On measures invariant under tori on quotients of semisimple groups

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Abstract

We classify invariant and ergodic probability measures on arithmetic homogeneous quotients of semisimple S-algebraic groups invariant under a maximal split torus in at least one simple local factor and show that the algebraic support of such a measure splits into the product of four homogeneous spaces: a torus, a homogeneous space on which the measure is (up to finite index) the Haar measure, a product of homogeneous spaces on each of which the action degenerates to a rank one action, and a homogeneous space in which every element of the action acts with zero entropy.

1. Introduction

1.1. *Background.* It is well known that orbits of one-parameter diagonal groups such as the group

$$\left\{ \begin{pmatrix} s & \\ & s^{-1} \end{pmatrix} : s \in \mathbb{R}^{\times} \right\}$$

on the quotient space $\mathrm{SL}(2, \mathbb{Z}) \backslash \mathrm{SL}(2, \mathbb{R})$ can have very irregular closures (this remains true even if one assumes the orbits are bounded), as well as the closely interlinked fact that this action has a profusion of invariant probability measures. This phenomenon occurs more generally for actions of rank one $\mathbb{R}$-split tori (embeddings of the multiplicative group of $\mathbb{R}$ in an $\mathbb{R}$-algebraic group $\mathbf{G}$) on quotient spaces $\Gamma \backslash \mathbf{G}(\mathbb{R})$ (see, e.g., [18]) as well as for $\mathbb{Q}_p$-groups.

This situation changes dramatically for the action of higher rank tori, and implicitly this has been observed already in the 1950's by Cassels and Swinnerton-Dyer [1] (though in a different, dual, language). In particular, [1, Hypothesis A] is equivalent to the existence of a bounded but nonperiodic orbit of the rank two diagonal group $A < \mathrm{SL}(3, \mathbb{R})$ on $\mathrm{SL}(3, \mathbb{Z}) \backslash \mathrm{SL}(3, \mathbb{R})$, and

Cassels and Swinnerton-Dyer state that they "tend rather to believe" that this hypothesis is false. (An interesting account of this insightful paper, as well as its connection to Margulis' proof of the Oppenheim Conjecture, has been given by Margulis [22].) This phenomenon was independently discovered and investigated by Furstenberg, who studied subsets of $\mathbb{R}/\mathbb{Z}$ invariant under nonvirtually cyclic multiplicative semigroups of integers [12]. Furstenberg also posed (though not in print) an influential conjecture regarding the possible invariant measures for such actions, a conjecture that is still open.

One of the highlights of the theory of flows on homogeneous spaces is the work of Ratner on the action of groups H generated by Ad-unipotent one-parameter subgroups (e.g., [29], [30], [31]), which was later extended by Ratner [32] and by Margulis and Tomanov [23] to the S-algebraic setting we consider in this paper. This important work has had numerous applications in number theory, geometry, and other areas in mathematics. The cornerstone of Ratner's approach to the study of the action of such groups H on quotient spaces $\Gamma \backslash G$ is the study of H-invariant and ergodic probability measures on such spaces, and in a series of papers culminating in [30] she gives a complete classification of these measures. Using this classification, Ratner has been able to resolve in full Raghunathan's Conjecture on orbit closures for the action of such groups, as well as establish the equidistribution of individual orbits for one-parameter unipotent groups [31]. Special cases of Raghunathan's Conjecture were established earlier by purely topological methods by Dani and Margulis, such as in the paper [2].

With regards to the action of higher rank tori and, in particular, the conjecture of Cassels and Swinnerton-Dyer quoted above, the most significant progress to date has also been achieved via the study of invariant measures, though to date a full classification of invariant measures remains elusive.

The first substantial results regarding measures invariant under higher rank abelian actions were in the context of actions on $\mathbb{R}/\mathbb{Z}$. Rudolph [34] (following some results of Lyons [21]) showed that Lebesgue measure is the only probability measure on $\mathbb{R}/\mathbb{Z}$ invariant and ergodic under the action of the multiplicative semigroup of integers generated by two relatively prime integers, which has positive ergodic theoretic entropy with respect to one of the generators of the semigroup. (His result was subsequently generalized to any nonvirtually cyclic semigroup by Johnson [14].) This result is substantial in part because the condition of having entropy larger than some given positive lower bound is stable under weak* limits, which allows one to deduce interesting corollaries from this partial measure classification results; perhaps the first instance where the strategy was utilized was in Johnson and Rudolph's paper [15]. By now there are several genuinely different proofs of Rudolph's theorem; all require in a crucial way the entropy assumption.

Katok and Spatzier [16], [17] were the first to give a partial measure classification result for the action of higher rank groups on homogeneous spaces, using an argument that is related to Rudolph's. However, in the context of homogeneous spaces these techniques seem to give less than they give on $\mathbb{R}/\mathbb{Z}$, and so in addition to an entropy assumption Katok and Spatzier needed to assume some mixing properties for the flow, an assumption that does not behave nicely under weak* limits.

The purpose of this paper is to give a meaningful, and usable, classification of measures invariant under higher-rank tori. This extends our earlier work with A. Katok [5] on measures invariant under the diagonal group in $\Gamma\backslash G$ for $G = \mathrm{SL}(n,\mathbb{R})$ and $\Gamma = \mathrm{SL}(n,\mathbb{Z})$, as well as the work of the second author [20] that treated the case of $G = \mathrm{SL}(2,\mathbb{Z}) \times H$. Both of these results have had applications beyond the theory of flows on homogeneous spaces: we mention in particular the proof of Arithmetic Quantum Unique Ergodicity in [20], an estimate on the dimension of the set of exception to Littlewood's Conjecture in [5], as well as extensions of results of Linnik and Duke to number fields of higher degree by Michel, Venkatesh and the authors [10], [11].

The classification we provide is less complete than that given by Ratner because of two main reasons, one of which is inherent to the action of higher rank tori, the other due to our inability to say anything meaningful about zero entropy measures:

- *Rank one subactions.* As mentioned above, there can be no meaningful measure classification for the action of rank one tori. Even if we are considering the action of higher rank tori on a quotient of a simple algebraic group we can encounter a situation in which the action essentially degenerates into a rank one action if the measure is not fully supported, as was pointed out first by M. Rees [33] [3, §] who constructed irregular orbit closures for the action of the full diagonal group on a compact quotient of $\mathrm{SL}(3,\mathbb{R})$.
- *Zero entropy subactions.* Similarly, even if there is some element of the acting group that acts with positive entropy, the measure may be supported on a product of homogeneous spaces on one of which there may be an exotic zero entropy measure.

1.2. *Statement of main results.* Before stating our main theorem we need to set up some notations. Let $\mathbb{G}$ be a semisimple linear algebraic group defined over $\mathbb{Q}$. Let S be a finite set of places including ∞ if $\mathbb{G}(\mathbb{R})$ is noncompact. We define $G = \mathbb{G}(\mathbb{Q}_S) = \prod_{\sigma \in S} \mathbb{G}(\mathbb{Q}_\sigma)$, where $\mathbb{G}(\mathbb{Q}_\sigma)$ is the group of $\mathbb{Q}_\sigma$-points of $\mathbb{G}$ for $\sigma \in S$ and $\mathbb{Q}_\infty = \mathbb{R}$. Furthermore, let $\Gamma < \mathbb{G}(\mathbb{Q})$ be a lattice in G commensurable with $\mathbb{G}(\mathcal{O}_S)$ with $\mathcal{O}_S$ denoting the ring $\mathbb{Z}[\frac{1}{p} : p \in S]$. We define the homogeneous space X to be the quotient $X = \Gamma\backslash G$.

For every $\sigma \in S$, we suppose $\mathbb{G}$ considered over $\mathbb{Q}_\sigma$ has the almost direct $\mathbb{Q}_\sigma$-almost simple factors $\mathbf{G}_{\sigma,f}$ for $f \in F_\sigma$. (We use $\mathbb{G}, \mathbb{L}, \ldots$ to denote algebraic

groups defined over the global field $\mathbb{Q}$ and $\mathbf{G}, \mathbf{L}, \ldots$ to denote algebraic groups defined over a local field.) Take $F = \bigsqcup_{\sigma \in S} F_\sigma$ to be the disjoint union of the index sets. We simply write $\mathbf{G}_f$ for the almost direct factor of $\mathbb{G}$ over $\mathbb{Q}_\sigma$ (with $\sigma = \sigma(f) \in S$ uniquely determined by the index f). Moreover, we write $G_f = \mathbf{G}_f(\mathbb{Q}_\sigma)$ for the group of $\mathbb{Q}_\sigma$-points of $\mathbf{G}_f$ for any $f \in F$.

Now let $F' \subset F$ be a nonempty subset of the set of factors. Let $A_f \subset G_f$, for $f \in F'$, be a subgroup of finite index of the group of $\mathbb{Q}_\sigma$-points of a maximal $\mathbb{Q}_\sigma$-split torus $\mathbf{A}_f < \mathbf{G}_f$. We define $A = \prod_{f \in F'} A_f$ and define the rank of A as

$$\mathrm{rank}(A) = \sum_{f \in F'} \dim_{\mathbb{Q}_{\sigma(f)}}(A_f).$$

We consider A as a subgroup of G — in particular, A acts by right translation on X.

THEOREM 1.1. *Let $\mathbb{G}$ be a semisimple linear algebraic group over $\mathbb{Q}$, and let $S, \Gamma, X, F, F' \subset F$, and A be as above. Let μ be an A-invariant and ergodic probability measure on X, and let $p \in G$ be such that $\Gamma p \in \mathrm{supp}\,\mu$. Then there exists a reductive linear algebraic subgroup $\mathbb{L}$ defined over $\mathbb{Q}$ so that the following holds*:

(S) (*Support*). *The measure μ is supported on the periodic orbit $\Gamma \mathbb{L}(\mathbb{Q}_S) p$, and $\mathbb{L}$ is the smallest $\mathbb{Q}$-group[1] so that some right translate of $\Gamma \mathbb{L}(\mathbb{Q}_S)$ supports μ.*

(D) (*Decomposition*). *As an algebraic group, the group $\mathbb{L}$ is the almost direct product of a $\mathbb{Q}$-anisotropic $\mathbb{Q}$-torus $\mathbb{L}_T$ and semisimple algebraic $\mathbb{Q}$-subgroups $\mathbb{L}_I, \mathbb{L}_R, \mathbb{L}_Z$ (where some of the subgroups may be trivial). Furthermore, if we set for $t \in \{T, I, R, Z\}$ the group A_t to be $A \cap p^{-1} \mathbb{L}_t(\mathbb{Q}_S) p$, then*

$$\check{A} = A_T A_I A_R A_Z$$

has finite index in A.

(T) (*Torus*). *The quotient $(\mathbb{L}_T(\mathbb{Q}_S) \cap \Gamma) \backslash \mathbb{L}_T(\mathbb{Q}_S)$ is a compact abelian group, and there exists a closed subgroup $T \subset \mathbb{L}_T(\mathbb{Q}_S)$ containing $p A_T p^{-1}$ so that μ is $p^{-1} T p$-invariant and $(T \cap \Gamma) \backslash T$ is compact.*

(I) (*Invariance*). *There exists a finite index subgroup $L_I < p^{-1} \mathbb{L}_I(\mathbb{Q}_S) p$ that is normalized by A_I such that μ is L_I-invariant and for μ-almost every x, the orbit $x L_I$ is periodic.*

(R) (*Rank one factor*). *The algebraic subgroup $\mathbb{L}_R$ is an almost direct product $\mathbb{L}_R = \prod_i \mathbb{L}_{R,i}$ of $\mathbb{Q}$-almost simple algebraic groups, and A_R contains the product of the subgroups $A_{R,i} = A \cap p^{-1} \mathbb{L}_{R,i}(\mathbb{Q}_S) p$ as a finite-index subgroup and $\mathrm{rank}(A_{R,i}) = 1$ for all i.*

(Z) (*Zero entropy*). $h_\mu(a) = 0$ *for all $a \in A_Z$.*

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[1]Reductive or not.

Using the information provided by Theorem 1.1, it is possible to provide a more explicit description of the possible A-invariant measures μ on $\Gamma\backslash G$:

COROLLARY 1.2. *Under the conditions of Theorem* 1.1, *and with notations as in the statement of that theorem, there are for $t \in \{T, I, R, Z\}$ probability measures μ_t on $(\mathbb{L}_t(\mathbb{Q})\cap\Gamma)\backslash\mathbb{L}_t(\mathbb{Q}_S)$, invariant and ergodic under pA_tp^{-1}, with*

(T) *There is a closed subgroup $T \subset \mathbb{L}_T(\mathbb{Q}_S)$ containing pA_Tp^{-1} so that μ_T is T-invariant, supported on a single T-orbit;*

(I) *μ_I is pL_Ip^{-1}-invariant, supported on a single pL_Ip^{-1}-orbit;*

(Z) *$h_{\mu_Z}(a) = 0$ for any $a \in pA_Zp^{-1}$*

so that μ is (up to translations) an almost direct product of $\mu_T, \mu_I, \mu_R, \mu_Z$.

More precisely, for $t \in \{T, I, R, Z\}$ let $\tilde{\mu}_t$ be the natural[2] lift of μ_t to $\mathbb{L}_t(\mathbb{Q}_S)$. Let $\tilde{\mu}'$ denote the push forward of the product measure $\tilde{\mu}_T\times\tilde{\mu}_I\times\tilde{\mu}_R\times\tilde{\mu}_Z$ on the direct product $\mathbb{L}_T(\mathbb{Q}_S) \times \mathbb{L}_I(\mathbb{Q}_S) \times \mathbb{L}_R(\mathbb{Q}_S) \times \mathbb{L}_Z(\mathbb{Q}_S)$ to $\mathbb{L}(\mathbb{Q}_S)p$ via the map $(g_T, g_I, g_R, g_Z) \mapsto g_Tg_Ig_Rg_Zp$. Then the natural lift $\tilde{\mu}$ of μ to $\mathbb{L}(\mathbb{Q}_S)p$ satisfies

$$\tilde{\mu} = \frac{1}{[A:\breve{A}]} \sum_{a\in A/\breve{A}} \tilde{\mu}'a.$$

Note that (T) and (I) above are not symmetrical: we have less control over the closed subgroup $T \subset \mathbb{L}_T(\mathbb{Q}_S)$ than we have on L_I, which we know (after conjugation) is a finite index subgroup of $\mathbb{L}_I(\mathbb{Q}_S)$. Determining which subgroups T may occur is linked with difficult Diophantine questions, closely connected to the conjectures discussed in [25]; see also [38, §4.4].

We shall deduce Theorem 1.1 from the following pleasantly concise special case of that theorem.

THEOREM 1.3. *Let $\mathbb{G}$ be a $\mathbb{Q}$-almost simple linear algebraic group, and let $S, \Gamma, X, F, F' \subset F$, and A be as above. Let μ be an A-invariant and ergodic probability measure on X. Suppose in addition that*

(1) *μ is not supported on any periodic orbit $\Gamma\mathbb{L}(\mathbb{Q}_S)p$ for any $p \in G$ and reductive $\mathbb{Q}$-subgroup $\mathbb{L} \lneqq \mathbb{G}$;*

(2) *$\mathrm{rank}(A) \geq 2$;*

(3) *$h_\mu(a) > 0$ for some $a \in A$.*

Then there is a finite index subgroup $L_I < G$ so that μ is L_I-invariant and supported on a single L_I-orbit.

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[2]That is, left $\Gamma\cap\mathbb{L}_t(\mathbb{Q})$-invariant.

Remark. For some groups $\mathbb{G}$ and $\mathbb{Q}_\sigma$, it may occur that there are restrictions on the possible reductive subgroup $\mathbb{L} < \mathbb{G}$ with the same $\mathbb{Q}_\sigma$-rank as $\mathbb{G}$ and with $\mathbb{Q}$-anisotropic center that may simplify the statements of the results above. The nicest situation is when any such $\mathbb{L} < \mathbb{G}$ will be $\mathbb{Q}$-almost simple. In such a case, if A has rank ≥ 2 and contains a maximal $\mathbb{Q}_\sigma$-split torus in $\mathbb{G}(\mathbb{Q}_\sigma)$, the statements of Theorem 1.1 and Corollary 1.2 simplify considerably: indeed, in such a case, if μ is an A-invariant and ergodic measure on $\Gamma\backslash\mathbb{G}(\mathbb{Q}_S)$ so that $h_\mu(a) > 0$ for some $a \in A$, then $\mathbb{L}_T, \mathbb{L}_R, \mathbb{L}_Z$ are all trivial, and the measure μ is L-invariant and supported on a single periodic orbit of L. This happens in particular for $\mathbb{G} = \mathrm{SL}(n)$ for all $\mathbb{Q}_\sigma$, or more generally for $\mathbb{G} = \mathrm{SL}(k, \mathbb{D})$ where $\mathbb{D}$ is a division algebra of degree ℓ over $\mathbb{Q}$ for all $\mathbb{Q}_\sigma$ for which $\mathbb{D}$ is unramified (i.e., $\mathbb{D} \otimes \mathbb{Q}_\sigma \cong M_{\ell\times\ell}(\mathbb{Q}_\sigma)$). These are precisely the cases considered in [5], [10]; cf. also [37].

1.3. *Some remarks about the proofs, and relation to prior works.* A key construction that was first used in the context of measure classification for diagonalizable actions by Katok and Spatzier [16], [17] (though implicitly can be found also in the proof of Rudolph's theorem [34] on invariant measures on $\mathbb{R}/\mathbb{Z}$) is the construction of leafwise measures for a measure μ on X: a system of measures that can be defined for orbits of arbitrary subgroups of G [20], but that is particularly informative for the orbits of A-normalized unipotent groups U that are contracted by some $a \in A$. Whether these systems of measures degenerate and become trivial is closely connected to the positivity of entropy of μ, and this is precisely why the condition of positive entropy is so useful.

In broad outline, our argument is similar to that of our joint paper with Katok [5]. Using the leafwise measures, one is able to isolate the contribution of each nondivisible root α of G to the entropy using the leafwise measures for the group U^α corresponding to the roots α and possibly 2α (if 2α is also a root). If there are noncommuting roots α, β that contribute nontrivially to the entropy (the "high entropy" case), then the work of Katok and the first author [3], [4] gives that μ is invariant under some one-parameter unipotent group (in which case one can then apply Ratner's Measure Classification Theorem or its S-algebraic extensions).

If, however, there is a root α that contributes nontrivially to the entropy but the high entropy assumption is not satisfied (the "low entropy" case), a completely different argument is used that uses nonmeasure preserving (even non measure-class preserving!) dynamics along U^α, using ideas developed in earlier works of Ratner on rigidity of horocycle flows [26], [27], [28] (where of course the unipotent flow was measure preserving). The measure preserving action of the group A merely exists in the background and ensures suitable

regularity. This argument was first applied in the paper [20] by the second author and was combined with the high entropy argument in our joint paper with Katok [5].

The most difficult part of extending the measure classification result of [5] to the substantially more general case we consider here is that in the case of the $\mathbb{R}$-split torus action on a quotient of $\mathrm{SL}(n, \mathbb{R})$, the groups U^α corresponding to the roots were one dimensional, whereas in our case these groups U^α are multidimensional and even (if we have double roots) noncommutative. As there is no invariance under the U^α, one cannot reduce to the one-dimensional case, and a detailed analysis of the leafwise measures on the U^α is needed. In the paper [4] the high entropy argument was generalized to the case of multidimensional root groups. Generalizing the low entropy argument proved to be quite tricky and was carried out, specifically for this purpose, in our paper [8], which should be considered as a technical first part of the present paper.

The tools we use give some information also for more general higher rank abelian actions on homogeneous spaces. In particular, in [4] (using only the high entropy argument) it is shown that in many cases any measure invariant and ergodic under a higher rank abelian action with entropy sufficiently close to that of uniform measure must coincide with the uniform measure on this homogeneous space. However, we believe we have pushed these tools to the limit if one wants to get relatively sharp measure classification results such as Theorem 1.1 and Corollary 1.2. Beyond the class of actions we consider here, new ideas are needed.

The results of this paper were announced (in slightly different, less arithmetic form) in [6]. We thank the Clay Institute for Mathematics for the generous support it provided to the authors during the time the ideas of this paper were obtained. This paper was finalized while both authors were Fellows of the Israel Institute of Advanced Studies special program on Arithmetic and Dynamics.

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2. **Some reductions**

The purpose of this section is to show how Theorem 1.1 can be reduced to its special case Theorem 1.3. We achieve this gradually in several steps.

2.0. *Reduction to the case Γe in the support of μ.* Mostly for notational convenience, by replacing μ with μp^{-1} (i.e., the push forward of μ under the map $x \mapsto xp^{-1}$) and A with pAp^{-1} for $p \in G$ with $\Gamma p \in \operatorname{supp} \mu$, we may assume that $\Gamma e \in \operatorname{supp} \mu$.

2.1. *Passage to finite index subgroups.* We continue our reduction by showing how Theorem 1.1 for some group $A = \prod_{f \in F'} A_f$ can be deduced from knowing that this theorem holds for a finite index subgroup $A' = \prod_{f \in F'} A'_f < A$.

Suppose μ is a measure on $\Gamma \backslash G$ invariant and ergodic under A. Then we may write μ as

$$(2.1) \qquad \mu = \frac{1}{[A : A']} \sum_{a \in A/A'} \mu' a,$$

with μ' invariant and ergodic under A'. Here and throughout the paper we use the suggestive notation $\mu' a$ for the push forward of μ' under right multiplication by $a \in A$.

Applying Theorem 1.1 to μ' and A' we obtain $\mathbb{Q}$-groups $\mathbb{L}$, $\mathbb{L}_T$, $\mathbb{L}_I$, $\mathbb{L}_R$, $\mathbb{L}_Z$ and finite index subgroups $L_I < \mathbb{L}_I(\mathbb{Q}_S)$ as in that theorem.

It follows from Theorem 1.1(S) applied to μ' that $A' < \mathbb{L}(\mathbb{Q}_S)$. Recall that for any $f \in F'$ both A_f and A'_f are finite index subgroups of the $\mathbb{Q}_{\sigma(f)}$-points of a maximally $\mathbb{Q}_{\sigma(f)}$-split torus $\mathbf{A}_f$ of $\mathbf{G}_f$. As A'_f is Zariski dense in $\mathbf{A}_f$ we may conclude that A_f (hence A) are contained in $\mathbb{L}(\mathbb{Q}_S)$. Equation (2.1) implies that μ also satisfies Theorem 1.1(S) for the same $\mathbb{L}$.

Theorem 1.1(D) and Theorem 1.1(R) are purely algebraic statements that do not involve μ', and moreover the validity of these statements for A' implies elementarily that they are valid for A as well. Using the relation between μ and μ' given in (2.1) and basic properties of the ergodic theoretic entropy it is easy to see that Theorem 1.1(Z) for μ' and A' implies the same for μ and A. Indeed, for any $a \in A_Z = A \cap \mathbb{L}_Z(\mathbb{Q}_S)$ there exists some $n \geq 1$ such that $a' = a^n \in A'$, hence also in $A'_Z = A' \cap \mathbb{L}_Z(\mathbb{Q}_S)$. From (2.1) and $h_{\mu'}(a') = 0$ it follows that $h_\mu(a') = 0$, hence $h_\mu(a) = 0$.

Theorem 1.1(T) for μ and A again follows from the same statement for μ' and A' — we only have to note that A commutes with $\mathbb{L}_T$.

Thus it only remains to verify Theorem 1.1(I) for μ and A. We note that if L_I satisfies Theorem 1.1(I) for μ' and A', then so does any finite index subgroup of L_I. Hence by replacing L_I with

$$(2.2) \qquad \bigcap_{a \in A/A'} a L_I a^{-1},$$

we may assume that L_I is normalized not just by A'_I but by A_I. Moreover since for μ'-almost every x, the orbit $x L_I$ is periodic, it follows from (2.1) that the same holds for μ-almost every x.

2.2. *Reduction to the case of $\mathbb{G}$ a $\mathbb{Q}$-almost simple group.* Suppose $\mathbb{G}$ is not $\mathbb{Q}$-almost simple. We reduce Theorem 1.1 for $\mathbb{G}$ from the $\mathbb{Q}$-almost simple case by writing $\mathbb{G}$ as an almost direct product $\prod_{i=1}^M \mathbb{H}_{(i)}$ of $\mathbb{Q}$-almost simple groups $H_{(i)}$ and passing to the space $\tilde{\Gamma} \backslash \tilde{\mathbb{G}}(\mathbb{Q}_S)$, where $\tilde{\mathbb{G}}$ is the direct product

of the groups $\mathbb{H}_{(i)}$ and $\tilde{\Gamma}$ is the product of the lattices $\Gamma \cap \mathbb{H}_{(i)}(\mathbb{Q}_S)$ in $\mathbb{H}_{(i)}(\mathbb{Q}_S)$. We also assume as we may (see Section 2.0) that $\Gamma e \in \operatorname{supp} \mu$.

Let $\psi : \tilde{\mathbb{G}} \to \mathbb{G}$ be the obvious isogeny. Setting $\tilde{G} = \tilde{\mathbb{G}}(\mathbb{Q}_S)$, it is clear that ψ also induces finite-to-one maps $\tilde{G} \to G$ and $\tilde{\Gamma} \backslash \tilde{G} \to \Gamma \backslash G$ that will also be denoted by ψ. To avoid confusion, we make a notational distinction between the algebraic $\mathbb{Q}$-subgroup $\mathbb{H}_{(i)}$ of $\mathbb{G}$ and the isomorphic subgroup $\tilde{\mathbb{H}}_{(i)}$ of $\tilde{\mathbb{G}}$. By construction, Section 2.1, and after relabeling indices if necessary, we may assume that the group A can be written as an almost direct product group $\prod_{i=1}^{M'} A_{(i)}$ with $A_{(i)}$ a nontrivial subgroup of $\mathbb{H}_{(i)}(\mathbb{Q}_S)$. Let $\tilde{A}$ be the corresponding direct product group in $\tilde{G}$. Note that while in general $\psi(\tilde{G})$ may have finite index in G, the way things have been set up assures that $\psi(\tilde{A}) = A$. Since μ is A-ergodic and $A \subset \psi(\tilde{G})$ it follows that μ is supported on a single orbit of $\psi(\tilde{G})$. Since $\Gamma e \in \operatorname{supp} \mu$ we conclude that $\operatorname{supp} \mu \subset \Gamma \psi(\tilde{G})$.

The collection of probability measures ν on $\tilde{\Gamma} \backslash \tilde{G}$ with $\psi_* \nu = \mu$ is a convex compact set, which is clearly nonempty since μ is supported on $\Gamma \psi(\tilde{G})$. By averaging over Følner sets in $\tilde{A}$ and taking a limit, we can find inside this collection an $\tilde{A}$-invariant measure, and by taking an ergodic component we deduce that there is an $\tilde{A}$-ergodic and invariant probability measure $\tilde{\mu}$ on $\tilde{\Gamma} \backslash \tilde{G}$ so that $\psi_* \tilde{\mu} = \mu$. Since $\tilde{\mu}$ is ergodic under the action of a product group of a product space, it must be of the form $\tilde{\mu} = \tilde{\mu}_{(1)} \times \cdots \times \tilde{\mu}_{(M)}$ with $\tilde{\mu}_{(i)}$ for $i \leq M'$ an $\tilde{A}_{(i)}$-ergodic and invariant measure on $\tilde{\Gamma}_{(i)} \backslash \tilde{\mathbb{H}}_{(i)}(\mathbb{Q}_S)$ and $\tilde{\mu}_{(i)}$ atomic, supported on a single point for $i > M'$.

Assuming that Theorem 1.1 has been established for $\mathbb{Q}$-almost simple groups, we have for each $i \leq M'$ four $\mathbb{Q}$-groups $\tilde{\mathbb{L}}_t^{(i)}$ of $\tilde{\mathbb{H}}_{(i)}$ for $t \in \{T, I, R, Z\}$ that satisfy all the conditions of Theorem 1.1 for $\tilde{\mu}_{(i)}$ on $\tilde{\Gamma}_{(i)} \backslash \mathbb{H}_{(i)}(\mathbb{Q}_S)$. The $\mathbb{Q}$-groups $\mathbb{L}_t = \psi(\prod_{i=1}^{M'} \tilde{\mathbb{L}}_t^{(i)})$ for $t \in \{T, I, R, Z\}$ are now natural candidates for satisfying the conditions of Theorem 1.1 for $\mathbb{G}$.

Verifying that this choice satisfies (S), (D), (T), (I), (R) of Theorem 1.1 is straightforward. (For (I), and in obvious notations, A_I may contain $\psi(\prod_i \tilde{A}_I^{(i)})$ as a nontrivial finite index subgroup and hence may hypothetically fail to normalize $L_I := \psi(\prod_i \tilde{L}_I^{(i)})$, but replacing L_I by a finite index subgroup if necessary as in (2.2) alleviates this minor nuisance.)

To see that (Z) holds, i.e., that for every $a \in A_Z$ the entropy $h_\mu(a) = 0$, we note first that it is enough to check this for a in the (at worst) finite index subgroup $\psi(\prod_i A_Z^{(i)})$, i.e., for $a = \psi((a_1, \ldots, a_{M'}, e, \ldots, e))$ with $a_i \in \tilde{A}_Z^{(i)}$. Since $\psi : \tilde{\Gamma} \backslash \tilde{G} \mapsto \Gamma \backslash G$ is finite-to-one and $\psi_* \tilde{\mu} = \mu$,

$$h_\mu(a) = h_{\tilde{\mu}}((a_1, \ldots, a_{M'}, e, \ldots, e)) = \sum_{i=1}^{M'} h_{\tilde{\mu}_{(i)}}(a_i) = 0,$$

where we used $\tilde{\mu} = \tilde{\mu}_{(1)} \times \cdots \times \tilde{\mu}_{(M)}$, $a_i \in \tilde{A}_Z^{(i)}$ and the fact that $\tilde{\mu}_{(i)}$ satisfies Theorem 1.1(Z).

2.3. *Completing the reduction.* Our strategy in deducing Theorem 1.1 from Theorem 1.3 is simple: we find in $\mathbb{G}$ a minimal reductive subgroup $\mathbb{L}$ over $\mathbb{Q}$ so that $\Gamma\mathbb{L}(\mathbb{Q}_S)p$ supports μ for some $p \in G$. Assuming as we may that $\Gamma e \in \operatorname{supp}\mu$, it is enough to take $p = e$. We show that $\mathbb{L}$ does not have any $\mathbb{Q}$-characters, split $\mathbb{L}$ into an almost direct product of a torus $\mathbb{L}_T$ and a semisimple group $\mathbb{L}_{\mathrm{ss}}$, and reduce the study of μ to the study of an $A\cap\mathbb{L}_{\mathrm{ss}}(\mathbb{Q}_S)$-ergodic component of μ on $(\Gamma\cap\mathbb{L}_{\mathrm{ss}}(\mathbb{Q}))\backslash\mathbb{L}_{\mathrm{ss}}(\mathbb{Q}_S)$, which we show also satisfies the assumptions of Theorem 1.1.

LEMMA 2.1. *Let $\mathbb{G}$ be a semisimple $\mathbb{Q}$-group, A, G, S and Γ as in Theorem 1.1 and μ an A-invariant and ergodic probability measure on $\Gamma\backslash G$ with $\Gamma e \in \operatorname{supp}\mu$. Let $\mathbb{L}$ be a $\mathbb{Q}$-subgroup of $\mathbb{G}$ so that $\mu(\Gamma\mathbb{L}(\mathbb{Q}_S)) = 1$. Then there is a normal $\mathbb{Q}$-subgroup $\mathbb{L}'\triangleleft\mathbb{L}$, without $\mathbb{Q}$-characters, for which $\operatorname{supp}\mu \subset \Gamma\mathbb{L}'(\mathbb{Q}_S)$.*

In particular, note that since normal algebraic subgroups of reductive algebraic groups are reductive, Lemma 2.1 implies that if $\mathbb{L}$ is the smallest reductive $\mathbb{Q}$-group so that $\operatorname{supp}\mu \subset \Gamma\mathbb{L}(\mathbb{Q}_S)$, then $\mathbb{L}$ does not have $\mathbb{Q}$-characters. For later purposes, we also stress that in the statement of the above lemma, $\mathbb{L}$ is *not* assumed to be reductive.

Proof. Suppose χ is a $\mathbb{Q}$-character of $\mathbb{L}$, and let $L = \mathbb{L}(\mathbb{Q}_S)$. If $\ell = (\ell_\sigma)_{\sigma\in S} \in L$ with $\ell_\sigma \in \mathbb{L}(\mathbb{Q}_\sigma)$, define

$$|\chi|\,(\ell) = \prod_{\sigma\in S} |\chi(\ell_\sigma)|_\sigma\,.$$

This is a group homomorphism $L \to \mathbb{R}^\times$, and moreover $|\chi|\,(\gamma) = 1$ for every $\gamma \in \mathbb{L}(\mathcal{O}_S)$. Since $\Gamma_L := \Gamma \cap \mathbb{L}(\mathbb{Q})$ is commensurable with $\mathbb{L}(\mathcal{O}_S)$, we see that Γ_L is in the kernel of $|\chi|$; hence $|\chi|$ is defined on the points[3] of ΓL. The group A has to be in the kernel of $|\chi|$ since if there were an element $a_0 \in A$ with $|\chi(a_0)| > 1$, then for any $x \in \Gamma L$, we would have $|\chi|\,(xa_0^n) \to \infty$ contradicting Poincare recurrence. It follows that χ itself must be trivial on $A \cap \mathbb{G}(\mathbb{Q}_\sigma)$ for every $\sigma \in S$, and hence A is contained in the proper normal $\mathbb{Q}$-subgroup $\mathbb{L}' = (\mathbb{L}\cap\ker\chi)\triangleleft\mathbb{L}$. By ergodicity of μ, it follows that μ has to be supported on a single orbit $\Gamma\mathbb{L}'(\mathbb{Q}_S)p = \Gamma p\mathbb{L}'(\mathbb{Q}_S)$ for some $p \in L$, establishing the lemma. □

We now continue with the reduction. Suppose $\mathbb{L}$ is a minimal reductive $\mathbb{Q}$-group with $\mu(\Gamma\mathbb{L}(\mathbb{Q}_S)) = 1$. We split this reductive $\mathbb{Q}$-group $\mathbb{L}$ as the semidirect product of a $\mathbb{Q}$-torus $\mathbb{L}_T$ and a semisimple $\mathbb{Q}$-group $\mathbb{L}_{\mathrm{ss}}$. By the discussion

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[3]We identify ΓL with the L-orbit of the identity coset Γe in $\Gamma\backslash G$; note that it is isomorphic to $(L\cap\Gamma)\backslash L$.

following the statement of Lemma 2.1 the group $\mathbb{L}$ has no $\mathbb{Q}$-characters; hence the torus $\mathbb{L}_T$ is $\mathbb{Q}$-anisotropic.

We aim to establish two important facts about how A lies in the product $\mathbb{L}_T(\mathbb{Q}_S)\mathbb{L}_{\mathrm{ss}}(\mathbb{Q}_S)$:

(1) $A_T := A \cap \mathbb{L}_T(\mathbb{Q}_S)$ and $A_{ss} := A \cap \mathbb{L}_{ss}(\mathbb{Q}_{\mathrm{ss}})$ satisfy that $A_T A_{\mathrm{ss}}$ is a finite index subgroup of A.

(2) If we decompose for each $\sigma \in S$ the group $\mathbb{L}_{\mathrm{ss}}$, considered now as a $\mathbb{Q}_\sigma$-group, as the almost direct product of almost simple $\mathbb{Q}_\sigma$-groups $\mathbf{L}_j$ with $j \in J_\sigma$ and set $J = \bigsqcup_{\sigma \in S} J_\sigma$ the group

$$\prod_{j \in J} \left(A \cap \mathbf{L}_j(\mathbb{Q}_{\sigma(j)}) \right)$$

is a finite index subgroup of A_{ss} satisfying the conditions of Theorem 1.1 for $\mathbb{L}_{\mathrm{ss}}$.

It will be helpful to make a notational distinction between the $\mathbb{Q}$-group $\mathbb{L}$ and the same group considered as a $\mathbb{Q}_\sigma$-group, which we denote by $\mathbf{L}$ (and similarly for $\mathbb{L}_T$ etc.); to make notation less cumbersome we keep the dependence on σ implicit.

For any $f \in F$, let $\sigma \in S$ be the associated valuation and consider the $\mathbb{Q}_\sigma$-group $\mathbf{G}_f \cap \mathbf{L}$. Since $\mathbf{G}_f$ is normal in $\mathbf{G}$, this intersection is a normal algebraic subgroup of a reductive algebraic group and hence reductive. Moreover, its decomposition into torus and semisimple parts is given by

$$(2.3) \qquad \mathbf{G}_f \cap \mathbf{L} = (\mathbf{G}_f \cap \mathbf{L}_T) \cdot (\mathbf{G}_f \cap \mathbf{L}_{\mathrm{ss}}).$$

Indeed, decompose $\mathbf{G}_f \cap \mathbf{L}$ as a product of a $\mathbb{Q}_\sigma$-torus $\tilde{\mathbf{L}}_T$ and a semisimple $\mathbb{Q}_\sigma$-group $\tilde{\mathbf{L}}_{\mathrm{ss}}$. The group $\mathbf{L}_{\mathrm{ss}} \cap \mathbf{G}_f$ is a normal subgroup of a semisimple group, hence semisimple, and hence $\mathbf{L}_{\mathrm{ss}} \cap \mathbf{G}_f \subset \tilde{\mathbf{L}}_{\mathrm{ss}}$. On the other hand, $\tilde{\mathbf{L}}_{\mathrm{ss}}$ is generated by commutators of elements $\tilde{\mathbf{L}}_{\mathrm{ss}} \subset \mathbf{L}$ and hence is contained in $\mathbf{L}_{\mathrm{ss}}$, and we conclude that $\mathbf{L}_{\mathrm{ss}} \cap \mathbf{G}_f = \tilde{\mathbf{L}}_{\mathrm{ss}}$. Similarly $\mathbf{L}_T \cap \mathbf{G}_f$ commutes with $\tilde{\mathbf{L}}_{\mathrm{ss}}$ and hence is contained in $\tilde{\mathbf{L}}_T$; on the other hand, $\tilde{\mathbf{L}}_T$ commutes with $\mathbf{L}_{ss} \cap \mathbf{G}_f$ as well as with all $\mathbf{G}_{f'}$ for $f' \in F_\sigma \setminus \{f\}$, hence $\tilde{\mathbf{L}}_T$ commutes with $\mathbf{L}_{ss}$, hence is contained in $\mathbf{L}_T$, and (2.3) follows.

Suppose now that $f \in F'$ and that $\sigma = \sigma(f)$ is the appropriate place. Let $\mathbf{A}_f$ be a maximal $\mathbb{Q}_\sigma$-split torus in $\mathbf{G}_f$ as in the setup of Theorem 1.1. By definition of $\mathbb{L}$ (and the assumptions that $\Gamma e \in \operatorname{supp} \mu$) it follows that $\mathbf{A}_f < \mathbf{L} \cap \mathbf{G}_f$. Equation (2.3) allows us to project[4] $\mathbf{A}_f$ to split $\mathbb{Q}_\sigma$-tori in $\mathbf{L}_T \cap \mathbf{G}_f$ and

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[4]Formally there is only a projection from $\mathbf{G}_f \cap \mathbf{L}$ to $\mathbf{G}_f \cap \mathbf{L}_t / Z$ for $t \in \{T, ss\}$ where $Z = (\mathbf{G}_f \cap \mathbf{L}_T) \cap (\mathbf{G}_f \cap \mathbf{L}_{\mathrm{ss}})$ is a finite subgroup of the center. The projected torus is then defined as the connected component of the preimage in $\mathbf{G}_f \cap \mathbf{L}_t$ of the projection to $(\mathbf{G}_f \cap \mathbf{L}_t)/Z$ for $t \in \{T, ss\}$.

$\mathbf{L}_{\mathrm{ss}} \cap \mathbf{G}_f$; since $\mathbf{A}_f$ is a maximal split torus, it must coincide with the product of these projections; hence $(A_f \cap L_T)(A_f \cap L_{\mathrm{ss}})$ has finite index in A_f. Taken over all $f \in F'$ this implies Claim (1) above. Similarly, for every $f \in F'$, and taking $J_f \subset J_{\sigma(f)}$ to be the indices of all almost simple $\mathbb{Q}_{\sigma(f)}$-groups appearing in the decomposition of $\mathbf{L}_{ss} \cap \mathbf{G}_f$, it follows from the maximality of rank of A_f that $\prod_{j \in J_f} \left(A_f \cap \mathbf{L}_j(\mathbb{Q}_{\sigma(f)}) \right)$ is a finite index subgroup of $A_f \cap L_{\mathrm{ss}}$, and moreover each $\mathbf{A}_f \cap \mathbf{L}_j$ is a maximal split $\mathbb{Q}_{\sigma(f)}$-torus in $\mathbf{L}_j$, establishing Claim (2).

In view of Claims (1) and (2) and Section 2.1, we may as well assume that $A = A_T A_{\mathrm{ss}}$ and that $A_{\mathrm{ss}} = \prod_{j \in J} \left(A \cap \mathbf{L}_j(\mathbb{Q}_{\sigma(j)}) \right)$. Let $\mu = \int_X \mu_x^{\mathcal{E}} \, d\mu(x)$ be the decomposition of μ into its ergodic components with respect to A_{ss}. For any $a_0 \in A$, the action of A_{ss} on the measure $\mu_x^{\mathcal{E}}$ is measure theoretically conjugate to the action on the translated measure $\mu_x^{\mathcal{E}} a_0$; moreover, by uniqueness of ergodic decomposition, we have that μ-almost surely, $\mu_{x a_0}^{\mathcal{E}} = \mu_x^{\mathcal{E}} a_0$. It follows that for any $a \in A_{\mathrm{ss}}$, we have almost everywhere an equality of entropies

$$h_{\mu_x^{\mathcal{E}}}(a) = h_{\mu_{x a_0}^{\mathcal{E}}}(a).$$

Hence by ergodicity of μ and using the fact that entropy is an affine function on the measure, i.e., $h_\mu(a) = \int h_{\mu_x^{\mathcal{E}}}(a) \, d\mu(x)$, we have that for any $a \in A_{\mathrm{ss}}$,

$$(2.4) \qquad\qquad h_{\mu_x^{\mathcal{E}}}(a) = h_\mu(a) \qquad \mu\text{-a.e.}$$

As a function of a, for any A_{ss}-invariant measure ν on X, the entropy $h_\nu(a)$ is continuous,[5] and hence there is a $x_0 \in \operatorname{supp} \mu$ so that $x_0 A$ is dense in $\operatorname{supp} \mu$ and for which (2.4) holds for all $a \in A_{\mathrm{ss}}$.

Decomposing $\mathbb{L}$ as an almost direct product of groups $\mathbb{H}_{(i)}$ as in Section 2.2 (with $\mathbb{L}$ taking the role of $\mathbb{G}$), and by applying the reduction in that section to $\mu_{x_0}^{\mathcal{E}}$, we obtain measures $\widetilde{(\mu_{x_0}^{\mathcal{E}})}_{(j)}$ on $\tilde{\Gamma}_{(i)} \backslash \tilde{\mathbb{H}}_{(i)}(\mathbb{Q}_S)$ that are not supported on orbits of smaller reductive algebraic groups over $\mathbb{Q}$ — for if one of them were supported on an orbit of a smaller reductive algebraic group, this and $\overline{x_0 A} = \operatorname{supp} \mu$ would contradict our assumption that $\mathbb{L}$ was already chosen minimally.

If this measure $\widetilde{(\mu_{x_0}^{\mathcal{E}})}_{(j)}$ on the quotient $\tilde{\Gamma}_{(i)} \backslash \tilde{\mathbb{H}}_{(i)}(\mathbb{Q}_S)$ does not satisfy the second or third assumption to Theorem 1.3, it satisfies Theorem 1.1 trivially (by setting one of the group in (R) or (Z) equal to $\mathbb{H}_{(i)}$). Assuming Theorem 1.3 has already been established, applying it to $\mu_{x_0}^{\mathcal{E}}$ we get groups $\tilde{\mathbb{L}}_T, \tilde{\mathbb{L}}_I, \tilde{\mathbb{L}}_R, \tilde{\mathbb{L}}_Z$. Using (2.4) it is easy to verify that $\mathbb{L}_T \cdot \tilde{\mathbb{L}}_T, \tilde{\mathbb{L}}_I, \tilde{\mathbb{L}}_R, \tilde{\mathbb{L}}_Z$ satisfy Theorem 1.1 for μ.

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[5]This is *not* an abstract property of entropy of individual element in a commutative group but is a well-known consequence of the relation between entropy and dimension of leafwise measures, a connection explained in Section 4.

2.4. *Proof of Corollary* 1.2. Corollary 1.2 follows from Theorem 1.1 using very similar arguments to those that were used above in the reduction of Theorem 1.1 to Theorem 1.3.

To begin with, it is clear that without loss of generality we may assume that $p = e$ and $\mathbb{L} = \mathbb{G}$ in both Corollary 1.2 and Theorem 1.1. Let $\breve{A}$ be as in (D) of Theorem 1.1. Since $[A : \breve{A}] < \infty$, we may write μ as

$$\mu = \frac{1}{[A : \breve{A}]} \sum_{a \in A/\breve{A}} \mu' a,$$

with μ' an $\breve{A}$-ergodic and invariant measure. Define for $t \in \{T, I, R, Z\}$ the discrete group $\Gamma_t = \mathbb{L}_t(\mathbb{Q}) \cap \Gamma$; as in Section 2.2, it is possible to lift μ' to an $\breve{A}$-invariant and ergodic measure μ'' on

$$\prod_{t \in \{T, I, R, Z\}} (\Gamma_t \backslash \mathbb{L}_t(\mathbb{Q}_S)).$$

Since $\breve{A} = \prod_{t \in \{T, I, R, Z\}} A_t$ is a product group acting ergodically on a product space, it follows that μ'' is a product of A_t-invariant and ergodic measure μ''_t on $\Gamma_t \backslash \mathbb{L}_t(\mathbb{Q}_S)$ where $t \in \{T, I, R, Z\}$.

The quotient $\Gamma_T \backslash \mathbb{L}_T(\mathbb{Q}_S)$ is a finite dimensional compact abelian group, and an A_T invariant and ergodic measure has to be the periodic measure on a single periodic orbit of a closed (but not necessarily algebraic) subgroup of $\mathbb{L}_T(\mathbb{Q}_S)$. Since every $a \in A_Z$ acts with zero entropy on μ, hence on μ', hence on μ'', the measure μ''_Z has entropy zero with respect to every $a \in A_Z$. Regarding μ''_I we already know it is invariant under a finite index subgroup $L_I < \mathbb{L}_I(\mathbb{Q}_S)$; since $A_I \cap L_I$ is of finite index in A_I, we can find a finite index subgroup $L'_I < L_I$ normalized by A_I and then μ''_I will be invariant under the group $A_I \cdot L'_I$. By A_I-ergodicity of μ''_I, the measure μ''_I is supported on a single orbit of this group. This decomposition we have established for μ'' is equivalent to the statement of the corollary (where in order to avoid passing to a finite index subgroup of $\Gamma \cap \mathbb{L}(\mathbb{Q})$, the structure on μ was given in terms of its lift $\tilde{\mu}$ to $\mathbb{L}(\mathbb{Q}_S)$).

2.5. *Some variations.* The reader may wonder why we insist on working over the minimal local fields $\mathbb{R}$ or $\mathbb{Q}_p$ (and not over $\mathbb{C}$ or finite field extensions of $\mathbb{Q}_p$). The following example shows that Theorem 1.1 and Corollary 1.2 as stated do not hold over larger local fields; indeed, essentially the best way to understand the action of these bigger tori is by reducing to the $\mathbb{R}$ or $\mathbb{Q}_p$ case.

Example 2.2. Let $X = \mathrm{SL}(k, \mathbb{Z}[i]) \backslash \mathrm{SL}(k, \mathbb{C})$, let T be the group of diagonal matrices, and let $A \subset T$ be the subgroup of real diagonal matrices. Let ν be the Haar measure of $X_\mathbb{R} = \mathrm{SL}(k, \mathbb{Z}) \backslash \mathrm{SL}(k, \mathbb{R})$. We view $X_\mathbb{R}$ as a subset of X, so that ν is a measure on X. Let $M \subset T$ be the maximal compact subgroup

that consists of all diagonal matrices with entries of absolute values one along the diagonal. Then $\mu = \int_M \nu h \, dm_M(h)$ is a T-invariant and ergodic measure. Note that $\mathrm{SL}(k, \mathbb{R})M$ is *not* a subgroup of $\mathrm{SL}(k, \mathbb{C})$.

Of course, this situation is not drastically different from that of Corollary 1.2. Instead of the finite average we had to take in that corollary, in Example 2.2 one takes an average over a compact group that commute with the $\mathbb{R}$-split torus A.

The analysis of this seemingly more general scenario is easily achieved by restriction of scalars — simply treating, e.g., in this particular case the group $\mathrm{SL}(k, \mathbb{C})$ as an algebraic group over $\mathbb{R}$. Then T becomes the group of $\mathbb{R}$-points of a maximal torus in $\mathrm{SL}(k, \mathbb{C})$ defined over $\mathbb{R}$ and A becomes the group of $\mathbb{R}$-points of a maximal $\mathbb{R}$-split subtorus. In this formulation the above example also shows why we restrict ourselves to the maximal $\mathbb{R}$-split subtorus.

We conclude that the classification of A-invariant and T-invariant measures are essentially equivalent: Any A-invariant and ergodic measure gives rise to a T-invariant and ergodic measure just as in the example, and any T-invariant and ergodic measure can be decomposed into A-invariant and ergodic measures, and a relatively straightforward analysis shows that these ergodic components fit together into a measure of a type similar to that presented in Example 2.2.

3. **Semi-simple linear algebraic groups and their roots**

3.1. *Semi-simple linear algebraic groups.* We recall some basic properties of algebraic groups and refer the reader to [35]. We use this theory as a natural framework that makes no distinction between the real and p-adic numbers.

Let $\mathbb{Q}_\infty = \mathbb{R}$, and let σ be ∞ or a prime p so that $\mathbb{Q}_\sigma$ is either $\mathbb{R}$ or $\mathbb{Q}_p$. Let $|\cdot|_\sigma$ denote the absolute value if $\sigma = \infty$ or the p-adic norm if $\sigma = p$.

Let $G_\sigma = \mathbf{G}(\mathbb{Q}_\sigma)$ be the $\mathbb{Q}_\sigma$-points of a semisimple linear algebraic group $\mathbf{G}$ defined over $\mathbb{Q}_\sigma$, and let $A_\sigma \subset G_\sigma$ be the subgroup of $\mathbb{Q}_\sigma$-points of a maximal $\mathbb{Q}_\sigma$-split torus in $\mathbf{G}$. Then $A_\sigma \cong (\mathbb{Q}_\sigma^\times)^k$, where k is the $\mathbb{Q}_\sigma$-rank of $\mathbf{G}$ and the group of rational characters $\chi : A_\sigma \to \mathbb{Q}_\sigma^\times$ is isomorphic to $\mathbb{Z}^k$. A subgroup $U < G_\sigma$ is *unipotent* if for every $g \in U$, $g - e$ is a nilpotent matrix, i.e., for some n, $(g - e)^n = 0$, where e is identity. A subgroup H is said to be *normalized by* $g \in G_\sigma$ if $gHg^{-1} = H$; H is normalized by $L < G_\sigma$ if it is normalized by every $g \in L$; and the *normalizer* $N_{G_\sigma}(H)$ of H is the group of all $g \in G_\sigma$ normalizing it. Furthermore, $N_{G_\sigma}^1(H)$ denotes the subgroup of the normalizer that in addition also preserves the Haar measure on H. Similarly, g *centralizes* H if $gh = hg$ for every $h \in H$, and we set $C_{G_\sigma}(H)$, the *centralizer* of H in G_σ, to be the group of all $g \in G_\sigma$ centralizing H.

Let $\mathfrak{g}$ be the Lie algebra of G_σ, and let

$$\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g} \text{ for } g \in G_\sigma$$

be the adjoint representation of G_σ on $\mathfrak{g}$. Let Φ denote the set of restricted roots, i.e., all nontrivial characters $\alpha : A_\sigma \to \mathbb{Q}_\sigma^\times$ such that there exists a nontrivial (restricted) root space $\mathfrak{u}^\alpha \subset \mathfrak{g}$ with

$$\mathrm{Ad}_a(u) = \alpha(a)u \text{ for } u \in \mathfrak{u}^\alpha \text{ and } a \in A_\sigma.$$

We write $\mathfrak{g}^0$ for the Lie algebra of the centralizer of A_σ, so that

$$\mathfrak{g} = \mathfrak{g}^0 \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{u}^\alpha.$$

Since $\mathbb{Q}_\sigma$ is not algebraically closed, it may happen, as for instance for the algebraic group $\mathrm{SU}(n, 1)$ over $\mathbb{R}$ with $n \geq 2$, that $\alpha, 2\alpha \in \Phi$ are roots; see also [35, 15.3.9]. We say a root α is *indivisible* if $\frac{1}{2}\alpha$ is not a root. For any indivisible root α, there exists an algebraic unipotent subgroup U^α whose Lie algebra is the sum of the root spaces $\mathfrak{u}^\alpha \oplus \mathfrak{u}^{2\alpha}$ (where $\mathfrak{u}^{2\alpha} = \{0\}$ is allowed); see [35, 15.4].

By embedding the group of characters into a real vector space V of dimension k as a lattice, we can identify the roots with elements of V. Furthermore, we equip V with an inner product $(\cdot, \cdot)$ such that the natural action of the Weyl group $W = W(G_\sigma, A_\sigma)$ of G_σ and A_σ leaves $(\cdot, \cdot)$ invariant. We need a few more fundamental properties of the Weyl group W. The Weyl group W acts naturally on A_σ, which induces an action on the characters of A_σ. (The action on V is the linear extension of this action.) The set Φ of roots is invariant. For every root α, there is an associated element $s_\alpha \in W$; its action on V is an orthogonal reflection defined by

(3.1) $$s_\alpha(v) = v - 2(\alpha, \alpha)^{-1}(\alpha, v)\alpha \text{ for } v \in V.$$

For $\alpha \neq \beta \in \Phi$ with $(\alpha, \beta) > 0$, the number $m = 2(\alpha, \alpha)^{-1}(\alpha, \beta)$ is an integer and $\beta, \beta - \alpha, \ldots, \beta - m\alpha = s_\alpha(\beta) \in \Phi$ are all roots. As is known (and we will recall below) that every root comes naturally with a subtorus of A_σ of which we will use one element as described in the next lemma, which will be proved in Section 3.3.

LEMMA 3.1. *For every root $\alpha \in \Phi$, there exists an element $a_\alpha \in A_\sigma$ with $\log |\beta(a_\alpha)|_\sigma = c_\alpha(\alpha, \beta)$ (for some constant $c_\alpha > 0$). In fact, one can find a homomorphism ϕ from SL_2 into $\mathbf{G}$ defined over $\mathbb{Q}_\sigma$ so that a given element of $\mathfrak{u}^\alpha$ is the image of an upper nilpotent element of $\mathfrak{sl}_2$ and so that $a_\alpha \in A_\sigma$ is the image of a diagonal element*

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Q}_\sigma)$$

with the first eigenvalue t satisfying $|t|_\sigma > 1$.

3.2. *The Lie algebra and the exponential map.* In this section we recall standard facts and notation from [19] (which as phrased below also hold for p-adic Lie groups). Let

$$[\cdot, \cdot] : \mathfrak{g}^2 \to \mathfrak{g}$$

be the $\mathbb{Q}_\sigma$-bilinear Lie bracket satisfying

$$(3.2) \qquad \mathrm{Ad}_g([u, v]) = [\mathrm{Ad}_g u, \mathrm{Ad}_g v]$$

for all $g \in G_\sigma$ and $u, v \in \mathfrak{g}$. Since $\mathbb{Q}_\sigma$ is a local field of characteristic zero, the exponential map $\exp(\cdot)$ is a local homeomorphism between the Lie algebra $\mathfrak{g}$ and G_σ such that

$$(3.3) \qquad g \exp(u) g^{-1} = \exp(\mathrm{Ad}_g(u))$$

whenever both sides are defined. Write $\log(\cdot)$ for the locally defined inverse map. Furthermore, we define $\mathrm{ad}_u(v) = [u, v]$ for any $u, v \in \mathfrak{g}$. Then

$$(3.4) \qquad \mathrm{Ad}_{\exp u} = \exp(\mathrm{ad}_u)$$

whenever both sides are defined; here the exponential map on the right is defined for sufficiently small endomorphisms of $\mathfrak{g}$.

Recall that

$$(3.5) \qquad [\mathfrak{u}^\alpha, \mathfrak{u}^\beta] \subset \mathfrak{u}^{\alpha+\beta}$$

for any roots α, β (which follows easily from (3.2)). If $\alpha \in \Phi$ is a root, then the exponential map is actually a polynomial map and so can be uniquely extended to the whole of $\mathfrak{u}^\alpha \oplus \mathfrak{u}^{2\alpha}$ such that (3.3) and (3.4) still hold. The image $U^\alpha = \exp(\mathfrak{u}^\alpha \oplus \mathfrak{u}^{2\alpha})$ is the unipotent subgroup mentioned earlier. More generally, let $\Psi \subset \Phi$ be a set of roots such that $(\Psi + \Psi) \cap \Phi \subset \Psi$ and $|\alpha(a)|_\sigma > 1$ for all $\alpha \in \Psi$ and some fixed $a \in A_\sigma$. Then

$$\mathfrak{u}^\Psi = \sum_{\alpha \in \Psi} \mathfrak{u}^\alpha$$

is a Lie algebra, $\exp(\cdot)$ can be uniquely extended to $\mathfrak{u}^\Psi$ such that (3.3) holds, and $U^\Psi = \exp \mathfrak{u}^\Psi$ is a unipotent subgroup (that is generated by the subgroups U^α for $\alpha \in \Psi$). We say that a subgroup $U' \subset U^\Psi$ is *connected* if there exists a Lie subalgebra $\mathfrak{u}' \subset \mathfrak{u}^\Psi$ such that $U' = \exp(\mathfrak{u}')$. Note that in the real case this notion agrees with the subgroup being connected with respect to the Hausdorff topology and that in general it is equivalent to U' being Zariski closed (and Zariski connected).

3.3. *The image of iterates of* ad_v *for* $v \in \mathfrak{u}^\alpha$. We now formulate and prove a corollary of the structure of semisimple groups and their maximal $\mathbb{Q}_\sigma$-split subtori. This will be the most important use of the assumption that A_σ is maximal — the structure provided by this result is crucial to our approach. For any subset $R \subset V$, we define
$$\mathfrak{u}^R = \bigoplus_{\alpha \in R} \mathfrak{u}^\alpha \subset \mathfrak{g},$$
where $\mathfrak{u}^\alpha = \{0\}$ if $\alpha \notin \Phi$.

PROPOSITION 3.2. *Let* $A_\sigma \subset G_\sigma$ *be the subgroup of* $\mathbb{Q}_\sigma$-*points of a maximal* $\mathbb{Q}_\sigma$-*split torus in a semisimple linear algebraic group* $\mathbf{G}$ *defined over a local field* $\mathbb{Q}_\sigma$ *of characteristic zero. Let* α *be a root, and let* $v \in \mathfrak{u}^\alpha \setminus \{0\}$. *Then for any root* β *with* $(\beta, \alpha) > 0$ *and* $s_\alpha(\beta) = \beta - m\alpha$, *the restriction of* ad_v^m *to* $\mathfrak{u}^{s_\alpha(\beta)}$ *is bijective.*

Proof of Lemma 3.1 *and Proposition* 3.2. Let $v \in \mathfrak{u}^\alpha$ be nonzero. As v is a nonzero nilpotent element in the semisimple part of the Lie algebra of $\mathbf{G}$, the Jacobson-Morozov theorem (see, e.g., [13, III, Lemma 7 and Th. 17]) implies that v must be one element of an $\mathfrak{sl}_2$-triple. More precisely, there exists some h and w so that $[h, v] = 2v$, $[h, w] = -2w$, and $[v, w] = h$ (which implies that the Lie algebra generated by v, h, w is isomorphic to $\mathfrak{sl}_2$). We claim that we may assume that h belongs to the Lie algebra of A_σ.

To see the claim, let $h = h_0 + h_1$ be a decomposition of h into an element $h_0 \in \mathfrak{g}^0$ and an element h_1 that belongs to the sum of the root spaces. By assumption, $[h, v] = 2v$, so that $2v = [h_0, v] + [h_1, v]$. However, by (3.5) this implies that $[h_1, v] = 0$. We also have that $h = [v, w] = h_0 + h_1 \in \mathrm{Im}(\mathrm{ad}(v))$. By (3.5) we see that the image $\mathrm{Im}(\mathrm{ad}(v))$ is invariant under the adjoint action of A_σ, which implies that $h_0 \in \mathrm{Im}(\mathrm{ad}(v))$. However, this implies again by the Jacobson-Morozov theorem [13, III, Lemma 7] that there exists some w' such that (v, h_0, w') is an $\mathfrak{sl}_2$-triple. If $w' \notin \mathfrak{u}^{-\alpha}$, then we can split w' into two components $w^{-\alpha} + w''$, with $w^{-\alpha} \in \mathfrak{u}^{-\alpha}$ and w'' belonging to the remaining weight spaces. Since
$$[h_0, w'] = -2w' = [h_0, w^{-\alpha} + w''] = -2(w^{-\alpha} + w''),$$
by comparing components from different weight spaces we get $[h_0, w^{-\alpha}] = -2w^{-\alpha}$. Moreover, $[v, w'] = h_0 = [v, w^{-\alpha} + w'']$, which by comparing components also implies $[v, w^{-\alpha}] = h_0$. Therefore, $(v, h_0, w^{-\alpha})$ is an $\mathfrak{sl}_2$-triple (and now one can show that actually $w' = w^{-\alpha}$).

From the existence of the above $\mathfrak{sl}_2$-triple it follows that there exists an algebraic subgroup $\mathbf{L}$ defined over $\mathbb{Q}_\sigma$ (defined, e.g., by the Zariski-closure of the image of a neighborhood of 0 in the span of the $\mathfrak{sl}_2$-triple under exp) with the span of the $\mathfrak{sl}_2$-triple as its Lie algebra. The subgroup $\mathbf{L}$ contains the split torus $\mathbf{T} = \mathbf{L} \cap C_\mathbf{G}(A_\sigma)$ whose Lie algebra is spanned by the element h_0.

However, as $\mathbf{T}$ commutes with A_σ and is split, this implies that $\mathbf{T}(\mathbb{Q}_\sigma) \subseteq A_\sigma$. To summarize and to simplify the notation, we have shown that there exists an $\mathfrak{sl}_2$-triple (v, h, w) for which h belongs to the Lie algebra of A_σ and $w \in \mathfrak{u}^{-\alpha}$.

If now $\beta \in \Phi$ is a root and $m \geq 0$ for which $\beta + \alpha, \beta - (m+1)\alpha \notin \Phi$ but $\beta - \alpha, \ldots, \beta - m\alpha \in \Phi$, then $\mathfrak{u}^{\{\beta, \ldots, \beta - m\alpha\}}$ is a finite-dimensional representation for $\mathbf{L}$ (resp. its Lie algebra). Hence $\mathfrak{u}^\beta$ is an eigenspace for ad_h with eigenvalue k for some k and $\mathfrak{u}^{\beta - m\alpha}$ an eigenspace of eigenvalue $-k$, which must then equal $k - 2m$. By the discussion preceding Lemma 3.1,

$$k = m = 2 \frac{(\alpha, \beta)}{(\alpha, \alpha)}.$$

Since α is fixed, we may also phrase this by saying that the eigenvalue of ad_h on $\mathfrak{u}^\beta$ equals the multiple of (α, β) by the constant $c_\alpha = \frac{2}{(\alpha, \alpha)}$. This clearly extends also to all roots $\beta' = \beta - \ell\alpha$ and hence holds for all roots $\beta \in \Phi$. Thus we have established the proposition.

In order to establish Lemma 3.1, we note that if $a \in \mathbf{T}(\mathbb{Q}_\sigma)$ is the image of $\left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix} \right) \in \mathrm{SL}_2(\mathbb{Q}_\sigma)$ under the map induced by the $\mathfrak{sl}_2$-triple, then Ad_a has the eigenvalue $t^{c_\alpha(\alpha, \beta)}$ on $\mathfrak{u}^\beta$. In particular, if we take $|t|_\sigma > 1$, then this element will satisfy the claim in Lemma 3.1. □

3.4. *The S-algebraic group G and the locally homogeneous spaces X.* Recall from the introduction and from Section 2 that we may assume $\mathbb{G}$ is a $\mathbb{Q}$-almost simple algebraic $\mathbb{Q}$-group. Let S be a finite set of places (finite primes and the symbol ∞), and define for every $\sigma \in S$ the group $G_\sigma = \mathbb{G}(\mathbb{Q}_\sigma)$. If $\mathbb{G}(\mathbb{R})$ is not compact, we require that $\infty \in S$. We also define $G = \mathbb{G}(\mathbb{Q}_S) = \prod_{\sigma \in S} G_\sigma$ and suppose that $\Gamma < G$ is an arithmetic lattice, i.e., a lattice commensurable with $\mathbb{G}(\mathcal{O}_S)$ as in the introduction.

The quotient $X = \Gamma\backslash G$ is a locally homogeneous space; let $d(\cdot, \cdot)$ be a left invariant metric on G_S, and denote the induced metric on X also by $d(\cdot, \cdot)$. The group G acts on X by right translations, i.e., $g.x = xg^{-1}$ for $g \in G, x \in X$.

For every $\sigma \in S$, we can write the algebraic group $\mathbb{G}$ considered as an algebraic group over $\mathbb{Q}_\sigma$ as an almost direct product of $\mathbb{Q}_\sigma$-almost simple groups $\prod_{f \in F_\sigma} \mathbf{G}_f$. We define $F = \bigsqcup_{\sigma \in S} F_\sigma$, and if $f \in F_\sigma$, we shall say that σ is the place attached to f; often, when it is clear which f we discuss, σ will be implicitly assumed to be the attached place to f. We also write $\mathfrak{g}_f$ for the Lie algebra of $\mathbf{G}_f$.

Let $F' \subset F$, and for every $f \in F'$, fix a maximal $\mathbb{Q}_\sigma$-split torus $\mathbf{A}_f$. Furthermore, we choose for every $f \in F'$ a finite index subgroup A_f of the group of $\mathbb{Q}_\sigma$-points of this torus. As in the introduction we set $A = \prod_{f \in F'} A_f$.

For a root α of G_{f_0} for $f_0 \in F'$, we define an associated *Lyapunov root* to be the group homomorphism $A \to \mathbb{R}^+$ given by

$$a = ((a_f)_{f \in F'}) \mapsto \log |\alpha(a_{f_0})|_{\sigma_0},$$

where σ_0 is the place corresponding to f_0 and the absolute value is taken in $\mathbb{Q}_{\sigma_0}$. We write Φ_{f_0} for the set of Lyapunov roots of G_{f_0}.

We denote by $\Phi' = \bigcup_{f \in F'} \Phi_f$ the set of all Lyapunov roots obtained in this way and will continue to use Greek letters to denote Lyapunov roots — in fact we will not distinguish between the root and its associated Lyapunov root. As discussed in Section 3.1, for every $f \in F'$, the Lyapunov roots for G_f span a vector space V_f over $\mathbb{R}$ with a natural Euclidean inner product that is preserved by the Weyl group; we define the inner product on the product of these Euclidean spaces so that the individual subspaces V_f for $f \in F'$ are orthogonal. The orthogonal reflections s_α for $\alpha \in \Phi'$ are naturally extended to all of V, acting as the identity on the orthogonal complement of the subspace where s_α was originally defined.

For a Lyapunov root $\alpha \in \Phi_f$, we will write U^α and $\mathfrak{u}^\alpha$ for the subgroup of G_f resp. the Lie subalgebra of $\mathfrak{g}_f$ corresponding to the root as in Section 3.1. Furthermore, we define $\mathfrak{g}^0 = \sum_{f \in F'} \mathfrak{g}_f^0 + \sum_{f \in F \setminus F'} \mathfrak{g}_f$ (which has the structure of an additive group, but for $|S| \geq 2$ is no longer a Lie algebra in the usual sense).

Finally we say a subgroup $U < G_S$ is unipotent and connected if U is a direct product of unipotent (Zariski) connected subgroups $U_\sigma < G_\sigma$ for $\sigma \in S$. For instance, we can define for the subset $\Psi = \{\alpha \in \Phi' : \alpha(a) < 0\}$ the connected unipotent Zariski-closed subgroup

$$U^\Psi = \prod_{f \in F'} U^{\Psi \cap \Phi_f},$$

where $U^{\Psi \cap \Phi_f} < G_f$ is the unipotent subgroup over $\mathbb{Q}_\sigma$ associated to all roots that are contracted by a_f. It will also be convenient to notice the following simple fact.

LEMMA 3.3. *Any A-normalized closed subgroup $U < U^\Psi$ is a product of Zariski-closed Zariski-connected unipotent subgroups U_f for $f \in F'$.*

This lemma can be deduced from the argument in the proof of Proposition 6.2 in [4]; for completeness we provide a proof below.

Proof. Let $u = (u_f)_{f \in F'} \in U < U^\Psi$. For any fixed $f' \in F'$, we can find some $a \in A$ that acts trivially in the factor corresponding to f' and contracts $U^\Psi \cap G_f$ for all $f \in F' \setminus \{f'\}$. Conjugating u by a^n and taking the limit we see that the element u' with $u'_f = e$ for all $f \in F' \setminus \{f'\}$ and $u'_{f'} = u_{f'}$ also belongs to U. Therefore, $U = \prod_{f \in F}(U \cap G_f)$ is a direct product.

Hence it remains to study the case where $U < U^\Psi \cap G_f$ for some $f \in F'$. Suppose first that G_f is a real Lie group (i.e., $\sigma(f) = \infty$) and $\mathfrak{u}$ is the Lie algebra of the connected component of U. Choose some $a \in A_f$ that contracts U^Ψ and only has positive eigenvalues. Using that U is normalized by a we see that $\mathfrak{u} = \sum_{j=1}^J \mathfrak{u} \cap \mathfrak{g}_j$ where Ad_a acts on $\mathfrak{g}_j$ with eigenvalue λ_j and $1 > \lambda_1 > \lambda_2 > \cdots > \lambda_J > 0$. Let now $u \in U$ be arbitrary and

consider $v = \log u = \sum_j v_j$ where $v_j \in \mathfrak{g}_j$. Suppose $v_1 = \cdots = v_{k-1} = 0$ but $v_k \neq 0$. Considering $m \operatorname{Ad}_a^n(v) \in \log U$ for $m \in \mathbb{Z}$ and $n \to \infty$ we may conclude that $v_k \in \mathfrak{u}$. Replacing u by $u \exp(-v_k)$ we obtain a new element in the same connected component as the original element (with an increased value of k). By induction it follows that $u \in \exp \mathfrak{u}$, and since $u \in U$ was arbitrary, we see that U is connected. Since exp is a polynomial map on the Lie algebra of the unipotent group U^Ψ, it follows that U is a Zariski closed and Zariski connected subgroup of U^Ψ.

Suppose now that G_f is a p-adic Lie group (i.e., $\sigma(f) = p$). Note that for any $u \in U$, we automatically have $\exp(t \log(u)) \in U$ for all $t \in \mathbb{Z}$ and so also for all $t \in \mathbb{Z}_p$. Let us write $\mathfrak{u}^\Psi$ for the Lie algebra of U^Ψ and define the Lie subalgebra $\mathfrak{u} = \{v \in \mathfrak{u}^\Psi : \exp(\mathbb{Q}_p v) \subset U\}$. Choose some $a \in A$ for which $\mathfrak{u}^\Psi$ is expanded under Ad_a and all eigenvalues are positive powers of $\lambda_p \in \mathbb{Q}_p$ with $|\lambda_p|_p > 1$. Hence we get a decomposition $\mathfrak{u}^\Psi = \sum_{j=1}^J \mathfrak{g}_j$ such that Ad_a restricted to $\mathfrak{g}_j$ equals multiplication by λ_p^j for all $j = 1, \ldots, J$. Let $u \in U$ be arbitrary, and let $v = \log u \in \mathfrak{u}^\Psi = \sum_{j=1}^J v_j$ with $v_j \in \mathfrak{g}_j$ for all $j \geq 1$. Suppose $v_k \neq 0$ but $v_{k+1} = v_{k+2} = \cdots = 0$. We now consider the elements

$$t \operatorname{Ad}_a^n(v) = \sum_{j=1}^k t \lambda_p^{nj} v_j \in \log U$$

for $t \in \mathbb{Z}_p$ and $n \geq 1$. Letting $n \to \infty$ and choosing $t = s \lambda_p^{-nk}$ for some arbitrary $s \in \mathbb{Q}_p$ gives $v_k \in \mathfrak{u}$.

Setting up an induction, we assume that $v = \log u = \sum_{j=1}^J v_j$ for some $u \in U$, and $v_j \in \mathfrak{g}_j$ for $j = 1, \ldots, J$ implies $v_j \in \mathfrak{u}$ for all $j = k, \ldots, J$. Note that the case $k = J$ follows at once from the above. As in the real case we may now consider the element $u \exp(-v_k) \in U$. By the Campbell-Baker-Hausdorff Formula, this element can be written as

$$\log(u \exp(-v_k)) = \sum_{j=1}^{k-1} v_j + \sum_{j=k+1}^J v_j',$$

where we must allow for $v_j' \in \mathfrak{g}_j$ being nonzero for $j > k$. However, by the inductive assumption we can deduce that $v_{k+1}' \in \mathfrak{u}$ and may consider

$$u \exp(-v_k) \exp(-v_{k+1}'),$$

and so on. Thus we obtain that

$$\exp \left(\sum_{j=1}^{k-1} v_j \right) \in U,$$

and hence by the preceding paragraph we have that $v_{k-1} \in \mathfrak{u}$, completing the induction. As $u \in U$ was arbitrary, we see that $U = \exp(\mathfrak{u})$; hence the group U is once more the image under a polynomial map of the Lie subalgebra $\mathfrak{u}$, from which we may conclude it is Zariski closed and Zariski connected. $\square$

4. **Leaf-wise measures and entropy**

Leafwise measures (which are also referred to as conditional measures on leaves of a foliation) were used in this context the first time by Katok and Spatzier in their pioneering work [16], [17], though implicitly they can also be found hidden in the arguments Rudolph's original paper on the subject [34]. A fairly general construction of leafwise measures is presented in [20, §3]. A self-contained and comprehensive construction of these measures (though somewhat less general, as it only treats the case of foliations whose leaves are given by orbits of groups) as well as the relation of these leafwise measures to entropy is given in [9]. Here we only summarize without proofs the main properties of these leafwise measures.

4.1. *Basic properties.* We will be working with connected subgroups of U^{Ψ} for $\Psi \subset \Phi'$ with $(\Psi + \Psi) \cap \Phi' \subset \Psi$ and $\alpha(a) < 0$ for some fixed $a \in A$ and all Lyapunov roots $\alpha \in \Psi$. These subgroups are automatically unipotent. If $U < G_S$ is normalized by A, then for every $x \in X$ and $a \in A$, $a.(U.x) = xUa^{-1} = xa^{-1}U = U.(a.x)$, so that the *foliation* of X into U-orbits is invariant under the action of A. We will say that $a \in A$ *expands* (*contracts*) the U-leaves, or simply U, if the absolute values of all eigenvalues of Ad_a restricted to the Lie algebra of U are greater (smaller) than one.

For any locally compact metric space Y, let $\mathcal{M}_{\infty}(Y)$ denote the space of Radon measures on Y equipped with the weak* topology, i.e., all locally finite Borel measures on Y with the coarsest topology for which $\rho \mapsto \int_Y f(y) \, d\rho(y)$ is continuous for every compactly supported continuous f. For two Radon measures ν_1 and ν_2 on Y, we write

$$\nu_1 \propto \nu_2 \text{ if } \nu_1 = C\nu_2 \text{ for some } C > 0$$

and say that ν_1 and ν_2 are proportional.

We let $B_{\varepsilon}^{Y}(y)$ (or $B_{\varepsilon}(y)$ if Y is understood) denote the ball of radius ε around $y \in Y$; if H is a group, we set $B_{\varepsilon}^{H} = B_{\varepsilon}^{H}(e)$; and if H acts on X and $x \in X$, we let $B_{\varepsilon}^{H}(x) = B_{\varepsilon}^{H}.x$.

Let μ be an A-invariant probability measure on X. For any unipotent subgroup $U < U^{\Psi} < G_S$ normalized by A, one has a system $\left\{\mu_x^U\right\}_{x \in X}$ of Radon measures on U and a co-null set $X' \subset X$ with the following properties:

(1) The map $x \mapsto \mu_x^U$ is measurable.
(2) For every $\varepsilon > 0$ and $x \in X'$, it holds that $\mu_x^U(B_{\varepsilon}^U) > 0$.
(3) For every $x \in X'$ and $u \in U$ with $u.x \in X'$, we have that $\mu_{x,U} \propto (\mu_{u.x}^U)u$, where $(\mu_{u.x}^U)u$ denotes the push forward of the measure $\mu_{u.x}^U$ under the map $v \mapsto vu$ for all $v \in U$.
(4) For every $a \in A$, and $x, a.x \in X'$, $\mu_{a.x}^U \propto (\theta_a)_*(\mu_x^U)$, where the latter is the push forward of the leafwise measure under the conjugation map $\theta_a(u) = aua^{-1}$.

In general, there is no canonical way to normalize the leafwise measures μ_x^U; we fix a specific normalization by requiring that $\mu_{x,U}(B_1^U) = 1$ for every $x \in X'$. This implies the next crucial property.

(5) If $U \subset C(a) = \{g \in G_S : ga = ag\}$ is centralized by $a \in A$, then $\mu_{a(x)}^U = \mu_x^U$ whenever $x, a(x) \in X'$.

(6) μ is U-invariant if, and only if, μ_x^U is a Haar measure on U almost everywhere (see [20, Prop. 4.3]).

The other extreme to invariance as above is where μ_x^U is atomic. If μ is A-invariant, then outside some set of measure zero if μ_x^U is atomic, then it is supported on the single point $e \in U$, in which case we say that μ_x^U is *trivial*. The leafwise measures for the unipotent subgroup U^α associated to a Lyapunov root $\alpha \in \Phi'$ we denote by μ_x^α, and more generally we write μ_x^Ψ for the leafwise measures on U^Ψ when $\Psi \subset \Phi'$ is a set of Lyapunov roots such that U^Ψ is a unipotent subgroup.

Fundamental to us is the following characterization of positive entropy for the action of an element $a \in A$ on an A-invariant and ergodic probability measure μ:

(7) Let $\Psi = \{\alpha \in \Phi' : \alpha(a) < 0\}$ so that U^Ψ is the *horospherical stable subgroup* defined by a. Then the measure theoretic entropy $h_\mu(a)$ is positive if and only if the leafwise measures μ_x^Ψ are nonatomic almost everywhere.

Hence positive entropy implies that certain leafwise measures are nontrivial almost everywhere.

4.2. *Entropy contribution.* In this section we refine property (7) from above to a more quantitative statement; for more details see [23, §9], [4, §9] or [9, §7].

Let U be a connected subgroup normalized by A such that $a \in A$ contracts U. We let
$$\theta_a(g) = aga^{-1}$$
be the inner automorphism defined by a. Then for any a-invariant probability measure μ on X, the limit

$$(4.1) \qquad \mathrm{vol}_\mu(a, U, x) = -\lim_{n \to +\infty} \frac{\log \mu_x^U\left(\theta_a^n(B_1^U)\right)}{n}$$

exists for almost every $x \in X$ by [4, Lemma 9.1]. Furthermore, if μ_x^U is supported by a subgroup $P \subseteq U$ that is connected and normalized by A, then

$$(4.2) \qquad \mathrm{vol}_\mu(a, U, x) \le \mathrm{mod}(a, P) = \sum_{\alpha \in \Phi'} \alpha(a)^- \dim_{\mathbb{Q}_{\sigma(\alpha)}}\left(\mathfrak{p} \cap \mathfrak{u}^\alpha\right)$$

for almost every $x \in X$ by [9, Th. 7.9]. Here we write $r^- = \max(0, -r)$, $\mathfrak{p}$ for the Lie algebra of P and $\mathbb{Q}_{\sigma(\alpha)}$ for the field over which the Lyapunov

root $\alpha \in \Phi'$ is defined. Note that $\mathfrak{p}$ is a direct sum of its subspaces $\mathfrak{p} \cap \mathfrak{u}^\alpha$ since P is normalized by A. In fact, $\mathrm{mod}(a, P)$ is the negative logarithm of the module of the restriction of θ_a to P. Using property (4) of the leafwise measures it is easy to check that $\mathrm{vol}_\mu(a, U, \cdot)$ is A-invariant and thus constant for an A-ergodic measure. We define

$$\mathrm{h}_\mu(a, U) = \int_X \mathrm{vol}_\mu(a, U, x) d\mu(x)$$

and will refer to $\mathrm{h}_\mu(a, U)$ as the *entropy contribution of U*.

In the case where $U = U^\alpha < G_f$ for $\alpha \in \Phi_f$ as in Section 3.2, the map $a \mapsto \mathrm{vol}_\mu(a, U, x)$ restricted to the half space $\alpha(a) \leq 0$ can be extended to a homomorphism of groups from A to the additive group on $\mathbb{R}$. To see this, note first that $\theta_a^n(B_1^U)$ only depends on the value of $n\alpha(a)$ since $a, a' \in A$ and $m, n \in \mathbb{Z}$ with $n\alpha(a) < m\alpha(a') < 0$ implies $\theta_a^n(B_1^U) \subset \theta_{a'}^m(B_1^U)$. Therefore, there exists $\delta_\mu^\alpha \geq 0$ with

(4.3) $$\mathrm{h}_\mu(a, U^\alpha) = \delta_\mu^\alpha \alpha(a)^- \text{ for all } a \in A \text{ with } \alpha(a) \leq 0;$$

we refer to δ_μ^α as the *dimension of μ along U^α*. Now (4.2) is equivalent to

(4.4) $$\delta_\mu^\alpha \leq \dim_{\mathbb{Q}_\sigma}\left(\mathfrak{p} \cap \mathfrak{u}^\alpha\right) + 2\dim_{\mathbb{Q}_\sigma}\left(\mathfrak{p} \cap \mathfrak{u}^{2\alpha}\right),$$

where we agree to set the second term to zero if 2α is not a root.

A σ-algebra $\mathcal{A}$ of Borel subsets of X is *subordinate to U* if $\mathcal{A}$ is countably generated, for every $x \in X$ the atom $[x]_\mathcal{A}$ of x with respect to $\mathcal{A}$ is contained in the leaf Ux, and for almost every x

$$B_\varepsilon^U x \subseteq [x]_\mathcal{A} \subseteq B_\rho^U x \text{ for some } \varepsilon > 0 \text{ and } \rho > 0.$$

A σ-algebra $\mathcal{A}$ is *a-decreasing* if $a^{-1}\mathcal{A} \subseteq \mathcal{A}$. By [4, Lemma 9.3] we have

$$\mathrm{H}_\mu(\mathcal{A}|a^{-1}\mathcal{A}) = \int \mathrm{vol}_\mu(a, U, x)\, \mathrm{d}\mu$$

whenever $\mathcal{A}$ is an a-decreasing σ-algebra that is subordinate to U.

It has been shown in [23, Prop. 9.3] that there exists an a-decreasing σ-algebra $\mathcal{A}$ subordinate to U provided μ is a-ergodic. Moreover, if U is the horospherical stable subgroup defined by a, then $\mathrm{h}_\mu(a) = \mathrm{H}_\mu(\mathcal{A}|a^{-1}\mathcal{A})$ by [23, Prop. 9.3(iii)]. This can be used to show that the entropy equals the entropy contribution of the horospherical stable subgroup

(4.5) $$\mathrm{h}_\mu(a) = \mathrm{h}_\mu(a, U)$$

even if μ in not a-ergodic; see [4, Prop. 9.4].

4.3. *The structure of leafwise measures for the horospherical subgroup.*
We recall and slightly simplify [4, Ths. 8.4, 8.5], which describe the structure
of the leafwise measure μ_x^U for the horospherical stable subgroup U defined
by a. Under the assumption that μ is an A-ergodic and invariant probability
measure on X, these theorems simplify to the following statements due to the
fact that the adjoint action restricted to A has its eigenvalues in $\mathbb{Q}_\sigma$.

Let $\Psi = \{\alpha \in \Phi' : U^\alpha \subset U$ and α is indivisible$\}$. Fix some order of the
elements of $\Psi = \{\alpha_1, \ldots, \alpha_\ell\}$, and define $\phi : \prod_{i=1}^\ell U^{\alpha_i} \to U$ by $\phi(u_1, \ldots, u_\ell) =$
$u_1 \cdots u_\ell$ for any $(u_1, \ldots, u_\ell) \in \prod_{i=1}^\ell U^{\alpha_i}$. Here $U^\alpha = \exp(\mathfrak{u}^\alpha \oplus \mathfrak{u}^{2\alpha})$; in the
terminology of [4, §4.4], these are precisely the coarse Lyapunov subgroups of
the action of A. Therefore, [4, Th. 8.4] gives

$$(4.6) \qquad \mu_x^U \propto \phi_* \big(\mu_x^{\alpha_1} \times \cdots \times \mu_x^{\alpha_\ell} \big) \text{ a.e.}$$

This, (4.5) and (4.1) readily imply that the entropy

$$(4.7) \qquad \mathrm{h}_\mu(a) = \sum_{i=1}^\ell \mathrm{h}_\mu(a, U^{\alpha_i})$$

is the sum over all entropy contributions of $U^\alpha \subset U$. In particular, positive
entropy implies that there exists an indivisible root α so that the leafwise
measure μ_x^α is nontrivial almost everywhere — indeed, so that the dimension
δ_μ^α of μ along U^α is positive.

The following is a version of the high entropy theorem contained in [4]
(and generalizes [3]); see also [9, §9] for a simplified derivation of this result.

THEOREM 4.1. *Let μ be an A-invariant and ergodic probability measure
on $X = \Gamma \backslash G_S$. Let $a \in A$ with stable horospherical subgroup $U = U_a$. Then
there exist two connected A-normalized subgroups $H < P < U$ such that*

(1) μ_x^U *is supported by P almost everywhere.*

(2) μ_x^U *is left- and right-invariant under multiplication with elements of H
almost everywhere.*

(3) H *is a normal subgroup of P. Moreover, setting $P^{\alpha_s} = P \cap U^{\alpha_s}$ for any
root α_s, we have that for any two distinct roots α_r and α_s, for any elements
$g \in P^{\alpha_r}$ and $h \in P^{\alpha_s}$, the cosets gH and hH commute with each other
in P/H.*

(4) $\mu_x^{\alpha_i}$ *is left- and right-invariant under multiplication with elements of $H \cap
U^{\alpha_i}$ for $i = 1, \ldots, \ell$ almost everywhere.*

Proof. By [4, Th. 8.5] there exist for almost every $x \in X$ two subgroups
$H_x < P_x < U$ with properties (1)–(4) of the theorem at the point x. Addition-
ally, these subgroups are connected and their Lie algebras are direct sums of
subspaces of root spaces $\mathfrak{u}^\alpha$; in the notation of [4, Th. 8.5] this is the same as
H_x and P_x allowing a weight decomposition. However, in our case this implies

that these subgroups are normalized by A. (This would not necessarily be so if we were working over field extensions of $\mathbb{Q}_\sigma$.) This together with Property (4) of Section 4.1 shows that for almost every $x \in X$, the subgroups $H_x < P_x$ also satisfy Properties (1)–(4) at $a.x$ for any $a \in A$. Using ergodicity of μ it is not difficult to find subgroups H and P as in the theorem; see, for instance, [7, Lemma 3.5] where this has been done for P. $\square$

For the Lie algebras $\mathfrak{h} \subset \mathfrak{p} \subset \mathfrak{u}$ of $H < P < U$, we get the following corollary. This actually appears in the proof of [4, Th. 8.5] but is also immediate from Theorem 4.1.(3) and (3.5).

COROLLARY 4.2. *For any two linearly independent roots* α, β *with* $\mathfrak{u}^\alpha$, $\mathfrak{u}^\beta < U$ *we have* $[\mathfrak{p} \cap \mathfrak{u}^\alpha, \mathfrak{p} \cap \mathfrak{u}^\beta] \subset \mathfrak{h} \cap \mathfrak{u}^{\alpha+\beta}$.

While the subgroup P gives an upper bound for the entropy contribution as we have seen in (4.2), the subgroup H gives a lower bound as follows.

LEMMA 4.3. *Let* $H < U$ *be connected subgroups normalized by* A *such that* $a \in A$ *contracts* U. *If* μ *is invariant under* H, *then*

$$\mathrm{h}_\mu(a, U) \geq \sum_{\alpha \in \Psi} \alpha(a)^- \dim_{\mathbb{Q}_\sigma}\big(\mathfrak{h} \cap \mathfrak{u}^\alpha\big),$$

where $\mathfrak{h}$ *is the Lie algebra of* H *and* Ψ *is the set of roots with* $\alpha(a) < 0$.

The lemma follows from the following two observations that only rely on the definitions: If $H < U$, then $\mathrm{h}_\mu(a, H) \leq \mathrm{h}_\mu(a, U)$. If μ is H-invariant, then μ_x^H is the Haar measure of H for almost every x and $\mathrm{h}_\mu(a, H) = \mathrm{mod}(a, H)$ is the negative logarithm of the module of conjugation by a restricted to H.

5. The high entropy roots

It will be convenient to divide the indivisible roots in Φ' into three classes depending on the measure μ:

(1) the roots with no attached entropy, i.e., for which $\delta_\mu^\alpha = 0$;
(2) the roots $\alpha \in \Phi'$ for which $\delta_\mu^\alpha > 0$ but $\delta_\mu^\beta = 0$ for every indivisible root $\beta \neq \pm\alpha$ with $(\alpha, \beta) \neq 0$;
(3) the roots $\alpha \in \Phi'$ for which $\delta_\mu^\alpha > 0$ and also $\delta_\mu^\beta > 0$ for some indivisible $\beta \neq \pm\alpha$ with $(\alpha, \beta) \neq 0$.

In the second case we say that α is a root of *low entropy*, and in the third α is a root of *high entropy*.

In this section we deal with the high entropy roots and show in particular that if α is a root of high entropy, then μ is invariant under a subgroup generated by unipotent elements. We also establish a symmetric property of the δ_μ^α, namely that δ_μ^α is positive if and only if $\delta_\mu^{-\alpha}$ is.

THEOREM 5.1. *Let μ be an A-invariant and ergodic probability measure on $X = \Gamma \backslash G$. Then the dimensions of μ along the subgroups associated to the roots are symmetric; i.e., $\delta_\mu^\alpha = \delta_\mu^{-\alpha}$ for any indivisible $\alpha \in \Phi$. Moreover, if α is a high entropy root for μ, then μ is invariant under a nontrivial subgroup of U^α — in fact, under the unipotent group P^α as in Theorem 4.1, which is nontrivial as $\delta_\mu^\alpha > 0$.*

Proof. We first prove $\delta_\mu^\alpha \leq \delta_\mu^{-\alpha}$ for an arbitrary indivisible Lyapunov root $\alpha \in \Phi'$, which implies the symmetry. If $\delta_\mu^\alpha = 0$, this is trivial. So suppose $\delta_\mu^\alpha > 0$, and choose $a_\alpha \in A$ as in Lemma 3.1 so that $\beta(a_\alpha) = c(\alpha, \beta)$ for all Lyapunov roots $\beta \in \Phi'$ and some $c > 0$. By (4.7) and (4.3) we can express the entropy of a_α as

$$(5.1) \qquad \mathrm{h}_\mu(a_\alpha) = c \sum_{\beta : (\alpha,\beta) < 0} \delta_\mu^\beta (\alpha, \beta)^-.$$

Since $\mathrm{h}_\mu(a_\alpha) = \mathrm{h}_\mu(a_\alpha^{-1})$, we also have

$$\mathrm{h}_\mu(a_\alpha) = c \sum_{\beta : (\alpha,\beta) > 0} \delta_\mu^\beta (\alpha, \beta).$$

However, $s_\alpha : \Phi' \to \Phi'$ maps all the roots β appearing in (5.1) to those appearing in the second sum without changing the absolute value of the inner products, and so the last formula can also be written as

$$(5.2) \qquad \mathrm{h}_\mu(a_\alpha) = c \sum_{\beta : (\alpha,\beta) < 0} \delta_\mu^{s_\alpha(\beta)} (\alpha, \beta)^-.$$

We claim that $\delta_\mu^\beta \leq \delta_\mu^{s_\alpha(\beta)}$ for any indivisible root $\beta \notin \langle \alpha \rangle$ with $(\alpha, \beta) < 0$. Assuming the claim, all terms in the sum in (5.1) are smaller than or equal to the corresponding terms in (5.2) except for the term corresponding to $\beta = -\alpha$. However, as the sums equal, this exceptional term must satisfy the opposite inequality $\delta_\mu^\alpha \leq \delta_\mu^{-\alpha}$.

For the proof of the claim fix an indivisible Lyapunov root β and choose $a \in A$ such that

$$\alpha(a), \beta(a) < 0,$$

which is possible since α, β are linearly independent and correspond to characters on $A_\sigma < A$. Let U be the stable horospherical subgroup defined by a so that $U^\alpha, U^\beta < U$. Recall that almost every μ_x^U is the product measure of all the leafwise measures associated to indivisible roots whose subgroups are contained in U. Now let $H < P < U$ be as in Theorem 4.1 so that P supports μ_x^U almost everywhere and both subgroups are normalized by A. Then $P \cap U^\alpha$ supports μ_x^α almost everywhere. Since $\delta_\mu^\alpha > 0$, these leafwise measures are nontrivial almost everywhere and there exists a nonzero $v \in \mathfrak{p} \cap \mathfrak{u}^{\alpha'}$ where α' is either α or 2α. Note that $s_{\alpha'} = s_\alpha$. Let m be such that $s_{\alpha'}(\beta) = \beta + m\alpha'$. Define $W = \mathfrak{p} \cap \mathfrak{u}^\beta$. Then Proposition 3.2 shows that ad_v^m maps W to a subspace of $\mathfrak{u}^{s_\alpha(\beta)}$ of the

same dimension. By Corollary 4.2 we have $\mathrm{ad}_v^m(W) \subset \mathfrak{h} \cap \mathfrak{u}^{s_\alpha(\beta)}$; hence

$$(5.3) \qquad \dim_{\mathbb{Q}_\sigma} \mathfrak{p} \cap \mathfrak{u}^\beta \leq \dim_{\mathbb{Q}_\sigma} \mathfrak{h} \cap \mathfrak{u}^{s_\alpha(\beta)} \,.$$

The same argument also shows that

$$\dim_{\mathbb{Q}_\sigma} \mathfrak{p} \cap \mathfrak{u}^{2\beta} \leq \dim_{\mathbb{Q}_\sigma} \mathfrak{h} \cap \mathfrak{u}^{2s_\alpha(\beta)} \,.$$

Lemma 4.3 and (4.4) now imply for an indivisible root $\beta \neq \alpha$ with $(\alpha, \beta) < 0$ that

$$\delta_\mu^{s_\alpha(\beta)} \geq \dim_{\mathbb{Q}_\sigma} \mathfrak{p} \cap \mathfrak{u}^\beta + 2 \dim_{\mathbb{Q}_\sigma} \mathfrak{p} \cap \mathfrak{u}^{2\beta} \geq \delta_\mu^\beta$$

as claimed. This implies the symmetry of the dimensions $\delta_\mu^\alpha = \delta_\mu^{-\alpha}$ for all indivisible roots α.

Suppose now β is a high entropy root and $\alpha \in \Phi'$ a distinct indivisible root with $(\alpha, \beta) \neq 0$ and $\delta_\mu^\alpha > 0$. We claim that

$$(5.4) \qquad \mathfrak{h} \cap (\mathfrak{u}^\beta \oplus \mathfrak{u}^{2\beta}) = \mathfrak{p} \cap (\mathfrak{u}^\beta \oplus \mathfrak{u}^{2\beta})$$

from which it follows that $H^\beta = P^\beta$; i.e., μ is P^β-invariant. By the symmetry properties of δ_μ^α, if β is a high entropy root, so is $-\beta$; hence μ will also be $P^{-\beta}$-invariant and the theorem follows.

It remains to establish (5.4). By symmetry of the dimensions we may assume that $(\alpha, \beta) < 0$. Then (5.3) implies

$$\dim_{\mathbb{Q}_\sigma} \mathfrak{h} \cap \mathfrak{u}^{s_\alpha(\beta)} \geq \dim_{\mathbb{Q}_\sigma} \mathfrak{p} \cap \mathfrak{u}^\beta \geq \dim_{\mathbb{Q}_\sigma} \mathfrak{h} \cap \mathfrak{u}^\beta,$$

and similarly for 2β. Also by symmetry we may apply the above to $-\alpha, \beta$ to get

$$\dim_{\mathbb{Q}_\sigma} \mathfrak{h} \cap \mathfrak{u}^\beta \geq \dim_{\mathbb{Q}_\sigma} \mathfrak{p} \cap \mathfrak{u}^{s_\alpha(\beta)} \geq \dim_{\mathbb{Q}_\sigma} \mathfrak{h} \cap \mathfrak{u}^{s_\alpha(\beta)},$$

and similarly for 2β. Together these imply (5.4). $\qquad \square$

6. The low entropy roots

To deal with the low entropy root, we need to apply the main theorem of [8]. In order to state this theorem, we first need to introduce some terminology.

Let X be a locally compact metric space with a Borel probability measure μ, let $a : X \to X$ be a measure preserving transformation of (X, μ), and let H be a locally compact metric group acting continuously and locally free on X. We denote the action by $h.x$ for $h \in H$ and $x \in X$. Let $F : X \to Y$ be a measurable map to a Borel space Y. We say μ is (F, H)-transient if there exists a set $X' \subset X$ of full measure such that there are no two different $x, y \in X'$ on the same H-orbit $H.x = H.y$ with $F(x) = F(y)$. We say μ is (F, H)-exceptional if for every $\varepsilon > 0$ and compact neighborhood O of the identity in H, there exists $B \subset X$ with $\mu(B) > 1 - \varepsilon$ and some $\delta > 0$ such that $x, y \in B$ with distance $d(x, y) < \delta$ and $F(x) = F(y)$ implies that $y = h.x$ for some $h \in O$.

We fix some semisimple $a \in G$ that is diagonalizable over the ground field $\mathbb{R}$ resp. $\mathbb{Q}_p$. More precisely, we require that the adjoint representation is diagonalizable over the ground field and that there are no two distinct eigenvalues for Ad_a with the same absolute value, and in particular that 1 is the only eigenvalue of absolute value one. We assume that a contracts the leaves of the foliation defined by the U-orbits of some nontrivial closed unipotent subgroup $U < G$ — in other words, that a normalizes U and $a^n u a^{-n} \to e$ for $n \to \infty$ and all $u \in U$. We also assume that U is a direct product of Zariski closed unipotent subgroups U_σ of G_σ for $\sigma \in S$.

To be able to apply the results of [8] we will consider the leafwise measures as a map $F(x) = [\mu_x^U]$ for $x \in X$, where the equivalence is taken up to proportionality. This map F defines a factor map with respect to the action of A and U, where A acts by push forward by conjugation (see property (4) of §4.1) and U acts by push forward by right multiplication (see property (3) of §4.1). These equivalence classes of leafwise measures can be considered as elements of a Borel space Y. In fact, the leafwise measures are Radon measures with some mild control on the growth of the measures of balls (e.g., because of the analogue of (4.1) for $n \to +\infty$), which allows us to find a continuous function $\rho(u) > 0$ with $\int \rho(u) \, d\mu_x^U(u) < \infty$ for almost every $x \in X$. We now let $Y = \{\nu \mid \nu$ is a locally finite measure with $\int \rho \, d\nu \leq 1\}$ equipped with the Borel structure arising from the weak*-topology and identify $[\mu_x^U]$ with $F(x) = \left(\int \rho \, d\mu_x^U \right)^{-1} \mu_x^U \in Y$.

We are going to use the following abbreviations for the *stable* respectively *unstable horospherical subgroup*:

$$G^- = \{g \in G : a^n g a^{-n} \to e \text{ for } n \to \infty\},$$
$$G^+ = \{g \in G : a^n g a^{-n} \to e \text{ for } n \to -\infty\},$$

and the *central subgroup*

$$G^0 = \{g \in G : ag = ga\}.$$

Recall that recurrence of U with respect to μ is equivalent to $\mu_x^U(U) = \infty$ by [20, Prop. 4.1]. We say μ is *faithfully U-recurrent* if for almost every x there does not exist a Zariski closed proper subgroup $U' < U$ such that μ_x^U is supported on U'. (In other words, we have $P = U$ for the subgroup P as in Theorem 4.1.)

In [8] we have shown the following dichotomy for μ holds in the low entropy case.[6]

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[6]In fact, the main result of [8] is slightly more general than what is quoted above. Note also that in Theorem 6.1 no assumptions are made regarding the characteristic of the global field $\mathbb{K}$; we plan to explore positive characteristic analogues of our main theorem elsewhere.

THEOREM 6.1. *Suppose $X = \Gamma \backslash G$, $a \in G$, and $U < G$ are as above, with G as in Section 3.4. Assume that*

- *for every $\sigma \in S$ with $a_\sigma \neq e$, there is a homomorphism $\phi_\sigma : \mathrm{SL}_2(\mathbb{K}_\sigma) \to G_\sigma$ such that $a_\sigma = \phi_\sigma \left(\begin{smallmatrix} t & \\ & t^{-1} \end{smallmatrix} \right)$ for some $t \in \mathbb{K}_\sigma^\times$ with $\|t\|_\sigma < 1$ and $\phi_\sigma \left(\begin{smallmatrix} 1 & * \\ & 1 \end{smallmatrix} \right) \subset U_\sigma$.*

Then any a-invariant and faithfully U-recurrent probability measure μ on X satisfies at least one of the following conditions:

(1) *μ is not $(\mu_x^U, C_G(U) \cap G^-)$-transient;*
(2) *μ is $(\mu_x^U, C_G(U))$-exceptional.*

In the context of the proof of Theorem 1.1 the above gives us the following corollary.

COROLLARY 6.2. *Let $X = \Gamma \backslash G$, A, and μ be as in Theorem 1.1. Suppose μ has low entropy for some indivisible Lyapunov root $\alpha \in \Phi'$. Let $P^\alpha \subset U^\alpha$ ($P^{-\alpha} \subset U^{-\alpha}$) be the minimal connected subgroup for which μ_x^α is supported on P^α ($\mu_x^{-\alpha}$ is supported on $P^{-\alpha}$) for almost every x. Then one of the following conditions must hold:*

(1) *(Invariance). μ is invariant under a nontrivial Zariski connected subgroup $H^\alpha \subset P^\alpha$ or under a nontrivial connected subgroup $H^{-\alpha} \subset P^{-\alpha}$;*
(2) *(Exceptional). μ-almost everywhere ergodic component $\mu_x^{\mathcal{E}}$ for the action of $A' = \{a' \in A | \alpha(a') = 0\}$ is concentrated on a single orbit of $C_G(P^\alpha) \cap C_G(P^{-\alpha}) \subset C_G(a_\alpha)$, where a_α is as in Lemma 3.1.*

Here $\mathcal{E}$ equals the σ-algebra that consists of A'-invariant sets. With this definition the conditional measures $\mu_x^{\mathcal{E}}$ indeed give the A'-ergodic components of μ. We say that a probability measure ν is *concentrated* on a set $B \subset X$ if $\nu(B) = 1$. With some more effort, it is possible to deduce that if (2) does not hold, *both H^α and $H^{-\alpha}$ are nontrivial* (cf. [5, §4.1]), though for our purposes the weaker conclusion given above suffice.

We shall make use of the following definition (cf. [23]).

Definition 6.3. We say that an element $a \in A$ is of class $\mathcal{A}$ if $a \in G_\sigma$ for some $\sigma \in S$ and all eigenvalues of a_σ are powers of some $t \in \mathbb{Q}_\sigma$ with $|t| \neq 1$.

The significance of this condition is that if a is of class $\mathcal{A}$, then for any algebraic representation ρ of G_σ on any $\mathbb{Q}_\sigma$-vector space V and any vector $v \in V$, the sequence $\rho(a^n)v$ converges in the projective sense to an eigendirection of $\rho(a)$.

Proof of Corollary 6.2. Let $\alpha \in \Phi'$ be as in the corollary. By Theorem 5.1 it follows that $\delta_\mu^\alpha = \delta_\mu^{-\alpha} > 0$. Essentially by construction, the subgroup $P^\alpha = P \cap U^\alpha$, with P as in Theorem 4.1, satisfies that μ is faithfully P^α-recurrent. Indeed, suppose that on a set of positive measure there would exist smaller subgroups on which $\mu_x^{P^\alpha} = \mu_x^\alpha$ is supported. Let P_x^α denote the smallest Zariski connected subgroup of U that supports μ_x^α. Let $a_1, \ldots, a_k \in A$ be elements of class $\mathcal{A}$ such that the group generated by them is Zariski dense

in A. We claim first that P_x^α is A-normalized almost everywhere. If this also does not hold, then there is a set Z of positive measure and some a_i from the above list such that P_x^α is not normalized by a_i. Applying (4) from Section 4.1 shows now that $P_{a_i^n.x}^\alpha = a_i^n P_x^\alpha a_i^{-n}$ almost everywhere. If $x \in Z$, then $a_i^n P_x^\alpha a_i^{-n}$ converges to an a_i-invariant subgroup, but this contradicts Poincaré recurrence. Therefore, P_x^α is A-normalized almost everywhere so that $P_{a.x}^\alpha = P_x^\alpha$ almost everywhere. By ergodicity of A this implies that P_x^α is constant. Hence we have $P^\alpha = P_x^\alpha$ almost everywhere and μ is faithfully P^α-recurrent.

By Lemma 3.1 and the above discussion it follows that the elements $a = a_{\pm\alpha}$ and corresponding unipotent group $U = P^{\mp\alpha}$ satisfy the assumption in Theorem 6.1. Therefore, Theorem 6.1 applies and we obtain one of two possible outcomes:

Case (1) *in Theorem* 6.1 *holds for at least one choice of* $a = a_{\pm\alpha}$ *and* $U = P^{\mp\alpha}$. Suppose (1) of Theorem 6.1 holds for $a = a_{-\alpha}, U = P^\alpha$. Recall that by assumption on α (i.e., that it is a low entropy root), by (4.6) applied to G^-, and by assumption on P^α (i.e., that it supports $\mu_x^{P^\alpha}$) we have that the measure $\mu_x^{G^-}$ is also supported on P^α almost everywhere. Therefore, (3) from Section 4.1 shows that there exists some set $X' \subset X$ of full measure so that $x, h.x \in X'$ with $h \in G^-$ implies $h \in P^\alpha$. By assumption, μ is not $(\mu_x^{P^\alpha}, C_G(P^\alpha) \cap G^-)$-transient, which together implies now that for every set of full measure $X'' \subset X$, there exist some $x \in X''$ and $u \in C_{P^\alpha}(P^\alpha)$ so that both $x, u.x \in X''$ and $\mu_x^{P^\alpha} = \mu_{u.x}^{P^\alpha}$. However, by (3) from Section 4.1 again, this shows that for any set of full measure, there exist some x in that set and some nontrivial $u \in C_{P^\alpha}(P^\alpha)$ with $\mu_x^{P^\alpha} \propto \mu_x^{P^\alpha} u = u\mu_x^{P^\alpha}$ (where the later equality holds since u centralizes P^α). This actually implies, almost surely, that in fact $\mu_x^{P^\alpha} = u\mu_x^{P^\alpha}$. There are two ways to show this: if $\mu_x^{P^\alpha} = cu\mu_x^{P^\alpha}$ for $c > 1$, then for every n, $\mu_{a.x}^{P^\alpha} = caua^{-1}\mu_{a.x}^{P^\alpha}$, which can be shown to contradict Poincaré recurrence for a as $a^n u a^{-n} \to e$ (see [4, Lemma 5.10] for details). Alternatively, $\mu_x^{P^\alpha} = cu\mu_x^{P^\alpha}$ for $c > 1$ implies that the $\mu_x^{P^\alpha}$-measure of a ball of radius t around $e \in P^\alpha$ grows exponentially in t, which contradicts [9, Th. 6.29].

Moreover, having found some nontrivial $u \in P^\alpha$ preserving $\mu_x^{P^\alpha}$ from both sides, we claim that, almost surely, $\mu_x^{P^\alpha}$ is invariant under left multiplication by a nontrivial connected A-normalized subgroup $H_x \subset C_{P^\alpha}(P^\alpha)$. In fact this follows from [4, Prop. 6.2] (which once more only relies on Poincaré recurrence) together with the fact that the eigenvalues of the adjoint action of elements of A belong to $\mathbb{Q}_\sigma$. For the same reason, H_x is normalized by A so that $\mu_{a.x}^{P^\alpha}$ is also invariant under left multiplication by elements of H_x. Using ergodicity of A, it follows that $\mu_y^{P^\alpha}$ is invariant under left multiplication by elements of a nontrivial connected A-normalized subgroup $H^\alpha \subset C_{P^\alpha}(P^\alpha)$ for almost every y. We have shown statement (1) of the corollary for a subgroup $H^\alpha \subset P^\alpha$. If (1) in Theorem 6.1 holds for the root $-\alpha$, then this gives as above invariance

of μ under a nontrivial connected subgroup $H^{-\alpha} \subset P^{-\alpha}$, which implies (1) of the corollary once more.

Case (2) *holds in Theorem* 6.1 *both for* a_α *and for* a_α^{-1}. In other words, μ is $(\mu_x^{P^\alpha}, C_G(P^\alpha))$-exceptional and $(\mu_x^{P^{-\alpha}}, C_G(P^{-\alpha}))$-exceptional. We claim that this implies property (2) of the corollary.

Let $\varepsilon > 0$. Now choose O to be a sufficiently small compact neighborhood of the identity in G so that the natural map $O \to O.x$ is injective for x belonging to a set of measure bigger than $1 - \varepsilon/3$. We also assume that O is sufficiently small so that $O \cap (C_G(P^\alpha) \cap C_G(P^{-\alpha})) = O \cap L$ where we set $L = (C_\mathbb{G}(P^\alpha) \cap C_\mathbb{G}(P^{-\alpha}))^\circ(\mathbb{Q}_S)$.

Then by definition of exceptional, there exist $\delta > 0$ and a set $B \subset X$ of measure $> 1 - \varepsilon/3$ so that $\mu_x^{P^\alpha} = \mu_y^{P^\alpha}$ with $x, y \in B$ and $d(x, y) < \delta$ implies $y \in (O \cap C_G(P^\alpha)).x$. Replacing B by an appropriately chosen subset of measure $> 1 - \varepsilon$, we may assume that the same holds similarly for $\mu_x^{P^{-\alpha}}$ and $C_G(P^{-\alpha})$, and moreover that the map $O \to O.x$ is injective for all $x \in B$.

Assume now z is such that $\mu_z^{\mathcal{E}}(B) > 0$, and $z \in B \cap \mathrm{supp}(\mu_z^{\mathcal{E}}|_B)$. Let $V = B_{\delta/2}^X(z)$. Then for $\mu_z^{\mathcal{E}}$-almost every $y \in V \cap B$, there exists a sequence $a_n' \in A'$ with $a_n'.y \in V \cap B$ and $a_n'.y \to x$ as $n \to \infty$. By property (5) of Section 4.1 this implies almost surely that $\mu_y^{P^\alpha} = \mu_{a_n'.y}^{P^\alpha}$ and $\mu_y^{P^{-\alpha}} = \mu_{a_n'.y}^{P^{-\alpha}}$. However, this implies that $a_n'.y = c_n.y$ for some

$$c_n \in O \cap C_G(P^\alpha) \cap C_G(P^{-\alpha}) = O \cap L.$$

As $a_n'.y \to x$ as $n \to \infty$, and since O is compact, we may choose a subsequence so that $c_n \to c \in L$. As $P^\alpha \subset U^\alpha$, we have $A' \subset C_G(P^\alpha)$ and similarly $A' \subset C_G(P^{-\alpha})$ and so $A' \subset L$. Therefore, $\mu_z^{\mathcal{E}}$-almost every $y \in V \cap B$ belong to $L.x$. This shows that the orbit $L.x$ is an A'-invariant set of positive $\mu_z^{\mathcal{E}}$-measure. Hence $\mu_z^{\mathcal{E}}$ is concentrated on the orbit $L.x$. Letting ε go to zero, we obtain that μ-almost everywhere ergodic component with respect to A' is concentrated on a single L-orbit.

Finally, note that by Proposition 3.2 the Lie algebra of $C_G(P^\alpha)$ is contained in $\mathfrak{g}^0 \oplus \sum_{(\beta,\alpha)\geq 0} \mathfrak{u}^\beta$ so the Lie algebra of L is contained in $\mathfrak{g}^0 \oplus \sum_{(\beta,\alpha)=0} \mathfrak{u}^\beta$. It follows that L is centralized by a_α. $\qquad\square$

7. **The exceptional case and rank one factors**

In this section we will prove the following step towards Theorem 1.3.

PROPOSITION 7.1. *In the setting of Theorem* 1.3 *we have that* (2) *of Corollary* 6.2 *cannot hold.*

Note that the assumption $\mathrm{rank}(A) \geq 2$ of Theorem 1.3 is essential here, for otherwise the group A' considered in Corollary 6 is the trivial group and (2) of this corollary is satisfied in a degenerate way.

We will need the following lemmas.

LEMMA 7.2. *Let* **G** *be a* $\mathbb{Q}_\sigma$-*almost simple algebraic group. Let* A *be a maximal* $\mathbb{Q}_\sigma$-*split subtorus of* **G**, *and let* α *be a Lyapunov root. Let* **L** $<$ **G** *be an algebraic subgroup that is* A-*normalized and contains both a one-parameter subgroup of* U^α *and a one-parameter subgroup of* $U^{-\alpha}$. *Then* **L** *also contains a power of the element* a_α *as in Lemma* 3.1. *Moreover, conjugation by* a_α *preserves Haar measure on* **L**.

Proof. As **L** is A-normalized we may assume that the one-parameter subgroups of **L** $\cap U^{\pm\alpha}$ are also A-normalized, i.e., that they are uniquely defined by some element v^+ of $\mathfrak{u}^{\alpha'}$ with $\alpha' \in \{1, 2\}\alpha$, resp. v^- of $\mathfrak{u}^{-\alpha''}$ with $\alpha'' \in \{1, 2\}\alpha$. Let $\mathfrak{l}$ be the Lie algebra of **L**, and let $\mathbf{H}_\alpha$ be the algebraic subgroup generated by U^α and $U^{-\alpha}$.

By Proposition 3.2 we know that for every β with $(\beta, \alpha) > 0$ and $s_\alpha(\beta) = \beta - m\alpha'$, the map $\mathrm{ad}_{v^+}^m$ restricted to $\mathfrak{u}^{s_\alpha(\beta)}$ is injective. Note that by definition $s_\alpha = s_{\alpha'} = s_{\alpha''}$ are all one and the same involution. It follows from the above injectivity of $\mathrm{ad}_{v^+}^m$ that for any root β with $(\beta, \alpha) > 0$,

(7.1) $\dim(\mathfrak{l} \cap \mathfrak{u}^{s_\alpha(\beta)}) \leq \dim(\mathfrak{l} \cap \mathfrak{u}^\beta).$

Applying this same to v^- we obtain the opposite inequality; hence equality holds in (7.1).

In particular, $\dim(\mathfrak{l} \cap \mathfrak{u}^\alpha) = \dim(\mathfrak{l} \cap \mathfrak{u}^{-\alpha})$, and if α is a double root, the same also holds for 2α. Moreover, $\mathrm{ad}_{v^+}^2$ restricted to $\mathfrak{l} \cap \mathfrak{u}^{-\alpha'}$ gives a bijection between this space and $\mathfrak{l} \cap \mathfrak{u}^{\alpha'}$; hence there exists $w^- \in \mathfrak{l} \cap \mathfrak{u}^{-\alpha'}$ with $\mathrm{ad}_{v^+}^2(w^-) = v^+$. It follows that we can choose the $\mathfrak{s}l_2$-triple as in the proof of Lemma 3.1 and Proposition 3.2 within $\mathfrak{l}$.

However, since $\mathbf{H}_\alpha$ is semisimple with $\mathbb{Q}_\sigma$-rank one (which follows since A is maximal), we see that the split element $[v^+, w] \in \mathfrak{g}^0$ must be in the Lie algebra of the torus generated by powers of a_α. This implies that a power of a_α belongs to **L** as claimed. The symmetry in dimensions given by (7.1) (with equality replacing the $\leq$) implies that a_α preserves the Haar measure on **L**.   □

LEMMA 7.3. *Let* $\mathbb{G}, S, F' \subset F, A, \Gamma, \mu$ *be as in Theorem* 1.1. *Suppose* μ *is concentrated on the orbit* $\Gamma L p$, *where* $L = \mathbb{L}(\mathbb{Q}_S)$ *for some* $\mathbb{Q}$-*group* $\mathbb{L}$ *and* $\Gamma p \in \mathrm{supp}\,\mu$. *Then there is a reductive* $\mathbb{Q}$-*subgroup* $\mathbb{L}' \subseteq \mathbb{L}$ *with* $\mathbb{Q}$-*anisotropic center so that* μ *is supported on the orbit* $\Gamma L' p$, *where* $L' = \mathbb{L}'(\mathbb{Q}_S)$.

Note that we do not assume $\Gamma L p$ is closed; it follows however from $\mathbb{L}'$ having no $\mathbb{Q}$-characters that $\Gamma L' p$ will be closed of finite volume.

Proof. Conjugating A if necessary and replacing μ by μp^{-1} we may as well assume that $p = e$. Applying Lemma 2.1, by passing to a normal subgroup of $\mathbb{L}$ if necessary, we may assume that $\mathbb{L}$ has no $\mathbb{Q}$-characters, so that ΓL is closed

of finite volume, and supports μ. It remains, however, to show that $\mathbb{L}$ can be taken to be reductive.

Suppose $\mathbb{L}$ has a unipotent radical $\mathbb{U}$, which is automatically defined over $\mathbb{Q}$. As $A < \mathbb{L}(\mathbb{Q}_S)$, we know that A normalizes $\mathbb{U}(\mathbb{Q}_S)$ and its (generalized) Lie algebra. For a given $\sigma \in S$, this implies that $\mathbb{U}(\mathbb{Q}_\sigma)$ must be a direct product of the subgroups $\mathbb{U}(\mathbb{Q}_\sigma) \cap G_f$ with $f \in F' \cap F_\sigma$ and a subgroup $U' \subset \prod_{f \in F_\sigma \setminus F'} G_f$. Indeed the Lie algebra $\mathfrak{f}$ of $\mathbb{U}$ (considered over $\mathbb{Q}_\sigma$) is A_f-normalized for all $f \in F' \cap F_\sigma$. Therefore,

$$\mathfrak{f} = \sum_\alpha \mathfrak{f} \cap \mathfrak{u}^\alpha + \mathfrak{f} \cap \mathfrak{g}^0.$$

However, $\mathfrak{f} \cap \mathfrak{g}^0$ projected to $\sum_{f \in F'} \mathfrak{g}_f^0$ must be trivial as the latter cannot contain the Lie algebra of a unipotent subgroup. Hence $\mathfrak{f} \cap \mathfrak{g}^0$ belongs to $\sum_{f \in F_\sigma \setminus F'} \mathfrak{g}_f$.

We claim that $\mathfrak{f} \cap \mathfrak{u}^\alpha$ is trivial for all $\alpha \in \Phi'$. Suppose this were false. Then the Lie algebra $\mathfrak{h} = \mathfrak{f} \cap \mathfrak{g}_f$ is A_f-normalized and the determinant of the adjoint representation of A_f on $\mathfrak{f}$ must be one (as this determinant is a $\mathbb{Q}$-character of $\mathbb{L}$, which is assumed to be trivial). However, this enables us to use the same arguments as in the proof of Theorem 5.1 (which relies on Proposition 3.2) to show that the dimensions of the intersections $\mathfrak{f} \cap \mathfrak{u}^\alpha$ are symmetric. Indeed if $v \in \mathfrak{f} \cap \mathfrak{u}^\alpha$, $\beta \in \Phi'$ and m satisfy $(\beta, \alpha) > 0$ and $s_\alpha(\beta) = \beta - m\alpha$, then by Proposition 3.2 we have

$$\mathrm{ad}_v^m(\mathfrak{f} \cap \mathfrak{u}^{s_\alpha(\beta)}) \subset f \cap u^\beta$$

and so

(7.2) $$\dim \mathfrak{f} \cap \mathfrak{u}^\beta \geq \dim \mathfrak{f} \cap \mathfrak{u}^{s_\alpha(\beta)}.$$

As a_α expands all root spaces $\mathfrak{u}^\beta$ with $(\alpha, \beta) > 0$ (including $\beta = \alpha$) and resp. contracts $\mathfrak{u}^{s_\alpha(\beta)}$ by the inverse factor, the fact that the determinant of the adjoint representation of A_f on $\mathfrak{f}$ is one implies equality must hold in (7.2), and so in particular $\mathfrak{f} \cap \mathfrak{u}^{-\alpha}$ has to be nontrivial. In view of Proposition 3.2, this implies that $[\mathfrak{f} \cap \mathfrak{u}^\alpha, \mathfrak{f} \cap \mathfrak{u}^{-\alpha}]$ is nontrivial. However, as this is a subset of $\mathfrak{f} \cap \mathfrak{g}^0 \cap \mathfrak{g}_f$, we get a contradiction to $\mathfrak{f} \cap \mathfrak{g}^0 \subset \sum_{f' \in F_\sigma \setminus F'} \mathfrak{g}_{f'}$.

It follows from the claim we have just established that $\mathbb{U}(\mathbb{Q}_\sigma) \cap G_f$ must be trivial for $f \in F' \cap F_\sigma$ and all $\sigma \in S$; hence $\mathbb{U}(\mathbb{Q}_\sigma)$ commutes with A. Now let $\mathbb{L}' < \mathbb{L}$ be a $\mathbb{Q}$-Levy subgroup of $\mathbb{L}$ and $L' = \mathbb{L}(\mathbb{Q}_S)$. Since $\mathbb{L}'$ is unique up to conjugation by $\mathbb{U}$ and A commutes with $\mathbb{U}$, we see that any choice gives that $\mathbb{L}'$ contains A. Since $\Gamma L'$ is closed and A-invariant, and since we assumed that $\Gamma e \in \mathrm{supp}\,\mu$, it follows from ergodicity of A that μ is supported on $\Gamma L'$. □

Proof of Proposition 7.1. Let $H = C_G(P^\alpha) \cap C_G(P^{-\alpha}) \subset C_G(a_\alpha)$ be as in case (2) of Corollary 6.2. Let $x = \Gamma g$ be a point for which the conclusion of

the corollary holds. We define $\Lambda_g = \Gamma \cap (gHg^{-1})$, take $\mathbb{M}_g$ to be the connected
component of its Zariski closure, i.e., $\mathbb{M}_g = (\bar{\Lambda}_g)^{\circ}$, and finally define

$$\mathbb{L}_g = [C_{\mathbb{G}}(\mathbb{M}_g)^{\circ}, C_{\mathbb{G}}(\mathbb{M}_g)^{\circ}] < C_{\mathbb{G}}(\mathbb{M}_g).$$

Note that both $\mathbb{M}_g$ and $\mathbb{L}_g$ are Zariski connected algebraic groups defined
over $\mathbb{Q}$.

By definition we have that $gP^{\alpha}g^{-1}, gP^{-\alpha}g^{-1}$ are subgroups of the
$\mathbb{Q}_{\sigma}$-points of $C_{\mathbb{G}}(\mathbb{M}_g)^{\circ}$ with $\sigma \in S$ the place corresponding to the root α,
which implies that a power of $ga_{\alpha}g^{-1}$ belongs to $C_{\mathbb{G}}(\mathbb{M}_g)^{\circ}$ by Lemma 7.2.
Therefore, $gP^{\alpha}g^{-1}, gP^{-\alpha}g^{-1} < \mathbb{L}_g(\mathbb{Q}_{\sigma})$ and (again by Lemma 7.2) the same
power of $ga_{\alpha}g^{-1}$ also belongs to $\mathbb{L}_g(\mathbb{Q}_{\sigma})$.

Let us study how these $\mathbb{Q}$-groups vary: Since A normalizes P^{α} and $P^{-\alpha}$,
it follows that A also normalizes H. Therefore, $gHg^{-1} = (ga)H(ga)^{-1}$ for any
$a \in A$. From this we get that $\Lambda_g = \Lambda_{ga}$ for any $g \in G$, and hence the same
holds for the groups $\mathbb{M}_g$ and $\mathbb{L}_g$ that are determined by Λ_g.

Choose in some measurable way a representative $g_x \in G$ for every $x \in X$
so that $x = \Gamma g_x$. Since $\mathbb{G}$ has only countably many $\mathbb{Q}$-subgroups, there is
a subset $Z \subset X$ of positive μ-measure on which $\mathbb{M}_{g_x}, \mathbb{L}_{g_x}$ are constant, say
$\mathbb{L}_{g_x} = \mathbb{L}$ and $\mathbb{M}_{g_x} = \mathbb{M}$. By ergodicity, for μ-almost every $x \in X$, there is some
$a \in A$ for which $xa \in Z$; hence we see that in fact almost every point x can be
written as $x = \Gamma g$ for some g for which $\mathbb{M} = \mathbb{M}_g$ and $\mathbb{L} = \mathbb{L}_g$.

Set $A' = \ker \alpha < A$. We claim that $gA'g^{-1} < \mathbb{M}(\mathbb{Q}_S)$ almost surely.
Indeed, assume that the A'-ergodic component of μ at $x = \Gamma g$ is concentrated
on the single H-orbit $xH = \Gamma gH$ as in case (2) in Corollary 6.2 — currently
we do not know whether this orbit is closed. Let $a \in A' \cap G_{\sigma}$ be an element
of class $\mathcal{A}$, and let $x \in \operatorname{supp} \mu$ satisfy Poincaré recurrence applied to a and
neighborhoods of x within the H-orbit. (This condition holds for μ-almost
every $x \in X$ by the above assumption on A'-ergodic components.) Then
there exists a sequence $\gamma_k g a^{n_k} = g h_k$ as $k \to \infty$ with $\gamma_k \in \Gamma$, $h_k \in H$, and
$h_k \to e$ as $k \to \infty$. As $a \in A' < H$, we get $\gamma_k \in \Lambda_g = \Gamma \cap gHg^{-1}$. By
Chevalley's theorem there exists an algebraic representation ρ of $\mathbb{G}$ over $\mathbb{Q}_{\sigma}$
and a vector v that together define the Zariski closure of Λ_g as the stabilizer
of the line generated by v. Notice that $\rho(ga^{-n_k}g^{-1})v$ converges projectively to
an eigendirection since a is of class $\mathcal{A}$. As h_k converges to the identity, we also
get that $\rho(gh_k a^{-n_k}g^{-1})v = \rho(\gamma_k)v$ converges projectively to an eigendirection
of $\rho(gag^{-1})$. However, as $\rho(\gamma_k)v \in \mathbb{Q}_{\sigma}v$ for all k, this shows that v is an
eigenvector of $\rho(gag^{-1})$ and that gag^{-1} belongs to the Zariski closure of Λ_g
as claimed. As this holds for all $a \in A'$ that are of class $\mathcal{A}$, we get $gA'g^{-1} <$
$\mathbb{M}(\mathbb{Q}_S)$. It follows that $g^{-1}C_{\mathbb{G}}(\mathbb{M})(\mathbb{Q}_S)g < C_{\mathbb{G}(\mathbb{Q}_S)}(A')$, and hence all Lyapunov
roots appearing in the Lie algebra of $g^{-1}C_{\mathbb{G}}(\mathbb{M})(\mathbb{Q}_S)g$ must be proportional
to α. As $\mathbb{L} = [C_{\mathbb{G}}(\mathbb{M})^{\circ}, C_{\mathbb{G}}(\mathbb{M})^{\circ}]^{\circ}$, we see that $A' \cap g^{-1}\mathbb{L}(\mathbb{Q}_S)g$ is finite.

Since the group $\mathbb{L}$ is defined over $\mathbb{Q}$, so is its unit normalizer $\mathbb{N} = N_{\mathbb{G}}^1(\mathbb{L})$. In fact, by its very definition, $\mathbb{N}$ fixes a vector in a $\mathbb{Q}$-representation of $\mathbb{G}$ (namely the vector corresponding to the Lie algebra of $\mathbb{L}$ in an appropriate wedge product of the Lie algebra of $\mathbb{G}$). Since Γ is commensurable with $\mathbb{G}(\mathcal{O}_S)$, it follows that $\Gamma \mathbb{N}(\mathbb{Q}_S)$ is closed in X. Moreover we have seen that $gA'g^{-1}$ commutes with $\mathbb{L}(\mathbb{Q}_S)$ (since it is contained in $\mathbb{M}$) and hence $gA'g^{-1} < \mathbb{N}(\mathbb{Q}_S)$. We have already noted that $ga_{\alpha}^{\ell}g^{-1} \in \mathbb{L}(\mathbb{Q}_S)$ for some ℓ, and it follows from Lemma 7.2 that conjugating by this element preserves Haar measure on $\mathbb{L}(\mathbb{Q}_S)$; hence $ga_{\alpha}^{\ell}g^{-1} \in \mathbb{N}(\mathbb{Q}_S)$. Thus a cocompact subgroup of gAg^{-1} is contained in $\mathbb{N}(\mathbb{Q}_S)$, which implies that $gAg^{-1} < \mathbb{N}(\mathbb{Q}_S)$.

Since μ is A-ergodic, we may choose $\Gamma g \in \operatorname{supp}\mu$ such that the A-orbit of Γg is dense in $\operatorname{supp}\mu$ while $gAg^{-1} < \mathbb{N}(\mathbb{Q}_S)$, which implies that ΓgA is contained in the closed homogeneous set $\Gamma \mathbb{N}(\mathbb{Q}_S)g$. Therefore, we may conclude that μ is supported on $\Gamma \mathbb{N}(\mathbb{Q}_S)g$. By Lemma 7.3 and the minimality assumption on $\mathbb{G}$ in Theorem 1.3, we may conclude that $\mathbb{N} = \mathbb{G}$. But $\mathbb{L}$ is a nontrivial normal subgroup of $\mathbb{N}$ (it is not equal to $\mathbb{N}$ since it does not contain A'), which contradicts the assumption in Theorem 1.3 that $\mathbb{G}$ is almost simple. $\square$

8. Unipotent invariance

In this section we will finish the proof of Theorem 1.3. By Proposition 7.1 we know that case (2) of Corollary 6.2 never takes place. Knowing that μ is invariant under a unipotent subgroup we are going to use the classification of measures that are invariant under unipotent one-parameter subgroups as proven by Ratner [29], [30] and extended by Ratner [32] and Margulis and Tomanov [23]. In fact, it will be convenient to use a refined version of these results by Tomanov [36] that is more adapted to the arithmetic case (i.e., when the lattice Γ is commensurable to $\mathbb{G}(\mathbb{Q}_S)$).

Proof of Theorem 1.3. In the case of a high entropy root α we have seen in Theorem 5.1 that μ is invariant under a nontrivial subgroup of U^{α}. In the case of a low entropy root α by Proposition 7.1 we know that the first possibility (invariance) in Corollary 6.2 holds, which says that μ is invariant either under a nontrivial subgroup of U^{α} or of $U^{-\alpha}$.

Now let H_u be the subgroup of G generated by all Zariski connected subgroups of U^{α} for any $\alpha \in \Phi'$ that preserve μ. As H_u is generated by unipotent one-parameter subgroups, we wish to apply the S-algebraic version of Ratner's measure classification theorem. However, we do not know whether μ is ergodic under the H_u-action, so we have to apply the classification to the H_u-ergodic components $\mu_x^{\mathcal{E}}$ where $\mathcal{E}$ is the σ-algebra of H_u-invariant Borel subsets of X and analyze how this affects the measure μ. (A similar, more general, analysis can be found in [24]; however, for completeness we include the argument here.)

Let $x = \Gamma g$ be such that the conditional measure $\mu_x^{\mathcal{E}}$ is an H_u-invariant and ergodic probability measure. By [32] or [23], $\mu_x^{\mathcal{E}}$ is the normalized L_x-invariant volume measure on a periodic orbit xL_x of a closed subgroup $L_x < G$ and [36] gives us the additional information that there exists a connected $\mathbb{Q}$-group $\mathbb{L}_g$ so that L_x is a finite index subgroup of $g^{-1}\mathbb{L}_g(\mathbb{Q}_S)g$ containing H_u. Note that the group L_x may not be unique as, e.g., a finite index subgroup would have the same property, but that its (generalized) Lie algebra $\mathfrak{l}_x$ is well defined.

Let us analyze the dependence of the above groups on the base points. Since H_u is A-normalized, we see that the σ-algebra $\mathcal{E}$ is invariant under A and so the conditional measure $\mu_x^{\mathcal{E}}$ is mapped almost surely under a to $\mu_{a.x}^{\mathcal{E}}$. This implies that aL_xa^{-1} is commensurable with $L_{a.x}$ and that $\mathrm{Ad}_a(\mathfrak{l}_x) = \mathfrak{l}_{a.x}$. Now choose $a \in A$ to be of class $\mathcal{A}$ and apply Poincaré recurrence. It follows that along some subsequence $\mathfrak{l}_{a^{n_k}.x}$ should converge to $\mathfrak{l}_x$, but since a is of class $\mathcal{A}$ any limit point of $\mathfrak{l}_{a^n.x} = \mathrm{Ad}_{a^n}(\mathfrak{l}_x)$ has to be normalized by a. Since A is generated by its elements of class $\mathcal{A}$, we see that $\mathfrak{l}_x$ is constant along the A-orbit and by ergodicity constant μ-almost everywhere. Let us denote the common value by $\mathfrak{l}_0$. This implies that every point $x \in X$ has a representative $g_x \in G$ so that $x = \Gamma g_x$ and such that the Lie algebra of $g_x^{-1}\mathbb{L}_{g_x}(\mathbb{Q}_S)g_x$ equals $\mathfrak{l}_0$; and it is moreover clear that we can make such a choice so that the map $x \mapsto g_x$ is measurable. Since there are only countably many $\mathbb{Q}$-subgroups of $\mathbb{G}$, there is some subset $Z \subset X$ of positive μ measure and a connected $\mathbb{Q}$-subgroup $\mathbb{L} < \mathbb{G}$ so that $\mathbb{L}_{g_x} = \mathbb{L}$ for every $x \in Z$. However by definition of g_x, the Lie algebra of $g_x^{-1}\mathbb{L}(\mathbb{Q}_S)g_x$ equals $\mathfrak{l}_0$ for every $x \in Z$. Set $\mathbb{M} = N_{\mathbb{G}}(\mathbb{L})$. Since the Lie algebra determines the group (for connected algebraic groups), it follows that $g_x \in \mathbb{M}(\mathbb{Q}_S)g_{x'}$ for every $x, x' \in Z$.

Since $\mathfrak{l}_0$ is normalized by A, we have that $g_xAg_x^{-1} < \mathbb{M}(\mathbb{Q}_S)$ for $x \in Z$. Hence fixing some $x_0 \in Z$, we conclude that $\Gamma\mathbb{M}g_{x_0}$ is an A-invariant set of positive μ measure (it contains Z), so by ergodicity μ is concentrated on this set.

Applying Lemma 7.3, and using the minimality condition on $\mathbb{G}$ in the statement of Theorem 1.3, the only possibility that remains is $\mathbb{G} = \mathbb{M}$. Since $\mathbb{G}$ is $\mathbb{Q}$-almost simple and $\mathbb{L}$ is a nontrivial $\mathbb{Q}$-group with $\mathbb{L} \lhd \mathbb{M} = \mathbb{G}$, we must have that $\mathbb{L} = \mathbb{G}$.

We conclude that each H_u-ergodic component $\mu_x^{\mathcal{E}}$ is invariant under a finite index subgroup L_x of $\mathbb{G}(\mathbb{Q}_S)$. Set

$$\tilde{L} = \bigcap_{a \in A/(A \cap L_x)} aL_xa^{-1};$$

note that the intersection is finite since $A \cap L_x$ is of finite index in A. Then $\tilde{L}$ is also a finite index subgroup of $\mathbb{G}(\mathbb{Q}_S)$ and for every $a \in A$, we have that $\mu_{a.x}^{\mathcal{E}}$ is $\tilde{L}$-invariant; hence by ergodicity μ is $\tilde{L}$ invariant. Since μ is also A-invariant, μ is $L = \tilde{L}A$-invariant. By ergodicity, μ is supported on a single L-orbit. $\square$

References

[1] J. W. S. CASSELS and H. P. F. SWINNERTON-DYER, On the product of three homogeneous linear forms and the indefinite ternary quadratic forms, *Philos. Trans. Roy. Soc. London. Ser. A.* **248** (1955), 73–96. MR 0070653. Zbl 0065. 27905. http://dx.doi.org/10.1098/rsta.1955.0010.

[2] S. G. DANI and G. A. MARGULIS, Orbit closures of generic unipotent flows on homogeneous spaces of SL(3, **R**), *Math. Ann.* **286** (1990), 101–128. MR 1032925. Zbl 0679.22007. http://dx.doi.org/10.1007/BF01453567.

[3] M. EINSIEDLER and A. KATOK, Invariant measures on G/Γ for split simple Lie groups G, *Comm. Pure Appl. Math.* **56** (2003), 1184–1221, Dedicated to the memory of Jürgen K. Moser. MR 1989231. Zbl 1022.22023. http://dx.doi.org/ 10.1002/cpa.10092.

[4] M. EINSIEDLER and A. KATOK, Rigidity of measures—the high entropy case and non-commuting foliations, *Israel J. Math.* **148** (2005), 169–238, Probability in mathematics. MR 2191228. Zbl 1097.37017. http://dx.doi.org/10.1007/ BF02775436.

[5] M. EINSIEDLER, A. KATOK, and E. LINDENSTRAUSS, Invariant measures and the set of exceptions to Littlewood's conjecture, *Ann. of Math.* **164** (2006), 513–560. MR 2247967. Zbl 1109.22004. http://dx.doi.org/10.4007/annals.2006.164.513.

[6] M. EINSIEDLER and E. LINDENSTRAUSS, Diagonalizable flows on locally homogeneous spaces and number theory, in *International Congress of Mathematicians. Vol. II*, Eur. Math. Soc., Zürich, 2006, pp. 1731–1759. MR 2275667. Zbl 1121.37028.

[7] M. EINSIEDLER and E. LINDENSTRAUSS, Joinings of higher-rank diagonalizable actions on locally homogeneous spaces, *Duke Math. J.* **138** (2007), 203–232. MR 2318283. Zbl 1118.37008. http://dx.doi.org/10.1215/ S0012-7094-07-13822-5.

[8] M. EINSIEDLER and E. LINDENSTRAUSS, On measures invariant under diagonalizable actions: the rank-one case and the general low-entropy method, *J. Mod. Dyn.* **2** (2008), 83–128. MR 2366231. Zbl 1146.37006.

[9] M. EINSIEDLER and E. LINDENSTRAUSS, Diagonal actions on locally homogeneous spaces, in *Homogeneous Flows, Moduli Spaces and Arithmetic, Clay Math. Proc.* **10**, Amer. Math. Soc., Providence, RI, 2010, pp. 155–241. MR 2648695. Zbl 1225.37005.

[10] M. EINSIEDLER, E. LINDENSTRAUSS, P. MICHEL, and A. VENKATESH, Distribution of periodic torus orbits on homogeneous spaces, *Duke Math. J.* **148** (2009), 119–174. MR 2515103. Zbl 1172.37003. http://dx.doi.org/10.1215/ 00127094-2009-023.

[11] M. EINSIEDLER, E. LINDENSTRAUSS, P. MICHEL, and A. VENKATESH, Distribution of periodic torus orbits and Duke's theorem for cubic fields, *Ann. of Math.* **173** (2011), 815–885. MR 2776363. Zbl 1248.37009. http://dx.doi.org/10.4007/ annals.2011.173.2.5.

[12] H. FURSTENBERG, Disjointness in ergodic theory, minimal sets, and a problem in Diophantine approximation, *Math. Systems Theory* **1** (1967), 1–49. MR 0213508. Zbl 0146.28502. http://dx.doi.org/10.1007/BF01692494.

[13] N. JACOBSON, *Lie Algebras*, *Intersci. Tracts Pure Appl. Math.* **10**, Interscience Publishers (a division of John Wiley & Sons), New York, 1962. MR 0143793. Zbl 0121.27504.

[14] A. JOHNSON, Measures on the circle invariant under multiplication by a non-lacunary subsemigroup of the integers, *Israel J. Math.* **77** (1992), 211–240. MR 1194793. Zbl 0790.28012. http://dx.doi.org/10.1007/BF02808018.

[15] A. JOHNSON and D. J. RUDOLPH, Convergence under $\times_q$ of $\times_p$ invariant measures on the circle, *Adv. Math.* **115** (1995), 117–140. MR 1351328. Zbl 0934.28010. http://dx.doi.org/10.1006/aima.1995.1052.

[16] A. KATOK and R. J. SPATZIER, Invariant measures for higher-rank hyperbolic abelian actions, *Ergodic Theory Dynam. Systems* **16** (1996), 751–778. MR 1406432. Zbl 0859.58021. http://dx.doi.org/10.1017/S0143385700009081.

[17] A. KATOK and R. J. SPATZIER, Corrections to: "Invariant measures for higher-rank hyperbolic abelian actions", *Ergodic Theory Dynam. Systems* **18** (1998), 503–507. MR 1619571. http://dx.doi.org/10.1017/S0143385798110969.

[18] D. Y. KLEINBOCK, Nondense orbits of flows on homogeneous spaces, *Ergodic Theory Dynam. Systems* **18** (1998), 373–396. MR 1619563. Zbl 0915.58078. http://dx.doi.org/10.1017/S0143385798100408.

[19] A. W. KNAPP, *Lie Groups Beyond an Introduction*, *Progr. Math.* **140**, Birkhäuser, Boston, 1996. MR 1399083. Zbl 0862.22006. http://dx.doi.org/10.1007/978-1-4757-2453-0.

[20] E. LINDENSTRAUSS, Invariant measures and arithmetic quantum unique ergodicity, *Ann. of Math.* **163** (2006), 165–219. MR 2195133. Zbl 1104.22015. http://dx.doi.org/10.4007/annals.2006.163.165.

[21] R. LYONS, On measures simultaneously 2- and 3-invariant, *Israel J. Math.* **61** (1988), 219–224. MR 0941238. Zbl 0654.28010. http://dx.doi.org/10.1007/BF02766212.

[22] G. A. MARGULIS, Oppenheim conjecture, in *Fields Medallists' Lectures*, *World Sci. Ser. 20th Century Math.* **5**, World Sci. Publ., River Edge, NJ, 1997, pp. 272–327. MR 1622909. http://dx.doi.org/10.1142/9789812385215_0035.

[23] G. A. MARGULIS and G. M. TOMANOV, Invariant measures for actions of unipotent groups over local fields on homogeneous spaces, *Invent. Math.* **116** (1994), 347–392. MR 1253197. Zbl 0816.22004. http://dx.doi.org/10.1007/BF01231565.

[24] G. A. MARGULIS and G. M. TOMANOV, Measure rigidity for almost linear groups and its applications, *J. Anal. Math.* **69** (1996), 25–54. MR 1428093. Zbl 0864.22005. http://dx.doi.org/10.1007/BF02787100.

[25] D. PRASAD, An analogue of a conjecture of Mazur: a question in Diophantine approximation on tori, in *Contributions to Automorphic Forms, Geometry, and Number Theory*, Johns Hopkins Univ. Press, Baltimore, MD, 2004, pp. 699–709. MR 2058623. Zbl 1154.11330.

[26] M. Ratner, Factors of horocycle flows, *Ergodic Theory Dynam. Systems* **2** (1982), 465–489 (1983). MR 0721735. Zbl 0536.58029. http://dx.doi.org/10.1017/S0143385700001723.

[27] M. Ratner, Rigidity of horocycle flows, *Ann. of Math.* **115** (1982), 597–614. MR 0657240. Zbl 0506.58030. http://dx.doi.org/10.2307/2007014.

[28] M. Ratner, Horocycle flows, joinings and rigidity of products, *Ann. of Math.* **118** (1983), 277–313. MR 0717825. Zbl 0556.28020. http://dx.doi.org/10.2307/2007030.

[29] M. Ratner, On measure rigidity of unipotent subgroups of semisimple groups, *Acta Math.* **165** (1990), 229–309. MR 1075042. Zbl 0745.28010. http://dx.doi.org/10.1007/BF02391906.

[30] M. Ratner, On Raghunathan's measure conjecture, *Ann. of Math.* **134** (1991), 545–607. MR 1135878. Zbl 0763.28012. http://dx.doi.org/10.2307/2944357.

[31] M. Ratner, Raghunathan's topological conjecture and distributions of unipotent flows, *Duke Math. J.* **63** (1991), 235–280. MR 1106945. Zbl 0733.22007. http://dx.doi.org/10.1215/S0012-7094-91-06311-8.

[32] M. Ratner, Raghunathan's conjectures for Cartesian products of real and p-adic Lie groups, *Duke Math. J.* **77** (1995), 275–382. MR 1321062. Zbl 0914.22016. http://dx.doi.org/10.1215/S0012-7094-95-07710-2.

[33] M. Rees, Some R^2-Anosov flows, 1982, unpublished.

[34] D. J. Rudolph, $\times 2$ and $\times 3$ invariant measures and entropy, *Ergodic Theory Dynam. Systems* **10** (1990), 395–406. MR 1062766. Zbl 0709.28013. http://dx.doi.org/10.1017/S0143385700005629.

[35] T. A. Springer, *Linear Algebraic Groups*, second ed., *Progr. Math.* **9**, Birkhäuser, Boston, 1998. MR 1642713. Zbl 0927.20024. http://dx.doi.org/10.1007/978-0-8176-4840-4.

[36] G. Tomanov, Orbits on homogeneous spaces of arithmetic origin and approximations, in *Analysis on Homogeneous Spaces and Representation Theory of Lie Groups* (Okayama-Kyoto, 1997), *Adv. Stud. Pure Math.* **26**, Math. Soc. Japan, Tokyo, 2000, pp. 265–297. MR 1770724. Zbl 0960.22006.

[37] G. Tomanov, Actions of maximal tori on homogeneous spaces, in *Rigidity in Dynamics and Geometry* (Cambridge, 2000), Springer-Verlag, New York, 2002, pp. 407–424. MR 1919414. Zbl 1012.22021.

[38] M. Waldschmidt, Open Diophantine problems, *Mosc. Math. J.* **4** (2004), 245–305, 312. MR 2074991. Zbl 1066.11030. Available at http://www.ams.org/distribution/mmj/vol4-1-2004/abst4-1-2004.html#waldschmidt_abstract.

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