2 dV_M[\Psi].$$

Similarly as before, we can replace smoothness of c_A in the above theorem by continuity.

3. Applications and main corollaries

In this section, we present applications and main corollaries of our main theorems, among others, solutions of a conjecture of Suita on the equality condition in Suita's conjecture, the L-conjecture, the extended Suita conjecture; a relation to Berndtsson's log-plurisubharmonicity of the Bergman kernel; optimal constant versions of various known L^2 extension theorems; an affirmative answer to a question by Ohsawa about a limiting case for the extension operators between the weighted Bergman spaces; and so on.

3.1. *A conjecture of Suita.* In this subsection, we present a corollary of Theorem 2.2, which solves a conjecture of Suita on the equality condition in Suita's conjecture on the comparison between the Bergman kernel and the logarithmic capacity.

Let Ω be an open Riemann surface, which admits a nontrivial Green function G_Ω. Let w be a local coordinate on a neighborhood V_{z_0} of $z_0 \in \Omega$ satisfying $w(z_0) = 0$. Let κ_Ω be the Bergman kernel for holomorphic $(1, 0)$ forms on Ω. We define

$$B_\Omega(z)|dw|^2 := \kappa_\Omega(z)|_{V_{z_0}}$$

and

$$B_\Omega(z, \bar{t})dw \otimes d\bar{t} := \kappa_\Omega(z, \bar{t})|_{V_{z_0}}.$$

Let $c_\beta(z)$ be the logarithmic capacity which is locally defined by

$$c_\beta(z_0) := \exp \lim_{z \to z_0} (G_\Omega(z, z_0) - \log |w(z)|)$$

on Ω (see [47]).

Suita's conjecture in [54] says that on any open Riemann surface Ω as above, $(c_\beta(z_0))^2 \leq \pi B_\Omega(z_0)$.

The above conjecture was first proved for bounded planar domains by Blocki [10], [11] and then by Guan-Zhou [26] for open Riemann surfaces. For earlier works, see [61].

In the same paper [54], Suita also conjectured a necessary and sufficient condition for the equality holding in his inequality:

A CONJECTURE OF SUITA. $(c_\beta(z_0))^2 = \pi B_\Omega(z_0)$ *for* $z_0 \in \Omega$ *if and only if* Ω *is conformally equivalent to the unit disc less a* (*possible*) *closed set of inner capacity zero.*

In fact, a closed set of inner capacity zero is a polar set (locally singularity set of a subharmonic function).

Using Theorem 2.2, we solve the conjecture of Suita:

THEOREM 3.1. *The above conjecture of Suita holds.*

3.2. *The L-conjecture.* In this subsection, we now give a proof of the L-conjecture. Let Ω be an open Riemann surface which admits a nontrivial Green function G_Ω and is not biholomorphic to the unit disc less a (possible) closed set of inner capacity zero.

Assume that $G_\Omega(\cdot, t)$ is an exhaustion function for any $t \in \Omega$. Associated to the Bergman kernel $\kappa_\Omega(z, \bar{t})$, one may define the adjoint L-kernel $L_\Omega(z, t) := \frac{2}{\pi}\frac{\partial^2 G_\Omega(z,t)}{\partial z \partial t}$ (see [48]). In [57], there is a conjecture on the zero points of the adjoint L-kernel as follows:

THE L-CONJECTURE (LC). *For any* $t \in \Omega$, $\exists z \in \Omega$, *we have* $L_\Omega(z, t) = 0$.

It is known that, for finite Riemann surface Ω, $G_\Omega(\cdot, t)$ is an exhaustion function for any $t \in \Omega$ (see [57]). By Theorem 6 in [57], the L-conjecture for finite Riemann surfaces is deduced from the above conjecture of Suita. Using Theorem 3.1, we solve the L-conjecture for any open Riemann surface with the exhaustion Green function:

THEOREM 3.2. *The above L-conjecture holds.*

The following example shows that the assumption that $G_\Omega(\cdot, t)$ is an exhaustion function for any $t \in \Omega$ is necessary.

Let m and p denote the numbers of the boundary contours and the genus of Ω, respectively (see [55]). In fact, for any finite Riemann surface Ω, which is not simply connected, the Bergman kernel $\kappa_\Omega(z, \bar{t})$ of Ω has exactly $2p+m-1$ zeros for suitable t (see [55]).

Let Ω be an annulus. Then we have $2p+m-1 = 1$ (see page 93, [48]). It is known that $\#\{z|L_\Omega(z,t) = 0\} + \#\{z|\kappa_\Omega(z,\bar{t}) = 0\} \leq 4p+2m-2 = 2$ for all $t \in \Omega$ (see [55]). Note that $\kappa_\Omega(z, \bar{t})$ has exactly $2p+m-1 = 1$ zeros for suitable t. Using Theorem 3.2, we have $\#\{z|L_\Omega(z,t) = 0\} = 1 = 4p+2m-2-1$ for suitable $t \in \Omega$. Let $t_1 \in \Omega$ satisfy $\#\{z|L_\Omega(z,t_1) = 0\} = 1$. Assume that $z_1 \in \{z|L_\Omega(z,t_1) = 0\}$. Note that $z_1 \neq t_1$. As $G_{\Omega\setminus\{z_1\}} = G_\Omega|_{\Omega\setminus\{z_1\}}$, then we have $\#\{z|L_{\Omega\setminus\{z_1\}}(z,t_1) = 0\} = 0$.

3.3. *Extended Suita Conjecture.* Let Ω be an open Riemann surface, which admits a nontrivial Green function G_Ω. Take $z_0 \in \Omega$ with a local coordinate z. Let $p : \Delta \to \Omega$ be the universal covering from unit disc Δ to Ω.

We call the holomorphic function f (resp. holomorphic $(1,0)$-form F) on Δ a multiplicative function (resp. multiplicative differential (Prym differential)) if there is a character χ, which is the representation of the fundamental group of Ω, such that $g^*f = \chi(g)f$ (resp. $g^*F = \chi(g)F$), where $|\chi| = 1$ and g is an element of the fundamental group of Ω which naturally acts on the universal covering of Ω (see [19]). Denote the set of such kinds of f (resp. F) by $\mathcal{O}^\chi(\Omega)$ (resp. $\Gamma^\chi(\Omega)$).

As p is a universal covering, then for any harmonic function h_Ω on Ω, there exist a χ_h and a multiplicative function $f_h \in \mathcal{O}^{\chi_h}(\Omega)$, such that $|f_h| = p^*e^{h_\Omega}$.

For the Green function $G_\Omega(\cdot, z_0)$, one can also find a χ_{z_0} and a multiplicative function $f_{z_0} \in \mathcal{O}^{\chi_{z_0}}(\Omega)$, such that $|f_{z_0}| = p^*e^{G_\Omega(\cdot,z_0)}$.

Because $g^*|f| = |g^*f| = |\chi(g)f| = |f|$ and $g^*(F \wedge \bar{F}) = g^*F \wedge \overline{g^*F} = \chi(g)F \wedge \overline{\chi(g)F} = F \wedge \bar{F}$, it follows that $|f|$ and $F \wedge \bar{F}$ are fibre constant with respect to p.

As $F \wedge \bar{F}$ is fibre constant, one can define the multiplicative Bergman kernel $\kappa^\chi(x, \bar{y})$ for $\Gamma^\chi(\Omega)$ on $\Omega \times \Omega$. Let $B^\chi_\Omega(z)|dz|^2 := \kappa^\chi_\Omega(z, \bar{z})$. The extended Suita conjecture is formulated as follows ([57]):

Extended Suita Conjecture $c^2_\beta(z_0) \leq \pi B^\chi_\Omega(z_0)$, *and equality holds if and only if* $\chi = \chi_{z_0}$.

The weighted Bergman kernel $\kappa_{\Omega,\rho}$ with weight ρ of holomorphic $(1,0)$-form on a Riemann surface Ω is defined by $\kappa_{\Omega,\rho} := \sum_i e_i \otimes \bar{e}_i$, where $\{e_i\}_{i=1,2,\ldots}$ are holomorphic $(1,0)$-forms on Ω and satisfy $\sqrt{-1}\int_\Omega \rho \frac{e_i}{\sqrt{2}} \wedge \frac{\bar{e}_j}{\sqrt{2}} = \delta^j_i$.

Let h_Ω be a harmonic function on Ω, and let $\rho = e^{-2h_\Omega}$. Related to the weighted Bergman kernel, there is an equivalent form of the extended Suita conjecture in [57]:

Conjecture $c^2_\beta(z_0) \leq \pi\rho(z_0)B_{\Omega,\rho}(z_0)$, *and the equality holds if and only if* $\chi_{-h} = \chi_{z_0}$.

The reason for the equivalence between the above two conjectures is as follows: By the above argument, we have $f_h^{-1}p^*e_j \in \Gamma^{\chi_{-h}}$. Note that the orthogonal basis of $\Gamma^{\chi_{-h}}$ is $\{f_h^{-1}p^*e_j\}_{j=1,2,\ldots}$; then we have $\rho(z_0)B_{\Omega,\rho}(z_0) = B^{\chi_{-h}}_\Omega(z_0)$.

It suffices to show that for any χ such that Γ^χ has a nonzero element F_0, there is a harmonic function h on Ω which satisfies $\chi = \chi_h$. As Ω is noncompact, for $F_0 \in \Gamma^\chi$, one can find a holomorphic function h_0 on Ω such that $F_0p^*h_0^{-1}$ does not have any zero point on Δ. As Ω is noncompact, one can find a holomorphic $(1,0)$-form H_0 on Ω such that H_0 does not have any zero point on Ω.

Then one obtains a holomorphic function $f_1 := \frac{F_0 p^* h_0^{-1}}{p^* H_0^{-1}}$, which does not have any zero point. It is clear that $\log|f_1|$ is harmonic and fibre constant, which can be seen as a harmonic function on Ω.

Note that $f_1 \in \mathcal{O}^{\chi}(\Omega)$. Set $h := \log|f_1|$. Then $\chi_h = \chi$. It is also easy to see that the equality part of two conjectures are also equivalent. Then we prove the equivalence of the two conjectures.

In [29], we have proved $c_\beta^2(z_0) \leq \pi\rho(z_0)B_{\Omega,\rho}(z_0)$. Combining this result in [29] with Theorem 2.2, we completely solve the extended Suita conjecture:

THEOREM 3.3 (a complete solution of the extended Suita conjecture). *$c_\beta^2(z_0) \leq \pi\rho(z_0)B_{\Omega,\rho}(z_0)$ holds, and the equality holds if and only if $\chi_{-h} = \chi_{z_0}$.*

3.4. *A question posed by Ohsawa.* Let Ω be a Stein manifold with a continuous volume form dV_Ω. Let D be a strongly pseudoconvex relatively compact domain in Ω, with a C^2 smooth plurisubharmonic defining function ρ. Let $\delta(z)$ be a distance induced by a Riemannian metric from z to the boundary ∂D of D.

Let H be a closed smooth complex hypersurface on Ω. Then there exists a continuous function s on Ω, which satisfies

(1) $H = \{s = 0\}$;
(2) s^2 is a smooth function on Ω;
(3) $\log|s|$ is a plurisubharmonic function on Ω;
(4) for any point $z \in H$, there exists a local holomorphic defining function e of H, such that $2\log|s| - 2\log|e|$ is continuous near z.

In fact, associated to the hypersurface H, there exists a holomorphic line bundle L_H on Ω with a smooth Hermitian metric h_H and there is a holomorphic section f of L_H, such that $\{f = 0\} = H$ and $df|_z \neq 0$ for any $z \in H$. As Ω is Stein, then there exits a smooth plurisubharmonic function s_1 on Ω, which satisfies that $s_1 + \log|f|_{h_H}$ is a plurisubharmonic function on Ω. Let $s := e^{s_1}|f|_{h_H}$. Then we obtain the existence of the function s.

Assume that ∂D intersects with H transversally. Let

$$A^2_{\alpha,\varphi}(D) := \left\{ f \in \Gamma(D, K_\Omega) | \int_D e^{-\varphi}\delta^\alpha |f|^2 dV_\Omega < \infty \right\}$$

and

$$\| f \|^2_{\alpha,\varphi} = (\alpha+1)\int_D e^{-\varphi}\delta^\alpha |f|^2 dV_\Omega.$$

We put

$$A^2_{-1,\varphi}(D) := \left\{ f \in \Gamma(D, K_\Omega) | \lim_{\alpha \searrow -1}(1+\alpha)\int_D e^{-\varphi}\delta^\alpha |f|^2 dV_\Omega < \infty \right\}$$

and

$$\| f \|^2_{-1,\varphi} := \lim_{\alpha \searrow -1}(1+\alpha)\int_D e^{-\varphi}\delta^\alpha |f|^2 dV_\Omega < \infty.$$

In [18], when Ω is $\mathbb{C}^n$ and H is a smooth complex hypersurface, Diederich and Herbort gave an L^2 extension theorem from $A^2_{\alpha+1,\varphi}(D\cap H)$ to $A^2_{\alpha,\varphi}(D)$, where $\alpha > -1$.

THEOREM 3.4 ([18]). *The extension operator from* $A^2_{\alpha+1,\varphi}(D\cap H)$ *to* $A^2_{\alpha,\varphi}(D)$ *is bounded for any* $\alpha > -1$.

In [43], Ohsawa gave an L^2 extension theorem from $A^2_{0,\varphi}(D\cap H)$ to $A^2_{-1,\varphi}(D)$, which is called a limiting case.

THEOREM 3.5 ([43]). *The extension operator from* $A^2_{0,\varphi}(D\cap H)$ *to* $A^2_{-1,\varphi}(D)$ *is bounded.*

In [43], Ohsawa posed a question about unifying Diederich and Herbort's theorem with his theorem.

Using Theorem 2.1, we give the Ohsawa's question an affirmative answer:

THEOREM 3.6. *Without assuming that* ∂D *intersects with* H *transversally, the extension operator from* $A^2_{\alpha+1,\varphi}(D\cap H)$ *to* $A^2_{\alpha,\varphi}(D)$ *for every* $\alpha > -1$ *has a bound* $C_0 \max\{C_1^\alpha, C_2^\alpha\}$, *where* C_0, C_1 *and* C_2 *are positive constants, which are independent of* α ($\alpha > -1$). *Consequently, the extension operator from* $A^2_{0,\varphi}(D\cap H)$ *to* $A^2_{-1,\varphi}(D)$ *is bounded.*

3.5. *Application to a log-plurisubharmonicity of the Bergman kernel.* In this subsection, we give a relation between Theorem 2.2 and Berndtsson's theorem on log-plurisubharmonicity of the Bergman kernel in the following framework: Let M be a complex $(n+m)$-dimensional manifold fibred over complex m-dimensional manifold Y with n-dimensional fibres, and let $p : M \to Y$ be the projection which satisfies, for any point $t \in Y$, that there exists a unit disc $\Delta_t \subset Y$ such that $(p^{-1}(\Delta_t), p^{-1}(t))$ satisfies condition (ab). Let (L, h) be a semipositive holomorphic line bundle on M with Hermitian metric h over M.

There are two such examples:

(1) M is a pseudoconvex domain in $\mathbb{C}^{n+m}$ with coordinate (z,t), where $z \in \mathbb{C}^n$, $t \in \mathbb{C}^m$, and Y is a domain in $\mathbb{C}^m$ with coordinate t,
$$p(z,t) = t;$$
(2) M is a projective family, Y is a complex manifold, and p is a projection map.

Let (z,t) be the coordinate of $S \times \mathbb{B}^m$, which is the local trivialization of the fibration p with fibre S, and let e be the local frame of L. Let κ_{M_t} be the Bergman kernel of $K_{M_t} \otimes L$ on M_t, and let $\kappa_{M_t} := B_t(z) dz \otimes e \otimes d\bar{z} \otimes \bar{e}$ locally.

In this section, we prove that $\log B_t(z)$ is plurisubharmonic with respect to (z,t), using our result on the L^2 extension with an optimal estimate. It should be noted that we cannot get the log-plurisubharmonicity without an optimal estimate.

Without loss of generality, we assume that Y is 1-dimensional. Then (z,t) is the coordinate of $S\times\Delta_1$.

In order to show that $\log B_t(z)$ is plurisubharmonic with respect to (z,t), we need to check that for any complex line L on (z,t), $\log B_t(z)|_L$ is subharmonic. As we can change the coordinate locally, we only need to check that for the complex lines $\{t|(z,t)\}$. Then it suffices to check the submean value inequality for a disc small enough (see Chapter 1 of [13]).

Consider the framework at the beginning of the present subsection. For any point $w_0\in M$, there is a unit disc $\Delta_{p(w_0)}\subset Y$, such that

$$(p^{-1}\Delta_{p(w_0)},p^{-1}(p(w_0)))$$

satisfies condition (ab). Then we have that $(p^{-1}(\Delta_1),p^{-1}(p(w))$ satisfies condition (ab) for any point $w\in S\times\Delta_1$, by choosing Δ_1 small enough.

For any given t, if $\kappa_{M_t}\not\equiv 0$, by the extremal property of the Bergman kernel, there exists a holomorphic section u_t of $K_{p^{-1}(t)}\otimes L$ on $p^{-1}(t)$ such that

$$B_t(z)=\frac{|g(z,t)|^2}{\int_{M_t}\frac{1}{2^n}\{u_t,u_t\}},$$

where $u_t=g(z,t)dz\otimes e$ on (z,t).

If $\log B_{t_0}(z)=-\infty$, we are done. Then we can assume that

$$B_{t_0}(z)=\frac{|g(z)|^2}{\int_{M_{t_0}}\frac{1}{2^n}\{u_{t_0},u_{t_0}\}_h},$$

where u_{t_0} is a holomorphic section of $K_{p^{-1}(t_0)}\otimes L$ on $p^{-1}(t_0)$, and $u_{t_0}=g(z)dz\otimes e$ on (z,t).

Let Δ_r be the unit disc with center (z,t_0) and radius r on the line $\{t|(z,t)\}$. In Theorem 2.2, let $\Psi=\log|t|^2$ and $c_A\equiv 1$, where $A=2\log r$. We obtain a holomorphic section $\tilde u$ on $p^{-1}(p(\Delta_r))$ such that

$$\int_{M_{t_0}}\{u_{t_0},u_{t_0}\}_h^2\geq\frac{1}{\pi r^2}\int_{\Delta_r}\int_{M_t}\left\{\frac{\tilde u}{dt}|_{M_t},\frac{\tilde u}{dt}|_{M_t}\right\}_h d\lambda_{\Delta_r}(t),\tag{3.1}$$

where $\tilde u=\tilde g(z,t)dz\wedge dt\otimes e$ on (z,t) and $\tilde g(z,t_0)=g(z)$.

Using the extremal property of the Bergman kernel, we have

$$B_t(z)\geq\frac{|\tilde g(z,t)|^2}{\int_{M_t}\frac{1}{2^n}\{\frac{\tilde u}{dt}|_{M_t},\frac{\tilde u}{dt}|_{M_t}\}_h},$$

for any $(z,t)\in\Delta_r$, if $\int_{M_t}\{\frac{\tilde u}{dt}|_{M_t},\frac{\tilde u}{dt}|_{M_t}\}_h\neq 0$.

Note that the Lebesgue measure of $\{t|\int_{M_t}\{\frac{\tilde u}{dt}|_{M_t},\frac{\tilde u}{dt}|_{M_t}\}_h=0\}$ is zero. Using convexity of function $y=e^x$ and inequality (3.1), we have

$$e^{2\log|g(z)|-\log B_{t_0}(z)}=\frac{|g(z)|^2}{B_{t_0}(z)}\geq e^{\frac{1}{\pi r^2}\int_{\Delta_r}(2\log|\tilde g(z,t)|-\log B_t(z))d\lambda_{\Delta_1}(t)}.\tag{3.2}$$

Since $\log|\tilde{g}|$ is a plurisubharmonic function, then we obtain the relation to log-plurisubharmonicity of the Bergman kernel:

COROLLARY 3.7. *$\log B_t(z)$ is a plurisubharmonic function with respect to (z,t).*

The above result is due to [5] and [3] in the case of example (1) and due to [7] in the case of example (2).

3.6. *L^p extension theorems with optimal estimates and Ohsawa's question.* Denote the smooth form $dV_M = e^{-\varphi}c_n dz\wedge d\bar{z}$ on the local coordinate $z = (z_1,\dots,z_n)$.

Using Theorem 2.1 (resp. Theorem 5.2) and a similar method as in the proof of Proposition 0.2 in [9] (see also [8]), we obtain an L^p $(0<p<2)$ extension theorem with an optimal estimate:

THEOREM 3.8. *Let M be a Stein manifold, and let S be a closed complex submanifold of M. Let h be a smooth metric on a holomorphic line bundle L on M (resp. holomorphic line bundle L with locally integrable singular metric h), which satisfies*

(1) $\sqrt{-1}\frac{p}{2}\Theta_h + \frac{2-p}{2}\sqrt{-1}\partial\bar\partial\varphi + \sqrt{-1}\partial\bar\partial\Psi \geq 0$ *on* $M\setminus S$;

(2) $a(-\Psi)(\frac{p}{2}\sqrt{-1}\Theta_h + \frac{2-p}{2}\sqrt{-1}\partial\bar\partial\varphi + \sqrt{-1}\partial\bar\partial\Psi) + \sqrt{-1}\partial\bar\partial\Psi \geq 0$ *on* $M\setminus S$, *where a and Ψ are as in Theorem* 2.1,

respectively,

(1) $\frac{p}{2}\sqrt{-1}\Theta_h + \frac{2-p}{2}\sqrt{-1}\partial\bar\partial\varphi + \sqrt{-1}\partial\bar\partial\Psi \geq 0$ *in the sense of currents on* $M\setminus S$;

(2) $\frac{p}{2}\sqrt{-1}\Theta_h + \frac{2-p}{2}\sqrt{-1}\partial\bar\partial\varphi + (1+\delta)\sqrt{-1}\partial\bar\partial\Psi \geq 0$ *in the sense of currents on* $M\setminus S$, *where Ψ is as in Theorem* 5.2.

Then for any holomorphic section f of $K_M\otimes L|_S$ on S satisfying

$$\sum_{k=1}^{n}\frac{\pi^k}{k!}\int_{S_{n-k}}|f|_h^p dV_M[\Psi] = 1, \tag{3.3}$$

there exists a holomorphic section F of $K_M\otimes L$ on M satisfying $F=f$ on S and

$$\int_M c_A(-\Psi)|F|_h^p dV_M \leq \frac{1}{\delta}c_A(-A)e^A + \int_{-A}^{\infty} c_A(t)e^{-t}dt, \tag{3.4}$$

where $c_A(t)$ is as in Theorem 2.1 (*resp. Theorem* 5.2).

By a similar method as in the proof of Theorem 3.6, assume

$$h = e^{-\frac{2}{p}(\varphi-\alpha\log(-r+\varepsilon_0|s|^2))}$$

and

$$dV_M = c_n e^{-\varphi_\Omega}dz\wedge d\bar{z},$$

where φ_Ω is a smooth plurisubharmonic function on Ω, we answer the above mentioned Ohsawa's question for any p $(0<p<2)$ as follows:

THEOREM 3.9. *Without assuming that ∂D intersects with H transversally, the extension operator from $A^p_{\alpha+1,\varphi}(D\cap H)$ to $A^p_{\alpha,\varphi}(D)$ for each $\alpha>-1$ has a bound $C_0\max\{C_1^\alpha, C_2^\alpha\}$, where C_0, C_1 and C_2 are positive constants, which are independent of α, $\alpha>-1$. Consequently, the extension operator from $A^p_{0,\varphi}(D\cap H)$ to $A^p_{-1,\varphi}(D)$ is bounded.*

3.7. *$L^{\frac{2}{m}}$ extension theorems with optimal estimates on Stein manifolds.* Replace L by $(m-1)K_M+L$. Take e^φ as the Hermitian metric on K_M. Let $p=\frac{2}{m}$.

Using Theorem 3.8, we give an optimal estimate of the $L^{\frac{2}{m}}$ extension theorem:

THEOREM 3.10. *Let M be a Stein manifold and S be a closed complex submanifold on M. Let h be a smooth metric on a holomorphic line bundle L on M (resp. holomorphic line bundle L with locally integrable singular metric h), which satisfies*

(1) $\frac{1}{m}\sqrt{-1}\Theta_h+\sqrt{-1}\partial\bar\partial\Psi\geq 0$ *on* $M\setminus S$;
(2) $a(-\Psi)(\frac{1}{m}\sqrt{-1}\Theta_h+\sqrt{-1}\partial\bar\partial\Psi)+\sqrt{-1}\partial\bar\partial\Psi\geq 0$ *on $M\setminus S$, where a is as in Theorem* 2.1,

respectively,

(1) $\frac{1}{m}\sqrt{-1}\Theta_h+\sqrt{-1}\partial\bar\partial\Psi\geq 0$ *in the sense of currents on* $M\setminus S$;
(2) $\frac{1}{m}\sqrt{-1}\Theta_h+(1+\delta)\sqrt{-1}\partial\bar\partial\Psi\geq 0$ *in the sense of currents on* $M\setminus S$.

Then for any holomorphic section f of $K_M^m\otimes L|_S$ on S satisfying

$$\sum_{k=1}^{n}\frac{\pi^k}{k!}\int_{S_{n-k}}|f|_h^{\frac{2}{m}}dV_M[\Psi]=1, \tag{3.5}$$

there exists a holomorphic section F of $mK_M\otimes L$ on M satisfying $F=f$ on S and

$$\int_M c_A(-\Psi)|F|_h^{\frac{2}{m}}dV_M\leq\frac{1}{\delta}c_A(-A)e^A+\int_{-A}^{\infty}c_A(t)e^{-t}dt, \tag{3.6}$$

where $c_A(t)$ is as in Theorem 2.1 (*resp. Theorem* 5.2).

Using the arguments in Section 3.5, we obtain the relation to log-plurisubharmonicity of the fiberwise m-Bergman kernels on Stein manifolds (see [9] or [8]).

3.8. *Interpolation hypersurfaces in Bargmann-Fock space.* In this subsection, we give an application of Theorem 3.8 to a generalization of interpolation hypersurfaces in Bargmann-Fock space (see [45]).

We say that W is a uniformly flat submanifold in $\mathbb{C}^n$ (the case of hypersurface is referred to [45]) if there exists T, which is a plurisubharmonic polar function of W on $\mathbb{C}^n$, such that $(\partial\bar{\partial}T * \frac{\mathbf{1}_{B(0,r)}}{\mathrm{Vol}(B(0,r))})(z)$ has a uniform upper bound on $\mathbb{C}^n$ which is independent of $z \in \mathbb{C}^n$ and r.

We say that W is an interpolation submanifold if for each $f \in \mathfrak{bf}^p_\varphi$ there exists $F \in \mathfrak{BF}^p_\varphi$ such that $F|_W = f$, where the plurisubharmonic function φ satisfies

$$\sqrt{-1}\partial\bar{\partial}\varphi \simeq \omega = \sqrt{-1}\partial\bar{\partial}|z|^2,$$

where

$$\mathfrak{bf}^p_\varphi := \left\{ f \in \mathcal{O}(W) : \int_W |f|^p e^{-p\varphi}\omega^{n-1} < +\infty \right\}$$

and

$$\mathfrak{BF}^p_\varphi := \left\{ F \in \mathcal{O}(\mathbb{C}^n) : \int_{\mathbb{C}^n} |F|^p e^{-p\varphi}\omega^{n} < +\infty \right\}.$$

Let T be a plurisubharmonic function in $\#(W) \cap C^\infty(\mathbb{C}^n \setminus W)$. For any $z \in \mathbb{C}^n$ and $r > 0$, consider the $(1,1)$-form

$$\Upsilon_{W,T}(z,r) := \sum_{i,j=1}^n \left(\frac{1}{\mathrm{Vol}(B(z,r))} \int_{B(z,r)} \frac{\partial^2 \log|T|}{\partial\xi^i \partial\bar{\xi}^j}\omega^n(\xi) \right) \sqrt{-1}dz^i \wedge d\bar{z}^j.$$

The density of W in the ball of radius r and center z is

$$D(W,T,z,r) := \sup \left\{ \frac{\Upsilon_{W,T}(z,r)(v,v)}{\sqrt{-1}\partial\bar{\partial}\varphi_r(v,v)} := v \in T_{\mathbb{C}^n,z} - \{0\} \right\},$$

where $\varphi_r := \varphi * \frac{\mathbf{1}_{B(0,r)}}{\mathrm{Vol}(B(0,r))}$. The upper density of W is

$$D^+(W) := \sup_T \limsup_{r\to\infty} \sup_{z\in\mathbb{C}^n} D(W,T,z,r).$$

In [45], one of the main results is

THEOREM 3.11 ([45]). *Let W be a uniformly flat hypersurface. Let $p \geq 2$. If $D^+ < 1$, then W is an interpolation hypersurface.*

Using Theorem 3.8, we obtain a sufficient condition for interpolation submanifold in Bargmann-Fock space for $p \leq 2$:

THEOREM 3.12. *Let W be a uniformly flat submanifold. Let $0 < p \leq 2$. If $D^+ < \frac{p}{2}$, then W is an interpolation submanifold.*

3.9. *Optimal estimate of the L^2 extension theorem of Ohsawa.* In this subsection, we give some applications of Theorem 2.2 by giving an optimal estimate of the L^2 extension theorem of Ohsawa in [36]. Assume that (M,S) satisfies condition (ab).

Let $c_\infty(t) := (1+e^{-\frac{t}{m}})^{-m-\varepsilon}$, where ε is a positive constant. It is clear that $\int_{-\infty}^{\infty} c_\infty(t)e^{-t}dt = m\sum_{j=0}^{m-1} C^j_{m-1}(-1)^{m-1-j}\frac{1}{m-1-j+\varepsilon} < \infty$.

Using Remark 4.12, we obtain that inequality (2.4) holds for any $t \in (-\infty, +\infty)$. Let $\Psi = m\log(|g_1|^2 + \cdots + |g_m|^2)$, where $S = \{g_1 = \cdots = g_m = 0\}$ and g_i are holomorphic functions on M, which satisfy $\wedge_{j=1}^m dg_j|_{S_{\mathrm{reg}}} \neq 0$.

Using Theorem 2.2 and Lemma 4.14, we obtain an optimal estimate version of the main result in [36] as follows:

COROLLARY 3.13. *For any holomorphic section f of $K_{S_{\mathrm{reg}}} \otimes E|_{S_{\mathrm{reg}}}$ on S_{reg} satisfying*

$$\frac{\pi^m}{m!}\int_{S_{\mathrm{reg}}} \{f, f\}_h < \infty,$$

there exists a holomorphic section F of $K_M \otimes E$ on M satisfying $F = f \wedge \bigwedge_{k=1}^m dg_k$ on S_{reg} and

$$\begin{aligned}&\int_M (1 + |g_1|^2 + \cdots + |g_m|^2)^{-m-\varepsilon}\{F, F\}_h \\ &\leq \mathbf{C}\left(m\sum_{j=0}^{m-1} C_{m-1}^j (-1)^{m-1-j}\frac{1}{m-1-j+\varepsilon}\right)\frac{(2\pi)^m}{m!}\int_{S_{\mathrm{reg}}}\{f, f\}_h,\end{aligned}$$

where the uniform constant $\mathbf{C} = 1$, which is optimal for any m.

When M is Stein, for any plurisubharmonic function φ on M, we can choose a sequence of smooth plurisubharmonic functions $\{\varphi_k\}_{k=1,2,\ldots}$, which is decreasingly convergent to φ. Then the above corollary gives an optimal estimate version of the main theorem in [36].

Let $c_\infty(t) := (1 + e^{-t})^{-1-\varepsilon}$, where ε is a positive constant. Using Remark 4.12, we obtain that inequality (2.4) holds for any $t \in (-\infty, +\infty)$. Let $\Psi = m\log(|g_1|^2 + \cdots + |g_m|^2)$, where $S = \{g_1 = \cdots = g_m = 0\}$ and g_i are the same as in Corollary 3.13.

Using Theorem 2.2 and Lemma 4.14, we can formulate a similar version to the above corollary with more concise estimate:

COROLLARY 3.14. *For any holomorphic section f of $K_{S_{\mathrm{reg}}} \otimes E|_{S_{\mathrm{reg}}}$ on S_{reg} satisfying*

$$\frac{\pi^m}{m!}\int_{S_{\mathrm{reg}}} \{f, f\}_h < \infty,$$

there exists a holomorphic section F of $K_M \otimes E$ on M satisfying $F = f \wedge \bigwedge_{k=1}^m dg_k$ on S and

$$\int_M (1 + (|g_1|^2 + \cdots + |g_m|^2)^m)^{-1-\varepsilon}\{F, F\}_h \leq \mathbf{C}\frac{1}{\varepsilon}\frac{(2\pi)^m}{m!}\int_{S_{\mathrm{reg}}}\{f, f\}_h,$$

where the uniform constant $\mathbf{C} = 1$, which is optimal.

Let M be a Stein manifold, and let S be an analytic hypersurface on M, which is locally defined by $\{w_j = 0\}$ on $U_j \subset M$, where $\{U_j\}_{j=1,2,\ldots}$ is an open covering of M, and functions $\{w_j\}_{j=1,2,\dots}$ together give a nonzero holomorphic section w of the holomorphic line bundle $[S]$ associated to S (see [25]).

Let $|\cdot|$ be a Hermitian metric on $[S]$ satisfying that $|\cdot|_{e^{-\psi}}$ is seminegative, where ψ is an upper-semicontinuous function on M. Assume that $\log(|w|^2) + \psi < 0$ on M. Let φ be a plurisubharmonic function.

Let $\Psi := \log(|w|^2)+\psi$. Note that Ψ is plurisubharmonic. By Lemma 4.14, we have

$$|F|^2 dV_M[\Psi] = 2\frac{c_{n-1}\frac{F}{dw} \wedge \frac{\bar{F}}{dw}}{|dw|^2} e^{-\psi}$$

for any continuous $(n,0)$-form F on M, where $|dw|$ is the Hermitian metric on $[-S]|_{S_{\mathrm{reg}}}$ induced by the Hermitian metric $|\cdot|$ on $[S]|_{S_{\mathrm{reg}}}$.

Let $c_0(t) := 1$. It is easy to see that $\int_0^\infty c_0(t)e^{-t}dt = 1 < \infty$ and $c_0(t)e^{-t}$ is decreasing with respect to t, and inequality (2.4) holds for any $t \in (-\infty, +\infty)$.

Using Theorem 2.2, we obtain another proof of the following result in [26], which is an optimal estimate version of main results in [40], [30] and [61], etc.

COROLLARY 3.15. [26] *For any holomorphic section f of $K_{S_{\mathrm{reg}}}$ on S_{reg} satisfying*

$$c_{n-1}\int_{S_{\mathrm{reg}}} \frac{f \wedge \bar{f}}{|dw|^2} e^{-\varphi-\psi} < \infty,$$

there exists a holomorphic section F of K_M on M satisfying $F = f \wedge dw$ on S_{reg} and

$$c_n \int_M F \wedge \bar{F} e^{-\varphi} \leq 2\pi \mathbf{C} c_{n-1} \int_{S_{\mathrm{reg}}} \frac{f \wedge \bar{f}}{|dw|^2} e^{-\varphi-\psi},$$

where the uniform constant $\mathbf{C} = 1$, which is optimal.

When w is a holomorphic function on M, the above corollary is an optimal estimate version of the L^2 extension theorems in [34], [40], [49], [50], [2], [15], [4], [16], [30], [61], [10], etc.

3.10. *The optimal constant version of the L^2 extension theorems of Manivel and Demailly.*

THEOREM 3.16 ([34] and [15]). *Let (X, g) be a Stein n-dimensional manifold possessing a Kähler metric g, and let L (resp. E) be a Hermitian holomorphic line bundle (resp. a Hermitian holomorphic vector bundle of rank r over X). Let w be a global holomorphic section of E. Assume that w is generically transverse to the zero section, and let*

$$H = \{x \in X : w(x) = 0, \wedge^r dw(x) \neq 0\}.$$

Moreover, assume that the $(1,1)$-form $\sqrt{-1}\Theta(L) + r\sqrt{-1}\partial\bar{\partial}\log|w|^2$ is semipositive and that there is a continuous function $\alpha \geq 1$ such that the following *two inequalities hold everywhere on* X:

(a) $\sqrt{-1}\Theta(L)+r\sqrt{-1}\partial\bar{\partial}\log|w|^2 \geq \frac{\{\sqrt{-1}\Theta(E)w,w\}}{\alpha|w|^2}$;

(b) $|w| \leq e^{-\alpha}$.

Then for every holomorphic section f *of* $\wedge^n T_X^* \otimes L$ *over* H, *such that*

$$\int_H |f|^2 | \wedge^r (dw)|^{-2} dV_H < +\infty,$$

there exists a holomorphic extension F *to* X *such that* $F\Big|_H = f$ *and*

$$\int_X \frac{|F|^2}{|w|^{2r}(-\log|w|)^2} dV_X \leq \mathbf{C}\frac{3r}{4}\frac{(2\pi)^r}{r!}\int_H \frac{|f|^2}{|\wedge^r (dw)|^2} dV_H, \tag{3.7}$$

where $\mathbf{C}$ *is a uniform constant depending only on* r.

Using Theorem 2.1, we obtain the following:

Corollary 3.17. *Theorem* 3.16 *holds with the optimal constant* $\mathbf{C}=1$ *in the estimate* (3.7).

3.11. *The optimal estimate for the* L^2 *extension theorems of McNeal and Varolin.* In [35], McNeal and Varolin defined a function class $\mathfrak{D}$.

Definition 3.18. The class $\mathfrak{D}$ consists of nonnegative functions with the following three properties:

(I) Each $g \in \mathfrak{D}$ is continuous and increasing.

(II) For each $g \in \mathfrak{D}$, the improper integral

$$C(g) = \int_1^\infty \frac{dt}{g(t)} < +\infty.$$

For $\delta > 0$, set

$$H_\delta(y) = \frac{1}{1+\delta}\left(1+\frac{\delta}{C(g)}\int_1^y \frac{dt}{g(t)}\right),$$

and note that this function takes values in $(0,1]$. Let

$$g_\delta(x) = \int_1^x \frac{1-H_\delta(y)}{H_\delta(y)} dy.$$

(III) For each $g \in \mathfrak{D}$, there exists a constant $\delta > 0$ such that

$$K_\delta(g) = \sup_{x\geq 1}\frac{x+g_\delta(x)}{g(x)} < +\infty.$$

The extension theorem proved by McNeal and Varolin is stated as below.

THEOREM 3.19. *Let X be a Kähler manifold of complex dimension n. Assume there exists a holomorphic function w on X, such that $\sup_X |w| = 1$ and dw is never zero on the set $H = \{w = 0\}$. Assume there exists an analytic subvariety $V \subset X$ such that H is not contained in V and $X \backslash V$ is Stein. Let L be a holomorphic line bundle over X together with a singular Hermitian metric. Let $\psi : X \longrightarrow [-\infty, +\infty]$ be a locally integrable function such that for any local representative $e^{-\varphi}$ of the metric of L over an open set U, the function $\psi + \varphi$ is not identically $+\infty$ or $-\infty$ on $H \cap U$. Let g be a function in $\mathfrak{D}$. Assume that ψ satisfies that for all $\gamma > 1$ and $\varepsilon > 0$ sufficiently small (depending on $\gamma - 1$),*

$$\begin{aligned}&\sqrt{-1}\partial\bar{\partial}(\varphi + \psi + \log|w|^2) \geq 0,\\ &g^{-1}\Big(e^{-\psi}g(1-\log|w|^2)\Big) \geq 1 \text{ and}\\ &\alpha - g^{-1}\Big(e^{-\psi}g(\alpha)\Big) \text{ is plurisubharmonic,}\end{aligned}$$

where $\alpha = \gamma - \log(|w|^2 + \varepsilon^2)$. Then for any holomorphic $(n-1)$-form $f \in C^\infty(H, \wedge^{n-1}T_H^* \otimes L)$ on H with values in L such that

$$\int_H \{f, f\}_{e^{-\varphi-\psi}}\, dV_H < +\infty,$$

there is a holomorphic n-form $F \in C^\infty(X, \wedge^n T_X^* \otimes L)$ with value in L such that $F\Big|_H = f \wedge dw$ and

$$\int_X \frac{\{F, F\}_{e^{-\varphi}}}{|w|^2 g\Big(\log\frac{e}{|w|^2}\Big)}\, dV_X \leq 2\frac{\pi}{e}\mathbf{C}C(g)\int_H \{f, f\}_{e^{-\varphi-\psi}}\, dV_H, \tag{3.8}$$

where $\mathbf{C} = 4\frac{\left(K_\delta(g) + \frac{1+\delta}{\delta}C(g)\right)}{C(g)}$

By Theorem 2.2, it follows that

COROLLARY 3.20. *Theorem* 3.19 *holds with the optimal constant $\mathbf{C} = 1$ in the estimate* (3.8), *and g only needs to satisfy* (I) *and* (II).

In Section 3 of [35], McNeal and Varolin gave various cases of extension theorems with gains. We give optimal estimates of their extension theorems: Let $g(t) := \frac{1}{c_A(t)e^{-t}}$, where $g : [1, +\infty] \to [0, +\infty]$ is as in [35] and $A = -1$. Let $\Psi = \log|w|^2$.

Using Corollary 3.20, we obtain optimal estimates for all extension theorems in Section 3 of [35].

3.12. *An optimal estimate for an L^2 extension theorem on projective families.* In [51], [46] and [6], one has an L^2 extension theorem on projective families:

THEOREM 3.21. *Let M be a projective family fibred over the unit ball in $\mathbb{C}^m$, with compact fibers M_t. Let (L,h) be a holomorphic line bundle on M with a smooth hermitian metric h of semipositive curvature. Let u be a holomorphic section of $K_{M_0}\otimes L$ over M_0 such that*

$$\int_{M_0}\{u,u\}_h\leq 1.$$

Then there is a holomorphic section $\tilde{u}$ of $K_M\otimes L$ over M such that $\tilde{u}|_{M_0}=u\wedge dt$, and

$$\int_M\{\tilde{u},\tilde{u}\}_h\leq C_b. \tag{3.9}$$

In [46], one can take $C_b<200$. In Theorem 2.2, take $c_A=1$ and let $\Psi:=2m\log|t|$. We obtain an optimal estimate of the above L^2 extension theorem:

COROLLARY 3.22. *Theorem 3.21 holds with $C_b=\frac{2^m\pi^m}{m!}$, which is optimal.*

3.13. *An optimal estimate for an L^2 extension theorem of Demailly, Hacon and Păun.* In [17], Demailly, Hacon and Păun gave an L^2 extension theorem in the following framework:

Let M be a Stein manifold and S be a closed complex submanifold with globally defining function w. Let φ_F, φ_{G_1} and φ_{G_2} be plurisubharmonic functions on M.

Let $\varphi_S:=\varphi_{G_1}-\varphi_{G_2}$. Assume that $\alpha\varphi_F-\varphi_S$ is plurisubharmonic on M, $|w|^2e^{-\varphi_S}\leq e^{-\alpha}$, and $\varphi_F\leq\varepsilon_0\varphi_{G_2}+C$ on M, where $\alpha\geq 1$, $\varepsilon_0>1$ and C are all real numbers. Let $\bar{\varphi}_S$ be a smooth function on M, such that $\max_M|w|^2e^{-\bar{\varphi}_S}<\infty$.

Demailly-Hacon-Păun's L^2 extension theorem is as follows:

THEOREM 3.23 ([17]). *Let u be a section of K_S satisfying*

$$\int_S\{u,u\}e^{-\varphi_F}<\infty.$$

Then there exists a section U of K_M, such that $U|_S=u\wedge dw$ and

$$\int_M\{U,U\}e^{-b\varphi_S-(1-b)\bar{\varphi}_S-\varphi_F}\leq C_b\int_S\{u,u\}e^{-\varphi_F}, \tag{3.10}$$

where $1\geq b>0$ is an arbitrary real number, and the constant

$$C_b=C_0b^{-2}\left(\max_M|w|^2e^{-\bar{\varphi}_S}\right)^{1-b},$$

where C_0 depends only on the dimension.

Using Theorem 2.1, we obtain an optimal estimate of the above extension theorem:

COROLLARY 3.24. *Theorem* 3.23 *holds with*

$$C_b = 2\pi\left(\alpha e^{-b\alpha} + \frac{1}{b}e^{-b\alpha}\right)\left(\max_M |w|^2 e^{-\bar{\varphi}_S}\right)^{1-b},$$

which is optimal, without assuming that $\varphi_F \le \varepsilon_0\varphi_{G_2} + C$ *on* M.

4. **Some results used in the proof of main results and applications**

In this section, we give some lemmas which will be used in the proofs of main theorems and corollaries of the present paper.

4.1. *Some results used in the proofs the main results.* In this subsection, we recall some lemmas on L^2 estimates for some $\bar\partial$ equations (for general cases, see [32], [53], etc.) and give some useful lemmas. Denote by $\bar\partial^*$ or D''^* the Hilbert adjoint operator of $\bar\partial$.

LEMMA 4.1 (see [40] or [44]). *Let* (X,ω) *be a Kähler manifold of dimension* n *with a Kähler metric* ω. *Let* (E,h) *be a hermitian holomorphic vector bundle. Let* $\eta, g > 0$ *be smooth functions on* X. *Then for every form* $\alpha \in \mathcal{D}(X, \Lambda^{n,q}T_X^* \otimes E)$, *which is the space of smooth differential forms with values in* E *with compact support, we have*

$$\begin{aligned}(4.1)\qquad &\|(\eta+g^{-1})^{\frac{1}{2}}D''^*\alpha\|^2 + \|\eta^{\frac{1}{2}}D''\alpha\|^2\\ &\ge \left\langle\left\langle[\eta\sqrt{-1}\Theta_E - \sqrt{-1}\partial\bar\partial\eta - \sqrt{-1}g\partial\eta\wedge\bar\partial\eta, \Lambda_\omega]\alpha, \alpha\right\rangle\right\rangle.\end{aligned}$$

LEMMA 4.2. *Let* X *and* E *be as in the above lemma and* θ *be a continuous* $(1,0)$*-form on* X. *Then we have*

$$[\sqrt{-1}\theta\wedge\bar\theta, \Lambda_\omega]\alpha = \bar\theta\wedge(\alpha\llcorner(\bar\theta)^\sharp)$$

for any $(n,1)$*-form* α *with value in* E. *Moreover, for any positive* $(1,1)$*-form* β, *we have* $[\beta,\Lambda_\omega]$ *is semipositive.*

Proof. For any $x \in X$, we choose a local coordinate $(z_1,\dots,z_n)$ near x, such that

(1) $\theta|_x = adz_1$,
(2) $\omega|_x = \sqrt{-1}dz_1\wedge d\bar z_1 + \cdots + \sqrt{-1}dz_n\wedge d\bar z_n$.

It suffices to prove $[\sqrt{-1}dz_1\wedge d\bar z_1, \Lambda_\omega]\alpha = d\bar z_1\wedge\left(\alpha\llcorner(d\bar z_1)^\sharp\right)$.

At x, we have $\Lambda_\omega\alpha = \left(\alpha\llcorner(\sqrt{-1}dz_1\wedge d\bar z_1 + \cdots + \sqrt{-1}dz_n\wedge d\bar z_n)^\sharp\right)$. It is clear that $\sqrt{-1}dz_1\wedge d\bar z_1\wedge\Lambda_\omega\alpha = \sqrt{-1}dz_1\wedge d\bar z_1\wedge\left(\alpha\llcorner(\sqrt{-1}dz_1\wedge d\bar z_1)^\sharp\right)$.

Let $\alpha|_x = \alpha^j e_j = \sum_{k=1}^n \alpha_k^j \bigwedge_{l=1}^n dz_l\wedge d\bar z_k\otimes e_j$, where $\{e_j\}$ is an orthonormal basis of E_x. Then we have

$$\left(\alpha\llcorner(\sqrt{-1}dz_1\wedge d\bar z_1)^\sharp\right)|_x = -\sqrt{-1}(-1)^{n-1}\alpha_1^j\bigwedge_{l=2}^n dz_l\otimes e_j$$

and

$$\begin{aligned}\left[\sqrt{-1}dz_1 \wedge d\bar{z}_1, \Lambda_\omega\right]\alpha =& \sqrt{-1}dz_1 \wedge d\bar{z}_1 \wedge (\Lambda_\omega \alpha)\\ =& \alpha_1^j \bigwedge_{l=1}^{n} dz_l \wedge d\bar{z}_1 \otimes e_j\\ =& d\bar{z}_1 \wedge \left(\alpha \llcorner (d\bar{z}_1)^\sharp\right).\end{aligned}$$ □

Lemma 4.3 (see [15], [16]). *Let X be a complete Kähler manifold equipped with a (not necessarily complete) Kähler metric ω, and let E be a Hermitian vector bundle over X. Assume that there are smooth and bounded functions η, $g > 0$ on X such that the (Hermitian) curvature operator*

$$B := \left[\eta\sqrt{-1}\Theta_E - \sqrt{-1}\partial\bar{\partial}\eta - \sqrt{-1}g\partial\eta \wedge \bar{\partial}\eta, \Lambda_\omega\right]$$

is positive definite everywhere on $\Lambda^{n,q}T_X^* \otimes E$ for some $q \geq 1$. Then for every form $\lambda \in L^2(X, \Lambda^{n,q}T_X^* \otimes E)$ such that $D''\lambda = 0$ and $\int_X \langle B^{-1}\lambda, \lambda\rangle dV_\omega < \infty$, there exists $u \in L^2(X, \Lambda^{n,q-1}T_X^* \otimes E)$ such that $D''u = \lambda$ and

$$\int_X (\eta + g^{-1})^{-1}|u|^2 dV_\omega \leq \int_X \langle B^{-1}\lambda, \lambda\rangle dV_\omega.$$

For any point $x \in S$, we have a neighborhood $U_x \subset M$ of x and a biholomorphic map p from U_x to Δ^n, such that $p(U_x \cap S) = \Delta^{\dim S_x}$, and $p(U_x \setminus S) = \Delta^{\dim S_x} \times (\Delta^{\operatorname{codim} S_x})^*$. Then we can use the following lemma to study high dimension cases:

Lemma 4.4. *Let Δ be the unit disc and Δ_r be the disc with radius r. Then for any holomorphic function f on Δ, which satisfies*

$$\int_\Delta |f|^2 d\lambda < \infty,$$

we have a uniformly constant $C_r = \frac{1}{1-r^2}$, which is only dependent on r, such that

$$\int_\Delta |f|^2 d\lambda \leq C_r \int_{\Delta\setminus\Delta_r} |f|^2 d\lambda,$$

where λ is the Lebesgue measure on $\mathbb{C}$.

Proof. By Taylor expansion at $o \in \mathbb{C}$, it suffices to check the lemma for $f = z^j$. By some simple calculations, the lemma follows. □

Let $L_h^2(M) := \{F | F \in \Lambda^{n,0}T^*M \otimes E, \int_M \{F, F\}_h < \infty\}$. We now discuss the convergence of holomorphic $(n,0)$-forms with values in E as follows:

Lemma 4.5. *Let M be a complex manifold with dimension n and a continuous volume form dV_M. Let E be a holomorphic vector bundle with rank r*

and h be a Hermitian metric on E. Let $\{F_j\}_{j=1,2,\dots}$ be a sequence of holomorphic $(n,0)$-forms with values in E. Assume that for any compact subset K of M, there exists a constant $C_K>0$, such that

$$\int_K |F_j|_h^2 dV_M \le C_K \tag{4.2}$$

holds for any $j=1,2,\dots$. Then we have a subsequence of $\{F_j\}_{j=1,2,\dots}$, which is uniformly convergent to a holomorphic section of $K_M\otimes E$ on any compact subset of M.

Proof. We can choose a covering $\{U_i\}_{i=1,2,\dots}$ of M, which satisfies

(1) $U_i \subset\subset M$, and $\exists K_i \subset\subset U_i$, such that $\cup_{i=1}^{\infty} K_i = M$;
(2) $E|_{U_i}$ is trivial with holomorphic basis $e_1^i,\dots,e_r^i$;
(3) $K_M|_{U_i}$ is trivial with holomorphic basis v^i.

Then we may write $F_j|_{U_i} = f_{j,i}^k e_k^i \otimes v^i$, where $f_{j,i}^k$ are holomorphic functions on U_i. As h is a Hermitian metric and $U_i\subset\subset M$, there exists a constant $B_K>0$, such that

$$\sum_{1\le k,l\le r} h(e_k^i,e_l^i) f_{j,i}^k \bar{f}_{j,i}^l \ge B_K \sum_{k=1}^r |f_{j,i}^k|^2.$$

By inequality (4.2), it follows that

$$\int_{U_j} \sum_{k=1}^r |f_{j,i}^k|^2 c_n v^i\wedge \bar{v}^i \le \frac{C_K}{B_K} \tag{4.3}$$

for any $j=1,2,\dots$.

We can obtain a subsequence of $\{F_j\}_{j=1,2,\dots}$ which is uniformly convergent on any compact subset of M by the following steps:

(1) On U_1, by inequality (4.3), we can obtain subsequence $\{F'_{1_j}\}_{j=1,2,\dots}$ of $\{F_j\}_{j=1,2,\dots}$ which is uniformly convergent on K_1;
(2) On U_2, by inequality (4.3), we can obtain subsequence $\{F'_{2_j}\}_{j=1,2,\dots}$ of $\{F'_{1,j}\}_{j=1,2,\dots}$ which is uniformly convergent on K_2;
(3) On U_3, by inequality (4.3), we can obtain subsequence $\{F'_{3_j}\}_{j=1,2,\dots}$ of $\{F'_{2,j}\}_{j=1,2,\dots}$ which is uniformly convergent on K_3 ... .

As the transition matrix of E is invertible, we see that $\{F'_{j_j}\}_{j=1,2,\dots}$ is uniformly convergent on any compact subset of M. Thus we have proved the lemma. □

LEMMA 4.6. *Let M be a complex manifold. Let S be a closed complex submanifold of M. Let $\{U_j\}_{j=1,2,\dots}$ be a sequence of open subsets on M, which satisfies*

$$U_1 \subset U_2 \subset \cdots \subset U_j \subset U_{j+1} \subset \cdots,$$

and $\bigcup_{j=1}^{\infty} U_j = M\setminus S$. Let $\{V_j\}_{j=1,2,\dots}$ be a sequence of open subsets on M, which satisfies

$$V_1 \subset V_2 \subset \cdots \subset V_j \subset V_{j+1} \subset \cdots,$$

$V_j \supset U_j$, and $\bigcup_{j=1}^{\infty} V_j = M$.

Let $\{g_j\}_{j=1,2,\ldots}$ be a sequence of positive Lebesgue measurable functions on U_k, which satisfies that g_j are almost everywhere convergent to g on any compact subset of U_k $(j \geq k)$, and g_j have uniformly positive lower and upper bounds on any compact subset of U_k $(j \geq k)$, where g is a positive Lebesgue measurable function on $M \setminus S$.

Let E be a holomorphic vector bundle on M, with Hermitian metric h. Let $\{F_j\}_{j=1,2,\ldots}$ be a sequence of holomorphic $(n,0)$-form on V_j with values in E. Assume that $\lim_{j\to\infty}\int_{U_j}\{F_j,F_j\}_h g_j = C$, where C is a positive constant.

Then there exists a subsequence $\{F_{j_l}\}_{l=1,2,\ldots}$ of $\{F_j\}_{j=1,2,\ldots}$, which satisfies that $\{F_{j_l}\}$ is uniformly convergent to an $(n,0)$-form F on M with value in E on any compact subset of M when $l \to +\infty$, such that

$$\int_M \{F,F\}_h g \leq C.$$

Proof. As $\liminf_{j\to\infty}\int_{U_j}\{F_j,F_j\}_h g_j = C < \infty$, it follows that there exists a subsequence of $\{F_j\}$, denoted still by F_j without ambiguity, such that $\lim_{j\to\infty}\int_{U_j}\{F_j,F_j\}_h g_j = C$.

By Lemma 4.4, for any compact set $K_k \subset\subset M$, it follows that there exists $\tilde{K}_{j_k} \subset\subset M \setminus S$, which satisfies $\tilde{K}_{j_k} \subset U_{j_k}$, and

$$\int_{K_k}\{F_j,F_j\}_h \leq C_k \int_{\tilde{K}_{j_k}}\{F_j,F_j\}_h g_j,$$

for any $j \geq j_k$, where C_k is a constant which is only dependent on k.

Using Lemma 4.5, we have a subsequence of F_j, which is uniformly convergent on K_k°, denoted still by F_j without ambiguity. Assume $\bigcup_{k=1}^\infty K_k^\circ = M$ and $K_k \subset\subset K_{k+1}$.

Using the diagonal method for k, we obtain a subsequence of F_j, denoted by F_j without ambiguity, which is uniformly convergent to a holomorphic $(n,0)$-form F with value in E on any compact subset of M.

Given $\tilde{K} \subset\subset M \setminus S$, as $\{F_j\}$ (resp. g_j) is uniformly convergent to F (resp. g) for $j \geq k_{\tilde{K}}$, we have $\int_K\{F,F\}_h g \leq \lim_{j\to\infty}\int_{U_j}\{F_j,F_j\}_h g_j$, where $k_{\tilde{K}}$ satisfies $U_{k_{\tilde{K}}} \supset \tilde{K}$. It is clear that $\int_M\{F,F\}_h g \leq \lim_{j\to\infty}\int_{U_j}\{F_j,F_j\}_h g_j$. □

We now give a remark to illustrate the extension properties of holomorphic sections of holomorphic vector bundles from $M \setminus X$ to M.

Remark 4.7. Let (M,S) satisfy condition (ab), and let h be a singular metric on a holomorphic line bundle L on M (resp. continuous metric on holomorphic vector bundle E on M with rank r) such that h has locally a positive lower bound. Let F be a holomorphic section of $K_{M\setminus X} \otimes E|_{M\setminus X}$, which satisfies $\int_{M\setminus X}|F|_h^2 < \infty$. As h has locally a positive lower bound and

M satisfies (a) of condition (ab), there is a holomorphic section $\tilde{F}$ of $K_M \otimes L$ on M (resp. $K_M \otimes E$), such that $\tilde{F}|_{M\setminus X} = F$.

We now give an approximation property of the function $c_A(t)$ $(A < +\infty)$ as follows:

Lemma 4.8. *Let $c_A(t)$ be a positive function in $C^\infty((-A,+\infty))$, which satisfies $\int_{-A}^{\infty} c_A(t)e^{-t}dt < \infty$ and inequality (2.4), for any $t \in (-A,+\infty)$. Then there exists a sequence of positive C^∞ smooth functions $\{c_{A,m}(t)\}_{m=1,2,\dots}$ on $(-A,+\infty)$, which satisfies*

(1) *$c_{A,m}(t)$ are continuous near $+\infty$ and $\lim_{t\to+\infty} c_{A,m}(t) > 0$;*
(2) *$c_{A,m}(t)$ are uniformly convergent to $c_A(t)$ on any compact subset of $(-A,+\infty)$, when m goes to ∞;*
(3) *$\int_{-A}^{\infty} c_{A,m}(t)e^{-t}dt$ is convergent to $\int_{-A}^{\infty} c_A(t)e^{-t}dt$ when m approaches to ∞;*
(4) *for any $t \in (-A,+\infty)$,*

$$\left(\int_{-A}^{t} c_{A,m}(t_1)e^{-t_1}dt_1\right)^2 > c_{A,m}(t)e^{-t}\int_{-A}^{t}\int_{-A}^{t_2} c_{A,m}(t_1)e^{-t_1}dt_1dt_2$$

holds.

Proof. We give a construction of $c_{A,m}$. First, we consider the case that $A < +\infty$. Let $g_B(t) := c_A(t)$ when $t \in (-A,-A+B]$. We can choose $g_B(t)$, which is a positive continuous decreasing function on $t \in [-A+B,\infty)$, and smooth on $(-A+B,\infty)$, which satisfies $\lim_{t\to+\infty} g_B(t) > 0$, such that

$$\int_{-A+B}^{\infty} g_B(t)e^{-t}dt < B^{-1}, \tag{4.4}$$

where $B > 0$.

As $g_B(t) = c_A(t)$ when $t \in (-A,-A+B)$, we have

$$\left(\int_{-A}^{t} g_B(t_1)e^{-t_1}dt_1\right)^2 > g_B(t)e^{-t}\int_{-A}^{t}\int_{-A}^{t_2} g_B(t_1)e^{-t_1}dt_1dt_2 \tag{4.5}$$

holds for any $t \in (-A,-A+B)$. As $g_B(t)$ is decreasing on $[-A+B,+\infty)$, it is clear that inequality (4.5) holds for any $t \in (-A,+\infty)$, and

$$\lim_{B\to+\infty}\int_{-A}^{\infty} g_B(t)e^{-t}dt = \int_{-A}^{\infty} c_A(t)dt$$

by inequality (4.4).

Given ε_B small enough, such that

$$[-A+B-\varepsilon_B, -A+B+\varepsilon_B] \subset\subset (-A,+\infty),$$

one can find a sequence of functions $\{g_{B,j}(t)\}_{j=1,2,\dots}$ in $C^\infty(-A,+\infty)$, satisfying $g_{B,j}(t) = g_B(t)$ when $t \notin [-A+B-\varepsilon_B, -A+B+\varepsilon_B]$, which is uniformly convergent to G_B. Then it is clear that for j big enough,

$$\left(\int_{-A}^{t} g_{B,j}(t_1)e^{-t_1}dt_1\right)^2 > g_{B,j}(t)e^{-t}\int_{-A}^{t}\int_{-A}^{t_2} g_{B,j}(t_1)e^{-t_1}dt_1dt_2$$

holds for any $t \in (-A, +\infty)$.

For any given B, we can choose j_B large enough such that

$$\begin{aligned}
&(1) \quad \left|\int_{-A}^{\infty} g_{B,j_B}(t)e^{-t}dt - \int_{-A}^{\infty} g_B(t)dt\right| < B^{-1};\\
&(2) \quad \max_{t\in(-A,+\infty)} |g_{B,j_B}(t) - g_B(t)| < B^{-1};\\
&(3) \quad \left(\int_{-A}^{t} g_{B,j_B}(t_1)e^{-t_1}dt_1\right)^2\\
&\qquad > g_{B,j_B}(t)e^{-t}\int_{-A}^{t}\int_{-A}^{t_2} g_{B,j_B}(t_1)e^{-t_1}dt_1dt_2 \ \ \forall t \in (-A, +\infty).
\end{aligned} \tag{4.6}$$

Let $c_{A,m} := g_{m,j_m}$; thus we have proved the case that $A < +\infty$. Secondly, we consider the case that $A = +\infty$. Let $g_B(t) := c_\infty(t)$ when $t \in (-\infty, B)$, $g_B(t) := c_\infty(B)$ when $t \in [B, \infty)$, where $B > 0$. Using the same construction as the case $A < +\infty$, we obtain the case that $A = +\infty$. □

Remark 4.9. Let $c_A(t)$ be the positive function in Theorems 2.1 and 5.2. By the construction in the proof of the above lemma, one can choose a sequence of positive smooth functions $\{c_{A,m}(t)\}_{m=1,2,\ldots}$ on $(-A, +\infty)$, which are continuous on $[-A, +\infty]$ and uniformly convergent to $c_A(t)$ on any compact subset of $(-A, +\infty)$, and satisfying the same conditions as $c_A(t)$ in Theorems 2.1 and 5.2, such that $\int_{-A}^{\infty} c_{A,m}(t)e^{-t}dt + \frac{1}{\delta}c_{A,m}(-A)e^{A}$ are convergent to $\int_{-A}^{\infty} c_A(t)e^{-t}dt + \frac{1}{\delta}c_A(-A)e^{A}$ when m goes to ∞.

In fact, we may replace smoothness of $c_A(t)$ by continuity:

Remark 4.10. Using partition of unity $\{\rho_j\}_j$ on $(-A, +\infty)$ and smoothing for $\rho_j c_A$, we can replace smoothness of $c_A(t)$ by continuity in Lemma 4.8.

Now we introduce a relationship between inequalities (2.4) and (2.1).

LEMMA 4.11. *Let $c_A(t)$ satisfy $\int_{-A}^{+\infty} c_A(t)e^{-t}dt < \infty$ and inequality* (2.4) $(A \in (-\infty, +\infty])$. *For each $A' < A$, there exist A'' and $\delta'' > 0$, such that $A > A'' > A'$ and there exists $c_{A''}(t) \in C^0([-A'', +\infty))$ satisfying*

(1) $c_{A''}(t) = c_A(t)|_{[-A',+\infty)}$;

(2) $\int_{-A''}^{+\infty} c_{A''}(t)e^{-t}dt + \frac{1}{\delta''}c_{A''}(-A'')e^{A''} = \int_{-A}^{+\infty} c_A(t)e^{-t}dt$;

(3) $\int_{-A''}^{t}(\frac{1}{\delta''}c_{A''}(-A'')e^{A''} + \int_{-A''}^{t_2} c_{A''}(t_1)e^{-t_1}dt_1)dt_2 + \frac{1}{\delta''^2}c_{A''}(-A'')e^{A''} <$ $\int_{-A}^{t}(\int_{-A}^{t_2} c_A(t_1)e^{-t_1}dt_1)dt_2$.

Proof. Given $A' < A$, let $g(t)|_{[-A',+\infty)} := c_A(t)|_{[-A',+\infty)}$. As $c_A(t)$ satisfies $\int_{-A}^{+\infty} c_A(t)e^{-t}dt < \infty$ and inequality (2.4) holds ($A \in (-\infty, +\infty]$), we can choose a continuous function $g(t)$ such that it is decreasing rapidly enough on $[A'', A']$ (A'' can be chosen near A' enough), and the following holds:

(1) $\int_{-A''}^{+\infty} c_{A''}(t)e^{-t}dt + \frac{1}{\delta''}c_{A''}(-A'')e^{A''} = \int_{-A}^{+\infty} c_A(t)e^{-t}dt$;

(2) $\int_{-A''}^{t}(\frac{1}{\delta''}c_{A''}(-A'')e^{A''} + \int_{-A''}^{t_2} c_{A''}(t_1)e^{-t_1}dt_1)dt_2 + \frac{1}{\delta''^2}c_{A''}(-A'')e^{A''} < \int_{-A}^{t}(\int_{-A}^{t_2} c_A(t_1)e^{-t_1}dt_1)dt_2$.

Thus we have proved the lemma. □

Since A may be chosen as positive infinity, we have a sufficient condition for inequality (2.4) holding:

Remark 4.12. Assume that $\frac{d}{dt}c_A(t)e^{-t} > 0$ for $t \in (-A, a)$, and $\frac{d}{dt}c_A(t)e^{-t} \leq 0$ for $t \in [a, +\infty)$, where $A = +\infty$ and $a > -A$. Assume $\frac{d^2}{dt^2}\log(c_A(t)e^{-t}) < 0$ for $t \in (-A, a)$. Then inequality (2.4) holds.

Proof. Let $H(t, f) := (\int_{-A}^{t} f(t_1)dt_1)^2 - f(t)\int_{-A}^{t}(\int_{-A}^{t_2} f(t_1)dt_1)dt_2$, where $f(t)$ is a positive smooth function on $(-A, +\infty)$. Inequality (2.4) becomes $H(t, c_A(t)e^{-t}) > 0$ for any $t \in (-A, +\infty)$; that is, $\frac{H(t,c_A(t)e^{-t})}{c_A(t)e^{-t}} > 0$ for any $t \in (-A, +\infty)$.

It suffices to prove $\frac{d}{dt}\frac{H(t,c_A(t)e^{-t})}{c_A(t)e^{-t}} > 0$ for any $t \in (-\infty, a)$, and therefore

$$H\left(t, \frac{d}{dt}(c_A(t)e^{-t})\right) > 0$$

for any $t \in (-\infty, a)$.

As $\frac{d}{dt}(c_A(t)e^{-t}) > 0$ for any $t \in (-\infty, a)$, it suffices to prove that

$$\frac{d}{dt}\frac{H(t, \frac{d}{dt}(c_A(t)e^{-t}))}{\frac{d}{dt}(c_A(t)e^{-t})} > 0$$

for any $t \in (-\infty, a)$, which is $H(t, \frac{d}{dt}\frac{d}{dt}(c_A(t)e^{-t})) > 0$ for any $t \in (-\infty, a)$.

Note that $H(t, \frac{d}{dt}\frac{d}{dt}(c_A(t)e^{-t})) = -(c_A(t)e^{-t})^2 \frac{d}{dt}\frac{d}{dt}\log(c_A(t)e^{-t})$. Thus we have proved the present remark. □

In the last part of this section, we recall a theorem of Fornæss and Narasimhan on the approximation property of plurisubharmonic functions of Stein manifolds.

LEMMA 4.13 ([20]). *Let X be a Stein manifold and $\varphi \in \mathrm{PSH}(X)$. Then there exists a sequence $\{\varphi_n\}_{n=1,2,\ldots}$ of smooth strongly plurisubharmonic functions such that $\varphi_n \downarrow \varphi$.*

4.2. *Properties of polar functions.* In this subsection, we give some lemmas on properties of polar functions.

LEMMA 4.14. *Let M be a complex manifold of dimension n and S be an $(n-l)$-dimensional closed complex submanifold. Let $\Psi \in \Delta(S)$. Assume that there exists a local coordinate $(z_1, \dots, z_n)$ on a neighborhood U of $x \in M$ such that $\{z_{n-l+1} = \cdots = z_n = 0\} = S \cap U$ and $\psi := \Psi - l\log(|z_{n-l+1}|^2 + \cdots + |z_n|^2)$ is continuous on U. Then we have $d\lambda_z[\Psi] = e^{-\psi} d\lambda_{z'}$, where $d\lambda_z$ and $d\lambda_{z'}$ denote the Lebesgue measures on U and $S \cap U$. Especially,*

$$|f \wedge dz_{n-l+1} \wedge \cdots \wedge dz_n|_h^2 d\lambda_z[\Psi] = 2^l \{f, f\}_h e^{-\psi},$$

where f is a continuous $(n-l, 0)$ form with value in the Hermitian vector bundle (E, h) on $S \cap U$.

Proof. Note that

$$d\lambda_z[l\log(|z_{n-l+1}|^2 + \cdots + |z_n|^2)] = d\lambda_{z'}$$

for $z = (z', z_{n-l+1}, \dots, z_n)$. According to the definition of generalized residue volume form $d\lambda_z[\Psi]$ and the continuity of ψ, the lemma follows. □

Using a similar method as in the proof of the above lemma, we obtain a remark as follows:

Remark 4.15. Let M be a complex manifold of dimension n and S be an $(n-l)$-dimensional closed complex submanifold. Let $\Psi \in \Delta(S)$. Assume that there exists a local coordinate $(z_1, \dots, z_l, w_{2l+1}, \dots, w_{2n})$ on a neighborhood U of $x \in M$ such that $\{w_{2l+1} = \cdots = w_{2n} = 0\} = S \cap U$ and $\psi := \Psi - l\log(|w_{2l+1}|^2 + \cdots + |w_{2n}|^2)$ is continuous on U, where $z' = (z_1, \dots, z_l)$ are complex coordinates, and $w_{2l+1}, \dots, w_{2n}$ are real coordinates. Then we have $dV_{z',w}[\Psi] = e^{-\psi} d\lambda_{z'}$, where $dV_{z',w}$ and $d\lambda_{z'}$ denote the Lebesgue measures on U and $S \cap U$. Especially,

$$|F|_S|_h^2 d\lambda_z[\Psi] = \frac{\{F, F\}_h}{dV_{z',w}} dV_{z',w}[\Psi] = \frac{\{F, F\}_h}{dV_{z',w}} e^{-\psi} d\lambda_{z'},$$

where F is a continuous $(n, 0)$-form with value in the Hermitian vector bundle (E, h) on U.

LEMMA 4.16. *Let $d_1(t)$ and $d_2(t)$ be two positive continuous functions on $(0, +\infty)$, which satisfy*

$$\int_0^{+\infty} d_1(t) e^{-t} dt = \int_0^{+\infty} d_2(t) e^{-t} dt < \infty,$$
$$d_1(t)|_{\{t>r_1\} \cup \{t<r_3\}} = d_2(t)|_{\{t>r_1\} \cup \{t<r_3\}},$$
$$d_1(t)|_{\{r_2<t<r_1\}} > d_2(t)|_{\{r_2<t<r_1\}},$$

and

$$d_1(t)|_{\{r_3<t<r_2\}} < d_2(t)|_{\{r_3<t<r_2\}},$$

where $0 < r_3 < r_2 < r_1 < +\infty$. *Let* f *be a holomorphic function on* Δ, *then we have*

$$\int_\Delta d_1(-\ln(|z|^2))|f|^2 d\lambda \leq \int_\Delta d_2(-\ln(|z|^2))|f|^2 d\lambda < +\infty,$$

where λ *is the Lebesgue measure on* Δ. *Moreover, the equality holds if and only if* $f \equiv f(0)$.

Proof. Set

$$f(z) = \sum_{k=0}^{\infty} a_k z^k,$$

a Taylor expansion of f at 0, which is uniformly convergent on any given compact subset of Δ.

As

$$\int_\Delta d_1(-\ln(|z|^2)) z^{k_1} \bar{z}^{k_2} d\lambda = 0$$

when $k_1 \neq k_2$, it follows that

$$\begin{aligned} \int_\Delta d_1(-\ln(|z|^2))|f|^2 d\lambda &= \int_\Delta \sum_{k=0}^{\infty} d_1(-\ln(|z|^2))|a_k|^2|z|^{2k} d\lambda \\ &= \pi \sum_{k=0}^{\infty} |a_k|^2 \int_0^{+\infty} d_1(t) e^{-kt} e^{-t} dt \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} \int_\Delta d_2(-\ln(|z|^2))|f|^2 d\lambda &= \int_\Delta \sum_{k=0}^{\infty} d_2(-\ln(|z|^2))|a_k|^2|z|^{2k} d\lambda \\ &= \pi \sum_{k=0}^{\infty} |a_k|^2 \int_0^{+\infty} d_2(t) e^{-kt} e^{-t} dt. \end{aligned} \tag{4.8}$$

As

$$\int_0^{+\infty} d_1(t) e^{-t} dt = \int_0^{+\infty} d_2(t) e^{-t} dt < \infty,$$
$$d_1(t)|_{\{r_2<t<r_1\}} > d_2(t)|_{\{r_2<t<r_1\}},$$

and

$$d_1(t)|_{\{r_3<t<r_2\}} < d_2(t)|_{\{r_3<t<r_2\}},$$

it follows that

$$\begin{aligned}\int_{r_3}^{r_2}(d_2(t)-d_1(t))e^{-kt}e^{-t}dt &> \int_{r_3}^{r_2}(d_2(t)-d_1(t))e^{-kr_2}e^{-t}dt\\ &= \int_{r_2}^{r_1}(d_1(t)-d_2(t))e^{-kr_2}e^{-t}dt\\ &> \int_{r_2}^{r_1}(d_1(t)-d_2(t))e^{-kt}e^{-t}dt;\end{aligned} \tag{4.9}$$

therefore

$$\int_{r_3}^{r_1} d_1(t)e^{-kt}e^{-t}dt < \int_{r_3}^{r_1} d_2(t)e^{-kt}e^{-t}dt$$

for every $k \geq 1$.

Since

$$d_1(t)|_{\{t>r_1\}\cup\{t<r_3\}} = d_2(t)|_{\{t>r_1\}\cup\{t<r_3\}},$$

we have

$$\int_0^{+\infty} d_1(t)e^{-kt}e^{-t}dt < \int_0^{+\infty} d_2(t)e^{-kt}e^{-t}dt$$

for every $k \geq 1$.

Comparing equalities (4.7) and (4.8), we obtain that the inequality in the lemma holds, and the equality in the lemma holds if and only if $a_k = 0$ for any $k \geq 1$; i.e., $f = f(0)$. Then we are done. $\square$

Let Ω be an open Riemann surface. Let $z_0 \in \Omega$, and let V_{z_0} be a neighborhood of z_0 with local coordinate w, such that $w(z_0) = 0$.

Using the above lemma, we have the following lemma on open Riemann surfaces:

LEMMA 4.17. *Assume that there is a negative subharmonic function Ψ on Ω, such that $\Psi|_{V_{z_0}} = \ln|w|^2$, and $\Psi|_{\Omega\setminus V_{z_0}} \geq \sup_{z\in V_{z_0}}\Psi(z)$. Let $d_1(t)$ and $d_2(t)$ be two positive continuous functions on $(0,+\infty)$ as in Lemma 4.16. Assume that $\{\Psi < -r_3+1\} \subset\subset V_{z_0}$ is a disc with the coordinate z. Let F be a holomorphic $(1,0)$-form, which satisfies $F|_{z_0} = dw$. Then we have*

$$\int_\Omega d_1(-\Psi)\sqrt{-1}F\wedge\bar{F} \leq \int_\Omega d_2(-\Psi)\sqrt{-1}F\wedge\bar{F} < +\infty.$$

Moreover, the equality holds if and only if $F|_{V_{z_0}} = dw$.

Proof. It is clear that

$$\begin{aligned}\int_\Omega d_1(-\Psi)\sqrt{-1}F\wedge\bar{F} &= \int_{\{\log|w|^2<-r_3+1\}} d_1(-\Psi)\sqrt{-1}|\frac{F}{dw}|^2 dw\wedge d\bar{w}\\ &\quad+\int_{\Omega\setminus\{\log|w|^2<-r_3+1\}} d_1(-\Psi)\sqrt{-1}F\wedge\bar{F},\end{aligned} \tag{4.10}$$

$$\begin{aligned}\int_\Omega d_2(-\Psi)\sqrt{-1}F\wedge\bar{F} &= \int_{\{\log|w|^2<-r_3+1\}} d_2(-\Psi)\sqrt{-1}|\frac{F}{dw}|^2 dw\wedge d\bar{w}\\ &\quad+\int_{\Omega\setminus\{\log|w|^2<-r_3+1\}} d_2(-\Psi)\sqrt{-1}F\wedge\bar{F}.\end{aligned}\tag{4.11}$$

Note that $-\Psi|_{\Omega\setminus\{\log|w|^2<-r_3+1\}}<r_3-1$. Then

$$\int_{\Omega\setminus\{\log|w|^2<-r_3+1\}} d_1(-\Psi)\sqrt{-1}F\wedge\bar{F}=\int_{\Omega\setminus\{\log|w|^2<-r_3+1\}} d_2(-\Psi)\sqrt{-1}F\wedge\bar{F}.$$

Applying Lemma 4.16 to the rest of the parts of equalities (4.10) and (4.11), we get the present lemma. □

Let Ω be an open Riemann surface with a Green function. Let $p:\Delta\to\Omega$ be the universal covering of Ω. We can choose V_{z_0} small enough, such that p restricted on any component of $p^{-1}(V_{z_0})$ is biholomorphic. Let h be a harmonic function on Ω and $\rho=e^{-2h}$. As h is harmonic on Ω, then there exists a multiplicative holomorphic function f_h on Δ, such that $|f_h|=e^{p^*h}=p^*e^h$. Let $f_{-h}:=f_h^{-1}$. Let $f_{-h,j}:=f_{-h}|_{U_j}$ and $p_j:=p|_{U_j}$, where U_j is a component of $p^{-1}(V_{z_0})$ for any fixed j.

Using Lemma 4.16, we obtain the following lemma:

Lemma 4.18. *Let Ω be an open Riemann surface with a Green function G_Ω. Let $z_0\in\Omega$, and let V_{z_0} be a neighborhood of z_0 with local coordinate w, such that $w(z_0)=0$. Assume that there is a negative subharmonic function Ψ on Ω, such that $\Psi|_{V_{z_0}}=\ln|w|^2$ and $\Psi|_{\Omega\setminus V_{z_0}}\geq\sup_{z\in V_{z_0}}\Psi(z)$. Let $d_1(t)$ and $d_2(t)$ be two positive continuous functions on $(0,+\infty)$ as in Lemma* 4.16. *Assume that $\{\Psi<-r_3+1\}\subset\subset V_{z_0}$, which is a disc with the coordinate w. Let F be a holomorphic $(1,0)$-form, which satisfies $((p_j)_*(f_{-h,j}))F|_{z_0}=dw$. Then we have*

$$\int_\Omega d_1(-\Psi)\sqrt{-1}\rho F\wedge\bar{F}\leq\int_\Omega d_2(-\Psi)\sqrt{-1}\rho F\wedge\bar{F}.$$

Moreover, the equality holds if and only if $((p_j)_*(f_{-h,j}))F|_{V_{z_0}}=dw$.

Proof. It is clear that

$$\begin{aligned}\int_\Omega d_1(-\Psi)\rho\sqrt{-1}F\wedge\bar{F} &= \int_{\{\log|w|^2<-r_3+1\}} d_1(-\Psi)\rho\sqrt{-1}|\frac{F}{dw}|^2 dw\wedge d\bar{w}\\ &\quad+\int_{\Omega\setminus\{\log|w|^2<-r_3+1\}} d_1(-\Psi)\rho\sqrt{-1}F\wedge\bar{F},\end{aligned}\tag{4.12}$$

$$\begin{aligned}\int_\Omega d_2(-\Psi)\rho\sqrt{-1}F\wedge\bar{F} &= \int_{\{\log|w|^2<-r_3+1\}} d_2(-\Psi)\rho\sqrt{-1}|\frac{F}{dw}|^2 dw\wedge d\bar{w}\\ &\quad+\int_{\Omega\setminus\{\log|w|^2<-r_3+1\}} d_2(-\Psi)\rho\sqrt{-1}F\wedge\bar{F}.\end{aligned}\tag{4.13}$$

Note that $-\Psi|_{\Omega\setminus\{\log|w|^2<-r_3+1\}}<r_3-1$. Then one has

$$\int_{\Omega\setminus\{\log|w|^2<-r_3+1\}} d_1(-\Psi)\rho\sqrt{-1}F\wedge\bar{F}=\int_{\Omega\setminus\{\log|w|^2<-r_3+1\}} d_2(-\Psi)\rho\sqrt{-1}F\wedge\bar{F}.$$

Applying Lemma 4.16 to the rest of the parts of equalities (4.12) and (4.13), we get the present lemma. □

4.3. *Basic properties of the Green function.* Let Ω be an open Riemann surface with a Green function G_Ω, and let z_0 be a point of Ω with a fixed local coordinate w on the neighborhood V_{z_0} of z_0, such that $w(z_0)=0$.

Remark 4.19 (see [47] or [56]). $G_\Omega(z,z_0)=\sup_{u\in\Delta_0(z_0)}u(z)$, where $\Delta_0(z_0)$ is the set of negative subharmonic functions on Ω satisfying that $u-\log|w|$ has a locally finite upper bound near z_0.

Remark 4.20 (see [47] or [56]). $G_\Omega(z,z_0)$ is harmonic on $\Omega\setminus\{z_0\}$, and $G_\Omega(z,z_0)-\log|w|$ is harmonic near z_0.

4.4. *Results used in the proofs of the conjecture of Suita, the L-conjecture and the extended Suita conjecture.* In this subsection, we give some results which are used to prove the conjecture of Suita, the L-conjecture and the extended Suita conjecture.

Using Theorem 2.2 and Lemma 4.17, we obtain the following proposition, which will be used in the proof of the conjecture of Suita.

PROPOSITION 4.21. *Let Ω be an open Riemann surface with a Green function G_Ω. Let $z_0\in\Omega$, and let V_{z_0} be a neighborhood of z_0 with local coordinate w, such that $w(z_0)=0$ and $G_\Omega|_{V_{z_0}}=\log|w|$. Assume that there is a unique holomorphic $(1,0)$-form F on Ω, which satisfies $F|_{z_0}=b_0dw$ (b_0 is a complex constant which is not 0), such that*

$$\int_\Omega\sqrt{-1}F\wedge\bar{F}\le\pi\int_{z_0}|b_0dw|^2dV_\Omega[2G(z,z_0)].$$

Then $F|_{V_{z_0}}=b_0dw$.

Remark 4.22. In Theorem 2.2, let $\Psi:=2G_\Omega(\cdot,z_0)+2G_\Omega(\cdot,z_2)$, where z_2 near z_0 and $z_0\neq z_2$, $c_A(t)\equiv1$ and $A=0$. Then we have F_2 such that $F_2|_{z_0}=b_0dw$, $F_2|_{z_2}=0$ and

$$\int_\Omega\sqrt{-1}F_2\wedge\bar{F}_2\le\pi\int_{z_0}|b_0dw|^2dV_\Omega[2G_\Omega(\cdot,z_0)+2G_\Omega(\cdot,z_2)]<+\infty.$$

If there exists a holomorphic $(1,0)$-form, which satisfies

$$\int_\Omega\sqrt{-1}F\wedge\bar{F}<\pi\int_{z_0}|b_0dw|^2dV_\Omega[2G(z,z_0)],$$

then there exists $\varepsilon_0 > 0$, such that for any $\varepsilon \in (0, \varepsilon_0)$,

$$\int_\Omega \sqrt{-1}((1-\varepsilon)F + \varepsilon F_2) \wedge \overline{((1-\varepsilon)F + \varepsilon F_2)} < \pi \int_{z_0} |b_0 dw|^2 dV_\Omega[2G(z, z_0)].$$

Since $((1-\varepsilon)F+\varepsilon F_2)|_o = b_0 dw$, and $((1-\varepsilon)F+\varepsilon F_2)$ also satisfies the inequality in Proposition 4.21, it is a contradiction to the uniqueness of F. Then we have

$$\int_\Omega \sqrt{-1}F \wedge \bar{F} = \pi \int_{z_0} |b_0 dw|^2 dV_\Omega[2G(z, z_0)].$$

Proof of Proposition 4.21. Let $\Psi := 2G_\Omega(\cdot, z_0)$. We can choose r_3 big enough, such that $\{\Psi < -r_3\} \subset\subset \{\Psi < -r_3 + 1\} \subset\subset V_{z_0}$, and $\{\Psi < -r_3 + 1\}$ is a disc with the coordinate w. Let $d_1(t) = 1$. One can find smooth $d_2(t)$ as in Lemma 4.17, such that $d_2(t)e^{-t}$ is decreasing with respect to t.

Using Theorem 2.2, we have a holomorphic $(1, 0)$ form F_1 on Ω, which satisfies $F_1|_{z_0} = b_0 dw$ and

$$\int_\Omega d_2(-\Psi)\sqrt{-1}F_1 \wedge \bar{F}_1 \leq \pi \int_{z_0} |b_0 dw|^2 dV_\Omega[\Psi].$$

Using Lemma 4.17, we have

$$\int_\Omega \sqrt{-1}F_1 \wedge \bar{F}_1 \leq \int_\Omega d_2(-\Psi)\sqrt{-1}F_1 \wedge \bar{F}_1.$$

Therefore,

$$\int_\Omega \sqrt{-1}F_1 \wedge \bar{F}_1 \leq \pi \int_{z_0} |b_0 dw|^2 dV_\Omega[\Psi].$$

According to the assumption of uniqueness of F and the above remark, it follows that

$$\int_\Omega d_1(-\Psi)\sqrt{-1}F_1 \wedge \bar{F}_1 = \int_\Omega d_2(-\Psi)\sqrt{-1}F_1 \wedge \bar{F}_1$$

and $F_1 = F$. Using Lemma 4.17, we have $F_1|_{V_{z_0}} = b_0 dw$, and therefore $F|_{V_{z_0}} = b_0 dw$. □

Let Ω be an open Riemann surface with a Green function G_Ω. Let $z_0 \in \Omega$, and let V_{z_0} be a neighborhood of z_0 with local coordinate w, such that $w(z_0) = 0$. Note that there exists a holomorphic function f_0 near z_0, which is locally injective near z_0, such that $|f_0| = e^{G_\Omega(\cdot, z_0)}$.

Let $w = f_0$. Then we have a local coordinate w, such that $G_\Omega(\cdot, z_0) = \log|w|$ near z_0. Using Theorem 2.2 and Lemma 4.18, we obtain the following proposition, which will be used in the proof of the extended Suita conjecture.

Proposition 4.23. *Let Ω be an open Riemann surface with a Green function G_Ω. Let $z_0 \in \Omega$, and let V_{z_0} be a neighborhood of z_0 with local coordinate w, such that $w(z_0) = 0$ and $G_\Omega(z, z_0)|_{V_{z_0}} = \log|w|$. Assume that there is a unique* *holomorphic* $(1,0)$*-form* F *on* Ω, *which satisfies* $((p_j)_*(f_{-h,j}))F|_{z_0} = b_0dw$ $(b_0$ *is a complex constant which is not* $0)$, *and*

$$\int_\Omega \sqrt{-1}\rho F \wedge \bar{F} \leq \pi \int_{z_0} \rho |b_0dw|^2 dV_\Omega[2G(z, z_0)].$$

Then $((p_j)_*(f_{-h,j}))F|_{V_{z_0}} = b_0dw$.

Remark 4.24. In Theorem 2.2, let $\Psi := 2G_\Omega(\cdot, z_0) + 2G_\Omega(\cdot, z_2)$, where z_2 near z_0 and $z_0 \neq z_2$, $c_A(t) \equiv 1$ and $A = 0$. Then we have F_2 such that $F_2|_{z_0} = b_0dw$, $F_2|_{z_2} = 0$, and

$$\int_\Omega \sqrt{-1}\rho F_2 \wedge \bar{F}_2 \leq \pi \int_{z_0} \rho |b_0dw|^2 dV_\Omega[2G(z, z_0) + 2G_\Omega(\cdot, z_2)] < +\infty.$$

If there exists a holomorphic $(1,0)$-form F, which satisfies

$$\int_\Omega \sqrt{-1}\rho F \wedge \bar{F} < \pi \int_{z_0} \rho |b_0dw|^2 dV_\Omega[2G(z, z_0)],$$

then there exists $\varepsilon_0 > 0$, such that for any $\varepsilon \in (0, \varepsilon_0)$,

$$\int_\Omega \sqrt{-1}\rho((1-\varepsilon)F + \varepsilon F_2) \wedge \overline{((1-\varepsilon)F + \varepsilon F_2)} < \pi \int_{z_0} \rho |b_0dw|^2 dV_\Omega[2G(z, z_0)].$$

Since $((1-\varepsilon)F + \varepsilon F_2)|_o = b_0dw$, and $(1-\varepsilon)F + \varepsilon F_2$ also satisfies the inequality in the present proposition, it is a contradiction to the uniqueness of F. Then we have

$$\int_\Omega \sqrt{-1}\rho F \wedge \bar{F} = \pi \int_{z_0} \rho |b_0dw|^2 dV_\Omega[2G(z, z_0)].$$

Proof of Proposition 4.23. Let $\Psi := 2G_\Omega(\cdot, z_0)$. We can choose r_3 big enough, such that $\{\Psi < -r_3\} \subset\subset \{\Psi < -r_3 + 1\} \subset\subset V_{z_0}$, and $\{\Psi < -r_3 + 1\}$ is a disc with the coordinate w.

Let $d_1(t) = 1$. One can find smooth $d_2(t)$ as in Lemma 4.18, which satisfies that $d_2(t)e^{-t}$ is decreasing with respect to t.

From Theorem 2.2, it follows that there exists a holomorphic $(1,0)$-form F_1 on Ω, which satisfies $F_1|_{z_0} = b_0dw$, and

$$\int_\Omega d_2(-\Psi)\sqrt{-1}\rho F_1 \wedge \bar{F}_1 \leq \pi \int_{z_0} \rho(z_0) |b_0dw|^2 dV_\Omega[\Psi].$$

Using Lemma 4.18, we have

$$\int_\Omega \sqrt{-1}\rho F_1 \wedge \bar{F}_1 \leq \int_\Omega d_2(-\Psi)\rho\sqrt{-1}F_1 \wedge \bar{F}_1.$$

Therefore,

$$\int_\Omega \sqrt{-1}\rho F_1 \wedge \bar{F}_1 \leq \pi \int_{z_0} \rho(z_0) |b_0dw|^2 dV_\Omega[\Psi].$$

From the assumption of uniqueness of F and the above remark, it follows that

$$\int_\Omega d_1(-\Psi)\rho\sqrt{-1}F_1 \wedge \bar{F}_1 = \int_\Omega d_2(-\Psi)\rho\sqrt{-1}F_1 \wedge \bar{F}_1$$

and $F_1 = F$.

Using Lemma 4.18, we have

$$((p_j)_*(f_{-h,j}))F_1|_{V_{z_0}} = b_0 dw;$$

therefore

$$((p_j)_*(f_{-h,j}))F|_{V_{z_0}} = b_0 dw.$$

We have thus proved the proposition. □

Let Ω be an open Riemann surface with a Green function G, and let z_0 be a point of Ω with a fixed local coordinate w on the neighborhood V_{z_0} of z_0, such that $w(z_0) = 0$.

Let $\mathscr{A}_{z_0}$ be a family of analytic functions f on Ω satisfying the normalization condition: $f|_{z_0} = 0$ and $df|_{z_0} = dw$. Analytic capacity c_B is defined as follows:

$$c_B := c_B(z_0) = \frac{1}{\min_{f\in\mathscr{A}_{z_0}} \sup_{z\in\Omega} |f(z)|}.$$

About a relation between c_β and c_B, it is well known that one has $c_\beta^2(z_0) \geq c_B^2(z_0)$. Furthermore, one has the following lemma:

LEMMA 4.25. *If there is a holomorphic function g on Ω, which satisfies $|g(z)| = \exp G(z, z_0)$, then we have $c_\beta^2(z_0) = c_B^2(z_0)$.*

Proof. For the sake of completeness, we give a proof of the inequality $c_\beta^2(z_0) \geq c_B^2(z_0)$.

Consider

$$\mathscr{A}_{z_0}^M := \mathscr{A}_{z_0} \cap \{f||f| \leq M\}.$$

As $|g(z)| = \exp G(z, z_0)$, then $\mathscr{A}_{z_0}^M$ is not empty.

As $\mathscr{A}_{z_0}^M$ is a normal family, there exists a holomorphic function $f_1 \in \mathscr{A}_{z_0}$, such that $\sup_{z\in\Omega} |f_1| = \min_{f\in\mathscr{A}_{z_0}} \sup_{z\in\Omega} |f(z)|$. That is, $|f_1(z)|c_B(z_0) < 1$ for any $z \in \Omega$. Note that $\log(|f_1(z)|c_B(z_0)) - \log|w(z)|$ is locally finite on V_{z_0}. Then by Remark 4.19, we have $\log|f_1(z)|c_B(z_0) \leq G(z, z_0)$; therefore,

$$\lim_{z\to z_0}(\log(|f_1(z)|c_B(z_0)) - \log|w(z)|) \leq \lim_{z\to z_0}(G(z, z_0) - \log|w(z)|). \tag{4.14}$$

As $df_1|_{z_0} = dw$, we have $\lim_{z\to z_0}(\log(|f_1(z)| - \log|w(z)|) = 0$. Then inequality (4.14) implies that $c_B(z_0) \leq \lim_{z\to z_0}(G(z, z_0) - \log|w(z)|) = c_\beta(z_0)$. Then we prove $c_\beta^2(z_0) = c_B^2(z_0)$ under the assumption in the present lemma.

Suppose that there is a holomorphic function g on Ω which satisfies $|g(z)| = \exp G(z, z_0)$. As $\sup_{z\in\Omega} |f_1| = \min_{f\in\mathscr{A}_{z_0}} \sup_{z\in\Omega} |f(z)|$, we have $\sup|f_1(z)| \leq$

$\sup \frac{|g(z)|}{|g'(z_0)|}$, and therefore

$$\log|f_1||g'(z_0)| \leq 0, \tag{4.15}$$

where $g'(z_0) = \frac{dg}{dw}|_{z_0}$.

As $\log|f_1(z))||g'(z_0)| - \log|w(z)|$ has a locally finite upper bound near z_0, we have $\log|f_1||g'(z_0)| \leq G(z, z_0) = \log|g|$ by Remark 4.19 (see [1] or [56]).

Note that

$$\lim_{z\to z_0} \log(|f_1(z)||g'(z_0)| - \log|w(z)|) = \log|g'(z_0)| = \lim_{z\to z_0}(\log|g(z)| - \log|w(z)|).$$

It follows that

$$\lim_{z\to z_0}(\log|f_1(z)||g'(z_0)| - \log|g(z)|) = 0,$$

and therefore

$$\lim_{z\to z_0}(\log|f_1(z)||g'(z_0)| - G(z, z_0)) = 0.$$

From inequality (4.15), it follows that $\log(|f_1(z)||g'(z_0)|) - G(z, z_0)$ is a negative subharmonic function on Ω.

Applying the maximal principle to $\log(|f_1(z)||g'(z_0)|) - G(z, z_0)$, since

$$\lim_{z\to z_0}(\log|f_1(z)||g'(z_0)| - G(z, z_0)) = 0,$$

we have

$$\log|f_1(z)||g'(z_0)| - G(z, z_0) = 0,$$

i.e.,

$$|f_1||g'(z_0)| = |g|.$$

Then it follows that

$$c_B(z_0) = \frac{1}{\sup_{z\in\Omega}|f_1|} = \frac{|g'(z_0)|}{\sup_{z\in\Omega}|g(z)|} = |g'(z_0)|.$$

As

$$\begin{aligned} c_\beta(z_0) &:= \exp\lim_{z\to z_0}(G(z, z_0) - \log|w(z)|) \\ &= \exp\lim_{z\to z_0}(\log|g(z)| - \log|w(z)|) = |g'(z_0)| = c_B(z_0), \end{aligned}$$

we have $c_\beta(z_0) = c_B(z_0)$. □

Let us recall the following result of Suita in [54]:

LEMMA 4.26 ([54]). *Assume that Ω admits a Green function. Then $\pi B_\Omega(z) \geq c_B^2(z)$ for any $z \in \Omega$. There exists $z_0 \in \Omega$ such that equality holds if and only if Ω is conformally equivalent to the unit disc less a (possible) closed set of inner capacity zero.*

Remark 4.27. We now present the relationship between the definition of $c_B(z_0) := \sup_{\{f|f_{z_0}=0\&|f|<1\}}|f'(z_0)|$ in [54] and the definition of $c_B(z_0)$ used in the present paper. When $\mathscr{A}^M_{z_0}$ is not empty, for any element g in $\mathscr{A}^M_{z_0}$, one can normalize the norm of g by $\sup_{z\in\Omega}|g|$ denoted by $\sup|g|$ for convenience. Then it is clear that

$$\begin{aligned}
\sup_{\{f|f(z_0)=0\&|f|<1\}}|f'(z_0)| &= \left(\min_{\{f|f(z_0)=0\&|f|<1\}}|f'(z_0)|^{-1}\right)^{-1}\\
&= \left(\min_{\{g|g(z_0)=0\&|\frac{g}{\sup|g|}|<1\}}|\frac{d}{dt}\frac{g}{\sup|g|}|_{z_0}|^{-1}\right)^{-1}\\
&= \left(\min_{\{g|g(z_0)=0\&|\frac{g}{\sup|g|}|<1\}}|\frac{\frac{dg}{dt}|_{z_0}}{\sup|g|}|^{-1}\right)^{-1}\\
&= \left(\min_{g\in\mathscr{A}^M_{z_0}}\sup|g|\right)^{-1}\\
&= \left(\min_{g\in\mathscr{A}_{z_0}}\sup|g|\right)^{-1},
\end{aligned} \tag{4.16}$$

where $g\in\mathscr{A}^M_{z_0}$. If $\mathscr{A}^M_{z_0}$ is not empty (i.e., $\{f|f(z_0)=0\,\&\,|f|<1\}$ does not only contain 0), the above two definitions of $c_B(z_0)$ are equivalent. If $\mathscr{A}^M_{z_0}$ is empty (i.e., $\{f|f(z_0)=0\,\&\,|f|<1\}$ only contains 0), the above two definitions of $c_B(z_0)$ are both 0. Then the above two definitions of $c_B(z_0)$ are the same.

Now we prove an identity theorem of holomorphic maps between complex spaces, which is useful.

LEMMA 4.28. *Let X be a irreducible complex space and Y be a complex space. Let $f,g:X\to Y$ be holomorphic maps. Assume that for a point $a\in X$, the germs f_a and g_a of holomorphic maps f and g satisfy $f_a=g_a$. Then we have $f=g$.*

Proof. Consider a map $(f,g):X\to Y\times Y$, which is $(f,g)(x)=(f(x),g(x))$. Denote that $A:=\{x\in X|f(x)=g(x)\}$. Note that $A=(f,g)^{-1}(\Delta_Y)$, where Δ_Y is the diagonal of $Y\times Y$. Then A is an analytic set. As $f_a=g_a$, there is a neighborhood U_a of a in X, such that $f|_{U_a}=g|_{U_a}$.

Using the Identity Lemma in [24], we obtain $A=X_a$, which is the irreducible component of X containing a. As X is irreducible, it is clear that $X=X_a=A$. Thus we have proved the lemma. □

Remark 4.29. By the above lemma, one can see that if two holomorphic maps f and g from irreducible complex space X to complex space Y, which satisfy $f|_S=g|_S$, where S is totally real with maximal dimension in X, then $f\equiv g$.

LEMMA 4.30. *Let g_1 and g_2 be two holomorphic functions on domain Ω in $\mathbb{C}$, such that $|g_1| = |g_2|$, and $dg_1 = dg_2$. Assume that $dg_1 = dg_2$ do not vanish identically. Then we have $g_1 = g_2$.*

Proof. As $|g_1| = |g_2|$, we have $g_1\bar{g}_1 = g_2\bar{g}_2$. Then $g_1\bar{\partial}\bar{g}_1 = g_2\bar{\partial}\bar{g}_2$. It is known that the zero sets of $\bar{\partial}\bar{g}_1$ and $\bar{\partial}\bar{g}_2$ are both analytic sets on Δ. From the assumption, it follows that $d\bar{g}_1 = \bar{\partial}\bar{g}_1 = \bar{\partial}\bar{g}_2 = d\bar{g}_2$.

As dg_1 and dg_2 do not vanish identically and so are $\bar{\partial}\bar{g}_1$ and $\bar{\partial}\bar{g}_2$, then $g_1 = g_2$ on an open subset of Δ. It is clear that $g_1 = g_2$ on Δ by the identity theorem of holomorphic functions. □

LEMMA 4.31 (see [4] and [60]). *Let $\mathcal{H}$ be a Hilbert space with norm $\|\cdot\|$, and let $\mathcal{C}$ be a convex subset of $\mathcal{H}$. Let $\alpha \in \mathcal{C}$, such that $\|\alpha\| = \inf_{\beta\in\mathcal{C}}\|\beta\|$. Then α is unique.*

Proof. If not, there are α_1 and α_2 in $\mathcal{C}$, such that

$$\|\alpha_1\| = \|\alpha_2\| = \inf_{\beta\in\mathcal{C}}\|\beta\|.$$

As

$$\|\frac{\alpha_1+\alpha_2}{2}\|^2 + \|\frac{\alpha_1-\alpha_2}{2}\|^2 = \frac{\|\alpha_1\|^2+\|\alpha_2\|^2}{2}$$

and $\|\frac{\alpha_1-\alpha_2}{2}\| > 0$, we have

$$\|\frac{\alpha_1+\alpha_2}{2}\| < \sqrt{\frac{\|\alpha_1\|^2+\|\alpha_2\|^2}{2}} = \inf_{\beta\in\mathcal{C}}\|\beta\|. \tag{4.17}$$

Note that $\frac{\alpha_1+\alpha_2}{2} \in \mathcal{C}$. Then inequality (4.17) contradicts $\|\alpha_1\| = \|\alpha_2\| = \inf_{\beta\in\mathcal{C}}\|\beta\|$. □

Remark 4.32. Let Ω be an open Riemann surface with a Green function G_Ω, and let z_0 be a point of Ω with a fixed local coordinate w on the neighborhood V_{z_0} of z_0, such that $w(z_0) = 0$. Let $c_0(t) = 1$, $\Psi = 2G_\Omega(\cdot, z_0)$.

From Theorem 2.2 and the definition

$$c_\beta := \exp\lim_{z\to z_0}(G_\Omega(z,z_0) - \log|w(z)|),$$

it follows that there is a holomorphic $(1,0)$-form F on Ω, which satisfies $F|_{z_0} = dw|_{z_0}$ and

$$\sqrt{-1}\int_\Omega F\wedge\bar{F} \le \pi\int_{z_0}|dw|^2 dV_\Omega[2G_\Omega(z,z_0)] = \frac{2\pi}{c_\beta^2(z_0)}.$$

Therefore,

$$\pi B_\Omega(z_0) \ge c_\beta^2(z_0),$$

by the extremal property of the Bergman kernel.

If there is another holomorphic $(1,0)$-form $\tilde{F}$ on Ω, which satisfies $\tilde{F}|_{z_0} = dw|_{z_0}$, and

$$\sqrt{-1}\int_{\Omega}\tilde{F}\wedge\bar{\tilde{F}} \leq \frac{2\pi}{c_{\beta}^2(z_0)},$$

then the holomorphic $(1,0)$-form $\frac{F+\tilde{F}}{2}$ on Ω satisfies $\frac{F+\tilde{F}}{2}|_{z_0} = dw|_{z_0}$. According to the proof of Lemma 4.31, it follows that

$$\sqrt{-1}\int_{\Omega}\frac{F+\tilde{F}}{2}\wedge\overline{\frac{F+\tilde{F}}{2}} < \frac{2\pi}{c_{\beta}^2(z_0)}.$$

Therefore,

$$\pi B_{\Omega}(z_0) > c_{\beta}^2(z_0),$$

by the extremal property of the Bergman kernel.

Remark 4.33. Let Ω be an open Riemann surface with a Green function G_{Ω}, and let z_0 be a point of Ω with a fixed local coordinate w on the neighborhood V_{z_0} of z_0, such that $w(z_0)=0$. Let $c_0(t)=1$, $\Psi = 2G_{\Omega}(\cdot, z_0)$, $h=\rho$. By Theorem 2.2 and $c_{\beta} := \exp\lim_{z\to z_0}(G_{\Omega}(z,z_0)-\log|w(z)|)$, there is a holomorphic $(1,0)$-form F on Ω, which satisfies $F|_{z_0} = dw|_{z_0}$, such that

$$\sqrt{-1}\int_{\Omega}\rho F\wedge\bar{F} \leq \pi\int_{z_0}\rho(z_0)|dw|^2 dV_{\Omega}[2G_{\Omega}(z,z_0)] = \frac{2\pi\rho(z_0)}{c_{\beta}^2(z_0)}.$$

Therefore,

$$\pi\rho(z_0)B_{\Omega,\rho}(z_0) \geq c_{\beta}^2(z_0),$$

by the extremal property of the Bergman kernel.

If there is another holomorphic $(1,0)$-form $\tilde{F}$ on Ω, which satisfies $\tilde{F}|_{z_0} = dw|_{z_0}$, and

$$\sqrt{-1}\int_{\Omega}\rho\tilde{F}\wedge\bar{\tilde{F}} \leq \frac{2\pi\rho(z_0)}{c_{\beta}^2(z_0)},$$

then $(1,0)$-form $\frac{F+\tilde{F}}{2}$ on Ω, which satisfies $\frac{F+\tilde{F}}{2}|_{z_0} = dw|_{z_0}$. From the proof of Lemma 4.31, it follows that

$$\sqrt{-1}\int_{\Omega}\rho\frac{F+\tilde{F}}{2}\wedge\overline{\frac{F+\tilde{F}}{2}} < \frac{2\pi\rho(z_0)}{c_{\beta}^2(z_0)}.$$

Therefore,

$$\pi\rho(z_0)B_{\Omega\rho}(z_0) > c_{\beta}^2(z_0),$$

by the extremal property of the Bergman kernel.

We now show a lemma which will be used to discuss the uniform bound of a sequence of holomorphic functions:

LEMMA 4.34. *Let φ be a plurisubharmonic function on $\Omega \subset\subset \mathbb{C}^n$, which is not identically $-\infty$. Let $\{f_n\}_{n=1,2,\ldots}$ be a sequence of holomorphic functions on Ω, such that $\int_\Omega |f_n|^2 e^{\varphi} < C$, where C is a positive constant which is independent of n. Then the sequence $\{f_n\}_{n=1,2,\ldots}$ has a uniform bound on any compact subset of Ω.*

Proof. Let K be a compact subset of Ω, such that $0 < 2r < \mathrm{dist}(K, \partial\Omega)$. Let $\Omega_0 := \{z|\mathrm{dist}(z, K) < r\}$. As φ is plurisubharmonic, then there is $N > 0$, such that $\int_\Omega e^{-\frac{\varphi}{N}} dV_\Omega < C_0 < +\infty$.

Note that

$$\left(\int_{\Omega_0} |f_n|^2 e^{\varphi} dV_\Omega\right)^{\frac{1}{N+1}} \left(\int_{\Omega_0} e^{-\frac{\varphi}{N}} dV_\Omega\right)^{\frac{N}{N+1}} \geq \int_{\Omega_0} |f_n|^{\frac{2}{N+1}} dV_\Omega \geq \frac{\pi^n r^{2n}}{n!} |f_n(w)|^{\frac{2}{N+1}}, \tag{4.18}$$

where $w \in K$. Then the lemma follows. $\square$

5. Proofs of the main theorems

In this section, we give proofs of the main theorems.

5.1. *Proof of Theorem* 2.1. By Remark 4.7, it suffices to prove the case that M is a Stein manifold. By Lemma 4.6 and Remark 4.9, it suffices to prove the case that c_A is smooth on $(A, +\infty)$ and continuous on $[A, +\infty]$, such that $\lim_{t\to+\infty} c_A(t) > 0$. Since M is a Stein manifold, there is a sequence of Stein manifolds $\{D_m\}_{m=1}^{\infty}$ satisfying $D_m \subset\subset D_{m+1}$ for all m and $\bigcup_{m=1}^{\infty} D_m = M$. It is known that all $D_m \setminus S$ are complete Kähler ([22]). Since M is Stein, there is a holomorphic section $\tilde{F}$ of K_M on M such that $\tilde{F}|_S = f$.

Let ds^2_M be a Kähler metric on M, and let dV_M be the volume form with respect to ds^2_M. Let $\{v_{t_0,\varepsilon}\}_{t_0\in\mathbb{R},\varepsilon\in(0,\frac{1}{4})}$ be a family of smooth increasing convex functions on $\mathbb{R}$, such that

(1) $v_{t_0,\varepsilon}(t) = t$, for $t \geq -t_0 - \varepsilon$; $v_{t_0,\varepsilon}(t)$ is a constant depending on t_0 and ε, for $t < -t_0 - 1 + \varepsilon$;
(2) the sequence $v''_{t_0,\varepsilon}(t)$ is pointwise convergent to $\mathbb{I}_{\{-t_0-1<t<-t_0\}}$ when $\varepsilon \to 0$, and $0 \leq v''_{t_0,\varepsilon}(t) \leq 2$ for any $t \in \mathbb{R}$;
(3) the sequence $v_{t_0,\varepsilon}(t)$ is C^1 convergent to $b_{t_0}(t)$ —

$$b_{t_0}(t) := \int_{-\infty}^{t} (\int_{-\infty}^{t_2} \mathbb{I}_{\{-t_0-1<t_1<-t_0\}} dt_1) dt_2 - \int_{-\infty}^{0} (\int_{-\infty}^{t_2} \mathbb{I}_{\{-t_0-1<t_1<-t_0\}} dt_1) dt_2$$

is also a C^1 function on $\mathbb{R}$ — when $\varepsilon \to 0$, and $0 \leq v'_{t_0,\varepsilon}(t) \leq 1$ for any $t \in \mathbb{R}$.

We can construct the family $\{v_{t_0,\varepsilon}\}_{t_0\in\mathbb{R},\varepsilon\in(0,\frac{1}{4})}$ by setting

$$
\begin{aligned}
v_{t_0,\varepsilon}(t) :=& \int_{-\infty}^{t}\int_{-\infty}^{t_1}\frac{1}{1-2\varepsilon}\mathbb{I}_{\{-t_0-1+\varepsilon<s<-t_0-\varepsilon\}} * \rho_{\frac{1}{4}\varepsilon} ds dt_1 \\
&- \int_{-\infty}^{0}\int_{-\infty}^{t_1}\frac{1}{1-2\varepsilon}\mathbb{I}_{\{-t_0-1+\varepsilon<s<-t_0-\varepsilon\}} * \rho_{\frac{1}{4}\varepsilon} ds dt_1,
\end{aligned}
\tag{5.1}
$$

where $\rho_{\frac{1}{4}\varepsilon}$ is the kernel of convolution satisfying $\mathrm{supp}(\rho_{\frac{1}{4}\varepsilon}) \subset (-\frac{1}{4}\varepsilon, \frac{1}{4}\varepsilon)$. Then we have

$$
v'_{t_0,\varepsilon}(t) = \int_{-\infty}^{t}\frac{1}{1-2\varepsilon}\mathbb{I}_{\{-t_0-1+\varepsilon<s<-t_0-\varepsilon\}} * \rho_{\frac{1}{4}\varepsilon} ds
$$

and

$$
v''_{t_0,\varepsilon}(t) = \frac{1}{1-2\varepsilon}\mathbb{I}_{\{-t_0-1+\varepsilon<t<-t_0-\varepsilon\}} * \rho_{\frac{1}{4}\varepsilon}.
$$

Let s and u be two undetermined real functions which will be determined later on after doing calculations based on Lemmas 4.1 and 4.2. Let $\eta = s(-v_{t_0,\varepsilon}\circ\Psi)$ and $\phi = u(-v_{t_0,\varepsilon}\circ\Psi)$, where $s \in C^{\infty}((-A,+\infty))$ satisfying $s \geq \frac{1}{\delta}$ and $u \in C^{\infty}((-A,+\infty))$ satisfying $\lim_{t\to+\infty} u(t)$ exists (which will be determined to be $= -\log(\frac{1}{\delta}c_A(-A)e^{A} + \int_{-A}^{\infty} c_A(t)e^{-t}dt)$). Let $\tilde{h} = he^{-\Psi-\phi}$.

Now let $\alpha \in \mathcal{D}(X, \Lambda^{n,1}T^*_{D_m\setminus S}\otimes E)$ be an E-valued smooth $(n,1)$-form with compact support on $D_m \setminus S$. Using Lemmas 4.1 and 4.2 and the assumption $\sqrt{-1}\Theta_{he^{-\Psi}} \geq 0$ on $D_m \setminus S$, we get

$$
\begin{aligned}
&\|(\eta+g^{-1})^{\frac{1}{2}}D''^{*}\alpha\|^2_{D_m\setminus S,\tilde{h}} + \|\eta^{\frac{1}{2}}D''\alpha\|^2_{D_m\setminus S,\tilde{h}} \\
&\geq \left\langle\left\langle[\eta\sqrt{-1}\Theta_{\tilde{h}} - \sqrt{-1}\partial\bar{\partial}\eta - \sqrt{-1}g\partial\eta\wedge\bar{\partial}\eta, \Lambda_{\omega}]\alpha,\alpha\right\rangle\right\rangle_{D_m\setminus S,\tilde{h}} \\
&= \left\langle\left\langle[\eta\sqrt{-1}\partial\bar{\partial}\phi + \eta\sqrt{-1}\Theta_{he^{-\Psi}} \right.\right. \\
&\qquad \left.\left. -\sqrt{-1}\partial\bar{\partial}\eta - \sqrt{-1}g\partial\eta\wedge\bar{\partial}\eta, \Lambda_{\omega}]\alpha,\alpha\right\rangle\right\rangle_{D_m\setminus S,\tilde{h}},
\end{aligned}
\tag{5.2}
$$

where g is a positive continuous function on $D_m \setminus S$.

We need some calculations to determine g. We have

$$
\partial\bar{\partial}\eta = -s'(-v_{t_0,\varepsilon}\circ\Psi)\partial\bar{\partial}(v_{t_0,\varepsilon}\circ\Psi) + s''(-v_{t_0,\varepsilon}\circ\Psi)\partial(v_{t_0,\varepsilon}\circ\Psi)\wedge\bar{\partial}(v_{t_0,\varepsilon}\circ\Psi)
\tag{5.3}
$$

and

$$
\partial\bar{\partial}\phi = -u'(-v_{t_0,\varepsilon}\circ\Psi)\partial\bar{\partial}v_{t_0,\varepsilon}\circ\Psi + u''(-v_{t_0,\varepsilon}\circ\Psi)\partial(v_{t_0,\varepsilon}\circ\Psi)\wedge\bar{\partial}(v_{t_0,\varepsilon}\circ\Psi).
\tag{5.4}
$$

Therefore,

$$
\begin{aligned}
&\eta\sqrt{-1}\partial\bar{\partial}\phi-\sqrt{-1}\partial\bar{\partial}\eta-\sqrt{-1}g\partial\eta\wedge\bar{\partial}\eta\\
&\quad=(s'-su')\sqrt{-1}\partial\bar{\partial}(v_{t_0,\varepsilon}\circ\Psi)\\
&\quad\quad+((u''s-s'')-gs'^2)\sqrt{-1}\partial(v_{t_0,\varepsilon}\circ\Psi)\wedge\bar{\partial}(v_{t_0,\varepsilon}\circ\Psi)\\
&\quad=(s'-su')((v'_{t_0,\varepsilon}\circ\Psi)\sqrt{-1}\partial\bar{\partial}\Psi+(v''_{t_0,\varepsilon}\circ\Psi)\sqrt{-1}\partial(\Psi)\wedge\bar{\partial}(\Psi))\\
&\quad\quad+((u''s-s'')-gs'^2)\sqrt{-1}\partial(v_{t_0,\varepsilon}\circ\Psi)\wedge\bar{\partial}(v_{t_0,\varepsilon}\circ\Psi).
\end{aligned}\tag{5.5}
$$

We omit the composite item $(-v_{t_0,\varepsilon}\circ\Psi)$ after $s'-su'$ and $(u''s-s'')-gs'^2$ in the above equalities.

It is natural to ask $u''s-s''>0$. Let $g=\frac{u''s-s''}{s'^2}\circ(-v_{t_0,\varepsilon}\circ\Psi)$. We have $\eta+g^{-1}=(s+\frac{s'^2}{u''s-s''})\circ(-v_{t_0,\varepsilon}\circ\Psi)$. Since $\sqrt{-1}\Theta_{he^{-\Psi}}\geq 0$, $a(-\Psi)\sqrt{-1}\Theta_{he^{-\Psi}}+\sqrt{-1}\partial\bar{\partial}\Psi\geq 0$ on $M\setminus S$, and $0\leq v'_{t_0,\varepsilon}\circ\Psi\leq 1$, we have

$$
\eta(1-v'_{t_0,\varepsilon}\circ\Psi)\sqrt{-1}\Theta_{he^{-\Psi}}+(v'_{t_0,\varepsilon}\circ\Psi)(\eta\sqrt{-1}\Theta_{he^{-\Psi}}+\sqrt{-1}\partial\bar{\partial}\Psi)\geq 0\tag{5.6}
$$

on $M\setminus S$ for t_0 big enough, which means that

$$
\eta\sqrt{-1}\Theta_{he^{-\Psi}}+(v'_{t_0,\varepsilon}\circ\Psi)\sqrt{-1}\partial\bar{\partial}\Psi\geq 0\tag{5.7}
$$

on $M\setminus S$.

From equality (5.5), in order to obtain the L^2 estimate, it is natural to let $s'-su'>0$; since to find s and u is an extremal problem, it is natural to let $s'-su'$ be a constant; by the boundary condition, the constant should be 1.

Using the inequality $v'_{t_0,\varepsilon}\geq 0$, Lemma 4.2, equality (5.5), and inequalities (5.2) and (5.7), one has

$$
\begin{aligned}
\langle B\alpha,\alpha\rangle_{\tilde{h}}&=\langle[\eta\sqrt{-1}\Theta_{\tilde{h}}-\sqrt{-1}\partial\bar{\partial}\eta-\sqrt{-1}g\partial\eta\wedge\bar{\partial}\eta,\Lambda_\omega]\alpha,\alpha\rangle_{\tilde{h}}\\
&\geq\langle[(v''_{t_0,\varepsilon}\circ\Psi)\sqrt{-1}\partial\Psi\wedge\bar{\partial}\Psi,\Lambda_\omega]\alpha,\alpha\rangle_{\tilde{h}}\\
&=\langle(v''_{t_0,\varepsilon}\circ\Psi)\bar{\partial}\Psi\wedge(\alpha\llcorner(\bar{\partial}\Psi)^\sharp),\alpha\rangle_{\tilde{h}}.
\end{aligned}\tag{5.8}
$$

Using the definition of contraction, the Cauchy-Schwarz inequality and inequality (5.8), we have

$$
\begin{aligned}
|\langle(v''_{t_0,\varepsilon}\circ\Psi)\bar{\partial}\Psi\wedge\gamma,\tilde{\alpha}\rangle_{\tilde{h}}|^2&=|\langle(v''_{t_0,\varepsilon}\circ\Psi)\gamma,\tilde{\alpha}\llcorner(\bar{\partial}\Psi)^\sharp\rangle_{\tilde{h}}|^2\\
&\leq\langle(v''_{t_0,\varepsilon}\circ\Psi)\gamma,\gamma\rangle_{\tilde{h}}(v''_{t_0,\varepsilon}\circ\Psi)|\tilde{\alpha}\llcorner(\bar{\partial}\Psi)^\sharp|^2_{\tilde{h}}\\
&=\langle(v''_{t_0,\varepsilon}\circ\Psi)\gamma,\gamma\rangle_{\tilde{h}}\langle(v''_{t_0,\varepsilon}\circ\Psi)\bar{\partial}\Psi\wedge(\tilde{\alpha}\llcorner(\bar{\partial}\Psi)^\sharp),\tilde{\alpha}\rangle_{\tilde{h}}\\
&\leq\langle(v''_{t_0,\varepsilon}\circ\Psi)\gamma,\gamma\rangle_{\tilde{h}}\langle B\tilde{\alpha},\tilde{\alpha}\rangle_{\tilde{h}}
\end{aligned}\tag{5.9}
$$

for any $(n,0)$-form γ and $(n,1)$-form $\tilde{\alpha}$.

Take $\lambda=\bar{\partial}[(1-v'_{t_0,\varepsilon}(\Psi))\tilde{F}]$, $\gamma=\tilde{F}$ and $\tilde{\alpha}=B^{-1}\bar{\partial}\Psi\wedge\tilde{F}$. It follows that

$$
\langle B^{-1}\lambda,\lambda\rangle_{\tilde{h}}\leq(v''_{t_0,\varepsilon}\circ\Psi)|\tilde{F}|^2_{\tilde{h}},
$$

and therefore

$$\int_{D_m\backslash S}\langle B^{-1}\lambda,\lambda\rangle_{\tilde{h}}dV_M \leq \int_{D_m\backslash S}(v''_{t_0,\varepsilon}\circ\Psi)|\tilde{F}|^2_{\tilde{h}}dV_M.$$

By Lemma 4.3, there exists an $(n,0)$-form $\gamma_{m,t_0,\varepsilon}$ with value in E on $D_m\backslash S$ satisfying

$$\bar{\partial}\gamma_{m,t_0,\varepsilon}=\lambda,$$

and

$$\int_{D_m\backslash S}|\gamma_{m,t_0,\varepsilon}|^2_{\tilde{h}}(\eta+g^{-1})^{-1}dV_M \leq \int_{D_m\backslash S}(v''_{t_0,\varepsilon}\circ\Psi)|\tilde{F}|^2_{\tilde{h}}dV_M. \tag{5.10}$$

Let $\mu_1=e^{v_{t_0,\varepsilon}\circ\Psi}$, $\mu=\mu_1 c_A(-v_{t_0,\varepsilon}\circ\Psi)e^{\phi}$. It is natural to ask η and ϕ to satisfy $\mu\leq \mathbf{C}(\eta+g^{-1})^{-1}$, which will be discussed at the end of this subsection, where $\mathbf{C}$ is just the constant in Theorem 2.1.

As $v_{t_0,\varepsilon}(\Psi)\geq\Psi$, we have

(5.11)
$$\int_{D_m\backslash S}|\gamma_{m,t_0,\varepsilon}|^2_{h}c_A(-v_{t_0,\varepsilon}\circ\Psi)dV_M \leq \int_{D_m\backslash S}|\gamma_{m,t_0,\varepsilon}|^2_{\tilde{h}}c_A(-v_{t_0,\varepsilon}\circ\Psi)\mu_1 e^{\phi}dV_M.$$

From inequalities (5.10) and (5.11), it follows that

$$\int_{D_m\backslash S}|\gamma_{m,t_0\varepsilon}|^2_{h}c_A(-v_{t_0,\varepsilon}\circ\Psi)dV_M \leq \mathbf{C}\int_{D_m\backslash S}(v''_{t_0,\varepsilon}\circ\Psi)|\tilde{F}|^2_{\tilde{h}}dV_M,$$

under the assumption $\mu\leq\mathbf{C}(\eta+g^{-1})^{-1}$.

For any given t_0, there exists a neighborhood U_0 of $\{\Psi=-\infty\}\cap\overline{D_m}$ on M, such that for any ε, $v''_{t_0,\varepsilon}\circ\Psi|_{U_0}=0$. Therefore, $\bar{\partial}\gamma_{m,t_0,\varepsilon}|_{U_0\backslash S}=0$. As Ψ is upper-semicontinuous and ϕ is bounded on D_m, it is easy to see that $\gamma_{m,t_0,\varepsilon}$ is locally L^2 integrable along S. Then $\gamma_{m,t_0,\varepsilon}$ can be extended to U_0 as a holomorphic function, which is denoted by $\tilde{\gamma}_{m,t_0,\varepsilon}$.

It follows from $\Psi\in\#(S)$ that $e^{-\Psi}$ is disintegrable near S. Then $\tilde{\gamma}_{m,t_0,\varepsilon}$ satisfies

$$\tilde{\gamma}_{m,t_0,\varepsilon}|_S=0,$$

and

$$\int_{D_m}|\tilde{\gamma}_{m,t_0,\varepsilon}|^2_{h}c_A(-v_{t_0,\varepsilon}\circ\Psi)dV_M \leq \frac{\mathbf{C}}{e^{A_{t_0}}}\int_{D_m}(v''_{t_0,\varepsilon}\circ\Psi)|\tilde{F}|^2_{he^{-\Psi}}dV_M, \tag{5.12}$$

where $A_{t_0}:=\inf_{t\geq t_0}\{u(t)\}$.

As

$$\lim_{t\to+\infty}u(t)=-\log\left(\frac{1}{\delta}c_A(-A)e^{A}+\int_{-A}^{+\infty}c_A(t)e^{-t}dt\right),$$

it is easy to see that

$$\lim_{t_0\to\infty}\frac{1}{e^{A_{t_0}}}=\frac{1}{\delta}c_A(-A)e^{A}+\int_{-A}^{+\infty}c_A(t)e^{-t}dt.$$

Let $F_{m,t_0,\varepsilon} := (1 - v'_{t_0,\varepsilon} \circ \Psi)\widetilde{F} - \tilde{\gamma}_{m,t_0,\varepsilon}$. By $\tilde{\gamma}_{m,t_0,\varepsilon}|_S = 0$, then $F_{m,t_0,\varepsilon}$ is a holomorphic $(n,0)$-form with value in E on D_m satisfying

$$F_{m,t_0,\varepsilon}|_S = \tilde{F}|_S,$$

and inequality (5.12) can be reformulated as follows:

$$\begin{aligned}&\int_{D_m} |F_{m,t_0,\varepsilon} - (1 - v'_{t_0,\varepsilon} \circ \Psi)\tilde{F}|^2_h c_A(-v_{t_0,\varepsilon} \circ \Psi) dV_M \\ \le&\frac{\mathbf{C}}{e^{A_{t_0}}} \int_{D_m} (v''_{t_0,\varepsilon} \circ \Psi)|\tilde{F}|^2_{he^{-\Psi}} dV_M.\end{aligned} \tag{5.13}$$

Given t_0 and D_m, it is easy to check that $(v''_{t_0,\varepsilon} \circ \Psi)|\tilde{F}|^2_{he^{-\Psi}}$ has a uniform bound on D_m independent of ε. Then

$$\int_{D_m} |(1 - v'_{t_0,\varepsilon} \circ \Psi)\tilde{F}|^2_h c_A(-v_{t_0,\varepsilon} \circ \Psi) dV_M,$$

and

$$\int_{D_m} v''_{t_0,\varepsilon} \circ \Psi |\tilde{F}|^2_{he^{-\Psi}} dV_M$$

has a uniform bound independent of ε for any given t_0 and D_m.

Using $\bar{\partial} F_{m,t_0,\varepsilon} = 0$ and Lemma 4.5, we can choose a subsequence of $\{F_{m,t_0,\varepsilon}\}_\varepsilon$, such that the chosen sequence is uniformly convergent on any compact subset of D_m, still denoted by $\{F_{m,t_0,\varepsilon}\}_\varepsilon$ without ambiguity.

For any compact subset K on D_m, it is easy to check that $F_{m,t_0,\varepsilon}$, $(1 - v'_{t_0,\varepsilon} \circ \Psi)\tilde{F}$ and $(v''_{t_0,\varepsilon} \circ \Psi)|\tilde{F}|^2_{he^{-\Psi}}$ have uniform bounds on K independent of ε.

By the dominated convergence theorem on any compact subset K of D_m and inequality (5.13), it follows that

$$\begin{aligned}&\int_K |F_{m,t_0} - (1 - b'_{t_0}(\Psi))\tilde{F}|^2_h c_A(-b_{t_0}(\Psi)) dV_M \\ &\qquad \le \frac{\mathbf{C}}{e^{A_{t_0}}} \int_{D_m} (\mathbb{I}_{\{-t_0-1<t<-t_0\}} \circ \Psi)|\tilde{F}|^2_{he^{-\Psi}} dV_M,\end{aligned} \tag{5.14}$$

which implies that

$$\begin{aligned}&\int_{D_m} |F_{m,t_0} - (1 - b'_{t_0}(\Psi))\tilde{F}|^2_h c_A(-b_{t_0}(\Psi)) dV_M \\ &\qquad \le \frac{\mathbf{C}}{e^{A_{t_0}}} \int_{D_m} (\mathbb{I}_{\{-t_0-1<t<-t_0\}} \circ \Psi)|\tilde{F}|^2_{he^{-\Psi}} dV_M.\end{aligned} \tag{5.15}$$

From the definition of $dV_M[\Psi]$ and the inequality $\sum_{k=1}^{n}\frac{\pi^k}{k!}\int_{S_{n-k}}|f|^2_h dV_M[\Psi] < \infty$, it follows that

$$\begin{aligned}
&\limsup_{t_0\to+\infty}\int_{D_m}(\mathbb{I}_{\{-t_0-1<t<-t_0\}}\circ\Psi_v)|\tilde{F}|^2_{he^{-\Psi}}dV_M \\
&\leq \limsup_{t_0\to+\infty}\int_M \mathbb{I}_{\overline{D}_m}(\mathbb{I}_{\{-t_0-1<t<-t_0\}}\circ\Psi)|\tilde{F}|^2_{he^{-\Psi}}dV_M \\
&\leq \sum_{k=1}^{n}\frac{\pi^k}{k!}\int_{S_{n-k}}\mathbb{I}_{\overline{D}_m}|f|^2_h dV_M[\Psi] \leq \sum_{k=1}^{n}\frac{\pi^k}{k!}\int_{S_{n-k}}|f|^2_h dV_M[\Psi] < \infty.
\end{aligned}\tag{5.16}$$

Then $\int_{D_m}(\mathbb{I}_{\{-t_0-1<t<-t_0\}}\circ\Psi)|\tilde{F}|^2_{he^{-\Psi}}dV_M$ has a uniform bound independent of t_0 for any given D_m, and

$$\begin{aligned}
&\limsup_{t_0\to+\infty}\int_{D_m}(\mathbb{I}_{\{-t_0-1<t<-t_0\}}\circ\Psi)|\tilde{F}|^2_{he^{-\Psi}}dV_M \\
&\qquad\leq \sum_{k=1}^{n}\frac{\pi^k}{k!}\int_{S_{n-k}}|f|^2_h dV_M[\Psi] < \infty.
\end{aligned}\tag{5.17}$$

It is clear that

$$\int_{D_m}|F_{m,t_0}-(1-b'_{t_0}(\Psi))\tilde{F}|^2_h c_A(-b_{t_0}(\Psi))dV_M$$

has a uniform bound independent of t_0 for any given D_m. Using the fact that

$$\int_{D_m}|(1-b'_{t_0}(\Psi))\tilde{F}|^2_h c_A(-b_{t_0}(\Psi))dV_M$$

has a uniform bound independent of t_0, inequality (5.15), and the following inequality,

$$\begin{aligned}
&\left(\int_{D_m}|F_{m,t_0}-(1-b'_{t_0}(\Psi))\tilde{F}|^2_h c_A(-b_{t_0}(\Psi))dV_M\right)^{\frac{1}{2}} \\
&\qquad+\left(\int_{D_m}|(1-b'_{t_0}(\Psi))\tilde{F}|^2_h c_A(-b_{t_0}(\Psi))dV_M\right)^{\frac{1}{2}} \\
&\geq\left(\int_{D_m}|F_{m,t_0}|^2_h c_A(-b_{t_0}(\Psi))dV_M\right)^{\frac{1}{2}},
\end{aligned}\tag{5.18}$$

we obtain that $\int_{D_m}|F_{m,t_0}|^2_h c_A(-b_{t_0}(\Psi))dV_M$ has a uniform bound independent of t_0.

Using $\bar{\partial}F_{m,t_0}=0$ and Lemma 4.5, we can choose a subsequence of $\{F_{m,t_0}\}_{t_0}$, such that the chosen subsequence is uniformly convergent on any compact subset of D_m, still denoted by $\{F_{m,t_0}\}_{t_0}$ without ambiguity.

For any compact subset K on D_m, it is clear that both F_{m,t_0} and $(1-b'_{t_0}\circ\Psi)\tilde{F}$ have uniform bounds on K independent of t_0.

By inequality (5.15), inequality (5.17), the equality

$$\lim_{t_0\to\infty}\frac{1}{e^{A_{t_0}}}=\frac{1}{\delta}c_A(-A)e^A+\int_{-A}^{+\infty}c_A(t)e^{-t}dt,$$

and the dominated convergence theorem on any compact subset K of D_m, it follows that

$$\begin{aligned}\int_{D_m}&\mathbb{I}_K|F_m|^2_h c_A(-\Psi)dV_M\\ &\leq \mathbf{C}\left(\frac{1}{\delta}c_A(-A)e^A+\int_{-A}^{+\infty}c_A(t)e^{-t}dt\right)\sum_{k=1}^{n}\frac{\pi^k}{k!}\int_{S_{n-k}}|f|^2_h dV_M[\Psi],\end{aligned}\tag{5.19}$$

which implies that

$$\begin{aligned}\int_{D_m}&|F_m|^2_h c_A(-\Psi)dV_M\\ &\leq \mathbf{C}\left(\frac{1}{\delta}c_A(-A)e^A+\int_{-A}^{+\infty}c_A(t)e^{-t}dt\right)\sum_{k=1}^{n}\frac{\pi^k}{k!}\int_{S_{n-k}}|f|^2_h dV_M[\Psi],\end{aligned}\tag{5.20}$$

where the Lebesgue measure of $\{\Psi=-\infty\}$ is zero.

Define $F_m=0$ on $M\backslash D_m$. Then the weak limit of some weakly convergent subsequence of $\{F_m\}_{m=1}^{\infty}$ gives a holomorphic section F of $K_M\otimes E$ on M satisfying $F|_S=\tilde{F}|_S$, and

$$\begin{aligned}\int_{M}&|F|^2_h c_A(-\Psi)dV_M\\ &\leq \mathbf{C}\left(\frac{1}{\delta}c_A(-A)+\int_{-A}^{+\infty}c_A(t)e^{-t}dt\right)\sum_{k=1}^{n}\frac{\pi^k}{k!}\int_{S_{n-k}}|f|^2_h dV_M[\Psi].\end{aligned}$$

To finish the proof of Theorem 2.1, it suffices to determine η and ϕ such that $(\eta+g^{-1})\leq \mathbf{C}c_A^{-1}(-v_{t_0,\varepsilon}\circ\Psi)e^{-v_{t_0,\varepsilon}\circ\Psi}e^{-\phi}=\mathbf{C}\mu^{-1}$ on D_m. Recall that $\eta=s(-v_{t_0,\varepsilon}\circ\Psi)$ and $\phi=u(-v_{t_0,\varepsilon}\circ\Psi)$. So we have $(\eta+g^{-1})e^{v_{t_0,\varepsilon}\circ\Psi}e^{\phi}=(s+\frac{s'^2}{u''s-s''})e^{-t}e^{u}\circ(-v_{t_0,\varepsilon}\circ\Psi)$.

Summarizing the above discussion about s and u, we are naturally led to a system of ODEs:

$$\begin{aligned}&(1)\qquad \left(s+\frac{s'^2}{u''s-s''}\right)e^{u-t}=\frac{\mathbf{C}}{c_A(t)},\\ &(2)\qquad s'-su'=1,\end{aligned}\tag{5.21}$$

where $t\in(-A,+\infty)$ and $\mathbf{C}=1$; $s\in C^{\infty}((-A,+\infty))$ satisfies $s\geq\frac{1}{\delta}$ and $u\in C^{\infty}((-A,+\infty))$ satisfies $\lim_{t\to+\infty}u(t)=-\log(\frac{1}{\delta}c_A(-A)e^A+\int_{-A}^{\infty}c_A(t)e^{-t}dt)$ such that $u''s-s''>0$.

We solve the above system of ODEs in Section 5.4 and get the solution of the system of ODEs (5.21):

$$\begin{aligned}
&(1) \qquad u = -\log\left(\frac{1}{\delta}c_A(-A)e^A + \int_{-A}^{t} c_A(t_1)e^{-t_1}dt_1\right),\\
(5.22)\quad &\\
&(2) \qquad s = \frac{\int_{-A}^{t}(\frac{1}{\delta}c_A(-A)e^A + \int_{-A}^{t_2} c_A(t_1)e^{-t_1}dt_1)dt_2 + \frac{1}{\delta^2}c_A(-A)e^A}{\frac{1}{\delta}c_A(-A)e^A + \int_{-A}^{t} c_A(t_1)e^{-t_1}dt_1}.
\end{aligned}$$

One can check that $s \in C^\infty((-A,+\infty))$, and $u \in C^\infty((-A,+\infty))$ with $\lim_{t\to+\infty} u(t) = -\log(\frac{1}{\delta}c_A(-A)e^A + \int_{-A}^{+\infty} c_A(t_1)e^{-t_1}dt_1)$.

It follows from $su'' - s'' = -s'u'$ and $u' < 0$ that $u''s - s'' > 0$ is equivalent to $s' > 0$. It is easy to see that the inequality (2.1) is just $s' > 0$. Therefore, $u''s - s'' > 0$. In conclusion, we have proved Theorem 2.1 with the constant $\mathbf{C} = 1$.

Remark 5.1. Both $\mathbf{C}$ and the power of δ in Theorems 2.1 and 5.2 are optimal on the ball $\mathbb{B}^m(0, e^{\frac{A}{2m}})$ with trivial holomorphic line bundle when $S = \{0\}$.

5.2. *A singular metric version of Theorem* 2.1. In this subsection, we formulate and prove the following singular metric version of Theorem 2.1:

THEOREM 5.2. *Let* (M,S) *satisfy condition* (ab), *and let h be a singular metric on a holomorphic line bundle L on M, which is locally integrable on M. Then, for any function Ψ on M such that $\Psi \in \Delta_{A,h,\delta}(S)$, there exists a uniform constant $\mathbf{C} = 1$, which is optimal, such that, for any holomorphic section f of $K_M \otimes L|_S$ on S satisfying the L^2 integrable condition* (2.2), *there exists a holomorphic section F of $K_M \otimes L$ on M satisfying $F = f$ on S and the optimal estimate* (2.3).

Proof. By Remark 4.7, it suffices to prove the case that M is a Stein manifold. By Lemma 4.6 and Lemma 4.8, it suffices to prove the case that c_A is smooth on $(A,+\infty)$ and continuous on $(A,+\infty]$, such that $\lim_{t\to+\infty} c_A(t) > 0$.

Since M is a Stein manifold, we can find a sequence of Stein manifolds $\{D_m\}_{m=1}^{\infty}$ satisfying $D_m \subset\subset D_{m+1}$ for all m and $\bigcup_{m=1}^{\infty} D_m = M$.

As $\varphi + \Psi$ and $\varphi + (1+\delta)\Psi$ are plurisubharmonic functions on M, then by Lemma 4.13, we have smooth functions φ_k and Ψ_k on M, such that $\varphi_k + \Psi_k$ and $\varphi_k + (1+\delta)\Psi_k$ are plurisubharmonic functions on M, which are deceasing convergent to $\varphi + \Psi$ and $\varphi + (1+\delta)\Psi$ respectively.

Since M is Stein, there is a holomorphic section $\tilde{F}$ of K_M on M such that $\tilde{F}|_S = f$. Let ds_M^2 be a Kähler metric on M and dV_M be the volume form with respect to ds_M^2. Let $\{v_{t_0,\varepsilon}\}_{t_0\in\mathbb{R},\varepsilon\in(0,\frac{1}{4})}$ be a family of smooth increasing convex functions on $\mathbb{R}$, such that

(1) $v_{t_0,\varepsilon}(t)=t$ for $t\geq -t_0-\varepsilon$, and $v_{t_0,\varepsilon}(t)$ is a constant for $t<-t_0-1+\varepsilon$ depending on t_0,ε;
(2) $v''_{t_0,\varepsilon}(t)$ is pointwise convergent to $\mathbb{I}_{\{-t_0-1<t<-t_0\}}$ when $\varepsilon\to 0$, and $0\leq v''_{t_0,\varepsilon}(t)\leq 2$ for any $t\in\mathbb{R}$;
(3) $v_{t_0,\varepsilon}(t)$ is C^1 convergent to $b_{t_0}(t)$ (and $b_{t_0}(t):=\int_{-\infty}^{t}(\int_{-\infty}^{t_2}\mathbb{I}_{\{-t_0-1<t_1<-t_0\}}dt_1)\, dt_2-\int_{-\infty}^{0}\left(\int_{-\infty}^{t_2}\mathbb{I}_{\{-t_0-1<t_1<-t_0\}}dt_1\right)dt_2$ is also a C^1 function on $\mathbb{R}$) when $\varepsilon\to 0$, and $0\leq v'_{t_0,\varepsilon}(t)\leq 1$ for any $t\in\mathbb{R}$.

As before, let $\eta=s(-v_{t_0,\varepsilon}\circ\Psi_k)$ and $\phi=u(-v_{t_0,\varepsilon}\circ\Psi_k)$, where $s\in C^\infty((-A,+\infty))$ satisfies $s\geq\frac{1}{\delta}$, and $u\in C^\infty((-A,+\infty))\cap C^\infty([-A,+\infty))$ satisfies $\lim_{t\to+\infty}u(t)=-\log(\frac{1}{\delta}c_A(-A)e^A+\int_{-A}^{\infty}c_A(t)e^{-t}dt)$, such that $u''s-s''>0$ and $s'-u's=1$. Let $\tilde{h}=e^{-\varphi_k-\Psi_k-\phi}$.

Now let $\alpha\in\mathcal{D}(X,\Lambda^{n,1}T^*_{D_m})$ be a smooth $(n,1)$-form with compact support on D_m. Using Lemmas 4.1 and 4.2, the inequality $s\geq\frac{1}{\delta}$ and the fact that $\varphi_k+\Psi_k$ is plurisubharmonic on D_m, we get

$$
\begin{aligned}
&\|(\eta+g^{-1})^{\frac{1}{2}}D''^{*}\alpha\|^2_{D_m,\tilde{h}}+\|\eta^{\frac{1}{2}}D''\alpha\|^2_{D_m,\tilde{h}}\\
&\quad\geq\left\langle\left\langle[\eta\sqrt{-1}\Theta_{\tilde{h}}-\sqrt{-1}\partial\bar{\partial}\eta-\sqrt{-1}g\partial\eta\wedge\bar{\partial}\eta,\Lambda_\omega]\alpha,\alpha\right\rangle\right\rangle_{D_m,\tilde{h}}\\
&\quad\geq\left\langle\left\langle[\eta\sqrt{-1}\partial\bar{\partial}\phi+\frac{1}{\delta}\sqrt{-1}\partial\bar{\partial}(\varphi_k+\Psi_k)\right.\right.\\
&\qquad\left.\left.-\sqrt{-1}\partial\bar{\partial}\eta-\sqrt{-1}g\partial\eta\wedge\bar{\partial}\eta,\Lambda_\omega]\alpha,\alpha\right\rangle\right\rangle_{D_m,\tilde{h}},
\end{aligned}
\tag{5.23}
$$

where g is a positive continuous function on D_m. We need some calculations to determine g.

We have

$$
\begin{aligned}
\partial\bar{\partial}\eta=&-s'(-v_{t_0,\varepsilon}\circ\Psi_k)\partial\bar{\partial}(v_{t_0,\varepsilon}\circ\Psi_k)\\
&+s''(-v_{t_0,\varepsilon}\circ\Psi_k)\partial(v_{t_0,\varepsilon}\circ\Psi_k)\wedge\bar{\partial}(v_{t_0,\varepsilon}\circ\Psi_k)
\end{aligned}
\tag{5.24}
$$

and

$$
\begin{aligned}
\partial\bar{\partial}\phi=&-u'(-v_{t_0,\varepsilon}\circ\Psi_k)\partial\bar{\partial}v_{t_0,\varepsilon}\circ\Psi_k\\
&+u''(-v_{t_0,\varepsilon}\circ\Psi_k)\partial(v_{t_0,\varepsilon}\circ\Psi_k)\wedge\bar{\partial}(v_{t_0,\varepsilon}\circ\Psi_k).
\end{aligned}
\tag{5.25}
$$

Therefore,

$$
\begin{aligned}
&\eta\sqrt{-1}\partial\bar{\partial}\phi-\sqrt{-1}\partial\bar{\partial}\eta-\sqrt{-1}g\partial\eta\wedge\bar{\partial}\eta\\
&\quad=(s'-su')\sqrt{-1}\partial\bar{\partial}(v_{t_0,\varepsilon}\circ\Psi_k)\\
&\qquad+((u''s-s'')-gs'^2)\sqrt{-1}\partial(v_{t_0,\varepsilon}\circ\Psi_k)\wedge\bar{\partial}(v_{t_0,\varepsilon}\circ\Psi_k)\\
&\quad=(s'-su')((v'_{t_0,\varepsilon}\circ\Psi_k)\sqrt{-1}\partial\bar{\partial}\Psi_k\\
&\qquad+(v''_{t_0,\varepsilon}\circ\Psi_k)\sqrt{-1}\partial(\Psi_k)\wedge\bar{\partial}(\Psi_k))\\
&\qquad+((u''s-s'')-gs'^2)\sqrt{-1}\partial(v_{t_0,\varepsilon}\circ\Psi_k)\wedge\bar{\partial}(v_{t_0,\varepsilon}\circ\Psi_k).
\end{aligned}
\tag{5.26}
$$

We omit the composite item $(-v_{t_0,\varepsilon}\circ\Psi_k)$ after $s'-su'$ and $(u''s-s'')-gs'^2$ in the above equalities.

Let $g=\frac{u''s-s''}{s'^2}\circ(-v_{t_0,\varepsilon}\circ\Psi_k)$. We have $\eta+g^{-1}=(s+\frac{s'^2}{u''s-s''})\circ(-v_{t_0,\varepsilon}\circ\Psi_k)$. Since $\varphi_k+\Psi_k$ and $\varphi_k+(1+\delta)\Psi_k$ are plurisubharmonic on M and $0\leq v'_{t_0,\varepsilon}\circ\Psi_k\leq 1$, we have

$$(1-v'_{t_0,\varepsilon}\circ\Psi_k)\sqrt{-1}\partial\bar\partial(\varphi_k+\Psi_k)+(v'_{t_0,\varepsilon}\circ\Psi_k)\sqrt{-1}\partial\bar\partial(\varphi_k+(1+\delta)\Psi_k)\geq 0 \tag{5.27}$$

on $M\setminus S$, which means that

$$\frac{1}{\delta}\sqrt{-1}\partial\bar\partial(\varphi_k+\Psi_k)+(v'_{t_0,\varepsilon}\circ\Psi_k)\partial\bar\partial\Psi_k\geq 0, \tag{5.28}$$

on M.

As $v'_{t_0,\varepsilon}\geq 0$ and $s'-su'=1$, using Lemma 4.2, equality (5.26), and inequalities (5.23) and (5.28), we have

$$\begin{aligned}\langle B\alpha,\alpha\rangle_{\tilde h}=&\langle[\eta\sqrt{-1}\Theta_{\tilde h}-\sqrt{-1}\partial\bar\partial\eta-\sqrt{-1}g\partial\eta\wedge\bar\partial\eta,\Lambda_\omega]\alpha,\alpha\rangle_{\tilde h}\\ \geq&\langle[(v''_{t_0,\varepsilon}\circ\Psi_k)\sqrt{-1}\partial\Psi_k\wedge\bar\partial\Psi_k,\Lambda_\omega]\alpha,\alpha\rangle_{\tilde h}\\ =&\langle(v''_{t_0,\varepsilon}\circ\Psi_k)\bar\partial\Psi_k\wedge(\alpha\llcorner(\bar\partial\Psi_k)^\sharp\big),\alpha\rangle_{\tilde h}.\end{aligned} \tag{5.29}$$

Using the definition of contraction, the Cauchy-Schwarz inequality and inequality (5.29), we have

(5.30)
$$\begin{aligned}|\langle(v''_{t_0,\varepsilon}\circ\Psi)\bar\partial\Psi\wedge\gamma,\tilde\alpha\rangle_{\tilde h}|^2=&|\langle(v''_{t_0,\varepsilon}\circ\Psi)\gamma,\tilde\alpha\llcorner(\bar\partial\Psi)^\sharp\rangle_{\tilde h}|^2\\ \leq&\langle(v''_{t_0,\varepsilon}\circ\Psi)\gamma,\gamma\rangle_{\tilde h}(v''_{t_0,\varepsilon}\circ\Psi)\left|\tilde\alpha\llcorner(\bar\partial\Psi)^\sharp\right|^2_{\tilde h}\\ =&\langle(v''_{t_0,\varepsilon}\circ\Psi)\gamma,\gamma\rangle_{\tilde h}\langle(v''_{t_0,\varepsilon}\circ\Psi)\bar\partial\Psi\wedge\big(\tilde\alpha\llcorner(\bar\partial\Psi)^\sharp\big),\tilde\alpha\rangle_{\tilde h}\\ \leq&\langle(v''_{t_0,\varepsilon}\circ\Psi)\gamma,\gamma\rangle_{\tilde h}\langle B\tilde\alpha,\tilde\alpha\rangle_{\tilde h}\end{aligned}$$

for any (n,q)-form γ and $(n,q+1)$-form $\tilde\alpha$ with values in E.

Take $\lambda=\bar\partial[(1-v'_{t_0,\varepsilon}(\Psi))\tilde F]$, $\gamma=\tilde F$, and $\tilde\alpha=B^{-1}\bar\partial\Psi\wedge\tilde F$. We have

$$\langle B^{-1}\lambda,\lambda\rangle_{\tilde h}\leq(v''_{t_0,\varepsilon}\circ\Psi)|\tilde F|^2_{\tilde h}.$$

Then it is easy to see that

$$\int_{D_m\setminus S}\langle B^{-1}\lambda,\lambda\rangle_{\tilde h}dV_M\leq\int_{D_m\setminus S}(v''_{t_0,\varepsilon}\circ\Psi)|\tilde F|^2_{\tilde h}dV_M.$$

From Lemma 4.3, it follows that there exists an $(n,0)$-form $\gamma_{m,t_0,\varepsilon,k}$ on D_m satisfying $\bar\partial\gamma_{m,t_0,\varepsilon,k}=\lambda$ and

$$\int_{D_m}|\gamma_{m,t_0,\varepsilon,k}|^2_{\tilde h}(\eta+g^{-1})^{-1}dV_M\leq\int_{D_m}(v''_{t_0,\varepsilon,k}\circ\Psi_k)|\tilde F|^2_{\tilde h}dV_M. \tag{5.31}$$

Let $\mu_1 = e^{v_{t_0,\varepsilon}\circ\Psi_k}$, $\mu = \mu_1 c_A(-v_{t_0,\varepsilon}\circ\Psi_k)e^{\phi}$. Claim that we can choose η and ϕ satisfying $\mu \leq \mathbf{C}(\eta + g^{-1})^{-1}$, which will be discussed at the end of this subsection, where $\mathbf{C}$ is just the constant in Theorem 5.2.

Let $F_{m,t_0,\varepsilon,k} := (1 - v'_{t_0,\varepsilon}\circ\Psi_k)\widetilde{F} - \gamma_{m,t_0,\varepsilon,k}$. Then inequality (5.31) means that

(5.32)
$$\int_{D_m} |F_{m,t_0,\varepsilon,k} - (1 - v'_{t_0,\varepsilon}\circ\Psi_k)\widetilde{F}|^2 e^{-\varphi_k-\Psi_k+v_{t_0,\varepsilon}\circ\Psi_k} c_A(-v_{t_0,\varepsilon}\circ\Psi_k)dV_M \leq \int_{D_m}(v''_{t_0,\varepsilon}\circ\Psi_k)|\tilde{F}|^2_{\tilde{h}}dV_M.$$

Note that for any compact subset K of D_m, we obtain

(5.33)
$$\begin{aligned}\Big(\int_K |F_{m,t_0,\varepsilon,k} - (1 - v'_{t_0,\varepsilon}\circ\Psi_k)\widetilde{F}|^2 e^{-\varphi_k-\Psi_k+v_{t_0,\varepsilon}\circ\Psi_k} c_A(-v_{t_0,\varepsilon}\circ\Psi_k)dV_M\Big)^{1/2} \\ + \Big(\int_K |(v'_{t_0,\varepsilon}\circ\Psi_k)\widetilde{F}|^2 e^{-\varphi_k-\Psi_k+v_{t_0,\varepsilon}\circ\Psi_k} c_A(-v_{t_0,\varepsilon}\circ\Psi_k)dV_M\Big)^{1/2} \\ \geq \Big(\int_K |F_{m,t_0,\varepsilon,k} - \widetilde{F}|^2 e^{-\varphi_k-\Psi_k+v_{t_0,\varepsilon}\circ\Psi_k} c_A(-v_{t_0,\varepsilon}\circ\Psi_k)dV_M\Big)^{1/2}.\end{aligned}$$

Note that

(1) $e^{-\varphi_k-\Psi_k}$, $e^{v_{t_0,\varepsilon}\circ\Psi_k}$ and $c_A(-v_{t_0,\varepsilon}\circ\Psi_k)$ have uniform positive lower bounds independent of k;
(2) $|v'_{t_0,\varepsilon}\circ\Psi_k)\widetilde{F}|^2e^{-\Psi}$ and $\int_{D_m}(v''_{t_0,\varepsilon}\circ\Psi_k)|\tilde{F}|^2_{\tilde{h}}dV_M$ have uniform positive upper bounds independent of k;
(3) $e^{-\varphi}$ is locally integrable on M, and the sequence $\varphi_k + \Psi_k$ is decreasing with respect to k.

According to inequality (5.33), it follows that $\int_K |F_{m,t_0,\varepsilon,k} - \widetilde{F}|^2 dV_M$ has a uniform bound independent of k for any compact subset K of D_m.

Using Lemma 4.5, we have a subsequence of $\{F_{m,t_0,\varepsilon,k}\}_k$, still denoted by $\{F_{m,t_0,\varepsilon,k}\}_k$, which is uniformly convergent to a holomorphic $(n,0)$-form $F_{m,t_0,\varepsilon}$ on any compact subset of D_m.

As all the terms $e^{v_{t_0,\varepsilon}\circ\Psi_k}$, $c_A(-v_{t_0,\varepsilon}\circ\Psi_k)$, $(1 - v'_{t_0,\varepsilon}\circ\Psi_k)\widetilde{F}$, and $(v''_{t_0,\varepsilon}\circ\Psi_k)|\tilde{F}|^2e^{-\varphi_k-\Psi_k-\phi}$ have uniform positive upper bounds independent of k, and $v_{t_0,\varepsilon}(\Psi_k) \geq \Psi_k$, it follows from the dominated convergence theorem that

$$\begin{aligned}\int_K |F_{m,t_0,\varepsilon} - (1 - v'_{t_0,\varepsilon}\circ\Psi)\widetilde{F}|^2 e^{-\varphi_k-\Psi_k+v_{t_0,\varepsilon}\circ\Psi} c_A(-v_{t_0,\varepsilon}\circ\Psi)dV_M \\ \leq \int_{D_m}(v''_{t_0,\varepsilon}\circ\Psi)|\tilde{F}|^2 e^{-\varphi-\Psi-u(-v_{t_0,\varepsilon}(\Psi))}dV_M\end{aligned} \tag{5.34}$$

for any compact subset K of D_m.

As the sequence $\varphi_k+\Psi_k$ is decreasing convergent to $\varphi+\Psi$, it follows from Levi's theorem that

$$\begin{aligned}\int_K |F_{m,t_0,\varepsilon}-(1-v'_{t_0,\varepsilon}\circ\Psi)\widetilde{F}|^2 e^{-\varphi-\Psi+v_{t_0,\varepsilon}\circ\Psi}c_A(-v_{t_0,\varepsilon}\circ\Psi)dV_M\\ \leq \int_{D_m}(v''_{t_0,\varepsilon}\circ\Psi)|\tilde{F}|^2 e^{-\varphi-\Psi-u(-v_{t_0,\varepsilon}(\Psi))}dV_M\end{aligned} \tag{5.35}$$

for any compact subset K of D_m, which means

$$\begin{aligned}\int_{D_m} |F_{m,t_0,\varepsilon}-(1-v'_{t_0,\varepsilon}\circ\Psi)\widetilde{F}|^2 e^{-\varphi-\Psi+v_{t_0,\varepsilon}\circ\Psi}c_A(-v_{t_0,\varepsilon}\circ\Psi)dV_M\\ \leq \int_{D_m}(v''_{t_0,\varepsilon}\circ\Psi)|\tilde{F}|^2 e^{-\varphi-\Psi-u(-v_{t_0,\varepsilon}(\Psi))}dV_M.\end{aligned} \tag{5.36}$$

Note that $e^{-\Psi}$ is not integrable along S, and $F_{m,t_0,\varepsilon}$ and $(1-v'_{t_0,\varepsilon}\circ\Psi)\widetilde{F}$ are both holomorphic near S. Then $(F_{m,t_0,\varepsilon}-(1-v'_{t_0,\varepsilon}\circ\Psi)\widetilde{F})|_S=0$, and therefore $F_{m,t_0,\varepsilon}|_S=\widetilde{F}|_S$. It is clear that $F_{m,t_0,\varepsilon}$ is an extension of f.

Note that $v_{t_0,\varepsilon}(\Psi)\geq\Psi$. Then the inequality (5.36) becomes

$$\begin{aligned}&\int_{D_m} |F_{m,t_0,\varepsilon}-(1-v'_{t_0,\varepsilon}\circ\Psi)\widetilde{F}|^2 e^{-\varphi}c_A(-v_{t_0,\varepsilon}\circ\Psi)dV_M\\ &\quad\leq \int_{D_m}(v''_{t_0,\varepsilon}\circ\Psi)|\tilde{F}|^2 e^{-\varphi-\Psi-u(-v_{t_0,\varepsilon}(\Psi))}dV_M\\ &\quad\leq \frac{1}{e^{A_{t_0}}}\int_{D_m}(v''_{t_0,\varepsilon}\circ\Psi)|\tilde{F}|^2 e^{-\varphi-\Psi}dV_M,\end{aligned} \tag{5.37}$$

where $A_{t_0}:=\inf_{t\geq t_0}\{u(t)\}$.

As

$$\lim_{t\to+\infty}u(t)=-\log(\frac{1}{\delta}c_A(-A)+\int_{-A}^{+\infty}c_A(t)e^{-t}dt),$$

it is clear that

$$\lim_{t_0\to\infty}\frac{1}{e^{A_{t_0}}}=\frac{1}{\delta}c_A(-A)+\int_{-A}^{+\infty}c_A(t)e^{-t}dt.$$

Given t_0 and D_m,

$$(v''_{t_0,\varepsilon}\circ\Psi)|\tilde{F}|^2e^{-\varphi-\Psi}$$

has a uniform bound on D_m independent of ε. Then both

$$\int_{D_m}|(1-v'_{t_0,\varepsilon}\circ\Psi)\tilde{F}|^2e^{-\varphi}c_A(-v_{t_0,\varepsilon}\circ\Psi)dV_M$$

and

$$\int_{D_m}v''_{t_0,\varepsilon}\circ\Psi|\tilde{F}|^2e^{-\varphi-\Psi}dV_M$$

have uniform bounds independent of ε for any given t_0 and D_m.

Using the equation $\bar\partial F_{m,t_0,\varepsilon}=0$ and Lemma 4.5, we can choose a subsequence of $\{F_{m,t_0,\varepsilon}\}_\varepsilon$, such that the chosen sequence is uniformly convergent on any compact subset of D_m, still denoted by $\{F_{m,t_0,\varepsilon}\}_\varepsilon$ without ambiguity.

For any compact subset K on D_m, all terms $F_{m,t_0,\varepsilon}$, $(1-v'_{t_0,\varepsilon}\circ\Psi)\tilde{F}$, $c_A(-v_{t_0,\varepsilon}\circ\Psi)$ and $(v''_{t_0,\varepsilon}\circ\Psi)|\tilde{F}|^2e^{-\varphi-\Psi}$ have uniform bounds on K independent of ε.

Using the dominated convergence theorem on any compact subset K of D_m and inequality (5.37), we have

$$\begin{aligned}\int_K &|F_{m,t_0}-(1-b'_{t_0}(\Psi))\tilde{F}|^2e^{-\varphi}c_A(-b_{t_0}(\Psi))dV_M\\ &\leq\frac{\mathbf{C}}{e^{At_0}}\int_{D_m}(\mathbb{I}_{\{-t_0-1<t<-t_0\}}\circ\Psi)|\tilde{F}|^2e^{-\varphi-\Psi}dV_M,\end{aligned}\tag{5.38}$$

which implies

$$\begin{aligned}\int_{D_m} &|F_{m,t_0}-(1-b'_{t_0}(\Psi))\tilde{F}|^2e^{-\varphi}c_A(-b_{t_0}(\Psi))dV_M\\ &\leq\frac{\mathbf{C}}{e^{At_0}}\int_{D_m}(\mathbb{I}_{\{-t_0-1<t<-t_0\}}\circ\Psi)|\tilde{F}|^2e^{-\varphi-\Psi}dV_M.\end{aligned}\tag{5.39}$$

According to the definition of $dV_M[\Psi]$ and the assumption

$$\sum_{k=1}^n\frac{\pi^k}{k!}\int_{S_{n-k}}|f|^2_hdV_M[\Psi]<\infty,$$

it follows that

$$\begin{aligned}\limsup_{t_0\to+\infty}&\int_{D_m}(\mathbb{I}_{\{-t_0-1<t<-t_0\}}\circ\Psi)|\tilde{F}|^2e^{-\varphi-\Psi}dV_M\\ &\leq\limsup_{t_0\to+\infty}\int_M\mathbb{I}_{\overline{D}_m}(\mathbb{I}_{\{-t_0-1<t<-t_0\}}\circ\Psi)|\tilde{F}|^2e^{-\varphi-\Psi}dV_M\\ &\leq\sum_{k=1}^n\frac{\pi^k}{k!}\int_{S_{n-k}}\mathbb{I}_{\overline{D}}|f|^2_hdV_M[\Psi]\leq\sum_{k=1}^n\frac{\pi^k}{k!}\int_{S_{n-k}}|f|^2_hdV_M[\Psi]<\infty.\end{aligned}\tag{5.40}$$

Then

$$\int_{D_m}(\mathbb{I}_{\{-t_0-1<t<-t_0\}}\circ\Psi)|\tilde{F}|^2e^{-\varphi-\Psi}dV_M$$

has a uniform bound independent of t_0 for any given D_m, and

$$\begin{aligned}\limsup_{t_0\to+\infty}&\int_{D_m}(\mathbb{I}_{\{-t_0-1<t<-t_0\}}\circ\Psi)|\tilde{F}|^2e^{-\varphi-\Psi}dV_M\\ &\leq\sum_{k=1}^n\frac{\pi^k}{k!}\int_{S_{n-k}}|f|^2e^{-\varphi}dV_M[\Psi]<\infty.\end{aligned}\tag{5.41}$$

Therefore,

$$\int_{D_m}|F_{m,t_0}-(1-b'_{t_0}(\Psi))\tilde{F}|^2e^{-\varphi}c_A(-b_{t_0}(\Psi))dV_M$$

has a uniform bound independent of t_0 for any given D_m.

Since

$$\int_{D_m}|(1-b'_{t_0}(\Psi))\tilde{F}|^2e^{-\varphi}c_A(-b_{t_0}(\Psi))dV_M$$

has a uniform bound independent of t_0, and

$$\begin{aligned}(5.42)\qquad &\left(\int_{D_m}|F_{m,t_0}-(1-b'_{t_0}(\Psi))\tilde{F}|^2e^{-\varphi}c_A(-b_{t_0}(\Psi))dV_M\right)^{\frac{1}{2}}\\ &+\left(\int_{D_m}|(1-b'_{t_0}(\Psi))\tilde{F}|^2e^{-\varphi}c_A(-b_{t_0}(\Psi))dV_M\right)^{\frac{1}{2}}\\ &\geq\left(\int_{D_m}|F_{m,t_0}|^2e^{-\varphi}c_A(-b_{t_0}(\Psi))dV_M\right)^{\frac{1}{2}},\end{aligned}$$

it follows from inequality (5.39) that $\int_{D_m}|F_{m,t_0}|^2e^{-\varphi}c_A(-b_{t_0}(\Psi))dV_M$ has a uniform bound independent of t_0.

Using the equation $\bar{\partial}F_{m,t_0}=0$ and Lemma 4.5, we can choose a subsequence of $\{F_{m,t_0}\}_{t_0}$, such that the chosen sequence is uniformly convergent on any compact subset of D_m, still denoted by $\{F_{m,t_0}\}_{t_0}$ without ambiguity.

For any compact subset K on D_m, both F_{m,t_0} and $(1-b'_{t_0}\circ\Psi)\tilde{F}$ have uniform bounds on K independent of t_0.

Using inequalities (5.39) and (5.41), the following equality,

$$\lim_{t_0\to\infty}\frac{1}{e^{At_0}}=\int_{-A}^{+\infty}c_A(t)e^{-t}dt,$$

and the dominated convergence theorem on any compact subset K of D_m, we have

$$\begin{aligned}(5.43)\qquad &\int_{D_m}\mathbb{I}_K|F_m|^2e^{-\varphi}c_A(-\Psi)dV_M\\ &\leq\mathbf{C}\left(\int_{-A}^{+\infty}c_A(t)e^{-t}dt\right)\sum_{k=1}^{n}\frac{\pi^k}{k!}\int_{S_{n-k}}|f|^2e^{-\varphi}dV_M[\Psi],\end{aligned}$$

which implies

$$\begin{aligned}(5.44)\qquad &\int_{D_m}|F_m|^2e^{-\varphi}c_A(-\Psi)dV_M\\ &\leq\mathbf{C}\left(\int_{-A}^{+\infty}c_A(t)e^{-t}dt\right)\sum_{k=1}^{n}\frac{\pi^k}{k!}\int_{S_{n-k}}|f|^2e^{-\varphi}dV_M[\Psi],\end{aligned}$$

where the Lebesgue measure of $\{\Psi=-\infty\}$ is zero.

Define $F_m=0$ on $M\backslash D_m$. Then the weak limit of some weakly convergent subsequence of $\{F_m\}_{m=1}^{\infty}$ gives a holomorphic section F of $K_M\otimes E$ on M satisfying $F|_S=\tilde{F}|_S$, and

(5.45)

$$\begin{aligned}&\int_M|F|^2e^{-\varphi}c_A(-\Psi)dV_M\\ &\leq\mathbf{C}\left(\frac{1}{\delta}c_A(-A)e^A+\int_{-A}^{+\infty}c_A(t)e^{-t}dt\right)\sum_{k=1}^{n}\frac{\pi^k}{k!}\int_{S_{n-k}}|f|^2e^{-\varphi}dV_M[\Psi].\end{aligned}$$

To finish the proof of Theorem 5.2, it suffices to determine η and ϕ such that $(\eta+g^{-1}) \leq \mathbf{C}c_A^{-1}(-v_{t_0,\varepsilon}\circ\Psi)e^{-v_{t_0,\varepsilon}\circ\Psi}e^{-\phi} = \mathbf{C}\mu^{-1}$ on D_v.

As $\eta = s(-v_{t_0,\varepsilon}\circ\Psi)$ and $\phi = u(-v_{t_0,\varepsilon}\circ\Psi)$, we have $(\eta+g^{-1})e^{v_{t_0,\varepsilon}\circ\Psi}e^{\phi} = (s+\frac{s'^2}{u''s-s''})e^{-t}e^{u}\circ(-v_{t_0,\varepsilon}\circ\Psi)$.

We naturally obtain the system of ODEs (5.21), where $t\in(-A,+\infty)$, $\mathbf{C}=1$, $s\in C^\infty((-A,+\infty))$ satisfying $s\geq\frac{1}{\delta}$, $u\in C^\infty((-A,+\infty))$ satisfying $\lim_{t\to+\infty}u(t) = -\log(\frac{1}{\delta}c_A(-A)e^A+\int_{-A}^{\infty}c_A(t)e^{-t}dt)$, and $u''s-s''>0$.

We solve the system of ODEs (5.21) in Subsection 5.4 and get the solution

$$
(5.46)\quad
\begin{aligned}
&(1) && u = -\log\left(\frac{1}{\delta}c_A(-A)e^A+\int_{-A}^{t}c_A(t_1)e^{-t_1}dt_1\right),\\
&(2) && s = \frac{\int_{-A}^{t}(\frac{1}{\delta}c_A(-A)e^A+\int_{-A}^{t_2}c_A(t_1)e^{-t_1}dt_1)dt_2+\frac{1}{\delta^2}c_A(-A)e^A}{\frac{1}{\delta}c_A(-A)e^A+\int_{-A}^{t}c_A(t_1)e^{-t_1}dt_1}.
\end{aligned}
$$

One can check that $s\in C^\infty((-A,+\infty))$, $\lim_{t\to+\infty}u(t) = -\log(\frac{1}{\delta}c_A(-A)e^A + \int_{-A}^{+\infty}c_A(t_1)e^{-t_1}dt_1)$, and $u\in C^\infty((-A,+\infty))$.

As $su''-s'' = -s'u'$ and $u'<0$, it is clear that $u''s-s''>0$ is equivalent to $s'>0$, and inequality (2.1) means that $s'>0$. Then we obtain $u''s-s''>0$. In conclusion, we have proved Theorem 5.2. $\square$

Using Remark 4.10 and Lemma 4.8, we may replace smoothness of c_A by continuity. When we take $c_A=1$, using the above Theorems 2.1 and 5.2, one obtains the main results in [27] and [29], which are the optimal estimate versions of the main theorems in [38], [41], [42].

5.3. *Proof of Theorem* 2.2. By Remark 4.7, it suffices to prove the case that M is a Stein manifold. By Lemmas 4.6 and 4.8, it is enough to prove the case that c_A is smooth on $(A,+\infty)$ and continuous on $(A,+\infty]$, such that $\lim_{t\to+\infty}c_A(t)$ exists and is bigger than 0.

Since M is a Stein manifold, we can find a sequence of Stein manifolds $\{D_m\}_{m=1}^\infty$ satisfying $D_m \subset\subset D_{m+1}$ for all m and $\bigcup_{m=1}^{\infty}D_m = M$. All $D_m\setminus S$ are complete Kähler ([22]).

As Ψ is a plurisubharmonic function on M, then

(1) when $A<+\infty$, $\sup_{z\in D_m}\Psi(z) < A-\varepsilon$, where $\varepsilon>0$;
(2) when $A=+\infty$, $\sup_{z\in D_m}\Psi(z) < A_m$, where $A_m<+\infty$ is sufficient large.

We just consider our proof for condition (1). (The case under condition (2) can be proved similarly.) By Lemma 4.11, for any given $A'<A$, it follows that there exists $c_{A''}$ and $\delta''>0$ satisfying conditions (1), (2) and (3) in Lemma 4.11, where $A''<A$ and $A''>A-\varepsilon$.

Note that $\sqrt{-1}\partial\bar\partial\Psi\geq 0$, and $\sqrt{-1}\Theta_{he^{-\Psi}}\geq 0$ on $M\setminus S$ implies conditions (1) and (2) in Theorem 2.1 for any $\delta''>0$.

Using Theorem 2.1, we obtain a holomorphic $(n,0)$-form $F_{m,A''}$ with value in E on D_m, which satisfies $F_m|_S = f$ and

$$\int_{D_m} c_{A''}(-\Psi)|F_{m,A''}|_h^2 dV_M \leq \mathbf{C}\int_{-A}^{\infty} c_A(t)e^{-t}dt \sum_{k=1}^{n} \frac{\pi^k}{k!} \int_{S_{n-k}} |f|_h^2 dV_M[\Psi].$$

Note that $c_{A''}(-\Psi)$ is uniformly convergent to $c_A(-\Psi)$ on any compact subset of D_m, as $A'' \to A$. Let $A' \to A$ $(A'' \to A)$, and then let $m \to +\infty$. Using Lemma 4.6, we prove the present theorem.

Remark 5.3. $\mathbf{C}$ is optimal on the ball $\mathbb{B}^m(0, e^{\frac{A}{2m}})$ for trivial holomorphic line bundle when $S=\{0\}$ and $\Psi=2m\log|z|$. When $A=+\infty$, $\mathbb{B}^m(0, e^{\frac{A}{2m}}):=\mathbb{C}^m$.

Using Theorem 2.2 and Corollary 4.17 by taking $d_2 = 1$, we obtain

COROLLARY 5.4. *Let Ω be an open Riemann surface which admits a Green function G, and let $\Psi := 2G(z,z_0)$. Let V_{z_0} be a neighborhood of z_0 with a local coordinate w, $(w(z_0) = 0)$, which satisfies $\Psi|_{V_{z_0}} \leq \Psi|_{\Omega \setminus V_{z_0}}$ and $\Psi|_{V_{z_0}} = \log|w|^2$.*

If there is a unique holomorphic $(1,0)$-form F, such that $F|_{z_0} = dw$ and

$$\int_{\Omega} \sqrt{-1}F \wedge \bar{F} \leq \pi \int_{z_0} |dw|^2 dV_{\Omega}[\Psi],$$

then we have $F|_{V_{z_0}} = dw$.

5.4. *Solution of the* ODE *system* (5.21). We now solve equations (5.21) as follows: By (2) of equation (5.21), it follows that $su'' - s'' = -s'u'$. Then (1) of equation (5.21) can be reformulated to

$$(s - \frac{s'}{u'})e^{u-t} = \frac{\mathbf{C}}{c_A(t)};$$

i.e.,

$$\frac{su' - s'}{u'}e^{u-t} = \frac{\mathbf{C}}{c_A(t)}.$$

By (2) of equation (5.21) again, it follows that

$$\frac{\mathbf{C}}{c_A(t)} = \frac{su' - s'}{u'}e^{u-t} = \frac{-1}{u'}e^{u-t},$$

and therefore

$$\frac{de^{-u}}{dt} = -u'e^{-u} = \frac{c_A(t)e^{-t}}{\mathbf{C}}.$$

Note that (2) of equation (5.21) is equivalent to $\frac{d(se^{-u})}{dt} = e^{-u}$.

As $s \geq 0$, we obtain the solution

$$\begin{cases} u = -\log(a + \int_{-A}^{t} c_A(t_1)e^{-t_1}dt_1), \\ s = \frac{\int_{-A}^{t}(a+\int_{-A}^{t_2} c_A(t_1)e^{-t_1}dt_1)dt_2+b}{a+\int_{-A}^{t} c_A(t_1)e^{-t_1}dt_1}, \end{cases}$$

when $\mathbf{C} = 1$, where $a \geq 0$ and $b \geq 0$.

As $\lim_{t\to+\infty} u(t) = -\log(\frac{1}{\delta}c_A(-A)e^A + \int_{-A}^{+\infty} c_A(t_1)e^{-t_1}dt_1)$, we have $a = \frac{1}{\delta}c_A(-A)e^A$. As $s \geq \frac{1}{\delta}$, we have $\frac{b}{a} \geq \frac{1}{\delta}$.

As $u' < 0$ and $su'' - s'' = -s'u'$, it is clear that $u''s - s'' > 0$ is equivalent to $s' > 0$. By inequality $s' > 0$, it follows that $a^2 \geq c_A(-A)e^A b$. Then we get $b = \frac{1}{\delta}a$.

5.5. *Verifications of Remarks* 5.1 *and* 5.3. Let $\mathbb{B}^m(0, e^{\frac{A}{2m}})$ be the unit ball with radius $e^{\frac{A}{2m}}$ on $\mathbb{C}^m$ ($\mathbb{B}^m(0,+\infty) := \mathbb{C}^m$), with coordinate $z = (z_1, \ldots, z_m)$. Let

$$\varphi(z) = (1+\delta)m \max\{\log|z|^2, \log|a|^2\}$$

and

$$\Psi(z) = -m \max\{\log|z|^2, \log|a|^2\} + m\log|z|^2 + A - \varepsilon,$$

where $a \in (0, +\infty)$ and $\varepsilon > 0$.

As both φ and $\varphi + (1+\delta)\Psi$ are plurisubharmonic, and

$$\varphi + \Psi = \frac{\delta\varphi + (\varphi + (1+\delta)\Psi)}{1+\delta},$$

it is clear that $\Psi(z) \in \Delta_{\varphi,\delta}(S)$, where $S = \{z = 0\}$.

For any $f(0) \neq 0$, it suffices to prove

$$\lim_{a\to 0} \frac{\min_{F\in Hol(\mathbb{B}^m(0,e^{\frac{A}{2m}}))} \int_{\mathbb{B}^m(0,e^{\frac{A}{2m}})} |F|^2 c_A(-\Psi)e^{-\varphi}d\lambda}{a^{-2\delta}e^{\varepsilon-A}|F(0)|^2} = \frac{\pi^m}{m!}\left(\int_{-A+\varepsilon}^{+\infty} c_A(t)e^{-t}dt + \frac{1}{\delta}c_A(-A+\varepsilon)e^{A-\varepsilon}\right), \tag{5.47}$$

where $F(0) = f(0)$.

Because $e^{-\varphi}d\lambda[\Psi] = a^{-2\delta}e^{\varepsilon}\delta_0$ (by Lemma 4.14), where δ_0 is the Dirac function at 0, let ε go to zero. Then we see that the constant of Theorem 2.2 is optimal.

Set the Taylor expansion of $F(z)$ at $0 \in \mathbb{C}^m$ of $F(z) = \sum_{k=0}^{\infty} a_k z^k$, where $k = \{k_1, \ldots, k_m\}$, a_k are complex constants, and $z^k = z_1^{k_1} \cdots z_m^{k_m}$.

Note that $\int_\Delta z^{k_1}\bar{z}^{k_2}e^{-\varphi}d\lambda = 0$ when $k_1 \neq k_2$, and $\int_\Delta z^{k_1}\bar{z}^{k_2}e^{-\varphi}d\lambda > 0$ when $k_1 = k_2$. It is clear that

$$\min_{F\in Hol(\mathbb{B}^m(0,e^{\frac{A}{2m}}))}\int_{\mathbb{B}^m(0,e^{\frac{A}{2m}})} c_A(-\Psi)|F|^2e^{-\varphi}d\lambda$$
$$= \int_{\mathbb{B}^m(0,e^{\frac{A}{2m}})} c_A(-\Psi)|F(0)|^2e^{-\varphi}d\lambda.$$

It is not hard to see that

$$\int_{\mathbb{B}^m(0,e^{\frac{A}{2m}})} c_A(-\Psi)e^{-\varphi}d\lambda$$
$$= \frac{\pi^m}{m!}\left(a^{-2\delta}e^{-A+\varepsilon}\int_{-A+\varepsilon}^{+\infty} c_A(t)e^{-t}dt + c_A(-A+\varepsilon)\frac{a^{-2\delta}-e^{-\delta A}}{\delta}\right)$$

and

$$\lim_{a\to 0}\frac{a^{-2\delta}-e^{-\delta A}}{\delta a^{-2\delta}} = \frac{1}{\delta}.$$

As $\int_{-A}^{\infty} c_A(t)e^{-t}dt < \infty$, $c_A(-A)e^A < \infty$ and $c_A(-A)e^A \neq 0$, then we have proved the equality (5.47).

Now we finish proving Remark 5.1. Let $\varphi = 0$ and $\Psi = m\log|z|^2$. Then we obtain Remark 5.3 on $\mathbb{B}^m(0,e^{\frac{A}{2m}})$, where $A \in (-\infty,+\infty]$.

6. **Proofs of the main corollaries**

In this section, we give proofs of the main corollaries including a conjecture of Suita on the equality conditions in Suita's conjecture and the extended Suita conjecture, optimal estimates of various known L^2 extension theorems, optimal estimate for L^p extension and for the $L^{\frac{2}{m}}$ extension, etc.

6.1. *Proof of Theorem* 3.1. It is well known that if Ω is conformally equivalent to the unit disc less a (possible) closed set of inner capacity zero, then

$$\pi B_\Omega(z_0) = c_\beta^2(z_0).$$

It suffices to prove that if $\pi B_\Omega(z_0) = c_\beta^2(z_0)$ holds, then Ω is conformally equivalent to the unit disc less a (possible) closed set of inner capacity zero.

As Ω is a noncompact Riemann surface, there exists a holomorphic function g_0 on Ω, which satisfies $dg_0|_{z_0} \neq 0$, $g_0|_{z_0} = 0$, and $g_0|_{\Omega\setminus\{z_0\}} \neq 0$.

Let $p : \Delta \to \Omega$ be the universal covering of Ω. We can choose a connected component V_{z_0} small enough, such that p is biholomorphic on any connected component of $p^{-1}(V_{z_0})$.

Since $p^*(G_\Omega(z,z_0) - \log|g_0(z)|)$ is a harmonic function on Δ (by Lemma 4.20), then there exists a holomorphic function f_1 on Δ, such that the real part of f_1 is $p^*(G_\Omega(z,z_0) - \log|w|)$.

We want to show that for any $z_1 \in \Omega$, $p^*(g_0)\exp f_1$ is constant along the fibre $p^{-1}(z_1)$. Note that

$$\log|p_*((p^*(g_0)\exp f_1)|_{U^0})| = G_\Omega(z,z_0)|_{V_{z_0}},$$

where U^0 is a fixed connected component of $p^{-1}(V_{z_0})$. By assumption $\pi B_\Omega(z_0) = c_\beta^2(z_0)$, and by Remark 4.32, there is a unique holomorphic $(1,0)$-form F on Ω, which satisfies $F|_{z_0} = dp_*((p^*(g_0)\exp f_1)|_{U^0})|_{z_0}$, and

$$\sqrt{-1}\int_\Omega F\wedge\bar{F} \le \pi\int_{z_0} |dp_*((p^*(g_0)\exp f_1)|_{U^0})|^2 dV_\Omega[2G_\Omega(z,z_0)].$$

Using Proposition 4.21, we have

$$dp_*(p^*(g_0)\exp f_1|_{U^0}) = F|_{V_{z_0}},$$

and therefore

$$d(p^*(g_0)\exp f_1|_{U^0}) = (p^*F)|_{U^0}.$$

Using Lemma 4.28, we have $d(p^*(g_0)\exp f_1) = p^*F$.

For $z_1 \in \Omega$, there exists V_{z_1}, a connected neighborhood small enough, such that p is biholomorphic on any connected component of $p^{-1}(V_{z_1})$, and U_1 and U_2 are any two connected components of $p^{-1}(V_{z_1})$. Let

$$g_1 = (p|_{U_1})_*((p^*(g_0)\exp f_1)|_{U_1})$$

and

$$g_2 = (p|_{U_2})_*((p^*(g_0)\exp f_1)|_{U_2});$$

they are holomorphic functions on V_{z_1}.

As $d(p^*(g_0)\exp f_1) = p^*F$, therefore

$$(p|_{U_1})_*(d(p^*(g_0)\exp f_1)|_{U_1}) = (p|_{U_2})_*((dp^*(g_0)\exp f_1)|_{U_2}),$$

i.e.,

$$dg_1 = dg_2.$$

As $|p^*(g_0)\exp f_1| = \exp(p^*G_\Omega(\cdot,z_0))$, which restricted on $p^{-1}(z)$ takes the same value, we have $|g_1| = |g_2|$, which are not constant on V_{z_1}.

Using Lemma 4.30, we have $g_1 = g_2$. Therefore $(p^*(g_0)\exp f_1)|_{p^{-1}(z)}$ is constant for any $z\in\Omega$. Then we obtain a well-defined holomorphic function

$$g(z) := (p^*(g_0)\exp f_1)|_{p^{-1}(z)}$$

on Ω, which satisfies $|g(z)| = \exp G_\Omega(z,z_0)$. Using Lemma 4.25, we have $c_B(z_0) = c_\beta(z_0)$. By the assumption $\pi B_\Omega(z_0) = c_\beta^2(z_0)$, it follows that $\pi B_\Omega(z_0) = c_B^2(z_0)$. Using Lemma 4.26, we obtain that Ω is conformally equivalent to the unit disc less a (possible) closed set of inner capacity zero.

6.2. *Proof of Theorem* 3.2. Let $\{\Omega_m\}_{m=1,2,\dots}$ be domains with smooth boundaries, which satisfy $\Omega_m \subset\subset \Omega_{m+1}$ and $\cup_{m=1}^{\infty}\Omega_m = \Omega$. Assume that $t \in \Omega_1$. Denote B_{Ω_m} by B_m, L_{Ω_m} by L_m, and G_{Ω_m} by G_m. Denote B_Ω by B, L_Ω by L, and G_Ω by G. Denote $\exp\lim_{z\to t}(G_m(z,t) - \log|z-t|)$ by $c_{\beta,m}(t)$, where z is the local coordinate near t.

It is known that $B_m = \frac{2}{\pi}\frac{\partial^2 G_m(z,t)}{\partial z\partial \bar{t}}$ and $L_m = \frac{2}{\pi}\frac{\partial^2 G_m(z,t)}{\partial z\partial t}$ by [48] (see also [55]). Note that $B_m(z,\bar{t})dz = -L_m(z,t)dz$ for $z \in \Omega$ and $t \in \partial\Omega_m$ (see [55]). If $L_m(z,t)$ has no zeros for a t, then we obtain a subharmonic function

$$H_{m,t}(z) := |\frac{B_m(z,\bar{t})}{-L_m(z,t)}\exp -2G_m(z,t)|$$

which is 1 at $\partial\Omega_m$.

By maximum principle, it follows that $H_{m,t}(z) \leq 1$ for any $z \in \Omega$. As $L_m(z,t) - \frac{1}{\pi(z-t)^2}$ is holomorphic near t (see [48, p. 92]), then we have

$$\begin{aligned}
&\lim_{z\to t}|L_m(z,t)|\exp 2G_m(z,t)\\
&\qquad= \lim_{z\to t}\frac{\exp(2G_m(z,t))}{\pi|z-t|^2}\\
&\qquad= \frac{1}{\pi}\exp 2\lim_{z\to t}(G_m(z,t) - \log|z-t|)\\
&\qquad= \frac{c^2_{\beta,m}(t)}{\pi}.
\end{aligned} \tag{6.1}$$

Note that $\lim_{m\to+\infty} c_{\beta,m}(t) = c_\beta(t)$ and $\lim_{m\to+\infty} B_m(t,\bar{t}) = B(t,\bar{t})$, and by Corollary 3.1, it follows that

$$B_m(t,\bar{t}) > \lim_{z\to t}|L_m(z,t)|\exp 2G_m(z,t) \tag{6.2}$$

for m big enough, therefore $H_{m,t}(t) > 1$. This contradicts that $H_m(z) \leq 1$ for any $z \in \Omega$ when m is big enough. Then Theorem 3.2 follows.

6.3. *Proof of Theorem* 3.3. Let $p : \Delta \to \Omega$ be the universal covering of Ω. We can choose V_{z_0} small enough, such that p is biholomorphic on any component U_j $(j = 1,2,\dots)$ of $p^{-1}(V_{z_0})$.

Let $z_0 \in \Omega$ with local coordinate $w = (p|_{U_j})_*(f_{z_0}|_{U_j})$ for a fixed j. It is known that if $\chi_{-h} = \chi_{z_0}$, then $c^2_\beta(z_0) = \pi\rho(z_0)B_{\Omega,\rho}(z_0)$ holds (see [57]). Then it suffices to prove that if

$$c^2_\beta(z_0) = \pi\rho(z_0)B_{\Omega,\rho}(z_0)$$

holds, then

$$\chi_{-h} = \chi_{z_0}.$$

By the assumption

$$c^2_\beta(z_0) = \pi\rho(z_0)B_{\Omega,\rho}(z_0)$$

and by Remark 4.33, it follows that there is a unique holomorphic $(1,0)$-form F on Ω, which satisfies $((p|_{U_j})_*(f_{-h}|_{U_j}))F|_{z_0} = dw$, and

$$\sqrt{-1}\int_\Omega F\wedge\bar{F} \le \pi\int_{z_0}|dw|^2 dV_\Omega[2G_\Omega(z,z_0)].$$

It follows from Proposition 4.23 that $((p|_{U_j})_*(f_{-h}|_{U_j}))F|_{V_{z_0}} = dw$. Then we have

$$f_{-h}(p^*F)|_{U_j} = (p|_{U_j})^*dw = df_{z_0}|_{U_j}.$$

It follows from Lemma 4.28 that $f_{-h}p^*F = df_{z_0}$. As p^*F is single-valued and $df_{z_0}\in\Gamma^{\chi_{z_0}}$, it is clear that $\chi_{-h} = \chi_{z_0}$.

6.4. *Proof of Theorem* 3.6. Note that $\frac{-r}{\delta}$ has uniform positive upper and lower bound on D. Then we can consider the function $-r$ instead of δ in the present theorem.

Let $\Psi := -\log(-\frac{r}{\varepsilon_0|s|^2}+1)|_D < 0$, where ε_0 is a positive constant small enough. As r is strictly plurisubharmonic (see [33]) on $\bar{D}$, we have $r-\varepsilon_0|s|^2$ is a plurisubharmonic function on D for ε_0 small enough. Note that $-\log(-t)$ is increasing convex when $t<0$. Then $-\log(-r+\varepsilon_0|s|^2)$ is a plurisubharmonic function on D. As $\log\varepsilon_0|s|^2$ is a plurisubharmonic function on D, then Ψ is a plurisubharmonic function on D.

Let $c_0(t)|_{0<t<1} := t^\alpha$ and $c_0(t)|_{t\ge 1} := 1$. Then we have $\int_0^{+\infty}c_0(t)e^{-t}dt < \frac{1}{1+\alpha}+1$. Let $h := e^{-(\varphi-\alpha\log(-r+\varepsilon_0|s|^2))}$. Then we have $\Theta_{he^{-\Psi}}\ge 0$.

Note that there are positive constants C_3 and C_4, which are independent of α, such that $\frac{c_0(-\Psi)e^{\alpha\log(-r+\varepsilon_0|s|^2)}}{r^\alpha}\le\max\{C_3^\alpha,C_4^\alpha\}$ on D.

By the similar method in the proof of Theorem 2.2, it follows that when h is C^2 smooth, Ψ is C^2 plurisubharmonic function, and $\Theta_{he^{-\Psi}}\ge 0$, then Theorem 2.2 also holds.

For any point $z\in H$, there exists a local holomorphic defining function e of H, such that $2\log|s|-2\log|e|$ is continuous near z. Then using Lemma 4.14, for any holomorphic section f on $H\cap D$, we have an extension F of f on D, such that

$$\begin{aligned}&\int_D|F|^2(-r)^\alpha e^{-\varphi}d\lambda\\ &\qquad\le C_{(D,H)}\max\{C_3^\alpha,C_4^\alpha\}\frac{2+\alpha}{1+\alpha}e^{-\varepsilon_0}\int_{D\cap H}|f|^2(-r)^{1+\alpha}e^{-\varphi}d\lambda_H,\end{aligned}$$

where $C_{(D,H)}$ only depends on D and H. As $\frac{-r}{\delta}$ has uniform upper and lower bounds on D, we have proved Theorem 3.6.

6.5. *Proof of Theorem* 3.8. As M is a Stein manifold, then for any given f, there exists a holomorphic section F_1 on $K_M\otimes L$, such that $F_1|_S = f$.

Note that

$$\sqrt{-1}\Theta_{he^{-(2-p)\log|F_1|_h}} \geq \sqrt{-1}\frac{p}{2}\Theta_h + \frac{2-p}{2}\sqrt{-1}\partial\bar{\partial}\varphi.$$

Then the metric $he^{-(2-p)\log|F_1|_h}$ and Ψ satisfy conditions (1) and (2) in Theorem 2.1 on the Stein manifold $M \setminus \{F_1 = 0\}$.

Since M is a Stein manifold, we can find a sequence of Stein subdomains $\{D_j\}_{j=1}^{\infty}$ satisfying $D_j \subset\subset D_{j+1}$ for all j and $\bigcup_{j=1}^{\infty} D_j = M$, and all $D_j \setminus S$ are complete Kähler ([22]).

Let $A_1 := \int_{D_j} c_A(-\Psi)|F_1|_h^p dV_M < +\infty$. By the upper semicontinuity of $\log|F_1|_h$ on M, it follows that there exists a new extension F_2 on M of f satisfying

$$\int_{D_j} c_A(-\Psi)|F_2|^2_{he^{-(2-p)\log|F_1|_h}} dV_M \leq \frac{1}{\delta}c_A(-A)e^A + \int_{-A}^{\infty} c_A(t)e^{-t}dt. \tag{6.3}$$

By Hölder's inequality, it follows that

(6.4)

$$\begin{aligned}\int_{D_j} c_A(-\Psi)|F_2|_h^p dV_M &= \int_{D_j} c_A(-\Psi)\frac{|F_2|_h^p}{|F_1|_h^{p-\frac{p^2}{2}}}|F_1|_h^{p-\frac{p^2}{2}} dV_M \\ &\leq \left(\int_{D_j} c_A(-\Psi)|F_2|^2_{he^{-(2-p)\log|F_1|_h}} dV_M\right)^{\frac{p}{2}} \left(\int_{D_j} c_A(-\Psi)|F_1|_h^p dV_M\right)^{1-\frac{p}{2}},\end{aligned}$$

which is smaller than

$$\max\left\{\left(\frac{1}{\delta}c_A(-A)e^A + \int_{-A}^{\infty} c_A(t)e^{-t}dt\right)^{\frac{p}{2}} A_1^{1-\frac{p}{2}}, \right.$$
$$\left. \frac{1}{\delta}c_A(-A)e^A + \int_{-A}^{\infty} c_A(t)e^{-t}dt\right\} =: A_2.$$

If

$$A_1 \leq \frac{1}{\delta}c_A(-A)e^A + \int_{-A}^{\infty} c_A(t)e^{-t}dt,$$

then we are done. We only need to consider the case that

$$A_1 > \frac{1}{\delta}c_A(-A)e^A + \int_{-A}^{\infty} c_A(t)e^{-t}dt.$$

In this case, $A_2 < A_1$.

We can repeat the same argument with F_1 replaced by F_2 etc., and get a decreasing sequence of numbers A_k, such that

$$A_{k+1} := \max\left\{\left(\frac{1}{\delta}c_A(-A)e^A + \int_{-A}^{\infty} c_A(t)e^{-t}dt\right)^{\frac{p}{2}} A_k^{1-\frac{p}{2}}, \right.$$
$$\left. \frac{1}{\delta}c_A(-A)e^A + \int_{-A}^{\infty} c_A(t)e^{-t}dt\right\}$$

for $k \geq 1$.

It is clear that

$$A_{k+1} > \frac{1}{\delta}c_A(-A)e^A + \int_{-A}^{\infty} c_A(t)e^{-t}dt$$

and $A_{k+1} < A_k$. Then $\lim_{k\to\infty} A_k$ exists. By the definition of A_k, it follows that

$$\lim_{k\to\infty} A_k = \frac{1}{\delta}c_A(-A)e^A + \int_{-A}^{\infty} c_A(t)e^{-t}dt.$$

Then the present theorem for D_j has been proved. Let j tend to ∞. Thus we have proved the present theorem.

6.6. *Proof of Theorem* 3.12. Let $h := e^{-\varphi_r}$, where

$$\varphi_r := \varphi * \frac{\mathbf{1}_{B(0,r)}}{\mathrm{Vol}(B(0,r))}.$$

Let

$$\Psi := 2(T - T * \frac{\mathbf{1}_{B(0,r)}}{\mathrm{Vol}(B(0,r))}),$$

where T is a plurisubharmonic polar function of W on $\mathbb{C}^n$, such that $(\partial\bar{\partial}T * \frac{\mathbf{1}_{B(0,r)}}{\mathrm{Vol}(B(0,r))})(z)$ has a uniformly upper bound on $\mathbb{C}^n$ which is independent of $z \in \mathbb{C}^n$ and r.

As $D^+(W) < \frac{p}{2}$, there exists T, for r large enough, we have $D(W,T,z,r) < (1-\epsilon)\frac{p}{2}$, $(\epsilon > 0)$, which implies that

$$\sqrt{-1}\partial\bar{\partial}((1+\delta)\Psi + \varphi_r) \geq 0$$

for positive δ small enough.

Note that Ψ has uniformly upper bound on $\mathbb{C}^n$. There exists positive constants C and C', such that $C\omega < \sqrt{-1}\partial\bar{\partial}\varphi < C'\omega$. Then we have $\varphi_r - \varphi < C_r < +\infty$, where $\omega = \sqrt{-1}\partial\bar{\partial}|z|^2$. Let $c_A = 1$. Using Theorem 3.8, we obtain the present theorem.

6.7. *Proof of Corollary* 3.17. Let $c_A(t) := e^t t^{-2}$. It is easy to see that $\int_{-A}^{\infty} c_A(t)e^{-t}dt < +\infty$ and $c_A(t)e^{-t}$ is decreasing with respect to t, where $t \in (-A, +\infty)$ and $A = -2r$.

Let $\Psi = r\log(|w|^2) < -2r$, where $S = \{s = 0\}$. Let $\delta = \frac{1}{r}$. Note that $s(t) = \frac{(1+\frac{1}{r})t - \log t - 1}{2-\frac{1}{t}} \geq \frac{t}{2}$ in Theorem 2.1, and

$$\frac{\{\sqrt{-1}\Theta(E)w,w\}}{|w|^2} \geq -\sqrt{-1}\partial\bar{\partial}\log|w|^2,$$

then condition (2) in Theorem 2.1 holds.

Note that

$$\frac{|f|^2}{|\wedge^r(dw)|^2}dV_H = \sqrt{-1}^{(n-r)^2}\left\{\frac{f}{\wedge^r(dw)}, \frac{f}{\wedge^r(dw)}\right\}_h e^{-\psi}$$

(see Remark 12.7 in [16]), and

$$2^r\sqrt{-1}^{(n-r)^2}\left\{\frac{f}{\wedge^r(dw)},\frac{f}{\wedge^r(dw)}\right\}_h e^{-\psi}=|f|_h^2 dV_M[\Psi].$$

Then it follows from Remark 4.15 and Theorem 2.1 that Corollary 3.17 holds. Then we illustrate that the estimate is optimal.

Let M be the disc $\Delta_{e^{-1}}\subset\mathbb{C}$. Let E be a trivial line bundle with Hermitian metric $h_{E,a}=e^{-\max\{\log|z|^2,\log|a|^2\}-2}$, and $w=z$. Let L be a trivial line bundle with Hermitian metric $h_{L,a}=e^{-2\max\{\log|z|^2,\log|a|^2\}}$. It is clear that $r=1$, $\delta=1$. Let $\alpha=1$. Then

$$|w|=|z|e^{\frac{1}{2}(-\max\{\log|z|^2,\log|a|^2\}-2)}\leq e^{-1}=e^{-\alpha}$$

satisfying inequality (b) in Theorem 3.16, and inequality (a) in Theorem 3.16 becomes

$$\sqrt{-1}\partial\bar{\partial}2\max\{\log|z|^2,\log|a|^2\}-2\sqrt{-1}\partial\bar{\partial}(\max\{\log|z|^2,\log|a|^2\}+2)\geq 0.$$

Note that

$$\begin{aligned}\sqrt{-1}\Theta(L)+r\sqrt{-1}\partial\bar{\partial}\log|w|^2&=\sqrt{-1}\partial\bar{\partial}2\max\{\log|z|^2,\log|a|^2\}\\&\quad-\sqrt{-1}\partial\bar{\partial}(\max\{\log|z|^2,\log|a|^2\}+2)\geq 0\end{aligned}\tag{6.5}$$

and $\frac{1}{\delta}c_A(-A)e^A+\int_2^{+\infty}c_A(t)e^{-t}dt=\frac{1}{4}+\int_2^{+\infty}t^{-2}dt=\frac{3}{4}$.

Let a go to zero, by arguments in the proof of Remark 5.1. It follows that the estimate in Corollary 3.17 is optimal.

6.8. *Proof of Corollary* 3.20. It is not hard to see that $\varphi+\psi$ and $\log\frac{|w|^2}{e}-g^{-1}(e^{-\psi}g(1-\log|w|^2))$ are plurisubharmonic functions. It suffices to prove the case that M is a Stein manifold and L is a trivial line bundle with singular metric $e^{-\varphi}$ globally.

Let $\varphi_n+\psi_n$ and $\tilde{\psi}_n$ be smooth plurisubharmonic functions, which are decreasingly convergent to $\varphi+\psi$ and $\log\frac{|w|^2}{e}-g^{-1}(e^{-\psi}g(1-\log|w|^2))$ respectively, when $n\to+\infty$.

Let $g(t):=\frac{1}{c_{-1}(t)e^{-t}}$, $\Psi:=\log\frac{e}{|w|^2}+\tilde{\psi}_{n_2}$, and $h=e^{-\varphi_{n_1}-\psi_{n_1}+\tilde{\psi}_{n_2}}$. Since M is a Stein manifold, we can find a sequence of Stein subdomains $\{D_m\}_{m=1}^{\infty}$ satisfying $D_m\subset\subset D_{m+1}$ for all m and $\bigcup_{m=1}^{\infty}D_m=M$.

Note that $\log\frac{|w|^2}{e}-g^{-1}(e^{-\psi}g(1-\log|w|^2))<0$. Given n_2, for m large enough, we have $\Psi|_{D_m}=-\log\frac{e}{|w|^2}+\tilde{\psi}_{n_2}|_{D_m}<1$.

$\sqrt{-1}\partial\bar{\partial}\Psi\geq 0$ and $\sqrt{-1}\Theta_{he^{-\Psi}}\geq 0$ on $M\setminus S$ imply conditions (1) and (2) in Theorem 2.2.

Using Theorem 2.2 and Lemma 4.14, we obtain a holomorphic $(n,0)$-form F_{m,n_1,n_2} on D_m, which satisfies $F_{m,n_1,n_2}|_S = f$ and

$$\begin{aligned}
&\int_{D_m} c_{-1}(\log\frac{e}{|w|^2}-\tilde{\psi}_{n_2})|F_{m,n_2,n_1}|^2_h dV_M\\
&\leq \mathbf{C}2\pi\int_{-A}^{\infty}c_{-1}(t)e^{-t}dt\int_S|f|^2e^{-\varphi_{n_1}-\psi_{n_1}}dV_S\\
&\leq \mathbf{C}2\pi\int_{-A}^{\infty}c_{-1}(t)e^{-t}dt\int_S|f|^2e^{-\varphi-\psi}dV_S,
\end{aligned}\tag{6.6}$$

and therefore

$$\int_{D_m}\frac{ee^{-\varphi_{n_1}-\psi_{n_1}}}{|w|^2g(\log\frac{e}{|w|^2}-\tilde{\psi}_{n_2})}|F_{m,n_2,n_1}|^2dV_M\leq \mathbf{C}2\pi C(g)\int_S|f|^2e^{-\varphi-\psi}dV_S.$$

As $\frac{ee^{-\varphi_{n_1}-\psi_{n_1}}}{|w|^2g(\log\frac{e}{|w|^2}-\tilde{\psi}_{n_2})}$ has a uniform lower bound for any compact subset of $D_m\setminus S$, which is independent of n_1, it follows from Lemma 4.6 that there exists a subsequence of $\{F_{m,n_2,n_1}\}_{n_1}$, which is uniformly convergent to a holomorphic $(n,0)$-form F_{m,n_2} on any compact subset of D_m.

By dominated convergence theorem, it follows that

$$\int_{D_m}\frac{ee^{-\varphi_{n_1}-\psi_{n_1}}}{|w|^2g(\log\frac{e}{|w|^2}-\tilde{\psi}_{n_2})}|F_{m,n_2}|^2dV_M\leq \mathbf{C}2\pi C(g)\int_S|f|^2e^{-\varphi-\psi}dV_S.$$

By Levi's theorem, it follows that

$$\int_{D_m}\frac{ee^{-\varphi-\psi}}{|w|^2g(\log\frac{e}{|w|^2}-\tilde{\psi}_{n_2})}|F_{m,n_2}|^2dV_M\leq \mathbf{C}2\pi C(g)\int_S|f|^2e^{-\varphi-\psi}dV_S.$$

As $\frac{ee^{-\varphi-\psi}}{|w|^2g(\log\frac{e}{|w|^2}-\tilde{\psi}_{n_2})}$ has a uniform lower bound for any compact subset of $D_m\setminus S$, which is independent of n_2, it follows from Lemma 4.6 that there exists a subsequence of $\{F_{m,n_2}\}_{n_2}$, which is uniformly convergent to a holomorphic $(n,0)$-form F_m on any compact subset of D_m.

Note that $\frac{ee^{-\psi}}{|w|^2g(\log\frac{e}{|w|^2}-\tilde{\psi}_{n_2})}$ is decreasingly convergent to $\frac{e}{|w|^2g(\log\frac{e}{|w|^2})}$. It follows from Levi's theorem that

$$\int_{D_m}\frac{ee^{-\max\{\varphi,K\}}}{|w|^2g(\log\frac{e}{|w|^2})}|F_{m,n_2}|^2dV_M\leq \mathbf{C}2\pi C(g)\int_S|f|^2e^{-\varphi-\psi}dV_S,$$

where K is a real number.

From dominated convergence theorem on $M\setminus S$, it follows that

$$\int_{D_m}\frac{ee^{-\max\{\varphi,K\}}}{|w|^2g(\log\frac{e}{|w|^2})}|F_m|^2dV_M\leq \mathbf{C}2\pi C(g)\int_S|f|^2e^{-\varphi-\psi}dV_S.$$

Using Lemma 4.6, we have a subsequence of $\{F_m\}_m$, which is uniformly convergent to a holomorphic $(n,0)$-form F on any compact subset of M. Using dominated convergence theorem on $M \setminus S$, we have

$$\int_{D_m} \frac{ee^{-\max\{\varphi,K\}}}{|w|^2 g(\log \frac{e}{|w|^2})}|F|^2 dV_M \leq \mathbf{C}2\pi C(g)\int_S |f|^2 e^{-\varphi-\psi} dV_S.$$

When K goes to $-\infty$, using Levi's theorem, we have

$$\int_{M} \frac{ee^{-\varphi}}{|w|^2 g(\log \frac{e}{|w|^2})}|F|^2 dV_M \leq \mathbf{C}2\pi C(g)\int_S |f|^2 e^{-\varphi-\psi} dV_S.$$

Thus the present corollary follows.

6.9. *Proof of Corollary* 3.24. Let $\Psi := \log(|s|^2 e^{-\varphi_S})$ and $h := e^{-\varphi_F-\varphi_S}$. Then it is clear that $\Psi \leq -\alpha$ and $A = -\alpha$. Let $c_{-\alpha}(t) := e^{(1-b)t}$ and $\delta = \frac{1}{\alpha}$. Then we have $c_{-\alpha}(\alpha)e^{-\alpha} = e^{-b\alpha}$ and $\int_\alpha^{+\infty} c_{-\alpha}(t)e^{-t}dt = \frac{1}{b}e^{-b\alpha}$.

When φ_S and φ_F are both smooth, using Theorem 2.1 and Remark 4.14, we obtain $C_b = 2\pi(\alpha e^{-b\alpha} + \frac{1}{b}e^{-b\alpha})(\max_M |s|^2 e^{-\bar{\varphi}_S})^{1-b}$.

Now we discuss the general case. (φ_S and φ_F may not be smooth.) As M is Stein, we can choose relatively compact strongly pseudoconvex domains $\{\Omega_n\}_{n=1,2,\ldots}$ of M exhausting M.

Note that $\varphi_F = \alpha\varphi_S - (\alpha\varphi_S - \varphi_F)$. By Lemma 4.13, it follows that there exist smooth functions $\{\varphi_{S,j}\}_{j=1,2,\ldots}$ and $\{\varphi_{F,j}\}_{j=1,2,\ldots}$, such that

(1) $\{\varphi_{S,j}\}_{j=1,2,\ldots}$ are plurisubharmonic functions;
(2) $\{\alpha\varphi_{S,j} - \varphi_{F,j}\}_{j=1,2,\ldots}$ are plurisubharmonic functions;
(3) $\{\varphi_{S,j}\}_{j=1,2,\ldots}$ and $\{\alpha\varphi_{S,j} - \varphi_{F,j}\}_{j=1,2,\ldots}$ are decreasingly convergent to φ_S and $\alpha\varphi_S - \varphi_F$ respectively;
(4) given n, there exists j_n such that for any $j \geq j_n$, $|w|^2 e^{-\varphi_{S,j}}|_{\Omega_n} \leq e^{-\alpha}$.

Using the smooth case which we have already discussed, we obtain holomorphic $(n,0)$-forms $\{U_{n,j}\}_{n,j}$ satisfying the optimal estimate (3.10) on Ω_n for $\varphi_{S,j}$ and $\varphi_{F,j}$.

Note that $b\varphi_{S,j} + \varphi_{F,j} = -b(\alpha\varphi_{F,j} - \varphi_{S,j}) + (b\alpha+1)\varphi_{F,j}$. While $\varphi_{F,j}$ is invariant, let $\alpha\varphi_{F,j} - \varphi_{S,j}$ go to $\alpha\varphi_F - \varphi_S$. From Lemma 4.34 it follows that there exists a subsequence of $\{U_{n,j}\}_j$, denoted by $\{U_{n,j}\}_j$, which is uniformly convergent on any compact subset of Ω_n.

First let $\alpha\varphi_{F,j} - \varphi_{S,j}$ go to $\alpha\varphi_F - \varphi_S$, and then let $\alpha\varphi_{F,j}$ go to $\alpha\varphi_F$. Using Levi's Theorem, we obtain that the limit U_n of $\{U_{n,j}\}_j$ satisfies the estimate (3.10) on Ω_n.

Using weak compactness of unit ball in the Hilbert space

$$L^2_{e^{-b\varphi_S-\varphi_F-(1-b)\bar{\varphi}_S}}(\Omega_n) \cap \{\text{holomorphic } (n,0)\text{-form}\},$$

Lemma 4.34 and the diagonal method, we have a subsequence of $\{U_n\}_n$, still denoted by $\{U_n\}_n$, uniformly convergent to a holomorphic $(n,0)$-form U on any compact subset of Ω for n large enough, such that U satisfies the estimate (3.10) on any Ω_n. Therefore, U satisfies the estimate (3.10) on Ω. Then Corollary 3.24 follows.

We conclude the present subsection by pointing out that C_b is optimal. Let M be the disc $\Delta_{e^{-\frac{\alpha}{2}}} \subset \mathbb{C}$. By Remark 5.1, and letting $e^{\bar{\varphi}_S}$ be decreasingly convergent to $|s|^2$, we can obtain that the estimate in Corollary 3.24 is optimal.

Acknowledgements. The authors would like to thank Professor Yum-Tong Siu, Professor J-P. Demailly and Professor Bo Berndtsson for giving series of talks at CAS and explaining us their works. The authors would also like to thank the referees for suggestions and comments. An announcement of the present paper appears in [28].

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(Received: July 31, 2013)
(Revised: December 9, 2013)

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