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Periodic approximations of irrational pseudo-rotations using pseudoholomorphic curves

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Abstract

We prove that every C^∞-smooth, area preserving diffeomorphism of the closed 2-disk having not more than one periodic point is the uniform limit of periodic C^∞-smooth diffeomorphisms. In particular, every smooth irrational pseudo-rotation can be C^0-approximated by integrable systems. This partially answers a long standing question of A. Katok regarding zero entropy Hamiltonian systems in low dimensions. Our approach uses pseudoholomorphic curve techniques from symplectic geometry.

Contents

1. Introduction

1.1. *The main result.* In this paper we prove a statement that is a significant first step towards answering the following general question of Katok.

In low dimensions is every conservative dynamical system with zero topological entropy a limit of integrable systems?

This is stated as Problem 1 in [27], but it relates also to the Anosov-Katok constructions [1] in 1970. Low dimensions means maps on surfaces or flows on 3-dimensional manifolds. The result in this article is concerned with Katok's question for area preserving disk maps.

Arguably one of the main obstacles to an affirmative answer to this question is the presence of ergodic components of positive measure. Indeed, ergodic maps (with respect to Lebesgue measure) exhibit strongly different dynamical behavior from integrable ones. In particular, almost every point is the initial condition for a dense orbit.

With regard to area preserving disk maps, the only known ergodic examples with zero entropy are so-called irrational pseudo-rotations. These are the area preserving disk maps with precisely one periodic point; see Definition 3. Using pseudoholomorphic curve methods we show that all irrational pseudo-rotations are in some sense limits of integrable systems as Katok's question suggests.

More precisely, for each $t \in \mathbb{R}$, let $R_{2\pi t} : D \to D$ denote the rigid rotation through angle $2\pi t$ on the disk $D = \{(x, y) \,|\, x^2 + y^2 \leq 1\}$. Let $\mathrm{Diff}^\infty_+(D)$ and $\mathrm{Diff}^\infty(D, \omega_0)$ be the spaces of C^∞-smooth diffeomorphisms of the disk that preserve orientation and the area form $\omega_0 = dx \wedge dy$ respectively.

Theorem 1 (Main result). *Suppose $\varphi \in \mathrm{Diff}^\infty(D, \omega_0)$ fixes the origin and has no other periodic points. Then it is the C^0-limit of a sequence of maps of the form $\varphi_k = g_k^{-1} R_{2\pi p_k/q_k} g_k$, for a sequence of conjugating maps $g_k \in \mathrm{Diff}^\infty_+(D)$ that fix the origin, and a sequence of rationals p_k/q_k converging to an irrational number.*

Concerning the convergence of the approximation maps in Theorem 1 it is natural to ask whether C^0-convergence is so weak as to allow "almost anything" to be obtained in the limit. A more natural topology to consider Katok's question, as discussed in [27], is at least a $C^{1,\varepsilon}$-topology, $\varepsilon > 0$, in which the topological entropy is lower semi-continuous. The author would therefore like to thank Patrice Le Calvez for pointing out the following and the idea of its proof using work of Franks. A slightly more general statement is proven in Appendix A.1.

Proposition 2. *Suppose $\varphi \in \mathrm{Diff}^\infty(D, \omega_0)$ is the C^0-limit of a sequence of maps of the form $\varphi_k = g_k^{-1} R_{2\pi p_k/q_k} g_k$, for a sequence of conjugating maps $g_k \in \mathrm{Diff}^\infty_+(D)$ fixing the origin, and a sequence of rationals p_k/q_k converging to an irrational number. Then φ necessarily has precisely one periodic point. In particular, it is an irrational pseudo-rotation.*

Thus C^0-convergence is still strong enough to guarantee that the limit object is an irrational pseudo-rotation — in particular, that it has zero entropy.

(This is due, for example, to Katok's theorem for $C^{1,\varepsilon}$ surface diffeomorphisms [26] that bounds entropy from above by the exponential growth rate of periodic points.)

The existence of ergodic area preserving disk maps with zero entropy was established back in 1970 by Anosov and Katok [1]. Previous to their constructions it was even an open question in the nonconservative setting; Shnirelman 1930 [36] found a (non area preserving) homeomorphism of the disk with a dense orbit, a more detailed discussion of which can be found in [10].

It is interesting to compare the Anosov-Katok construction to the statement of Theorem 1. They construct an ergodic map $\varphi : D \to D$ as the C^∞-limit of a sequence of maps $\varphi_k : D \to D$ that are inductively constructed with the following form. For each $k \in \mathbb{N}$, there exists $(p_k, q_k) \in \mathbb{Z} \times \mathbb{N}$ relatively prime, and $g_k \in \mathrm{Diff}^\infty(D, \omega_0)$, also fixing the origin, so that

$$\varphi_k = g_k^{-1} \circ R_{2\pi p_k/q_k} \circ g_k. \tag{1}$$

The maps g_k are arranged so that the orbits of φ_k increasingly spread out over the disk as $k \to \infty$. Consequently, the sequence $\{g_k\}$ blows up in every C^r-topology. But by iteratively choosing $q_{k+1} - q_k$ sufficiently large depending on the size of $\|g_k\|_{C^k}$, the C^k norm of φ_k can be controlled. A limiting subsequence converges to a map φ with the desired "pathological" behavior such as a dense orbit, or ergodicity, or even weak mixing. More details of this method, other results and questions are in Fayad-Katok [10]. See also Fayad-Saprykina [11].

In some sense then, Theorem 1 reverses the limiting process just described above. However our conclusions are in two respects weaker than a word for word converse to the Anosov-Katok construction. Firstly, the convergence $\varphi_k \to \varphi$ in [1] is in the C^∞-sense. Secondly, each φ_k in [1] preserves the standard area form. We do not show this for the approximation maps in Theorem 1. This raises natural questions for further investigation.

A remark on integrability: The notion of integrability for a map on a surface that appears to be referred to in [27] is that the map should admit a "first integral" — that is, a smooth real valued function on the surface that is not constant on any open set but is constant on the orbits of the given map. It is obviously in this sense that each of our approximation maps in Theorem 1 is integrable. A natural question is whether approximation maps can be found that are integrable in the Liouville-Arnold sense. This would follow if they were area preserving.

1.2. *Outline of the proof.* Let $\varphi : D \to D$ be an irrational pseudo-rotation. That is, φ is a smooth area and orientation preserving diffeomorphism that fixes the origin and has no other periodic points. As a consequence the circle map on the boundary $\varphi : \partial D \to \partial D$ has irrational rotation number $[\alpha] \in \mathbb{R}/\mathbb{Z}$ say.

We pick a closed loop of Hamiltonians $H_t : D \to D$, over $t \in \mathbb{R}/\mathbb{Z}$, that generate a symplectic isotopy whose time-one map is φ. Denote the 1-periodic path of Hamiltonian vector fields on the disk by X_{H_t}. For each $n \in \mathbb{N}$, equip the solid torus

$$Z_n := \mathbb{R}/n\mathbb{Z} \times D$$

with coordinates (τ, z). Then the vector field $R_n(\tau, z) = \partial_\tau + X_{H_\tau}(z)$ defines a flow on Z_n with time-one map φ and first return map φ^n. For each $n \in \mathbb{N}$, there is a unique simple periodic orbit in Z_n (corresponding to the unique fixed point of φ^n at the origin). We can choose $t \mapsto H_t$ so that this periodic orbit in Z_n passes through the center of each disk slice and indeed is parametrized by

$$\begin{aligned} \gamma_n : \mathbb{R}/n\mathbb{Z} &\to Z_n, \\ t &\mapsto (t, 0). \end{aligned}$$

Consider the 4-manifold $W_n := \mathbb{R} \times Z_n$ with the unique almost complex structure satisfying

$$\begin{cases} J_n \partial_{\mathbb{R}} = R_n, \\ J_n|_{TD} = i, \end{cases} \tag{2}$$

where $\partial_{\mathbb{R}}$ is the vector field dual to the $\mathbb{R}$-coordinate on W_n. Then (W_n, J_n) is a so-called cylindrical, symmetric, almost complex manifold. That is, it is compatible in a precise way with the necessary symplectic structures for the compactness framework from symplectic field theory [4] to apply to J_n-holomorphic curves. In Section 7 we adapt techniques developed by Hofer, Wysocki, and Zehnder [18], [21], [22], [25], [24] to construct, for each $n \in \mathbb{N}$, a pair of finite energy foliations $\mathcal{F}_n^+$ and $\mathcal{F}_n^-$ of the almost complex manifold (W_n, J_n). In this introduction we will only need to refer to the sequence $(\mathcal{F}_n^+)$, which we therefore abbreviate by $(\mathcal{F}_n)$. We describe these foliations in a moment. However, having constructed the foliations, the proof of Theorem 1 consists of three steps:

(1) For each $n \in \mathbb{N}$, we use the leaves in $\mathcal{F}_n$ to define a disk map $\varphi_n = \varphi_{\mathcal{F}_n} : D \to D$. Since this is defined by following a path along the leaves, we will refer to it as the *holonomy map of* $\mathcal{F}_n$. Each map φ_n inherits the smoothness of the foliation.

(2) A symmetry in the leaves of $\mathcal{F}_n$ implies that φ_n is integrable in the sense that $(\varphi_n)^n = \mathrm{id}_D$. Indeed, by classical results this implies that φ_n is at least topologically conjugate to a rotation $R_{2\pi q/n} : D \to D$ for some $q \in \{0, 1, \ldots, n-1\}$.

(3) There exists a subsequence n_j such that φ_{n_j} converges to the given irrational pseudo-rotation φ.

We now outline these steps in more detail.

Step 1. For each $n \in \mathbb{N}$, $\mathcal{F}_n$ is a 2-dimensional foliation of the 4-manifold $\mathbb{R}\times Z_n = \mathbb{R}\times\mathbb{R}/n\mathbb{Z}\times D$ by surfaces called the leaves of $\mathcal{F}_n$. Each leaf $F \in \mathcal{F}_n$ is the image of a properly embedded, noncompact J_n-holomorphic map satisfying a finite energy condition. For precise definitions, see Section 3. Here let us describe the geometric properties of the leaves in $\mathcal{F}_n$. There is a distinguished leaf $C_n \in \mathcal{F}_n$ that is homeomorphic to $\mathbb{R} \times S^1$, which we call the *cylinder* in $\mathcal{F}_n$; see Figure 1. This has the following simple description:

$$C_n = \left\{(a, \gamma_n(t)) \in \mathbb{R} \times Z_n \,\middle|\, a \in \mathbb{R},\ t \in S^1\right\}.$$

Thus C_n intersects each slice $\{c\}\times Z_n$ in the trace of the unique simple periodic orbit γ_n. The other leaves are homeomorphic to $[0,\infty)\times S^1$ and will be referred to as *half cylinders*. Each half cylinder F can be parametrized by a smooth embedding $\tilde{u} : [0,\infty) \times \mathbb{R}/n\mathbb{Z} \to W_n = \mathbb{R} \times Z_n = \mathbb{R} \times \mathbb{R}/n\mathbb{Z} \times D$ of the form $(s,t) \mapsto (s - s_0, t, z(s,t))$ for some constant s_0 depending on F. The loops $s \mapsto (\cdot, z(s,\cdot))$ begin in the boundary of Z_n at $s = 0$ and converge in $C^\infty(\mathbb{R}/n\mathbb{Z}, Z_n)$ to the unique simple periodic orbit γ_n as $s \to +\infty$.

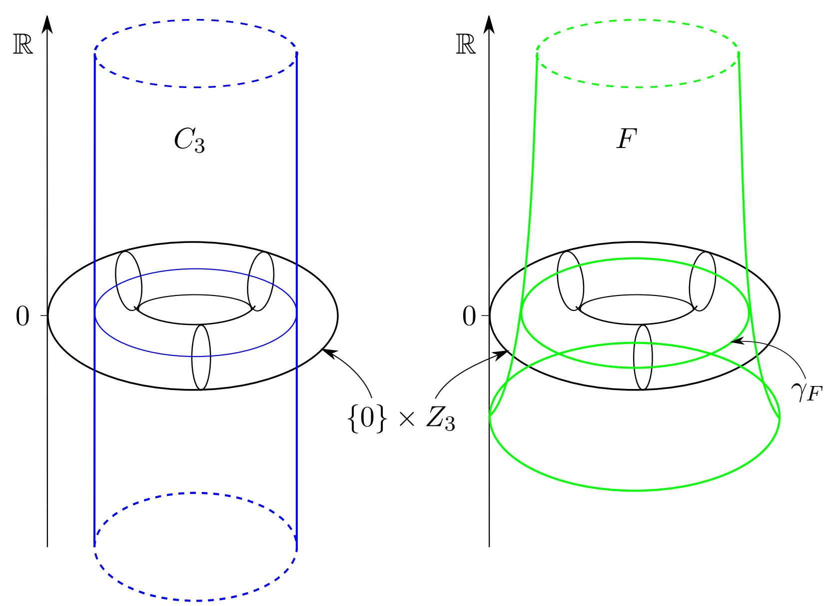

Figure 1. Illustrating $n = 3$: both figures separately represent leaves in $\mathbb{R} \times Z_3$ coming from the foliation $\mathcal{F}_3$. On the left the leaf is the cylinder leaf C_3, and on the right is a typical half cylinder leaf F. Both leaves intersect the hypersurface $\{0\} \times Z_3$ transversely in embedded closed loops.

In fact the half cylinders come in a 2-parameter family. One parameter comes from translations in the $\mathbb{R}$-direction: if $F \in \mathcal{F}_n$ is a half cylinder, then for each $c \in \mathbb{R}$, the translated set

$$F_c := \Big\{(a+c, m) \in \mathbb{R} \times Z_n \,\Big|\, (a,m) \in F\Big\}$$

will also be a half cylinder in $\mathcal{F}_n$.

For a half cylinder $F \in \mathcal{F}_n$, consider the intersection

$$\gamma_F := F \cap \big(\{0\} \times Z_n\big).$$

Identifying the hypersurface $\{0\} \times Z_n$ with Z_n, we can view

$$\gamma_F \subset Z_n.$$

It turns out that γ_F is either empty or is an embedded circle that closes up after going "once" around the solid torus. For any point $p \in Z_n$, there exists a leaf $F \in \mathcal{F}_n$ such that $p \in \gamma_F$. Moreover, if G is another leaf for which $p \in \gamma_G$, then G and F are related by an $\mathbb{R}$-translation in $\mathbb{R} \times Z_n$, and so $\gamma_G = \gamma_F$. Thus the set

$$\Delta(\mathcal{F}_n) := \Big\{\gamma_F \subset Z_n \,\Big|\, F \in \mathcal{F}_n\Big\}$$

is some kind of filling of the mapping torus Z_n by pairwise disjoint embedded circles. Note that the trace of the periodic orbit γ_n also lies in $\Delta(\mathcal{F}_n)$ since

$$\gamma_{C_n} = \operatorname{image}(\gamma_n).$$

It turns out that each circle $\gamma \in \Delta(\mathcal{F}_n)$ intersects each disk slice $\{\tau\} \times D \subset Z_n$ transversely and uniquely. This is a consequence of the positivity of intersections phenomenon between pairs of distinct pseudoholomorphic curves; the disks $\{a, \tau\} \times D \subset \mathbb{R} \times Z_n$ are also images of embedded J_n-holomorphic maps. We can therefore define a new disk map

$$\varphi_n = \varphi_{\mathcal{F}_n} : D \to D$$

that maps the point $\xi \in D$ to $\xi' \in D$ if there exists a loop $\gamma \in \Delta(\mathcal{F}_n)$ connecting the points $(0, \xi)$ and $(1, \xi)$; see Figure 2. We refer to φ_n as the *holonomy map associated to* $\mathcal{F}_n$.

From the properties above the holonomy map is a bijection between the disks. Since $\operatorname{image}(\gamma_n) \in \Delta(\mathcal{F}_n)$, we know that $\varphi_{\mathcal{F}_n}$ fixes the origin. This completes our definition and geometric description of the holonomy maps.

Step 2. The integrability of φ_n comes from the following symmetry within the foliation $\mathcal{F}_n$. Our 3-manifold Z_n has a natural $\mathbb{Z}_n$-action on it given by the deck transformations. The corresponding $\mathbb{Z}_n$-action on W_n generated by the map

$$\begin{aligned} \mathcal{T} : \mathbb{R} \times Z_n &\to \mathbb{R} \times Z_n, \\ (a, \tau, z) &\mapsto (a, \tau + 1, z) \end{aligned} \tag{3}$$

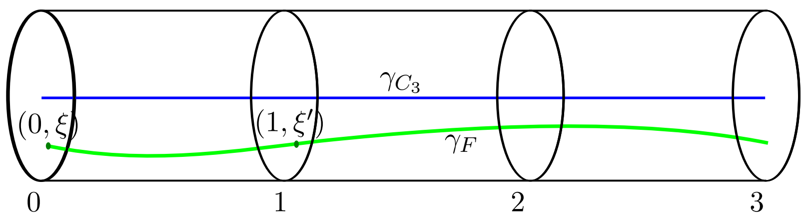

Figure 2. Illustrating the definition of the holonomy map $\varphi_3 : D \to D$. The tube represents the mapping torus Z_3 opened out. Identify this space with the hypersurface $\{0\}\times Z_3$ inside $\mathbb{R}\times Z_3$. Then the curves γ_{C_3} and γ_F are the closed loops from Figure 1 where the leaves C_3 and F respectively intersect $\{0\} \times Z_3$. We then set $\varphi_3(\xi) = \xi'$.

also preserves the almost complex structure J_n defined in (2). Thus if F is a leaf in $\mathcal{F}_n$, then $\mathcal{T}(F)$ is the image of another J_n-holomorphic map, and we can ask whether $\mathcal{T}(F)$ is also a leaf in $\mathcal{F}_n$. This turns out to be the case, the key reason being the positivity of intersections phenomenon for pseudoholomorphic curves, plus certain convenient features of the dynamics. We conclude then that

$$\mathcal{T}(\mathcal{F}_n) := \left\{\mathcal{T}(F) \,\middle|\, F \in \mathcal{F}_n\right\} = \mathcal{F}_n$$

for each $n \in \mathbb{N}$. We say that $\mathcal{F}_n$ has *deck transformation invariance*. The deck transformation invariance of $\mathcal{F}_n$ implies that the set of embedded loops $\Delta(\mathcal{F}_n) \subset Z_n$ is invariant under

$$\begin{aligned} \hat{\mathcal{T}} &: Z_n \to Z_n, \\ &(\tau, z) \mapsto (\tau + 1, z). \end{aligned} \tag{4}$$

That is, if $\gamma \in \Delta(\mathcal{F}_n)$, then $\mathcal{T}(\gamma) \in \Delta(\mathcal{F}_n)$. As a consequence the holonomy map is n-periodic. That is,

$$\left(\varphi_{\mathcal{F}_n}\right)^n = \mathrm{id}_D\,.$$

To see this consider the disk maps

$$\psi_0,\ \psi_1, \ldots, \psi_{n-1},$$

where for each $k \in \mathbb{Z}_n$, ψ_k is defined to take $\xi \in D$ to $\xi' \in D$ if the loop $\gamma \in \Delta(\mathcal{F}_n)$ that contains (k, ξ) also passes through $(k + 1, \xi')$. In particular, $\psi_0 = \varphi_{\mathcal{F}_n}$. Now fix any point $\xi \in D$. Consider the unique loop $\gamma \in \Delta(\mathcal{F}_n)$ that contains $(0, \xi) \in Z_n$. Following γ around the solid torus Z_n it must intersect the disk slice $\{1\}\times D$ at the point $(1, \psi_0(\xi))$, the slice $\{2\}\times D$ at $(2, \psi_1\circ\psi_0(\xi))$, and so on. Continuing, going around Z_n once, γ will intersect $\{0\} \times D$ again at the point $(0, \psi_{n-1} \circ \cdots \circ \psi_0(\xi))$. On the other hand γ closes up after going

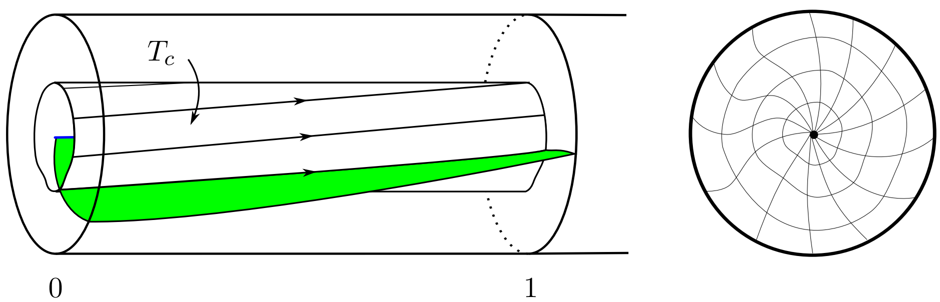

Figure 3. Illustrating a 2-torus in Z_n, which corresponds to an invariant circle for the n-th holonomy map $\varphi_{\mathcal{F}_n}$. The shaded strip is a portion of the projection into Z_n of a leaf $F \in \mathcal{F}_n$. This strip would close up in time n. For each $c \in (-\infty, 0]$, we obtain a 2-torus $T_c \in Z_n$ of which one is illustrated. The set T_c is defined as the union of all closed loops γ_G as $G \in \mathcal{F}_n$ runs over all leaves with boundary condition $\partial G \in \{c\} \times \partial Z_n$ for the given $c \in (-\infty, 0]$. (For $c > 0$, this defines an empty set.) As $c \to -\infty$, the tori T_c collapse onto the periodic orbit γ_n at the center. Note that although T_c is comprised of n-periodic loops, T_c itself is 1-periodic due to the deck transformation invariance of $\mathcal{F}_n$. The concentric circles in the figure on the right illustrate how these 2-tori intersect the disk $\{0\} \times D$ in circles invariant under $\varphi_{\mathcal{F}_n}$. Each radial arc would get mapped to another radial arc under $\varphi_{\mathcal{F}_n}$.

once around Z_n and so this point is also $(0, \xi)$. Thus,

$$\psi_{n-1} \circ \cdots \circ \psi_0 = \mathrm{id}_D \,.$$

However it follows from the deck transformation invariance of $\Delta(\mathcal{F}_n)$ that each of these maps is the same, and so

$$\psi_k = \varphi_{\mathcal{F}_n}$$

for all $0 \leq k \leq n-1$. Therefore $\left(\varphi_{\mathcal{F}_n}\right)^n = \mathrm{id}_D$.

Step 3. We claim that there exists a subsequence of the holonomy maps $\varphi_{\mathcal{F}_{n_k}} : D \to D$ that converges to the given pseudo-rotation φ as $k \to \infty$. To explain this heuristically, start with any mapping torus (Z_n, R_n) and consider the following way in which a trajectory $\gamma : \mathbb{R} \to Z_n$, that is a solution to $\dot{\gamma}(t) = R_n(\gamma(t))$, trivially gives rise to an immersed surface F_γ in $W_n = \mathbb{R} \times Z_n$ by taking the product space

$$F_\gamma := \mathbb{R} \times \gamma(\mathbb{R}) \subset W_n. \tag{5}$$

Since J_n couples the $\mathbb{R}$-direction with the R_n-direction, this surface is a (immersed but not in general embedded) J_n-holomorphic curve parametrized by $\tilde{u} : \mathbb{C} \to W_n$, $\tilde{u}(s,t) = (s, \gamma(t))$. Moreover, for two trajectories γ_1 and γ_2, the surfaces F_{γ_1} and F_{γ_2} are either equal or disjoint, and indeed the union

$$\mathcal{F}^{\text{vert}} := \left\{ F_\gamma \,\middle|\, \gamma \right\}$$

over all trajectories is a foliation of $\mathbb{R} \times Z_n$ by J_n-holomorphic curves. We will call this the (unique) vertical foliation of (W_n, J_n), because each leaf is invariant under translations in the $\mathbb{R}$-direction in $W_n = \mathbb{R} \times Z_n$ and it is standard to draw the $\mathbb{R}$-axis in this context pointing vertically. The holonomy map associated to this foliation is clearly just the time-one map of the flow of R_n that is the irrational pseudo-rotation φ. So we may formally write

$$\varphi_{\mathcal{F}^{\text{vert}}} = \varphi.$$

Amongst all J_n-holomorphic curves in (W_n, J_n), the ones that take the form of a product with a trajectory, as in (5), are also distinguished by the vanishing of a certain integral over the surface. More precisely, let ω be the exact differential 2-form $dx \wedge dy + d\tau \wedge dH$ on $Z_n = \mathbb{R}/n\mathbb{Z} \times D$, where (x, y) are standard coordinates on D, and τ is the $\mathbb{R}/n\mathbb{Z}$-coordinate, and $H : Z_n \to \mathbb{R}$ is the Hamiltonian. Then for any vector v in the tangent space to W_n, one can check that $\omega(v, Jv) \geq 0$ with equality if and only if $v \in \operatorname{span}\{\partial_{\mathbb{R}}, R_n\}$. Thus if $S \subset W_n$ is an immersed surface with J_n-invariant tangent bundle, then the integral

$$\int_S \omega \geq 0$$

is nonnegative and vanishes if and only if S is tangent to the 2-plane distribution with fibers $\operatorname{span}\{\partial_{\mathbb{R}}, R_n\}$. That is, if and only if S is contained in a "vertical" surface of the form $\mathbb{R} \times \gamma(\mathbb{R})$ for an R_n-trajectory γ, as in (5).

We can therefore view the value of the ω-integral (also called the ω-energy) of each leaf in the foliation $\mathcal{F}_n$ as a measure of how close to the vertical foliation it is, or alternatively how close the loop γ_F is to an R_n-trajectory. Using Stokes' theorem the ω-integral for the leaves can in fact be calculated. For the cylinder leaf the integral vanishes, while for each half cylinder leaf $F \in \mathcal{F}_n$ one gets the more interesting quantity

$$\int_F \omega = \pi\{n\alpha\},$$

where α is the rotation number of the pseudo-rotation on the boundary of the disk, and $\{x\} \in [0, 1)$ denotes the fractional part of $x \in \mathbb{R}$. x

Since α is irrational, there exists a subsequence $n_k \in \mathbb{N}$ for which $\{n_k\alpha\} \to 0$. Thus for the corresponding subsequence of foliations $(\mathcal{F}_{n_k})$, the above integrals over the leaves decays uniformly to zero as $k \to \infty$. Compactness properties of pseudoholomorphic curves imply that these foliations converge in some sense to the vertical foliation $\mathcal{F}^{\text{vert}}$, because the latter is the unique foliation for which the ω-integral vanishes on all leaves. From this one can conclude that the subsequence of holonomy maps $(\varphi_{\mathcal{F}_{n_k}})$ converges to the holonomy map of $\mathcal{F}^{\text{vert}}$. That is,

$$\lim_{k\to\infty} \varphi_{\mathcal{F}_{n_k}} = \varphi_{\mathcal{F}^{\text{vert}}}.$$

However, as we observed above, the right-hand side is simply the pseudo-rotation φ.

1.3. *Results in the literature of related interest.* Using different techniques, Le Calvez proved in 2004 (see [30, Th. 1.9]) that every minimal C^1-diffeomorphism of the 2-torus that is homotopic to the identity can be C^0-approximated by periodic C^1-diffeomorphisms. Recall that a diffeomorphism is minimal if every point is the initial condition for a dense orbit. Thus, in this result also strongly nonintegrable maps are approximated by, in some sense, integrable ones.

An interesting result about irrational pseudo-rotations in the class of homeomorphisms of the open and closed annulus homotopic to the identity was obtained by Béguin-Crovisier-LeRoux-Patou [3] and Béguin-Crovisier-LeRoux [2]. Stated for maps on the closed disk this is as follows. Let φ be an orientation preserving, measure preserving, homeomorphism of the disk with a single periodic point and boundary rotation number $\alpha \in \mathbb{R}/\mathbb{Z}$. Then the rigid rotation $R_{2\pi\alpha}$ is the C^0-limit of maps (not necessarily area preserving) conjugate to φ. The authors of these papers note that one does not know from their approach that φ is in the closure of the set of maps conjugate to $R_{2\pi\alpha}$.

Note added in Proof. Patrice Le Calvez recently announced in [32] a proof of a version of Theorem 1 using generating functions. The result he obtains is for C^1-pseudo-rotations that satisfy a condition on the boundary.

1.4. *Acknowledgements.* I would particularly like to thank Helmut Hofer for his interest, encouragement, and valuable discussions, and also for constructive comments on an earlier version of this paper. I also wish to thank Anatole Katok for his attention to this work and for pointing out to me the relevance of his question to the main result; see Problem 1 in [27]. I thank Patrice LeCalvez for his interest, for pointing out that Proposition 2 should hold, and for explaining the idea of its proof (also as stated in Appendix A.1), and Richard Siefring and Chris Wendl for many helpful discussions about pseudoholomorphic curves. This article was written at the IAS in Princeton; it is a pleasure to acknowledge the wonderful working environment there. Finally I thank the anonymous referees for many helpful comments.

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2. **Irrational pseudo-rotations**

Definition 3. A (smooth) *irrational*, *pseudo-rotation* is a C^∞-diffeomorphism $\varphi : D \to D$ of the closed 2-disk D with the following properties:

(1) φ preserves the volume form $dx \wedge dy$.
(2) $\varphi(0) = 0$.
(3) φ has no periodic points in $D \backslash \{0\}$.

There are equivalent definitions that admit generalizations to rational pseudo-rotations, which we will not need; see, for example, [31] and [3].

If $\varphi : D \to D$ is an irrational pseudo-rotation, then the restriction of φ to the boundary is an orientation preserving circle diffeomorphism without periodic points. It therefore has irrational rotation number on the boundary.

More precisely, let $\pi : \mathbb{R} \to \partial D$ be the projection map $x \mapsto e^{2\pi i x}$. Then for any lift $f : \mathbb{R} \to \mathbb{R}$ of $\varphi|_{\partial D}$ via π, the limit

$$\tau(f) := \lim_{n\to\infty} \frac{f^n(x) - x}{n} \in \mathbb{R} \tag{6}$$

exists and is independent of x (see, for example, [28]) and is called the translation number of f. Furthermore, the element $[\tau(f)] \in \mathbb{R}/\mathbb{Z}$ in the quotient space is even independent of the choice of lift f and is called the rotation number of $\varphi|_{\partial D}$.

Definition 4. Let $\varphi : D \to D$ be an irrational pseudo-rotation. Then we define the *rotation number* of φ to be the value on the circle

$$\mathrm{Rot}(\varphi) := [\tau(f)] \in \mathbb{R}/\mathbb{Z}$$

for any lift $f : \mathbb{R} \to \mathbb{R}$ of the restriction $\varphi : \partial D \to \partial D$.

A preferred homotopy $\{\varphi_t\}_{t\in[0,1]}$ from $\varphi_0 = \mathrm{id}_{\partial D}$ to $\varphi_1 = \varphi|_{\partial D}$ gives us a preferred lift of $\varphi|_{\partial D}$ — namely, the terminal map of the unique lift to a homotopy in the universal covering space that begins at $\mathrm{id}_{\mathbb{R}}$. In particular, any Hamiltonian generating φ as its time-one map restricts to a homotopy on the boundary of the disk from $\mathrm{id}_{\partial D}$ to $\varphi|_{\partial D}$ and thus determines a canonical lift of the latter. Using this, we define

Definition 5. Let $\varphi : D \to D$ be an irrational pseudo-rotation. Let $H_t \in C^\infty(D, \mathbb{R})$ be a path of Hamiltonians on $(D, \omega_0 = dx \wedge dy)$ generating φ as its time-one map. Then we define the *rotation number of* φ *with respect to* H to be the real number

$$\mathrm{Rot}(\varphi; H) := \tau(f) \in \mathbb{R},$$

where $f : \mathbb{R} \to \mathbb{R}$ is the canonical lift of $\varphi : \partial D \to \partial D$ determined by H.

If φ is an irrational pseudo-rotation, then the unique periodic point is nondegenerate in the sense that for all $n \in \mathbb{N}$, the linearization $D\varphi^{(n)}(0)$ does not have eigenvalue 1. The proof is a well-known application of the Poincaré-Birkhoff fixed point theorem; see Appendix A.2.

3. Finite energy foliations

For any $H \in C^\infty(\mathbb{R}/\mathbb{Z}\times D, \mathbb{R})$ with $H_t := H(t, \cdot)$ constant on the boundary of D for each $t \in \mathbb{R}/\mathbb{Z}$, the smooth time-dependent vector field $X_H(t, \cdot) := X_{H_t}$ on D defined by

$$\omega_0(X_{H_t}(z), \cdot) = -dH_t(z)$$

for all $z = (x, y) \in D$ is tangent to ∂D and therefore generates a 1-parameter family of diffeomorphisms $\phi^t : D \to D$ over $t \in \mathbb{R}$. Using that the disk is simply connected, it is well known that one may find an H for any element $\varphi \in \mathrm{Diff}^\infty(D, \omega_0)$ so that $\varphi = \phi^1$. Then H is said to generate φ.

From now on let $\varphi : D \to D$ be a fixed irrational pseudo-rotation. Unless stated otherwise, $H \in C^\infty(\mathbb{R}/\mathbb{Z} \times D, \mathbb{R})$ is a 1-periodic time-dependent Hamiltonian generating φ.

Remark 1. By precomposing H with a suitable closed loop in $\mathrm{Diff}^\infty(D, \omega_0)$ based at the identity, we may assume that the unique 1-periodic orbit of X_{H_t} corresponding to the fixed point $0 \in D$ of φ is the constant trajectory $t \mapsto 0 \in D$ for all $t \in \mathbb{R}$. This is not necessary, but it makes the proof of Theorem 8 slightly easier.

Define a smooth vector field R_H on the solid torus $Z := \mathbb{R}/\mathbb{Z} \times D$ by

$$R_H(\tau, z) := \partial_\tau + X_H(\tau, z) \tag{7}$$

for all $\tau \in \mathbb{R}/\mathbb{Z}$, $z \in D$. The first return map on $\{0\}\times D$ is canonically identified with the pseudo-rotation φ.

For each $n \in \mathbb{N}$, let Z_n be the 3-manifold-with-boundary $\mathbb{R}/n\mathbb{Z}\times D$ and R_n the vector field on Z_n that projects down to R_H under the natural projection $\pi_n : Z_n \to Z$. Clearly the first return map of the flow generated by R_n is the n-th iterate $\varphi^n : D \to D$. We will refer to the pair (Z_n, R_n) as *the mapping torus of length-n associated to H*. It will also be useful to denote by $Z_\infty := \mathbb{R} \times D$ the universal covering of each Z_n.

All the dynamical information on (Z_n, R_n) can be captured by an almost complex structure on the 4-manifold $\mathbb{R} \times Z_n$ as follows. For each $n \in \mathbb{N}\cup\{\infty\}$, define J_n on $\mathbb{R} \times Z_n$ by

$$\begin{cases} J_n(a, \tau, z)\partial_{\mathbb{R}} = R_n, \\ J_n(a, \tau, z)|_{T_zD} = i \end{cases} \tag{8}$$

for all $(a, \tau, z) \in \mathbb{R} \times Z_n$. Here, $\partial_{\mathbb{R}}$ is the vector field dual to the $\mathbb{R}$-coordinate on $\mathbb{R} \times Z_n$, and i denotes the constant almost complex structure on the disk coming from the standard integrable complex structure on $\mathbb{C}$.* In other words, $i\partial_x = \partial_y$ and $i\partial_y = -\partial_x$. Observe that J_n is independent of the $\mathbb{R}$-coordinate on $\mathbb{R} \times Z_n$, referred to as $\mathbb{R}$-*invariance*. This idea of coupling a suitable conservative vector field in an odd-dimensional manifold with the $\mathbb{R}$-direction in the product 4-manifold by an almost complex structure is due to Hofer [16].

For each $c \in \mathbb{R}$, let

$$L_c := \{c\} \times \partial Z_n,$$

which is a 2-torus. The union $\cup_{c\in\mathbb{R}} L_c$ is equal to the boundary $\mathbb{R} \times \partial Z_n$ of the 4-manifold. Each L_c is totally real with respect to the almost complex structure J_n; that is,

$$TL_c \oplus J_n T(L_c)$$

is the full 4-dimensional tangent space at each point of L_c. These will form the boundary conditions for our pseudoholomorphic curves with boundary.

Let us describe the J_n-holomorphic half infinite cylinders with totally real boundary conditions that we are interested in. Let $\mathbb{R}^+ = [0, \infty)$ and $\mathbb{R}^- = (-\infty, 0]$. For $n \in \mathbb{N}$, we consider maps $\tilde{u} = (a, \tau, z) \in C^\infty(\mathbb{R}^\pm \times \mathbb{R}/n\mathbb{Z}, \mathbb{R} \times Z_n = \mathbb{R} \times \mathbb{R}/n\mathbb{Z} \times D)$ for which there exists $c \in \mathbb{R}$ such that

$$\begin{cases} \partial_s \tilde{u}(s,t) + J_n(\tilde{u}(s,t))\partial_t \tilde{u}(s,t) = 0 & \text{for all } (s,t) \in \mathbb{R}^\pm \times \mathbb{R}/n\mathbb{Z}, \\ \tilde{u}(0,t) \in L_c & \text{for all } t \in \mathbb{R}/n\mathbb{Z}, \\ \tau(0,\cdot) : \mathbb{R}/n\mathbb{Z} \to \mathbb{R}/n\mathbb{Z} & \text{has degree } +1 \end{cases} \tag{9}$$

having so-called finite total energy, which we define in a moment.

This setting is a special case of that described in [4]. In particular, $(\mathbb{R} \times Z_n, J_n)$ is a cylindrical symmetric almost complex manifold, and the almost complex structure J_n is compatible with the stable Hamiltonian structure (ω_n, λ_n) on Z_n given by

$$\begin{cases} \omega_n = dx \wedge dy + d\tau \wedge dH, \\ \lambda_n = d\tau \end{cases} \tag{10}$$

in coordinates $(\tau, (x, y))$ on $\mathbb{R}/n\mathbb{Z} \times D$. Recall that this means that $\lambda_n \wedge \omega_n > 0$ and $\ker(\omega_n) \subset \ker(d\lambda_n)$; see, for example, [4] or [7]. The compactness theory in [4] leads us to consider the following two quantities for a solution to (9) which we will refer to as the ω-energy, the λ-energy, and the sum of them as the total energy. In our context the λ-energy of a solution $\tilde{u} = (a, \tau, z) \in \mathbb{R} \times \mathbb{R}/n\mathbb{Z} \times D$

*For convenience we use this “nongeneric” almost complex structure, although it is not necessary. The pseudoholomorphic curves we encounter are either orbit cylinders or embedded with genus zero, one boundary component, and Fredholm index 2. Such curves are automatically regular.

to (9) is the quantity

$$E_\lambda(\tilde{u}) := \sup_{\psi\in\mathcal{C}} \int_{\mathbb{R}^+\times\mathbb{R}/n\mathbb{Z}} \tilde{u}^*\Big(\psi(a)da\wedge d\tau\Big) \in [0,+\infty], \tag{11}$$

where $\mathcal{C}$ is the set of smooth functions $\psi : \mathbb{R}\to[0,\infty)$ for which $\int_{\mathbb{R}}\psi(s)ds = 1$. The second energy, that which in the more general context of [4] is called the ω-energy, is

$$E_\omega(\tilde{u}) := \int_{\mathbb{R}^+\times\mathbb{R}/n\mathbb{Z}} \tilde{u}^*\omega_n \in [0,+\infty]. \tag{12}$$

In Section 6 we will prove the following.

Lemma 6. *Let $n\in\mathbb{N}$. Suppose $\tilde{u}\in C^\infty(\mathbb{R}^+\times\mathbb{R}/n\mathbb{Z}, \mathbb{R}\times Z_n)$ is a solution to (9) with $E_\lambda(\tilde{u})<\infty$. Then there exists $(s_0,t_0)\in\mathbb{R}\times\mathbb{R}/n\mathbb{Z}$ so that*

$$\tilde{u}(s+s_0, t+t_0) = (s,t,z(s,t)) \tag{13}$$

for all $(s,t)\in[-s_0,\infty)\times\mathbb{R}/n\mathbb{Z}$, where $z\in C^\infty([-s_0,\infty)\times\mathbb{R}/n\mathbb{Z}, D)$ satisfies the Floer equation

$$\partial_s z(s,t) + i\Big(\partial_t z(s,t) - X_H(t,z(s,t))\Big) = 0 \tag{14}$$

for all $(s,t)\in[-s_0,\infty)\times\mathbb{R}/n\mathbb{Z}$.

This is a converse to "Gromov's trick" [14]. It follows from this lemma that if a solution $\tilde{u} = (a,\tau,z)$ to (9) has finite λ-energy, then the two energies have rather nice expressions:

$$E_\lambda(\tilde{u}) = n \tag{15}$$

and

$$E_\omega(\tilde{u}) = \frac{1}{2}\int_{s=-s_0}^{+\infty}\int_{t=0}^{n} \Big|\partial_s z(s,t)\Big|^2 + \Big|\partial_t z(s,t) - X_H(t,z(s,t))\Big|^2 ds dt. \tag{16}$$

In particular, the ω-energy of $\tilde{u}$ becomes equal to the Floer energy of z. Let us verify (15) and (16) now. First,

$$\begin{aligned} E_\lambda(\tilde{u}) &= \sup_{\psi\in\mathcal{C}} \int_{\mathbb{R}^+\times\mathbb{R}/n\mathbb{Z}} \psi(s-s_0)ds\wedge dt \\ &= \sup_{\psi\in\mathcal{C}} \int_{s=-s_0}^{\infty}\psi(s)ds\cdot\int_{\mathbb{R}/n\mathbb{Z}} dt \\ &\le n, \end{aligned}$$

and clearly there exists $\psi\in\mathcal{C}$ such that $\int_{s=-s_0}^{\infty}\psi(s)ds = 1$, so $E_\lambda(\tilde{u}) = n$. Similarly (16) follows from Lemma 6 as follows. Abbreviating $\omega_0 = dx\wedge dy$ for the area form on the disk,

$$E_\omega(\tilde{u}) := \int_{\mathbb{R}^+\times\mathbb{R}/n\mathbb{Z}} \tilde{u}^*\omega = \int_{[-s_0,\infty)\times\mathbb{R}/n\mathbb{Z}} z^*\omega_0 - z^*dH_t\wedge dt.$$

The integrand simplifies to

$$
\begin{aligned}
&= \Big(\omega_0(z_s, z_t) - dH_t(z)[z_s]\Big) ds \wedge dt \\
&= \Big(\omega_0(z_s, iz_s) + \omega_0(z_s, X_H(t,z)) - dH_t(z)[z_s]\Big) ds \wedge dt \\
&= \Big(|z_s|^2 + dH_t(z)[z_s] - dH_t(z)[z_s]\Big) ds \wedge dt \\
&= |z_s|^2 ds \wedge dt.
\end{aligned}
$$

The expression (16) then follows immediately using (13) again.

Definition 7. For a solution $\tilde{u} = (a, \tau, z) \in C^\infty(\mathbb{R}^\pm \times \mathbb{R}/n\mathbb{Z}, \mathbb{R} \times Z_n)$ to (9), we refer to the degree of the circle map

$$z(0,\cdot) : \mathbb{R}/n\mathbb{Z} \to \partial D$$

as the *boundary index* of $\tilde{u}$.

For $x \in \mathbb{R}$, we write

$$
\begin{aligned}
\lceil x \rceil &:= \min\Big\{k \in \mathbb{Z} \,\Big|\, k > x\Big\}, \\
\lfloor x \rfloor &:= \max\Big\{k \in \mathbb{Z} \,\Big|\, k \leq x\Big\}.
\end{aligned}
$$

In Section 7 we will prove the following existence result concerning solutions to (9). Each $\mathcal{F}_n^+$ is the n-th finite energy foliation referred to in Section 1.2.

THEOREM 8. *Let $H \in C^\infty(\mathbb{R}/\mathbb{Z} \times D, \mathbb{R})$ be a Hamiltonian generating an irrational pseudo-rotation φ. Let $(Z_1, R_1), (Z_2, R_2), \ldots$ be the corresponding sequence of mapping tori. For each $n \in \mathbb{N}$, let $\gamma_n : \mathbb{R}/n\mathbb{Z} \to Z_n$ be the unique n-periodic orbit of R_n, parametrized so that $\gamma_n(0) \in \{0\} \times D$. Assume H was chosen so that $\gamma(t) = (t, 0)$ for all $t \in \mathbb{R}/n\mathbb{Z}$ (see Remark* 1*). Let $\alpha :=$* $\operatorname{Rot}(\varphi; H) \in \mathbb{R}$, *which is necessarily irrational.*

Then for each $n \in \mathbb{N}$ there exist two foliations $\mathcal{F}_n^+, \mathcal{F}_n^-$ of $\mathbb{R} \times Z_n$ by smoothly embedded surfaces, with the following properties:

- ***Cylinder leaf:*** *The cylinder $C_n := \mathbb{R} \times \gamma_n(\mathbb{R}/n\mathbb{Z}) \subset \mathbb{R} \times Z_n$ is a leaf in both $\mathcal{F}_n^+$ and $\mathcal{F}_n^-$.*
- ***Pseudo-holomorphic:*** *If $F \in \mathcal{F}_n^+$ (resp. $F \in \mathcal{F}_n^-$) is not C_n, then F is parametrized by a solution $\tilde{u}$ to* (9) *with J_n as in* (8), *with $E_\lambda(\tilde{u}) + E_\omega(\tilde{u}) < \infty$ and boundary index $\lfloor n\alpha \rfloor$ (resp. $\lceil n\alpha \rceil$).*
- $\mathbb{R}$***-invariance:*** *If $F \in \mathcal{F}_n^+$ (resp. $F \in \mathcal{F}_n^-$) is a leaf and $c \in \mathbb{R}$, the set $F + c := \{(a + c, \tau, z) \,|\, (a, \tau, z) \in F\}$ is also a leaf in $\mathcal{F}_n^+$ (resp. in $\mathcal{F}_n^-$).*
- ***Uniqueness:*** *$\mathcal{F}_n^+$ and $\mathcal{F}_n^-$ are uniquely determined by the above properties.*
- ***Smooth foliation:*** *$\mathcal{F}_n^+$ and $\mathcal{F}_n^-$ are C^∞-smooth* 2*-dimensional foliations at each point on the complement of C_n.*

This a special case of a much more general result to appear in [5].

Remark 2. For each leaf $F \in \mathcal{F}_n^+$ (resp. $F \in \mathcal{F}_n^-$) that is not the cylinder, any parametrization $\tilde{u}$ satisfying (9) has domain $\mathbb{R}^+ \times \mathbb{R}/n\mathbb{Z}$ (resp. $\mathbb{R}^- \times \mathbb{R}/n\mathbb{Z}$). Hence the superscripts in $\mathcal{F}_n^\pm$. In either case, as the unique n-periodic orbit γ_n is nondegenerate, the finite energy of $\tilde{u}$ implies that the Z_n component $u_\pm(s,\cdot) : \mathbb{R}/n\mathbb{Z} \to Z_n$ converges to γ_n uniformly in $C^\infty(\mathbb{R}/n\mathbb{Z}, Z_n)$ as $s \to +\infty$ (resp. $s \to -\infty$). This can be seen in two ways: either as a consequence of the compactness results in [16] applied to $\tilde{u}$ (as generalized in [20], [4]), or the original work of Floer [12] applied to the disk component z since, by Lemma 6, z is a finite energy solution to Floer's equation.

The following formula will be crucial to our application. Recall that $\alpha := \mathrm{Rot}(\varphi; H) \in \mathbb{R}$. In particular, α is irrational.

Lemma 9. *Let $n \in \mathbb{N}$. For any half cylinder leaf $F \in \mathcal{F}_n^+$,*

$$E_\omega(F) = \{n\alpha\}\pi.$$

where $\{\cdot\}$ denotes the fractional part of a real number — that is, for $x \in \mathbb{R}$, $\{x\} := x - \lfloor x \rfloor \in [0,1)$.

This is proven in Section 5.

4. **Proof of Theorem 1**

We use the finite energy foliations of Theorem 8 to define new disk maps.

Definition 10. For each $n \in \mathbb{N}$, define $\varphi_n : D \to D$ as follows. For $\xi \in D$, there is a unique leaf $F \in \mathcal{F}_n^+$ containing $(0,0,\xi) \in \mathbb{R} \times \mathbb{R}/n\mathbb{Z} \times D$. Define $\varphi_n(\xi) = \xi'$ where $\xi' \in D$ is unique such that $(0,1,\xi') \in F$. We will refer to φ_n as the *holonomy map* of the foliation $\mathcal{F}_n^+$.

Remark 3. The holonomy maps are well defined. Indeed, for each $n \in \mathbb{N}$, by Lemma 6 if a leaf $F \in \mathcal{F}_n^+$ intersects the hypersurface

$$\hat{Z}_n := \{0\} \times Z_n,$$

then it does so transversally, and for each $\tau \in \mathbb{R}/n\mathbb{Z}$, it will intersect the disk slice $\{\tau\} \times D \subset \hat{Z}_n$ in a unique point.

Remark 4. We could as easily define maps in terms of the foliations $\mathcal{F}_n^-$.

Remark 5. The inverse map φ_n^{-1} exists and can be defined similarly in terms of $\mathcal{F}_n^+$.

Lemma 11. *Each holonomy map $\varphi_n : D \to D$ is n-periodic*; *that is*, $(\varphi_n)^n = \mathrm{id}_D$.

Proof. We could define n many disk maps using $\mathcal{F}_n^+$ — say $\varphi_{n,i}$ for $i = 0, 1, \ldots, n-1$ — by requiring that $\varphi_{n,i}$ takes the point $\xi \in D$ to $\xi' \in D$ if

$(0, i, \xi)$ and $(0, i+1, \xi')$ lie on the same leaf in $\mathcal{F}_n^+$. Since each leaf in the foliation closes up after going once around in the $\mathbb{R}/n\mathbb{Z}$ direction, it follows that the composition $\varphi_{n,n-1}\circ, \ldots, \circ\varphi_{n,0}$ is the identity map.

We now exploit a symmetry in $\mathcal{F}_n^+$ to see that each of the maps $\varphi_{n,i}$ is equal to φ_n. Consider the $\mathbb{Z}_n$ action on $\mathbb{R} \times Z_n$ generated by the deck transformation

$$\mathcal{T} : \mathbb{R} \times Z_n \to \mathbb{R} \times Z_n,$$
$$\mathcal{T}(a, \tau, z) = (a, \tau - 1, z)$$

that preserves the almost complex structure; $\mathcal{T}^* J_n = J_n$. From the uniqueness part of Theorem 8 we conclude that the foliation $\mathcal{T}(\mathcal{F}_n^+) := \{\mathcal{T}(F) \mid F \in \mathcal{F}_n^+\}$ is equal to $\mathcal{F}_n^+$. Hence $\varphi_{n,i} = \varphi_n$ for each i. □

Note that $\varphi_n(0) = 0$ as the cylinder C_n passes through the center of the disk slices $\{0\} \times D$ and $\{1\} \times D$ in $\hat{Z}_n = \{0\} \times Z_n$.

Lemma 12. *Each holonomy map $\varphi_n : D \to D$ is C^∞-smooth on $D\backslash\{0\}$.*

Proof. This is basically a consequence of the fact that by Theorem 8, $\mathcal{F}_n^+$ is a C^∞-smooth foliation on the complement of the leaf C_n, not only in the local sense that a neighborhood of each point admits a foliation chart in which the leaves are "flat," but in the slightly more global sense that a neighborhood of each *leaf* admits a foliation chart.

Fix a point $\xi_0 \in D\backslash\{0\}$. Then the unique leaf $F \in \mathcal{F}_n^+$ that contains the point $p_0 = (0, 0, \xi_0) \in \mathbb{R} \times Z_n$ cannot be the cylinder C_n. Thus by Proposition 38, an open neighborhood of F is foliated by leaves of $\mathcal{F}_n^+$. More precisely (see [23, Ths. 1.5 and 5.7] and their generalizations to surfaces with boundary [38]), there exists a C^∞-smooth map

$$\Phi : B_\varepsilon^2 \times \mathbb{R}^+ \times \mathbb{R}/n\mathbb{Z} \to \mathbb{R} \times Z_n,$$

where $\varepsilon > 0$ and B_ε^2 denotes the open ε-ball in $\mathbb{R}^2$ about the origin, having the following properties:

(1) Φ is an injective immersion.
(2) For each $c \in B_\varepsilon^2$, the map $\Phi(c, \cdot)$ parametrizes a leaf F_c in $\mathcal{F}_n^+$ where $F_0 = F$.
(3) Φ takes the following form:

$$\Phi(c, s, t) = (s - s(c), t, z(c, s, t)) \in \mathbb{R} \times \mathbb{R}/n\mathbb{Z} \times D,$$

where $s(c)$ depends smoothly on c, and $z(c, s, t)$ depends smoothly on c, s, t.

Consider the two smooth immersions

$$i_0 : B_\varepsilon^2 \to D, \qquad c \mapsto z(c, s(c), 0),$$
$$i_1 : B_\varepsilon^2 \to D, \qquad c \mapsto z(c, s(c), 1).$$

Recall that the holonomy map $\varphi_n : D \to D$ of $\mathcal{F}_n^+$ is defined by the property that $\varphi_n(\xi) = \xi'$ if and only if $(0, 0, \xi)$ and $(0, 1, \xi')$ lie on the same leaf in $\mathcal{F}_n^+$. Observe that for each $c \in B_\varepsilon^2$,

$$(0, 0, i_0(c)) = \Phi(c, s(c), 0) \in F_c$$

and

$$(0, 1, i_1(c)) = \Phi(c, s(c), 1) \in F_c.$$

Thus $(0, 0, i_0(c))$ and $(0, 1, i_1(c))$ lie on the same leaf $F_c \in \mathcal{F}_n^+$ and so $\varphi_n(i_0(c)) = i_1(c)$. Thus, on some neighborhood of $i_0(0) = \xi_0$, we have $\varphi_n(c) = i_1 \circ (i_0)^{-1}(c)$ and, in particular, φ_n is smooth near ξ_0. □

LEMMA 13. *Each holonomy map $\varphi_n : D \to D$ is continuous at the origin.*

Proof. Let $\xi_j \in D\backslash\{0\}$ be a sequence of points converging to $0 \in D$, and $p_j := (0, 0, \xi_j) \in \mathbb{R} \times Z_n$. By Lemma 22 there exists a sequence of parametrizations $\tilde{u}_j$ of the unique leaf $F_j \in \mathcal{F}_n^+$ containing p_j that converges in the C^∞_{loc}-topology to C_n. By Lemma 6 we may choose each $\tilde{u}_j$ to take the form $\tilde{u}_j : [S_j, \infty) \times \mathbb{R}/n\mathbb{Z} \to \mathbb{R} \times Z_n$ for some $S_j \in \mathbb{R}$, with

$$\tilde{u}_j(s, t) = (s, t, z_j(s, t))$$

for some sequence z_j. It follows that the sequence of points

$$F_j \cap \Big(\{(0, 1)\} \times D\Big) = (0, 1, z_j(0, 1))$$

converges to $(0, 1, 0)$, which means that $\varphi_n(\xi_j) = z_j(0, 1) \to 0$ as $j \to \infty$. □

Fix a subsequence $\{n_j\}_{j\in\mathbb{N}}$ for which the sequence of fractional parts

$$\lim_{j\to\infty} \{n_j\alpha\} = 0. \tag{17}$$

From Lemma 9 this implies that the ω-energies of the leaves in the foliations $\mathcal{F}_{n_j}^+$ tends to zero, as $j \to \infty$, uniformly over all leaves.

For maps $f, g : D \to D$, define $d_{C^0}(f, g)$ using the linear structure and Euclidean norm on $\mathbb{R}^2$ by

$$d_{C^0}(f, g) := \sup_{\xi\in D} |f(\xi) - g(\xi)|.$$

PROPOSITION 14. *The subsequence φ_{n_j} converges to the pseudo-rotation φ in the following sense*:

$$d_{C^0}(\varphi_{n_j}, \varphi) + d_{C^0}(\varphi_{n_j}^{-1}, \varphi^{-1}) \to 0$$

as $j \to \infty$.

Proof. We show that $d_{C^0}(\varphi_{n_j}, \varphi) \to 0$ as $j \to \infty$, as the same argument will work for the inverses. Arguing indirectly, there exists a sequence of points $\xi_j \in D$ and $\delta > 0$ such that $|\varphi_{n_j}(\xi_j) - \varphi(\xi_j)| \geq \delta$ for all $j \in \mathbb{N}$, where $|\cdot|$ is the Euclidean norm on $\mathbb{R}^2$. Restricting to a subsequence we may assume that $\xi_j \to$

ξ for some $\xi \in D$, and $\delta \leq |\varphi_{n_j}(\xi_j) - \varphi(\xi_j)| \leq |\varphi_{n_j}(\xi_j) - \varphi(\xi)| + |\varphi(\xi_j) - \varphi(\xi)|$ for all $j \in \mathbb{N}$. Therefore as φ is continuous,

$$\frac{1}{2}\delta \leq |\varphi_{n_j}(\xi_j) - \varphi(\xi)| \tag{18}$$

for all j sufficiently large.

For each $j \in \mathbb{N}$, let $F_j \in \mathcal{F}^+_{n_j}$ be the unique leaf containing the point $(0, 0, \xi_j) \in \mathbb{R} \times \mathbb{R}/n_j\mathbb{Z} \times D$. Let us assume that each F_j is a half cylinder; otherwise the argument is even easier. There exists a solution $\tilde{u}_j$ to (9) parametrizing F_j. After a holomorphic reparametrization we may assume that

$$\begin{aligned} \tilde{u}_j : [S_j, \infty) \times \mathbb{R}/n_j\mathbb{Z} \to \mathbb{R} \times Z_{n_j}, \\ \tilde{u}_j(0, 0) =& (0, 0, \xi_j) \end{aligned}$$

for some $S_j \leq 0$. For each j, $E_\lambda(\tilde{u}_j) < \infty$, so by Lemma 6, $\tilde{u}_j$ takes the form

$$\tilde{u}_j(s, t) = (s, t, z_j(s, t)) \tag{19}$$

for some $z_j : [S_j, \infty) \times \mathbb{R}/n_j\mathbb{Z} \to D$. Moreover, the sequence $\{\tilde{u}_j\}_{j \in \mathbb{N}}$ satisfies all the criterion for the compactness result Theorem 25. In particular,

$$\lim_{j \to \infty} E_\omega(\tilde{u}_j) = 0$$

due to our choice of subsequence satisfying (17). We conclude that the sequence $\tilde{u}_j$ converges in the following sense: for each j, let $\bar{u}_j : [S_j, \infty) \times \mathbb{R} \to \mathbb{R} \times Z_\infty$ be the unique lift of $\tilde{u}_j$ to the universal covering, satisfying

$$\bar{u}_j(0, 0) = (0, 0, \xi_j). \tag{20}$$

After restricting to a further subsequence we can assume that $\bar{u}_j \to \bar{u}_\infty$ in the $C^\infty_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R} \times Z_\infty)$ topology — that is, uniformly on compact sets — where $\bar{u}_\infty$ takes the form

$$\bar{u}_\infty(s, t) = (s, t, \gamma(t))$$

for some $\gamma \in C^\infty(\mathbb{R}, D)$ solving $\dot{\gamma}(t) = X_H(t, \gamma(t))$ for all $t \in \mathbb{R}$. From (20) we have $\bar{u}_\infty(0, 0) = (0, 0, \xi)$. We conclude that for all j,

$$(0, 1, \varphi_{n_j}(\xi_j)) = \bar{u}_j(0, 1).$$

The right-hand side converges to

$$\begin{aligned} \bar{u}_\infty(0, 1) &= (0, 1, \gamma(1)) \\ &= (0, 1, \varphi(\gamma(0))) \\ &= (0, 1, \varphi(\xi)). \end{aligned}$$

This contradicts (18), and we are done. □

Combining the results of this section we have proven the following statement, which is almost Theorem 1.

Theorem 15. *Suppose $\varphi \in \mathrm{Diff}^\infty(D,\omega_0)$ fixes the origin and has no other periodic points. For each $j \in \mathbb{N}$, let $\varphi_j : D \to D$ be the holonomy map of the foliation $\mathcal{F}^+_{n_j}$. Then for each $j \in \mathbb{N}$, $\varphi_j(0) = 0$,*

$$\varphi_j \in \mathrm{Homeo}_+(D) \cap \mathrm{Diff}^\infty(D\backslash\{0\}),$$
$$\varphi_j^{n_j} = \mathrm{id}_D\,.$$

Moreover, $d_{C^0}(\varphi_j, \varphi) + d_{C^0}(\varphi_j^{-1}, \varphi^{-1}) \to 0$ as $j \to \infty$.

There are presumably nicer ways to go from this conclusion to the final statement; for example, using changes of coordinates from the pseudoholomorphic curves themselves. This will presumably follow from a more serious analysis of the asymptotic properties of the curves.

Proof of Theorem 1. It is a classical result [6], [9], [29] that if an orientation preserving homeomorphism $f : D \to D$ satisfies $f^n = \mathrm{id}_D$ for some $n \in \mathbb{N}$, then there exists $g \in \mathrm{Homeo}_+(D)$ and $q \in \{0, 1, \ldots, n-1\}$ so that

$$g \circ f \circ g^{-1} = R_{2\pi q/n}.$$

Moreover, if $f(0) = 0$, then $g(0) = 0$. Indeed, if $q = 0$, then $f = \mathrm{id}_D$, while if $q \neq 0$, then $R_{2\pi q/n}$ has a unique fixed point at the origin, and so $g(0) = g(f(0)) = R_{2\pi q/n}(g(0))$ implies $g(0) = 0$.

Applying this result to each holonomy map φ_j we find $g_j \in \mathrm{Homeo}_+(D)$, fixing the origin, and $p_j \in \mathbb{Z}$ such that

$$\varphi_j = g_j^{-1} \circ R_{2\pi p_j/n_j} \circ g_j.$$

Now we replace g_j by a C^0-close smooth approximation. More precisely, let $(\hat{g}_j)$ be a sequence in $\mathrm{Diff}^\infty(D)$, each map fixing the origin, with $d_{C^0}(\hat{g}_j, g_j) + d_{C^0}(\hat{g}_j^{-1}, g_j^{-1}) \to 0$ as $j \to \infty$. Then the maps $\hat{\varphi}_j := \hat{g}_j^{-1} \circ R_{2\pi p_j/n_j} \circ \hat{g}_j$ are C^∞-diffeomorphisms that converge in the C^0-sense to the irrational pseudo-rotation φ. The maps $\hat{\varphi}_j$ satisfy the conclusions of Theorem 1. □

5. Calculation of the ω-energy

The aim of this section is to prove Lemma 9, recalled as Lemma 18. It is convenient to fix a 1-form λ_0 on the disk so that $d\lambda_0 = \omega_0 = dx \wedge dy$. For each $n \in \mathbb{N}$, define the action functional $\mathbf{A}_n : C^\infty(\mathbb{R}/n\mathbb{Z}, Z_n) \to \mathbb{R}$ (associated to λ_0) by

$$\mathbf{A}_n(\sigma) := \int_{\mathbb{R}/n\mathbb{Z}} \sigma^*\lambda_0 - \int_0^n H(\sigma(t))dt.$$

We may rewrite this as

$$\mathbf{A}_n(\sigma) := \int_{\mathbb{R}/n\mathbb{Z}} \sigma^*\eta_n,$$

where $\eta_n := \lambda_0 - H d\tau$ is a primitive of ω_n the 2-form used to define the ω-energy. Note that η_n restricts to a closed 1-form on ∂Z_n since R_n is tangent to ∂Z_n and $d\eta_n(R_n, \cdot) = \omega_n(R_n, \cdot) = 0$. Hence $\mathbf{A}_n$ restricted to $C^\infty(\mathbb{R}/n\mathbb{Z}, \partial Z_n)$ descends to a map on homology.

It is convenient to introduce the closed loops

$$\begin{aligned} 1_{\mathbb{R}/n\mathbb{Z}} &: \mathbb{R}/n\mathbb{Z} \to \partial Z_n, & t &\mapsto (t, 1), \\ 1_{\partial D} &: \mathbb{R}/n\mathbb{Z} \to \partial Z_n, & t &\mapsto ([0], e^{2\pi i t/n}), \end{aligned}$$

which represent homology classes that generate $H_1(\partial Z_n)$.

Lemma 16. *For each* $n \in \mathbb{N}$,

$$\mathbf{A}_n(1_{\mathbb{R}/n\mathbb{Z}}) + \lfloor n\alpha \rfloor \mathbf{A}_n(1_{\partial D}) \leq \mathbf{A}_n(\gamma_n) \leq \mathbf{A}_n(1_{\mathbb{R}/n\mathbb{Z}}) + \lceil n\alpha \rceil \mathbf{A}_n(1_{\partial D}).$$

Proof. Let $\tilde{u}_\pm = (a_\pm, u_\pm) : \mathbb{R}^\pm \times \mathbb{R}/n\mathbb{Z} \to \mathbb{R} \times Z_n$ be parametrizations of leaves $F^\pm \in \mathcal{F}_n^\pm$ respectively that satisfy (9). In either case $u_\pm(s, \cdot)$ converges uniformly in $C^\infty(\mathbb{R}/n\mathbb{Z}, Z_n)$ to a parametrization $\gamma_n(\mathrm{const}_\pm + \cdot)$ as $s \to \pm\infty$ respectively. Applying Stokes theorem,

$$E_\omega(\tilde{u}_+) = \int_{\mathbb{R}^+ \times \mathbb{R}/n\mathbb{Z}} u_+^* \omega_n = \mathbf{A}_n(\gamma_n) - \mathbf{A}_n(u_+(0, \cdot))$$

and

$$E_\omega(\tilde{u}_-) = \int_{\mathbb{R}^- \times \mathbb{R}/n\mathbb{Z}} u_-^* \omega_n = \mathbf{A}_n(u_-(0, \cdot)) - \mathbf{A}_n(\gamma_n).$$

Therefore, as the energies are nonnegative,

$$\mathbf{A}_n(u_+(0, \cdot)) \leq \mathbf{A}_n(\gamma_n) \leq \mathbf{A}_n(u_-(0, \cdot)).$$

We observed that the action $\mathbf{A}_n$ of a closed loop in ∂Z_n depends only on its homology class. From Theorem 8, $u_+(0, \cdot) : \mathbb{R}/n\mathbb{Z} \to \mathbb{R}/n\mathbb{Z} \times \partial D$ and $u_-(0, \cdot) : \mathbb{R}/n\mathbb{Z} \to \mathbb{R}/n\mathbb{Z} \times \partial D$ are homologous to $\mathbb{R}/n\mathbb{Z} \ni t \mapsto (t, e^{2\pi i (\lfloor n\alpha \rfloor / n) t})$ and $\mathbb{R}/n\mathbb{Z} \ni t \mapsto (t, e^{2\pi i (\lceil n\alpha \rceil / n) t})$ respectively. This gives us the desired inequalities. $\square$

Corollary 17. *The unique* 1*-periodic orbit* $\gamma_1 : \mathbb{R}/\mathbb{Z} \to Z_1$ *has action*

$$\mathbf{A}_1(\gamma_1) = \mathbf{A}_1(1_{\mathbb{R}/\mathbb{Z}}) + \alpha \mathbf{A}_1(1_{\partial D}). \tag{21}$$

Proof. From the definition of $\mathbf{A}_n$,

$$\begin{aligned} \mathbf{A}_n(\gamma_n) &= n \mathbf{A}_1(\gamma_1), \\ \mathbf{A}_n(1_{\mathbb{R}/n\mathbb{Z}}) &= n \mathbf{A}_1(1_{\mathbb{R}/\mathbb{Z}}), \\ \mathbf{A}_n(1_{\partial D}) &= \mathbf{A}_1(1_{\partial D}) \end{aligned}$$

for all $n \in \mathbb{N}$. Substituting these into the inequalities in Lemma 16, dividing through by n, and letting $n \to +\infty$ gives (21). $\square$

Lemma 18. *Let $n \in \mathbb{N}$. Every leaf $F \in \mathcal{F}_n^+$ with boundary has ω-energy*

$$E_\omega(F) = \{n\alpha\}\pi,$$

where $\{\cdot\}$ applied to any real number denotes its fractional part.

Proof. By Stokes theorem as in the last lemma,

$$E_\omega(F) = \mathbf{A}_n(\gamma_n) - \mathbf{A}_n(u(0,\cdot))$$

where $\tilde{u} = (a, u) : \mathbb{R}^+ \times \mathbb{R}/n\mathbb{Z} \to \mathbb{R} \times Z_n$ is a parametrization of F. Using Corollary 17 and that $u(0,\cdot) : \mathbb{R}/n\mathbb{Z} \to \mathbb{R}/n\mathbb{Z} \times \partial D$ is homologous to $\mathbb{R}/n\mathbb{Z} \ni t \mapsto (t, e^{2\pi i(\lfloor n\alpha \rfloor/n)t})$, this becomes

$$\begin{aligned}
E_\omega(F) &= n\Big(\mathbf{A}_1(1_{\mathbb{R}/\mathbb{Z}}) + \alpha\mathbf{A}_1(1_{\partial D})\Big) \\
&\quad - \Big(\mathbf{A}_n(1_{\mathbb{R}/n\mathbb{Z}}) + \lfloor n\alpha \rfloor \mathbf{A}_n(1_{\partial D})\Big) \\
&= \Big(n\mathbf{A}_1(1_{\mathbb{R}/\mathbb{Z}}) + n\alpha\mathbf{A}_1(1_{\partial D})\Big) \\
&\quad - \Big(n\mathbf{A}_1(1_{\mathbb{R}/\mathbb{Z}}) + \lfloor n\alpha \rfloor \mathbf{A}_1(1_{\partial D})\Big) \\
&= (n\alpha - \lfloor n\alpha \rfloor)\mathbf{A}_1(1_{\partial D}) \\
&= \{n\alpha\} \cdot \int_D dx \wedge dy.
\end{aligned}$$

□

6. Compactness

Here we prove some compactness statements that were used in the proof of Theorem 1. Our tool for doing so are straightforward versions of the rescaling arguments of Gromov [14] that were inspired by Sacks-Uhlenbeck [35]. As we are dealing with noncompact domains and targets, we use the concept of energy originating with Hofer [16]. We encounter one small novelty here however, which is that the sequence of target manifolds are symplectizations of a sequence of Hamiltonian energy surfaces $Z_{n_j} = \mathbb{R}/n_j\mathbb{Z} \times D$ that also lose compactness as $j \to \infty$. As a result, the pseudoholomorphic curves we consider have unbounded total energy, which means that the standard compactness theory does not immediately apply. The purpose of this section then is largely to handle this difficulty.

Let us briefly recall how ellipticity of the Cauchy-Riemann equations reduces compactness questions for sequences of pseudoholomorphic maps to local uniform bounds on their gradients. The fundamental elliptic estimates for the linear operator $\bar{\partial} = \partial_s + i\partial_t$ on maps from a 2-dimensional domain into $\mathbb{R}^{2d}$, $d \in \mathbb{N}$, are as follows; see [8]. For each $p \in (1, \infty)$ and $k \in \mathbb{N}$, there exists a constant $c = c(k, p) \in (0, \infty)$ such that

$$\|\nabla\varphi\|_{W^{k,p}(D)} \leq c\|\bar{\partial}\varphi\|_{W^{k-1,p}(D)}$$

for all $\varphi \in C_c^\infty(D, \mathbb{R}^{2d})$, where the latter means that φ has compact support in the interior of the closed unit 2-disk D. These estimates can be used to prove the following interior estimates for the nonlinear Cauchy-Riemann equation. For a self contained exposition of this, see [17, Ths. 2.1 and 2.3]. For $r > 0$, we denote by $D_r \subset \mathbb{C}$ the closed disk of radius r.

THEOREM 19. *Let J be a smooth almost complex structure on $\mathbb{R}^{2d}$, $d \in \mathbb{N}$, and let $C_0, C_1 \in (0, \infty)$ be constants. Let $\Omega(J, C_0, C_1) \subset C^\infty(D_1, \mathbb{R}^{2d})$ be the set of maps f that satisfy*

$$\begin{cases} \partial_s f(s,t) + J(f(s,t))\partial_t f(s,t) = 0 \text{ for all } (s,t) \in D, \\ |f(0)| \le C_0, \\ \|\nabla f\|_{C^0(D)} \le C_1. \end{cases}$$

Then there exists a sequence $(c_k) \subset (0, \infty)$ over $k \in \mathbb{N}$ such that for all $f \in \Omega(J, C_0, C_1)$,

$$\|\nabla f\|_{C^k(D_{1/2})} \le c_k$$

for all $k \in \mathbb{N}$.

Remark 6. Replacing $D_1, D_{1/2}$ by half disks $D_1^+, D_{1/2}^+$, where $D_r^+ := \{(x,y) \in \mathbb{R}^2 | x^2 + y^2 \le r,\ y \ge 0\}$, the same statement holds for maps f that take the boundary points $[-1,1] \times \{0\}$ into a smooth path of J-totally real subspaces in $\mathbb{R}^{2d}$. A straightforward reflection argument reduces this to the statement above.

Remark 7. Norms, such as $\|\nabla \tilde{u}\|_{L^\infty}, \|\nabla \tilde{u}\|_{C^r}$ etc., will be implicitly with respect to the Riemannian metric $dx^2 + dy^2 + d\tau^2 + da^2$ on $\mathbb{R} \times Z_n$, where (x,y) are the standard Euclidean coordinates on the disk, τ is the "coordinate" on $\mathbb{R}/n\mathbb{Z}$, and a is the $\mathbb{R}$-coordinate. This metric is J_n-invariant.

We divide the rest of this section into two parts. In 6.1 we consider the situation when $n_j = n$ is fixed and the total energy of the curves is uniformly bounded. In Section 6.2 we consider situations when $n_j \to +\infty$ and the total energy of the curves is also unbounded.

6.1. *Compactness when n is uniformly bounded.* The aim of this section is to prove Lemma 6 stated earlier. We break this into the following two lemmas.

LEMMA 20 (The Floer equation from the Cauchy-Riemann equations). *Let $\tilde{u} = (a, \tau, z) : \mathbb{R}^+ \times \mathbb{R}/n\mathbb{Z} \to \mathbb{R} \times \mathbb{R}/n\mathbb{Z} \times D$ be a solution to (9) for which $\|\nabla a\|_{L^\infty} < \infty$ (or $\|\nabla \tau\|_{L^\infty} < \infty$). Then there exist $(a_0, \tau_0) \in \mathbb{R} \times \mathbb{R}/n\mathbb{Z}$ such that*

$$\begin{cases} a(s,t) = s + a_0, \\ \tau(s,t) = t + \tau_0 \end{cases} \tag{22}$$

for all $(s,t) \in \mathbb{R}^+ \times \mathbb{R}/n\mathbb{Z}$ *and, moreover,* $z : \mathbb{R}^+ \times \mathbb{R}/n\mathbb{Z} \to D$ *satisfies the following Floer equation*:

$$\partial_s z(s,t) + i\Big(\partial_t z(s,t) - X_H(t+\tau_0, z(s,t))\Big) = 0 \tag{23}$$

for all $(s,t) \in \mathbb{R}^+ \times \mathbb{R}/n\mathbb{Z}$.

Proof. Writing out in coordinates what it means for $\tilde{u}$ to satisfy (9) gives us

$$(a_s - \tau_t)\partial_a + (a_t + \tau_s)\partial_\tau + \Big(a_t X_H(\tau,z) + z_s + i\big(z_t - \tau_t X_H(\tau,z)\big)\Big) = 0. \tag{24}$$

The boundary condition on $\tilde{u}$ in (9) implies $a_t(0,t) = 0$ for all $t \in \mathbb{R}/n\mathbb{Z}$. From (24),

$$\begin{aligned} a_s(s,t) &= \tau_t(s,t), \\ a_t(s,t) &= -\tau_s(s,t) \end{aligned} \tag{25}$$

for all $(s,t) \in \mathbb{R}^+ \times \mathbb{R}/n\mathbb{Z}$. In particular, both functions a, τ lift to harmonic functions on the upper half plane with gradient bounded in L^∞. The boundary conditions on a allow a smooth extension by reflection to the whole plane, still with gradient in L^∞, and therefore by Liouville the partial derivatives of a are constant. So there exists $b, c, a_0 \in \mathbb{R}$ so that $a(s,t) = cs + bt + a_0$ for all $(s,t) \in \mathbb{R}^+ \times \mathbb{R}/n\mathbb{Z}$. Putting this into (25) there exists $\tau_0 \in \mathbb{R}$ so that $\tau(s,t) = ct - bs + \tau_0$ for all $(s,t) \in \mathbb{R}^+ \times \mathbb{R}/n\mathbb{Z}$. The n-periodicity of a in the t variable implies $b = 0$. By assumption the degree of the map $\tau(0,\cdot) : \mathbb{R}/n\mathbb{Z} \to \mathbb{R}/n\mathbb{Z}$ is 1, and therefore $c = 1$. This proves (22).

From (24) we also have $a_t X_H(\tau,z) + z_s + i\big(z_t - \tau_t X_H(\tau,z)\big) = 0$. But we have shown that $\tau_t \equiv 1$ and $a_t \equiv 0$. Substituting these in we obtain

$$z_s + i\big(z_t - X_H(t+\tau_0, z)\big) = 0$$

as required. □

The next statement says that we can use the above relation between the Cauchy-Riemann and Floer equations if (and only if) the λ-energy is finite.

Lemma 21. *Let* $\tilde{u} = (a,\tau,z)$ *be a solution to* (9). *Then* $E_\lambda(\tilde{u}) < \infty$ *implies* $\|\nabla a\|_{L^\infty} < \infty$ (*equivalently* $\|\nabla \tau\|_{L^\infty} < \infty$).

Proof. Equations (25) in the last lemma are valid here as they did not require the bounds on the gradient. Therefore, bounding ∇a is equivalent to bounding $\nabla \tau$, and the map $f : \mathbb{R}^+ \times \mathbb{R}/n\mathbb{Z} \to \mathbb{R} \times \mathbb{R}/n\mathbb{Z}$ given by $f(s,t) := (a(s,t), \tau(s,t))$ in terms of the a and τ components of $\tilde{u}$ is holomorphic with respect to the obvious complex structure on $\mathbb{R} \times \mathbb{R}/n\mathbb{Z}$. This is useful because the λ-energy of $\tilde{u}$ depends only on f; indeed, we may write

$$E_\lambda(\tilde{u}) = \sup_{\psi\in\mathcal{C}} \int_{\mathbb{R}^+\times\mathbb{R}/n\mathbb{Z}} f^*\Big(\psi(a)da \wedge d\tau\Big).$$

We can therefore view this as an energy of the map f. Recall from (11) that $\mathcal{C}$ is the set of all $\psi \in C^\infty(\mathbb{R}, [0,\infty))$ for which $\int_{\mathbb{R}} \psi(x)dx = 1$.

Arguing indirectly, suppose that the gradient of a, equivalently the gradient of f, is unbounded. Then there exists a sequence $(\xi_j) \subset \mathbb{R}^+ \times \mathbb{R}/n\mathbb{Z}$ for which $|\nabla f(\xi_j)| \to \infty$. Clearly the points ξ_j leave every compact subset of the domain; in particular, they do not converge to the boundary. Therefore a standard rescaling argument applied to f yields a holomorphic plane $g : \mathbb{C} \to \mathbb{R}\times\mathbb{R}/n\mathbb{Z}$ for which

$$0 < \|\nabla g\|_{L^\infty(\mathbb{C})} \le 2, \tag{26}$$

$$\sup_{\psi\in\mathcal{C}} \int_{\mathbb{C}} g^*\Big(\psi(a)da \wedge d\tau\Big) < \infty. \tag{27}$$

The first property implies that g has constant, nonzero gradient through Liouville's theorem (after, for example, considering a lift $\tilde{g} : \mathbb{C} \to \mathbb{C}$). It therefore follows from a direct calculation that each integral $\int_{\mathbb{C}} g^*(\psi(a)da \wedge d\tau)$, with $\psi \in \mathcal{C}$, is nonfinite, contradicting the second conclusion. Thus no such sequence (ξ_j) can exist and the lemma is proven.

Let us briefly elaborate on the rescaling procedure that produced g. Applying a well-known lemma due to Hofer (see, for example, [33, Lemma 4.6.4]) we may slightly modify the sequence (ξ_j) to a sequence (ξ_j') that retains the gradient blowup property $R_j := |\nabla f(\xi_j')| \to \infty$ and acquires the additional property that there exists a sequence of positive numbers $\varepsilon_j \to 0$ such that

$$\lim_{j\to\infty} \varepsilon_j R_j = \infty,$$
$$\|\nabla f\|_{L^\infty(B_{\varepsilon_j}(\xi_j'))} \le 2R_j.$$

Define a sequence of maps

$$g_j : B_{\varepsilon_j R_j}(0) \to \mathbb{R}\times\mathbb{R}/n\mathbb{Z},$$
$$g_j(\xi) := f(\xi_j' + \xi/R_j) - (a(\xi_j'), 0).$$

These "rescaled maps" are also holomorphic and, moreover, satisfy $|g_j(0)| \le n+1$ and $\|g_j\|_{L^\infty} \le 2$ uniformly in $j \in \mathbb{N}$. Hence the sequence (g_j) has uniformly bounded C^1-norm on each compact subset of $\mathbb{C}$. From the ellipticity of the Cauchy-Riemann equations in the form of Theorem 19, we deduce uniform bounds on (g_j) in the $C^r(K)$ norm for each $r \in \mathbb{N}$ and $K \subset \mathbb{C}$ compact. By the Arzela-Ascoli theorem there exists a subsequence, which we denote also by (g_j), which converges in a C^∞_{loc}-sense to a holomorphic map $g : \mathbb{C} \to \mathbb{R}\times\mathbb{R}/n\mathbb{Z}$ also satisfying $\|\nabla g\|_{L^\infty(\mathbb{C})} \le 2$. Since $|\nabla g(0)| = \lim_{j\to\infty} |\nabla g_j(0)| = 1$, we have that g is nonconstant. Another consequence of the Cauchy-Riemann equations is that for each $\psi \in \mathcal{C}$, the integrand $g_j^*(\psi(a)da \wedge d\tau) = |\nabla g_j|^2(\psi \circ g_j)ds \wedge dt$, which we see has nonnegative density. Therefore,

$$\int_{\mathbb{C}} g^*(\psi(a)da \wedge d\tau) \leq \liminf_{j\to\infty} \int_{B_{\varepsilon_j R_j}} g_j^*(\psi(a)da \wedge d\tau).$$

Each term on the right-hand side is bounded above by $E_\lambda(\tilde{u}) < \infty$. We have verified that the map g satisfies (26) and (27). □

In our proof that the maps φ_n are continuous we used the following compactness property of each foliation.

LEMMA 22. *For $n \in \mathbb{N}$, suppose that $(F_j) \subset \mathcal{F}_n^+$ is a sequence of leaves, and $p_j \in F_j$ is a sequence of points, over $j \in \mathbb{N}$. Suppose that $p_j \to p$ for some $p \in \mathbb{R} \times Z_n$, and let $F \in \mathcal{F}_n^+$ be the unique leaf containing p. Then there exists a sequence of J_n-holomorphic parametrizations ϕ_j of F_j that converge in a C^∞_{loc}-sense to a J_n-holomorphic parametrization of F.*

Proof. This is a standard property of finite energy foliations from positivity of intersections, used many times in [25]. The uniform bounds on the λ-energy and ω-energy ensure existence of a convergent subsequence. Positivity of intersections ensures that the limit is a leaf in the foliation. (The whole leaf as all leaves are connected and have no nodal points.) □

6.2. *Compactness as $n \to \infty$.* In our proof of convergence of the holonomy maps φ_n in Proposition 14, we used a compactness statement for a sequence of J_n-holomorphic maps $\tilde{u}_n : [S_n, \infty) \times \mathbb{R}/n\mathbb{Z} \to \mathbb{R} \times Z_n$ for which

$$\begin{aligned} E_\lambda(\tilde{u}_n) &= n \to +\infty, \\ E_\omega(\tilde{u}_n) &= \{n\alpha\}\pi \end{aligned}$$

for some irrational real number α. Hence the total energy $E(\tilde{u}_n) = E_\lambda(\tilde{u}_n) + E_\omega(\tilde{u}_n)$ diverges to $+\infty$. In general, for a sequence of maps $\{\tilde{u}_n\}$ for which the total energy is unbounded, one cannot expect uniform bounds on the gradient in L^∞. However if the λ-energy grows at most linearly with n and the ω-energy is bounded, then indeed uniform bounds on $\|\nabla\tilde{u}_n\|_{L^\infty}$ can be achieved. (Actually much weaker assumptions suffice, but we will not need to explore these here.) Our arguments will be further simplified since we restricted to a subsequence for which the ω-energy of the sequence decays to zero.

Consider a sequence $\{\tilde{u}_n\}_{n\in\mathbb{N}}$ of smooth J_n-holomorphic maps, for J_n as in (8), with numbers $c_n, S_n \in \mathbb{R}$, $S_n \leq 0$, satisfying for each n,

$$\begin{cases} \tilde{u}_n = (a_n, \tau_n, z_n) : [S_n, \infty) \times \mathbb{R}/n\mathbb{Z} \to \mathbb{R} \times Z_n, \\ \partial_s \tilde{u}_n(s,t) + J_n(\tilde{u}_n(s,t))\partial_t \tilde{u}_n(s,t) = 0, \\ \tilde{u}_n(S_n, t) \in L_{c_n}, \\ \tau_n : (S_n, \cdot) : \mathbb{R}/n\mathbb{Z} \to \mathbb{R}/n\mathbb{Z} \qquad \text{has degree 1} \end{cases} \tag{28}$$

for all $(s,t) \in [S_n, \infty) \times \mathbb{R}/n\mathbb{Z}$.

LEMMA 23. *Suppose that $E_\lambda(\tilde{u}_n) < \infty$ for each $n \in \mathbb{N}$ and $\lim_{n\to\infty} E_\omega(\tilde{u}_n) = 0$. Then there exists $C \in (0, \infty)$ such that*

$$\|\nabla \tilde{u}_n\|_{L^\infty([S_n,\infty)\times\mathbb{R}/n\mathbb{Z})} \leq C$$

for all $n \in \mathbb{N}$.

Note that we do not assume uniform bounds on the λ-energy.

Proof. Since $E_\lambda(\tilde{u}_n) < \infty$ for each n, Lemma 21 implies $\|\nabla a_n\|_{L^\infty} < \infty$ (for each n). Therefore, since also each τ_n has degree 1, Lemma 20 applies so

$$\begin{cases} a_n(s,t) = s + a_n, \\ \tau_n(s,t) = t + \tau_n \end{cases}$$

for all $(s,t) \in [S_n, \infty) \times \mathbb{R}/n\mathbb{Z}$ for some constants $(a_n, \tau_n) \in [S_n, \infty) \times \mathbb{R}/n\mathbb{Z}$. Thus

$$\|\nabla a_n\|_{L^\infty} \leq 1 \qquad \text{and} \qquad \|\nabla \tau_n\|_{L^\infty} \leq 1$$

for all $n \in \mathbb{N}$.

It therefore remains to show that the gradients of the map $z_n : [S_n, \infty) \times \mathbb{R}/n\mathbb{Z} \to D$ are uniformly bounded. Arguing indirectly we find a sequence $\xi_n \in [S_n, \infty) \times \mathbb{R}/n\mathbb{Z}$ for which $|\nabla \tilde{u}_n(\xi_n)| \geq |\nabla z_n(\xi_n)| \to \infty$ as $n \to \infty$. A standard rescaling argument produces a J_∞-holomorphic plane or half plane in $\mathbb{R} \times Z_\infty$. For example, suppose the sequence ξ_n is uniformly bounded away from the boundary; that is, $\xi_n = (s_n, t_n)$ where $\liminf s_n - S_n > 0$. Then one produces a pseudoholomorphic plane as follows. After possibly modifying the sequence (ξ_n) to (ξ_n') using Hofer's lemma (see proof of Lemma 21), one obtains a sequence $(\varepsilon_n) \subset (0, +\infty)$ with $\varepsilon_n \to 0$ such that $R_n := |\nabla \tilde{u}_n(\xi_n')| \to \infty$ and

$$\lim_{n\to\infty} \varepsilon_n R_n = \infty,$$
$$\|\nabla \tilde{u}_n\|_{L^\infty(B_{\varepsilon_n}(0))} \leq 2R_n.$$

Define "rescaled maps"

$$\tilde{v}_n : B_{\varepsilon_n R_n}(0) \to \mathbb{R} \times \mathbb{R}/n\mathbb{Z} \times D,$$
$$\tilde{v}_n(\xi) := \tilde{u}_n(\xi_n' + \xi/R_n) - (a_n(\xi_n'), \lfloor \tau_n(\xi_n') \rfloor, 0)$$

for each $n \in \mathbb{N}$. Each $\tilde{v}_n$ is J_n-holomorphic because the almost complex structure J_n on $\mathbb{R} \times \mathbb{R}/n\mathbb{Z} \times D$ is invariant under $\mathbb{R}$-translations and under the $\mathbb{Z}_n$-action on the $\mathbb{R}/n\mathbb{Z}$-coordinate generated by the deck transformation $[\tau] \mapsto [\tau + 1]$. Moreover, each $\tilde{v}_n$ satisfies $\|\nabla \tilde{v}_n\|_{L^\infty} \leq 2$ and $|\tilde{v}_n(0)| \leq 1$ and, therefore, has C^1-bounds on each compact $K \subset \mathbb{C}$ uniform in n. Ellipticity in the form of Theorem 19 allow these C^1-bounds to be bootstrapped to $C^r(K)$-bounds for each fixed $r \in \mathbb{N}$, and $K \subset \mathbb{C}$ compact, uniform in n. Therefore there exists a convergent subsequence $\tilde{v}_{n_j} \to \tilde{v}$ in the $C^\infty_{\text{loc}}(\mathbb{C})$-topology to a J_∞-holomorphic map

$$\tilde{v} : \mathbb{C} \to \mathbb{R} \times Z_\infty.$$

Let us write $\tilde{v} = (a, \tau, z) \in \mathbb{R} \times \mathbb{R} \times D$. Since each $|\nabla \tilde{v}_{n_j}(0)| = 1$, we have $|\nabla \tilde{v}(0)| = 1$, and so $\tilde{v}$ is nonconstant. Moreover, as a result of the rescaling process, the uniform bounds $\|\nabla a_{n_j}\|_{L^\infty} \leq 1$ and $\|\nabla \tau_{n_j}\|_{L^\infty} \leq 1$ are "killed" in the limit, and so $\nabla a \equiv 0 \equiv \nabla \tau$. Finally, each integrand $\tilde{v}_n^* \omega = |\partial_s z_n|^2 ds \wedge dt$ has nonnegative density, and so for each $\Omega \subset \mathbb{C}$ bounded,

$$0 \leq \int_\Omega \tilde{v}^* \omega \leq \liminf_{n \to \infty} \int_\Omega \tilde{v}_n^* \omega \leq \lim_{n \to \infty} E_\omega(\tilde{u}_n) = 0.$$

Thus $E_\omega(\tilde{v}) = 0$.

If the sequence (ξ_n) instead converged to the boundary, and sufficiently fast, then a similar rescaling procedure would produce a J_∞-holomorphic map on the upper half plane

$$\tilde{v} : \mathbb{H} \to \mathbb{R} \times Z_\infty$$

with totally real boundary conditions $\tilde{v}(\partial \mathbb{H}) \subset \{c\} \times \partial Z_\infty$ for some $c \in \mathbb{R}$, and similar features to the plane above. In either case our limiting map $\tilde{v} = (a, \tau, z)$, with domain $\mathbb{H}$ or $\mathbb{C}$, has the following properties:

$$\begin{aligned} \nabla a &\equiv 0, \\ \nabla \tau &\equiv 0, \\ |\nabla \tilde{v}(0)| &> 0, \\ E_\omega(\tilde{v}) &= 0. \end{aligned}$$

Thus there exist constants $a_0, \tau_0 \in \mathbb{R}$ such that

$$\tilde{v}(s, t) = (a_0, \tau_0, z(s, t)) \in \mathbb{R} \times \mathbb{R} \times D$$

for all $(s, t) \in \mathbb{C}$ (resp. all $(s, t) \in \mathbb{H}$). That $\tilde{v}$ is J_∞-holomorphic translates into $z : \mathbb{C} \to D$ or $z : \mathbb{H} \to D$ satisfying the equation

$$a_t X_H(\tau, z) + z_s + i\big(z_t - \tau_t X_H(\tau, z)\big) = 0;$$

see (24). So a and τ constant implies $z_s + i z_t = 0$. (We could alternatively have just rescaled the sequence of maps $\{z_n\}$ as in Floer theory to get the same conclusion.) Therefore,

$$0 = E_\omega(\tilde{v}) = \int \tilde{v}^*\omega = \int \frac{1}{2}\left(|z_s|^2 + |z_t|^2\right) dsdt,$$

and so z is also constant. Thus we have shown that $\tilde{v}$ is constant, contradicting $|\nabla\tilde{v}(0)| > 0$. □

COROLLARY 24. *Suppose that $\{\tilde{u}_n\}_{n\in\mathbb{N}}$ is a sequence of solutions to* (28) *such that $E_\lambda(\tilde{u}_n) < \infty$ for each $n \in \mathbb{N}$, and $\lim_{n\to\infty} E_\omega(\tilde{u}_n) = 0$. Then there exists a sequence $c_k \in (0,\infty)$ over $k \in \mathbb{N}$ such that*

$$\|\nabla\tilde{u}_n\|_{C^k([S_n,\infty)\times\mathbb{R}/n\mathbb{Z})} < c_k \tag{29}$$

for all $k \in \mathbb{N}$ and $n \in \mathbb{N}$.

Proof. From Lemma 23 the first bound c_1 exists. The remaining bounds then follow from Theorem 19. Because the latter is a local result, it is important to use that the almost complex structures J_n satisfy

(1) they are invariant under the $\mathbb{R}$ and $\mathbb{Z}_n$ actions on $\mathbb{R}\times\mathbb{R}/n\mathbb{Z}\times D$, and
(2) they each lift to the same almost complex structure J_∞ on the universal covering $\mathbb{R}\times\mathbb{R}\times D$. □

We can now prove the goal of Section 6.2.

THEOREM 25. *Let $\bar{u}_n : [S_n,\infty)\times\mathbb{R} \to \mathbb{R}\times Z_\infty$ be a sequence of J_∞-holomorphic maps over $n \in \mathbb{N}$, where each $\bar{u}_n$ is a lift of a solution $\tilde{u}_n$ to* (28). *Suppose that $\bar{u}_n(0,0)$ is uniformly bounded in n, that $E_\lambda(\tilde{u}_n) < \infty$ for each $n \in \mathbb{N}$, and that $\lim_{n\to\infty} E_\omega(\tilde{u}_n) = 0$. Then there exists a subsequence $\{\tilde{u}_{n_j}\}_{j\in\mathbb{N}}$ such that $\bar{u}_{n_j}$ converges in $C^\infty_{\mathrm{loc}}(\mathbb{C},\mathbb{R}\times Z_\infty)$ to a J_∞-holomorphic map $\tilde{u}_\infty$ having domain either $\Sigma = \mathbb{C}$ or $\Sigma = [S,\infty)\times\mathbb{R} \subset \mathbb{C}$ for some $S \in (-\infty, 0]$. Moreover, $\tilde{u}_\infty$ takes the following form. There exist constants $a_0, \tau_0 \in \mathbb{R}$ such that*

$$\begin{aligned} \tilde{u}_\infty :&\Sigma \to \mathbb{R}\times\mathbb{R}\times D, \\ \tilde{u}_\infty(s,t) &= (s + a_0, t + \tau_0, \gamma(t)) \end{aligned} \tag{30}$$

for all $(s,t) \in \Sigma$, where $\gamma \in C^\infty(\mathbb{R}, D)$ satisfies

$$\dot{\gamma}(t) = X_{H_{t+\tau_0}}(\gamma(t)) \tag{31}$$

for all $t \in \mathbb{R}$.

Proof. Taking a subsequence we may assume that $\tilde{u}_{n_j}(0,0)$ converges, and that S_{n_j} converges to some $S \in (-\infty, 0] \cup \{-\infty\}$. The assumptions on the two energies allow us to use Corollary 24 and obtain uniform bounds on $\|\nabla\bar{u}_n\|_{C^k}$ for each $k \in \mathbb{N}$ and, therefore, also C^0-bounds on $\bar{u}_n$ on compact subsets, uniform in n. Repeated use of the Arzela-Ascoli theorem yields a subsequence converging uniformly with all derivatives on each compact subset of $\mathbb{C}$ to a smooth map $\tilde{u}_\infty : \Sigma \to \mathbb{R} \times Z_\infty$, where $\Sigma = [S,\infty)\times\mathbb{R}$ if S is finite and $\Sigma = \mathbb{C}$ otherwise. From Lemma 20 each map in the sequence $\bar{u}_{n_j}$ takes the form

$$\bar{u}_{n_j}(s,t) = (s + a_j, t + \tau_j, z_j(s,t))$$

for constants $a_j, \tau_j \in \mathbb{R}$, with $z_j : [S_{n_j}, \infty) \times \mathbb{R} \to D$ satisfying

$$\partial_s z_j(s,t) + i\Big(\partial_t z_j(s,t) - X_H(t+\tau_j, z_j(s,t))\Big) = 0$$

for all $(s,t) \in [S_{n_j}, \infty) \times \mathbb{R}$. Therefore $\tilde{u}_\infty$ takes the form

$$\tilde{u}_\infty(s,t) = (s + a_\infty, t + \tau_\infty, z_\infty(s,t))$$

for constants $a_\infty, \tau_\infty \in \mathbb{R}$ and some $z_\infty : \Sigma \to D$ satisfying

$$\partial_s z_\infty + i\Big(\partial_t z_\infty - X_H(t+\tau_\infty, z_\infty)\Big) = 0.$$

Let $\omega_\infty := dx \wedge dy + d\tau \wedge dH$ on Z_∞. Then

$$0 \le \int_{\mathbb{R}^2} \tilde{u}_\infty^* \omega_\infty \le \lim_{j\to\infty} E_\omega(\tilde{u}_{n_j}) = 0.$$

Thus

$$\frac{1}{2}\iint \left|\frac{\partial z_\infty}{\partial s}(s,t)\right|^2 + \left|\frac{\partial z_\infty}{\partial t}(s,t) - X_H(t+\tau_\infty, z_\infty(s,t))\right|^2 dsdt$$
$$= \int_{\mathbb{R}^2} \tilde{u}_\infty^* \omega_\infty = 0.$$

Hence $z_\infty(s,t) = \gamma(t)$ for some solution $\gamma : \mathbb{R} \to D$ to (31). □

7. Construction of the finite energy foliations

In this final section we give a terse proof of Theorem 8. The approach is along by now fairly standard lines; the only minor complication arises from the presence of the boundary of the almost complex manifold. A more general construction will appear in [5]. We will assume more familiarity with terminology from [4] than elsewhere in this article and with the intersection theory in [37].

To recall the statement of Theorem 8, let $H \in C^\infty(\mathbb{R}/\mathbb{Z} \times D, \mathbb{R})$ be a Hamiltonian generating an irrational pseudo-rotation φ. Let (Z, R) be the corresponding Hamiltonian mapping torus. That is, $Z = \mathbb{R}/\mathbb{Z} \times D$ with coordinates $(\tau, z = (x,y))$, and R is the vector field

$$R(\tau, z) = \partial_\tau + X_{H_\tau}(z).$$

Denote by $\gamma : \mathbb{R}/\mathbb{Z} \to Z$ the unique 1-periodic orbit of R for which $\gamma(0) \in \{0\} \times D$, and assume H was chosen so that $\gamma(t) = (t, 0)$ for all $t \in \mathbb{R}/\mathbb{Z}$ (see Remark 1). Let

$$\alpha := \mathrm{Rot}(\varphi; H) \in \mathbb{R}$$

denote the *real valued*, irrational, rotation number of φ with respect to the isotopy between φ and id_D induced by H; see Definition 5. Let J be the the associated $\mathbb{R}$-invariant almost complex structure on the 4-manifold $W := \mathbb{R}\times Z$ characterized by the conditions

$$\begin{cases} J\partial_{\mathbb{R}} = R, \\ J|_{TD} = i. \end{cases} \tag{32}$$

We will now prove Theorem 8 for $n = 1$, the general case being identical,* which states the following.

THEOREM 26. *With the above assumptions on $H \in C^\infty(\mathbb{R}/\mathbb{Z} \times D, \mathbb{R})$ there exist two foliations $\mathcal{F}^+, \mathcal{F}^-$ of $\mathbb{R} \times Z$ by J-holomorphic curves with the following properties*:

- ***Cylinder leaf:*** *The cylinder $C := \mathbb{R} \times \gamma(\mathbb{R}/\mathbb{Z}) \subset \mathbb{R} \times Z$ is a leaf in both $\mathcal{F}^+$ and $\mathcal{F}^-$.*
- ***Pseudo-holomorphic:*** *If $F \in \mathcal{F}^+$ (resp. $F \in \mathcal{F}^-$) is not C, then F is parametrized by a solution $\tilde{u}$ to* (9), *with $E_\lambda(\tilde{u}) + E_\omega(\tilde{u}) < \infty$ and boundary index $\lfloor\alpha\rfloor$ (resp. $\lceil\alpha\rceil$).*
- ***$\mathbb{R}$-invariance:*** *If $F \in \mathcal{F}^+$ (resp. $F \in \mathcal{F}^-$) is a leaf and $c \in \mathbb{R}$, the set $F + c := \{(a + c, \tau, z) \,|\, (a, \tau, z) \in F\}$ is also a leaf in $\mathcal{F}^+$ (resp. in $\mathcal{F}^-$).*
- ***Uniqueness:*** *$\mathcal{F}^+$ and $\mathcal{F}^-$ are uniquely determined by the above properties.*
- ***Smooth foliation:*** *$\mathcal{F}^+$ and $\mathcal{F}^-$ are C^∞-smooth foliations at each point on the complement of C.*

In Section 7.1 we prove the statement for $\mathcal{F}^+$ assuming restrictions on the boundary. Section 7.2 removes the boundary restrictions. In Section 7.3 we explain how the construction of $\mathcal{F}^-$ differs from $\mathcal{F}^+$. We defer two technical points to the end: Section 7.4 fills in a detail regarding compactness, and in Section 7.5 we explain how to find the filling of the boundary used in Section 7.1.

Remark 8. In our proof of Theorem 26 we do not actually use that φ is a pseudo-rotation. We only use the much weaker assumption that φ has only a single fixed point and that it is nondegenerate. We will exploit this in Section 7.2.

*For maps more general than irrational pseudo-rotations, the proof for $n > 1$ is a little more involved as the almost complex structure has additional symmetry that makes transversality less obvious. But for irrational pseudo-rotations, automatic transversality suffices and the same proof works for all $n \geq 1$.

7.1. *Construction of $\mathcal{F}^+$ when $\varphi|_{\partial D}$ is conjugate to a rigid rotation.* In this section suppose that there exists $g \in \mathrm{Diff}_+^\infty(\partial D)$ so that

$$g \circ \varphi|_{\partial D} \circ g^{-1} = R_{2\pi\alpha}, \tag{33}$$

where $R_{2\pi\alpha} : \partial D \to \partial D$ is the rigid rotation $z \mapsto ze^{2\pi i\alpha}$.

Let $H_-, H_+ \in C^\infty(Z, \mathbb{R})$ be as follows. Define $H_- = H$, where H is as in the statement of Theorem 26. Define H_+ by

$$H_+(\tau, z) := \pi\alpha|z|^2 + C \tag{34}$$

for some constant $C \leq 0$ chosen so that $\max H_+ < \min H_-$.

The function H_+ on the solid torus defines a closed loop of Hamiltonians on the disk, which we denote by $H_+^t := H_+(t, \cdot) : D \to \mathbb{R}$ over $t \in \mathbb{R}/\mathbb{Z}$. Similarly, H_- defines $H_-^t := H_-(t, \cdot) : D \to \mathbb{R}$ over $t \in \mathbb{R}/\mathbb{Z}$. Let $X_{H_+^t}(z)$ and $X_{H_-^t}(z)$ be the corresponding time-dependent Hamiltonian vector fields on the disk D, with respect to the symplectic form $\omega_0 = dx \wedge dy$. The time-one map generated by $X_{H_+^t}(z)$ is the rigid rotation $R_{2\pi\alpha}$, while the time-one map generated by $X_{H_-^t}(z)$ is the pseudo-rotation φ. From (34), $\mathrm{Rot}(R_{2\pi\alpha}, H_+) = \alpha$, and therefore $\mathrm{Rot}(R_{2\pi\alpha}, H_+) = \mathrm{Rot}(\varphi, H_-)$.

The two pairs of differential forms $\mathcal{H}_\pm = (\omega_\pm, \lambda_\pm)$ on Z, given by

$$\omega_\pm = dx \wedge dy + d\tau \wedge dH_\pm, \qquad \lambda_\pm = d\tau,$$

define stable Hamiltonian structures on Z. Their associated (stable Hamiltonian) Reeb vector fields R_+ and R_- are defined by the conditions $\omega_\pm(R_\pm, \cdot) = 0$ and $\lambda_\pm(R_\pm) = 1$. These are calculated to be

$$R_\pm(\tau, z) = \partial_\tau + X_{H_\pm^\tau}(z).$$

Thus, the first return map of R_+ on the initial disk slice $\{0\} \times D \subset Z$ is the rigid rotation $R_{2\pi\alpha}$, while the first return map of R_- on this disk slice is the pseudo-rotation φ. The flows generated by R_+ and R_- each have a unique 1-periodic orbit. Let $\gamma_\pm : \mathbb{R}/\mathbb{Z} \to Z$ be the parametrization of each, uniquely satisfying $\gamma_\pm(0) \in \{0\} \times D$. These have equal Conley-Zehnder indices (using any conventions).

Along similar lines to [18], we construct a "cobordism" between the two Hamiltonian energy surfaces (Z, X_{H_+}) and (Z, X_{H_-}) on the 4-manifold $W := \mathbb{R} \times Z$. To do this we choose a function $H \in C^\infty(\mathbb{R} \times Z, \mathbb{R})$ interpolating between H_+ and H_- in the following way:

$$\begin{cases} H(a, m) = H_+(m) & \text{for all } a \geq 1, \\ \partial_a H(a, m) < 0 & \text{for } -1 < a < 1, \\ H(a, m) = H_-(m) & \text{for all } a \leq -1. \end{cases}$$

For example, let us assume we chose $H(a,m) = \chi(a)H_+(m)+(1-\chi(a))H_-(m)$ for some $\chi \in C^\infty(\mathbb{R},[0,1])$ with $\chi \equiv 0$ on $(-\infty,-1]$ and $\chi \equiv 1$ on $[1,\infty)$ and such that $\chi'(a) > 0$ for all $a \in (-1,1)$.

Define an almost complex structure $\hat{J}$ on $W := \mathbb{R} \times Z$ by

$$\begin{cases} \hat{J}(a,\tau,z)\partial_{\mathbb{R}} = \partial_\tau + X_{H_a^\tau}(z), \\ \hat{J}|_{TD} = i, \end{cases}$$

where $X_{H_a^\tau}$ is the Hamiltonian vector field of $H_a^\tau := H(a,\tau,\cdot) : D \to \mathbb{R}$. Then $(W,\hat{J})$ is an almost complex manifold with cylindrical ends $E_+ = [1,\infty) \times Z$ and $E_- = (-\infty,-1] \times Z$, adjusted to the stable Hamiltonian structures $\mathcal{H}_\pm = (\omega_\pm,\lambda_\pm)$ on these ends. In the "cobordim region" $(-1,1) \times Z$, $\hat{J}$ tames the exact symplectic 2-form

$$\Omega := dx \wedge dy + d\tau \wedge dH.$$

Although $\hat{J}$ is not $\mathbb{R}$-invariant, the $\mathbb{R}$-invariant cylinder

$$C := \{(a,\tau,0) \in \mathbb{R} \times \mathbb{R}/\mathbb{Z} \times D \mid a \in \mathbb{R}, \tau \in \mathbb{R}/\mathbb{Z}\} \tag{35}$$

is a pseudoholomorphic curve in $(W,\hat{J})$. Its positive puncture is asymptotic to γ_+ and its negative puncture is asymptotic to γ_-. Although C is not an orbit cylinder, it has the following properties in common with one:

$$C \cdot C = -1, \tag{36}$$

where $\cdot$ refers to the generalized intersection number in the sense of Siefring [37], C is embedded, has Fredholm index $\operatorname{ind}(C) = 0$, and is Fredholm regular. Indeed (36) follows from the adjunction formula, Theorem 4.6 in Siefring [37].

We will show in Section 7.5 that after a small perturbation of $\hat{J}$ on a neighborhood of the boundary points $[-1,1] \times \partial Z$ in W, $\hat{J}$ is Levi-flat on the boundary of W. More precisely, $\mathbb{R} \times \partial Z$ is filled by a set $\mathcal{S}$ of immersed $\hat{J}$-holomorphic planes that in the ends $E_\pm \cap (\mathbb{R} \times \partial Z)$ coincide with the product of the $\mathbb{R}$-component and a Reeb trajectory of $R_\pm$. Let us assume that such a perturbation has been made.

We proceed to construct $\mathcal{F}^+$ in four steps.

Step 1. The almost complex structure $\hat{J}$ on W satisfies $\hat{J}|_{E_+} = J_+|_{E_+}$, where J_+ is the cylindrical almost complex structure

$$\begin{cases} J_+\partial_{\mathbb{R}} = \partial_\tau + 2\pi\alpha\partial_\theta, \\ J_+|_{TD} = i \end{cases} \tag{37}$$

on $\mathbb{R} \times Z$, in standard polar coordinates (r, θ) on the disk. For each $c \in \mathbb{R}$ and $z \in \partial D$, the map

$$\tag{38} \begin{gathered} \tilde{u}_{c,z} : \mathbb{R}^+ \times \mathbb{R}/\mathbb{Z} \to \mathbb{R} \times Z, \\ \tilde{u}_{c,z}(s,t) = (s + c, t, ze^{2\pi(\lfloor\alpha\rfloor - \alpha)s} e^{2\pi i \lfloor\alpha\rfloor t}) \end{gathered}$$

is J_+-holomorphic. The combined images of these maps, along with the cylinder

$$\tag{39} C_+ := \{(a, \tau, 0) \in \mathbb{R} \times \mathbb{R}/\mathbb{Z} \times D \,|\, a \in \mathbb{R}, \tau \in \mathbb{R}/\mathbb{Z}\},$$

defines an $\mathbb{R}$-invariant finite energy foliation for $(\mathbb{R} \times Z, J_+)$ with boundary index $\lfloor\alpha\rfloor$. A direct calculation shows that if F_1 and F_2 are the images of two curves in (38), then

$$\tag{40} F_1 \cdot F_2 = 0 \qquad \text{and} \qquad F_1 \cdot C_+ = 0.$$

See Appendix A.3 for a discussion of the generalized intersection number for pairs of curves with boundary.

Step 2. We return to the manifold $(W = \mathbb{R} \times Z, \hat{J})$ with cylindrical ends. Let $\mathcal{M}$ denote the moduli space of all finite energy $\hat{J}$-holomorphic curves $F \subset W$ that admit a $\hat{J}$-holomorphic parametrization by a map $\tilde{u} = (a, \tau, z) \in C^\infty(\mathbb{R}^+ \times \mathbb{R}/\mathbb{Z}, \mathbb{R} \times Z)$ satisfying

$$\tag{41} \begin{cases} \tilde{u}(0, \cdot) \in L_c, \\ \tau(0, \cdot) : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z} \text{ has degree } +1, \\ z(0, \cdot) : \mathbb{R}/\mathbb{Z} \to \partial D \text{ has degree } \lfloor\alpha\rfloor \end{cases}$$

for some $c \in \mathbb{R}$. Equip $\mathcal{M}$ with the topology coming from convergence in $C^\infty_{\mathrm{loc}} \cap C^0([0, \infty) \times \mathbb{R}/\mathbb{Z}, W)$. Note that $\mathcal{M}$ is nonempty as it contains the image of each curve $\tilde{u}_{c,z}$ from (38) for which $c \geq 1$, as $\hat{J} = J_+$ on the positive end $E_+ = [1, \infty) \times Z$. Let

$$\mathcal{M}_0 \subset \mathcal{M}$$

be the connected component containing the curves $\tilde{u}_{c,z}$ from (38) for which $c \geq 1$.

Recall that the boundary of W has a filling $\mathcal{S}$ by injective immersed $\hat{J}$-holomorphic planes. Positivity of intersections between the curves in $\mathcal{M}_0$ and those in $\mathcal{S}$ imply the following.

Lemma 27. *Suppose $F \in \mathcal{M}_0$. Then*

(1) *F meets ∂W transversely along ∂F.*
(2) *$F \backslash \partial F$ lies in $W \backslash \partial W$.*
(3) *∂F is an embedded closed loop.*

Remark 9. Note that property (3) in the lemma implies that F is also embedded near the boundary. That is, it is parametrized by a solution $\tilde{u} : \mathbb{R}^+ \times \mathbb{R}/\mathbb{Z} \to \mathbb{R} \times Z$ to (41) that restricts to an embedding on some neighborhood $[0, \varepsilon) \times \mathbb{R}/\mathbb{Z}$. This is because the differential can only have even-dimensional kernel, and by (3) $\dim \ker D\tilde{u}(0,t) \leq 1$ for all $t \in \mathbb{R}/\mathbb{Z}$.

Proof. The three conditions are open in the topology on $\mathcal{M}$. It suffices to show then that they are also closed. That is, suppose that $F_k \in \mathcal{M}_0$ is a sequence converging to $F \in \mathcal{M}_0$ and that each F_k satisfies conditions (1), (2), and (3). We wish to show that F also satisfies these properties.

Take any point $q \in \partial F$. This is a limit of points $q_k \in \partial F_k$. By assumption each q_k is an isolated transverse intersection point between F_k and a leaf $S_k \in \mathcal{S}$. The local intersection index at q_k is $+1$ (or $+1/2$ depending on conventions) as transverse intersections immersed curves.

Let $S \in \mathcal{S}$ be the unique leaf containing q. It is not too hard to show from this that q must also be an isolated intersection point of F with S and that the local intersection index is $+1$. Indeed, as each leaf in $\mathcal{S}$ is locally embedded, one can get away with using the similarity principle after viewing each F_k as a graph over S_k near q_k. The only slightly tricky point is to show that other intersections between F_k and S_k do not accumulate at q. In the target they will indeed accumulate, but in the domains of the curves they will remain isolated uniformly in k. This latter can be seen by considering the pairs of 1-dimensional parametrized curves f_k and s_k in ∂Z where F_k and S_k intersect L_{c_k} respectively. The curves f_k and s_k intersect transversely and all intersections have the same sign, because each F_k is assumed to be in $\mathcal{M}_0$. For purely topological reasons, this makes it impossible for intersections between f_k and s_k to accumulate, from the point of view of their domains.

The local intersection index of $+1$ at $q \in \partial F$, between F and S, then implies that F, viewed locally as a graph over S, corresponds to a zero of the graph of order 1. This implies that F is immersed at q and is transverse to S at q and, therefore, also transverse to ∂W at q.

Similar arguments, again using positivity of intersections between the curves in $\mathcal{M}_0$ and those in $\mathcal{S}$, allow us to conclude that F can have no boundary-boundary double points and that $F \backslash \partial F$ is disjoint from ∂W. This shows that F satisfies conditions (1), (2), and (3). □

Now that our curves have these nice properties in relation to the boundary of W, we can use the homotopy invariant intersection number and adjunction formula from Appendix A.3.

The explicit curves $\tilde{u}_{c,z}$ from Step 1 have Fredholm index 2 and are clearly embedded. Therefore by the adjunction formula (62), $\tilde{u}_{c,z} \cdot \tilde{u}_{c,z} = 0$. For all $F \in \mathcal{M}_0$, F is homotopic to the image of $\tilde{u}_{c,z}$ for any $c \geq 1$. More precisely, they are homotopic within the space $C^\infty(\gamma_+, \partial W, A)$ described in Appendix A.3, where in the case at hand $A \in H_1(\partial Z, \mathbb{Z})$ is the homology class represented by the loop $\mathbb{R}/\mathbb{Z} \ni t \mapsto (t, e^{2\pi i \lfloor \alpha \rfloor t})$. Note that this is the right homology class because for any curve $G \in \mathcal{M}$, its boundary circle ∂G represents this class A, which case been seen from (41). Thus

$$F \cdot F = 0$$

for each $F \in \mathcal{M}_0$. Hence by Proposition 38 each $F \in \mathcal{M}_0$ is globally embedded.

LEMMA 28. *The set* $\mathcal{E} := \{w \in W \,|\, w \in F \textit{ for some } F \in \mathcal{M}_0\}$ *is a nonempty open and closed subset of* $W \backslash C$.

Proof. By definition, $\mathcal{M}_0$ is nonempty.

Openness: Each curve F in $\mathcal{M}_0$ has Fredholm index $\operatorname{ind}(F) = 2$. The last lemma showed that F is embedded on its boundary (see Remark 9) and meets ∂W transversely. Moreover, we just saw that $F \cdot F = 0$. Thus Proposition 38 applies, and the curves in $\mathcal{M}_0$ fill up an open neighborhood of F in W.

Closedness: Suppose that $p_k \in \mathcal{E}$ is a sequence of points converging to a point $p_\infty \in W\backslash C$. Let $F_k \in \mathcal{M}_0$ be a curve containing p_k, and view p_k as the image of a marked point. The total energy of F_k is uniformly bounded in k, and we find a converging subsequence in the sense of [4]. In Proposition 32 the limit is found to be a height-1 building F_∞ say, that is also a curve in $\mathcal{M}_0$. F_∞ must contain p_∞, so $p_\infty \in \mathcal{E}$. □

Thus each point in $W\backslash C$ contains a curve in $\mathcal{M}_0$.

Step 3. Let $p_k \in W\backslash C$ be a sequence of points that converge to a point $p_\infty \in C$. Let $F_k \in \mathcal{M}_0$ be a sequence of curves with $p_k \in F_k$. As in Step 2 these curves have uniformly bounded total energy because the asymptotic data γ_+ is fixed, as is the homology class in ∂W in which the boundaries of the curves lie. View each p_k as the image of a marked point of F_k. Applying the compactness theory in [4] we get a convergent subsequence to a stable nodal holomorphic building $\bar{F}$ say. By Proposition 32 $\bar{F}$ is a height-2 building, with nonempty lower level a half cylinder F_- in the cylindrical manifold $(\mathbb{R} \times Z, J_-)$. The half cylinder F_- is asymptotic to γ_- as a positive puncture and has boundary index $\lfloor \alpha \rfloor$. Moreover,

$$F_- \cdot F_- = 0 \qquad \text{and} \qquad F_- \cdot C_- = 0,$$

where C_- is the orbit cylinder in $(\mathbb{R} \times Z, J_-)$ over γ_-. From Lemma 6, F_- is embedded and transverse to $\mathbb{R} \times \partial Z$.

Step 4. We may now argue in the manner of Step 2, and conclude that F_- lies in a nonempty moduli space of half cylinders $\mathcal{M}_-$ that fills an open and closed subset of $(\mathbb{R} \times Z)\backslash C_-$. By Lemma 6, each curve in $\mathcal{M}_-$ is embedded and meets the boundary $\mathbb{R} \times \partial Z$ transversely. By the homotopy invariance of the generalized intersection number, the properties

$$F_0 \cdot F_1 = 0 \qquad \text{and} \qquad F_0 \cdot C_- = 0 \tag{42}$$

for all $F_0, F_1 \in \mathcal{M}_-$ are inherited from F_-. By Proposition 38 then, $\mathcal{M}_-$ is a smooth foliation of $(\mathbb{R} \times Z)\backslash C_-$ by embedded curves. Now we set $\mathcal{F}^+ := \mathcal{M}_- \cup C_-$.

It remains to show that $\mathcal{F}^+$ is $\mathbb{R}$-invariant and unique in the sense of Theorem 26. Both properties follow from (42). Indeed, the moduli space $\mathcal{M}_-$ is by definition $\mathbb{R}$-invariant, and to prove uniqueness it suffices to show that if G is any J_--holomorphic half cylinder in $\mathbb{R} \times Z$ that is asymptotic to γ_- as a positive puncture, and has boundary behavior as in (41), then G is in $\mathcal{M}_-$. First, by Lemma 6, G is embedded and transverse to $\mathbb{R} \times \partial Z$. Therefore it is homotopic to any leaf F in $\mathcal{M}_-$ through curves along which the intersection number remains constant. Thus,

$$F \cdot G = F \cdot F = 0$$

for all $F \in \mathcal{M}_-$. In particular, choose $F \in \mathcal{M}_-$ sharing a point with G. Then F and G are not disjoint but have $F \cdot G = 0$. Therefore they are equal, and so $G \in \mathcal{F}^+$.

7.2. *Construction of $\mathcal{F}^+$ without boundary restrictions.* This completes the construction for any irrational pseudo-rotation $\varphi : D \to D$ that restricts to a circle diffeomorphism $\varphi|_{\partial D}$ on the boundary that is smoothly conjugate to a rigid rotation. A deep result of Herman implies that the set of such boundary conditions is dense, and a further limiting step can remove these restrictions. We explain how this works now.

THEOREM 29 (Herman [15]). *There is a subset $\mathcal{D} \subset \mathbb{R}/\mathbb{Z}$ of irrational numbers of full measure with the following property. If $f \in \mathrm{Diff}^\infty_+(\partial D)$ has rotation number in $\mathcal{D}$, then there exists $g \in \mathrm{Diff}^\infty_+(\partial D)$ such that $g^{-1} f g = R$ where $R : \partial D \to \partial D$ is a rigid rotation.*

Denote by $\mathrm{Diff}_{\mathcal{D}}(\partial D)$ those orientation preserving C^∞-smooth circle diffeomorphisms that have rotation number in $\mathcal{D}$. The following would hold for any dense subset $\mathcal{D} \subset \mathbb{R}/\mathbb{Z}$ of irrational numbers.

LEMMA 30. *$\mathrm{Diff}_{\mathcal{D}}(\partial D)$ is dense in $\mathrm{Diff}^\infty_+(\partial D)$ with the C^∞-topology.*

Proof. Fix any $f \in \mathrm{Diff}_{\mathcal{D}}(\partial D)$. Consider the continuous path $f_t := R_{2\pi t} \circ f \in \mathrm{Diff}^\infty_+(\partial D)$ over $t \in [0, 1/2]$. The rotation numbers $\mathrm{Rot}(f_t)$ vary continuously with t. Moreover, there is a continuous family of lifts $\tilde{f}_t : \mathbb{R} \to \mathbb{R}$ with the monotonicity property that for $t > 0$, $\tilde{f}_t(x) > \tilde{f}_0(x)$ for all $x \in \mathbb{R}$. Therefore, since $\mathrm{Rot}(f_0)$ is *irrational*, it follows that for all $t \in (0, 1/2]$, $\mathrm{Rot}(f_t) > \mathrm{Rot}(f_0)$.

See, for example, Proposition 11.1.9 in [28]. As $\mathcal{D}$ is dense in $\mathbb{R}/\mathbb{Z}$, we find a sequence $t_j \in (0, 1/2]$ converging to zero such that $\mathrm{Rot}(f_{t_j}) \in \mathcal{D}$. □

Now suppose that $\varphi \in \mathrm{Diff}^\infty(D, \omega_0)$ is any smooth irrational pseudo-rotation. Let $H \in C^\infty(\mathbb{R}/\mathbb{Z} \times D, \mathbb{R})$ be a Hamiltonian with time-one map φ. Using the lemma one can find smooth perturbations $H_j \in C^\infty(\mathbb{R}/\mathbb{Z} \times D, \mathbb{R})$ of H near the boundary $\mathbb{R}/\mathbb{Z} \times \partial D$, so that

$$H_j \to H \text{ in } C^\infty,$$
$$\mathrm{Rot}(\varphi_j|_{\partial D}) \in \mathcal{D}$$

for all j, where φ_j is the time-one map of H_j. For j sufficiently large, φ_j has no fixed points besides the origin, which of course remains nondegenerate.

Remark 10. We do *not* claim that we can find perturbed disk maps that are also irrational pseudo-rotations. In general, a perturbation will create new periodic points with high period.

By Herman's theorem, $\varphi_j|_{\partial D}$ is smoothly conjugate to a rigid rotation. Therefore, by Steps 1 to 4 in Section 7.1 we can find a finite energy foliation $\mathcal{F}_{H_j}$ of the cylindrical almost complex manifold $(\mathbb{R} \times Z, J_{H_j})$.

Remark 11. Note that we are applying Remark 8 here. Indeed, this allows us to conclude that each foliation $\mathcal{F}_{H_j}$ exists even though φ_j is not exactly an irrational pseudo-rotation.

The almost complex structures J_{H_j} converge uniformly to J_H in the C^∞-topology as $j \to \infty$. The energies of leaves in $\mathcal{F}_{H_j}$ will be uniformly bounded in j: indeed the λ-energy of each leaf will remain 1, and the ω-energy will be bounded above, for example, by $\pi\{\alpha\} + \varepsilon$ for some $\varepsilon > 0$. These are sufficient conditions to obtain a limiting finite energy foliation for $(\mathbb{R} \times Z, J_H)$. For example, we can take a limit of a single sequence of half cylinder leaves $F_j \in \mathcal{F}_{H_j}$ to obtain a single J_H-holomorphic half cylinder F_∞ disjoint from the orbit cylinder C and with vanishing self-intersection number. The same argument as in Step 4 then shows that a moduli space containing F_∞ fills up the complement of C by embedded curves that combine to form a smooth $\mathbb{R}$-invariant foliation. Uniqueness is also as in Step 4.

7.3. *Constructing* $\mathcal{F}^-$. The construction of $\mathcal{F}^-$ is along exactly the same lines as for $\mathcal{F}^+$, but beginning with a different model foliation in Step 1. Indeed, in place of the curves in (38), each of which is bounded from below, we use curves of the form

$$\begin{aligned} v_{c,z} :& \mathbb{R}^- \times \mathbb{R}/\mathbb{Z} \to \mathbb{R} \times Z, \\ v_{c,z}(s,t) &= (s + c, t, z e^{2\pi(\lceil\alpha\rceil - \alpha)s} e^{2\pi i \lceil\alpha\rceil t}) \end{aligned} \tag{43}$$

over $c \in \mathbb{R}$ and $z \in \partial D$. (These maps are also pseudoholomorphic with respect to the almost complex structure in (37).) Each $v_{c,z}$ has image bounded from above, so we have to insert one of them into the negative end of our almost complex manifold $(W = \mathbb{R} \times Z, \hat{J})$. So the main difference is that from the start we reverse the roles of H_- and H_+, this time picking a positive constant C in (34) so that $\min H_- > \max H_+$ still holds. Then the remaining steps are exactly analogous, and in the final foliation the curves with boundary have boundary index $\lceil \alpha \rceil$ instead, as the curves in (43) do.

7.4. *Compactness.* In Section 7.1 we applied the compactness theory in [4] in the form of Proposition 32 below. We state and justify this now.

Recall that $(W = \mathbb{R} \times Z, \hat{J})$ from Section 7.1 has cylindrical ends and that the Reeb flow on the positive and negative ends each has a unique simply covered periodic orbit γ_+ and γ_- respectively. Although $\hat{J}$ is not $\mathbb{R}$-invariant, the cylinder

$$C = \{(a, \tau, 0) \in \mathbb{R} \times \mathbb{R}/\mathbb{Z} \times D \,|\, a \in \mathbb{R},\ \tau \in \mathbb{R}/\mathbb{Z}\}$$

is a pseudoholomorphic curve in $(W, \hat{J})$. Its positive puncture is asymptotic to γ_+, and its negative puncture is asymptotic to γ_-. We observed in (36) that

$$C \cdot C = -1. \tag{44}$$

Consider a sequence F_k of $\hat{J}$-holomorphic curves in $(W, \hat{J})$ with a single marked point whose image we denote by $p_k \in W$. We suppose that each F_k is the image of a $\hat{J}$-holomorphic embedding $\tilde{u}_k = (a_k, \tau_k, z_k) \in C^\infty(\mathbb{R}^+ \times \mathbb{R}/\mathbb{Z}, \mathbb{R} \times Z)$ that is asymptotic to γ_+ and has the boundary conditions described by (41). Furthermore, we assume that

$$F_k \cdot F_k = 0 \qquad \text{and} \qquad F_k \cdot C = 0$$

for each k.

Suppose that F_k converges to a generalized nodal holomorphic building $\bar{F}$ in the sense of [4].

Lemma 31. *The building $\bar{F}$ has no disk components.*

Proof. Arguing indirectly suppose that $\bar{F}$ has a disk component D. Suppose first that D is in the middle level $(W, \hat{J})$. Then ∂D must be a closed loop in one of the totally real surfaces $L_c \simeq \partial Z$ say. As ∂D is contractible in W it must lie in a homology class $m1_{\partial D} \in H_1(\partial Z)$ for some $m \in \mathbb{Z}$. The topological count of intersections between D and the cylinder C is m. That the building is stable with at most one marked point implies that the component D is not just a point. Therefore it has nonzero energy, which implies that $m \neq 0$. Therefore D and C are distinct holomorphic curves with interior intersections. Therefore F_k and C intersect for large k, contradicting that they are in fact disjoint for all k.

If instead D is in an upper or lower level, a similar argument proves that D has isolated interior intersections with the unique orbit cylinder in the relevant level. This again implies that F_k and C intersect for large k. $\square$

Proposition 32. *Suppose the marked points p_k converge to $p_\infty \in W$.*

(1) *If $p_\infty \notin C$, then $\bar{F}$ is a height-1 building (i.e., there is no breaking). This building has a single component, F. This component F is a $\hat{J}$-holomorphic half cylinder with positive puncture asymptotic to γ_+ and the same boundary index as each F_k.*
(2) *If $p_\infty \in C$, then $\bar{F}$ has precisely two nonempty levels. The middle level is equal to the cylinder C, the top levels are empty, and the lower level is a single component curve F_-. The component F_- is a J_--holomorphic half cylinder with positive puncture asymptotic to γ_- and the same boundary index as each F_k. Moreover, $F_- \cdot C_- = 0$, where C_- is the orbit cylinder over γ_- and $F_- \cdot F_- = 0$.*

Proof. Any bubbling off on the boundary would result in $\bar{F}$ having a disk component, which is ruled out by Lemma 31. Any interior bubbling would result in a component of $\bar{F}$ that is a finite energy plane, which is impossible as there are no contractible periodic orbits. Any nodes in $\bar{F}$ would imply $\bar{F}$ has a component that is a plane or a closed curve. But any closed curve would be constant as the symplectic form Ω in the cobordism region is exact. However, there are not enough marked points for $\bar{F}$ to have any constant components and yet be stable. We are therefore reduced to the following two scenarios.

Case 1: Suppose no breaking occurs. Then $\bar{F}$ is a height-1 holomorphic building, with a single component, and this component must be a half cylinder asymptotic to γ_+. The boundary index must be the same for each F_k as no disks bubbled off.

It follows that each F_k is homotopic to the limiting curve F (through curves asymptotic to γ_+). Thus $F \cdot C = F_k \cdot C = 0$. Similarly, $F \cdot F = F_k \cdot F_k = 0$ for all k.

Case 2: Suppose breaking does occur. Then, as there is only one periodic orbit in the negative end, the middle level F_0 of $\bar{F}$ must be a single component that is a cylinder with punctures of opposite sign. The positive puncture is asymptotic to γ_+ and the negative puncture asymptotic to γ_-. Therefore F_0 is homotopic to C through curves with fixed end points, and so $F_0 \cdot C = C \cdot C = -1$ by (44), which implies that F_0 is a covering of C and therefore equals C.

The marked point must be on F_0. Therefore there are no orbit cylinders in the lower or upper levels. Thus, $\bar{F}$ is a height-2 holomorphic building, with middle level F_0 as described and lower level F_- a single component that is a half cylinder asymptotic to connecting orbit γ_- as a positive puncture. Again the boundary index of F_- must be the same as for each F_k as no disks bubbled off.

By assumption, F_k converges to the building $\bar{F}$ in the sense of [4]. The constant sequence $C_k := C$ also converges to the (unstable) building $\bar{C}$ that consists of a single middle level equal to the cylinder C and a single lower level equal to the orbit cylinder C_- and empty upper level. In particular, both buildings have a unique connecting orbit γ_-, and this is simply covered.

The proof of Proposition 4.3, part (4) in Siefring [37] yields

$$\lim_{k\to\infty} F_k \cdot C_k = C \cdot C + F_- \cdot C_- + p(\gamma_-), \tag{45}$$

where $p(\gamma_-) \in \{0,1\}$ denotes the parity of the Conley-Zehnder index of γ_-. Due to Lemma 36, γ_- is elliptic, so $p(\gamma_-) = 1$. By assumption, $F_k \cdot C_k = 0$ for each k, and we observed in (44) that $C \cdot C = -1$. We then conclude from (45) that $F_- \cdot C_- = 0$.

The formula that we applied to F_k and C_k to obtain (44) holds equally well for self intersections. Applied to F_k this yields

$$0 = \lim_{k\to\infty} F_k \cdot F_k = C \cdot C + F_- \cdot F_- + p(\gamma_-). \tag{46}$$

The left-hand side vanishes, $C\cdot C = -1$ (44), and $p(\gamma_-) = 1$. So $F_-\cdot F_- = 0$. □

7.5. *Foliating the boundary.* Finally we prove the existence of the foliation $\mathcal{S}$ of the boundary of $\mathbb{R} \times Z$ that was used in Section 7.1. For each $a \in \mathbb{R}$, consider for a fixed value of a the resulting time-dependent Hamiltonian on the disk H_a given by

$$H_a^t := H(a,t,\cdot) : D \to \mathbb{R}$$

over $t \in \mathbb{R}/\mathbb{Z}$. By modifying H_a on any arbitrarily small neighborhood of the boundary of the disk we may arrange that the time-one map of the path of generated Hamiltonian disk maps is any prescribed orientation preserving diffeomorphism on the boundary of the disk. By extension, given any smooth path $a \mapsto f_a \in \mathrm{Diff}_+^\infty(\partial D)$ over $a \in \mathbb{R}$, satisfying

$$f_a = \begin{cases} R_{2\pi\alpha}|_{\partial D} & \text{if } a \geq 1, \\ \varphi|_{\partial D} & \text{if } a \leq -1, \end{cases} \tag{47}$$

the function $H : \mathbb{R} \times Z \to \mathbb{R}$ may be modified on any small neighborhood of $[-1,1] \times \partial Z$ so that for each $a \in \mathbb{R}$, the time-one map of the modified Hamiltonian H_a now coincides with f_a on the boundary of the disk.

Suppose that we can find a smooth path $a \mapsto f_a \in \mathrm{Diff}_+^\infty(\partial D)$ satisfying (47), and which additionally has the property that each f_a is smoothly conjugate to the rigid rotation $R_{2\pi\alpha}|_{\partial D}$. More precisely, suppose that we find a smooth map $g \in C^\infty(\mathbb{R} \times \partial D, \partial D)$, so that for each $a \in \mathbb{R}$, the map $g_a := g(a,\cdot)$ is an element of $\mathrm{Diff}_+^\infty(\partial D)$, and with the property that the path $f_a := g_a R_{2\pi\alpha} g_a^{-1}$ satisfies (47). Then we may modify $H \in C^\infty(\mathbb{R} \times Z, \mathbb{R})$ near $[-1,1] \times \partial Z$ so that for each $a \in \mathbb{R}$, the time-one map of the time-dependent Hamiltonian $H_a := H(a,\cdot)$ equals f_a on the boundary of the disk. For each

$a \in \mathbb{R}$, let

$$\begin{aligned}\phi_a :&\mathbb{R} \times \partial Z \to \partial Z,\\ &(t, z) \mapsto \phi_a^t(z)\end{aligned}$$

denote the 1-parameter family of maps generated by $X_{H_{a,t}}$ on ∂D. So, in particular, $\phi_a^1 = f_a$ for all $a \in \mathbb{R}$. Now for each $z \in \partial D$,

$$S_z := \Big\{ \big(a, t, g_a(\phi_a^t(z))\big) \in \mathbb{R} \times \mathbb{R}/\mathbb{Z} \times \partial D \,|\, a \in \mathbb{R},\ t \in \mathbb{R} \Big\}$$

is an immersed surface in $\mathbb{R} \times \partial Z$, and the union

$$\mathcal{S} := \bigcup_{z \in \partial D} S_z$$

is a foliation of $\mathbb{R} \times \partial Z$. As α is irrational, each S_z is dense in $\mathbb{R} \times \partial Z$. However, the relation

$$f_a := g_a R_{2\pi\alpha} g_a^{-1} \tag{48}$$

for all $a \in \mathbb{R}$ enables us to find a C^∞-smooth almost complex structure J' on $\mathbb{R} \times Z$, prescribed at points on $\mathbb{R} \times \partial Z$ so that each S_z has J'-invariant tangent bundle. Indeed, differentiating the expression $\big(a, t, g_a(\phi_a^t(z))\big)$ in a gives a vector field V_1 say, while differentiating it in t results in a vector field V_2. Both are nonvanishing and transverse as we will see, so we can set $J'V_1 = V_2$. That V_1 is indeed a well-defined vector field uses (48). Moreover, one finds that

$$V_2 = \partial_\tau + X_{H_a^\tau},$$

while

$$V_1 = \partial_a + V_3,$$

for some V_3 on $\mathbb{R} \times \partial Z$ that has no ∂_a component, and tends to zero in C^0 as $\|\partial_a f_a\|_{C^0}$ tends to zero. We can arrange that $\|\partial_a f_a\|_{C^0}$ is as small as we wish by "slowing everything down" — that is, replacing the interval $[-1, 1] \times Z$ by $[-N, N] \times Z$ for sufficiently large $N > 0$. Then, from these expressions for V_1, V_2 we see that J' extends to an almost complex structure J'' on $\mathbb{R} \times Z$ with the following properties if $\|\partial_a f_a\|_{C^0}$ is sufficiently small:

(1) J'' coincides with the almost complex structure J outside of a small neighborhood of $[-N+1, N-1] \times \partial Z$.
(2) Each surface $L_c := \{c\} \times \partial Z$ is totally real with respect to J''.
(3) J'' is tamed by the symplectic form Ω on $(-N, N) \times Z$.

And finally, of course, $\mathcal{S}$ is a J''-holomorphic filling of the boundary $\mathbb{R} \times \partial Z$.

The only remaining question is when relation (48) can be arranged for all $a \in \mathbb{R}$. But this holds if and only if the circle maps $\varphi|_{\partial D}$ and $R_{2\pi\alpha}|_{\partial D}$ are conjugate by an orientation preserving C^∞-smooth diffeomorphism. Necessity is obvious; let us show sufficiency. Suppose that there exists $g \in \mathrm{Diff}_+^\infty(\partial D)$ such that $\varphi|_{\partial D} = g R_{2\pi\alpha}|_{\partial D} g^{-1}$. Since g has degree $+1$, it is smoothly isotopic to the identity and we may find a smooth isotopy $g_a \in \mathrm{Diff}^\infty_+(\partial D)$ over $a \in \mathbb{R}$, satisfying $g_a = \mathrm{id}$ for all $a \geq 1$ and $g_a = g$ for all $a \leq -1$. Thus the smooth path $f_a \in \mathrm{Diff}^\infty_+(\partial D)$ over $a \in \mathbb{R}$ defined by $f_a := g_a R_{2\pi\alpha} g_a^{-1}$ satisfies

$$f_a = \begin{cases} R_{2\pi\alpha} & \text{if } a \geq 1, \\ \varphi|_{\partial D} & \text{if } a \leq -1 \end{cases}$$

and therefore has the properties we require.

Appendix A.

A.1. *Proof of Proposition* 2. In this appendix we prove the following statement, which implies Proposition 2. The idea of the proof was explained to me by Patrice LeCalvez. We write $R_\theta : D \to D$ to denote the rigid rotation $z \mapsto e^{i\theta} z$ through angle $\theta \in \mathbb{R}$.

PROPOSITION 33. *Consider a sequence $\varphi_k \in \mathrm{Homeo}_+(D)$ converging in the C^0-topology to $\varphi \in \mathrm{Homeo}_+(D)$, where all maps fix $0 \in D$. Under the following additional assumptions it follows that φ has no periodic points in $D\backslash\{0\}$:*

(1) *For each $k \in \mathbb{N}$, there exists $g_k \in \mathrm{Homeo}_+(D)$, fixing the origin, such that $\varphi_k = g_k^{-1} R_{2\pi\theta_k} g_k$ for some $\theta_k \in \mathbb{R}$.*
(2) *$\theta_k \to \theta$ as $k \to \infty$, where θ is irrational.*

Remark 12. Regarding Proposition 33, note that it is sufficient for us to prove the weaker statement that under conditions (1) and (2) the limiting map φ has no *fixed* points in $D\backslash\{0\}$. Indeed, this weaker statement will then apply to each iterate of φ, and we will be able to conclude that each iterate of φ has no fixed points in $D\backslash\{0\}$ and, therefore, that φ has no *periodic* points in $D\backslash\{0\}$.

To prove this we need to recall the notions of positively and negatively returning disks due to Franks [13]. Let $\mathring{D} := D\backslash\partial D$. Denote by $A := \mathring{D}\backslash\{0\}$ and $\tilde{\mathbb{A}} := (0,1)\times\mathbb{R}$ the open annulus and its universal covering via the covering map

$$\pi : (0,1)\times\mathbb{R} \to \mathring{D}\backslash\{0\},$$
$$(x,y) \mapsto xe^{2\pi i y}.$$

Let $T : \tilde{\mathbb{A}} \to \tilde{\mathbb{A}}$ be the deck transformation $T(x,y) = (x, y+1)$. By an open disk $U \subset \tilde{\mathbb{A}}$ is meant an open subset homeomorphic to $\mathring{D}$ with the subspace topology.

Definition 34. Let $f : \mathbb{A} \to \mathbb{A}$ be a homeomorphism homotopic to the identity, and let $\tilde{f} : \tilde{\mathbb{A}} \to \tilde{\mathbb{A}}$ be a lift via π. Consider an open disk $U \subset \tilde{\mathbb{A}}$ for which

$$\tilde{f}(U) \cap U = \emptyset. \tag{49}$$

If there exist integers $n > 0$, $p \neq 0$ such that

$$\tilde{f}^n(U) \cap T^p U \neq \emptyset, \tag{50}$$

then U is called a *positively*, respectively *negatively*, *returning disk* for $\tilde{f}$ if $p > 0$, respectively $p < 0$.

Remark 13. Consider a disk $U \subset \tilde{\mathbb{A}}$ satisfying (49) and (50). Moreover, if the closure of the disk satisfies (49), that is, $\tilde{f}(\bar{U}) \cap \bar{U} = \emptyset$, then for any sufficiently C^0-small perturbation of $\tilde{f}$, U will satisfy (49) and (50) for the perturbed map also.

The key result for us is the following strong generalization of the Poincaré-Birkhoff fixed point theorem, Theorem 2.1 in [13].

THEOREM 35 (Franks). *Let $f : \mathbb{A} \to \mathbb{A}$ be a homeomorphism of the open annulus homotopic to the identity, and for which every point is nonwandering. If there exists a lift $\tilde{f} : \tilde{\mathbb{A}} \to \tilde{\mathbb{A}}$ having a positively returning disk that is a lift of a disk in $\mathbb{A}$ and a negatively returning disk that is a lift of a disk in $\mathbb{A}$, then f has a fixed point.*

Recall that a point $x \in \mathbb{A}$ is *nonwandering for f* if for every open neighborhood U of x there exists $n > 0$ such that $f^n(U) \cap U \neq \emptyset$. Using Poincaré's Recurrence Theorem every point is nonwandering for a homeomorphism f if f preserves a finite measure that is positive on open sets.

We now use Theorem 35 to prove the proposition.

Proof of Proposition 33. Arguing indirectly, suppose that φ has a periodic point $z_0 \in D\backslash\{0\}$. Indeed, by Remark 12 we may assume that z_0 is a fixed point of φ. Then we will show that some iterate $\psi = \varphi^n$ has a lift via the covering map

$$\pi : (0,1] \times \mathbb{R} \to D\backslash\{0\},$$
$$(x, y) \mapsto x e^{2\pi i y}$$

to a map of the half closed infinite strip $\tilde{\psi} : (0,1] \times \mathbb{R} \to (0,1] \times \mathbb{R}$, having disks $U_-, U_+ \subset (0,1) \times \mathbb{R}$ that satisfy the following:

(1) U_+ is a positively returning disk for $\tilde{\psi}$. U_- is a negatively returning disk for $\tilde{\psi}$.
(2) The closures satisfy $\tilde{\psi}(\bar{U}_+) \cap \bar{U}_+ = \emptyset$ and $\tilde{\psi}(\bar{U}_-) \cap \bar{U}_- = \emptyset$.
(3) U_+ and U_- are lifts of disks in $\mathring{D}\backslash\{0\}$.

Suppose for a moment that we have established this. Let us write $\psi_k := (\varphi_k)^n : D \to D$. Then ψ_k converges uniformly in the C^0-topology to $\psi = \varphi^n$ as $k \to \infty$. Therefore there exists a sequence of lifts

$$\tilde{\psi}_k : (0,1] \times \mathbb{R} \to (0,1] \times \mathbb{R}$$

of ψ_k that converges uniformly in the C^0-topology to the given lift $\tilde{\psi}$. Hence by Remark 13 there exists $K \in \mathbb{N}$ such that for all $k \geq K$, U_+ is a positively returning disk for $\tilde{\psi}_k$, and U_- is a negatively returning disk for $\tilde{\psi}_k$.

Recall that U_+ and U_- lie in the interior $(0,1) \times \mathbb{R}$ and both are lifts of disks in $\mathring{D}\backslash\{0\}$. Every point is nonwandering for ψ_k because it is conjugate to a rigid rotation. Therefore we may apply Franks' theorem, Theorem 35, to each map of the open annulus $\psi_k : \mathring{D}\backslash\{0\} \to \mathring{D}\backslash\{0\}$ for $k \geq K$. We conclude that for all $k \geq K$, ψ_k has a fixed point in $D\backslash\{0\}$. But $\psi_k = (\varphi_k)^n$ is conjugate to the rigid rotation $(R_{2\pi\theta_k})^n = R_{2\pi n\theta_k}$. It follows that for this fixed $n \in \mathbb{N}$, we have $n\theta_k \in \mathbb{Z}$ for all $k \geq K$, which contradicts the convergence of the sequence θ_k to an irrational number.

With this contradiction we will be done, and so it remains to establish that some iterate of φ has a lift for which we can find disks $U_+, U_- \subset (0,1) \times \mathbb{R}$ satisfying conditions (1), (2), and (3) listed above.

We are assuming that φ has a fixed point $z_0 \in D\backslash\{0\}$. From assumption (2) in Proposition 33 we know that the rotation number of φ on the boundary is irrational, and so φ has no periodic points on the boundary of the disk. Thus $z_0 \in \mathring{D}\backslash\{0\}$ and φ has no fixed points on ∂D.

Fix a lift $\tilde{z}_0 \in (0,1) \times \mathbb{R}$, meaning that $\pi(\tilde{z}_0) = z_0$. Let $\tilde{\varphi} : (0,1] \times \mathbb{R} \to (0,1] \times \mathbb{R}$ be the unique lift of φ satisfying $\tilde{\varphi}(\tilde{z}_0) = \tilde{z}_0$. Then for each $n \in \mathbb{N}$, $(\tilde{\varphi})^n$ is the unique lift of φ^n that fixes $\tilde{z}_0$. Let $h : \mathbb{R} \to \mathbb{R}$ be the map characterized by

$$\tilde{\varphi}(1, y) = (1, h(y))$$

for all $y \in \mathbb{R}$. In particular, h is a lift of the circle map $\varphi : \partial D \to \partial D$. Since φ has no fixed points on ∂D, it follows that h has no fixed points. Thus there exists $\varepsilon > 0$ such that one of the following two cases occurs:

(A) $h(y) > y + \varepsilon$ for all $y \in \mathbb{R}$, or
(B) $h(y) < y - \varepsilon$ for all $y \in \mathbb{R}$.

From here on *let us assume we are in case* (A), as the argument for case (B) is the same with obvious modifications. It follows inductively that for each $n \in \mathbb{N}$, $h^n(y) > y + n\varepsilon$. Now fix $n_0 \in \mathbb{N}$ sufficiently large that

$$h^{n_0}(y) > y + 3 \tag{51}$$

for all $y \in \mathbb{R}$. Note that it follows that $h^n(y) > y + 3$ for all $n \geq n_0$. Consider the lift $\tilde{\varphi}^{n_0} : (0,1] \times \mathbb{R} \to (0,1] \times \mathbb{R}$. We have $\tilde{\varphi}^{n_0}(1, y) = (1, h^{n_0}(y))$ for all

$y \in \mathbb{R}$. Thus, for each $y \in \mathbb{R}$ and $n \geq n_0$,

$$\tilde{\varphi}^n(1, y) = (1, y')$$

for some

$$y' > y + 3. \tag{52}$$

As ∂D is a compact invariant set for φ^{n_0}, there exists a point $z_1 \in \partial D$ that is nonwandering for the circle map $\varphi^{n_0} : \partial D \to \partial D$ (e.g., take any point in the ω-limit set of another point in ∂D). We write d for the Euclidean metric on D or $[0, 1] \times \mathbb{R}$. Let $I \subset \partial D$ be an open neighborhood of z_1 sufficiently small that for some $\varepsilon_0 > 0$, the set

$$V_0 := \{z \in \mathring{D} \backslash \{0\} \,|\, d(z, I) < \varepsilon_0\}$$

is an open disk. That z_1 is nonwandering for $\varphi^{n_0} : \partial D \to \partial D$ means that there exists $m \in \mathbb{N}$ such that

$$\varphi^{mn_0}(I) \cap I \neq \emptyset. \tag{53}$$

Let $\tilde{z}_1 = (1, y_1) \in (0, 1] \times \mathbb{R}$ be a lift of z_1, and let $\tilde{I} \subset \{1\} \times \mathbb{R}$ be the lift of I that contains $\tilde{z}_1$. We chose I sufficiently small above that $\tilde{I}$ will be disjoint from all of its translates. That is,

$$\tilde{I} \cap T^q(\tilde{I}) = \emptyset \tag{54}$$

for all $q \in \mathbb{Z} \backslash \{0\}$. From (53) we know that $\tilde{\varphi}^{mn_0}(\tilde{I})$ must intersect one of the translates of $\tilde{I}$. That is, there exists $p \in \mathbb{Z}$ such that

$$\tilde{\varphi}^{mn_0}(\tilde{I}) \cap T^p(\tilde{I}) \neq \emptyset. \tag{55}$$

From (52) we have $p \geq 3$.

Now set $\psi := \varphi^{n_0}$ and let $\tilde{\psi} : (0, 1] \times \mathbb{R} \to (0, 1] \times \mathbb{R}$ be the lift

$$\tilde{\psi} := T^{-1} \circ \tilde{\varphi}^{n_0}. \tag{56}$$

It is for this iterate φ^{n_0}, and with respect to this lift $\tilde{\psi}$, that we will construct positively and negatively returning disks $U_+, U_- \subset (0, 1) \times \mathbb{R}$.

Let us begin by listing the following properties of $\tilde{\psi}$ that follow immediately from the properties of $\tilde{\varphi}^{n_0}$ we accumulated above:

(i) $\tilde{\psi}(\tilde{z}_0) = T^{-1}(\tilde{z}_0)$.
(ii) For each $y \in \mathbb{R}$, $\tilde{\psi}(1, y) = (1, y')$ where $y' > y + 2$.
(iii) There exists $m \in \mathbb{N}$ and an integer $p \geq 2$ such that

$$\tilde{\psi}^m(\tilde{I}) \cap T^p(\tilde{I}) \neq \emptyset. \tag{57}$$

Note that a consequence of the second property is that

$$\tilde{\psi}(\mathrm{cl}(\tilde{I})) \cap \mathrm{cl}(\tilde{I}) = \emptyset, \tag{58}$$

where $\mathrm{cl}(\tilde{I})$ is the closure of $\tilde{I}$ in $(0, 1] \times \mathbb{R}$.

First consider the negatively returning disk. Write $(x, y) := \tilde{z}_0 \in (0,1)\times\mathbb{R}$. Then $\tilde{\psi}(x, y) = (x, y-1)$ by (i). Thus any sufficiently small disk neighborhood $U_- \subset (0,1)\times\mathbb{R}$ of (x, y) is a lift of a disk in $\mathring{D}\backslash\{0\}$ that satisfies $\tilde{\psi}(\bar{U}_-)\cap\bar{U}_- = \emptyset$. Moreover, $\tilde{\psi}(U_-) \cap T^{-1}(U_-)$ is nonempty as it contains the point $(x, y-1)$. Thus U_- is a negatively returning disk for $\tilde{\psi}$ satisfying conditions (1), (2), and (3) listed at the beginning.

It remains to find a suitable positively returning disk for $\tilde{\psi}$. Consider the open neighborhoods of $\tilde{I}$ in $(0,1] \times \mathbb{R}$ of the form

$$V := \{z \in (0,1] \times \mathbb{R} \,|\, d(z, I) < \varepsilon\}$$

for $\varepsilon > 0$. From (57) there exist integers $p \geq 2$ and $m \geq 1$ such that

$$\tilde{\psi}^m(V) \cap T^p(V) \neq \emptyset. \tag{59}$$

For $\varepsilon > 0$ sufficiently small, we have

$$\tilde{\psi}(\bar{V}) \cap \bar{V} = \emptyset. \tag{60}$$

This is because $\tilde{\psi}$ moves points on $\{1\} \times \mathbb{R}$ in one direction a distance greater than 2, and $\tilde{I} \subset \{1\} \times \mathbb{R}$ was chosen sufficiently small.

Set $U_+ := V \cap \tilde{\mathbb{A}} = V \cap \big((0,1) \times \mathbb{R}\big)$. From (59), and bearing in mind that $\tilde{\psi}^m(V) \cap T^p(V)$ is also open in $(0,1] \times \mathbb{R}$, we must have

$$\tilde{\psi}^m(U_+) \cap T^p(U_+) \neq \emptyset.$$

From (60),

$$\tilde{\psi}(\bar{U}_+) \cap \bar{U}_+ = \emptyset$$

as $\bar{U}_+ \subset \bar{V}$. Moreover, if we chose $0 < \varepsilon \leq \varepsilon_0$, then U_+ is a disk in $\tilde{\mathbb{A}} = (0,1)\times\mathbb{R}$ and is a lift of a disk in $\mathring{D}\backslash\{0\}$. Thus, as $p \geq 2$, U_+ is a positively returning disk for $\tilde{\psi}$ satisfying the required conditions (1), (2), and (3). □

A.2. *Nondegeneracy.* Let $\varphi : D \to D$ be an irrational pseudo-rotation as in Definition 3. In this appendix we prove that the unique periodic point is nondegenerate for every iterate. More precisely,

LEMMA 36. *For each $n \in \mathbb{N}$, the linearization $D\varphi^n(0)$ does not have eigenvalue* 1.

Remark 14. In fact the eigenvalues of $D\varphi^n(0)$ are $\{e^{2\pi i n\alpha}, e^{-2\pi i n\alpha}\}$, where $[\alpha] \in \mathbb{R}/\mathbb{Z}$ is the rotation number of φ. We see these are distinct from 1 for each n because α is irrational.

Proof. By applying the statement to iterates it suffices to prove the statement for $n = 1$. Arguing indirectly we find $v \in \partial D \subset \mathbb{R}^2$ such that $D\varphi(0)v = v$. Blowing up the fixed point $\varphi(0) = 0$ we obtain a map $\hat{\varphi} : [0,1] \times \partial D \to [0,1] \times \partial D$ that is an orientation preserving homeomorphism, and whose restriction to $(0,1] \times \partial D$ is a diffeomorphism that preserves the area form $dt \wedge d\theta$

and is smoothly conjugate to $\varphi : D\backslash\{0\} \to D\backslash\{0\}$, and which on the circle $\{0\} \times \partial D$ is the projection of the linear map $D\varphi(0) : \mathbb{R}^2 \to \mathbb{R}^2$; more precisely,

$$\hat{\varphi}(0, x) = \left(0\,, \frac{D\varphi(0)[x]}{|D\varphi(0)[x]|}\right)$$

for each $x \in \partial D \subset \mathbb{R}^2$. For example, one can take the unique continuous extension of the map $\pi^{-1} \circ \varphi \circ \pi : (0, 1] \times \partial D \to (0, 1] \times \partial D$, where $\pi : (0, 1] \times \partial D \to D\backslash\{0\}$ is the smooth diffeomorphism $\pi(t, x) := \sqrt{t}x$.

It follows that $\hat{\varphi}(0, v) = (0, v)$, and so the rotation number of $\hat{\varphi}$ on the boundary component $\{0\} \times \partial D$ is $[0] \in \mathbb{R}/\mathbb{Z}$. The rotation number of $\hat{\varphi}$ on the other boundary component $\{1\} \times \partial D$ is the irrational number $[\alpha] \in \mathbb{R}/\mathbb{Z}$ because it must equal the rotation number of $\varphi : \partial D \to \partial D$. Thus $\hat{\varphi}$ has distinct rotation numbers on the two boundary circles. Therefore some iterate has a twist in the sense of Poincaré and Birkhoff and so their fixed point theorem applies. We conclude that $\hat{\varphi}$ has periodic points in the interior $(0, 1) \times \partial D$ and so φ has periodic points in $D\backslash(\partial D \cup \{0\})$ contradicting that φ is an irrational pseudo-rotation. □

A.3. *Intersection theory.* The intersection number $F \cdot G$ between pairs of closed oriented surfaces F and G in a closed four manifold is of course a homological invariant. For pseudoholomorphic curves, it is especially useful since its vanishing implies the two curves are either equal or disjoint due to the positivity phenomenon in [34]. Moreover, via the so-called adjunction formula, the embeddedness properties of a surface are related to its self-intersection number, and the latter is homotopy invariant.

Siefring showed that a homotopy invariant intersection number can also be associated to pseudoholomorphic curves with punctures asymptotic to non-degenerate periodic orbits [37], which retains many of the properties that hold for closed curves.

In this appendix we state without proof the properties of an intersection number that we used in Section 7 for curves with a boundary component. We also state a corresponding adjunction formula. A more detailed explanation will appear in [5].

We use the notation from Section 7.1. Fix an homology class $A \in H_1(\partial Z, \mathbb{Z})$. Let $C^\infty(\gamma_+, \partial W, A)$ denote the set of maps $\tilde{u} \in C^\infty(\mathbb{R}^+ \times \mathbb{R}/\mathbb{Z}, W)$ equipped with the $C^\infty_{\mathrm{loc}} \cap C^0$ topology that satisfy

- $\tilde{u}(0, \cdot) \in L_c$ for some $c \in \mathbb{R}$ (not fixed), and represents A.
- $\tilde{u}$ is asymptotically cylindrical in the sense of [37], with asymptotic orbit γ_+.
- $\tilde{u}$ takes interior points to $W\backslash\partial W$.
- $\tilde{u}$ meets ∂W transversely along the loop $\tilde{u}(0, \cdot)$.

Remark 15. Recall that γ_+ lies in the positive end of $(W, \hat{J})$ and that it is nondegenerate and simply covered.

The homotopy invariant intersection index of Siefring can be extended to these curves with boundary. This allows us to associate to any pair of curves $\tilde{u}, \tilde{v} \in C^\infty(\gamma_+, \partial W, A)$ a half integer $\tilde{u} \cdot \tilde{v} \in \frac{1}{2}\mathbb{Z}$ with the following properties:

(1) *Symmetry*: $\tilde{u} \cdot \tilde{v} = \tilde{v} \cdot \tilde{u}$.
(2) *Homotopy invariance*: For a continuous path $\tilde{u}_\tau$ in $C^\infty(\gamma_+, \partial W, A)$, $\tilde{u}_\tau \cdot \tilde{v}$ is independent of τ.
(3) *Positivity*: Suppose $\tilde{u}$ and $\tilde{v}$ are also $\hat{J}$-holomorphic, embedded on their boundary circles, and do not have identical images. Then, $\tilde{u} \cdot \tilde{v} \geq 0$, and equality implies that the two curves have disjoint images.

Remark 16. This is more delicate for pairs of curves $\tilde{u} \in C^\infty(\gamma_+, \partial W, A)$ and $\tilde{v} \in C^\infty(\gamma_+, \partial W, B)$ for which $A \neq B$. We will not consider these cases.

Following [40], [41], which is for curves without boundary, we define the *normal Chern number* of a $\hat{J}$-holomorphic map $\tilde{u} \in C^\infty(\gamma_+, \partial W, A)$ to be the integer

$$c_N(\tilde{u}) = \operatorname{ind}(\tilde{u}) - 1 - p(\gamma_+). \tag{61}$$

Here $p(\gamma_+)$ denotes the parity of the Conley-Zehnder index of γ_+, and $\operatorname{ind}(\tilde{u})$ is the Fredholm index of the linearized Cauchy-Riemann operator at $\tilde{u}$ viewed as a solution of the free boundary problem.

In our simple situation expression (61) seems hardly to warrant a name. But it allows us to draw parallels with the general situation for curves without boundary. The terminology arises (see [40]) because if $\tilde{u}$ is immersed, then $c_N(\tilde{u})$ has an interpretation as a relative first Chern number of a normal bundle over $\tilde{u}$. In the case at hand this can be described as follows.

Let $\zeta \subset T(\partial W)$ denote the unique $\hat{J}$-invariant 2-plane distribution in the boundary of W. Each 2-torus $L \subset \partial W$ is totally real and therefore transverses to ζ. To the nondegenerate periodic orbit γ_+ one can associate an operator, the so-called asymptotic operator [19], with discrete real spectrum. Each eigenspace of this operator yields a homotopy class of sections of $\gamma_+^* TD \to \mathbb{R}/\mathbb{Z}$. Let Φ_+ denote the class associated to the maximal negative eigenvalue of this operator. Let Φ_0 denote the homotopy class of sections of $\tilde{u}(0,\cdot)^*\zeta \to \mathbb{R}/\mathbb{Z}$ that corresponds to the orientable line bundle $l := \zeta \cap TL_c$. Let $N_{\tilde{u}} \to \mathbb{R}^+ \times \mathbb{R}/\mathbb{Z}$ be a $\hat{J}$-invariant normal bundle to $\tilde{u}$ in W that, outside of a compact set, is equal to the tangent planes to the disks, and that over the boundary points of $\tilde{u}$ coincides with ζ. Then (61) can be interpreted as the first Chern number of the complex line bundle $(N_{\tilde{u}}, \hat{J}) \to \mathbb{R}^+ \times \mathbb{R}/\mathbb{Z}$ relative to the homotopy classes Φ_+ and Φ_0.

With this interpretation of the normal Chern number, the adjunction formula in our situation for curves with embedded boundary takes the following form.

THEOREM 37. *Suppose $\tilde{u} \in C^\infty(\gamma_+, \partial W, A)$ is $\hat{J}$-holomorphic and embedded on its boundary circle. Then*

$$\tilde{u} \cdot \tilde{u} = 2\delta(\tilde{u}) + c_N(\tilde{u}) \in \mathbb{Z}, \tag{62}$$

where $\delta(\tilde{u}) \geq 0$ is an integer that vanishes if and only if $\tilde{u}$ is embedded.

Remark 17. $\delta(\tilde{u})$ is the count of double and singular points of $\tilde{u}$ as defined in Siefring [37] and up to a factor of 2 is the same as in Micallef-White [34]. (There is no contribution to $\delta(\tilde{u})$ from the periodic orbit because it is simply covered.) Note that there are no additional contributions to consider from singular or double points on the boundary as we assume $\tilde{u}$ is embedded on the boundary. There are no interior-boundary double points because $\tilde{u} \in C^\infty(\gamma_+, \partial W, A)$.

Remark 18. For general punctured pseudoholomorphic curves, the adjunction formula is due to Siefring [37]. The expression above in (62) is however closer to (A.6) in [41]. The additional terms there have vanished here because the asymptotic periodic orbit γ_+ is simply covered.

Remark 19. The adjunction formula usually requires the curve to be somewhere injective. This is automatically the case here.

The utility of this formula is most apparent when $c_N(\tilde{u}) \geq 0$. Then the vanishing of the left-hand side guarantees that $\tilde{u}$ is embedded. This is convenient as $\tilde{u} \cdot \tilde{u}$ is a homotopy invariant.

PROPOSITION 38. *Suppose that $\tilde{u} \in C^\infty(\gamma_+, \partial W, A)$ is $\hat{J}$-holomorphic, embedded on its boundary circle, that* $\operatorname{ind}(\tilde{u}) = 2$ *as a solution to the free boundary problem, and that* $\tilde{u} \cdot \tilde{u} = 0$. *Then*

(1) *$\tilde{u}$ is embedded.*
(2) *$\tilde{u}$ is Fredholm regular.*
(3) *An open neighborhood of the image of $\tilde{u}$ in W is foliated by a smooth family of embedded $\hat{J}$-holomorphic curves in $C^\infty(\gamma_+, \partial W, A)$.*

By Fredholm regular we mean that the linearized Cauchy-Riemann operator at $\tilde{u}$, viewed as a solution of the free boundary problem, is surjective.

Proof. It follows from (61) that $c_N(\tilde{u}) = 0$. Since $\tilde{u}$ is embedded on the boundary, we can apply the adjunction formula, Theorem 37, and conclude from $\tilde{u} \cdot \tilde{u} = 0$ that $\tilde{u}$ is embedded.

Automatic transversality arguments in [39], [38] then show that $\tilde{u}$ is Fredholm regular. More precisely, this is because $\tilde{u}$ is immersed and $c_N(\tilde{u}) < \mathrm{ind}(\tilde{u})$. Thus $\tilde{u}$ is embedded and Fredholm regular, has Fredholm index 2, meets the boundary transversely in an embedded totally real submanifold, and has interior disjoint from ∂W. These conditions suffice to apply an implicit function theorem in [39], [38] (a slight generalization of the result in [23]) to produce a local 2-dimensional family of $\hat{J}$-holomorphic curves in $C^\infty(\gamma_+, \partial W, A)$. Finally, the condition $c_N(\tilde{u}) = 0$ implies that these curves foliate an open neighborhood of the image of $\tilde{u}$. □

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