

Properly embedded minimal planar domains

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Abstract

In 1997, Collin proved that any properly embedded minimal surface in \mathbb{R}^3 with finite topology and more than one end has finite total Gaussian curvature. Hence, by an earlier result of López and Ros, catenoids are the only nonplanar, nonsimply connected, properly embedded, minimal planar domains in \mathbb{R}^3 of finite topology. In 2005, Meeks and Rosenberg proved that the only simply connected, properly embedded minimal surfaces in \mathbb{R}^3 are planes and helicoids. Around 1860, Riemann defined a one-parameter family of periodic, infinite topology, properly embedded, minimal planar domains \mathcal{R}_t in \mathbb{R}^3 , $t \in (0, \infty)$. These surfaces are called the *Riemann minimal examples*, and the family $\{\mathcal{R}_t\}_t$ has natural limits being a vertical catenoid as $t \rightarrow 0$ and a vertical helicoid as $t \rightarrow \infty$. In this paper we complete the classification of properly embedded, minimal planar domains in \mathbb{R}^3 by proving that the only connected examples with infinite topology are the Riemann minimal examples. We also prove that the limit ends of Riemann minimal examples are model surfaces for the limit ends of properly embedded minimal surfaces $M \subset \mathbb{R}^3$ of finite genus and infinite topology, in the sense that such an M has two limit ends, each of which has a representative that is naturally asymptotic to a limit end representative of a Riemann minimal example with the same associated flux vector.

Contents

1. Introduction	474
2. Analytic definition of the Riemann minimal examples	479
3. Reduction of the proof of Theorem 1.2 to the case of two limit ends	481

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3.1. Properly embedded minimal surfaces with finite topology and one end	482
3.2. Properly embedded minimal surfaces with finite topology and more than one end	482
3.3. Properly embedded minimal surfaces with finite genus and an infinite number of ends	484
4. Jacobi functions on a minimal surface	491
4.1. The Shiffman Jacobi function	494
4.2. The space of allowed Gauss maps and their infinitesimal deformations	495
5. Holomorphic integration of the Shiffman function	502
5.1. Algebro-geometric potentials of the KdV equation	503
5.2. The Shiffman hierarchy associated to a Riemann type minimal surface	510
6. The proofs of Theorems 1.1 and 1.2	517
6.1. Minimal surfaces of Riemann type whose Shiffman function is linear	518
6.2. The linearity of the Shiffman function for every surface $M \in \mathcal{M}$	519
7. Linearity of bounded Jacobi functions on Riemann minimal examples	521
8. Asymptotic behavior of finite genus minimal surfaces	525
9. Appendix 1: Finite dimensionality of the space of bounded Jacobi functions	533
10. Appendix 2: Uniqueness of the Riemann minimal examples in the two-ended periodic case	538
References	542

1. Introduction

The history of minimal surfaces in \mathbb{R}^3 begins with the discovery of the classical examples found in the 18-th and 19-th centuries. The first important result in this direction is due to Euler [16], who proved in 1741 that when a small arc on the catenary $x_1 = \cosh x_3$ is rotated around the x_3 -axis, then one obtains a surface that minimizes area among all surfaces of revolution with the same boundary circles. The entire surface of revolution of $x_1 = \cosh x_3$ was initially called the *alysseid* but since Plateau's time, called the catenoid.

In 1776, Meusnier [41] observed that the plane, the catenoid and the helicoid all have zero mean curvature. In this same paper, he proved that the condition on the mean curvature of the graph G of a function $u = u(x, y)$ over a domain in the plane to vanish identically can be expressed by the following

quasilinear, second order, elliptic partial differential equation. This equation was found in 1762 by Lagrange¹ [28], who proved that it is equivalent to G being a *minimal surface*, i.e., a critical point of the area functional with respect to variations fixing the boundary of the graph:

$$(1) \quad (1 + u_x^2)u_{yy} - 2u_xu_yu_{xy} + (1 + u_y^2)u_{xx} = 0.$$

It follows that the plane, the catenoid and the helicoid are examples of properly embedded, minimal planar domains in \mathbb{R}^3 . (A *planar domain* is a connected surface that embeds in the plane, or equivalently that is noncompact, connected and has genus zero.)

Around 1860, Riemann discovered (and posthumously published, Hattendorf and Riemann [47], [48]) other interesting examples of properly embedded, periodic, minimal planar domains in \mathbb{R}^3 . These examples, called the *Riemann minimal examples*, appear in a one-parameter family \mathcal{R}_t , $t \in (0, \infty)$, and satisfy the property that, after a rotation, each \mathcal{R}_t intersects every horizontal plane in a circle or in a line. The \mathcal{R}_t have natural limits being a vertical catenoid as $t \rightarrow 0$ and a vertical helicoid as $t \rightarrow \infty$. The main theorem of this manuscript states that these beautiful surfaces of Riemann are unique in a particularly simple way. This result, which was conjectured in [35], is motivated by our earlier proof of it under the additional hypothesis that the minimal planar domain is periodic [34] and by partial results in our previous papers [37], [38], [35], [36].

THEOREM 1.1. *After a homothety and a rigid motion, any connected, properly embedded, minimal planar domain in \mathbb{R}^3 with an infinite number of ends is a Riemann minimal example.*

Understanding properly embedded minimal surfaces in \mathbb{R}^3 is the key for understanding the local structure of embedded minimal surfaces in any Riemannian three-manifold. Inside this family of surfaces, the case of genus zero is the most important, both because it is the starting point for the general theory and because it gives the local picture for any finite genus minimal surface in a three-manifold away from a finite collection of points where the genus concentrates. Since we will focus on surfaces with genus zero, the only topological information comes from the ends (i.e., the ways to go to infinity). For example, the plane and the helicoid have only one end, the catenoid is topologically a cylinder and thus has two ends, and the Riemann minimal examples are topologically cylinders with a periodic set of punctures and thus have infinitely many ends.

¹In reality, Lagrange arrived at a slightly different formulation, and equation (1) was derived five years later by Borda [3].

Meeks and Rosenberg [40] proved that the only simply connected, properly embedded, minimal planar domains in \mathbb{R}^3 are planes and helicoids. Earlier results of Collin [12] and of López and Ros [30] demonstrated that the only nonsimply connected, properly embedded, minimal planar domains in \mathbb{R}^3 with a finite number of ends are catenoids. Consequently, Theorem 1.1 gives the following final classification result.

THEOREM 1.2 (Classification Theorem for Minimal Planar Domains). *Up to scaling and rigid motion, any connected, properly embedded, minimal planar domain in \mathbb{R}^3 is a plane, a helicoid, a catenoid or one of the Riemann minimal examples. In particular, for every such surface there exists a foliation of \mathbb{R}^3 by parallel planes, each of which intersects the surface transversely in a connected curve that is a circle or a line.*

In a series of pioneering papers, Colding and Minicozzi [7], [8], [9], [10] gave a rather complete local description of properly embedded minimal disks in a ball, showing that any such surface is either graphical (like the plane) or contains a double-spiral staircase (like the helicoid). Building on these results by Colding-Minicozzi, Meeks and Rosenberg [40] characterized the plane and the helicoid as the only simply connected, properly embedded, minimal surfaces in \mathbb{R}^3 . The present paper relies on Colding-Minicozzi theory, which will be used to reduce the proofs of Theorems 1.1 and 1.2 to Assertion 1.3 below. This reduction is based on our previously published papers [35] and [36], where we appealed to certain results in the series [7], [8], [9], [10] and to further structure results by Colding-Minicozzi for genus-zero surfaces that appear in the fifth paper of their series [11]. Besides these crucial applications of results by Colding-Minicozzi in our papers [35] and [36], no other use will be made in the present paper of their results. We will make clear in Section 3 which results of Colding-Minicozzi theory are needed and where in [35] and [36] they are applied.

The results of Collin [12], López-Ros [30] and Meeks-Rosenberg [40] not only lead to the classification of all properly embedded, minimal planar domains in \mathbb{R}^3 of finite topology, but these results and work of Bernstein and Breiner [2] also characterize the asymptotic behavior of the annular ends of any properly embedded minimal surface in \mathbb{R}^3 ; namely, each such end contains a representative that is asymptotic to the end of a plane, a catenoid or a helicoid. We will apply Theorem 1.1 to obtain in Theorem 8.1 a similar characterization of the asymptotic behavior of any properly embedded minimal surface in \mathbb{R}^3 with finite genus and infinite topology. We remark that Hauswirth and Pacard [23] have found many interesting examples of such surfaces for any finite positive genus.

We next outline the organization of the paper and at the same time describe the strategy for proving Theorems 1.1 and 1.2. In Section 2 we give a brief geometric and analytic description of the Riemann minimal examples and present images of some of these surfaces. In Section 3 we outline the basic definitions and theory that reduce the proof of Theorem 1.2 to Theorem 1.1, and the proof of Theorem 1.1 to Assertion 1.3 below; see Theorems 3.1 and 3.2 for these reductions, which are crucial in the proof of our main results and which depend on our previously published papers [35], [36]; in particular, these reductions depend in an essential manner on Colding-Minicozzi theory.

Before stating Assertion 1.3, it is worth introducing some notation. Suppose that $M \subset \mathbb{R}^3$ is a properly embedded minimal planar domain with two limit ends (limit ends are defined in Section 3.3), such as one of the Riemann minimal examples. After a possible rotation of the surface, any horizontal plane P intersects M in a simple closed curve or in a proper Jordan arc γ_P . (See Theorem 3.1 for this property.) If we let η denote the unitary upward pointing conormal to M along γ_P , then the *flux vector* of M is defined to be

$$F_M = \int_{\gamma_P} \eta ds$$

(here ds stands for the length element), and F_M is independent of the choice of P . We proved in [35] that, after a rotation around the x_3 -axis and a homothety, F_M can be assumed to be $(h, 0, 1)$ for some $h > 0$. We remark that in our definition of the Riemann minimal examples, $F_{\mathcal{R}_t} = (t, 0, 1)$.

ASSERTION 1.3. *Let \mathcal{M} be the space of properly embedded, minimal planar domains $M \subset \mathbb{R}^3$ with two limit ends, normalized so that every horizontal plane intersects M in a simple closed curve or a proper arc and that the flux vector is $F_M = (h, 0, 1)$ for some $h = h(M) > 0$. Then, \mathcal{M} is the set of Riemann minimal examples $\{\mathcal{R}_t\}_{t \in (0, \infty)}$.*

The strategy to prove Assertion 1.3 is by means of the classical *Shiffman function*, which is a Jacobi function that adapts particularly well to the problem under consideration. Since minimal surfaces can be viewed as critical points for the area functional A , the nullity of the hessian of A at a minimal surface M contains valuable information about the geometry of M . Normal variational fields for M can be identified with functions (we will always consider orientable surfaces), and the second variation of area tells us that the functions in the nullity of the hessian of A coincide with the kernel of the Schrödinger operator $\Delta - 2K$ (called the *Jacobi operator*), where Δ denotes the intrinsic Laplacian on M and K is the Gaussian curvature function on M . Any function v satisfying $\Delta v - 2Kv = 0$ on M is called a *Jacobi function* and corresponds to an infinitesimal deformation of M by minimal surfaces. A particularly useful Jacobi function in our proof of Assertion 1.3 is the Shiffman

function S_M , defined for any surface $M \in \mathcal{M}$. In Section 4 we will study the Shiffman function, as well as basic properties of Jacobi functions on a minimal surface which will be applied in our paper. In the 1950s, Shiffman [52] defined and applied S_M to detect when a minimal surface is foliated by circles and straight lines in parallel planes; this remarkable property was known to characterize the surfaces \mathcal{R}_t since Riemann's times [47], [48]. By [47], [52], S_M vanishes for a surface $M \in \mathcal{M}$ if and only if M is a Riemann minimal example. Thus, a possible approach to proving Assertion 1.3 would be to verify that S_M vanishes for any $M \in \mathcal{M}$, although we will not prove this fact directly. Instead, we will demonstrate that S_M is *linear* (i.e., it is the normal part of a parallel vector field in \mathbb{R}^3), a weaker property that is enough to conclude M is a Riemann minimal example (Proposition 6.2).

The desired linearity of S_M for every $M \in \mathcal{M}$ will follow from the fact that S_M can always be integrated in the following sense. For an arbitrary $M \in \mathcal{M}$, there exists a one-parameter family $\{M_t\}_t \subset \mathcal{M}$ with $M_0 = M$ such that the normal part of the variational vector field for this variation, when restricted to each M_t , is the Shiffman Jacobi function S_{M_t} multiplied by the unit normal vector field to M_t . Moreover, in our integration of S_M by $\{M_t\}_t$, the parameter t can be extended to be a complex parameter, and $t \mapsto M_t$ can be viewed as the real part of a complex valued *holomorphic* curve in a certain complex variety; we will refer to this integrability property by saying that the Shiffman function can be *holomorphically integrated*. (See Definition 5.12 and Remark 5.13.)

Assume for the moment that S_M can be holomorphically integrated for any $M \in \mathcal{M}$; we will explain why S_M is linear. The basic idea is to fix a flux vector $F = (h, 0, 1)$ and then extremize the spacing between the planar ends among all examples in $\mathcal{M}_F = \{M \in \mathcal{M} \mid F_M = F\}$. (This requires a compactness result in \mathcal{M}_F that uses the uniform geometric estimates from Section 3.) Then one considers the complex deformation $t \mapsto M_t$ around an extremizer $M_0 \in \mathcal{M}_F$, given by holomorphic integration of the Shiffman function S_{M_0} of M_0 , and proves that the entire deformation is contained in \mathcal{M}_F and that the spacing between planar ends depends harmonically on the complex parameter t ; this harmonic dependence together with the maximum principle for harmonic functions applies to give that the spacing remains constant along the deformation $t \mapsto M_t$, which can be interpreted as the linearity of the Shiffman function of M_0 . From here we conclude that any minimizer and any maximizer of the spacing between planar ends in \mathcal{M}_F is a Riemann minimal example. As there is only one Riemann minimal example with each flux, then the maximizer and minimizer are the same, and thus every surface in \mathcal{M}_F is both a maximizer and minimizer, which implies that \mathcal{M}_F consists of a single

surface that is a Riemann minimal example. The purpose of Section 6 is to give the details of the arguments in this paragraph.

To finish our overview of the proof of Assertion 1.3, it remains to briefly explain how the Shiffman function S_M can be holomorphically integrated for any $M \in \mathcal{M}$, which will be the main task of Section 5. The approach is through the Korteweg-de Vries (KdV) equation and its hierarchy. A change of variables transforms the holomorphic integration of the Shiffman function into solving a Cauchy problem for a meromorphic KdV equation on the cylinder. The key step for the solvability of this meromorphic KdV Cauchy problem amounts to proving that the initial data is an *algebro-geometric potential* for KdV, which is a finiteness condition in the hierarchy associated to the KdV equation that will be established in Corollary 5.10. This finiteness condition depends crucially on the fact that the space $\mathcal{J}_\infty(M)$ of bounded Jacobi functions on any surface $M \in \mathcal{M}$ is finite dimensional. Finite dimensionality of $\mathcal{J}_\infty(M)$ could be deduced from a paper by Colding, de Lellis and Minicozzi [6], although we will include a self-contained proof in Appendix 1.

In Section 7 we will prove that all functions in $\mathcal{J}_\infty(\mathcal{R})$ are linear for any Riemann minimal example \mathcal{R} . This result could be seen as the linearization of our main classification theorem, although it does not directly follow from the uniqueness of the Riemann minimal examples as there might be a bounded Jacobi function on \mathcal{R} that does not integrate to an actual variation. Finally, in Section 8, we will apply this characterization of $\mathcal{J}_\infty(\mathcal{R})$ and Theorem 1.1 to prove Theorem 8.1, which describes the asymptotic behavior of the limit ends of properly embedded minimal surfaces in \mathbb{R}^3 with finite genus.

The authors would like to thank the referee for his valuable comments and suggestions to improve the exposition.

2. Analytic definition of the Riemann minimal examples

In [48], Riemann classified the compact minimal annuli in \mathbb{R}^3 that are foliated by circles in some family of horizontal planes. He proved that besides the catenoid and after a homothety and a translation of \mathbb{R}^3 , each such annulus is contained in a properly embedded, minimal planar domain $\mathcal{R}(t)$ for some $t \in (0, \infty)$, described as follows. Consider the rectangular torus $\mathbb{T}_t = \mathbb{C}/\Lambda_t$, where $\Lambda_t = \{tm + in \mid m, n \in \mathbb{Z}\}$ and $i = \sqrt{-1}$, and let \mathcal{P}_t denote the Weierstrass \mathcal{P} -function on \mathbb{T}_t . Let $\sigma_t: \mathbb{C}/\langle i \rangle \rightarrow \mathbb{T}_t$ denote the related \mathbb{Z} -cover of \mathbb{T}_t , where $\langle i \rangle = i\mathbb{Z}$. Let g_t denote the meromorphic function $a_t \mathcal{P}_t \circ \sigma_t: \mathbb{C}/\langle i \rangle \rightarrow \mathbb{C} \cup \{\infty\}$, where $a_t > 0$ is chosen so that the branch values of g_t are $0, \infty$ and one pair of antipodal points on the real axis; a simple calculation shows that $a_t = (\sqrt{-\mathcal{P}_t(i/2)\mathcal{P}_t(t/2)})^{-1}$. Let dz be the holomorphic differential on $\mathbb{C}/\langle i \rangle$ coming from the coordinate $z = x + iy$ in \mathbb{C} .

In order to complete our description of $\mathcal{R}(t)$, it is convenient to use the fact that $\mathbb{C}/\langle i \rangle$ is isometric to the cylinder $\mathbb{S}^1 \times \mathbb{R} \subset \mathbb{R}^2 \times \mathbb{R}$, where $\mathbb{S}^1 = i\mathbb{R}/\langle i \rangle$ is considered to be the circle of circumference one centered at the origin in \mathbb{R}^2 . Let Z_t be the set of zeroes of g_t , P_t be the set of poles of g_t and $E_t = Z_t \cup P_t$. With this meromorphic data and with $z_0 = 0 + \frac{1}{2}i \in \mathbb{C}/\langle i \rangle$, the minimal planar domain $\mathcal{R}(t)$ is defined analytically to be the image of the conformal harmonic map $X_t: (\mathbb{S}^1 \times \mathbb{R}) - E_t \rightarrow \mathbb{R}^3$ defined at $z = x + iy \in (\mathbb{C}/\langle i \rangle) - E_t$ by the Weierstrass formula given in equation (4) of Section 4:

$$X_t(z) = \Re \int_{z_0}^z \left(\frac{1}{2} \left(\frac{1}{g_t} - g_t \right), \frac{i}{2} \left(\frac{1}{g_t} + g_t \right), 1 \right) dz,$$

where $\Re(w)$ is the real part of a complex vector $w \in \mathbb{C}^3$.

When we view $\mathcal{R}(t)$ as being parametrized by the punctured flat cylinder $(\mathbb{S}^1 \times \mathbb{R}) - E_t$, then the level set circle $\mathbb{S}^1 \times \{s\}$ at a height s different from nt or $(n + \frac{1}{2})t$ for $n \in \mathbb{Z}$ has image curve in \mathbb{R}^3 that is a circle in the horizontal plane at height s . Note that $L_n = X_t[(\mathbb{S}^1 \times \{nt\}) - \{(0, nt)\}]$ is a line parallel to the x_2 -axis, placed at height nt . (Here we identify \mathbb{S}^1 with $i\mathbb{R}/\langle i \rangle$.) Similarly, $L_n^{\frac{1}{2}} = X_t[(\mathbb{S}^1 \times \{(n + \frac{1}{2})t\}) - \{(\frac{i}{2}, (n + \frac{1}{2})t)\}]$ is a line parallel to the x_2 -axis and placed at height $(n + \frac{1}{2})t$. The isometry group of $\mathcal{R}(t)$ is generated by the reflection in the (x_1, x_3) -plane and the rotations of angle π around the lines L_n, L'_n , where L'_n is the line parallel to L_n that is equidistant to L_n and $L_n^{\frac{1}{2}}$ (L'_n intersects $\mathcal{R}(t)$ orthogonally at two points), $n \in \mathbb{Z}$. In particular, the surface $\mathcal{R}(t)$ is periodic under the orientation-preserving translation v_t given by the composition of the rotation by angle π around L_0 with rotation by angle π around $L_0^{\frac{1}{2}}$. Thus, v_t lies in the (x_1, x_3) -plane and has vertical component t ; see Figure 1.

With respect to the above parametrization $X_t: (\mathbb{C}/\langle i \rangle) - E_t \rightarrow \mathcal{R}(t) \subset \mathbb{R}^3$, the points in E_t correspond to those ends of $\mathcal{R}(t)$ that can be represented by annuli asymptotic to horizontal planes at heights in $H_t = \{nt, (n + \frac{1}{2})t \mid n \in \mathbb{Z}\}$. Since $\mathbb{C}/\langle i \rangle$ is naturally conformally $\mathbb{C} - \{0\} \subset \mathbb{C} \cup \{\infty\} = \mathbb{S}^2$, we see that $\mathcal{R}(t)$ is conformally diffeomorphic to $\mathbb{S}^2 - \mathcal{E}(\mathcal{R}(t))$, where $\mathcal{E}(\mathcal{R}(t))$ is the union of the set of planar ends E_t of $\mathcal{R}(t)$ with $e_{-\infty}$ (resp. e_{∞}), corresponding to the bottom (resp. top) end of the cylinder $\mathbb{C}/\langle i \rangle = \mathbb{S}^1 \times \mathbb{R}$ viewed as being the south (resp. north) pole of \mathbb{S}^2 . The set of points $\mathcal{E}(\mathcal{R}(t)) \subset \mathbb{S}^2$ with the subspace topology can be naturally identified with the space of ends of the surface $\mathcal{R}(t)$. By the topological classification of noncompact genus-zero surfaces, $\mathcal{R}(t)$ is the unique (up to homeomorphism) planar domain with two limit ends. Finally, we remark that the holomorphic function $g_t|_{(\mathbb{C}/\langle i \rangle) - E_t}$ can be identified with the stereographic projection of the Gauss map of $\mathcal{R}(t)$ when we view the Gauss map as being defined on the parameter space $(\mathbb{C}/\langle i \rangle) - E_t$.

We now are in position to define a normalization of the Riemann minimal examples that we will use in the sequel. For any s , let η be the upward pointing, unitary conormal vector along the boundary curve γ_s of $\mathcal{R}(t) \cap \{x_3 \leq s\}$. The flux of $\mathcal{R}(t)$ is

$$F_{\mathcal{R}(t)} = \int_{\gamma_s} \eta ds$$

and has the form $(h(t), 0, 1)$ for some positive $h(t)$. It turns out that $h(t)$ determines the Riemann minimal example $\mathcal{R}(t)$ and, moreover, $h(t) \rightarrow \infty$ as $t \rightarrow 0$ and $h(t) \rightarrow 0$ as $t \rightarrow \infty$. Define $\mathcal{R}_t = \mathcal{R}(h^{-1}(t))$ for each $t > 0$; thus the flux of \mathcal{R}_t is $(t, 0, 1)$. Then we obtain the normalization $\{\mathcal{R}_t\}_{t>0}$ of the family of Riemann minimal examples to which we will refer throughout this paper. With this notation, the limit of suitable translations of the \mathcal{R}_t as $t \rightarrow 0$ is a vertical catenoid, and the limit of suitable translations and homotheties of the \mathcal{R}_t as $t \rightarrow \infty$ is a vertical helicoid; see Figure 2.

3. Reduction of the proof of Theorem 1.2 to the case of two limit ends

Before proceeding with a discussion of the theoretical results that reduce the proof of Theorem 1.2 to the case of two-limit-end examples of genus zero, we make a few comments that can suggest to the reader a visual idea of what is going on. The most natural motivation for understanding this theorem and other results presented in this section is to try to answer the following general question: *What are the possible shapes of a complete embedded surface $M \subset \mathbb{R}^3$ that satisfies a variational principle and has a given topology?* In our case, the

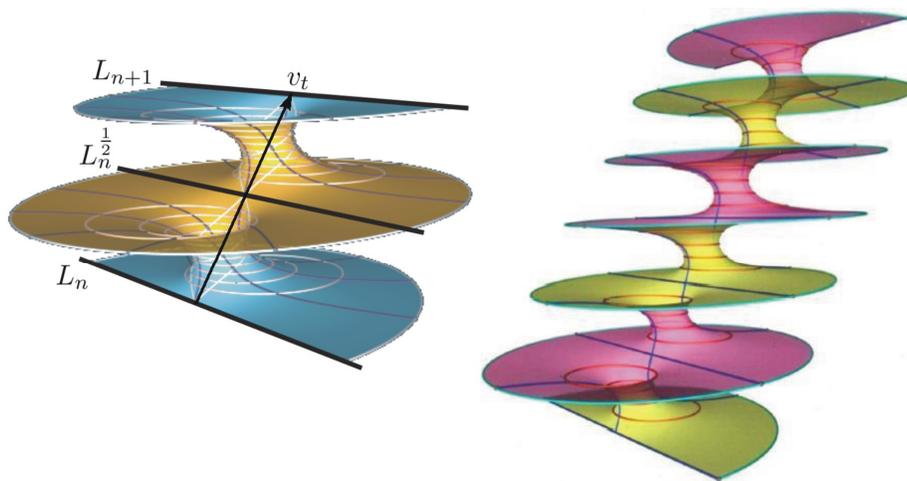


Figure 1. Two Riemann minimal examples (courtesy of M. Weber).

variational equation expresses the critical points of the area functional. We will describe the situation according to the topology of M .

3.1. *Properly embedded minimal surfaces with finite topology and one end.*

If the requested topology for M is the simplest one of a disk, then the classification theorem of Meeks and Rosenberg [40] states that the possible shapes for complete examples are the trivial one given by a plane and (after a rotation) an infinite double-spiral staircase, which is a visual description of a vertical helicoid. A more precise description of the double-spiral staircase nature of a vertical helicoid is that this surface is the union of two infinitely sheeted multigraphs, which are glued along a vertical axis. Crucial in the proof of this classification result are the results of Colding and Minicozzi [7], [8], [9], [10] that describe both the local structure of compact, embedded minimal disks as essentially being modeled by either graphs or pairs of finitely sheeted multigraphs glued along an “axis,” and global properties of limits of these shapes.

More generally, if we allow our complete minimal surface $M \subset \mathbb{R}^3$ to be topologically a disk with a finite positive number of handles, then it turns out that M is conformally a closed Riemann surface \bar{M} of positive genus punctured in a single point; see Bernstein and Breiner [2] and also see Meeks and Pérez [33], where they prove that M is asymptotic to a helicoid and that it can be defined analytically in terms of meromorphic data on \bar{M} . Using different approaches, Hoffman, Weber and Wolf [54] and Hoffman and White [26] proved the existence of such a genus-one helicoid; also see Hoffman, Traizet and White [25], where they construct properly embedded minimal surfaces with arbitrary positive genus $g \in \mathbb{N}$ and one end. Meeks and Rosenberg [40] have conjectured that there exists a unique genus g helicoid for each positive finite g .

3.2. *Properly embedded minimal surfaces with finite topology and more than one end.* We now describe the special geometry of any properly embedded

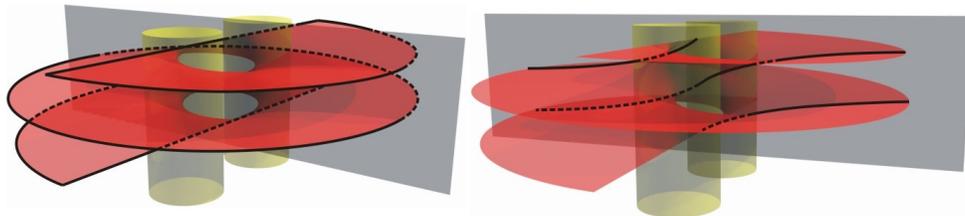


Figure 2. Two views of a Riemann minimal example close to the helicoidal limit. Two vertical helicoids are forming at opposite sides of the vertical plane of symmetry. (For the reader’s convenience, we have also represented vertical cylinders containing significant parts of the forming helicoids.)

minimal surface $M \subset \mathbb{R}^3$ that has finite topology and more than one end. In this situation we find many beautiful examples and even large dimensional families of examples, and so it is not possible to obtain a general classification result for all of these surfaces. However, the asymptotic behavior of the ends of M is well understood by Collin's Theorem [12], which states that each end of M is asymptotic to the end of a plane or catenoid. In particular, we find that M is conformally a compact Riemann surface \bar{M} punctured in a finite number of points and, by a theorem of Osserman [43], M can be defined analytically in terms of meromorphic data on \bar{M} . For example, one sees by a simple application of Picard's theorem that the stereographic projection of the Gauss map $g: M \rightarrow \mathbb{C} \cup \{\infty\}$ extends to a meromorphic function $G: \bar{M} \rightarrow \mathbb{C} \cup \{\infty\}$. The Gauss-Bonnet formula then implies that the degree of G is equal to the absolute total curvature of M divided by 4π .

A fundamental classification theorem of López and Ros [30] states that the plane and the catenoid are the only complete, embedded, minimal planar domains in \mathbb{R}^3 with finite total curvature. Thus, Collin's theorem and the López-Ros theorem together imply that the catenoid is the only connected, properly embedded, minimal planar domain of finite topology in \mathbb{R}^3 with more than one end. Summarizing, *the plane, the helicoid and the catenoid are the only properly embedded, minimal planar domains in \mathbb{R}^3 with finite topology.*

In order to better understand the asymptotic behavior of general finite topology examples with more than one end (i.e., finite genus not necessarily zero), it is helpful to consider the following question: *What is the visual picture for a connected, properly embedded minimal surface $M \subset \mathbb{R}^3$ with finite topology and at least two ends?* By Collin's theorem, each end of M is asymptotic to the end of a catenoid or to a plane. It then follows from the embeddedness of M that after a fixed rotation of M and for some large $R_M > 0$, $M \cap \{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 \geq R_M^2\}$ consists of a finite number E_1, E_2, \dots, E_n of graphs over the annulus $A(R_M) = (\mathbb{R}^2 \times \{0\}) - \{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 < R_M^2\}$. These graphs have logarithmic growths $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$, which are linearly ordered by the relative heights of the graphs over $A(R_M)$, and the collection $\{E_1, E_2, \dots, E_n\}$ represents the (annular) ends of M . Note that when $\lambda_i = 0$, then E_i is asymptotic to a horizontal plane. The Half-space Theorem by Hoffman and Meeks [24] implies that M cannot be contained in a half-space of \mathbb{R}^3 , and so $\lambda_1 < 0$ and $0 < \lambda_n$. After this rotation, to have graphical ends on the exterior of a disk on the (x_1, x_2) -plane, M is said to have *horizontal limit tangent plane at infinity*.

As in the case of one-ended minimal surfaces of finite topology, there is a precise conjecture on the topological types allowed in the class of properly embedded minimal surfaces with finite topology and more than one end. In 1982, Hoffman and Meeks conjectured that a necessary and sufficient condition for a

surface with finite genus g and a finite number $k > 2$ of ends to admit a proper minimal embedding into \mathbb{R}^3 is that $g+2 \geq k$. The case $k=1$ reduces to the plane, and Schoen [50] proved that if $k=2$, then the surface is a catenoid (hence $g=0$). López-Ros [30] proved that $g=0$ implies that the surface is a catenoid or a plane, and hence the last inequality also holds in this case. In the remaining cases, this conjecture is supported by existence theorems of Traizet [53] and of Weber and Wolf [55]. Along these lines, the authors proved that for each fixed genus g , there is an upper bound $k(g)$ for the number of ends of a properly embedded minimal surface with genus g and finitely many ends [37].

3.3. Properly embedded minimal surfaces with finite genus and an infinite number of ends. Any properly embedded minimal surface $M \subset \mathbb{R}^3$ with more than one end has an associated plane passing through the origin, which is called *the limit tangent plane at infinity* of M , defined as follows. Firstly, it can be shown that $\mathbb{R}^3 - M$ contains the end E of a plane or catenoid. Such an end E has a limiting normal vector v_E at infinity, which turns out to not depend on the choice of E in $\mathbb{R}^3 - M$. The plane passing through the origin and orthogonal to v_E is the limit tangent plane at infinity to M ; see [4] for further details. We will generally assume that the limit tangent plane at infinity to M is horizontal, or equivalently, it is the (x_1, x_2) -plane. A fundamental aid in discussing the asymptotic geometry of M is the Ordering Theorem of Frohman and Meeks [18], which states that the space of ends $\mathcal{E}(M)$ of M has a natural linear ordering by their relative heights over the (x_1, x_2) -plane, similar to the way in which the ends of a finite topology minimal surface with more than one end can be linearly ordered.

With this linear ordering on $\mathcal{E}(M)$ in mind and using the fact that $\mathcal{E}(M)$ has a natural topology induced by an order preserving embedding as a compact, totally disconnected subspace of the unit interval $[0, 1]$ (see Section 2.7 in [32]), we find that there exist unique elements $e_T, e_B \in \mathcal{E}(M)$ that are maximal and minimal elements in the linear ordering on $\mathcal{E}(M)$, respectively. e_T is called the *top end* and e_B the *bottom end* of M . The other ends in $\mathcal{E}(M) - \{e_B, e_T\}$ are called the *middle ends* of M . By a result of Collin, Kusner, Meeks and Rosenberg [13], the only possible limit ends of M (limit points of $\mathcal{E}(M)$ in its natural topology) are e_B or e_T .

The above discussion implies that the classification of the properly embedded minimal planar domains in \mathbb{R}^3 reduces to the classification of examples with two limit ends and to ruling out the case of one limit end. We will start by describing the geometry of any surface in the two limit end case. Note that Theorem 3.1 below uses the notation in Assertion 1.3.

THEOREM 3.1. *Given any $M \in \mathcal{M}$, we have*

- (1) M can be conformally parametrized by the cylinder $\mathbb{C}/\langle i \rangle$ punctured at an infinite discrete set of points $\{p_j, q_j\}_{j \in \mathbb{Z}}$.
- (2) The stereographically projected Gauss map of M , considered to be a meromorphic function g on $\mathbb{C}/\langle i \rangle$ after attaching the planar ends of M , has order-two zeros at the points p_j and order-two poles at the q_j .
- (3) The height differential of M is $dh = dz$, and so its height function is $x_3(z) = \Re(z)$. In particular, the middle ends p_j, q_j of M are planar, and they are naturally ordered by heights by $\Re(p_j) < \Re(q_j) < \Re(p_{j+1})$ for all $j \in \mathbb{Z}$, with $\Re(p_j) \rightarrow \infty$ (resp. $\Re(p_j) \rightarrow -\infty$) when $j \rightarrow \infty$ (resp. $j \rightarrow -\infty$).
- (4) Every horizontal plane intersects M in a simple closed curve when its height is not in $H = \{\Re(p_j), \Re(q_j) \mid j \in \mathbb{Z}\}$ and in a single properly embedded arc when its height is in H ; in particular, the principal divisor of g is $(g) = \prod_{j \in \mathbb{Z}} p_j^2 q_j^{-2}$.
- (5) M has bounded Gaussian curvature, and this bound only depends on an upper bound of h . (Recall that the flux of M along a compact horizontal section is $F_M = (h, 0, 1)$ with $h > 0$.)
- (6) The vertical spacings between consecutive ends are bounded from above and below by positive constants that only depend on h . Also, M admits an embedded regular neighborhood of fixed radius $r = r(h) > 0$.
- (7) For every divergent sequence $\{z_k\}_k \subset \mathbb{C}/\langle i \rangle$, there exists a subsequence of the meromorphic functions $g_k(z) = g(z + z_k)$ that converges uniformly on compact subsets of $\mathbb{C}/\langle i \rangle$ to a nonconstant meromorphic function $g_\infty: \mathbb{C}/\langle i \rangle \rightarrow \mathbb{C} \cup \{\infty\}$. In fact, g_∞ corresponds to the Gauss map of a surface $M_\infty \in \mathcal{M}$, which is the limit of a related subsequence of translations of M by vectors whose x_3 -components are $\Re(z_k)$.

Proof. Most of the arguments in this proof can be found in our previous paper [35]; for the sake of completeness and also in order to clarify the dependence of the results in [35] on Colding-Minicozzi theory, we will include some details about this proof. By Theorem 1.1 in [13], the middle ends of M are not limit ends. As M has genus zero, then these middle ends can be represented by annuli. By Collin’s theorem [12], every annular end of M is asymptotic to the end of a plane or catenoid. By Theorem 3.5 in [13], there exists a sequence of horizontal planes $\{P_i\}_{i \in \mathbb{N}}$ with increasing heights such that M intersects each plane P_i transversely in a compact set, every middle end of M has an end representative that is the closure of the intersection of M with the slab bounded by $P_i \cup P_{i+1}$, and every such slab contains exactly one of these middle end representatives. By the Halfspace Theorem [24], the restriction of the harmonic third coordinate function x_3 to the portion $M(+)$ of M above P_0 is not bounded from above and extends smoothly across the

middle ends. By Theorem 3.1 in [13], $M(+)$ has a parabolic conformal structure. After compactification of $M(+)$ by adding its middle ends and their limit point p_∞ corresponding to the top end in $M(+)$, we obtain a conformal parameterization of this compactification defined on the unit disk $\mathbb{D} = \{|z| \leq 1\}$, so that $p_\infty = 0$, the middle ends in $M(+)$ correspond to a sequence of points $p_i \in \mathbb{D} - \{0\}$ converging to zero and $x_3|_{M(+)}(z) = -\lambda \ln |z| + c$ for some $\lambda, c \in \mathbb{R}$, $\lambda > 0$. Also note that different planar ends cannot have the same height above P_0 (since they lie in different slabs bounded by the planes P_i). In particular, $M(+)$ intersects every plane P' above P_0 in a simple closed curve if the height of P' does not correspond to the height of any middle end, while P' intersects $M(+)$ in a Jordan arc when the height of P' equals the height of a middle end. This implies that the zeros and poles of the stereographically projected Gauss map g of M at the middle ends of $M(+)$ have order two. Since the behavior of $M(-) = M - [M(+) \cup P_0]$ can be described analogously, then items (1), (2), (3) and (4) of the theorem are proved.

The fact that the Gaussian curvature K_M of M is bounded with the bound depending only on an upper bound of the horizontal part of the flux vector F_M (item (5) of the theorem) was proven in Theorem 5 of [35] in the more general case of a sequence $\{M(i)\}_i \subset \mathcal{M}$ such that $\{h(i)\}_i$ is bounded, where $F_{M(i)} = (h(i), 0, 1)$ for each $i \in \mathbb{N}$. In this setting, the conclusion of Theorem 5 of [35] is that the sequence of Gaussian curvature functions $\{K_{M(i)}\}_i$ of the $M(i)$ is uniformly bounded. Since later we will use this stronger version of the curvature estimates (namely, in Proposition 6.3), we now sketch its proof. The argument is by contradiction, so assume that $\{K_{M(i)}\}_i$ is not uniformly bounded.

The first step consists of finding points $p(i) \in M(i)$ and positive numbers $\lambda(i) \rightarrow \infty$ such that after passing to a subsequence, the surfaces $M'(i) = \lambda(i)(M(i) - p(i))$ converge uniformly on compact subsets of \mathbb{R}^3 with multiplicity one to a vertical helicoid H passing through the origin $\vec{0}$, with $|K_H| \leq 1$ and $|K_H|(\vec{0}) = 1$. This is done in Lemma 5 of [35], whose proof uses a standard blow-up argument on the scale of curvature; this blow-up process creates a subsequential limit of the $M'(i)$, which is simply connected by a flux argument. (Nonzero fluxes of the limit surface must come from nonzero arbitrarily small fluxes on the $M(i)$, which is impossible.) This limit of the $M'(i)$ is a helicoid H by the classification by Meeks and Rosenberg of the simply connected, properly embedded minimal surfaces [40]. (We remind the reader that this classification depends on Colding-Minicozzi results contained in [7], [8], [9], [10].) The axis of H is vertical as the Gauss maps of the $M(i)$ omit the vertical directions by the already proven item (3) of the theorem, and the normalization of K_H follows directly from construction. This finishes the first step.

The second step consists of renormalizing the surfaces $M'(i)$ by rescaling and rotation around the x_3 -axis, so that

- (P1) $\vec{0} \in M'(i)$.
- (P2) The horizontal section of $M'(i)$ at height zero is a simple closed curve.
- (P3) $\{M'(i)\}_i$ converges on compact subsets of \mathbb{R}^3 to a vertical helicoid with axis passing through $\vec{0}$.

Property (P3) above insures that one can find an open arc $\alpha'(i) \subset M'(i) \cap \{x_3 = 0\}$ centered at $\vec{0}$ so that the Gauss map of $M'(i)$ takes values at the end points of $\alpha'(i)$ in different hemispheres determined by the horizontal equator. By continuity, this allows us to find a point $q'(i) \in [M'(i) \cap \{x_3 = 0\}] - \alpha'(i)$ closest to the origin where the Gauss map of $M'(i)$ is horizontal and then to renormalize the $M'(i)$ by rescaling and rotation around the x_3 -axis to define new surfaces $\widetilde{M}(i)$ so that $q'(i)$ is independent of i and has the form $\vec{q} = (0, 6\tau, 0) \in \widetilde{M}(i)$, where $\tau > \tau_0$ is fixed but arbitrary, and $\tau_0 > 1$ is defined by the following auxiliary property. (This is Lemma 4 in [35], whose proof only uses the curvature estimates for stable minimal surfaces by Schoen [49].)

- (P4) There exists $\tau_0 > 1$ such that given a properly embedded, noncompact, orientable, stable minimal surface Δ contained in a horizontal slab of width not greater than 1, and such that the boundary $\partial\Delta$ lies inside a vertical cylinder of radius 1, the portion of Δ at distance greater than τ_0 from the axis of the cylinder consists of a finite number of graphs over the complement of a disk of radius τ_0 in the (x_1, x_2) -plane.

The third step consists of finding embedded closed curves $\delta(\tau, i) \subset \widetilde{M}(i)$ so that the flux of $\widetilde{M}(i)$ along $\delta(\tau, i)$ decomposes as

$$(2) \quad \text{Flux}(\widetilde{M}(i), \delta(\tau, i)) = V(\tau, i) + W(\tau, i),$$

where $V(\tau, i), W(\tau, i) \in \mathbb{R}^3$ are vectors such that $\lim_{i \rightarrow \infty} V(\tau, i) = (12\tau, 0, 0)$ and $\|W(\tau, i)\|$ is bounded by a constant independent of i, τ . Assuming equation (2), the desired contradiction that will give item (5) of the theorem comes from the fact that the angle between the flux vector $F_{M(i)}$ of the $M(i)$ and its horizontal projection $h(i)$ is invariant under translations, homotheties and rotations around the x_3 -axis. But (2) implies that the corresponding angles for the flux vectors of the surfaces $\widetilde{M}(i)$ tend to zero as $i \rightarrow \infty$ and $\tau \rightarrow \infty$, which contradicts that the $h(i)$ were assumed to be bounded. To finish this sketch of the proof of item (5) of Theorem 3.1 we will give details on how to construct the connection loops $\delta(\tau, i)$. Each of these connection loops consists of four consecutive arcs $\alpha_1(\tau, i), L(\tau, i), \alpha_2(\tau, i), \widetilde{L}(\tau, i)$ contained in $\widetilde{M}(i)$ with the following properties:

- (P5) $\alpha_1(\tau, i)$ is a short curve close to the axis of the highly sheeted vertical helicoid that is forming nearby $\vec{0}$ by property (P3) above, which goes down exactly one level in the double staircase structure occurring around $\vec{0}$.

- (P6) $L(\tau, i)$, $\tilde{L}(\tau, i)$ are approximations of a horizontal segment from $\vec{0}$ to \tilde{q} , whose extrema near $\vec{0}$ coincide with the extrema of $\alpha_1(\tau, i)$, and such that $\tilde{L}(\tau, i)$ lies directly above $L(\tau, i)$. These almost segments $L(\tau, i)$, $\tilde{L}(\tau, i)$ are defined in pages 23, 24 of [35], and their existence is insured by Lemma 8 in [35], which is a delicate application of Colding-Minicozzi theory; this lemma asserts that the sequence $\{\tilde{M}(i)\}_i$ is *uniformly simply connected in \mathbb{R}^3* (ULSC), which means that there exists $r > 0$ such that every component of the intersection of $\tilde{M}(i)$ with any ball in \mathbb{R}^3 of radius r is a disk. This property allows one to use Theorem 0.9 in Colding and Minicozzi [11] to conclude that after passing to a subsequence, the $\tilde{M}(i)$ converges to the foliation \mathcal{L} of \mathbb{R}^3 by horizontal planes with singular set of convergence $S(\mathcal{L}) = \Gamma \cup \Gamma'$ being the x_3 -axis Γ and the vertical straight line Γ' passing through \tilde{q} . Once this limit foliation result is established, the almost-straight-line, almost-horizontal segments $L(\tau, i)$, $\tilde{L}(\tau, i)$ satisfying (P6), as well as the fourth arc $\alpha_2(\tau, i)$ in $\delta(\tau, i)$ satisfying the following property, are easy to construct; for similar constructions, see the Lamination Metric Theorem by Meeks (Theorem 2 in [31]).
- (P7) $\alpha_2(\tau, i)$ is an embedded arc connecting the end points of $L(\tau, i)$, $\tilde{L}(\tau, i)$ nearby \tilde{q} , with length bounded from above by a constant that does not depend either on i or on τ .

Assuming the connection loops $\delta(\tau, i) = \alpha_1(\tau, i) \cup L(\tau, i) \cup \alpha_2(\tau, i) \cup \tilde{L}(\tau, i)$ are constructed verifying (P5), (P6), (P7), then the decomposition in (2) reduces to defining $V(\tau, i)$ as the flux of $\tilde{M}(i)$ along $L(\tau, i) \cup \tilde{L}(\tau, i)$, and $W(\tau, i)$ as the flux of $\tilde{M}(i)$ along $\alpha_1(\tau, i) \cup \alpha_2(\tau, i)$. In summary, to conclude the proof of item (5) of Theorem 3.1 one needs to check that the sequence $\{\tilde{M}(i)\}_i$ satisfies the hypotheses of Theorem 0.9 in [11] (i.e., that $\{\tilde{M}(i)\}_i$ is ULSC on every compact subset of \mathbb{R}^3 ; see equation (0.1) in [11] for this notion). This proof of this property starts by demonstrating the following statement (Lemma 6 in [35]):

- (P8) There exist $a(i) < 0 < b(i)$ such that for every extrinsic ball B of radius 1, the intersection of $\tilde{M}(i)$ with the portion of B inside a horizontal slab $S(a(i), b(i)) = \{(x_1, x_2, x_3) \mid a(i) < x_3 < b(i)\}$ is simply connected.

It is worth explaining how the numbers $a(i), b(i)$ in (P8) are chosen in order to understand why property (P8) holds. We denote by

$$\overline{\mathbb{B}}(x_0, r) = \{x \in \mathbb{R}^3 \mid \|x - x_0\| \leq r\}$$

the closed Euclidean ball centered at $x_0 \in \mathbb{R}^3$ with radius $r > 0$. We choose $a(i) < 0 < b(i)$ so that $\tilde{M}(i) \cap \overline{\mathbb{B}}(\vec{0}, 1)$ contains an open arc $\beta(i)$ passing through $\vec{0}$ connecting $\{x_3 = a(i)\} \cap \partial\mathbb{B}(\vec{0}, 1)$ to $\{x_3 = b(i)\} \cap \partial\mathbb{B}(\vec{0}, 1)$, and $\tilde{M}(i) \cap \overline{\mathbb{B}}(\tilde{q}, 1)$ contains another open arc passing through \tilde{q} connecting $\{x_3 = a(i)\} \cap \partial\mathbb{B}(\tilde{q}, 1)$

to $\{x_3 = b(i)\} \cap \partial \mathbb{B}(\tilde{q}, 1)$, and $S(a(i), b(i))$ is maximal with this property. Statement (P8) is proven by contradiction: the existence of a homotopically non-trivial curve $\gamma(i) \subset \widetilde{M}(i) \cap S(a(i), b(i)) \cap B$ for some extrinsic ball B of radius 1 and a standard area-minimization construction using $\widetilde{M}(i)$ as a barrier allows us to find a properly embedded, noncompact, orientable stable minimal surface $\Delta(i)$ in $S(a(i), b(i)) - \widetilde{M}(i)$ with $\partial \Delta(i) = \gamma(i)$. As the distance from $\vec{0}$ to \tilde{q} is $6\tau > 6\tau_0$ (this τ_0 was defined in Property (P4)), then the distance from B to at least one of the vertical cylinders $C(\vec{0}, 1)$, $C(\tilde{q}, 1)$ of radius 1 with axes passing through $\vec{0}, \tilde{q}$ respectively, is larger than τ_0 . (We can assume $\text{dist}(B, C(\vec{0}, 1)) > \tau_0$ as the other case can be solved similarly.) By Property (P4), $\Delta(i) \cap C(\vec{0}, 1)$ is a union of horizontal graphs, all contained in $S(a(i), b(i))$. These graphs cross the arc $\beta(i)$, which is a contradiction that proves Property (P8).

With Property (P8) in hand, the next step shows that there exists some $\varepsilon > 0$ independent of i so that $S(-\varepsilon, \varepsilon) = \{(x_1, x_2, x_3) : |x_3| < \varepsilon\}$ is contained in $S(a(i), b(i))$ (Lemma 7 in [35]); this is essentially a consequence of the one-sided curvature estimates in [10]. From here we conclude

(P9) The origin is a *singular point* for the sequence $\{\widetilde{M}(i)\}_i$ (i.e., the Gaussian curvatures of the $\widetilde{M}(i)$ near $\vec{0}$ blow-up) and there exist constants $r, \delta > 0$ so that for every extrinsic ball B of radius r whose center is closer than δ from $\vec{0}$, the intersection of $\widetilde{M}(i)$ with B consists of compact disks with boundary in ∂B . With the notation in Colding-Minicozzi [11], this can be abbreviated by saying that $\vec{0} \in \mathcal{S}_{\text{ulsc}}$.

In this situation one can apply the no mixing Theorem 0.4 in [11] to conclude that every singular point for the sequence $\{\widetilde{M}(i)\}_i$ is in $\mathcal{S}_{\text{ulsc}}$, which in turn implies by Theorem 0.9 in [11] the desired convergence of the $\widetilde{M}(i)$ to the foliation \mathcal{L} of \mathbb{R}^3 by horizontal planes with singular set consisting of $\Gamma \cup \Gamma'$.

We remark that in order to apply Theorem 0.9 in [11], one needs to check that some component of the intersection of $\widetilde{M}(i)$ with a ball of fixed size centered at $\vec{0}$ is not a disk; this property holds since otherwise, Theorem 0.1 in [10] would lead to the convergence of (a subsequence of) the $\widetilde{M}(i)$ to the same horizontal foliation \mathcal{L} , but with singular set consisting solely of Γ . This contradicts that the tangent plane of the $\widetilde{M}(i)$ at \tilde{q} is vertical for every i .

We also remark that the above argument does not really need that $\{\widetilde{M}(i)\}_i$ is ULSC in \mathbb{R}^3 , but only that $\{\widetilde{M}(i)\}_i$ is ULSC on every compact subset of \mathbb{R}^3 , as defined in equation (0.1) in [11]; our argument in [35] to prove that $\{\widetilde{M}(i)\}_i$ is ULSC in \mathbb{R}^3 is different from the one presented here, as it does not use the Non-Mixing Theorem 0.4 in [11] but instead, uses a blow-up argument on the scale of topology to produce a new limit object of a blow-up sequence of the $\widetilde{M}(i)$ and then finds a contradiction in all possible such limits (proof of Assertion 2 of [35]). This finishes our sketch of the proof of item (5) of Theorem 3.1.

As for the proof of item (6) of the theorem, the fact that the Gaussian curvature function K_M of a surface $M \in \mathcal{M}$ is bounded implies that M admits an embedded regular neighborhood of radius $1/\sup \sqrt{|K_M|}$ (see Meeks and Rosenberg [39]). This clearly gives that the vertical spacing between consecutive ends is bounded from below. To see why the spacing is bounded from above, one first checks that for every two consecutive ends of M asymptotic to horizontal planes Π_n, Π_{n+1} , there exists a point $p_n \in M \cap \{x_3(\Pi_n) < x_3 < x_3(\Pi_{n+1})\}$ such that the tangent plane to M at p_n is vertical. Next one shows that given $\varepsilon > 0$ fixed and sufficiently small, there exists a point $q_n \in M$ at intrinsic distance less than 2 from p_n such that $|K_M(q_n)| > \varepsilon$. (This is a flux argument, since the tangent plane at p_n is vertical and the vertical component of the flux of M is normalized to be 1.) As M has bounded Gaussian curvature and admits an embedded regular neighborhood of fixed radius, then the translated surfaces $M - q_n$ converge (after passing to a subsequence) with multiplicity one to a connected, nonflat, properly embedded minimal planar domain M_∞ , whose (nonconstant) Gauss map omits the vertical directions. If the spacing between consecutive middle ends of M is unbounded, then one can produce such a limit surface M_∞ with a top or bottom end that is of catenoidal type with vertical limiting normal vector; this implies that M_∞ has vertical flux, and in this situation one can use a variation of the López-Ros deformation argument [30] on M to find a contradiction. For details, see page 36 of [35]. Similar reasoning shows that the spacing between consecutive ends for a sequence of surfaces $\{M_n\}_n \subset \mathcal{M}$ can be bounded from above and below by positive constants that only depend on upper and nonzero lower bounds of the horizontal component of the flux vector of the M_n . Now item (6) of the theorem is proved.

Finally, we explain the proof of item (7) of the theorem. Take a divergent sequence $\{z_k\}_k \subset \mathbb{C}/\langle i \rangle$, and call $g_k(z) = g(z + z_k)$, where g is the stereographically projected extension of the Gauss map of a surface $M \in \mathcal{M}$. Recall that the family of functions $\{g_k\}_k$ is normal if and only if on every compact set C of $\mathbb{C}/\langle i \rangle$, the sequence of numbers $\{S_k(C)\}_k$ is bounded from above, where

$$S_k(C) = \sup \left\{ \frac{|g'_k(z)|}{1 + |g_k(z)|^2} \mid z \in C \right\}.$$

As the height differential of M is dz , then the spherical derivative $\frac{|g'(z)|}{1+|g(z)|^2}$ of g is, up to a constant, the square root of the Gaussian curvature of M at the point corresponding to z , which is bounded by item (5) of the theorem. Thus, there exists a meromorphic function $g_\infty: \mathbb{C}/\langle i \rangle \rightarrow \mathbb{C} \cup \{\infty\}$ so that after passing to a subsequence, the g_k converge uniformly on compact subsets of $\mathbb{C}/\langle i \rangle$ to g_∞ . Note that g_∞ cannot be constant since the z_k is at bounded distance in $\mathbb{C}/\langle i \rangle$ from consecutive ends of M by item (6), where g has zeros and poles.

As g_∞ is not constant, then g_∞ has only second order zeros and poles by Hurwitz's theorem. The fact that g_∞ corresponds to the Gauss map of a surface $M_\infty \in \mathcal{M}$ is straightforward; in fact, M_∞ is a limit of an appropriately chosen sequence of translations of M . This completes our discussion of the proof of Theorem 3.1. \square

Coming back to our discussion of minimal planar domains in \mathbb{R}^3 , we must rule out the case of one limit end. This was done in Theorem 1 of [36], which we state below.

THEOREM 3.2. *If M is a connected, properly embedded minimal surface in \mathbb{R}^3 with finite genus, then one of the following possibilities holds:*

- (1) M is a plane;
- (2) M has one end and is asymptotic to the end of a helicoid;
- (3) M has a finite number of ends greater than one, has finite total curvature and each end of M is asymptotic to a plane or to the end of a catenoid;
- (4) M has two limit ends.

Furthermore, M has bounded Gaussian curvature and is conformally diffeomorphic to a compact Riemann surface punctured in a countable closed subset that has exactly two limit points if the subset is infinite.

The proof of Theorem 3.2 depends on the previous Theorem 3.1, as well as on the Limit Lamination Theorem 0.9 of Colding and Minicozzi [11].

The discussion in this section completes the reduction of the proof of the main Theorem 1.2 to that of Assertion 1.3.

4. Jacobi functions on a minimal surface

The first variation of area allows one to consider a minimal surface $M \subset \mathbb{R}^3$ to be a critical point for the area functional acting on compactly supported (normal) variations. The second variation of area is governed by the *stability* or *Jacobi operator* $L = \Delta + |\sigma|^2 = \Delta - 2K$, where Δ denotes the intrinsic Laplacian on M , $|\sigma|^2$ is the square of the norm of its second fundamental form and K is its Gaussian curvature function. L is a linear Schrödinger operator whose potential $|\sigma|^2 = |\nabla N|^2$ is associated to the Gauss map $N: M \rightarrow \mathbb{S}^2$ of M . The holomorphicity of N is crucial in understanding the functions in the kernel $\mathcal{J}(M)$ of L (so-called *Jacobi functions*), which correspond to normal parts of infinitesimal deformations of M through minimal surfaces. In terms of the stereographically projected Gauss map g of M , the Jacobi equation $Lv = 0$ can be written as

$$(3) \quad v_{z\bar{z}} + 2 \frac{|g'|^2}{(1 + |g|^2)^2} v = 0,$$

where z is a local conformal coordinate on M . Note that since $N: M \rightarrow \mathbb{S}^2$ is harmonic, then $\Delta N + |\nabla N|^2 N = 0$ and thus, $\mathcal{J}(M)$ always contains the space $L(N)$ of linear functions of the components of N (which we will refer to as *linear Jacobi functions*):

$$L(N) = \{\langle N, a \rangle \mid a \in \mathbb{R}^3\} \subset \mathcal{J}(M).$$

Next we briefly recall some well-known facts about the Weierstrass representation; see, e.g., Osserman [43], [44]. Besides the (meromorphic) stereographic projection g of the Gauss map, we can associate to every minimal surface $M \subset \mathbb{R}^3$ a holomorphic differential dh (not necessarily exact) so that M can be parametrized as $X: M \rightarrow \mathbb{R}^3$, $X(z) = \Re \int^z \Psi$, where

$$(4) \quad \Psi = \left(\frac{1}{2} \left(\frac{1}{g} - g \right), \frac{i}{2} \left(\frac{1}{g} + g \right), 1 \right) dh;$$

we call (g, dh) the *Weierstrass pair* of M . The so-called *period problem* for (g, dh) amounts to checking that $\Re \int_{\Gamma} \Psi = 0$ for each closed curve $\Gamma \subset M$. A simple algebraic calculation demonstrates that this vanishing period condition is equivalent to the following one for all closed curves $\Gamma \subset M$:

$$(5) \quad \overline{\int_{\Gamma} \frac{dh}{g}} = \int_{\Gamma} g dh, \quad \Re \int_{\Gamma} dh = 0.$$

Suppose that $t \mapsto M_t$ is a (smooth) deformation of $M_0 = M$ by minimal surfaces. Away from the set $B(N)$ of branch points of the Gauss map of M , we can use g as a local conformal coordinate for M_t , which gives a Weierstrass pair $(g, dh(t))$ with $dh(0) = dh$. Since the set of meromorphic differentials on $M - B(N)$ is a linear space and the \mathbb{C}^3 -valued differential form Ψ in (4) depends linearly on dh , a formal derivation in (4) with respect to t at $t = 0$ gives rise to a Weierstrass pair

$$\left(g, \hat{dh} = \left. \frac{d}{dt} \right|_0 dh(t) \right).$$

The pair (g, \hat{dh}) turns out to solve the period problem, defining a branched minimal immersion X_v (possibly constant) with the same Gauss map as M . After identification of the space of infinitesimal deformations of M by minimal surfaces with the space $\mathcal{J}(M)$ of Jacobi functions, we have a correspondence

$$(6) \quad v \in \mathcal{J}(M) \mapsto X_v = \Re \int^z \left(\frac{1}{2} \left(\frac{1}{g} - g \right), \frac{i}{2} \left(\frac{1}{g} + g \right), 1 \right) \hat{dh}.$$

This correspondence was studied by Montiel and Ros [42] (see also Ejiri and Kotani [15]), who wrote down explicitly X_v, \hat{dh} in terms of v as

$$(7) \quad \left. \begin{aligned} X_v &= vN + \frac{1}{|N_z|^2} \{v_z N_{\bar{z}} + v_{\bar{z}} N_z\} : M - B(N) \rightarrow \mathbb{R}^3, \\ \hat{dh} &= \frac{g}{g'} \left(v_{zz} + \left(\frac{2\bar{g}g'}{1+|g|^2} - \frac{g''}{g'} \right) v_z \right) dz, \end{aligned} \right\}$$

where z is any local conformal coordinate on M . The Gauss map of X_v is N and its support function $\langle X_v, N \rangle$ is v . Linear Jacobi functions $v \in L(N)$ produce constant maps X_v (and vice versa), and the correspondence $v \mapsto X_v$ is a linear isomorphism from the linear space of Jacobi functions on M modulo $L(N)$ onto the linear space of all branched minimal immersions $X : M - B(N) \rightarrow \mathbb{R}^3$ with Gauss map N modulo the constant maps.

A direct consequence of the derivation in $t = 0$ of (4) is that the map $v \in \mathcal{J}(M) \mapsto X_v$ behaves well with respect to fluxes, as stated in the following lemma. Note that since the flux of a minimal surface is a homological invariant, we can take the curve Γ described below away from the branch points of the Gauss map.

LEMMA 4.1. *In the above setting, let $\{\psi_t : M \rightarrow \mathbb{R}^3\}_{|t| < \epsilon}$ be a smooth deformation of M by minimal surfaces, and denote by $v \in \mathcal{J}(M)$ the normal part of its variational field. For any fixed closed curve $\Gamma \subset M$, we have*

$$(8) \quad \left. \frac{d}{dt} \right|_{t=0} \text{Flux}(\psi_t, \Gamma) = \text{Flux}(X_v, \Gamma).$$

Definition 4.2. Given a minimal surface M , the conjugate Jacobi function v^* of a Jacobi function $v \in \mathcal{J}(M)$ is defined (locally) as the support function $\langle (X_v)^*, N \rangle$ of the conjugate minimal immersion $(X_v)^*$. Recall that such a conjugate minimal immersion is an isometric minimal immersion of the underlying Riemannian surface, whose coordinate functions are the harmonic conjugates to the ones of X_v . Note that v^* is defined up to additive constants, and v^* is globally well defined precisely when $(X_v)^*$ is globally well defined. We define

$$\mathcal{J}_{\mathbb{C}}(M) = \{v + iv^* \mid v \in \mathcal{J}(M) \text{ and } v^* \text{ is globally defined}\}.$$

In other words, $\mathcal{J}_{\mathbb{C}}(M)$ is the space of support functions $\langle X, N \rangle$ of holomorphic maps $X : M - B(N) \rightarrow \mathbb{C}^3$ whose real and imaginary parts are orthogonal to N . (Now $\langle X, N \rangle$ denotes the usual bilinear complex product on \mathbb{C}^3 .) Since $(X_v)^{**} = -X_v$, we deduce that the conjugate Jacobi of v^* is $-v$, which endows $\mathcal{J}_{\mathbb{C}}(M)$ with a structure of complex vector space. A simple observation is that if $v \in L(N)$, then X_v is constant, which means that $(X_v)^*$ is also constant and so v^* is a (globally defined) function in $L(N)$. In other

words,

$$(9) \quad L_{\mathbb{C}}(N) := \{ \langle N, a \rangle \mid a \in \mathbb{C}^3 \} \subseteq \mathcal{J}_{\mathbb{C}}(M).$$

For a general $v \in \mathcal{J}(M)$, the map $(X_v)^*$ is globally well defined provided that all its period vectors along closed curves on M vanish (equivalently, when $\text{Flux}(X_v, \Gamma) = 0$ for all closed curves $\Gamma \subset M$). As a direct consequence of Lemma 4.1, we have the following statement.

LEMMA 4.3. *Given a minimal surface $M \subset \mathbb{R}^3$ and $v \in \mathcal{J}(M)$, the conjugate Jacobi function v^* of v is globally defined on M if and only if v preserves infinitesimally the flux vector along every closed curve on M .*

Remark 4.4. As we will see in Sections 4.1 and 4.2, if M is a minimal surface satisfying the hypotheses of Assertion 1.3 and M is not a Riemann minimal example, then it admits a nonzero Jacobi function called its *Shiffman function* S_M , whose Jacobi conjugate S_M^* is globally defined. In particular, if there exists a smooth deformation M_t of M by minimal surfaces such that for every t , M_t also admits a Shiffman function S_{M_t} and the normal part of $\frac{d}{dt}M_t$ equals the Shiffman function S_{M_t} , then the flux of M_t along any closed curve will be independent of t .

4.1. *The Shiffman Jacobi function.* Next we recall the definition and some basic properties of the Shiffman function. In 1956, Shiffman [52] introduced a Jacobi function that incorporates the curvature variation of the parallel sections of a minimal surface. This function can be defined locally: assume that $(g(z), dh = dz)$ is the Weierstrass data of a minimal surface $M \subset \mathbb{R}^3$, where z is a local conformal coordinate in M . (In particular, g has no zeros or poles and any minimal surface admits such a local representation around a point with nonvertical normal vector.) The induced metric ds^2 by the inner product of \mathbb{R}^3 is $ds^2 = \Lambda^2|dz|^2$, where $\Lambda = \frac{1}{2}(|g| + |g|^{-1})$. The horizontal level curves $x_3 = c$ correspond to $z_c(y) = c + iy$ in the z -plane (here $z = x + iy$ with $x, y \in \mathbb{R}$), and the planar curvature of this level curve is

$$(10) \quad \kappa_c(y) = \left[\frac{|g|}{1 + |g|^2} \Re \left(\frac{g'}{g} \right) \right] \Big|_{z=z_c(y)},$$

where the prime stands for derivative with respect to z .

Definition 4.5. We define the *Shiffman function* of M as

$$(11) \quad S_M = \Lambda \frac{\partial \kappa_c}{\partial y} = \Im \left[\frac{3}{2} \left(\frac{g'}{g} \right)^2 - \frac{g''}{g} - \frac{1}{1 + |g|^2} \left(\frac{g'}{g} \right)^2 \right],$$

where \Im stands for imaginary part.

Since Λ is a positive function, the zeros of S_M coincide with the critical points of $\kappa_c(y)$. Thus, S_M vanishes identically if and only if M is foliated by pieces of circles and straight lines in horizontal planes. In a posthumously published paper, Riemann [47], [48] classified all minimal surfaces with such a foliation property: they reduce to the plane, catenoid, helicoid and the one-parameter family of surfaces defined in Section 2. A crucial property is that $\Delta S_M + |\sigma|^2 S_M = 0$; i.e., S_M is a Jacobi function on M . Shiffman himself exploited this property when he proved that if a minimal annulus M is bounded by two circles in parallel planes, then M is foliated by circles in the intermediate planes; see also Fang [17] for other applications of the Shiffman function.

Coming back to our properly embedded minimal surface $M \subset \mathbb{R}^3$ in the family \mathcal{M} described in Assertion 1.3, we deduce from Theorem 3.1 (with the notation in that theorem) that M intersects transversally every horizontal plane and hence, its Shiffman function S_M can be defined globally on $M = (\mathbb{C}/\langle i \rangle) - \{p_j, q_j\}_j$. Expressing g locally around a zero p_j , it is straightforward to check that S_M is bounded around p_j , with continuous extension $S_M(p_j) = -\frac{1}{6} \Im \left(\frac{g^{(5)}(p_j)}{g''(p_j)} \right)$, and a similar result holds for poles of g . Hence S_M can be viewed as a continuous function on the cylinder $\mathbb{C}/\langle i \rangle$. Since $v = S_M$ solves the Jacobi equation (3) and when expressed around p_j or q_j the Jacobi equation has the form $\Delta v + qv = 0$ for q smooth (here Δ refers to the Laplacian in the flat metric on $\mathbb{C}/\langle i \rangle$), elliptic regularity implies that S_M extends smoothly to $\mathbb{C}/\langle i \rangle$. In fact, Corollary 4.15 below implies that S_M is bounded on $\mathbb{C}/\langle i \rangle$.

4.2. *The space of allowed Gauss maps and their infinitesimal deformations.* Our method to prove that every $M \in \mathcal{M}$ is a Riemann minimal example is based on the fact that the Shiffman function can be integrated at any $M \in \mathcal{M}$, in a similar manner as a vector field on a manifold admits integral curves passing through any point. To construct a framework in which this last sentence makes sense, we need some definitions. Since surfaces $M \in \mathcal{M}$ have Weierstrass data (g, dz) on $\mathbb{C}/\langle i \rangle$, all the information we need for understanding M is contained in its Gauss map g . We start by defining the appropriate space of functions where these Gauss maps naturally reside.

Definition 4.6. A meromorphic function $g: \mathbb{C}/\langle i \rangle \rightarrow \mathbb{C} \cup \{\infty\}$ will be called *quasiperiodic* if it satisfies the following two conditions:

- (1) There exists a constant $C > 0$ such that the distance between any two distinct points in $g^{-1}(\{0, \infty\}) \subset \mathbb{C}/\langle i \rangle$ is at least C and given any $p \in g^{-1}(\{0, \infty\})$, there exists at least one point in $g^{-1}(\{0, \infty\}) - \{p\}$ of distance less than $1/C$ from p .
- (2) For every divergent sequence $\{z_k\}_k \subset \mathbb{C}/\langle i \rangle$, there exists a subsequence of the meromorphic functions $g_k(z) = g(z + z_k)$ that converges uniformly on

compact subsets of $\mathbb{C}/\langle i \rangle$ to a nonconstant meromorphic function $g_\infty : \mathbb{C}/\langle i \rangle \rightarrow \mathbb{C} \cup \{\infty\}$. (In particular, g_∞ is quasiperiodic as well.)

Remark 4.7. A direct consequence of the last definition is that if $g : \mathbb{C}/\langle i \rangle \rightarrow \mathbb{C} \cup \{\infty\}$ is a quasiperiodic meromorphic function, then there is a bound on the order of the zeros and a bound on the order of the poles of g , as well as uniform bounds away from zero and from above for the coefficients of z^k (resp. z^{-k}) in the series expansion of g and its derivatives around any zero (resp. pole) of order k of g .

We consider the space of meromorphic functions

$$\mathcal{W} = \left\{ g : \mathbb{C}/\langle i \rangle \rightarrow \mathbb{C} \cup \{\infty\} \text{ quasiperiodic} : (g) = \prod_{j \in \mathbb{Z}} p_j^2 q_j^{-2} \right\},$$

where (g) denotes the divisor of zeros and poles of g on $\mathbb{C}/\langle i \rangle$. Statement (7) of Theorem 3.1 implies that the Gauss map of every $M \in \mathcal{M}$ lies in \mathcal{W} . We endow \mathcal{W} with the topology of uniform convergence on compact sets of $\mathbb{C}/\langle i \rangle$. Given $g \in \mathcal{W}$, it follows from Remark 4.7 that any limit g_∞ of (a subsequence of) $g_k(z) = g(z + z_k)$ with $\{z_k\}_k \subset \mathbb{C}/\langle i \rangle$ being a divergent sequence satisfies that g_∞ lies in \mathcal{W} . If $g \in \mathcal{W}$ has divisor of zeros $Z = \prod_j p_j^2$, then the set $\{p_j\}_j$ is quasiperiodic in the sense that for every divergent sequence $\{z_k\}_k \subset \mathbb{C}/\langle i \rangle$, there exists a subsequence of $\{Z + z_k\}_k$ that converges in the Hausdorff distance on compact subsets of $\mathbb{C}/\langle i \rangle$ to a divisor Z_∞ in $\mathbb{C}/\langle i \rangle$ (analogously for poles).

Reciprocally, two disjoint quasiperiodic divisors $Z = \prod_j p_j^2$, $P = \prod_j q_j^2$ in $\mathbb{C}/\langle i \rangle$ define a unique quasiperiodic meromorphic function g (up to multiplicative nonzero constants) whose principal divisor is $(g) = Z/P$: existence follows from Douady and Douady [14]:

$$g(z) = \prod_{n \in \mathbb{Z}} c(n) \frac{\cosh \frac{2\pi z - p_j}{2} \cosh \frac{2\pi z - p_{j+1}}{2}}{\sinh^2 \frac{2\pi z - q_j}{2}},$$

where $c(n)$ is a nonzero complex number such that the above infinite product converges, while uniqueness can be shown as follows. Suppose $g_1, g_2 \in \mathcal{W}$ have $(g_1) = (g_2)$. Then the function $f = g_1/g_2$ is holomorphic and has no zeros in $\mathbb{C}/\langle i \rangle$. If f is unbounded on $\mathbb{C}/\langle i \rangle$, then there exists $\{z_k\}_k \subset \mathbb{C}/\langle i \rangle$ such that $f(z_k)$ diverges. Furthermore, $\{z_k\}_k$ is a divergent sequence since f has no poles. By quasiperiodicity of g_1 and g_2 , after extracting a subsequence we can assume that $f_k(z) = f(z + z_k)$ converges uniformly on compact subsets of $\mathbb{C}/\langle i \rangle$ to a meromorphic function $f_\infty : \mathbb{C}/\langle i \rangle \rightarrow \mathbb{C} \cup \{\infty\}$ that is not constant infinity. Then f_∞ has no poles by Hurwitz’s theorem, but $f_k(0) = f(z_k) \rightarrow \infty$ as $k \rightarrow \infty$, which is a contradiction. Thus f must be bounded, and so f is constant by Liouville’s theorem.

To each $g \in \mathcal{W}$ we associate the quasiperiodic set of its zeros p_j and poles q_j in $\mathbb{C}/\langle i \rangle$ (we choose an ordering for this set of zeros and poles), together

with the value of g at a prescribed point $z_0 \in (\mathbb{C}/\langle i \rangle) - g^{-1}(\{0, \infty\})$. The bijective correspondence

$$(12) \quad g \mapsto (p_j, q_j, g(z_0)) \in [\prod_{j \in \mathbb{Z}} (\mathbb{C}/\langle i \rangle)] \times (\mathbb{C} - \{0\})$$

allows us to identify \mathcal{W} (endowed with the uniform topology on compact sets on $\mathbb{C}/\langle i \rangle$) as the space $[\prod_{j \in \mathbb{Z}} (\mathbb{C}/\langle i \rangle)] \times (\mathbb{C} - \{0\})$ (endowed with its metrizable product topology).

Given $\varepsilon > 0$, we denote by $\mathbb{D}(\varepsilon) = \{t \in \mathbb{C} \mid |t| < \varepsilon\}$. We say that a curve $t \in \mathbb{D}(\varepsilon) \rightarrow g_t \in \mathcal{W}$ with $g_0 = g$ is *holomorphic* if the corresponding functions $p_j(t), q_j(t), g_t(z_0)$ depend holomorphically on t . In this case, the function $\dot{g}: \mathbb{C}/\langle i \rangle \rightarrow \mathbb{C} \cup \{\infty\}$ given by $z \in \mathbb{C}/\langle i \rangle \mapsto \left. \frac{d}{dt} \right|_{t=0} g_t(z)$ is meromorphic on $\mathbb{C}/\langle i \rangle$. We will call \dot{g} the *infinitesimal deformation of g associated to the curve $t \mapsto g_t$* .

If \dot{g} is the infinitesimal deformation of $g = g_0 \in \mathcal{W}$ associated to the curve $t \mapsto g_t$ and g has principal divisor $(g) = \prod_j p_j^2 q_j^{-2}$, then the principal divisor of \dot{g} clearly satisfies

$$(13) \quad (\dot{g}) \geq \prod_j p_j q_j^{-3}.$$

In particular, if \dot{g} is constant, then $\dot{g} = 0$. Reciprocally, if f is a meromorphic function on $\mathbb{C}/\langle i \rangle$ and its principal divisor verifies $(f) \geq \prod_j p_j q_j^{-3}$, then f is the infinitesimal deformation of g associated to a holomorphic curve $t \mapsto g_t \in \mathcal{W}$ with $g_0 = g$. (Construct g_t up to a multiplicative constant $a(t) \in \mathbb{C} - \{0\}$ from its quasiperiodic principal divisor $(g_t) = \prod_j p_j(t)^2 q_j(t)^{-2}$, where $p_j(t), q_j(t)$ are holomorphic curves in $\mathbb{C}/\langle i \rangle$ such that $p_j(0) = p_j, q_j(0) = q_j$ and the order of $t \mapsto p_j(t)$ at p_j is chosen according to the order of f at p_j for each j ; then choose the constant $a(t)$ depending holomorphically on t such that $a(0) = g(z_0)$.)

We will denote the set of infinitesimal deformations of g associated to holomorphic curves by

$$T_g \mathcal{W} = \left\{ f: \mathbb{C}/\langle i \rangle \rightarrow \mathbb{C} \cup \{\infty\} \text{ meromorphic} \mid (f) \geq \prod_j p_j q_j^{-3} \right\}.$$

By the above arguments, $T_g \mathcal{W}$ is a complex linear space.

Remark 4.8. Note that $g, g' \in T_g \mathcal{W}$ are respectively the infinitesimal deformations at $t = 0$ associated to the holomorphic curves $t \mapsto (t + 1)g(z)$, $t \mapsto g(z + t)$. (From now on, we will denote by prime ' the derivation with respect to the conformal coordinate z .)

Let $\gamma = \{it \mid t \in [0, 1]\}$ be the generator of the homology of the cylinder $\mathbb{C}/\langle i \rangle$. Given $g \in \mathcal{W}$, the pair $(g, dh = dz)$ is the Weierstrass data of a complete, immersed minimal surface in \mathbb{R}^3 with embedded horizontal planar ends (each one considered separately) at the zeros and poles of g if and only

if the corresponding period problem (5) can be solved. In our setting, this is equivalent to solving the following equations:

$$(14) \quad \overline{\int_{\gamma} \frac{dz}{g}} = \int_{\gamma} g dz, \quad \text{Res}_{p_j} \left(\frac{dz}{g} \right) = \text{Res}_{q_j}(g dz) = 0 \quad \forall j \in \mathbb{Z}.$$

The above equalities suggest defining the *period map* $\text{Per}: \mathcal{W} \rightarrow \mathbb{C}^2 \times \mathbb{C}^{\mathbb{Z}} \times \mathbb{C}^{\mathbb{Z}}$ by

$$(15) \quad \text{Per}(g) = \left(\int_{\gamma} \frac{dz}{g}, \int_{\gamma} g dz, \left\{ \text{Res}_{p_j} \left(\frac{dz}{g} \right) \right\}_j, \{ \text{Res}_{q_j}(g dz) \}_j \right).$$

Inside \mathcal{W} we have the space of *immersed* minimal surfaces, i.e., those $g \in \mathcal{W}$ such that (g, dz) solves the period problem:

$$(16) \quad \mathcal{M}_{\text{imm}} = \text{Per}^{-1}\{(a, \bar{a}, 0, 0) \mid a \in \mathbb{C}\}.$$

Definition 4.9. A *quasiperiodic, immersed minimal surface of Riemann type* is a minimal surface $M \subset \mathbb{R}^3$ that admits a Weierstrass pair of the form (g, dz) on $(\mathbb{C}/\langle i \rangle) - g^{-1}(\{0, \infty\})$ where g lies in \mathcal{M}_{imm} .

Remark 4.10. Since $\text{Residue}_{p_j} \left(\frac{dz}{g} \right) = -\frac{2}{3} \frac{g'''(p_j)}{g''(p_j)^2}$ and $\text{Residue}_{q_j}(g dz) = -\frac{2}{3} \frac{(1/g)'''(q_j)}{(1/g)''(q_j)^2}$, the fact that the pair (g, dz) closes the period at a zero p_j (resp. at a pole q_j) of g can be stated equivalently as $g'''(p_j) = 0$ (resp. $(1/g)'''(q_j) = 0$).

Definition 4.11. A *Jacobi function* associated to an element $g \in \mathcal{W}$ is a map $v: (\mathbb{C}/\langle i \rangle) - g^{-1}(\{0, \infty\}) \rightarrow \mathbb{R}$ that satisfies equation (3) on $(\mathbb{C}/\langle i \rangle) - g^{-1}(\{0, \infty\})$. The linear space of real-valued Jacobi functions associated to g will be denoted by $\mathcal{J}(g)$. By equations (6) and (7), every $v \in \mathcal{J}(g)$ gives rise to a branched minimal immersion $X_v: (\mathbb{C}/\langle i \rangle) - B(g) \rightarrow \mathbb{R}^3$ with (complex) Gauss map g , where $B(g)$ is the branch locus of g . The *conjugate Jacobi function* v^* of $v \in \mathcal{J}(g)$ is the (locally defined) support function of the conjugate minimal immersion $(X_v)^*$ of X_v .

We consider the complex linear space

$$\mathcal{J}_{\mathbb{C}}(g) = \{v + iv^* \mid v \in \mathcal{J}(g) \text{ and } v^* \text{ is globally defined}\}.$$

$\mathcal{J}_{\mathbb{C}}(g)$ is the space of support functions $\langle X, N \rangle$ of holomorphic maps

$$X: (\mathbb{C}/\langle i \rangle) - B(g) \rightarrow \mathbb{C}^3$$

whose real and imaginary parts are orthogonal to $N = \left(\frac{2g}{|g|^2+1}, \frac{|g|^2-1}{|g|^2+1} \right) \in \mathbb{C} \times \mathbb{R} \equiv \mathbb{R}^3$. The linear functions of g form a complex linear subspace of $\mathcal{J}_{\mathbb{C}}(g)$:

$$L_{\mathbb{C}}(g) := \{\langle N, a \rangle \mid a \in \mathbb{C}^3\} \subseteq \mathcal{J}_{\mathbb{C}}(g).$$

For later purposes, it is useful to recognize a basis of $L_{\mathbb{C}}(g)$. Writing $a = (a_1, a_2, a_3)$ with $a_i \in \mathbb{C}$ and using that g is the stereographic projection of N

from the north pole, we have

$$\begin{aligned}
 (17) \quad \langle N, a \rangle &= \frac{2}{1 + |g|^2} [a_1 \Re(g) + a_2 \Im(g)] + a_3 \frac{|g|^2 - 1}{|g|^2 + 1} \\
 &= \frac{1}{|g|^2 + 1} (Ag + B\bar{g}) + a_3 \frac{|g|^2 - 1}{|g|^2 + 1},
 \end{aligned}$$

where $A, B \in \mathbb{C}$ are determined by the equations $2a_1 = A + B$, $2a_2 = i(A - B)$. In particular, $\frac{g}{|g|^2 + 1}, \frac{\bar{g}}{|g|^2 + 1}, \frac{|g|^2 - 1}{|g|^2 + 1}$ is a basis of $L_{\mathbb{C}}(g)$. We will use this fact in the proof of Corollary 4.15 below.

Definition 4.12. A Jacobi function $v \in \mathcal{J}(g)$ (resp. $\mathcal{J}_{\mathbb{C}}(g)$) is said to be *quasiperiodic* if for every divergent sequence $\{z_k\}_k \subset \mathbb{C}/\langle i \rangle$, there exists a subsequence of the functions $v_k(z) = v(z + z_k)$ that converges uniformly on compact subsets of $(\mathbb{C}/\langle i \rangle) - g_{\infty}^{-1}(\{0, \infty\})$ to a function v_{∞} , where $g_{\infty} \in \mathcal{W}$ is the limit of (a subsequence of) $\{g_k(z) = g(z + z_k)\}_k$, which exists since g is quasiperiodic. Note that $v_{\infty} \in \mathcal{J}(g_{\infty})$ (resp. $\mathcal{J}_{\mathbb{C}}(g_{\infty})$) and that if v_{∞} is constant, then $v_{\infty} = 0$.

Next we give a condition for a Jacobi function to have a globally defined conjugate Jacobi function.

PROPOSITION 4.13. *Given $g \in \mathcal{M}_{\text{imm}}$, we have*

- (1) *Let $h: \mathbb{C}/\langle i \rangle \rightarrow \mathbb{C} \cup \{\infty\}$ be a meromorphic function that is a rational expression of g and its derivatives with respect to z up to some order such that*

$$(18) \quad \dot{g}(h) = \left(\frac{g^3 h'}{2g'} \right)'$$

belongs to $T_g \mathcal{W}$. Then, the map

$$(19) \quad f(h) = \frac{g^2 h'}{g'} + \frac{2gh}{1 + |g|^2}$$

lies in $\mathcal{J}_{\mathbb{C}}(g)$, is quasiperiodic and bounded on $\mathbb{C}/\langle i \rangle$. Furthermore, for every closed curve $\Gamma \subset \mathbb{C}/\langle i \rangle$,

$$(20) \quad \int_{\Gamma} \frac{\dot{g}(h)}{g^2} dz = \int_{\Gamma} \dot{g}(h) dz = 0.$$

- (2) *Reciprocally, if $\dot{g} \in T_g \mathcal{W}$ satisfies (20), then there exists a meromorphic function h on $\mathbb{C}/\langle i \rangle$ such that (18) holds.*

Proof. We first demonstrate item (1). Since $\dot{g}(h) \in T_g \mathcal{W}$, there exists a holomorphic curve $t \mapsto g_t \in \mathcal{W}$ such that $g_0 = g$ and $\frac{d}{dt} \Big|_{t=0} g_t = \dot{g}(h)$.

Therefore $\langle \frac{d}{dt} \Big|_0 \int^z \Psi_t, N \rangle \in \mathcal{J}_{\mathbb{C}}(g)$, where $\Psi_t = \left(\frac{1}{2}(\frac{1}{g_t} - g_t), \frac{i}{2}(\frac{1}{g_t} + g_t), 1 \right) dz$ and $N = \left(\frac{2\Re(g)}{|g|^2+1}, \frac{2\Im(g)}{|g|^2+1}, \frac{|g|^2-1}{|g|^2+1} \right)$.

A simple calculation gives

$$(21) \quad \int^z \frac{\dot{g}(h)}{g^2} dz = \frac{gh'}{2g'} + h, \quad \int^z \dot{g}(h) dz = \frac{g^3 h'}{2g'}$$

up to additive complex numbers, and then for some $a \in \mathbb{C}^3$, we have

$$\begin{aligned} \left\langle \frac{d}{dt} \Big|_0 \int^z \Psi_t, N \right\rangle &= \left\langle \int^z \left(\frac{1}{2}(-\frac{\dot{g}(h)}{g^2} - \dot{g}(h)), \frac{i}{2}(-\frac{\dot{g}(h)}{g^2} + \dot{g}(h)), 0 \right) dz, N \right\rangle \\ &= \left\langle \left(\frac{1}{2}(-\frac{gh'}{2g'} - h - \frac{g^3 h'}{2g'}), \frac{i}{2}(-\frac{gh'}{2g'} - h + \frac{g^3 h'}{2g'}), 0 \right), N \right\rangle + \langle a, N \rangle \\ &= \frac{1}{|g|^2 + 1} \left\langle \left(\frac{gh'}{2g'} + h \right) (-1, -i, 0) + \frac{g^3 h'}{2g'} (-1, i, 0), (\Re(g), \Im(g), 0) \right\rangle + \langle a, N \rangle \\ &= \frac{1}{|g|^2 + 1} \left[- \left(\frac{gh'}{2g'} + h \right) g - \frac{g^3 h'}{2g'} \bar{g} \right] + \langle a, N \rangle = -\frac{1}{2} f(h) + \langle a, N \rangle. \end{aligned}$$

In summary,

$$(22) \quad \left\langle \frac{d}{dt} \Big|_0 \int^z \Psi_t, N \right\rangle = -\frac{1}{2} f(h) + \langle a, N \rangle.$$

From (22) we deduce that f is a Jacobi function and lies in $\mathcal{J}_{\mathbb{C}}(g)$. Quasi-periodicity of $f(h)$ follows directly from the quasiperiodicity of g since h is a rational function of g and its derivatives. In order to prove that $f(h)$ is bounded on $\mathbb{C}/\langle i \rangle$, we first check that $f(h)$ is bounded around every zero and pole of g and around every zero of g' .

- (A) Suppose $z = 0$ is a zero of g . It suffices to prove that $\frac{g^2 h'}{g'} + 2gh$ is bounded around $z = 0$. Since $\frac{g^2 h'}{g'} + 2gh = \frac{(g^2 h)'}{g'}$, we must check that $(g^2 h)'$ has a zero at $z = 0$. From equation (18) we have $h' = \frac{2g'}{g^3} \dot{G}$, where \dot{G} is a primitive of $\dot{g}(h)$ defined in a neighborhood of $z = 0$. Thus, $h = \int^z \frac{2g'}{g^3} \dot{G} dz = -\frac{1}{g^2} \dot{G} + \int^z \frac{\dot{g}(h)}{g^2}$ and $(g^2 h)' = -\dot{g}(h) + \left(g^2 \int^z \frac{\dot{g}(h)}{g^2} dz \right)'$. As $\dot{g}(h)$ vanishes at $z = 0$ (here we are using that $\dot{g}(h) \in T_g \mathcal{W}$ and formula (13)), it suffices to show that $\left(g^2 \int^z \frac{\dot{g}(h)}{g^2} dz \right)'$ vanishes at $z = 0$, which clearly follows from the fact that g has an order-two zero and $\dot{g}(h)$ vanishes at $z = 0$. (Also note that this property does not depend on the constant of integration since $g(0) = 0$.)
- (B) Suppose $z = 0$ is a pole of g . Since $\dot{g}(h) \in T_g \mathcal{W}$, then (13) implies that $\left(\frac{g^3 h'}{2g'} \right)'$ has at most a order-three pole at $z = 0$. Thus, $\frac{g^2 h'}{g'}$ is bounded at $z = 0$ and so h' vanishes at $z = 0$. Now we deduce that $\frac{2gh}{1+|g|^2}$ also vanishes at $z = 0$.

- (C) Suppose $g(0) \neq 0$ and $g'(0) = 0$. Then (13) implies that $\left(\frac{g^3 h'}{2g'}\right)'$ is holomorphic at $z = 0$. Hence the branching order of h at $z = 0$ is not less than the branching order of g at the same point. Then trivially both $\frac{g^2 h'}{g'}$, $\frac{2gh}{1+|g|^2}$ are bounded at $z = 0$.

The discussion in items (A), (B) and (C) shows that if we consider the discrete set $A = g^{-1}(\{0, \infty\}) \cup (g')^{-1}(0)$, then for every $Q_j \in A$, there exist a disk $D(Q_j) \subset \mathbb{C}/\langle i \rangle$ and a positive number C_j such that $|f(h)| \leq C_j$ in $D(Q_j)$. The quasiperiodicity of g and Remark 4.7 insure that both C_j and the radius of $D(Q_j)$ can be taken independently of j . Hence to deduce that $f(h)$ is bounded on $\mathbb{C}/\langle i \rangle$, it suffices to prove that $f(h)$ is bounded in $(\mathbb{C}/\langle i \rangle) - \cup_j D(Q_j)$. This last property holds because g is quasiperiodic, h is a rational expression of g and its derivatives, and f is given in $(\mathbb{C}/\langle i \rangle) - \cup_j D(Q_j)$ in terms of g, h by the formula (19). Hence, $f(h)$ is bounded on $\mathbb{C}/\langle i \rangle$. Finally, (20) is a direct consequence of (21), and item (1) of the proposition is proved.

Concerning item (2), equation (21) together with the hypothesis (20) allow us to find a meromorphic function h on $\mathbb{C}/\langle i \rangle$ such that (18) holds. This finishes the proof. \square

Remark 4.14.

- (1) Equation (20) could be interpreted as the fact that \dot{g} lies in the kernel of the differential $d\text{Per}_g$ of the period map at $g \in \mathcal{M}_{\text{imm}}$, defined as in (15).
- (2) If one takes $h = c_1 + \frac{c_2}{g}$ in (18) with $c_1, c_2 \in \mathbb{C}$, then $\dot{g}(h) = 0$ (and vice versa). Furthermore, $f(h)$ is a complex linear combination of $\frac{g}{1+|g|^2}, \frac{\bar{g}}{1+|g|^2}$, which can be viewed as a horizontal linear function of the ‘‘Gauss map’’ g . Taking $h = \frac{1}{g}$ in (18), then $\dot{g}(h) = -\frac{1}{2}g'$ and $f(h) = \frac{1-|g|^2}{1+|g|^2}$, which is a vertical linear function of g .

COROLLARY 4.15. *Let M be a quasiperiodic, immersed minimal surface of Riemann type. Then, its Shiffman function S_M given by (11) admits a globally defined conjugate Jacobi S_M^* , and $S_M + iS_M^* = f$ is given by equation (19) for*

$$(23) \quad h = h_S = \frac{i}{2} \frac{(g')^2}{g^3}.$$

In particular,

- (1) *Both S_M, S_M^* are bounded on the cylinder $M \cup g^{-1}(\{0, \infty\})$.*
- (2) *The corresponding infinitesimal deformation $\dot{g}_S = \dot{g}(h_S) \in T_g \mathcal{W}$ is given by*

$$(24) \quad \dot{g}_S = \frac{i}{2} \left(g''' - 3 \frac{g'g''}{g} + \frac{3(g')^3}{g^2} \right).$$

- (3) *If $\dot{g}_S = 0$ on M , then both S_M, S_M^* are linear.*

Proof. Note that h defined by (23) is a rational expression of g and g' . A direct computation gives that plugging (23) into (18) we obtain (24), and that this last expression has the correct behavior expressed in (13). In particular, \dot{g}_S is the tangent vector associated to a holomorphic curve in \mathcal{W} passing through g at $t = 0$, i.e., $\dot{g}_S \in T_g\mathcal{W}$. Using Proposition 4.13, we deduce item 1 of this corollary. It only remains to check item (3): If $\dot{g}(h_S) = 0$, then (18) gives $h_S = b - \frac{c}{g^2}$ for $b, c \in \mathbb{C}$. After substitution in (19), we obtain $S_M + iS_M^* = 2c\frac{\bar{g}}{1+|g|^2} + 2b\frac{g}{1+|g|^2}$. Hence, both S_M, S_M^* are linear. \square

5. Holomorphic integration of the Shiffman function

In this section we prove that the Shiffman function S_M of a quasiperiodic, immersed minimal surface M of Riemann type can be holomorphically integrated (Theorem 5.14 below), in the sense that M can be deformed by a complex family $t \mapsto M_t$ where t moves in a disk $\mathbb{D}(\varepsilon) \subset \mathbb{C}$ centered at the origin, $M_0 = M$, such that each M_t is a quasiperiodic, immersed minimal surface of Riemann type and at any $t \in \mathbb{D}(\varepsilon) \cap \mathbb{R}$, the normal component of the variational field of $t \mapsto M_t$ is the Shiffman function of M_t . This property will be crucial in our proof of Assertion 1.3.

The approach to prove the holomorphic integration of S_M is by means of meromorphic KdV theory, as we next briefly explain. By Corollary 4.15, we can associate to S_M an infinitesimal deformation \dot{g}_S given by equation (24), which can be considered to be an evolution equation in complex time t involving certain quasiperiodic meromorphic functions in the cylinder $\mathbb{C}/\langle i \rangle$ (namely, elements in the space \mathcal{W}). The change of variables

$$(25) \quad u = -\frac{3(g')^2}{4g^2} + \frac{g''}{2g}$$

transforms (24) into the meromorphic KdV equation

$$(26) \quad \dot{u} = -u''' - 6uu'.$$

In Remark 5.6 we will motivate the reason for the change of variables (25), which could seem to be mysterious at first sight. Therefore, we are interested in finding a solution $u(z, t)$ of (26) with initial data $u(z, 0) = u(z)$ given by (25). KdV theory insures that this Cauchy problem admits a unique solution if the initial data $u(z)$ is an *algebro-geometric potential* for KdV. (See the paragraph just before Theorem 5.1 for this notion.) Although this integrability result appears to be rather standard in KdV theory, the reader interested in minimal surface theory might be unfamiliar with it. Since to our knowledge this is the first time that this theory is applied to minimal surfaces, we will include a self-contained proof of the integrability of the Cauchy problem for KdV with algebro-geometric initial data (Proposition 5.3).

Suppose for the moment that the meromorphic function $u(z)$ given by (25) for $g \in \mathcal{M}_{\text{imm}}$ is algebro-geometric, and so the solution $u(z, t)$ of (26) with $u(z, 0) = u(z)$ exists. In order to construct the desired complex family M_t of quasiperiodic, immersed minimal surfaces of Riemann type, or equivalently, their Gauss maps $g_t \in \mathcal{M}_{\text{imm}}$, we argue as follows. First note that $g = 1/y^2$ defines a meromorphic function $y(z)$ on \mathbb{C} and that (25) implies that the following Schrödinger equation in the variable z is satisfied:

$$(27) \quad y'' + uy = 0.$$

Now replace $u(z)$ by $u(z, t)$ in (27), with the unknown $y(z, t)$. We will couple this Schrödinger equation with an evolution equation in y in such a way that the integrability condition of the corresponding system of PDEs is that $u(z, t)$ satisfies (26). Thus, there exists a solution $y(z, t)$ of this coupled system of PDEs with $y(z, 0) = y(z)$. It turns out that letting $g_t(z) = 1/y(z, t)^2$, then g_t solves (24). Of course, there are many technical aspects of this construction that must be taken into account in order for g_t to define an element in \mathcal{M}_{imm} .

We finish this summary of the results in this section by indicating why $u = u(z)$ given by (25) for $g \in \mathcal{M}_{\text{imm}}$ is an algebro-geometric potential for KdV. Equation (26) is just the second term in a sequence of infinitesimal flows, called the *KdV hierarchy*. By definition, $u(z)$ is algebro-geometric for KdV if this hierarchy stops at some level in the sense that the n -th flow in the hierarchy is a linear combination of the preceding flows. The idea here is to associate to each flow in the KdV hierarchy a bounded Jacobi function on the initial surface M associated to $g \in \mathcal{M}_{\text{imm}}$ (Theorem 5.8). Then, the fact that $u(z)$ is algebro-geometric for KdV will follow from the finite dimensionality of the linear space of bounded Jacobi functions on M , a result that will be proven in Appendix 1. (This finite dimensionality also follows from the more general results in [6].)

5.1. *Algebro-geometric potentials of the KdV equation.* We first introduce some background properties of the Korteweg-de Vries equation KdV. A presentation of the KdV theory close to the viewpoint we will need here can be found in Gesztesy and Weikard [19] and Joshi [27]. For a quick introduction, one can read Goldstein and Petrich [21], where the related mKdV equation (*modified* Korteweg-de Vries) is interpreted as a flow of the curvature of a planar curve; for other applications of the KdV equation in geometry, see Chern and Peng [5]. In the literature one can find different normalizations of the KdV equation (given by different coefficients for u''' , uu' in equation (28) below); all of them are equivalent up to a change of variables. We will follow here the normalization that appears in [27].

Given a meromorphic function $u(z)$, where z belongs to an open set $O \subset \mathbb{C}$, we consider the *KdV infinitesimal flow*, which is the infinitesimal deformation

$$(28) \quad \frac{\partial u}{\partial t} = -u''' - 6uu',$$

where as usual, $u', u'', u''', u^{(4)}, \dots$ denote the successive derivatives of u with respect to z . Associated to (28) we have the *KdV equation*, an evolution equation where we look for a meromorphic function $u(z, t)$, with $z \in O$ and $t \in \mathbb{D}(\varepsilon) = \{t \in \mathbb{C} \mid |t| < \varepsilon\}$, satisfying (28). The *Cauchy problem* for the KdV equation consists of finding a solution $u(z, t)$ of (28) with prescribed initial condition $u(z, 0) = u(z)$. In fact, the KdV infinitesimal flow is one of the terms in a sequence of infinitesimal flows of u , called the *KdV hierarchy*:

$$(29) \quad \left\{ \frac{\partial u}{\partial t_n} = -\partial_z \mathcal{P}_{n+1}(u) \right\}_{n \geq 0},$$

where $\mathcal{P}_{n+1}(u)$ is a differential operator given by a polynomial expression of u and its derivatives up to order $2n$. These operators are defined by the recurrence law

$$(30) \quad \begin{cases} \partial_z \mathcal{P}_{n+1}(u) = (\partial_{zzz} + 4u \partial_z + 2u') \mathcal{P}_n(u), \\ \mathcal{P}_0(u) = \frac{1}{2}. \end{cases}$$

In particular, the first operators and infinitesimal flows of the KdV hierarchy are given by

$$(31) \quad \begin{array}{l} \mathcal{P}_1(u) = u \\ \mathcal{P}_2(u) = u'' + 3u^2 \quad (\text{KdV}) \\ \mathcal{P}_3(u) = u^{(4)} + 10u u'' + 5(u')^2 + 10u^3 \\ \vdots \end{array} \quad \left| \quad \begin{array}{l} \frac{\partial u}{\partial t_0} = -u' \\ \frac{\partial u}{\partial t_1} = -u''' - 6u u' \quad (\text{KdV}) \\ \frac{\partial u}{\partial t_2} = -u^{(5)} - 10u u''' - 20u' u'' - 30u^2 u' \\ \vdots \end{array} \right.$$

The Cauchy problem for the n -th equation of the KdV hierarchy consists of finding a solution $u(z, t)$ of $\frac{\partial u}{\partial t_n} = -\partial_z \mathcal{P}_{n+1}(u)$ with prescribed initial condition $u(z, 0) = u(z)$.

A function $u(z)$ is said to be an *algebro-geometric potential of the KdV equation* (or simply *algebro-geometric*) if there exists an infinitesimal flow $\frac{\partial u}{\partial t_n}$ that is a linear combination of the lower order infinitesimal flows:

$$(32) \quad \frac{\partial u}{\partial t_n} = c_0 \frac{\partial u}{\partial t_0} + \dots + c_{n-1} \frac{\partial u}{\partial t_{n-1}},$$

with $c_0, \dots, c_{n-1} \in \mathbb{C}$. The next statement collects some important properties of algebro-geometric potentials.

THEOREM 5.1. *Let $u(z)$ be an algebro-geometric potential. Then*

- (1) u extends to a meromorphic function $u: \mathbb{C} \rightarrow \mathbb{C} \cup \{\infty\}$.

(2) If u has a pole at $z = z_0$, then its Laurent expansion around z_0 is given by

$$u(z) = \frac{-k(k+1)}{(z-z_0)^2} + \text{holomorphic}(z)$$

for a suitable positive integer k .

(3) All the solutions of the linear Schrödinger equation $y'' + uy = 0$ are meromorphic functions $y: \mathbb{C} \rightarrow \mathbb{C} \cup \{\infty\}$.

Item (1) is due to Segal and Wilson [51] and can be found also in Weikard [56] and Gesztesy and Weikard [19]. Items (2) and (3) are proved in [19] and [56].

Another fundamental property of algebro-geometric potentials is that the Cauchy problem for any infinitesimal flow of the KdV hierarchy is uniquely solvable in the class of algebro-geometric potentials (with fixed coefficients c_j in equation (32)). This integrability follows from the commutativity of any two infinitesimal flows of the KdV hierarchy. We now give a direct proof of this well-known fact in the particular case of the KdV infinitesimal flow (28), which we will use later.

The infinitesimal flow $\frac{\partial u}{\partial t_n}$ defines naturally a differential operator $\frac{\partial}{\partial t_n}$ that acts on differential expressions of u and its derivatives. For instance, we have

$$\frac{\partial}{\partial t_n}(u') = \left(\frac{\partial u}{\partial t_n}\right)' \quad \text{and} \quad \frac{\partial}{\partial t_n}\left(\frac{u''}{u} + u^2\right) = \frac{1}{u}\left(\frac{\partial u}{\partial t_n}\right)'' + \left(2u - \frac{u''}{u}\right)\frac{\partial u}{\partial t_n}.$$

LEMMA 5.2. *The KdV infinitesimal flow $\frac{\partial}{\partial t} = \frac{\partial}{\partial t_1}$ commutes with any other infinitesimal flow $\frac{\partial}{\partial t_n}$ in the KdV hierarchy:*

$$(33) \quad \frac{\partial}{\partial t} \frac{\partial u}{\partial t_n} = \frac{\partial}{\partial t_n} \frac{\partial u}{\partial t}.$$

Sketch of the proof. The proof is by induction on n . The lemma clearly holds for $n = 1$. Assuming that $\frac{\partial}{\partial t}$ commutes with $\frac{\partial}{\partial t_{n-1}}$, we want to prove (33). We will simply write \mathcal{P}_n instead of $\mathcal{P}_n(u)$. It follows from equations (29) and (31) that

$$(34) \quad \frac{\partial u}{\partial t} = -(u'' + 3u^2)'.$$

Thus, the induction hypothesis implies that the two expressions below have the same value:

$$\begin{aligned} \frac{\partial}{\partial t_{n-1}} \frac{\partial u}{\partial t} &= -\frac{\partial}{\partial t_{n-1}}(u'' + 3u^2)' = -\left(\left(\frac{\partial u}{\partial t_{n-1}}\right)'' + 6u\frac{\partial u}{\partial t_{n-1}}\right)', \\ \frac{\partial}{\partial t} \frac{\partial u}{\partial t_{n-1}} &= -\frac{\partial}{\partial t}\mathcal{P}'_n = -\left(\frac{\partial}{\partial t}\mathcal{P}_n\right)'. \end{aligned}$$

Therefore, there exists $c \in \mathbb{C}$ such that $\frac{\partial}{\partial t} \mathcal{P}_n = \left(\frac{\partial u}{\partial t_{n-1}}\right)'' + 6u \frac{\partial u}{\partial t_{n-1}} + c$. We claim that $c = 0$: since \mathcal{P}_n is a polynomial expression of u and its derivatives, if we differentiate \mathcal{P}_n with respect to t and we make the substitution $\frac{\partial u}{\partial t} = -u''' - 6u'$, we will obtain another polynomial expression in the variables u and its derivatives without independent term, which gives our claim. Therefore $\frac{\partial}{\partial t} \mathcal{P}_n = \left(\frac{\partial u}{\partial t_{n-1}}\right)'' + 6u \frac{\partial u}{\partial t_{n-1}}$ which, using that $\frac{\partial u}{\partial t_{n-1}} = -\mathcal{P}'_n$, transforms into

$$(35) \quad \frac{\partial}{\partial t} \mathcal{P}_n = -\mathcal{P}_n''' - 6u\mathcal{P}'_n.$$

We are now ready to prove the commutativity at the n -th level: Using equations (29) and (30), we have

$$\begin{aligned} \frac{\partial}{\partial t} \frac{\partial u}{\partial t_n} &= -\frac{\partial}{\partial t} [(\partial_{zzz} + 4u\partial_z + 2u') \mathcal{P}_n] \\ &= -(\partial_{zzz} + 4u\partial_z + 2u') \frac{\partial \mathcal{P}_n}{\partial t} - 4 \frac{\partial u}{\partial t} \mathcal{P}'_n - 2 \left(\frac{\partial u}{\partial t}\right)' \mathcal{P}_n. \end{aligned}$$

Substituting (34) and (35) in the last expression, we find a polynomial expression E_1 of u , \mathcal{P}_n and their derivatives with respect to z , for $\frac{\partial}{\partial t} \frac{\partial u}{\partial t_n}$. On the other hand, (34) gives

$$\frac{\partial}{\partial t_n} \frac{\partial u}{\partial t} = -\frac{\partial}{\partial t_n} [(u'' + 3u^2)'] = -\left(\left(\frac{\partial u}{\partial t_n}\right)'' + 6u \frac{\partial u}{\partial t_n}\right)',$$

which combined with the recurrence law $\frac{\partial u}{\partial t_n} = -(\partial_{zzz} + 4u\partial_z + 2u')\mathcal{P}_n$ gives a polynomial expression E_2 in the variables u , \mathcal{P}_n and its derivatives, too. Comparing both expressions E_1, E_2 , a lengthy but direct computation shows that

$$\frac{\partial}{\partial t} \frac{\partial u}{\partial t_n} - \frac{\partial}{\partial t_n} \frac{\partial u}{\partial t} = 0,$$

which proves the lemma. □

Next we prove the integrability of the KdV infinitesimal flow for an algebro-geometric initial condition.

PROPOSITION 5.3. *Let $u = u(z) : \mathbb{C} \rightarrow \mathbb{C} \cup \{\infty\}$ be an algebro-geometric potential of the KdV equation so that $\frac{\partial u}{\partial t_n} = c_0 \frac{\partial u}{\partial t_0} + \dots + c_{n-1} \frac{\partial u}{\partial t_{n-1}}$. Then, there exist $\varepsilon > 0$ and a unique map $u = u(z, t) : \mathbb{C} \times \mathbb{D}(\varepsilon) \rightarrow \mathbb{C} \cup \{\infty\}$ such that the following properties hold:*

- (1) $u(z, 0) = u(z)$, and $u_t(z) = u(z, t)$ is algebro-geometric for each $t \in \mathbb{D}(\varepsilon)$.
- (2) $u(z, t)$ is holomorphic in $\{(z, t) \in \mathbb{C} \times \mathbb{D}(\varepsilon) : |u(z, t)| < \infty\}$ and is a solution of the system of partial differential equations

$$(36) \quad \left. \begin{aligned} \text{(A-G)} \quad & \frac{\partial u}{\partial t_n} - c_0 \frac{\partial u}{\partial t_0} - \dots - c_{n-1} \frac{\partial u}{\partial t_{n-1}} = 0, \\ \text{(KdV)} \quad & \frac{\partial u}{\partial t} = -u''' - 6uu' \end{aligned} \right\}$$

where, as usual, prime denotes derivative with respect to z .

- (3) If there exists $\omega \in \mathbb{C}$ such that $u(z + \omega) = u(z)$ for all $z \in \mathbb{C}$, then $u(z + \omega, t) = u(z, t)$ for all $z \in \mathbb{C}$.
- (4) If the jet

$$J(z_0) = (u(z_0), u'(z_0), \dots, u^{(2n)}(z_0)) \in \mathbb{C}^{2n+1}$$

is bounded by a constant $C > 0$, then there exist $\delta > 0$ and $C_1 = C_1(\delta, C) > 0$ such that $u(z, t)$, $u'(z, t)$ and $\frac{\partial u}{\partial t}(z, t)$ are holomorphic functions bounded by C_1 in $\{z \in \mathbb{C} : |z - z_0| < \delta\} \times \mathbb{D}(\delta)$.

Proof. We will use the notation $\frac{\partial}{\partial s} = \frac{\partial}{\partial t_n} - c_0 \frac{\partial}{\partial t_0} - \dots - c_{n-1} \frac{\partial}{\partial t_{n-1}}$. Hence the system (36) can be equivalently written as

$$(37) \quad \begin{cases} \frac{\partial u}{\partial s} = 0, \\ \frac{\partial u}{\partial t} = -u''' - 6uu'. \end{cases}$$

According to the Frobenius Theorem, the integrability condition of (37) is given by the commutativity $\frac{\partial}{\partial s} \frac{\partial u}{\partial t} = \frac{\partial}{\partial t} \frac{\partial u}{\partial s}$, which follows from Lemma 5.2. Therefore given any $z_0 \in \mathbb{C}$ that is not a pole of $u(z)$, there exists a positive number δ and a unique solution $u(z, t)$, $(z, t) \in \{|z - z_0| < \delta\} \times \mathbb{D}(\delta)$ of the system (36) with initial conditions

$$(38) \quad \frac{\partial^k u}{\partial z^k}(z_0, 0) = u^{(k)}(z_0), \quad k = 0, \dots, 2n.$$

(Note that the operator $\frac{\partial}{\partial s}$ involves derivatives with respect to z up to order $2n + 1$.) As u_t satisfies (A-G), then it is algebro-geometric. Thus part (1) of Theorem 5.1 insures that u_t extends meromorphically to the whole plane \mathbb{C} . As equation (A-G) is an ODE in the variable z and $\frac{\partial u}{\partial t_n}$ involves derivatives with respect to z up to order $2n + 1$, it follows from the initial condition (38) that $u(z, 0) = u(z)$. This proves items (1) and (2) of Proposition 5.3.

Item (3) of the proposition follows easily from the uniqueness part, and the local estimate in item (4) is the standard dependence of the solution of an initial value problem on the initial data. \square

Our next result describes the evolution in time of the poles of a solution of the Cauchy problem for the KdV equation for a special case that we will find when applying this machinery to a quasiperiodic, properly immersed minimal surface of Riemann type.

THEOREM 5.4. *Let $u = u(z): \mathbb{C}/\langle i \rangle \rightarrow \mathbb{C} \cup \{\infty\}$ be a quasiperiodic algebro-geometric potential on the cylinder, whose Laurent expansion around each pole z_0 of u is given by*

$$(39) \quad u(z) = \frac{-2}{(z - z_0)^2} + \text{holomorphic}(z).$$

Let $u = u(z, t): \mathbb{C}/\langle i \rangle \times \mathbb{D}(\varepsilon) \rightarrow \mathbb{C} \cup \{\infty\}$ be the solution of the system (36) with initial data $u(z)$. Then, the following properties hold:

- (1) $u(z, t)$ is meromorphic (as a function of two variables), and $u_t(z) = u(z, t)$ is quasiperiodic for each t .
- (2) Given a pole z_0 of $u(z)$, there exists a holomorphic curve $t \in \mathbb{D}(\varepsilon) \mapsto z_0(t)$ with $z_0(0) = z_0$ such that in a neighborhood of $(z_0, 0)$, we have

$$(40) \quad u(z, t) = \frac{-2}{(z - z_0(t))^2} + \text{holomorphic}(z, t).$$

Moreover, all the poles of $u_t(z)$ are obtained in this way.

Proof. Since $u(z)$ is algebro-geometric, Proposition 5.3 gives a unique solution of (36) with $u(z, 0) = u(z)$, $z \in \mathbb{C}$. Furthermore, $u(z, t)$ is holomorphic in $\{(z, t) \in \mathbb{C} \times \mathbb{D}(\varepsilon) : |u(z, t)| < \infty\}$ and $u_t(z) = u(z, t)$ descends to $\mathbb{C}/\langle i \rangle$.

We next prove that every pole z_0 of $u(z)$ propagates holomorphically in t to a curve of poles $z_0(t)$ of $u_t(z)$ with the desired Laurent expansion. Let D be a closed disk centered at z_0 such that $u(z)$ does not vanish in $D - \{z_0\}$. By continuity, $u_t(z)$ has no zeros in ∂D for $|t|$ sufficiently small. Recall that u_t is meromorphic in \mathbb{C} since it is algebro-geometric. By the argument principle,

$$(41) \quad \#(u_t^{-1}(\infty) \cap D) - \#(u_t^{-1}(0) \cap D) = \#(u^{-1}(\infty) \cap D) - \#(u^{-1}(0) \cap D) = 1.$$

Let a_1, \dots, a_m be the poles of u_t in D . (Both m and the a_j may depend on t .) As u_t is algebro-geometric, part (2) of Theorem 5.1 insures that there exist positive integers k_1, \dots, k_m such that

$$u_t(z) = \frac{-k_j(k_j + 1)}{(z - a_j)^2} + \text{holomorphic}(z, t)$$

in a neighborhood of a_j . Since the residue of u_t at a_j is zero for all j , there exists a meromorphic function $v_t(z)$ defined on D such that $v'_t = u_t$ in D . Moreover, v_t is unique up to an additive constant, which we choose so that v_t has value 1 at some point $p_0 \in \partial D$ and $v_t|_{\partial D}$ has no zeros. The Laurent expansion of v_t around its poles is

$$v_t(z) = \frac{k_j(k_j + 1)}{z - a_j} + \text{holomorphic}(z, t).$$

Since $S(t) = \sum_j k_j(1 + k_j) = \frac{1}{2\pi i} \int_{\partial D} v_t(z) ds$ is continuous and integer-valued, $S(t)$ is constant. As $S(0) = 2$, we conclude that v_t has just one pole that is simple; i.e., $m = 1$ and $k_1 = 1$. Thus, u_t has just one pole $z_0(t)$ in D (which has order two) with the coefficient -2 for the term in $(z - z_0(t))^{-2}$ in its Laurent expansion; i.e., equation (40) holds. By (41), u_t has no zeros in D for $|t| < \varepsilon$.

Next we prove that the curve $t \mapsto z_0(t)$ is holomorphic. (Note that we cannot use the implicit function theorem since the function w below is not known

to be holomorphic as a function of two variables around $(z_0, 0)$.) Consider the function

$$w(z, t) = \frac{1}{v_t(z)}, \quad (z, t) \in D \times \mathbb{D}(\varepsilon).$$

Note that $w(z, t)$ is holomorphic in z since v_t does not vanish in D . (This follows because $A(t) = \#(v_t^{-1}(\infty)) - \#(v_t^{-1}(0))$ is constant, $A(0) = 1$ and v_t has a unique pole in D counted with multiplicities.) For t fixed, $w(\cdot, t)$ has a unique zero $z_0(t)$ in D , and $z_0(0) = z_0$. As a function of two variables, $w(z, t)$ is holomorphic outside $\{(z_0(t), t) \mid t \in \mathbb{D}(\varepsilon)\}$. Now the holomorphicity of $t \mapsto z_0(t)$ is a consequence of the following observation. The function $w_t(z) = w(z, t)$ is holomorphic in the closure \overline{D} of D , has a simple zero at $z_0(t) \in D$ and no more zeros in \overline{D} . Hence, $\frac{zw'_t(z)}{w_t(z)}dz$ is a meromorphic differential in \overline{D} with a simple pole at $z = z_0(t)$ and thus,

$$\int_{\partial D} \frac{zw'_t(z)}{w_t(z)}dz = 2\pi i \operatorname{Res}_{z_0(t)} \left(\frac{zw'_t(z)}{w_t(z)}dz \right) = 2\pi iz_0(t).$$

Since the integrand in the left-hand-side of the above formula depends holomorphically on t , the same holds for $z_0(t)$. This argument proves the following property, which we state separately for future reference.

ASSERTION 5.5. *Let $h(z, t)$ be a holomorphic function in $\{(z, t) \in \mathbb{D}(\varepsilon) \times \mathbb{D}(\varepsilon) : |h(z, t)| < \infty\}$, such that $z \mapsto h_t(z) = h(z, t)$ has exactly one zero in $\mathbb{D}(\varepsilon)$ counting multiplicities, and $h(0, 0) = 0$. Then, there exists a holomorphic curve $\alpha(t)$, $|t| < \varepsilon$, such that the zeros of h in a neighborhood of $(0, 0)$ are given by the trace of α .*

We now return to the proof of Theorem 5.4. To prove the first part of item (1), we only need to check that u is meromorphic around the points in $\Gamma = \{(z_0(t), t) \mid |t| < \varepsilon\}$ where $u = \infty$. This follows from equation (40): as $u(z, t) + 2(z - z_0(t))^{-2}$ is holomorphic and bounded outside of the analytic subset Γ , it extends holomorphically through Γ .

It remains to check that $u_t(z)$ is quasiperiodic for $|t|$ sufficiently small. This fact will hold if we prove the following inequality for the spherical gradient of $u_t(z)$:

$$(42) \quad \frac{|u'_t(z)|}{1 + |u_t(z)|^2} \leq C$$

for all $z \in \mathbb{C}$, where $C > 0$ is independent of z . Repeating the arguments above at every pole $z_{0,j}$ of u , we obtain a sequence of pairwise disjoint closed disks $\{D_j\}_j$ such that each D_j is centered at $z_{0,j}$, for $|t|$ small (independently of j), $u_t(z)$ has a unique pole at $z_{0,j}(t) \in D_j$, and the curve $t \mapsto z_{0,j}(t)$ is holomorphic in t . Note that since $u(z)$ is quasiperiodic, the radii of the D_j can be taken independently of j . Since the jet $J(z_1) = (u(z_1), u'(z_1), \dots, u^{(2n)}(z_1))$ is

uniformly bounded in \mathbb{C}^{2n+1} for $z_1 \in (\mathbb{C}/\langle i \rangle) - \cup_j D_j$ (because u is quasiperiodic), part (4) of Proposition 5.3 implies that both $u(z, t), u'(z, t)$ are uniformly bounded for $(z, t) \in [(\mathbb{C}/\langle i \rangle) - \cup_j D_j] \times \mathbb{D}(\varepsilon)$ for ε sufficiently small. Therefore, (42) holds outside $\cup_j D_j$ with C uniform in t . Now consider one of the disks D_j . For t fixed, $u_t|_{D_j}$ omits a neighborhood of zero that is independent of t . (This property needs estimates for u_t in a slightly bigger disk, which we may assume.) By Montel's theorem, $\{u_t|_{D_j}\}_t$ form a normal family, which implies that (42) holds for $z \in D$ uniformly in t . Now the proof is complete. \square

5.2. *The Shiffman hierarchy associated to a Riemann type minimal surface.* Let $M \in \mathcal{M}$ be a quasiperiodic, immersed minimal surface of Riemann type, with Gauss map $g \in \mathcal{M}_{\text{imm}}$. In this section we will associate to g a sequence of infinitesimal deformations $\frac{\partial g}{\partial t_n}$ that generalizes the tangent vector $\dot{g}_S \in T_g \mathcal{W}$ associated to the complex Shiffman function, which was given in equation (24). For this reason, we call this sequence the *Shiffman hierarchy*. In order to define the Shiffman hierarchy, we will first define a related hierarchy associated to a linear Schrödinger equation.

Consider meromorphic functions $y, u, g: \mathbb{C} \rightarrow \mathbb{C} \cup \{\infty\}$ related by the equations

$$y'' + uy = 0 \quad \text{and} \quad g = \frac{1}{y^2}.$$

From these relations we obtain

$$(43) \quad u = -\frac{3(g')^2}{4g^2} + \frac{g''}{2g}.$$

Remark 5.6. The reader may wonder why the KdV equation appears in connection to the Shiffman function. The change of variables $x = g'/g$ transforms the expression (24) for \dot{g}_S into an equation of mKdV type, namely, $\dot{x} = \frac{i}{2}(x''' - \frac{3}{2}x^2x')$. It is well known that mKdV equations in x can be transformed into KdV equations in u through the so-called *Miura transformations*, $x \mapsto u = ax' + bx^2$ with a, b suitable constants (see, for example, [19, p. 273]). Equation (43) is nothing but the composition of $g \mapsto x$ and a Miura transformation. Since the KdV theory is more standard than the mKdV theory, we have opted to deal only with the KdV equation and avoid dealing with the mKdV equation.

The *Schrödinger hierarchy* is defined as a sequence of infinitesimal flows of y given by

$$(44) \quad \left\{ \frac{\partial y}{\partial t_n} = \mathcal{P}_n(u)'y - 2\mathcal{P}_n(u)y' \right\}_{n \geq 0},$$

where $\mathcal{P}_n(u)$ is the polynomial expression of u and its derivatives given by equation (30). The connection between the Schrödinger and the KdV hierarchies comes from the fact that the integrability conditions for the system of partial differential equations

$$(45) \quad \begin{cases} y'' + uy = 0, \\ \frac{\partial y}{\partial t_n} = \mathcal{P}_n(u)'y - 2\mathcal{P}_n(u)y' \end{cases}$$

are precisely that $u(z, t)$ satisfies the n -th equation of the KdV hierarchy; see Joshi [27]. Both hierarchies are related by

$$(46) \quad \frac{\partial u}{\partial t_n} = -\frac{\partial}{\partial t_n} \left(\frac{y''}{y} \right).$$

If we rewrite these infinitesimal flows in terms of g , we obtain the following sequence of infinitesimal flows of g , which we call the *Shiffman hierarchy*:

$$\frac{\partial g}{\partial t_n} = \frac{\partial}{\partial t_n} \left(\frac{1}{y^2} \right) = -\frac{2}{y^3} \frac{\partial y}{\partial t_n} = -2 \frac{\mathcal{P}_n(u)'y - 2\mathcal{P}_n(u)y'}{y^3} = -2 \partial_z \left(\frac{\mathcal{P}_n(u)}{y^2} \right).$$

By construction, we have the following statement.

LEMMA 5.7. *The Shiffman hierarchy is given by*

$$(47) \quad \left\{ \frac{\partial g}{\partial t_n} = -2 \partial_z (g \mathcal{P}_n(u)) \right\}_{n \geq 0}.$$

If we compute explicitly these infinitesimal flows solely in terms of g (by substitution of (30) and (43) in (47)), each right-hand-side is a rational expression in g and its derivatives. The first infinitesimal flow in this hierarchy is the infinitesimal deformation in \mathcal{W} given by translations in the parameter domain, and the second one is, up to a multiplicative constant, the infinitesimal deformation \dot{g}_S given by (24), which corresponds to the Shiffman function. (Recall from Remark 4.8 and Corollary 4.15 that both infinitesimal deformations lie in $T_g \mathcal{W}$.) We also provide the expression for the third infinitesimal flow:

$$\begin{aligned} \frac{\partial g}{\partial t_0} &= -g', \\ \frac{\partial g}{\partial t_1} &= -g''' + 3 \frac{g'g''}{g} - \frac{3}{2} \frac{(g')^3}{2g^2}, \\ \frac{\partial g}{\partial t_2} &= -g^{(5)} + 5 \frac{g'g^{(4)}}{g} + 10 \frac{g''g'''}{g} - \frac{35}{2} \frac{(g')^2 g'''}{g^2} - \frac{55}{2} \frac{g'(g'')^2}{g^2} + \frac{95}{2} \frac{(g')^3 g''}{g^3} - \frac{135}{8} \frac{(g')^5}{g^4}, \\ &\vdots \end{aligned}$$

Another key reason why we are interested in the KdV equation is that the associated Shiffman hierarchy provides a sequence of bounded Jacobi functions on any quasiperiodic, immersed minimal surface of Riemann type, as we now explain. If we let $g \in \mathcal{M}_{\text{imm}}$ be the complex Gauss map of such a surface, observe that g has order-two zeroes and order-two poles without residues. Thus

there exists a meromorphic function $y: \mathbb{C} \rightarrow \mathbb{C} \cup \{\infty\}$ such that $g = 1/y^2$. Moreover y is either periodic, $y(z+i) = y(z)$, or antiperiodic $y(z+i) = -y(z)$. The later one is the case for the Riemann minimal examples \mathcal{R}_h , $h > 0$. (This follows since the Gauss map g of \mathcal{R}_h restricts to each compact horizontal section with degree one.)

THEOREM 5.8. *If $g \in \mathcal{M}_{\text{imm}}$, then each of the infinitesimal flows $\frac{\partial g}{\partial t_n}$ in the Shiffman hierarchy produces a Jacobi function $f(h_n) \in \mathcal{J}_{\mathbb{C}}(g)$, which is bounded and quasiperiodic on $\mathbb{C}/\langle i \rangle$.*

Proof. First observe that equation (43) and the fact that $\mathcal{P}_n(u)$ is a polynomial expression of u and its derivatives imply that $\frac{\partial g}{\partial t_n}$ is meromorphic and quasiperiodic, with poles only at (some of) the zeroes and poles of g . To prove the theorem we will use Proposition 4.13; hence it suffices to demonstrate the following statement.

ASSERTION 5.9. *Under the hypotheses of Theorem 5.8, for any n , there exists a meromorphic function h_n on $\mathbb{C}/\langle i \rangle$ that is a rational expression of g and its derivatives up to some order (depending on n) such that $\frac{\partial g}{\partial t_n} = \partial_z \left(\frac{g^3 h'_n}{2g'} \right)$ and $\frac{\partial g}{\partial t_n} \in T_g \mathcal{W}$.*

Proof of Assertion 5.9. Viewing the equation $-2\partial_z(g\mathcal{P}_n(u)) = \partial_z \left(\frac{g^3 h'_n}{2g'} \right)$ as an ODE for the unknown h_n and substituting $g = 1/y^2$, we have

$$\frac{1}{4}h'_n = (y^2)'\mathcal{P}_n(u) + c(y^4)' = (y^2\mathcal{P}_n(u))' - y^2\mathcal{P}_n(u)' + c(y^4)',$$

where $c \in \mathbb{C}$ is a constant of integration. Therefore, the existence of the desired h_n will follow if we see that $y^2\mathcal{P}_n(u)'$ has a global primitive on $\mathbb{C}/\langle i \rangle$ that is meromorphic. By the recurrence law (30) for the operators \mathcal{P}_n , rewritten using the function y instead of u , we have

$$\mathcal{P}'_n = \mathcal{P}'''_{n-1} - \frac{4y''}{y}\mathcal{P}'_{n-1} + 2\frac{y'y'' - yy'''}{y^2}\mathcal{P}_{n-1}.$$

Hence, by direct computation,

$$\begin{aligned} y^2\mathcal{P}'_n &= y^2\mathcal{P}'''_{n-1} - 4yy''\mathcal{P}'_{n-1} + 2(y'y'' - yy''')\mathcal{P}_{n-1} \\ &= \partial_z (y^2\mathcal{P}''_{n-1} - 2yy'\mathcal{P}'_{n-1} + 2((y')^2 - yy'')\mathcal{P}_{n-1}), \end{aligned}$$

from where we conclude the existence of a global primitive of $y^2\mathcal{P}_n(u)'$ in $\mathbb{C}/\langle i \rangle$. Furthermore, such a global primitive is meromorphic on $\mathbb{C}/\langle i \rangle$ by the same property that holds for y, u and $\mathcal{P}_{n-1}(u)$. Therefore, the existence of h_n is proved. Since $u, \mathcal{P}_n(u)$ are rational expressions of g and its derivatives, then the same holds for h_n . (In particular, h is meromorphic on $\mathbb{C}/\langle i \rangle$.)

In order to see that the meromorphic function $\frac{\partial g}{\partial t_n}$ lies in $T_g\mathcal{W}$, we must check that its principal divisor D satisfies $D \geq \prod_j p_j q_j^{-3}$, where $(g) = \prod_j p_j^2 q_j^{-2}$ is the principal divisor of g in $\mathbb{C}/\langle i \rangle$. Since $\frac{\partial g}{\partial t_n}$ is holomorphic outside from zeros and poles of g , we only need to study the behavior of $\frac{\partial g}{\partial t_n}$ around the points p_j, q_j .

Claim 1: $\frac{\partial g}{\partial t_n}$ has a zero at every zero p_j of g .

Proof of Claim 1. We may assume $p_j = 0$. Since $g \in \mathcal{M}_{\text{imm}}$, the Weierstrass pair (g, dz) closes periods at p_j . Thus $g'''(0) = 0$, which gives a series expansion $g(z) = az^2 + z^4 f_1(z)$, where a, b, \dots will denote complex numbers and f_1, f_2, \dots will represent holomorphic functions around $z = 0$ during this proof and that of Claim 2 below. Using (43), we obtain

$$(48) \quad u(z) = -\frac{2}{z^2} + b + z^2 f_2(z).$$

Assume we have proved that $\mathcal{P}_n(u)$ has an order-two pole without residue at each zero of g ; i.e.,

$$(49) \quad \mathcal{P}_n(u) = \frac{c}{z^2} + f_3(z).$$

Then we conclude that

$$g(z)\mathcal{P}_n(u) = ac + z^2 f_4(z),$$

which implies that $\frac{\partial g}{\partial t_n} = -2\partial_z(g\mathcal{P}_n(u))$ has a zero at the origin, as we wanted. It remains to check that $\mathcal{P}_n(u)$ has an order-two pole without residue at the origin, which will be proved by induction. Since $\mathcal{P}_1(u) = u$, the case $n = 1$ follows from equation (48). Assuming (49) we next study the Laurent expansion for $\mathcal{P}_{n+1}(u)$ around $z = 0$. The recurrence law (30), equations (48) and (49) and a direct computation give

$$\partial_z \mathcal{P}_{n+1}(u) = (\partial_{zzz} + 4u \partial_z + 2u')\mathcal{P}_n(u) = \frac{d}{z^3} + \frac{e}{z} + f_5(z).$$

As the left-hand-side has a well-defined primitive, we obtain $e = 0$ and thus, $\mathcal{P}_{n+1}(u)$ has the correct behavior at the origin. Now Claim 1 is proved.

Claim 2: $\frac{\partial g}{\partial t_n}$ has at most an order-three pole at every pole q_j of g .

Proof of Claim 2. Again we can suppose $q_j = 0$. First observe that, as g has a pole at $z = 0$ without residue, then u is holomorphic at $z = 0$ (direct computation). Since $\mathcal{P}_1(u) = u$ and $\partial_z \mathcal{P}_n(u) = (\partial_{zzz} + 4u \partial_z + 2u')\mathcal{P}_{n-1}(u)$, we deduce that $\partial_z \mathcal{P}_n(u)$ is holomorphic at $z = 0$. It follows that $\mathcal{P}_n(u)$ is holomorphic at $z = 0$ for all n . As g has an order-two pole at $z = 0$, we deduce that $\frac{\partial g}{\partial t_n} = -2\partial_z(g\mathcal{P}_n(u))$ has at most an order-three pole at $z = 0$. This completes the proofs of Claim 2 and of Theorem 5.8. \square

COROLLARY 5.10. *For every $g \in \mathcal{M}_{\text{imm}}$, the function $u = -\frac{3(g')^2}{4g^2} + \frac{g''}{2g}$ is an algebro-geometric potential of the KdV equation.*

Proof. Using Theorems 5.8 and 9.1 in Appendix 1, we deduce that there exists $n \in \mathbb{N}$ such that the Jacobi function $f(h_n) \in \mathcal{J}_{\mathbb{C}}(g)$ associated to the infinitesimal flow $\frac{\partial g}{\partial t_n}$ is a linear combination of $f(h_0), \dots, f(h_{n-1})$ associated to $\frac{\partial g}{\partial t_0}, \dots, \frac{\partial g}{\partial t_{n-1}}$, respectively. Note that the linear map $h \mapsto f(h)$ given by equation (19) is injective. Therefore, h_n is a linear combination of h_0, \dots, h_{n-1} , and (18) implies that the n -th infinitesimal flow $\frac{\partial g}{\partial t_n}$ of the Shiffman hierarchy is a linear combination of $\frac{\partial g}{\partial t_0}, \dots, \frac{\partial g}{\partial t_{n-1}}$. By equations (46) and (47), each of the infinitesimal flows $\frac{\partial u}{\partial t_n}$ of the KdV hierarchy can be expressed in terms of the ones of the Shiffman hierarchy as

$$\frac{\partial u}{\partial t_n} = \frac{\partial}{\partial t_n} \left(-\frac{3(g')^2}{4g^2} + \frac{g''}{2g} \right),$$

from where we conclude that $\frac{\partial u}{\partial t_n}$ depends linearly on the lower order infinitesimal flows in the KdV hierarchy. \square

LEMMA 5.11. *Let $u: \mathbb{C} \rightarrow \mathbb{C} \cup \{\infty\}$ be a meromorphic function with Laurent expansion given by (39) around any of its poles, and let $y: \mathbb{C} \rightarrow \mathbb{C} \cup \{\infty\}$ be a meromorphic solution of the equation $y'' + uy = 0$. Then, the following properties hold:*

- (1) *Outside of the poles of u , the function y is holomorphic and its zeros are simple.*
- (2) *At a pole of u , the function y has either a simple pole or an order-two zero.*

Proof. First suppose that y has a pole of order $k \geq 1$ at $z = 0$. Then locally $y(z) = z^{-k}f(z)$ with $f(0) \neq 0$, from where we conclude that

$$u(z) = -\frac{y''(z)}{y(z)} = -\frac{k(k+1)}{z^2} + \frac{2kf'(z)}{f(z)} + \text{holomorphic}(z).$$

This implies that every pole of y is also a pole of u , which is the first part of item (1). By equation (39) we have $k(k+1) = 2$; thus $k = 1$, and so all poles of y are simple.

We now deal with the zeros of y . If y has a zero at a point a where u is finite, then it must be a simple zero of y (because the solutions of $y'' + uy = 0$ are locally determined by $(y(a), y'(a))$). Thus it suffices to study the behavior of y at a pole z_0 of u such that y is holomorphic around z_0 . In this case, we can write locally $y(z) = (z - z_0)^k f(z)$ for some nonnegative integer k and some holomorphic function f with $f(z_0) \neq 0$. Then,

$$u(z) = -\frac{y''(z)}{y(z)} = -\frac{k(k-1)}{(z-z_0)^2} - \frac{2kf'(z)}{(z-z_0)f(z)} + \text{holomorphic}(z).$$

Again equation (39) implies $k(k - 1) = 2$; hence $k = 2$ and the lemma is proved. \square

Definition 5.12. Let M be a quasiperiodic, immersed minimal surface of Riemann type, with Weierstrass pair (g, dz) on $(\mathbb{C}/\langle i \rangle) - g^{-1}(\{0, \infty\})$. Let $(g) = \prod_{j \in \mathbb{Z}} p_j^2 q_j^{-2}$ be the principal divisor of g and let $z_0 \in \mathbb{C}/\langle i \rangle$ be a point different from p_j and q_j for all j . The Shiffman function S_M of M is said to be *holomorphically integrated* if there exist $\varepsilon > 0$ and families $\{p_j(t)\}_j, \{q_j(t)\}_j \subset \mathbb{C}/\langle i \rangle, a(t) \in \mathbb{C} - \{0\}$ such that

- (i) For each $j \in \mathbb{Z}$, the functions $t \in \mathbb{D}(\varepsilon) \mapsto p_j(t), t \mapsto q_j(t) \in \mathbb{C}/\langle i \rangle$ are holomorphic with $p_j(0) = p_j, q_j(0) = q_j$. Also, the function $t \mapsto a(t)$ is holomorphic as well.
- (ii) For any $t \in \mathbb{D}(\varepsilon)$, the divisor $\prod_j p_j(t)^2 q_j(t)^{-2}$ defines an element $g_t \in \mathcal{M}_{\text{imm}}$ with $g_0 = g$ and $g_t(z_0) = a(t)$. Let M_t be the quasiperiodic, immersed minimal surface of Riemann type with Weierstrass pair (g_t, dz) .
- (iii) For $t \in \mathbb{D}(\varepsilon)$, the derivative of $t \mapsto g_t$ with respect to t equals

$$\frac{d}{dt}g_t = \frac{i}{2} \left(g_t''' - 3 \frac{g_t' g_t''}{g_t} + \frac{3 (g_t')^3}{2 g_t^2} \right) \quad \text{on } \mathbb{C}/\langle i \rangle.$$

Remark 5.13. Definition 5.12 is motivated by the following fact. With the notation in that definition, suppose that the Shiffman function S_M of M can be holomorphically integrated. Consider the Weierstrass pair $(g_t, dz), t \in \mathbb{C}, |t| < \varepsilon$. Fix a point $z_0 \in (\mathbb{C}/\langle i \rangle) - g^{-1}(\{0, \infty\})$. Applying equation (22) to $\Psi_t = \left(\frac{1}{2}(\frac{1}{g_t} - g_t), \frac{i}{2}(\frac{1}{g_t} + g_t), 1 \right) dz$ at every t , we obtain

$$(50) \quad \left\langle \frac{d}{dt} \Big|_t \int_{z_0}^z \Psi_t, N_t \right\rangle = -\frac{1}{2} f(h_t) + \langle a(t), N_t \rangle,$$

where $f(h_t) = S_{M_t} + iS_{M_t}^*$ is the (complex valued) Jacobi function of M_t . Writing $t = t_1 + t_2 i$ with $t_1, t_2 \in \mathbb{R}$ and taking real parts in the last displayed equation at $t = t_1 + 0i$, we obtain

$$\left\langle \frac{\partial}{\partial t_1} \Big|_{t_1} \Re \int_{z_0}^z \Psi_{t_1}, N_{t_1} \right\rangle = -\frac{1}{2} S_{M_{t_1}} + \langle \Re(a(t_1)), N_{t_1} \rangle.$$

Calling $\psi_{t_1} = \Re \int_{z_0}^z \Psi_{t_1} - \int_0^{t_1} \Re(a(s)) ds$, we have that the normal component of the variational field of the deformation $t_1 \mapsto \psi_{t_1}$ of M equals (up to a multiplicative constant) the (real valued) Shiffman function of M_{t_1} .

THEOREM 5.14. *Let $M = (\mathbb{C}/\langle i \rangle, g, dz)$ be a quasiperiodic, immersed minimal surface of Riemann type (i.e., $g \in \mathcal{M}_{\text{imm}}$). Then, its Shiffman function can be holomorphically integrated. Furthermore, if M is embedded, then its related surfaces M_t are also embedded for $|t|$ sufficiently small.*

Proof. Choose a meromorphic function $y: \mathbb{C} \rightarrow \mathbb{C} \cup \{\infty\}$ such that $g(z) = y(z)^{-2}$, and let $u: \mathbb{C} \rightarrow \mathbb{C} \cup \{\infty\}$ be given by $u(z) = -y''(z)/y(z)$, which is also meromorphic. Note that $y(z+i) = \pm y(z)$ and $u(z+i) = u(z)$ and that each of the three functions g, y and u is quasiperiodic. By Corollary 5.10, u is algebro-geometric. A direct computation (using that the zeros and poles of g have order two) shows that around each pole z_0 of u , the Laurent expansion of u is

$$u(z) = \frac{-2}{(z - z_0)^2} + \text{holomorphic}(z).$$

Using Theorem 5.4, we can solve the Cauchy problem for the KdV equation with initial condition $u(z)$ and we get a meromorphic function $u_t(z) = u(z, t)$, $z \in \mathbb{C}/\langle i \rangle$ and $|t| < \varepsilon$, which is quasiperiodic for each t . Moreover the poles of u_t are given by holomorphic curves $t \mapsto z_0(t)$, and

$$(51) \quad u(z, t) = \frac{-2}{(z - z_0(t))^2} + \text{holomorphic}(z, t)$$

around $z_0(t)$. Consider the differential system in (45) for $n = 1$ with unknown $y(z, t)$,

$$(52) \quad \begin{cases} y'' + u(z, t)y = 0, \\ \frac{\partial y}{\partial t_1} = \mathcal{P}_1(u)'y - 2\mathcal{P}_1(u)y' = u'(z, t)y - 2u(z, t)y'. \end{cases}$$

(The second line in (52) corresponds to the Shiffman flow in the Schrödinger hierarchy.) The compatibility condition of (52) is just the KdV equation for u ; see Appendix A in Joshi [27]. By the Frobenius theorem, (52) admits a unique solution $y = y(z, t)$ with initial condition $y(z, 0) = y(z)$. Since $z \mapsto u(z, t)$ is algebro-geometric for every t , part (3) of Theorem 5.1 implies that $y(z, t)$ is defined on $\mathbb{C} \times \mathbb{D}(\varepsilon)$ (for some $\varepsilon > 0$) and is meromorphic in z . The uniqueness of solution of an initial value problem together with the fact that $y(z+i) = \pm y(z)$ give that $y(z+i, t) = \pm y(z, t)$, with the same choice of signs as for $y(z)$.

By Lemma 5.11 applied to $u_t(z) = u(z, t)$ and $y_t(z) = y(z, t)$, we find that y_t is holomorphic with simple zeros outside of the poles of u_t and that at a pole of u_t , either y_t has a simple pole or y_t has an order-two zero. We claim that this last possibility cannot occur. To see the claim, let D be a closed disk centered at a pole $z_0 = z_0(0)$ of $u(z)$, and let $\varepsilon > 0$ such that $z_0(t) \in \text{Int}(D)$ is the unique pole of $u_t(z)$ in D whenever $|t| < \varepsilon$; see the proof of Theorem 5.4 for details. Note that since $g(z)$ has order-two zeros and poles, then the zeros and poles of $y(z)$ are simple. Since $u(z)$ has a pole at $z_0 \in D$, then Lemma 5.11 demonstrates that $y(z)$ has a simple pole at z_0 . We can also assume without loss of generality that $y(z)$ has no other zeros or poles in D and, by continuity, $y_t(z)$ has no zeros or poles in ∂D for t sufficiently close to zero. Arguing by contradiction, assume there exists \hat{t} with $|\hat{t}| < \varepsilon$ such that $y_{\hat{t}}$ has an order-two

zero at $z_0(\hat{t})$. Then,

$$(53) \quad \#(y_t^{-1}(\infty) \cap D) - \#(y_t^{-1}(0) \cap D) = \#(y^{-1}(\infty) \cap D) - \#(y^{-1}(0) \cap D).$$

The right-hand-side of (53) is 1, while the left-hand-side for $t = \hat{t}$ equals -2 . This contradiction proves our claim.

Since the zeros $p_j(t)$ and poles $q_j(t)$ of $y_t(z)$ are simple, Assertion 5.5 insures that $p_j(t), q_j(t)$ depend holomorphically on t . Furthermore, the fact that $y(z, 0) = y(z)$ implies that $p_j(0) = p_j, q_j(0) = q_j$, where $(y) = \prod_j p_j q_j^{-1}$ is the principal divisor of $y(z)$. The same arguments in the proof of Theorem 5.4 now give that $y_t(z)$ is quasiperiodic. Finally, define the quasiperiodic meromorphic function

$$g(z, t) = \frac{1}{y^2(z, t)}, \quad (z, t) \in (\mathbb{C}/\langle i \rangle) \times \mathbb{D}(\varepsilon).$$

As a function of z , $g_t(z) = g(z, t)$ has only order-two zeros and poles, and $t \mapsto g_t$ is a holomorphic curve in \mathcal{W} . The complex periods of the Weierstrass pair $((\mathbb{C}/\langle i \rangle) - g_t^{-1}(\{0, \infty\}), g_t, dz)$ along every closed curve $\Gamma \subset \mathbb{C}/\langle i \rangle$ are constant in t provided that we prove that the following integrals vanish:

$$(54) \quad \frac{d}{dt} \int_{\Gamma} \frac{dz}{g_t} = - \int_{\Gamma} \frac{\frac{\partial g_t}{\partial t}}{g_t^2} dz, \quad \frac{d}{dt} \int_{\Gamma} g_t dz = \int_{\Gamma} \frac{\partial g_t}{\partial t} dz.$$

By Assertion 5.9 together with equation (20), both integrals in (54) vanish if we check that $\frac{\partial g}{\partial t} = \frac{\partial g}{\partial t_1}$, where the right-hand-side in the last equation is the flow of the Shiffman hierarchy for $n = 1$. Also note that once we know that the complex periods of (g_t, dz) on $(\mathbb{C}/\langle i \rangle) - g_t^{-1}(\{0, \infty\})$ do not depend on t , we can easily deduce that this pair is the Weierstrass data of a quasiperiodic, immersed minimal surface of Riemann type $M_t \subset \mathbb{R}^3$.

Next we prove that $\frac{\partial g}{\partial t} = \frac{\partial g}{\partial t_1}$. Since $y_t'' + u_t y_t = 0$ and $g_t = y_t^{-2}$, we have $u_t = -\frac{3(g_t')^2}{4g_t^2} + \frac{g_t''}{2g_t}$, which is equation (43) for time t . Using that $y(z, t)$ satisfies (52) and comparing with (45) and (47), we deduce the desired equality. Note that $\frac{\partial g}{\partial t_1}$ is a constant multiple of the (complex) Shiffman Jacobi function. Hence, we have proved that the Shiffman function can be holomorphically integrated on M .

Finally, the fact that M_t is embedded for $|t|$ sufficiently small provided that M is embedded follows from the previous arguments together with the maximum principle for minimal surfaces. \square

6. The proofs of Theorems 1.1 and 1.2

The goal of this section is to prove Assertion 1.3 stated in the introduction. Recall from Section 3 that Assertion 1.3 implies our main Theorem 1.1 and its consequence, Theorem 1.2. Our strategy to prove Assertion 1.3 is as follows. (We follow the notation in that assertion.) First, we will prove in

Proposition 6.2 that if the Shiffman function S_M of a surface $M \in \mathcal{M}$ is linear, then M is a Riemann minimal example. Second, we show that for every surface $M \in \mathcal{M}$, its Shiffman function is linear (item (2) of Proposition 6.3).

6.1. *Minimal surfaces of Riemann type whose Shiffman function is linear.*

LEMMA 6.1. *Suppose that the Shiffman function S_M of a quasiperiodic, immersed minimal surface of Riemann type $M \subset \mathbb{R}^3$ is linear. Then, M is singly-periodic, and its smallest orientable quotient surface M_1 is a torus punctured in two points with total curvature -8π . Furthermore, M_1 is properly and minimally immersed in $\mathbb{R}^3/\langle v \rangle$, where $v \in \mathbb{R}^3 - \{0\}$ is a translation vector of M , and the punctures of M_1 correspond to planar ends of M .*

Proof. Let N be the Gauss map of M . Since $S_M \in L(N)$, its conjugate Jacobi function S_M^* is also linear; thus $S_M + iS_M^* \in L_{\mathbb{C}}(N)$. By equations (9) and (11), this linearity of $S_M + iS_M^*$ implies that there exists $a \in \mathbb{C}^3$ such that

$$(55) \quad \frac{3}{2} \left(\frac{g'}{g}\right)^2 - \frac{g''}{g} - \frac{1}{1+|g|^2} \left(\frac{g'}{g}\right)^2 = \langle N, a \rangle.$$

After writing $a = (a_1, a_2, a_3)$, $2a_1 = A + B$, $2a_2 = i(A - B)$ and plugging the equation (17) into (55), we obtain the following ODE for g :

$$\frac{3}{2} \left(\frac{g'}{g}\right)^2 - \frac{g''}{g} - \frac{1}{|g|^2 + 1} \left(\frac{g'}{g}\right)^2 = \frac{1}{|g|^2 + 1} (Ag + B\bar{g}) + a_3 \frac{|g|^2 - 1}{|g|^2 + 1}.$$

An algebraic manipulation in the last expression leads to

$$\bar{g} \left(\frac{3}{2} \frac{(g')^2}{g} - g'' - B - a_3g \right) = \frac{g''}{g} - \frac{1}{2} \left(\frac{g'}{g}\right)^2 + Ag - a_3.$$

Since g is holomorphic and not constant, we deduce that

$$\frac{3}{2} \frac{(g')^2}{g} - g'' - B - a_3g = 0, \quad \frac{g''}{g} - \frac{1}{2} \left(\frac{g'}{g}\right)^2 + Ag - a_3 = 0.$$

After elimination of g'' in both equations, we arrive at $(g')^2 = g(-Ag^2 + 2a_3g + B)$. Hence we have a (possibly branched) holomorphic covering $\pi = (g, g')$ from the cylinder $M \cup \{\text{planar ends}\} \cong \mathbb{C}/\langle i \rangle$ onto the compact Riemann surface $\Sigma = \{(\xi, w) \in (\mathbb{C} \cup \{\infty\})^2 \mid w^2 = \xi(-A\xi^2 + 2a_3\xi + B)\}$. Clearly, Σ is either a sphere or a torus. We claim that Σ cannot be a sphere. Otherwise, consider the meromorphic differential $\frac{d\xi}{w}$ on Σ , whose pullback by π is $\pi^*\left(\frac{d\xi}{w}\right) = \frac{dg}{g'} = dz$. Given a pole $P \in \Sigma$ of $\frac{d\xi}{w}$, choose a point $z_0 \in \mathbb{C}/\langle i \rangle$ such that $\pi(z_0) = P$. The residue of $\frac{d\xi}{w}$ at P can be computed as the integral of $\frac{d\xi}{w}$ along a small closed curve $\Gamma_P \subset \Sigma$ that winds once around P . After lifting Γ_P through π locally around z_0 , we obtain a closed curve $\tilde{\Gamma}_P \subset \mathbb{C}/\langle i \rangle$ that winds a positive integer number of times around z_0 , depending on the branching order of π

at z_0 . Hence the residue of $\frac{d\xi}{w}$ at P equals a positive integer multiple of the residue of dz at z_0 , which is zero. Therefore, $\frac{d\xi}{w}$ has residue zero at all its poles, and so it is exact on Σ . This implies that dz is also exact on $\mathbb{C}/\langle i \rangle$, which is a contradiction. Thus, Σ is a torus. Now consider the following Weierstrass pair on Σ :

$$\left(g_1(\xi, w) = \xi, dh_1 = \frac{d\xi}{w} \right).$$

The associated metric to this pair is $\left(\frac{1}{2}(|\xi| + |\xi|^{-1}) \frac{|d\xi|}{|w|} \right)^2$, which can be easily proven to be positive definite and complete in $\Sigma - \{(0, 0), (\infty, \infty)\}$. Note that $g_1 \circ \pi = g$ and $\pi^*\left(\frac{d\xi}{w}\right) = dz$. This implies that the Weierstrass pair (g, dz) of M can be induced on the twice punctured torus $\Sigma - \{(0, 0), (\infty, \infty)\}$. From here one easily deduces that M is singly-periodic. Note that since the degree of the extended Gauss map g_1 on Σ is 2, then the total curvature of the quotient minimal surface is -8π . \square

PROPOSITION 6.2. *If the Shiffman function S_M of an embedded surface $M \in \mathcal{M}$ is linear, then M is a Riemann minimal example.*

Proof. Let $\mathcal{M}_1 \subset \mathcal{M}$ be the subset of surfaces that are singly-periodic; their smallest orientable quotient is a properly embedded, twice-punctured minimal torus in a quotient of \mathbb{R}^3 by a translation. By Lemma 6.1, our proposition reduces to proving that \mathcal{M}_1 coincides with the family $\mathcal{R} = \{\mathcal{R}_t\}_t$ of Riemann minimal examples. This result is implied by the main theorem in [34]; see also Appendix 2 for a self-contained proof. Up to this reduction, the proposition is proved. \square

6.2. The linearity of the Shiffman function for every surface $M \in \mathcal{M}$. Recall that \mathcal{M} is the space of properly embedded, minimal planar domains $M \subset \mathbb{R}^3$ with two limit ends and flux $F = F_M = (h, 0, 1)$, $h = h(M) > 0$. \mathcal{M} is endowed with its natural topology of uniform convergence on compact sets. According to the notation in Theorem 3.1, the heights of the planar ends of every $M \in \mathcal{M}$ are

$$(56) \quad \dots < \mathfrak{R}(p_{-1}) < \mathfrak{R}(q_{-1}) < \mathfrak{R}(p_0) < \mathfrak{R}(q_0) < \mathfrak{R}(p_1) < \mathfrak{R}(q_1) < \dots.$$

Recall from Section 4.2 that we view \mathcal{M} as a subset of \mathcal{W} by the map $M \mapsto g$ that associates to each $M \in \mathcal{M}$ its meromorphic Gauss map $g \in \mathcal{W}$. For this inclusion to make sense we must identify surfaces in \mathcal{M} up to horizontal translations in \mathbb{R}^3 . (Two elements in \mathcal{W} that differ in a vertical translation are considered as different elements in \mathcal{W} , and the same holds for two surfaces in \mathcal{M} that differ in a vertical translation.) Indeed, $\mathcal{M} \subset \mathcal{M}_{\text{imm}} \subset \mathcal{W}$, where \mathcal{M}_{imm} is defined by (16). This identification of \mathcal{M} as a subset of \mathcal{W} is consistent with the topology of both sets (\mathcal{W} was equipped with the uniform

convergence of compact sets of $\mathbb{C}/\langle i \rangle$) because the convergence of surfaces in \mathcal{M} produces convergence of the corresponding Gauss maps in \mathcal{W} . (For this property, one needs to extend uniform convergence across the planar ends $g^{-1}(\{0\}) = \{p_j\}_j$, $g^{-1}(\{\infty\}) = \{q_j\}_j$.) Also recall that the topology of \mathcal{W} is equivalent to the product topology on $[\prod_{j \in \mathbb{Z}} (\mathbb{C}/\langle i \rangle)] \times (\mathbb{C} - \{0\})$ by the bijection $g \mapsto (p_j, q_j, g(z_0))$ defined in (12).

Next we want to define the functions that map each surface $M \in \mathcal{M}$ to the relative height of each of its planar ends with respect to the end corresponding to p_0 . We define the positive functions $h_j: \mathcal{M} \rightarrow \mathbb{R}$ for $j \in \mathbb{N}$ by

$$\begin{aligned} h_1(M) &= \Re(q_0 - p_0), & h_2(M) &= \Re(p_0 - q_{-1}), \\ h_3(M) &= \Re(p_1 - p_0), & h_4(M) &= \Re(p_0 - p_{-1}) \cdots . \end{aligned}$$

PROPOSITION 6.3. *Given $F = (h, 0, 1)$ with $h > 0$, let $\mathcal{M}_F = \{M \in \mathcal{M} \mid F_M = F\}$. Then*

- (1) *There exists a surface $M_{\max} \in \mathcal{M}_F$ that maximizes each of the functions h_{j+1} in $\mathcal{M}_F(j) = \{M \in \mathcal{M}_F(j-1) \mid h_j(M) = \max_{\mathcal{M}_F(j-1)} h_j\}$ for all $j \geq 1$, where $\mathcal{M}_F(0) = \mathcal{M}_F$. Also, there exists a surface $M_{\min} \in \mathcal{M}_F$ that minimizes each h_{j+1} in $\widetilde{\mathcal{M}}_F(j) = \{M \in \widetilde{\mathcal{M}}_F(j-1) \mid h_j(M) = \min_{\widetilde{\mathcal{M}}_F(j-1)} h_j\}$ for all $j \geq 1$. Furthermore, the Shiffman function of every such surface M_{\min}, M_{\max} is linear.*
- (2) *The Shiffman function of every surface $M \in \mathcal{M}$ is linear.*

Proof. By the uniform curvature estimates in Theorem 3.1 and subsequent uniform local area estimates, we deduce that \mathcal{M}_F is compact where \mathcal{M}_F is considered as a topological subspace of $[\prod_{j \in \mathbb{Z}} (\mathbb{C}/\langle i \rangle)] \times (\mathbb{C} - \{0\})$ with its metrizable product topology. Note that h_j is continuous. Thus, there exists a maximum of h_1 in \mathcal{M}_F . Now consider the restriction of h_2 to the nonempty subset $\mathcal{M}_F(1) = \{M \in \mathcal{M}_F \mid h_1(M) = \max_{\mathcal{M}_F} h_1\}$. As before, we can maximize h_2 on $\mathcal{M}_F(1)$, which implies that the space $\mathcal{M}_F(2)$ defined in the statement of the proposition is nonempty and maximize h_3 in $\mathcal{M}_F(2)$. Repeating the argument, induction lets us maximize h_{j+1} in $\mathcal{M}_F(j) \neq \emptyset$ for each $j \in \mathbb{N}$. Since the compact subsets $\mathcal{M}_F(j)$ satisfy $\mathcal{M}_F(j) \supset \mathcal{M}_F(j+1)$ for all j , this collection of closed sets satisfies the finite intersection property. By the compactness of \mathcal{M}_F , we conclude that $\bigcap_{j \in \mathbb{N}} \mathcal{M}_F(j) \neq \emptyset$. Thus there exists a surface $M_{\max} \in \mathcal{M}_F$ that maximizes each of the functions h_{j+1} in $\mathcal{M}_F(j)$ for all $j \geq 1$. In the same way, we find a surface $M_{\min} \in \mathcal{M}_F$ that minimizes the function h_{j+1} on $\widetilde{\mathcal{M}}_F(j)$ for all $j \geq 1$. This proves the first statement of item (1).

Next we prove that if $M_0 \in \mathcal{M}_F$ maximizes each of the functions h_{j+1} in $\mathcal{M}_F(j)$ for all $j \geq 1$, then its Shiffman function is linear. (For minimizing surfaces, the argument is similar.) By Theorem 5.14, the Shiffman function S_{M_0} of M_0 can be holomorphically integrated. Thus we find a curve

$t \in \mathbb{D}(\varepsilon) \mapsto g_t \in \mathcal{M}_{\text{imm}} \subset \mathcal{W}$ whose zeros $p_j(t)$ and poles $q_j(t)$ depend holomorphically on t , satisfying items (i), (ii) and (iii) of Definition 5.12 for $M = M_0$. With the notation of that definition, let $\psi_t: (\mathbb{C}/\langle i \rangle) - \{p_j(t), q_j(t)\}_j \rightarrow \mathbb{R}^3$ be the parametrization of M_t given by

$$\psi_t(z) = \Re \int_{z_0}^z \left(\frac{1}{2} \left(\frac{1}{g_t} - g_t \right), \frac{i}{2} \left(\frac{1}{g_t} + g_t \right), 1 \right) dz,$$

where z_0 has been chosen in $(\mathbb{C}/\langle i \rangle) - \{p_j(t), q_j(t) \mid j \in \mathbb{Z}, |t| < \varepsilon\}$. By equation (50), the normal part of the variational field of $t \in \mathbb{D}(\varepsilon) \mapsto \psi_t$ is (up to a multiplicative constant) the complex valued Shiffman function $S_{M_t} + iS_{M_t}^*$ of M_t plus a linear function of the Gauss map N_t of M_t ; we conclude from Remark 4.4 that $M_t \in \mathcal{M}_F$ for all t . Therefore, the harmonic function $t \in \mathbb{D}(\varepsilon) \mapsto h_1(M_t) = \Re(q_0(t) - p_0(t))$ attains a maximum at $t = 0$, and hence it is constant. From here we conclude that the holomorphic function $t \in \mathbb{D}(\varepsilon) \mapsto q_0(t) - p_0(t)$ is also constant. The same argument applies to each function $t \mapsto h_j(M_t)$, concluding that for any t , all the planar ends $p_j(t), q_j(t)$ of M_t are placed at

$$p_j(t) = p_0(t) + p_j - p_0, \quad q_j(t) = p_0(t) + q_j - p_0.$$

Geometrically, this means that the maps ψ_t coincide with ψ_0 up to translations in the parameter domain and in \mathbb{R}^3 . Therefore, the normal part of the variational field of $t \mapsto \psi_t$ is linear, as desired. This proves the second statement in item (1) of the proposition.

Finally, we prove item (2) of the proposition. Given $M \in \mathcal{M}$, let $F = (h, 0, 1)$ be the flux vector of M . By item (1), there exist surfaces $M_{\text{max}}, M_{\text{min}} \in \mathcal{M}_F$ such that M_{max} maximizes each of the functions h_{j+1} in $\mathcal{M}_F(j)$ (resp. M_{min} minimizes h_{j+1} in $\widetilde{\mathcal{M}}_F(j)$) for all $j \geq 1$. Furthermore, the Shiffman functions of $M_{\text{max}}, M_{\text{min}}$ are linear. By Proposition 6.2, both $M_{\text{max}}, M_{\text{min}}$ are Riemann minimal examples. Since the flux parametrizes the space of Riemann minimal examples, it follows that the Riemann minimal example \mathcal{R}_F in \mathcal{M}_F is unique up to translation. Thus, M_{max} and M_{min} are translations of \mathcal{R}_F . On the other hand, the vertical distance between the ends p_0, q_0 of M (with the notation in (56)) is bounded above (resp. by below) by the distance between the corresponding ends of M_{max} (resp. of M_{min}). Therefore, the vertical distance between the ends p_0, q_0 of M is maximal, or equivalently, M maximizes h_1 on \mathcal{M}_F . Analogously, M maximizes each of the functions h_{j+1} in $\mathcal{M}_F(j)$ for all $j \geq 1$ and applying item (1), its Shiffman function S_M is linear. Now the proof of the proposition is complete. \square

7. Linearity of bounded Jacobi functions on Riemann minimal examples

We devote this section to describing the set of bounded Jacobi functions on every Riemann minimal example, which is the goal of Theorem 7.1 below.

This result plays a central role in our proof in Section 8 that any limit end of a properly embedded minimal surface with finite genus and horizontal limit tangent plane at infinity converges exponentially in height to a limit end of one of the Riemann minimal examples.

THEOREM 7.1. *Let $\mathcal{R} = \mathcal{R}_t \subset \mathbb{R}^3$ be a Riemann minimal example. Then, any bounded Jacobi function on \mathcal{R} is linear.*

Proof. We first homothetically rescale \mathcal{R} so that its middle ends are placed at integer heights. Let N be the Gauss map of \mathcal{R} and Γ the zero set of the linear Jacobi function $\langle N, e_2 \rangle$, where $e_2 = (0, 1, 0)$. Γ consists of the horizontal straight lines in \mathcal{R} plus the intersection of \mathcal{R} with the (x_1, x_3) -plane (of reflective symmetry). Viewing the parameter domain as a cylinder $\mathbb{S}^1 \times \mathbb{R}$ punctured at integer heights, the reflection in the (x_1, x_3) -plane produces a reflection symmetry of the cylinder by a plane passing through its axis, and Γ is represented in the cylinder by the wider circles and straight lines in Figure 3 left.

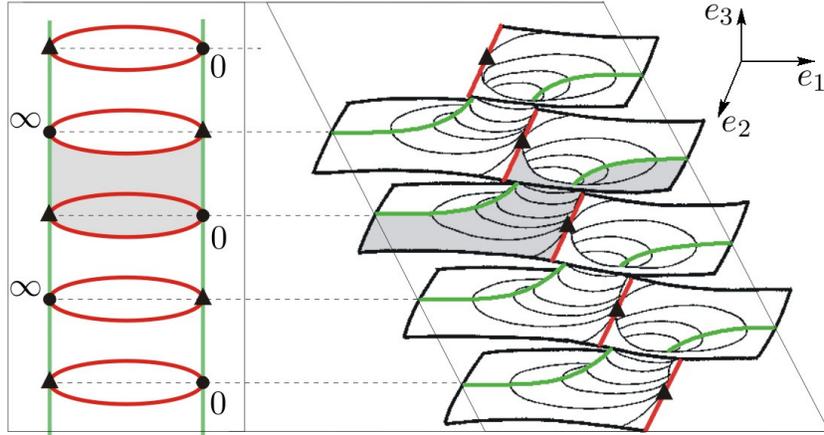


Figure 3. Left: The conformal model $\mathbb{S}^1 \times \mathbb{R}$. Right: A Riemann minimal example. Triangles denote finite branch points, and dots denote ends. The planar curves of symmetry together with the straight lines form the zero set of $\langle N, e_2 \rangle$, and we have shadowed one of its nodal domains.

Γ divides \mathcal{R} into infinitely many components Ω_i , $i \in \mathbb{Z}$, which we call *nodal domains*. (One of these nodal domains is shaded in Figure 3.) The branch values of N lie in the great circle $\mathbb{S}^2 \cap \{x_2 = 0\}$. Thus N restricts to Ω_i as a biholomorphism onto one of the open hemispheres in which $\{x_2 = 0\}$ divides \mathbb{S}^2 for each $i \in \mathbb{Z}$. Furthermore, such a biholomorphism has continuous

extension to the boundaries of these domains. Since the induced metric on Ω_i by the inner product of \mathbb{R}^3 and the spherical metric are conformally related by N , we can identify Ω_i with the hemisphere $N(\Omega_i)$ and express the stability form associated to the Jacobi operator as the quadratic form $\int_{\Omega_i} (|\nabla w|^2 - 2w^2) dA$ for any function in the Sobolev space $W^{1,2}(\Omega_i)$ with the spherical metric.

ASSERTION 7.2. *Let $v \in W^{1,2}(\Omega)$ be a bounded solution of $\Delta v + 2v = 0$ on a hemisphere $\Omega \subset \mathbb{S}^2$. Then*

$$\int_{\partial\Omega} v \frac{\partial v}{\partial \eta} ds \geq 0,$$

with equality if and only if v linear; i.e., $v(x) = \langle x, a \rangle$ for some $a \in \mathbb{R}^3$. (Here η stands for the exterior conormal unit vector to Ω along its boundary.)

Proof of Assertion 7.2. Clearly we can assume that Ω is one of the two hemispheres in $\mathbb{S}^2 - \{x_2 = 0\}$. Recall that the Neumann problem for the spherical Laplacian on Ω has first eigenvalue 0 (whose eigenfunctions are constant) and second eigenvalue 2, with eigenfunctions being of the type $x \in \Omega \mapsto \langle x, b \rangle$ with $b \in \mathbb{R}^3$ orthogonal to e_2 . In particular, for every function $w \in W^{1,2}(\Omega)$ with $\int_{\Omega} w dA = 0$, we have

$$(57) \quad \int_{\Omega} (|\nabla w|^2 - 2w^2) dA \geq 0,$$

and equality holds if and only if $w(x) = \langle x, b \rangle$ for any b orthogonal to e_2 .

Let φ be the restriction of $x_2 = \langle x, e_2 \rangle$ to \mathbb{S}^2 . The function φ has constant nonzero sign on Ω , and we can consider, for any bounded function $v \in W^{1,2}(\Omega)$, the real number $c = \frac{\int_{\Omega} v dA}{\int_{\Omega} \varphi dA}$. After applying (57) to $w = v - c\varphi$ on Ω , we obtain

$$(58) \quad \begin{aligned} 0 &\leq \int_{\Omega} [|\nabla(v - c\varphi)|^2 - 2(v - c\varphi)^2] dA \\ &= \int_{\Omega} (|\nabla v|^2 - 2v^2) dA \\ &\quad + c^2 \int_{\Omega} (|\nabla \varphi|^2 - 2\varphi^2) dA - 2c \int_{\Omega} (\langle \nabla v, \nabla \varphi \rangle - 2v\varphi) dA. \end{aligned}$$

Since $\varphi = 0$ on $\partial\Omega$, integration by parts gives

$$\int_{\Omega} (\langle \nabla v, \nabla \varphi \rangle - 2v\varphi) dA = - \int_{\Omega} \varphi (\Delta v + 2v) dA = 0.$$

In the same way, the second integral in (58) also vanishes, and the first integral in (58) is

$$(59) \quad \int_{\Omega} (|\nabla v|^2 - 2v^2) dA = \int_{\Omega} [\operatorname{div}(v\nabla v) - v(\Delta v + 2v)] dA = \int_{\partial\Omega} v \frac{\partial v}{\partial \eta} ds,$$

which gives the inequality in Assertion 7.2. If equality holds, then (57) implies that $v - c\varphi$ is linear on Ω . Hence, v is linear in Ω , which proves the

necessary condition in Assertion 7.2. The proof of the sufficient condition is straightforward.

We next continue with the proof of Theorem 7.1. Let $T = (T_1, 0, 2) \in \mathbb{R}^3 - \{0\}$ be the smallest orientation-preserving translation vector of \mathcal{R} . (Recall that the planar ends are placed at integer heights.) Take a bounded Jacobi function v on \mathcal{R} . We will prove that v is linear. For $j \in \mathbb{Z}$ fixed, denote by v_j the function on \mathcal{R} given by

$$v_j(p) = v(p + jT), \quad p \in \mathcal{R}.$$

Since $\{v_j\}_j$ is a sequence of bounded Jacobi functions on \mathcal{R} , after extracting a subsequence we can assume that as $j \rightarrow +\infty$, the sequence $\{v_j\}_j$ converges smoothly to a bounded Jacobi function $v_\infty: \mathcal{R} \rightarrow \mathbb{R}$ on compact subsets of \mathcal{R} ; see, for instance, [20]. In fact, both v_j, v_∞ extend smoothly through each planar end of \mathcal{R} and the smooth convergence $\{v_j\}_j \rightarrow v_\infty$ extends to the planar ends as well.

ASSERTION 7.3. v_∞ is linear on \mathcal{R} .

Proof of Assertion 7.3. By the convergence above, there exists a sequence of straight lines $r_k = \mathcal{R} \cap \{x_3 = j_k\}$ with $j_k \in \mathbb{N}$, $j_k \nearrow +\infty$ such that the values of v and its derivatives along r_k converge uniformly as $k \rightarrow +\infty$ to the values of v_∞ and its derivatives along the straight line $r = \mathcal{R} \cap \{x_3 = 0\}$. By Assertion 7.2 and equation (59), we have

$$(60) \quad 0 \leq \sum_j \int_{\Omega_j} (|\nabla v|^2 - 2v^2) dA = \int_{r_{k+1}} v \frac{\partial v}{\partial \eta} ds - \int_{r_k} v \frac{\partial v}{\partial \eta} ds,$$

where the sum runs in those nodal domains Ω_j between the heights of r_k and r_{k+1} . Taking $k \rightarrow +\infty$ in (59), we deduce that

$$(61) \quad \lim_{j \rightarrow +\infty} \int_{\Omega_j} (|\nabla v|^2 - 2v^2) = 0$$

for every sequence of nodal domains Ω_j with heights going to $+\infty$. Finally, the convergence $\{v_j\}_j \rightarrow v_\infty$ and (61) together with Assertion 7.2 imply that v_∞ is linear on any nodal domain of \mathcal{R} . By analyticity, v_∞ is linear on \mathcal{R} and Assertion 7.3 is proved.

The argument above can be repeated when $j \rightarrow -\infty$, from where we deduce that after passing to a subsequence, v_j converges as $j \rightarrow -\infty$ to a linear function $v_{-\infty}$ on \mathcal{R} (possibly distinct of v_∞). Furthermore, there exists a sequence of straight lines $r_k = \mathcal{R} \cap \{x_3 = j_k\}$ with $k \in \mathbb{Z} \mapsto j_k \in \mathbb{Z}$ increasing, such that the values of v and its derivatives along r_k converge uniformly as $k \rightarrow +\infty$ (resp. as $k \rightarrow -\infty$) to the values of v_∞ (resp. of $v_{-\infty}$) and its derivatives along the straight line $r = \mathcal{R} \cap \{x_3 = 0\}$.

Next consider the piece of \mathcal{R} bounded by the straight lines r_k, r_{-k} with $k \in \mathbb{N}$. The same arguments above demonstrate that for the nodal domains Ω_j between the heights of r_k and r_{-k} , we have

$$0 \leq \sum_j \int_{\Omega_j} (|\nabla v|^2 - 2v^2) dA = \int_{r_k} v \frac{\partial v}{\partial \eta} ds - \int_{r_{-k}} v \frac{\partial v}{\partial \eta} ds$$

$$\xrightarrow{(k \rightarrow +\infty)} \int_r v_\infty \frac{\partial v_\infty}{\partial \eta} ds - \int_r v_{-\infty} \frac{\partial v_{-\infty}}{\partial \eta} ds.$$

Thus it only remains to check that if w is a linear function on \mathcal{R} , then

$$(62) \quad \int_r w \frac{\partial w}{\partial \eta} ds = 0.$$

To prove (62) we again use the spherical geometry: the straight line r corresponds via the Gauss map N of \mathcal{R} to a twice covered geodesic arc $\gamma \subset \mathbb{S}^2 \cap \{x_2 = 0\}$ starting at the north or south pole and ending at a nonvertical branch value of N (when we view r in \mathbb{R}^3 , this description corresponds to traveling along r from one of its ends to the finite branch point P of N along r , and from P to the other end of r), and w corresponds to the height function $w(x) = \langle x, a \rangle$ for certain $a \in \mathbb{R}^3$. Then (62) holds since the conormal vector η at a point $x \in r$, viewed at one of the two halves of r , is opposite to the value of η at the same point x , viewed in the other half of r . Now the theorem is proved. \square

8. Asymptotic behavior of finite genus minimal surfaces

In this section, we will give the following descriptive result for the asymptotic behavior of every properly embedded minimal surface of finite genus and infinitely many ends.

THEOREM 8.1 (Asymptotic Limit End Property). *Let M be a properly embedded minimal surface in \mathbb{R}^3 with finite genus g and an infinite number of ends. Then, M has bounded curvature and after a possible rotation and a homothety, the following statements hold:*

- (1) *M has two limit ends. In fact, M is conformally diffeomorphic to $\overline{M} - \mathcal{E}_M$, where \overline{M} is a compact Riemann surface of genus g and $\mathcal{E}_M = \{e_n \mid n \in \mathbb{Z}\} \cup \{e_{-\infty}, e_\infty\}$ is a countable closed subset of \overline{M} with exactly two limit points e_∞ and $e_{-\infty}$. Furthermore, $\lim_{n \rightarrow -\infty} e_n = e_{-\infty}$ and $\lim_{n \rightarrow \infty} e_n = e_\infty$. The set of points \mathcal{E}_M is called the **space of ends** of M , the point $e_{-\infty}$ is called the **bottom end**, the point e_∞ is called the **top end** and every point e_n with $n \in \mathbb{Z}$ is called a **middle end** of M .*
- (2) *For each $n \in \mathbb{Z}$, there exists a punctured disk neighborhood $E_n \subset M \subset \overline{M}$ of e_n that is asymptotic in \mathbb{R}^3 to a horizontal plane Π_n and that is a graph over its projection to Π_n . Furthermore, the usual linear ordering on the*

index set \mathbb{Z} respects the linear ordering of the heights of the related planes. The ordered set of heights $H = \{h_n = x_3(\Pi_n) \mid n \in \mathbb{Z}\}$ of these planes naturally corresponds to the set of heights of the middle ends of M .

- (3) There exists a positive constant C_M such that if $|t| > C_M$, then the horizontal plane $\{x_3 = t\}$ intersects M in a proper arc when $t \in H$, or otherwise, $\{x_3 = t\}$ intersects M in a simple closed curve.
- (4) Let η denote the unitary outward conormal along the boundary of $M_t = M \cap \{x_3 \leq t\}$. Then the flux vector of M , which is defined to be

$$F_M = \int_{\partial M_t} \eta \, ds$$

(here ds stands for the length element), is independent of the choice of t and has the form $F_M = (h, 0, 1)$ for some $h > 0$.

- (5) Let $\mathcal{R}_h \subset \mathbb{R}^3$ be the Riemann minimal example with horizontal tangent plane at infinity and flux vector $F = (h, 0, 1)$ along a compact horizontal section. Then, there exists a translation vector $v_\infty \in \mathbb{R}^3$ such that as $t \rightarrow \infty$, the function

$$d_+(t) = \sup \{ \text{dist}(p, \mathcal{R}_h + v_\infty) \mid p \in M \cap \{x_3 \geq t\} \}$$

is finite and decays exponentially to zero. In a similar manner, there exists $v_{-\infty} \in \mathbb{R}^3$ such that as $t \rightarrow -\infty$, the function

$$d_-(t) = \sup \{ \text{dist}(p, \mathcal{R}_h + v_{-\infty}) \mid p \in M \cap \{x_3 \leq t\} \}$$

is finite and decays exponentially to zero. Furthermore, $x_2(v_\infty) = x_2(v_{-\infty})$ and for t large, $M \cap \{x_3 \geq t\}$ (resp. $M \cap \{x_3 \leq -t\}$) can be expressed as a small (with arbitrarily small C^k -norm for any k) normal graph over its projection on $\mathcal{R}_h + v_\infty$ (resp. on $\mathcal{R}_h + v_{-\infty}$).

Proof. Let $M \subset \mathbb{R}^3$ be a properly embedded minimal surface satisfying the hypotheses of Theorem 8.1. M has exactly two limit ends by Theorem 1 in [36]. In this situation, Theorem 3.5 in [13] gives that between any two middle ends of M , there is a plane that intersects M transversely in a compact set. It follows that all middle ends of M are planar. Now items (1), (2), (3) and (4) of the theorem follow from similar arguments as those giving items (1), (2), (3) and (4) of Theorem 3.1.

Item (3) of Theorem 8.1 allows us to reduce the proof of the property that M has bounded curvature to proving it for the three regions $M \cap \{|x_3| \leq C_M\}$, $M \cap \{x_3 > C_M\}$ and $M \cap \{x_3 \leq -C_M\}$. The region $M \cap \{|x_3| \leq C_M\}$ consists of a finite number of graphs outside of a compact set, and thus it has bounded Gaussian curvature. The other two regions have genus zero, and one can argue similarly as we explained in the proof of Theorem 3.1.

It remains to prove item (5) of the theorem. The already proven items (1), (2), (3) and (4) of the theorem imply that there exists a translation vector

$T \in \mathbb{R}^3$ such that $M^+ = (M - T) \cap \{x_3 \geq 0\}$ can be conformally parametrized by the half-cylinder $\mathbb{C}^+/\langle i \rangle$ (here $\mathbb{C}^+ = \{x + iy \mid x \geq 0\}$) punctured in an infinite discrete set of interior points $\{p_j, q_j\}_{j \in \mathbb{N}}$, which represent respectively those ends of M^+ where its Gauss map points to the north and south poles. In this setting, the proofs of items (5), (6) and (7) of Theorem 3.1 remain valid with $M, \mathbb{C}, \mathbb{Z}$ replaced by $M^+, \mathbb{C}^+, \mathbb{N}$, with the only change being in the statement of item (7), where the last two instances of $\mathbb{C}/\langle i \rangle$ are left unchanged, by choosing the third coordinate of T sufficiently large. We remark that these arguments rely solely on Colding-Minicozzi theory for planar domains.

Consider the Shiffman function S_{M^+} of M^+ , which exists by item (3) of the theorem. Reasoning as in the last paragraph of Section 4.1, we deduce that S_{M^+} extends smoothly through the points p_j, q_j to a function on $\mathbb{C}^+/\langle i \rangle$, which we also denote by S_{M^+} . Note that S_{M^+} is asymptotic to zero on the end of $\mathbb{C}^+/\langle i \rangle$, because any sequence of translations of M^+ diverging in height, up to a horizontal translation (which may depend upon the sequence), converges by Theorem 1.1 to the Riemann minimal example \mathcal{R}_h whose flux vector is $(h, 0, 1)$. In particular, S_{M^+} is bounded and can be considered to be a function in $\mathbb{S}^1 \times [0, \infty)$. The next lemma gives a control on the decay of $S_{M^+}: \mathbb{S}^1 \times [0, \infty) \rightarrow \mathbb{R}$.

LEMMA 8.2. *There exist $C, a > 0$ so that $|S_{M^+}(\theta, t)| \leq Ce^{-at}$ for all $(\theta, t) \in \mathbb{S}^1 \times [0, \infty)$.*

Proof. We first prove the following assertion.

ASSERTION 8.3. *Let $f = f(t): [0, \infty) \rightarrow [0, \frac{1}{2}]$ be a continuous function such that*

- (1) $f(t) \rightarrow 0$ as $t \rightarrow \infty$.
- (2) For any $a > 0$, there exists $t(a) \geq 0$ such that $f(t(a)) \geq 2^{-at(a)}$.

Then for each $n \in \mathbb{N}$, there exists $T_n \geq n$ such that $f(t) \leq 2f(T_n)$ for $t \in [T_n - n, \infty)$.

Proof of Assertion 8.3. Fix $n \in \mathbb{N}$. Let $t_n \in [0, \infty)$ be the smallest t such that $f(t) = 2^{-\frac{t}{n}}$. (The existence of t_n follows from $f(0) \leq \frac{1}{2}$ and from hypothesis (2).) Since $f(t_n) \leq \frac{1}{2}$, then $t_n \geq n$. Let $T_n \in [t_n, \infty)$ be a point where f attains its maximum value. (T_n exists by hypothesis (1).) We now prove that $f(t) \leq 2f(T_n)$ for all $t \in [T_n - n, \infty)$ by discussing two possibilities.

Assume $T_n - n \leq t_n$. If $t \in [t_n, \infty)$, then $f(t) \leq f(T_n) \leq 2f(T_n)$, and if $t \in [T_n - n, t_n]$, then $f(t) \leq 2^{-t/n} \leq 2^{-\frac{(t_n - n)}{n}} = 2 \cdot 2^{-t_n/n} = 2f(t_n) \leq 2f(T_n)$.

Assume $T_n - n > t_n$. Then $[T_n - n, \infty) \subset [t_n, \infty)$, so we apply the first two inequalities in the case above. This completes the proof of Assertion 8.3. \square

We now continue the proof of the lemma. Arguing by contradiction, suppose that $S_{M^+}(\theta, t)$ does not decay exponentially. Choose a constant $C > 0$ such that the function

$$f(t) = C \left| \max_{\theta \in \mathbb{S}^1} S_{M^+}(\theta, t) \right|$$

satisfies $f(t) \leq \frac{1}{2}$ for all t . By Assertion 8.3, there exist sequences $T_n \in [0, \infty)$, $\theta_n \in \mathbb{S}^1$ such that $T_n \geq n$ and $C|S_{M^+}(\theta, t)| \leq 2f(T_n)$ for every $(\theta, t) \in \mathbb{S}^1 \times [T_n - n, \infty)$. For any $n \in \mathbb{N}$, consider the function

$$h_n(\theta, t) = \frac{C}{f(T_n)} S_{M^+}(\theta, t + T_n),$$

defined on $\mathbb{S}^1 \times (-n, \infty)$. By construction, $|h_n| \leq 2$ and $|h(\theta(n), 0)| = 1$ for some $\theta(n) \in \mathbb{S}^1$. Therefore, after extracting a subsequence, the h_n converge to a bounded Jacobi function h_∞ on the Riemann minimal example \mathcal{R}_h , considered to be a function defined on the cylinder $\mathbb{S}^1 \times \mathbb{R}$.

By Theorem 7.1, h_∞ is linear, and so $h_\infty = \langle N_\infty, a \rangle$ for some $a \in \mathbb{R}^3 - \{0\}$, where N_∞ is the Gauss map of \mathcal{R}_h . By the Four Vertex Theorem, the Shiffman Jacobi function S_{M^+} has at least four zeros at each compact horizontal section of M^+ , and so the same holds for each of the functions h_n , which contradicts the assertion below and completes the proof of the lemma. \square

ASSERTION 8.4. *Given a Riemann minimal example \mathcal{R} with Gauss map N and a vector $a \in \mathbb{R}^3 - \{0\}$, there is a horizontal circle $\Gamma = \mathcal{R} \cap \{x_3 = t\}$ such that the linear function $v = \langle N, a \rangle$ has at most two zeros on Γ . Moreover, these zeros are nondegenerate.*

Proof. Consider the great circle $\gamma_a = \mathbb{S}^2 \cap \{x \in \mathbb{R}^3 \mid \langle x, a \rangle = 0\}$ and a horizontal line $L \subset \mathcal{R}$. We can assume that $L = \mathcal{R} \cap \{x_3 = 0\}$ and that $N(p_0) = (0, 0, 1)$, where p_0 denotes the end of \mathcal{R} at level $x_3 = 0$. The Gaussian image $N(L)$ consists of a twice covered geodesic arc in $\mathbb{S}^2 \cap \{x_2 = 0\}$ whose extrema are $(0, 0, 1)$ with $N(p) \in \mathbb{S}^2$, where $\{p\} = L \cap \{x_2 = 0\}$ is the unique branch point of N along L .

If a is horizontal, then γ_a passes through the north and south poles. As the Gauss image of any compact horizontal circle Γ on \mathcal{R} winds once around the north and south poles of \mathbb{S}^2 , it follows that Γ intersects γ_a transversely into two points. Thus, the assertion holds in this case.

If a is not horizontal, then we discuss the following cases:

(1) *The great circle γ_a and the geodesic arc $N(L)$ are disjoint.* In this case, we have that v does not vanish along $\Gamma_t = \mathcal{R} \cap \{x_3 = t\}$ for $t > 0$ small enough.

(2) *The great circle γ_a meets the interior of the geodesic arc $N(L)$.* Then $v(p) \neq 0$, $v(p_0) \neq 0$ and v has exactly two zeros along L , which are nondegenerate. It follows that v has just two nondegenerate zeros along the nearby circle Γ_t for any small positive t .

(3) *The great circle γ_a passes through the point $N(p)$.* If we parametrize L by $\gamma(s) = p + se_2$ where $e_2 = (0, 1, 0)$, then we have that $(v \circ \gamma)(0) = \langle N(p), a \rangle = 0$. Also note that $(v \circ \gamma)'(0) = \langle (N \circ \gamma)'(0), a \rangle = 0$ since p is a branch point of N . We claim that $(v \circ \gamma)''(0) \neq 0$: otherwise, $(N \circ \gamma)''(0) = \lambda N(p) + \mu a \times N(p)$ for certain $\lambda, \mu \in \mathbb{R}$, where \times denotes cross product. But $(N \circ \gamma)''(0) \times N(p) \neq 0$ (because this is the tangent component to \mathcal{R} of $(N \circ \gamma)''(0)$, and N has ramification order 1 at p); hence $\mu \neq 0$, which in turn implies that $a \times N(p)$ is orthogonal to e_2 (because both $(N \circ \gamma)''(0)$ and $N(p)$ are orthogonal to e_2). Since e_2 is also orthogonal to $N(p)$, we deduce that e_2 is parallel to a , a contradiction. Therefore, $(v \circ \gamma)''(0) \neq 0$. From here we conclude that, for small positive t , either v does not vanish along Γ_t or it has just two distinct simple zeros along Γ_t . \square

Next we prove the first statement in item (5) of Theorem 8.1, namely, that the exponential convergence of the top end of M to a translated image of the top end of the Riemann minimal example $\mathcal{R} = \mathcal{R}_h$ with flux $(h, 0, 1)$ equal to the flux of M . (The corresponding property for bottom ends follows similarly.) During this proof, we will make clear that the graphing property in the last statement of item (5) of the theorem also holds. Since we will use the notion of surface written as a graph over another surface, we first make this notion precise. We will consider minimal surfaces Σ in a horizontal slab $\{a \leq x_3 \leq b\}$, bounded by two Jordan curves, one in each boundary plane of the slab. (In particular, Σ is transversal to the boundary of the slab.) Furthermore, Σ will have genus zero and (possibly) finitely many horizontal planar ends. Thus, after compactification at the planar ends, we obtain $\bar{\Sigma}$, which is conformally a cylinder $\mathbb{S}^1(r) \times [a, b]$ for certain $r > 0$. We take on $\bar{\Sigma}$ a unitary, smooth, transversal vector field ν such that

- $\nu|_{\partial\Sigma}$ coincides with one of the two horizontal, normal vector fields to the planar curves that bound Σ .
- If Σ is noncompact, then $\nu = \pm(0, 0, 1)$ in a neighborhood of each of the planar ends of Σ .

Note that ν can be thought as a deformation of the Gauss map of Σ . Although ν is not unique, we will assume that given a surface Σ we have made a choice of this transversal vector field. If we have a second surface Σ' under the same conditions as Σ , then we say that Σ' is a *graph over Σ* if it can be written as the graph over Σ of a function $u \in C^2(\bar{\Sigma})$, in the direction of ν ; i.e.,

$$p \in \Sigma \mapsto p + u(p)\nu(p) \in \Sigma'$$

The notation $\|\Sigma - \Sigma'\|_{C^{k,\alpha}}$ that appear below will stand for $\|u\|_{C^{k,\alpha}}$ (taken with respect to the flat metric in the cylinder $\bar{\Sigma}$), where we are assuming that Σ' is the graph of a function u over Σ in the sense above.

All the above observations can be easily translated to minimal surfaces in a half-space above or below a horizontal plane. We leave the details to the reader.

By quasiperiodicity of M^+ , we can choose a sequence of positive numbers $\{b_n\}_{n \in \mathbb{N}} \nearrow \infty$ with $M^+ = M \cap \{x_3 \geq b_1\}$, satisfying the following properties:

- $\{b_{n+1} - b_n\}_n$ is bounded away from zero and bounded above.
- The surface M^+ intersects transversely the horizontal plane $\{x_3 = b_n\}$ in a Jordan curve Γ_n , whose length is bounded above independently of n .
- The surface $\Sigma_n = M \cap \{b_n \leq x_3 \leq b_{n+1}\}$ is either a compact annulus or has just one planar end. Furthermore, Σ_n has total curvature smaller than π .

As a consequence of Lemma 8.2 and linear elliptic theory, we have that the derivatives of S_{M^+} of any order also decay exponentially and, from the definition of the Shiffman function, we deduce that for n large, the curves Γ_n can be exponentially approximated by horizontal circles $\Gamma'_n \subset \{x_3 = b_n\}$, in the sense that

$$\|\Gamma_n - \Gamma'_n\|_{C^{4,\alpha}} \leq C_1 e^{-ab_n},$$

for certain constant $C_1 > 0$ independently of n .

As the total curvature of Σ_n is less than π , then Σ_n is stable and there are no bounded Jacobi functions on Σ_n vanishing at its boundary. An application of the Implicit Function Theorem in the Banach space context (see, for instance, White [57] for the compact case and Pérez and Ros [46] and Pérez [45] for the necessary modifications in the case Σ_n has a planar end) implies that there exists an embedded minimal surface R_n , described as a graph over Σ_n whose boundary is $\partial R_n = \Gamma'_n \cup \Gamma'_{n+1}$, and such that

$$(63) \quad \|\Sigma_n - R_n\|_{C^{2,\alpha}} \leq C_2 e^{-ab_n}.$$

Recall that the notion of graph over Σ_n depends on the choice of a transversal vector field ν_n along Σ_n . By the quasiperiodicity of M^+ , we can assume that both Σ_n and ν_n have geometry uniformly bounded in n , and $\nu_{n-1} = \nu_n$ along Γ_n . In particular, the constant C_2 in (63) can be chosen independently of n . In the sequel, we will find other positive constants independent of n , which will be denoted by C_3, C_4, \dots .

Also note that R_n is compact when Σ_n is compact and R_n has a horizontal planar end when Σ_n has a planar end. By the maximum principle, $R_n \subset \{b_n \leq x_3 \leq b_{n+1}\}$. Furthermore, the horizontal sections of R_n are either closed curves or an open arc. (This last case occurs at the height of the planar end of R_n , if it exists.) It also follows from (63) that the total curvature of R_n is smaller than $3\pi/2$, and so R_n is strictly stable [1]. Since the Shiffman function is well defined and bounded on R_n and vanishes at ∂R_n , the stability of R_n implies that its Shiffman function vanishes identically. Thus, R_n is a piece of a Riemann minimal example. Furthermore, (63) implies that the flux of the

Riemann minimal example that contains R_n is exponentially close to the flux $F_{M^+} = F_{\mathcal{R}}$.

As a consequence, there exists a piece $R'_n = (\mathcal{R} + v_n) \cap \{b_n \leq x_3 \leq b_{n+1}\}$ of a translated image of the Riemann minimal example \mathcal{R} such that $\|R_n - R'_n\|_{C^{2,\alpha}} \leq C_3 e^{-ab_n}$ for n large. By the triangle inequality, we have

$$(64) \quad \|\Sigma_n - R'_n\|_{C^{2,\alpha}} \leq C_4 e^{-ab_n}.$$

We next explain how to conclude all statements in item (5) of Theorem 8.1, except the property that $x_2(v_\infty) = x_2(v_{-\infty})$, which will be proven later. It is enough to prove the following statement.

ASSERTION 8.5. *There exists a vector $v_\infty \in \mathbb{R}^3$ such that if $W_n = \{b_n \leq x_3 \leq b_{n+1}\}$, then*

$$\|M^+ - (\mathcal{R} + v_\infty)\|_{C^{2,\alpha}(W_n)} \leq C_5 e^{-ab_n}.$$

Proof. By (64) applied to Σ_{n-1} and Σ_n that share the common boundary curve Γ_n (recall that the transversal vector fields ν_{n-1}, ν_n both coincide along Γ_n), we have that both $(\mathcal{R} + v_n) \cap \{x_3 = b_n\}$, $(\mathcal{R} + v_{n-1}) \cap \{x_3 = b_n\}$ are exponentially close to Γ_n in the norm $\|\cdot\|_{C^{2,\alpha}}$. Since \mathcal{R} is a periodic surface, we can choose v_n so that $\|v_n - v_{n-1}\| \leq C_6 e^{-ab_n}$. The triangle inequality gives

$$(65) \quad \|v_{n+k} - v_n\| \leq C_6 \sum_{j=n+1}^{n+k} e^{-ab_j}.$$

The convergence of the series $\sum_{j=1}^\infty e^{-ab_j}$ shows that $\{v_n\}_n$ is a Cauchy sequence, and so it converges to a vector $v_\infty \in \mathbb{R}^3$. Finally,

$$\begin{aligned} \|M^+ - (\mathcal{R} + v_\infty)\|_{C^{2,\alpha}(W_n)} &\leq \|M^+ - (\mathcal{R} + v_n)\|_{C^{2,\alpha}(W_n)} \\ &\quad + \|(\mathcal{R} + v_\infty) - (\mathcal{R} + v_n)\|_{C^{2,\alpha}(W_n)} \\ &\leq C_4 e^{-ab_n} + C_7 \|v_\infty - v_n\|, \end{aligned}$$

where in the last equality we have used (64) for the first summand and the fact that $\mathcal{R} + v_\infty$ and $\mathcal{R} + v_n$ differ by a small translation (namely, $v_\infty - v_n$). Finally, (65) implies

$$\|M^+ - (\mathcal{R} + v_\infty)\|_{C^{2,\alpha}(W_n)} \leq C_4 e^{-ab_n} + C_6 \sum_{j=n+1}^\infty e^{-ab_j} \leq C_5 e^{-ab_n},$$

which completes the proof of the assertion. (Consequently, item (5) of Theorem 8.1 is proved except for the property stated in the following lemma.) \square

LEMMA 8.6. $x_2(v_\infty) = x_2(v_{-\infty})$.

Proof. Recall that $\mathcal{R} = \mathcal{R}_h \subset \mathbb{R}^3$ is the Riemann minimal example with the same flux vector $(h, 0, 1)$ as M . Let T be the smallest orientation-preserving

translation vector of \mathcal{R} , with $x_3(T) > 0$. Also, assume that \mathcal{R} is normalized by a translation so that the (x_1, x_2) -plane intersects \mathcal{R} in a circle γ . By the first statement in item (5) of Theorem 8.1, for $n \in \mathbb{N}$ large, the curve $\gamma(n) = M \cap \{x_3 = x_3(v_\infty + nT)\}$ is closely approximated by the horizontal circle $\gamma + v_\infty + nT \subset \mathcal{R} + v_\infty$. Similarly, for n large, $\gamma(-n) = M \cap \{x_3 = x_3(v_{-\infty} - nT)\}$ is closely approximated by the horizontal circle $\gamma + v_{-\infty} - nT \subset \mathcal{R} + v_{-\infty}$.

As M is minimal, the \mathbb{R}^3 -valued one-form $\alpha: M \rightarrow \mathbb{R}^3$ given by $\alpha_p(v) = p \times v$ for all $p \in M$ and $v \in T_p M$ has divergence zero. The Divergence Theorem applied to α on a compact subdomain $\Omega \subset M$ gives

$$(66) \quad \vec{0} = \int_{\partial\Omega} \alpha(\eta) ds = \int_{\partial\Omega} p \times \eta ds,$$

where η denotes the exterior conormal field to Ω along its boundary. We now choose a domain Ω adapted to our setting: For n large, label by $A(n)$ the component of $M - [\gamma(n) \cup \gamma(-n)]$ whose boundary is $\partial A(n) = \gamma(n) \cup \gamma(-n)$. The proper domain $A(n)$ contains a finite positive number $l(n)$ of planar ends. For each end e_k in $A(n)$, choose an embedded curve $\beta_k \subset A(n)$ around this end, the β_k curves being disjoint. Finally, define $\Omega(n)$ to be the compact subdomain of $A(n)$ bounded by $\gamma(n) \cup \gamma(-n) \cup \beta_1 \cup \dots \cup \beta_{l(n)}$. Then, (66) can be written as

$$(67) \quad \vec{0} = \int_{\gamma(n)} p \times \eta ds + \int_{\gamma(-n)} p \times \eta ds + \sum_{k=1}^{l(n)} \int_{\beta_k} p \times \eta ds.$$

Each integral along β_k in the summation above is the torque vector associated to the end e_k . This is a horizontal vector pointing to the direction of the straight line asymptotic to the (noncompact) level section at the height of the end e_k . Thus, the third term in (67) will disappear after taking inner products with $e_3 = (0, 0, 1)$. Concerning the first integral in (67), we can estimate it as the corresponding integral over the translated Riemann minimal example $\mathcal{R} + v_\infty$ up to an error ε_n such that $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$:

$$\begin{aligned} \int_{\gamma(n)} p \times \eta ds &= \varepsilon_n + \int_{\gamma + v_\infty + nT} p(\mathcal{R} + v_\infty) \times \eta(\mathcal{R} + v_\infty) ds(\mathcal{R} + v_\infty) \\ &= \varepsilon_n + \int_{\gamma} [p(\mathcal{R}) + v_\infty + nT] \times \eta(\mathcal{R}) ds(\mathcal{R}) \\ &= \varepsilon_n + \int_{\gamma} p(\mathcal{R}) \times \eta(\mathcal{R}) ds(\mathcal{R}) + [v_\infty + nT] \times \text{Flux}(M), \end{aligned}$$

where $\eta(\mathcal{R} + v_\infty)$ is the unitary conormal vector to $\mathcal{R} + v_\infty$ along $\gamma + v_\infty + nT$ (we follow a similar notation for $p(\mathcal{R}), ds(\mathcal{R})$), and we have used that $\int_{\gamma} \eta(\mathcal{R}) ds(\mathcal{R}) = \text{Flux}(\mathcal{R}) = \text{Flux}(M)$.

If we repeat the same argument along $\gamma(-n)$, then we must take into account that the exterior conormal vectors to $\Omega(n)$ along $\gamma(n)$ and $\gamma(-n)$ are

almost opposite; thus, after taking limits, the conormal vector $\eta(\mathcal{R})$ along γ in the integral of the right-hand-side of the last displayed expression associated to $\gamma(n)$ is opposite to the corresponding one for $\gamma(-n)$. Therefore, equation (67) implies

$$\begin{aligned} 0 &= \lim_{n \rightarrow \infty} \left\langle \int_{\gamma(n)} p \times \eta \, ds + \int_{\gamma(n)} p \times \eta \, ds, e_3 \right\rangle \\ &= \langle [(v_\infty - v_{-\infty}) + 2nT] \times \text{Flux}(M), e_3 \rangle \\ &= \langle [(v_\infty - v_{-\infty}) + 2nT], \text{Flux}(M) \times e_3 \rangle. \end{aligned}$$

Since $\text{Flux}(M) \times e_3$ is a nonzero vector pointing to the x_2 -axis, we deduce that $x_2(v_\infty) = x_2(v_{-\infty})$, which completes the proof of the lemma. (Thus, the proof of Theorem 8.1 is complete.) \square

Remark 8.7. By Theorem 1.1, any properly embedded with genus zero and an infinite number of ends is a Riemann minimal example. In the statement of this result, it is natural to replace the hypothesis of properness by completeness. The authors have found a proof of this result in the complete setting under the additional hypothesis that the surface has countably many ends [38]. We remark that every properly embedded minimal surface in \mathbb{R}^3 has a countable number of ends, regardless of its genus [13].

9. Appendix 1: Finite dimensionality of the space of bounded Jacobi functions

We saw in Corollary 4.15 that the Shiffman function S_M associated to a quasiperiodic, immersed minimal surface M of Riemann type extends smoothly through the planar ends of M to a bounded, smooth function on the cylinder $\mathbb{C}/\langle i \rangle$. Given such an immersed minimal surface M , in Section 5.2 we produced a sequence $\{v_n\}_n$ of (complex valued) Jacobi functions on M , one of whose terms is $S_M + iS_M^*$. A key point in the holomorphic integration of S_M is that the sequence $\{v_n\}_n$ only has a finite number of linearly independent functions. This property follows from two facts: firstly, the fact that each Jacobi function v_n extends smoothly to a bounded function on $\mathbb{C}/\langle i \rangle$ (Theorem 5.8) and secondly, that the bounded kernel of the Jacobi operator on a quasiperiodic, immersed minimal surface of Riemann type is finite dimensional. This finite dimensionality can be deduced from Theorem 0.5 in Colding, de Lellis and Minicozzi [6]. For the sake of completeness, we provide a direct proof of this property communicated to us by to Frank Pacard.

THEOREM 9.1. *Let $M \subset \mathbb{R}^3$ be a quasiperiodic, immersed minimal surface of Riemann type. Then, the linear space $\mathcal{J}_\infty(M) = \{v \in \mathcal{J}(M) \mid v \text{ is bounded}\}$ is finite dimensional.*

Proof. By definition, M is conformally equivalent to $(\mathbb{C}/\langle i \rangle) - g^{-1}(\{0, \infty\})$, where $g \in \mathcal{M}_{\text{imm}}$ is the Gauss map of M . Recall that $\mathbb{C}/\langle i \rangle$ is isometric to $\mathbb{S}^1 \times \mathbb{R}$. Take global coordinates (θ, t) on $\mathbb{S}^1 \times \mathbb{R}$, and consider the product metric $d\theta^2 \times dt^2$, which is conformal to the metric ds^2 on M induced by the usual inner product of \mathbb{R}^3 . If we write N for the Gauss map of M and $ds^2 = \lambda^2(d\theta^2 \times dt^2)$, then the Jacobi operator $L = \Delta + |\sigma|^2 = \Delta + |\nabla N|^2$ of M is $L = \lambda^{-2}L_M$, where $L_M = (\Delta_{\mathbb{S}^1} + \partial_t^2) + V_M$ is a Schrödinger operator on $\mathbb{S}^1 \times \mathbb{R}$ with potential V_M given by the square of the norm of the differential of $N = N(\theta, t)$ (with respect to $d\theta^2 \times dt^2$). Since M is quasiperiodic, V_M is globally bounded on $\mathbb{S}^1 \times \mathbb{R}$.

By elliptic regularity, any bounded Jacobi function v on M extends through the zeros and poles of g to a smooth function \hat{v} in the kernel of the operator L_M , such that \hat{v} is bounded at both ends of $\mathbb{S}^1 \times \mathbb{R}$. Therefore, the space $\mathcal{J}_\infty(M)$ of bounded Jacobi functions identifies naturally with the bounded kernel of L_M . Now the theorem is a consequence of part (1) of a more general result proved by Colding, de Lellis and Minicozzi, namely, Theorem 0.5 in [6]. For the sake of completeness, we provide a direct proof of the finite dimensionality of the bounded kernel of L_M in a simpler setting, which is sufficient for our purposes; see Assertion 9.3 below. We thank Frank Pacard, who communicated this argument to us. Modulo Assertion 9.3 and Remark 9.4 below, Theorem 9.1 is proved. \square

Let Σ be a compact manifold endowed with a Riemannian metric h , and let $\lambda_0 = 0 < \lambda_1 < \lambda_2 < \dots$ be the eigenvalues of the Laplacian $-\Delta_h$. Our goal is to give a sufficient condition under which the bounded kernel of the Schrödinger operator on the metric cylinder $(\Sigma \times \mathbb{R}, h \times dt^2)$ given by $(\Delta_h + \partial_t^2) + V$, where $V \in L^\infty(\Sigma \times \mathbb{R})$, has finite dimension. Such a condition will relate the spectral gaps $\{\lambda_{j+1} - \lambda_j\}_j$ of $-\Delta_h$ and $\|V\|_{L^\infty(\Sigma \times \mathbb{R})}$; see Assertion 9.3 below.

We first study the operator $\Delta_h + \partial_t^2$ acting on functions belonging to the weighted space $e^{\delta t}L^2(\Sigma \times \mathbb{R})$, where δ is a real number. (We will assume from now on that $\Sigma \times \mathbb{R}$ is endowed with the product metric $h \times dt^2$.) The following result is a refinement of some ideas in the paper of Lockhart and McOwen [29].

ASSERTION 9.2. *Assume that $\delta \in \mathbb{R}$ is chosen so that $\delta^2 \neq \lambda_j$ for all $j \geq 0$. Then, if $(\Delta_h + \partial_t^2)U = F$ with $U, F \in e^{\delta t}L^2(\Sigma \times \mathbb{R})$, we have*

$$(68) \quad \|e^{-\delta t}U\|_{L^2(\Sigma \times \mathbb{R})} \leq \frac{1}{\inf_j |\delta^2 - \lambda_j|} \|e^{-\delta t}F\|_{L^2(\Sigma \times \mathbb{R})}.$$

Proof. First observe that the functions $u = e^{-\delta t}U$, $f = e^{-\delta t}F$ belong to $L^2(\Sigma \times \mathbb{R})$ and $f = e^{-\delta t}(\Delta_h + \partial_t^2)(e^{\delta t}u)$. We perform the Fourier transform of $t \mapsto u(y, t)$ and $t \mapsto f(y, t)$ for $y \in \Sigma$ fixed, defining the complex valued functions

$$\hat{u}(y, s) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} u(y, t)e^{ist} dt, \quad \hat{f}(y, s) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(y, t)e^{ist} dt$$

for all $(y, s) \in \Sigma \times \mathbb{R}$. To keep notations short, we set $z := \delta - is \in \mathbb{C}$. It is straightforward to check that given an $s \in \mathbb{R}$, the functions $\widehat{u}(\cdot, s)$, $\widehat{f}(\cdot, s)$ satisfy (in the sense of distributions) the equation

$$(\Delta_h + z^2)\widehat{u}(\cdot, s) = \widehat{f}(\cdot, s) \quad \text{on } \Sigma.$$

Given $z \in \mathbb{C}$, consider the linear Schrödinger operator $\widehat{B}_z = \Delta_h + z^2$, acting on complex valued functions on Σ . Let us denote by E_0, E_1, E_2, \dots the eigenspaces of $-\Delta_h$ corresponding to the eigenvalues $\lambda_0 = 0 < \lambda_1 < \lambda_2 < \dots$, respectively. By classical elliptic theory, if $z^2 \neq \lambda_j$ for all $j \in \mathbb{N} \cup \{0\}$, then there exists a bounded operator $\widehat{R}_z : L^2(\Sigma) \rightarrow L^2(\Sigma)$ that is a right inverse of \widehat{B}_z , i.e., $\widehat{B}_z \circ \widehat{R}_z$ is the identity on $L^2(\Sigma)$, where we keep the notation $L^2(\Sigma)$ for L^2 -complex valued functions on Σ . Also note that the condition $z^2 \neq \lambda_j$ holds for all j since $|z^2 - \lambda_j| \geq |\delta^2 - \lambda_j| > 0$.

Using the orthogonal eigendata decomposition $\widehat{f}(\cdot, s) = \sum_{j \geq 0} \sum_{\widehat{f}_h \in E_j} \widehat{f}_h$, it is easy to check that

$$\widehat{R}_z(\widehat{f}(\cdot, s)) = \sum_{j \geq 0} \frac{1}{z^2 - \lambda_j} \sum_{\widehat{f}_h \in E_j} \widehat{f}_h.$$

Plancherel’s formula then implies

$$\|\widehat{R}_z(\widehat{f}(\cdot, s))\|_{L^2(\Sigma)}^2 = \sum_{j \geq 0} \frac{1}{|z^2 - \lambda_j|^2} \sum_{\widehat{f}_h \in E_j} \|\widehat{f}_h\|_{L^2(\Sigma)}^2.$$

Using the inequality $|z^2 - \lambda_j| \geq |\delta^2 - \lambda_j|$, we obtain

$$\begin{aligned} (69) \quad \|\widehat{R}_z(\widehat{f}(\cdot, s))\|_{L^2(\Sigma)}^2 &\leq \frac{1}{\inf_j |z^2 - \lambda_j|^2} \sum_{j \geq 0} \sum_{\widehat{f}_h \in E_j} \|\widehat{f}_h\|_{L^2(\Sigma)}^2 \\ &\leq \frac{1}{\inf_j |\delta^2 - \lambda_j|^2} \|\widehat{f}(\cdot, s)\|_{L^2(\Sigma)}^2. \end{aligned}$$

Note that $\widehat{R}_z(\widehat{f}(\cdot, s)) = \widehat{u}(\cdot, s)$ (because $\widehat{B}_z \widehat{u} = \widehat{f}$ and $z^2 \neq \lambda_j$ for all j). Since the Fourier transform is an isometry of $L^2(\Sigma)$, one has

$$\begin{aligned} \|u\|_{L^2(\Sigma \times \mathbb{R})}^2 &= \|\widehat{u}\|_{L^2(\Sigma \times \mathbb{R})}^2 = \int_{\mathbb{R}} \|\widehat{u}(\cdot, s)\|_{L^2(\Sigma)}^2 ds \\ &= \int_{\mathbb{R}} \|\widehat{R}_z(\widehat{f}(\cdot, s))\|_{L^2(\Sigma)}^2 ds \\ &\stackrel{(69)}{\leq} \frac{1}{\inf_j |\delta^2 - \lambda_j|^2} \int_{\mathbb{R}} \|\widehat{f}(\cdot, s)\|_{L^2(\Sigma)}^2 ds \\ &= \frac{1}{\inf_j |\delta^2 - \lambda_j|^2} \|\widehat{f}\|_{L^2(\Sigma \times \mathbb{R})}^2 \\ &= \frac{1}{\inf_j |\delta^2 - \lambda_j|^2} \|f\|_{L^2(\Sigma \times \mathbb{R})}^2, \end{aligned}$$

from which one deduces directly the inequality (68). Hence, the assertion is proved. \square

ASSERTION 9.3. *Let (Σ, h) be a compact Riemannian manifold and $V \in L^\infty(\Sigma \times \mathbb{R})$. Assume that there exists $j_0 \in \mathbb{N}$ such that*

$$(70) \quad 4\|V\|_{L^\infty(\Sigma \times \mathbb{R})} \leq \lambda_{j_0+1} - \lambda_{j_0},$$

where $\lambda_0 = 0 < \lambda_1 < \lambda_2 < \dots$ is the spectrum of $-\Delta_h$ on Σ . Then, the bounded kernel of $\Delta_h + \partial_t^2 + V$ on $\Sigma \times \mathbb{R}$ is finite dimensional.

Remark 9.4. In the case where $\Sigma = \mathbb{S}^1$ with its standard metric, then $\lambda_j = j^2$ and $\lambda_{j+1} - \lambda_j = 2j + 1$, so the hypothesis (70) is always fulfilled.

Proof. Again the proof is essentially borrowed from the paper by Lockhart and McOwen [29] (see Section 2 on pages 420, 421). First consider a function w in the bounded kernel of $\Delta_h + \partial_t^2 + V$. Let $\chi \in C^\infty(\Sigma \times \mathbb{R})$ be a cutoff function only depending on t , equal to 0 for $t < -1$ and equal to 1 for $t > 1$. We choose $\delta > 0$ such that

$$\delta^2 = \frac{1}{2}(\lambda_{j_0+1} + \lambda_{j_0});$$

in particular, $\delta^2 \neq \lambda_j$ for all $j \geq 0$.

Applying the result in Assertion 9.2 to this value of δ and to the functions $U := \chi w$, $F := (\Delta_h + \partial_t^2)(\chi w) = -V\chi w + w\partial_t^2\chi + 2\partial_t\chi\partial_t w$, we obtain

$$\begin{aligned} \|e^{-\delta t}\chi w\|_{L^2(\Sigma \times \mathbb{R})} &\leq \frac{1}{\inf_j |\delta^2 - \lambda_j|} \|e^{-\delta t}F\|_{L^2(\Sigma \times \mathbb{R})} \\ &\leq \frac{1}{\inf_j |\delta^2 - \lambda_j|} \left(\|e^{-\delta t}V\chi w\|_{L^2(\Sigma \times \mathbb{R})} \right. \\ &\quad \left. + \|e^{-\delta t}w\partial_t^2\chi\|_{L^2(\Sigma \times \mathbb{R})} + \|e^{-\delta t}2\partial_t\chi\partial_t w\|_{L^2(\Sigma \times \mathbb{R})} \right) \\ &\leq \frac{1}{\inf_j |\delta^2 - \lambda_j|} \left(\|V\|_{L^\infty(\Sigma \times \mathbb{R})} \|e^{-\delta t}\chi w\|_{L^2(\Sigma \times \mathbb{R})} + c\|w\|_{W^{1,2}(\Sigma \times (-1,1))} \right) \end{aligned}$$

for some constant $c > 0$ that only depends on the choice of the cutoff function χ . (Note that we have used that $\partial_t\chi = 0$ outside $\Sigma \times (-1, 1)$ in the last inequality.)

On the other hand, (70) gives that

$$\inf_j |\delta^2 - \lambda_j| = \frac{1}{2}(\lambda_{j_0+1} - \lambda_{j_0}) \geq 2\|V\|_{L^\infty(\Sigma \times \mathbb{R})}.$$

Thus, $\|e^{-\delta t}\chi w\|_{L^2(\Sigma \times \mathbb{R})} \leq \frac{1}{2}\|e^{-\delta t}\chi w\|_{L^2(\Sigma \times \mathbb{R})} + \frac{2c}{\lambda_{j_0+1} - \lambda_{j_0}}\|w\|_{W^{1,2}(\Sigma \times (-1,1))}$, which implies

$$(71) \quad \|e^{-\delta t}\chi w\|_{L^2(\Sigma \times \mathbb{R})} \leq \frac{4c}{\lambda_{j_0+1} - \lambda_{j_0}}\|w\|_{W^{1,2}(\Sigma \times (-1,1))} \leq C\|w\|_{W^{1,2}(\Sigma \times (-1,1))}$$

for some constant $C > 0$ that depends on $\lambda_{j_0}, \lambda_{j_0+1}$ and on χ . (But these data have been fixed once for all.)

Next we apply Assertion 9.2 to the functions $U := (1 - \chi)w$ and $F := (\Delta_h + \partial_t^2)((1 - \chi)w) = -V(1 - \chi)w - w\partial_t^2\chi - 2\partial_t\chi\partial_t w$, and this time we change δ to $-\delta$ to get

$$\|e^{\delta t}(1 - \chi)w\|_{L^2(\Sigma \times \mathbb{R})} \leq \frac{1}{\inf_j |\delta^2 - \lambda_j|} \left(\|V\|_{L^\infty(\Sigma \times \mathbb{R})} \|e^{\delta t}(1 - \chi)w\|_{L^2(\Sigma \times \mathbb{R})} + c\|w\|_{W^{1,2}(\Sigma \times (-1,1))} \right).$$

Arguing as above, we conclude that

$$(72) \quad \|e^{\delta t}(1 - \chi)w\|_{L^2(\Sigma \times \mathbb{R})} \leq C\|w\|_{W^{1,2}(\Sigma \times (-1,1))}.$$

Collecting (71) and (72) and using the triangle inequality, we deduce that

$$(73) \quad \|e^{-\delta|t|}w\|_{L^2(\Sigma \times \mathbb{R})} \leq 2C\|w\|_{W^{1,2}(\Sigma \times (-1,1))}$$

for certain $C > 0$, and this estimate holds for any function w in the bounded kernel of $\Delta_h + \partial_t^2 + V$.

Another ingredient we will use in proving the assertion is a $W^{2,2}$ -estimate valid for any function in the kernel of $\Delta_h + \partial_t^2 + V$. Namely, the classical L^p -estimates applied to the solution w of $(\Delta_h + \partial_t^2)w = -Vw$ (see, for instance, Theorem 9.11 of Gilbarg and Trudinger [20]), imply that

$$(74) \quad \|w\|_{W^{2,2}(\Sigma \times (-1,1))} \leq C \left(1 + \|V\|_{L^\infty(\Sigma \times \mathbb{R})} \right) \|w\|_{L^2(\Sigma \times (-2,2))}.$$

The final ingredient is the compactness of the Sobolev embedding (Rellich's theorem; see, for instance, [20] Theorem 7.26):

$$(75) \quad W^{2,2}(\Sigma \times (-1,1)) \hookrightarrow W^{1,2}(\Sigma \times (-1,1)).$$

We finally prove the finite dimensionality of the bounded kernel K of $\Delta_h + \partial_t^2 + V$. Arguing by contradiction, suppose that K were infinite dimensional. There would exist a sequence of linearly independent functions $w_n \in K$, which could be normalized so that

$$(76) \quad \int_{\Sigma \times \mathbb{R}} w_n w_m e^{-2\delta|t|} dt dh = \begin{cases} 0 & \text{if } n \neq m, \\ 1 & \text{if } n = m. \end{cases}$$

Recall that $\delta > 0$, and hence the integrals are well defined since the functions w_n are bounded. As $e^{-\delta|t|}$ is bounded away from zero in $(-2, 2)$, we conclude from (76) that the sequence $\{w_n\}_n$ is bounded in $L^2(\Sigma \times (-2, 2))$, and (74) then implies that it is also bounded in $W^{2,2}(\Sigma \times (-1, 1))$. Now, using the compactness of the embedding (75), there exists a subsequence (still denoted by $\{w_n\}_n$) that converges in $W^{1,2}(\Sigma \times (-1, 1))$.

The convergence of $\{w_n\}_n$ in $W^{1,2}(\Sigma \times (-1, 1))$ together with (73) applied to $w_n - w_m$ implies that $\{e^{-\delta|t|}w_n\}_n$ is a Cauchy sequence in $L^2(\Sigma \times \mathbb{R})$. But, $L^2(\Sigma \times \mathbb{R})$ being a Banach space, this sequence converges in $L^2(\Sigma \times \mathbb{R})$ to some

function $W \in L^2(\Sigma \times \mathbb{R})$. Passing to the limit as $n \rightarrow \infty$ in (76) (keeping m fixed in the first integral), we have

$$(77) \quad \int_{\Sigma \times \mathbb{R}} W w_m e^{-\delta|t|} dt dh = 0, \quad \int_{\Sigma \times \mathbb{R}} W^2 dt dh = 1.$$

Finally, passing to the limit as $m \rightarrow \infty$ in the left integral above, one obtains

$$\int_{\Sigma \times \mathbb{R}} W^2 dt dh = 0,$$

which contradicts the right integral in (77). This finishes the proof of the assertion. \square

From the descriptions in Theorems 3.1 and 3.2, it follows that every properly embedded minimal surface M in \mathbb{R}^3 with finite genus g and infinite topology is conformally diffeomorphic to a compact Riemann surface of genus g with a countable set of points removed, and this set of points has exactly two limit points on the compact surface. Furthermore, M has bounded curvature, each middle end is planar and M has two limit ends of Riemann type. (See Theorem 8.1 for an improved description of such an M .) In particular, M has a partial conformal compactification \overline{M} by adding its nonlimit ends and a complete metric on \overline{M} with two cylindrical ends. As in the previously considered case when the genus of M was zero, the bounded Jacobi functions of \overline{M} can be identified with the bounded kernel of $\Delta_{\overline{M}} + V$, where V is a bounded potential. In this case, the arguments in the proof of Theorem 9.1 can be easily modified and imply the result below. We remark that when M has $0 < k < \infty$ ends, then M has finite total curvature and the statement below is well known. We also note that Theorem 9.5 is a particular case of the more general result in Theorem 0.5 in Colding, de Lellis and Minicozzi [6].

THEOREM 9.5. *Let $M \subset \mathbb{R}^3$ be a properly embedded minimal surface with finite genus and more than one end. Then, the linear space of bounded Jacobi functions on M is finite dimensional.*

10. Appendix 2: Uniqueness of the Riemann minimal examples in the two-ended periodic case

Recall that $\mathcal{M}_1 \subset \mathcal{M}$ is the subset of singly-periodic surfaces that define a two-ended torus in their orientable quotient by a translation with smallest absolute total curvature. In Proposition 6.2 of this paper we showed that if S_M is linear for a surface $M \in \mathcal{M}$, then $M \in \mathcal{M}_1$. At this point we can conclude that $M \in \mathcal{R}$ by quoting our previous characterization in [34]. Below, we give an alternative proof of the characterization $\mathcal{M}_1 = \mathcal{R}$ based on the Four Vertex Theorem.

Consider the flux map $h_1 = h|_{\mathcal{M}_1} : \mathcal{M}_1 \rightarrow (0, \infty)$ where $F_M = (h(M), 0, 1)$ is the flux vector of $M \in \mathcal{M}_1$. We first prove that \mathcal{R} is a connected component of \mathcal{M}_1 . Since \mathcal{R} is a path connected closed set in \mathcal{M}_1 , it remains to prove that $\mathcal{M}_1 - \mathcal{R}$ is closed in \mathcal{M}_1 . Otherwise, there exists a sequence of surfaces $\{\Sigma_n\}_n \subset \mathcal{M}_1 - \mathcal{R}$ that converges on compact subsets of \mathbb{R}^3 to some $\mathcal{R}_t \in \mathcal{R}$. Note that the Shiffman functions S_{Σ_n} of the Σ_n are not identically zero, and after the normalization $\widehat{S}_{\Sigma_n} = \frac{1}{\sup_{\Sigma_n} |S_{\Sigma_n}|} S_{\Sigma_n}$, we find a bounded sequence of Jacobi functions that converges (up to extracting a subsequence) to a periodic Jacobi function \widehat{S}_∞ on \mathcal{R}_t . By the Four Vertex Theorem, \widehat{S}_{Σ_n} has at least four zeros on each compact horizontal section of Σ_n (counted with multiplicity), and the same holds for \widehat{S}_∞ on each compact horizontal section of \mathcal{R}_t . On the other hand, the only periodic Jacobi functions on \mathcal{R}_t are the linear ones. (This follows, for instance, from Montiel and Ros [42] and also follows from our more general result in Theorem 7.1.) This contradicts Assertion 8.4 and proves that \mathcal{R} is a connected component of \mathcal{M}_1 .

Since \mathcal{R} is a connected component of \mathcal{M}_1 and $h|_{\mathcal{R}} : \mathcal{R} \rightarrow (0, \infty)$ is bijective, to deduce that $\mathcal{R} = \mathcal{M}_1$, it suffices to prove that the following three properties:

- (1) h_1 is a proper map.
- (2) h_1 is an open map.
- (3) There exists $\varepsilon > 0$ such that if $h_1(M) < \varepsilon$, then $M \in \mathcal{R}$.

The properness of h_1 in point (1) above follows from the curvature estimates in Theorem 5 of [35], which in fact insures properness of $h : \mathcal{M} \rightarrow (0, \infty)$. Both the openness point (2) and the local uniqueness point (3) above follow from arguments in [34], but we give simpler arguments below; once we prove these two points, then Proposition 6.2 will hold.

We first prove the openness of h_1 in point (2) above. Consider the space $\mathcal{W}_1 = \{(\Sigma, g, [\alpha])\}$, where Σ is a compact Riemann surface of genus one, $g : \Sigma \rightarrow \mathbb{C} \cup \{\infty\}$ is a meromorphic function of degree two with an order-two zero p and an order-two pole q , and $[\alpha]$ is a homology class in $H_1(\Sigma - \{p, q\}, \mathbb{Z})$ that is nontrivial in $H_1(\Sigma, \mathbb{Z})$. We denote the elements in \mathcal{W}_1 simply by g . The space \mathcal{W}_1 is a two-dimensional complex manifold, with local charts given by $g \mapsto (a_1 + a_2, a_1 \cdot a_2)$, where $a_1, a_2 \in \mathbb{C} - \{0\}$ are the (possibly equal) branch values of $g \in \mathcal{W}_1$ close to a given element $g_0 \in \mathcal{W}_1$. (In a chart, we can forget about the homology class associated to g after identification with that of g_0 .) Given $g \in \mathcal{W}_1$, we associate a unique holomorphic differential ϕ on Σ by the equation $\int_\alpha \phi = 2\pi i$. Consider the *period map* $\text{Per}_1 : \mathcal{W}_1 \rightarrow \mathbb{C}^2$ given by

$$\text{Per}_1(g) = \left(\int_\alpha \frac{1}{g} \phi, \int_\alpha g \phi \right).$$

Then, the space of elements $g \in \mathcal{W}_1$ such that (g, ϕ) is the Weierstrass pair of an immersed minimal surface are $\mathcal{M}_1^{\text{imm}} = \text{Per}_1^{-1}(\{(a, \bar{a}) \mid a \in \mathbb{C}\})$. (Note

that we do not need to impose any residue condition at the ends, since g has a unique zero and a unique pole and the sum of residues of a meromorphic differential on a compact Riemann surface is zero.) Since Per_1 is holomorphic, for $a \in \mathbb{C}$ fixed, the set $\mathcal{M}_1^{\text{imm}}(a) = \text{Per}_1^{-1}(a)$ is a complex analytic subvariety of \mathcal{W}_1 . Since the limit of embedded surfaces is embedded, we have that the subset $\mathcal{M}_1 \subset \mathcal{M}_1^{\text{imm}}$ of *embedded* surfaces is closed in $\mathcal{M}_1^{\text{imm}}$. An application of the maximum principle at infinity [39] gives that \mathcal{M}_1 is also open in $\mathcal{M}_1^{\text{imm}}$. In particular, the set $\mathcal{M}_1(a) = \mathcal{M}_1^{\text{imm}}(a) \cap \mathcal{M}_1$ is a complex analytic subvariety of \mathcal{W}_1 . By our uniform curvature estimates in Theorem 5 of [35] and subsequent uniform local area estimates, $\mathcal{M}_1(a)$ is compact. As the only compact, complex analytic subvarieties of \mathcal{W}_1 are finite sets (see Lemma 4 in [34]), we deduce that $\mathcal{M}_1(a)$ is finite. Thus, given $M \in \mathcal{M}_1$, there exists an open neighborhood U of M in \mathcal{W}_1 such that $U \cap \mathcal{M}_1(a) = U \cap \mathcal{M}_1^{\text{imm}}(a) = \{M\}$. In this setting, the openness theorem for finite holomorphic maps (Chapter 5.2 of Griffiths and Harris [22]) gives that Per_1 is an open map locally around M . Finally, the relationship between the period map Per_1 and the flux map $h_1: \mathcal{M}_1 \rightarrow (0, \infty)$ gives the desired *openness* for h_1 .

Our next statement is the local uniqueness point (3) in the list of properties of h_1 . (And thus, it finishes the proof of Proposition 6.2.) In fact, the argument below does not use periodicity for the surfaces in question, so it can be stated in \mathcal{M} instead of in \mathcal{M}_1 .

THEOREM 10.1. *There exists $\varepsilon > 0$ such that if $M \in \mathcal{M}$ has flux vector $F_M = (h, 0, 1)$ with $0 < h < \varepsilon$, then M is a Riemann minimal example.*

Proof. Here we will present a different proof from the one we gave in [34]. Arguing by contradiction, assume we have a sequence $\{M_n\}_n \subset \mathcal{M}$ with flux vector $F_{M_n} = (h(M_n), 0, 1)$ and $h(M_n) \rightarrow 0$ as $n \rightarrow \infty$, and assume that none of the M_n are Riemann minimal examples. Point 5 in Theorem 3.1 insures that there exists a uniform bound for the Gaussian curvature of the surfaces M_n , $n \in \mathbb{N}$. A suitable modification of the arguments in the proof of Lemma 3 in [34] can be used to show that as $n \rightarrow \infty$, the surfaces M_n become arbitrarily close to an infinite discrete collection of larger and larger translated pieces of a vertical catenoid with flux $e_3 = (0, 0, 1)$ joined by flatter and flatter graphs containing the ends of M_n . For each n , let \overline{M}_n be the conformal cylinder obtained by attaching the middle ends to M_n , and let S_{M_n} be the Shiffman function of M_n .

By Corollary 4.15, S_{M_n} extends smoothly to a bounded function on \overline{M}_n . Note that for fixed n , the function $|S_{M_n}|$ needs not attain its maximum on M_n , but in that case we can exchange each M_n by a limit of suitable translations of M_n (hence such limit also belongs to \mathcal{M}), so that the Shiffman function in absolute value reaches its maximum on this limit. Since the flux of a surface in

\mathcal{M} does not change under translations, we do not lose generality by assuming that for all n large, $|S_{M_n}|$ attains its maximum at a point $p_n \in M_n$. We now define $v_n = \frac{1}{|S_{M_n}(p_n)|} S_{M_n}$.

Take a sequence $\{\delta(n)\}_n \subset (0, 1)$ converging to 1. For n large, let $C_n \subset M_n$ be one of the connected components of $\langle N_n, e_3 \rangle^{-1}[-\delta(n), \delta(n)]$ that contains p_n or is adjacent to a horizontal graphical region containing p_n . By our previous arguments, C_n is arbitrarily close to a translated image of the intersection of a vertical catenoid C_∞ of vertical flux e_3 centered at the origin with a ball of arbitrarily large radius also centered at the origin.

ASSERTION 10.2. $\{\sup_{C_n} |v_n|\}_n$ tends to zero as $n \rightarrow \infty$.

Proof of Assertion 10.2. Since $\{v_n|_{C_n}\}_n$ is a bounded sequence of Jacobi functions on the forming catenoidal pieces C_n and suitable translations of the C_n converge to the catenoid C_∞ , it is not difficult to check that a subsequence of $\{v_n|_{C_n}\}_n$ (denoted in the same way) converges to a bounded Jacobi function on C_∞ . Since bounded Jacobi functions on a catenoid are linear, we conclude that $\{v_n|_{C_n}\}_n$ converges to a linear Jacobi function v on C_∞ . (Or by identifying the compactification of C_∞ with the sphere \mathbb{S}^2 through the Gauss map of C_∞ , we can view v as a linear function on \mathbb{S}^2 .) We now check that v is identically zero on \mathbb{S}^2 .

Arguing by contradiction, suppose v is not identically zero on \mathbb{S}^2 . Recall that the Shiffman function $(S_{M_n})|_{C_n}$ measures the derivative of the curvature of each planar section of C_n with respect to a certain parameter times a positive function. By the Four Vertex Theorem, each horizontal section of C_n contains at least four zeros of S_{M_n} , and so also at least four zeros of v_n . Since horizontal sections of the C_n (suitably translated) converge to horizontal sections of C_∞ and any nontrivial linear function on \mathbb{S}^2 has at most two zeros on each horizontal section (with a possible exceptional horizontal section if the linear function is the vertical coordinate, but this does not affect our argument by taking a different horizontal section), we conclude that at least two zeros of v_n in a certain horizontal section must collapse into a zero of v ; hence the gradient of v will vanish at such a collapsing zero. But the gradient of a nontrivial linear function on \mathbb{S}^2 never vanishes at a zero of the function. This contradiction proves Assertion 10.2.

Recall that $|v_n(p_n)| = 1$ for all n . By Assertion 10.2, $N_n(p_n)$ must converge to the vertical or equivalently, p_n must lie in one of the graphical components of the complement of all the catenoidal pieces in M_n , a noncompact minimal graph, which we will denote by Ω_n . Note that Ω_n is a graph over an unbounded domain in the plane $\{x_3 = 0\}$, $\partial\Omega_n$ consists of two almost-circular, almost-horizontal curves with $\langle N_n, e_3 \rangle|_{\partial\Omega_n} = \pm\delta(n)$ and Ω_n contains exactly one end of M_n . Hence we can apply Lemma 10.3 below to the minimal surface Ω_n and

to the bounded Jacobi function $v_n|_{\Omega_n}$, contradicting that $v_n|_{\partial\Omega_n}$ converges to zero (Assertion 10.2) but $|v_n(p_n)| = 1$. This contradiction finishes the proof of Theorem 10.1. \square

LEMMA 10.3. *Let $\delta \in (0, 1)$, and let $\Omega \subset \mathbb{R}^3$ be a complete, noncompact minimal surface with nonempty compact boundary and finite total curvature, whose Gauss map N satisfies $N_3 = \langle N, e_3 \rangle \geq 1 - \delta$ in Ω . Then, for every bounded Jacobi function v on Ω ,*

$$(1 - \delta) \sup_{\Omega} |v| \leq \sup_{\partial\Omega} |v|.$$

Proof. Since Ω has finite total curvature, Ω compactifies after attaching its ends to a compact Riemann surface $\bar{\Omega}$ with boundary. As v is bounded on Ω , v extends smoothly across the punctures to a Jacobi function on $\bar{\Omega}$. We will let $a = \sup_{\partial\Omega} |v|$. Since $N_3 \geq 1 - \delta > 0$ in Ω and N_3 is a Jacobi function, we conclude that Ω is strictly stable, and so $a > 0$. Now, $v + \frac{a}{1-\delta}N_3 \geq 0$ on $\partial\Omega$, and $v + \frac{a}{1-\delta}N_3$ is again a Jacobi function on Ω . Thus by stability, $v + \frac{a}{1-\delta}N_3 \geq 0$ in Ω . Analogously, $v - \frac{a}{1-\delta}N_3 \leq 0$ in $\partial\Omega$, and hence $v - \frac{a}{1-\delta}N_3 \leq 0$ in Ω . These inequalities together with $N_3 \leq 1$ give $|v| \leq \frac{a}{1-\delta}$ in Ω , as desired. \square

References

- [1] J. L. BARBOSA and M. DO CARMO, On the size of a stable minimal surface in R^3 , *Amer. J. Math.* **98** (1976), 515–528. MR 0413172. Zbl 0332.53006. <http://dx.doi.org/10.2307/2373899>.
- [2] J. BERNSTEIN and C. BREINER, Conformal structure of minimal surfaces with finite topology, *Comment. Math. Helv.* **86** (2011), 353–381. MR 2775132. Zbl 1213.53011. <http://dx.doi.org/10.4171/CMH/226>.
- [3] J. C. BORDA, Eclaircissement sur les méthodes de trouver les courbes qui jouissent de quelque propriété du maximum ou du minimum, *Mém. Acad. Roy. Sci. Paris* (1770), 551–565, presented in 1767.
- [4] M. CALLAHAN, D. HOFFMAN, and W. H. MEEKS, III, The structure of singly-periodic minimal surfaces, *Invent. Math.* **99** (1990), 455–481. MR 1032877. Zbl 0695.53005. <http://dx.doi.org/10.1007/BF01234428>.
- [5] S. S. CHERN and C. K. PENG, Lie groups and KdV equations, *Manuscripta Math.* **28** (1979), 207–217. MR 0535702. Zbl 0408.35074. <http://dx.doi.org/10.1007/BF01647972>.
- [6] T. H. COLDING, C. DE LELLIS, and W. P. MINICOZZI, II, Three circles theorems for Schrödinger operators on cylindrical ends and geometric applications, *Comm. Pure Appl. Math.* **61** (2008), 1540–1602. MR 2444375. Zbl 05358518. <http://dx.doi.org/10.1002/cpa.20232>.
- [7] T. H. COLDING and W. P. MINICOZZI, II, The space of embedded minimal surfaces of fixed genus in a 3-manifold. I. Estimates off the axis for disks, *Ann. of Math.* **160** (2004), 27–68. MR 2119717. Zbl 1070.53031. <http://dx.doi.org/10.4007/annals.2004.160.27>.

- [8] T. H. COLDING and W. P. MINICOZZI, II, The space of embedded minimal surfaces of fixed genus in a 3-manifold. II. Multi-valued graphs in disks, *Ann. of Math.* **160** (2004), 69–92. MR 2119718. Zbl 1070.53032. <http://dx.doi.org/10.4007/annals.2004.160.69>.
- [9] T. H. COLDING and W. P. MINICOZZI, II, The space of embedded minimal surfaces of fixed genus in a 3-manifold. III. Planar domains, *Ann. of Math.* **160** (2004), 523–572. MR 2123932. Zbl 1076.53068. <http://dx.doi.org/10.4007/annals.2004.160.523>.
- [10] T. H. COLDING and W. P. MINICOZZI, II, The space of embedded minimal surfaces of fixed genus in a 3-manifold. IV. Locally simply connected, *Ann. of Math.* **160** (2004), 573–615. MR 2123933. Zbl 1076.53069. <http://dx.doi.org/10.4007/annals.2004.160.573>.
- [11] T. H. COLDING and W. P. MINICOZZI, II, The space of embedded minimal surfaces of fixed genus in a 3-manifold V; Fixed genus, *Ann. of Math.* **181** (2015), 1–153. <http://dx.doi.org/10.4007/annals.2015.181.1.1>.
- [12] P. COLLIN, Topologie et courbure des surfaces minimales proprement plongées de \mathbf{R}^3 , *Ann. of Math.* **145** (1997), 1–31. MR 1432035. Zbl 0886.53008. <http://dx.doi.org/10.2307/2951822>.
- [13] P. COLLIN, R. KUSNER, W. H. MEEKS, III, and H. ROSENBERG, The topology, geometry and conformal structure of properly embedded minimal surfaces, *J. Differential Geom.* **67** (2004), 377–393. MR 2153082. Zbl 1098.53006. Available at <http://projecteuclid.org/euclid.jdg/1102536205>.
- [14] A. DOUADY and R. DOUADY, Changements de cadres á partir des surfaces minimales, *Cahier de DIDIREM* **23** (1994), edited by IREM de Paris7.
- [15] N. EJIRI and M. KOTANI, Index and flat ends of minimal surfaces, *Tokyo J. Math.* **16** (1993), 37–48. MR 1223287. Zbl 0856.53013. <http://dx.doi.org/10.3836/tjm/1270128981>.
- [16] L. EULER, *Methodus Inveniendi Lineas Curvas Maximi Minimive Proprietate Gaudeates Sive Solutio Problematis Isoperimetrici Latissimo Sensu Accepti*, Harvard Univ. Press, Cambridge, MA, 1969, *Opera omnia*(1), **24**, Fussli, Turici, 1952 (a source book in mathematics, partially translated by D. J. Struik); see pages 399–406.
- [17] Y. FANG, On minimal annuli in a slab, *Comment. Math. Helv.* **69** (1994), 417–430. MR 1289335. Zbl 0819.53006. <http://dx.doi.org/10.1007/BF02564495>.
- [18] C. FROHMAN and W. H. MEEKS, III, The ordering theorem for the ends of properly embedded minimal surfaces, *Topology* **36** (1997), 605–617. MR 1422427. Zbl 0878.53008. [http://dx.doi.org/10.1016/S0040-9383\(96\)00019-5](http://dx.doi.org/10.1016/S0040-9383(96)00019-5).
- [19] F. GESZTESY and R. WEIKARD, Elliptic algebro-geometric solutions of the KdV and AKNS hierarchies—an analytic approach, *Bull. Amer. Math. Soc.* **35** (1998), 271–317. MR 1638298. Zbl 0909.34073. <http://dx.doi.org/10.1090/S0273-0979-98-00765-4>.
- [20] D. GILBARG and N. S. TRUDINGER, *Elliptic Partial Differential Equations of Second Order*, second ed., *Grundl. Math. Wissen.* **224**, Springer-Verlag, New York, 1983. MR 0737190. Zbl 0562.35001.

- [21] R. E. GOLDSTEIN and D. M. PETRICH, The Korteweg-de Vries hierarchy as dynamics of closed curves in the plane, *Phys. Rev. Lett.* **67** (1991), 3203–3206. MR 1135964. Zbl 0990.37519. <http://dx.doi.org/10.1103/PhysRevLett.67.3203>.
- [22] P. GRIFFITHS and J. HARRIS, *Principles of Algebraic Geometry, Pure Appl. Math.*, Wiley-Interscience [John Wiley & Sons], New York, 1978. MR 0507725. Zbl 0408.14001.
- [23] L. HAUSWIRTH and F. PACARD, Higher genus Riemann minimal surfaces, *Invent. Math.* **169** (2007), 569–620. MR 2336041. Zbl 1129.53009. <http://dx.doi.org/10.1007/s00222-007-0056-z>.
- [24] D. HOFFMAN and W. H. MEEKS, III, The strong halfspace theorem for minimal surfaces, *Invent. Math.* **101** (1990), 373–377. MR 1062966. Zbl 0722.53054. <http://dx.doi.org/10.1007/BF01231506>.
- [25] D. HOFFMAN, M. TRAZET, and B. WHITE, Helicoidal minimal surfaces of prescribed genus II. arXiv 1304.6180.
- [26] D. HOFFMAN and B. WHITE, Genus-one helicoids from a variational point of view, *Comment. Math. Helv.* **83** (2008), 767–813. MR 2442963. Zbl 1161.53009. <http://dx.doi.org/10.4171/CMH/143>.
- [27] N. JOSHI, The second Painlevé hierarchy and the stationary KdV hierarchy, *Publ. Res. Inst. Math. Sci.* **40** (2004), 1039–1061. MR 2074710. Zbl 1063.33030. <http://dx.doi.org/10.2977/prims/1145475502>.
- [28] J. L. LAGRANGE, Essai d’une nouvelle méthode pour déterminer les maxima et les minima des formules integrales indefinies, *Miscellanea Taurinensia* 2 **325** (1760), 173–199.
- [29] R. B. LOCKHART and R. C. MCOWEN, Elliptic differential operators on non-compact manifolds, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **12** (1985), 409–447. MR 0837256. Zbl 0615.58048. Available at http://www.numdam.org/item?id=ASNSP_1985_4_12_3_409_0.
- [30] F. J. LÓPEZ and A. ROS, On embedded complete minimal surfaces of genus zero, *J. Differential Geom.* **33** (1991), 293–300. MR 1085145. Zbl 0719.53004. Available at <http://projecteuclid.org/euclid.jdg/1214446040>.
- [31] W. H. MEEKS, III, The limit lamination metric for the Colding-Minicozzi minimal lamination, *Illinois J. Math.* **49** (2005), 645–658. MR 2164355. Zbl 1087.53058. Available at <http://projecteuclid.org/euclid.ijm/1258138037>.
- [32] W. H. MEEKS, III and J. PÉREZ, Conformal properties in classical minimal surface theory, in *Surveys in Differential Geometry. Vol. IX*, Int. Press, Somerville, MA, 2004, pp. 275–335. MR 2195411. Zbl 1086.53007. <http://dx.doi.org/10.4310/SDG.2004.v9.n1.a8>.
- [33] W. H. MEEKS, III and J. PÉREZ, Embedded minimal surfaces of finite topology. Available at <http://wdb.ugr.es/~jperez/publications-by-joaquin-perez/>.
- [34] W. H. MEEKS, III, J. PÉREZ, and A. ROS, Uniqueness of the Riemann minimal examples, *Invent. Math.* **133** (1998), 107–132. MR 1626477. Zbl 0916.53004. <http://dx.doi.org/10.1007/s002220050241>.
- [35] W. H. MEEKS, III, J. PÉREZ, and A. ROS, The geometry of minimal surfaces of finite genus. I. Curvature estimates and quasiperiodicity, *J. Differential Geom.* **66**

- (2004), 1–45. MR 2128712. Zbl 1068.53012. Available at <http://projecteuclid.org/euclid.jdg/1090415028>.
- [36] W. H. MEEKS, III, J. PÉREZ, and A. ROS, The geometry of minimal surfaces of finite genus. II. Nonexistence of one limit end examples, *Invent. Math.* **158** (2004), 323–341. MR 2096796. Zbl 1070.53003. <http://dx.doi.org/10.1007/s00222-004-0374-3>.
- [37] W. H. MEEKS, III, J. PÉREZ, and A. ROS, Bounds on the topology and index of classical minimal surfaces. Available at <http://wdb.ugr.es/~jperez/publications-by-joaquin-perez/>.
- [38] W. H. MEEKS, III, J. PÉREZ, and A. ROS, The embedded Calabi-Yau conjectures for finite genus. Available at <http://wdb.ugr.es/~jperez/publications-by-joaquin-perez/>.
- [39] W. H. MEEKS, III and H. ROSENBERG, The maximum principle at infinity for minimal surfaces in flat three manifolds, *Comment. Math. Helv.* **65** (1990), 255–270. MR 1057243. Zbl 0713.53008. <http://dx.doi.org/10.1007/BF02566606>.
- [40] W. H. MEEKS, III and H. ROSENBERG, The uniqueness of the helicoid, *Ann. of Math.* **161** (2005), 727–758. MR 2153399. Zbl 1102.53005. <http://dx.doi.org/10.4007/annals.2005.161.727>.
- [41] J. B. MEUSNIER, Mémoire sur la courbure des surfaces, *Mém. Mathém. Phys. Acad. Sci. Paris, prés. par div. Savans* **10** (1785), 477–510, presented in 1776.
- [42] S. MONTIEL and A. ROS, Schrödinger operators associated to a holomorphic map, in *Global Differential Geometry and Global Analysis* (Berlin, 1990), *Lecture Notes in Math.* **1481**, Springer-Verlag, New York, 1991, pp. 147–174. MR 1178529. Zbl 0744.58007. <http://dx.doi.org/10.1007/BFb0083639>.
- [43] R. OSSERMAN, Global properties of minimal surfaces in E^3 and E^n , *Ann. of Math.* **80** (1964), 340–364. MR 0179701. Zbl 0134.38502. <http://dx.doi.org/10.2307/1970396>.
- [44] R. OSSERMAN, *A Survey of Minimal Surfaces*, second ed., Dover Publications, New York, 1986. MR 0852409. Zbl 0209.52901.
- [45] J. PÉREZ, On singly-periodic minimal surfaces with planar ends, *Trans. Amer. Math. Soc.* **349** (1997), 2371–2389. MR 1407709. Zbl 0882.53007. <http://dx.doi.org/10.1090/S0002-9947-97-01911-9>.
- [46] J. PÉREZ and A. ROS, The space of properly embedded minimal surfaces with finite total curvature, *Indiana Univ. Math. J.* **45** (1996), 177–204. MR 1406689. Zbl 0864.53008. <http://dx.doi.org/10.1512/iumj.1996.45.2053>.
- [47] B. RIEMANN, Über die Fläche vom kleinsten Inhalt bei gegebener Begrenzung, *Abh. Königl. d. Wiss. Göttingen, Mathem. Cl.* **13** (1867), 3–52. JFM 01.0218.01.
- [48] B. RIEMANN, *Oeuvres Mathématiques de Riemann*, Gauthiers-Villars, Paris, 1898.
- [49] R. SCHOEN, Estimates for stable minimal surfaces in three-dimensional manifolds, in *Seminar on Minimal Submanifolds*, *Ann. of Math. Stud.* **103**, Princeton Univ. Press, Princeton, NJ, 1983, pp. 111–126. MR 0795231. Zbl 0532.53042.

- [50] R. M. SCHOEN, Uniqueness, symmetry, and embeddedness of minimal surfaces, *J. Differential Geom.* **18** (1983), 791–809 (1984). MR 0730928. Zbl 0575.53037. Available at <http://projecteuclid.org/euclid.jdg/1214438183>.
- [51] G. SEGAL and G. WILSON, Loop groups and equations of KdV type, *Inst. Hautes Études Sci. Publ. Math.* **61** (1985), 5–65. MR 0783348. Zbl 0592.35112. <http://dx.doi.org/10.1007/BF02698802>.
- [52] M. SHIFFMAN, On surfaces of stationary area bounded by two circles, or convex curves, in parallel planes, *Ann. of Math.* **63** (1956), 77–90. MR 0074695. Zbl 0070.16803. <http://dx.doi.org/10.2307/1969991>.
- [53] M. TRAISET, An embedded minimal surface with no symmetries, *J. Differential Geom.* **60** (2002), 103–153. MR 1924593. Zbl 1054.53014. Available at <http://projecteuclid.org/euclid.jdg/1090351085>.
- [54] M. WEBER, D. HOFFMAN, and M. WOLF, An embedded genus-one helicoid, *Ann. of Math.* **169** (2009), 347–448. MR 2480608. Zbl 1213.49049. <http://dx.doi.org/10.4007/annals.2009.169.347>.
- [55] M. WEBER and M. WOLF, Teichmüller theory and handle addition for minimal surfaces, *Ann. of Math.* **156** (2002), 713–795. MR 1954234. Zbl 1028.53009. <http://dx.doi.org/10.2307/3597281>.
- [56] R. WEIKARD, On rational and periodic solutions of stationary KdV equations, *Doc. Math.* **4** (1999), 109126. MR 1683290. Zbl 0972.35121. Available at <http://www.math.uiuc.edu/documenta/vol-04/04.html>.
- [57] B. WHITE, The space of m -dimensional surfaces that are stationary for a parametric elliptic functional, *Indiana Univ. Math. J.* **36** (1987), 567–602. MR 0905611. Zbl 0770.58005. <http://dx.doi.org/10.1512/iumj.1987.36.36031>.

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