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The Hodge theory of Soergel bimodules

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Abstract

We prove Soergel's conjecture on the characters of indecomposable Soergel bimodules. We deduce that Kazhdan-Lusztig polynomials have positive coefficients for arbitrary Coxeter systems. Using results of Soergel one may deduce an algebraic proof of the Kazhdan-Lusztig conjecture.

Contents

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1. **Introduction**

In 1979 Kazhdan and Lusztig introduced the Kazhdan-Lusztig basis of the Hecke algebra of a Coxeter system [KL79]. The definition of the Kazhdan-Lusztig basis is elementary, however it appears to enjoy remarkable positivity properties. For example, it is conjectured in [KL79] that Kazhdan-Lusztig polynomials (which express the Kazhdan-Lusztig basis in terms of the standard basis of the Hecke algebra) have positive coefficients. The same paper also proposed the Kazhdan-Lusztig conjecture, a character formula for simple highest weight modules for a complex semi-simple Lie algebra in terms of Kazhdan-Lusztig polynomials associated to its Weyl group.

In a sequel [KL80], Kazhdan and Lusztig established that their polynomials give the Poincaré polynomials of the local intersection cohomology of Schubert varieties (using Deligne's theory of weights), thus establishing their positivity conjectures for finite and affine Weyl groups. In 1981 Beilinson and Bernstein [BB81] and Brylinski and Kashiwara [BK81] established a connection between highest weight representation theory and perverse sheaves, using D-modules and the Riemann-Hilbert correspondence, thus proving the Kazhdan-Lusztig conjecture. Since their introduction Kazhdan-Lusztig polynomials have become ubiquitous throughout highest weight representation theory, giving character formulae for affine Lie algebras, quantum groups at a root of unity, rational representations of algebraic groups, etc.

In 1990 Soergel [Soe90] gave an alternate proof of the Kazhdan-Lusztig conjecture, using certain modules over the cohomology ring of the flag variety.[1] In a subsequent paper [Soe92] Soergel introduced equivariant analogues of these modules, which have come to be known as *Soergel bimodules*.

Soergel's approach is remarkable in its simplicity. Using only the action of the Weyl group on a Cartan subalgebra, Soergel associates to each simple reflection a graded bimodule over the regular functions on the Cartan subalgebra. He then proves that the split Grothendieck group of the monoidal category generated by these bimodules (the category of Soergel bimodules) is isomorphic to the Hecke algebra. Moreover, the Kazhdan-Lusztig conjectures (as well as several positivity conjectures) are equivalent to the existence of certain bimodules whose classes in the Grothendieck group coincide with the Kazhdan-Lusztig basis. Despite its elementary appearance, this statement is difficult to verify.

[1][Soe90, §1.1, Bermerkung 5]. This seems not to be as well known as it should be.

For finite Weyl groups, Soergel deduces the existence of such bimodules by applying the decomposition theorem of Beilinson, Bernstein, Deligne and Gabber [BBD82] to identify the indecomposable Soergel bimodules with the equivariant intersection cohomology of Schubert varieties. This approach was extended by Härterich to the setting of Weyl groups of symmetrizable Kac-Moody groups [Här99]. Except for his appeal to the decomposition theorem, Soergel's approach is entirely algebraic. (The decomposition theorem relies on the base field having characteristic 0, which will be an important assumption below.)

In [Soe92] and [Soe07] Soergel pointed out that the algebraic theory of Soergel bimodules can be developed for an arbitrary Coxeter system. Starting with an appropriate representation of the Coxeter group (the substitute for the Weyl group's action on a Cartan subalgebra) one defines the monoidal category of Soergel bimodules by mimicking the Weyl group case. Surprisingly, one again obtains a monoidal category whose split Grothendieck group is canonically identified with the Hecke algebra. Soergel then conjectures the existence (over a field of characteristic 0) of indecomposable bimodules whose classes coincide with the Kazhdan-Lusztig basis of the Hecke algebra. At this level of generality there is no known recourse to geometry. One does not have a flag variety or Schubert varieties associated to arbitrary Coxeter groups, and so one has no geometric setting in which to apply the decomposition theorem. Soergel's conjecture was established for dihedral groups by Soergel [Soe92] and for "universal" Coxeter systems (where each product of simple reflections has infinite order) by Fiebig [Fie08] and Libedinsky. However, in both these cases there already existed closed formulas for the Kazhdan-Lusztig polynomials.

In this paper we prove Soergel's conjecture for an arbitrary Coxeter system. We thus obtain a proof of the positivity of Kazhdan-Lusztig polynomials (as well as several other positivity conjectures). We also obtain an algebraic proof of the Kazhdan-Lusztig conjecture, completing the program initiated by Soergel. In some sense we have come full circle: the original paper of Kazhdan and Lusztig was stated in the generality of an arbitrary Coxeter system; this paper returns Kazhdan-Lusztig theory to this level of generality.

Our proof is inspired by two papers of de Cataldo and Migliorini ([dCM02] and [dCM05]) which give Hodge-theoretic proofs of the decomposition theorem. In essence, de Cataldo and Migliorini show that the decomposition theorem for a proper map (from a smooth space) is implied by certain Hodge theoretic properties of the cohomology groups of the source, under a Lefschetz operator induced from the target. We discuss their approach in more detail below. Thus they are able to transform a geometric question on the target into an algebraic question on the source. They then use classical Hodge theory and some ingenious arguments to complete the proof. For Weyl groups, Soergel bimodules are the equivariant intersection cohomology of Schubert varieties, and as such they have a number of remarkable Hodge-theoretic properties which seem not to have been made explicit before. In fact, these properties hold for any Coxeter group; Soergel bimodules always behave as though they were intersection cohomology spaces of projective varieties! In this paper, we give an algebraic proof of these Hodge-theoretic properties for any Coxeter group, and we adapt the proof that these Hodge-theoretic properties imply the "decomposition theorem," at least insofar as Soergel's conjecture is concerned.

The following are some highlights of de Cataldo and Migliorini's proof from [dCM02]:

(1) "Local intersection forms" (which control the decomposition of the direct image of the constant sheaf) can be embedded into "global intersection forms" on the cohomology of smooth varieties.
(2) The Hodge-Riemann bilinear relations can be used to conclude that the restriction of a form to a subspace (i.e., the image of a local intersection form) stays definite.
(3) One should first prove the hard Lefschetz theorem and then deduce the Hodge-Riemann bilinear relations via a limiting argument from a family of known cases, using that the signature of a nondegenerate symmetric real form cannot change in a family.

It is this outline that we adapt to our algebraic situation. However, the translation of their results into the language of Soergel bimodules is by no means automatic. The biggest obstacle is to find a replacement for the use of hyperplane sections and the weak Lefschetz theorem. We believe that our use of the Rouquier complex to overcome this difficulty is an important observation and may have other applications.

There already exists a formidable collection of algebraic machinery, developed by Soergel [Soe92], [Soe08], Andersen-Jantzen-Soergel [AJS94] and Fiebig [Fie06], [Fie11], which provides algebraic proofs of many deep results in representation theory once Soergel's conjecture is known. These include the Kazhdan-Lusztig conjecture for affine Lie algebras (in noncritical level), the Lusztig conjecture for quantum groups at a root of unity, and the Lusztig conjecture on modular characters of reductive algebraic groups in characteristic $p \gg 0$.

There are many formal similarities between the theory we develop here and the theory of intersection cohomology of nonrational polytopes, which was developed to prove Stanley's conjecture on the unimodularity of the generalized h-vector [BL03], [Kar04], [BKBF07]. In both cases one obtains spaces which look like the intersection cohomology of a (in many cases nonexistent) projective algebraic variety. Dyer [Dyea], [Dyeb] has proposed a conjectural framework for understanding both of these theories in parallel. It would be interesting to know whether the techniques of this paper shed light on this more general theory.

1.1. *Results.* Fix a Coxeter system (W,S). Let $\mathcal{H}$ denote the Hecke algebra of (W,S), a $\mathbb{Z}[v^{\pm 1}]$-algebra with standard basis $\{H_x\}_{x\in W}$ and Kazhdan-Lusztig basis $\{\underline{H}_x\}_{x\in W}$ as in Section 3.2. We fix a reflection faithful (in the sense of [Soe07, Def. 1.5]) representation $\mathfrak{h}$ of W over $\mathbb{R}$ and let R denote the regular functions on $\mathfrak{h}$, graded with $\deg \mathfrak{h}^* = 2$. We denote by $\mathcal{B}$ the category of Soergel bimodules; it is the full additive monoidal Karoubian subcategory of graded R-bimodules generated by $B_s := R \otimes_{R^s} R(1)$ for all $s \in S$. Here, $R^s \subset R$ denotes the subalgebra of s-invariants and (1) denotes the grading shift which places the element $1 \otimes 1$ in degree -1. For any x there exists up to isomorphism a unique indecomposable Soergel bimodule B_x which occurs as a direct summand of the Bott-Samelson bimodule $\mathrm{BS}(\underline{x}) = B_s \otimes_R B_t \otimes_R \cdots \otimes_R B_u$ for any reduced expression $\underline{x} = st\cdots u$ for x, but does not occur as a summand of any Bott-Samelson bimodule for a shorter expression. The bimodules B_x for $x \in W$ give representatives for the isomorphism classes of all indecomposable Soergel bimodules up to shifts. The split Grothendieck group $[\mathcal{B}]$ of the category of Soergel bimodules is isomorphic to $\mathcal{H}$. The character $\mathrm{ch}(B) \in \mathcal{H}$ of a Soergel bimodule B is a $\mathbb{Z}_{\geq 0}[v^{\pm}]$-linear combination of standard basis elements $\{H_x\}$ given by counting ranks of subquotients in a certain canonical filtration; it realizes the class of B under the isomorphism $[\mathcal{B}] \xrightarrow{\sim} \mathcal{H}$.

THEOREM 1.1 (Soergel's conjecture). *For all* $x \in W$ *we have* $\mathrm{ch}(B_x) = \underline{H}_x$.

Because $\mathrm{ch}(B)$ is manifestly positive we obtain

COROLLARY 1.2 (Kazhdan-Lusztig positivity conjecture).

(1) *If we write* $\underline{H}_x = \sum_{y \leq x} h_{y,x} H_y$, *then* $h_{y,x} \in \mathbb{Z}_{\geq 0}[v]$.
(2) *If we write* $\underline{H}_x \underline{H}_y = \sum \mu_{x,y}^z \underline{H}_z$, *then* $\mu_{x,y}^z \in \mathbb{Z}_{\geq 0}[v^{\pm 1}]$.

(See Remark 3.2 for the relation between our notation and that of [KL79].)

We prove that indecomposable Soergel bimodules have all of the algebraic properties known for intersection cohomology. Given a Soergel bimodule B, we denote by $\overline{B} := B \otimes_R \mathbb{R}$ the quotient by the image of positive degree polynomials acting on the right. We let $(\overline{B})^i$ denote the degree i component of $\overline{B}$. The self-duality of Soergel bimodules implies that $\dim_{\mathbb{R}}(\overline{B_x})^{-i} = \dim_{\mathbb{R}}(\overline{B_x})^i$ for all i. For the rest of the paper we fix a degree two element $\rho \in \mathfrak{h}^*$ which is strictly positive on any simple coroot $\alpha_s^\vee \in \mathfrak{h}$ (see Section 3.1).

THEOREM 1.3 (Hard Lefschetz for Soergel bimodules). *The action of* ρ *on* B_x *by left multiplication induces an operator on* $\overline{B_x}$ *which satisfies the hard Lefschetz theorem. That is, left multiplication by* ρ^i *induces an isomorphism*

$$\rho^i : (\overline{B_x})^{-i} \xrightarrow{\sim} (\overline{B_x})^i.$$

We say that a graded R-valued form

$$\langle -,- \rangle : B_x \times B_x \to R$$

is *invariant* if it is bilinear for the right action of R and if $\langle rb, b' \rangle = \langle b, rb' \rangle$ for all $b, b' \in B$ and $r \in R$. Theorem 1.1 and Soergel's hom formula (see Theorem 3.6) imply that the degree zero endomorphisms of B_x consist only of scalars, i.e., $\mathrm{End}(B_x) = \mathbb{R}$. Combining this with the self-duality of indecomposable Soergel bimodules, we see that there exists an invariant form $\langle -,- \rangle_{B_x}$ on B_x which is unique up to a scalar. We write $\langle -,- \rangle_{\overline{B_x}}$ for the $\mathbb{R}$-valued form on $\overline{B_x}$ induced by $\langle -,- \rangle_{B_x}$. We fix the sign on $\langle -,- \rangle_{B_x}$ by requiring that $\langle \overline{c}, \rho^{\ell(x)} \overline{c} \rangle_{\overline{B_x}} > 0$, where c is any generator of $B_x^{-\ell(x)} \cong \mathbb{R}$. With this additional positivity constraint, we call $\langle -,- \rangle_{B_x}$ the *intersection form* on B_x. It is well defined up to a positive scalar.

THEOREM 1.4 (Hodge-Riemann bilinear relations). *For all $i \geq 0$ the Lefschetz form on $(\overline{B_x})^{-i}$ defined by*

$$(\alpha, \beta)^{\rho}_{-i} := \langle \alpha, \rho^i \beta \rangle_{\overline{B_x}}$$

is $(-1)^{(-\ell(x)+i)/2}$-definite when restricted to the primitive subspace

$$P_{\rho}^{-i} = \ker(\rho^{i+1}) \subset (\overline{B_x})^{-i}.$$

Note that $B_x^{-i} = 0$ unless i and $\ell(x)$ are congruent modulo 2. Throughout this paper we adopt the convention that if m is odd, then a space is $(-1)^{\frac{m}{2}}$-definite if and only it is zero. The reader need not worry too much about the sign in this and other Hodge-Riemann statements. Throughout the introduction the form on the lowest nonzero degree will be positive definite and the signs on primitive subspaces will alternate from there upwards.

As an example of our results, consider the case when W is finite. If $w_0 \in W$ denotes the longest element of W, then $B_{w_0} = R \otimes_{R^W} R(\ell(w_0))$, where R^W denotes the subalgebra of W-invariants in R. Hence

$$\overline{B_{w_0}} = (R \otimes_{R^W} R) \otimes_R \mathbb{R}(\ell(w_0)) = R/((R^W)^+)(\ell(w_0))$$

is the coinvariant ring, shifted so as to have Betti numbers symmetric about zero. (Here $((R^W)^+)$ denotes the ideal of R generated by elements of R^W of positive degree.) The coinvariant ring is equipped with a canonical symmetric form, and Theorems 1.3 and 1.4 yield that left multiplication by any ρ in the interior of the dominant chamber of $\mathfrak{h}^*$ satisfies the hard Lefschetz theorem and Hodge-Riemann bilinear relations.

If W is a Weyl group of a compact Lie group G, then the coinvariant ring above is isomorphic to the real cohomology ring of the flag variety of G and the hard Lefschetz theorem and Hodge-Riemann bilinear relations follow from classical Hodge theory, because the flag variety is a projective algebraic variety. On the other hand, if W is not a Weyl group (e.g., a noncrystallographic dihedral group or a group of type H_3 or H_4), then there is no obvious geometric reason why the hard Lefschetz theorem or Hodge-Riemann bilinear relations should hold. The hard Lefschetz property for coinvariant rings has been studied by a number of authors ([MNW11], [NW07], [McD11]) but even for the coinvariant rings of H_3 and H_4 the fact that the Hodge-Riemann bilinear relations hold seems to be new.

1.2. *Outline of the proof.*

1.2.1. *Setup.* Our proof is by induction on the Bruhat order, and the hard Lefschetz property and Hodge-Riemann bilinear relations play an essential role along the way. Throughout this paper we employ the following abbreviations for any $x \in W$:

$\mathrm{S}(x)$:	Soergel's conjecture for B_x; Theorem 1.1 holds for x.
$\mathrm{hL}(x)$:	hard Lefschetz for $\overline{B_x}$; Theorem 1.3 holds for x.
$\mathrm{HR}(x)$:	the Hodge-Riemann bilinear relations for $\overline{B_x}$; $\mathrm{S}(x)$ holds and Theorem 1.4 holds for x.

The abbreviation $\mathrm{hL}(< x)$ means that $\mathrm{hL}(y)$ holds for all $y < x$. Similar interpretations hold for abbreviations like $\mathrm{S}(\leq x)$, etc.

In the statement of $\mathrm{HR}(x)$ it is necessary to assume $\mathrm{S}(x)$ to ensure the uniqueness (up to positive scalar) of the intersection form on B_x. However, we need not assume $\mathrm{S}(x)$ in order to ask whether a given form on B_x (not necessarily the intersection form) induces a form on $\overline{B_x}$ satisfying the Hodge-Riemann bilinear relations. Now B_x appears as a summand of the Bott-Samelson bimodule $\mathrm{BS}(\underline{x})$ for any reduced expression $\underline{x}$ for x. Bott-Samelson bimodules are equipped with an explicit symmetric nondegenerate *intersection form* defined using the ring structure and a trace on $\mathrm{BS}(\underline{x})$ (just as the intersection form on the cohomology of a smooth projective variety is given by evaluating the fundamental class on a product). The following stronger version of $\mathrm{HR}(x)$ is more useful in induction steps, as it can be posed without assuming $\mathrm{S}(x)$:

$\mathrm{HR}(\underline{x})$:	for any embedding $B_x \subset \mathrm{BS}(\underline{x})$ the Hodge-Riemann bilinear relations hold; the conclusions of Theorem 1.4 hold for the restriction of the intersection form on $\mathrm{BS}(\underline{x})$ to B_x.

(Here and elsewhere an "embedding" of Soergel bimodules means an "embedding as a direct summand.") Together, $\mathrm{S}(x)$ and $\mathrm{HR}(\underline{x})$ imply that the restriction of the intersection form on $\mathrm{BS}(\underline{x})$ to B_x agrees with the intersection form on B_x up to a positive scalar for any choice of embedding. (See Lemma 3.11 for the proof.) In other words,

(1.1) If $\mathrm{S}(x)$ holds, then $\mathrm{HR}(x)$ and $\mathrm{HR}(\underline{x})$ are equivalent for any reduced expression $\underline{x}$ of x.

We now give the structure of the proof. In Sections 1.2.2, 1.2.3 and 1.2.4 we introduce and explain the implications between statements needed to perform the induction. In Section 1.2.5 we give a summary of the induction.

We make the following assumption:

(1.2) In Sections 1.2.2, 1.2.3 and 1.2.4 we fix $x \in W$ and $s \in S$ with $xs > x$ and assume that $\mathrm{S}(<xs)$ holds.

1.2.2. *Soergel's conjecture and the local intersection form.* By Soergel's hom formula (see Theorem 3.6), $\mathrm{S}(<xs)$ is equivalent to assuming $\mathrm{End}(B_y) = \mathbb{R}$ for all $y < xs$. Consider the form given by composition

$$(-,-)_y^{x,s} : \mathrm{Hom}(B_y, B_x B_s) \times \mathrm{Hom}(B_x B_s, B_y) \to \mathrm{End}(B_y) = \mathbb{R}.$$

Soergel's hom formula gives an expression for the dimension of these hom spaces in terms of an inner product on the Hecke algebra. Applying this formula one sees that $\mathrm{S}(xs)$ is equivalent to the nondegeneracy of this form for all $y < xs$ (see [Soe07, Lemma 7.1(2)]). Now B_y and $B_x B_s$ are naturally equipped with symmetric invariant bilinear forms (see Section 4) so there is a canonical identification ("take adjoints")

$$\mathrm{Hom}(B_y, B_x B_s) = \mathrm{Hom}(B_x B_s, B_y).$$

Hence we can view $(-,-)_y^{x,s}$ as a form on the real vector space $\mathrm{Hom}(B_y, B_x B_s)$. We call this form the *local intersection form.* We consider "Soergel's conjecture with signs":

$$\mathrm{S}_\pm(y,x,s): \quad \text{the form } (-,-)_y^{x,s} \text{ is } (-1)^{(\ell(x)+1-\ell(y))/2}\text{-definite.}$$

This is *a priori* stronger than Soergel's conjecture. By the above discussion,

$$\mathrm{S}(<xs) \text{ and } \mathrm{S}_\pm(<xs, x, s) \text{ imply } \mathrm{S}(xs). \tag{1.3}$$

1.2.3. *From the local to the global intersection form.* To prove $\mathrm{S}_\pm(y,x,s)$, we must digress and discuss hard Lefschetz and the Hodge-Riemann bilinear relations for $\overline{B_x B_s}$. The connection is explained by (1.4) below. Recall that we have fixed a degree two element $\rho \in R$ such that $\rho(\alpha_s^\vee) > 0$ for all simple coroots $\alpha_s^\vee$. Consider the "hard Lefschetz" condition

$$\mathrm{hL}(x,s): \quad \rho^i : (\overline{B_x B_s})^{-i} \to (\overline{B_x B_s})^i \text{ is an isomorphism.}$$

Because B_{xs} is a direct summand of $B_x B_s$, $\mathrm{hL}(x,s)$ implies $\mathrm{hL}(xs)$. They are equivalent if we know $\mathrm{hL}(<xs)$, since every other indecomposable summand of $B_x B_s$ is of the form B_y for $y < xs$ (a consequence of our standing assumption $\mathrm{S}(<xs)$).

If we fix a reduced expression $\underline{x}$ for x and an embedding $B_x \subset \mathrm{BS}(\underline{x})$, then B_x inherits an invariant form from $\mathrm{BS}(\underline{x})$ as discussed above. Similarly, $B_x B_s$ is a summand of $\mathrm{BS}(\underline{x}s)$ and inherits an invariant form, which we denote

$\langle -,-\rangle_{B_xB_s}$. We formulate the Hodge-Riemann bilinear relations for $\overline{B_xB_s}$ as follows:

HR($\underline{x}, s$): for any embedding $B_x \subset \mathrm{BS}(\underline{x})$ the Lefschetz form $(\alpha,\beta)_\rho^{-i} := \langle \alpha, \rho^i\beta\rangle_{\overline{B_xB_s}}$ is $(-1)^{(\ell(x)+1-i)/2}$-definite on the primitive subspace $P_\rho^{-i} := \ker(\rho^{i+1}) \subset (\overline{B_xB_s})^{-i}$.

Once again, using that $B_xB_s \cong B_{xs} \oplus \bigoplus B_y^{\oplus m_y}$ for some $m_y \in \mathbb{Z}_{\geq 0}$ one may deduce easily that HR($\underline{x}, s$) implies HR($\underline{x}s$) (see Lemma 2.2). However, HR($\underline{x}, s$) is stronger than assuming HR($\underline{x}s$) and HR(y) for all $y < xs$ with $m_y \neq 0$, because it fixes the sign of the restricted form. Indeed, HR($\underline{x}, s$) is equivalent to the statement that the restriction of $\langle -,-\rangle_{B_xB_s}$ to any summand B_y of B_xB_s is $(-1)^{(\ell(xs)-\ell(y))/2}$ times a positive multiple of the intersection form on B_y. For later use, we employ the following abbreviation:

HR(x, s): HR($\underline{x}, s$) holds for all reduced expressions $\underline{x}$ of x.

Recall that the space $\mathrm{Hom}(B_y, B_xB_s)$ is equipped with the local intersection form $(-,-)_y^{x,s}$ and that $(\overline{B_xB_s})^{-\ell(y)}$ is equipped with the Lefschetz form $(-,-)_\rho^{-\ell(y)}$. The motivation for introducing the condition HR($\underline{x}, s$) is the following (see Theorem 4.1): for any ρ as above there exists an embedding

$$\iota : \mathrm{Hom}(B_y, B_xB_s) \hookrightarrow P_\rho^{-\ell(y)} \subset (\overline{B_xB_s})^{-\ell(y)}.$$

Moreover, this embedding is an isometry up to a positive scalar.

Because the restriction of a definite form to a subspace is definite, we obtain

$$\text{S}(<xs) \text{ and HR}(\underline{x}, s) \text{ imply } \text{S}_\pm(<xs, x, s). \tag{1.4}$$

Combining (1.4) and (1.3) and the above discussion, we arrive at the core statement of our induction:

$$\text{S}(<xs) \text{ and HR}(\underline{x}, s) \text{ imply S}(\leq xs) \text{ and HR}(\underline{x}s). \tag{1.5}$$

It remains to show that S($\leq x$) and HR($\leq x$) implies HR($\underline{x}, s$). This reduces Soergel's conjecture to a statement about the modules $\overline{B_xB_s}$ and their intersection forms.

1.2.4. *Deforming the Lefschetz operator.* The reader might have noticed that hL seems to have disappeared from the picture. Indeed, HR is stronger than hL, and one might ask why we wish to treat hL separately. The reason is that it seems extremely difficult to attack HR($\underline{x}, s$) directly. As we noted earlier, de Cataldo and Migliorini's method of proving HR consists in proving hL first for a family of operators, and using a limiting argument to deduce HR.

We adapt their limiting argument as follows. For any real number $\zeta \geq 0$, consider the Lefschetz operator

$$L_\zeta := (\rho\cdot -)\mathrm{id}_{B_s} + \mathrm{id}_{B_x}(\zeta\rho\cdot -),$$

which we view as an endomorphism of B_xB_s. Here $(\rho \cdot -)$ (resp. $(\zeta\rho \cdot -)$) denotes the operator of left multiplication on B_x (resp. B_s) by ρ (resp. $\zeta\rho$) and juxtaposition denotes tensor product of operators. Now consider the following "ζ-deformations" of the above statements:

$\mathrm{hL}(x,s)_\zeta$:	$L_\zeta^i : (\overline{B_xB_s})^{-i} \to (\overline{B_xB_s})^i$ is an isomorphism.
$\mathrm{HR}(\underline{x},s)_\zeta$:	for any embedding $B_x \subset \mathrm{BS}(\underline{x})$ the Lefschetz form $(\alpha,\beta)^\rho_{-i} := \langle \alpha, L_\zeta^i\beta\rangle_{\overline{B_xB_s}}$ is $(-1)^{(\ell(x)+1-i)/2}$-definite on the primitive subspace $P^{-i}_{L_\zeta} := \ker(L_\zeta^i) \subset (\overline{B_xB_s})^{-i}$.
$\mathrm{HR}(x,s)_\zeta$:	$\mathrm{HR}(\underline{x},s)_\zeta$ holds, for all reduced expressions $\underline{x}$ of x.

Note that L_0 is simply left multiplication by ρ, and hence $\mathrm{hL}(x,s)_0 = \mathrm{hL}(x,s)$, $\mathrm{HR}(\underline{x},s)_0 = \mathrm{HR}(\underline{x},s)$ and $\mathrm{HR}(x,s)_0 = \mathrm{HR}(x,s)$. The signature of a family of nondegenerate symmetric real forms cannot change. Therefore, if $\mathrm{hL}(x,s)_\zeta$ holds for all $\zeta \geq 0$ and $\mathrm{HR}(\underline{x},s)_\zeta$ holds for any single nonnegative value of ζ, then $\mathrm{HR}(\underline{x},s)_0$ also holds. (This is the essence of de Cataldo and Migliorini's limiting argument.)

The first hint that this deformation is promising is Theorem 5.1:

$$\mathrm{HR}(\underline{z}) \text{ implies } \mathrm{HR}(\underline{z},s)_\zeta \text{ for } \zeta \gg 0 \tag{1.6}$$

(which holds regardless of whether $zs > z$ or $zs < z$). Therefore, we have

$$\mathrm{HR}(\underline{x}) \text{ and } \mathrm{hL}(x,s)_\zeta \text{ for all } \zeta \geq 0, \text{ implies } \mathrm{HR}(\underline{x},s)_\zeta \text{ for all } \zeta \geq 0. \tag{1.7}$$

In particular, the fact that $\mathrm{hL}(z,s)_\zeta$ and $\mathrm{HR}(\underline{z},s)_\zeta$ hold for all $\zeta \geq 0$ and all $z < x$ with $sz > z$ is something we may inductively assume when trying to prove the same facts for x.

We have reduced our problem to establishing $\mathrm{hL}(x,s)_\zeta$ for $\zeta \geq 0$. In de Cataldo and Migliorini's approach this is established using the weak Lefschetz theorem and the Hodge-Riemann bilinear relations in smaller dimension. In our setting the weak Lefschetz theorem is missing, and a key point is the use of Rouquier complexes as a replacement. (See the first few paragraphs of Section 6 for more details.) The usual proof of hL for a vector space V is to find a map $V \to W$ of degree 1, injective on V^{-i} for $i > 0$ and commuting with the Lefschetz operator, where HR is known to hold for W. The Rouquier complex yields a map of degree 1 from B_xB_s, injective on negative degrees and commuting with L, to a direct sum of B_x and terms of the form B_zB_s for summands B_z of $\mathrm{BS}(\underline{x})$ with $z < x$. This target space does not satisfy the Hodge-Riemann bilinear relations, but nevertheless we are able to prove the hard Lefschetz theorem.

When $\zeta = 0$, we have an argument which shows

$$\text{S}(\leq x),\ \text{hL}(<xs),\ \text{HR}(x) \text{ and } \text{HR}(z,t) \text{ for all } z < x \text{ with } zt > z \\ \text{together imply } \text{hL}(x,s). \tag{1.8}$$

This is Theorem 6.21. One feature of the proof is that, whenever $zs < z$, the decomposition $B_z B_s \cong B_z(1) \oplus B_z(-1)$ commutes with the Lefschetz operator L_0. This decomposition allows one to bypass the fact that $\text{HR}(z,s)$ fails if $zs < z$.

When $\zeta > 0$, the decomposition $B_z B_s \cong B_z(1) \oplus B_z(-1)$ for $zs < z$ does not commute with L_ζ. However, proving $\text{hL}(z,s)_\zeta$ for $\zeta > 0$ and $zs < z$ using $\text{hL}(z)$ is a straightforward computation (Theorem 6.19). Our inductive hypotheses and the limiting argument above now yield $\text{HR}(\underline{z},s)_\zeta$ for all $z < x$. A similar argument to the previous case shows

$$\text{For } \zeta > 0,\ \text{S}(\leq x),\ \text{HR}(\leq x),\ \text{HR}(<x,s)_\zeta \\ \text{and } \text{HR}(z,t) \text{ for all } z < x \text{ with } zt > z \text{ imply } \text{hL}(x,s)_\zeta. \tag{1.9}$$

This is Theorem 6.20.

1.2.5. *Structure of the proof.* Let us summarize the overall inductive proof. Let $X \subset W$ be an ideal in the Bruhat order (i.e., $z \leq x \in X \Rightarrow z \in X$) and assume

(1) $\text{HR}(z,t)_\zeta$ for all $\zeta \geq 0$, $z < zt \in X$ and $t \in S$;
(2) $\text{HR}(z,t)_\zeta$ for all $\zeta > 0$, $zt < z \in X$ and $t \in S$.

We have already explained why (1) implies $\text{S}(X)$, $\text{hL}(X)$ and $\text{HR}(X)$.

Now choose a minimal element x' in the complement of X, and choose $s \in S$ and $x \in X$ with $x' = xs$. As we just discussed, (1.8) and (1.9) imply that $\text{hL}(x,s)_\zeta$ holds for all $\zeta \geq 0$. Using $\text{HR}(\underline{x})$ and (1.6) we deduce $\text{HR}(x,s)_\zeta$ for all $\zeta \geq 0$. Therefore, (1) holds with X replaced by $X \cup \{x'\}$, and thus $\text{S}(x')$, $\text{hL}(x')$, and $\text{HR}(x')$ all hold.

As above, the straightforward calculations of Theorem 6.19 show that $\text{hL}(x',t)_\zeta$ holds for $\zeta > 0$ when $t \in S$ satisfies $x't < x'$. Again by $\text{HR}(x')$ and (1.6) we have $\text{HR}(x',t)_\zeta$ for all $\zeta > 0$ in this case. Thus (2) holds for $X \cup \{x'\}$ as well.

By inspection, (1) and (2) hold for the set $X = \{w \in W \mid \ell(w) \leq 2\}$. Hence by induction we obtain (1) and (2) for $X = W$. We have already explained why this implies all of the theorems in Section 1.1.

1.3. *Note to the reader.* In order to keep this paper short and have it cite only available sources, we have written it in the language of [Soe07]. However, [Soe07] is not an easy paper, and we make heavy use of its results. We did not discover the results of this paper in this language, but rather in the diagrammatic language of [EWa] and [EWb]. These papers also provide alternative proofs of the requisite results from [Soe07].

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The second author dedicates his work to the memory of Leigh, who would not have given two hoots if certain polynomials have positive coefficients!

2. **Lefschetz linear algebra**

Let $H = \bigoplus_{i \in \mathbb{Z}} H^i$ be a graded finite-dimensional real vector space equipped with a nondegenerate symmetric bilinear form

$$\langle -, - \rangle_H : H \otimes_{\mathbb{R}} H \to \mathbb{R}$$

which is *graded* in the sense that $\langle H^i, H^j \rangle = 0$ unless $i = -j$.

Let $L \colon H^\bullet \to H^{\bullet+2}$ denote an operator of degree 2. We may also write $L \in \mathrm{Hom}(H, H(2))$, where (2) indicates a grading shift. We say that L is a *Lefschetz operator* if $\langle Lh, h' \rangle = \langle h, Lh' \rangle$ for all $h, h' \in H$. We assume from now on that L is a Lefschetz operator. We say that L *satisfies the hard Lefschetz theorem* if $L^i : H^{-i} \to H^i$ is an isomorphism for all $i \in \mathbb{Z}_{\geq 0}$. For $i \geq 0$ set

$$P_L^{-i} := \ker L^{i+1} \subset H^{-i}.$$

We call P_L^{-i} the *primitive subspace* of H^{-i} (with respect to L). If L satisfies the hard Lefschetz theorem, then we have a decomposition

$$H = \bigoplus_{\substack{i \geq 0 \\ 0 \leq j \leq i}} L^j P_L^{-i}.$$

This is the *primitive decomposition* of H.

For each $i \geq 0$ we define the *Lefschetz form* on H^{-i} via

$$(h, h')_L^{-i} := \langle h, L^i h' \rangle.$$

All Lefschetz forms are nondegenerate if and only if L satisfies the hard Lefschetz theorem, because $\langle -, - \rangle$ is nondegenerate by assumption. Because L is a Lefschetz operator, we have $(h, h')_L^{-i} = (Lh, Lh')_L^{-i+2}$ for all $i \geq 2$ and $h, h' \in H^{-i}$. If L satisfies the hard Lefschetz theorem, then the primitive decomposition is orthogonal with respect to the Lefschetz forms.

We say that H is *odd* (resp. *even*) if $H^{\mathrm{even}} = 0$ (resp. $H^{\mathrm{odd}} = 0$). Recall that a bilinear form $(-, -)$ on a real vector space is said to be $+1$ *definite* (resp. -1 *definite*) if (v, v) is strictly positive (resp. negative) for all nonzero vectors v.

Let H and L be as above, and assume that L satisfies the hard Lefschetz theorem. Assume that H is either even or odd, and set $j = 0$ if H is even, and $j = 1$ if H is odd. We say that H and L *satisfy the Hodge-Riemann bilinear relations* if there exists $\varepsilon \in \{\pm 1\}$ such that the restriction of $(-,-)_L^{-i}$ to each primitive component $P_L^{-i} \subset H^{-i}$ is $\varepsilon(-1)^{(-i+j)/2}$ definite for all $i \leq 0$. We fix the ambiguity of global sign as follows. If H and L satisfy the Hodge-Riemann bilinear relations, we say that the Hodge-Riemann bilinear relations are satisfied with the *standard sign* if the Lefschetz form is positive definite on the lowest nonzero degree of H (which is necessarily primitive).

In order to avoid having to always specify if a vector space is even or odd we will adopt the following convention: the statement that a form on a space P is $(-1)^{m/2}$ definite has the above meaning if m is even and means that $P = 0$ if m is odd.

If H and L satisfy the Hodge-Riemann bilinear relations then, in particular, each Lefschetz form $(-,-)_L^{-i}$ is nondegenerate. Moreover, its signature is easily determined from the graded rank of H. (Use the fact that the primitive decomposition is orthogonal.) In fact, the Hodge-Riemann bilinear relations are equivalent to a statement about the signatures of all Lefschetz forms.

In the sequel, we will need to consider families of Lefschetz operators (keeping H and the form $\langle -,- \rangle$ fixed). It will be important to be able to decide whether any or all members of the family are Hodge-Riemann. The following elementary lemma will provide an invaluable tool:

LEMMA 2.1. *Let $a < b$ in $\mathbb{R}$, and let $\phi : [a,b] \to \mathrm{Hom}(H, H(2))$ be a continuous map* (*in the standard Euclidean topologies*) *such that $\phi(t)$ is a Lefschetz operator satisfying the hard Lefschetz theorem for all $t \in [a,b]$. If there exists $t_0 \in [a,b]$ such that $\phi(t_0)$ satisfies the Hodge-Riemann bilinear relations, then all $\phi(t)$ for $t \in [a,b]$ satisfy the Hodge-Riemann bilinear relations.*

Proof. This follows from the fact that the signature of a continuous family of nondegenerate symmetric bilinear forms is constant. □

In general it is difficult to decide whether the restriction of a nondegenerate bilinear form to a subspace stays nondegenerate. However, it is obvious that the restriction of a definite form is nondegenerate. This basic fact plays a crucial role in this paper. The following lemma extends this observation to certain L-stable subspaces of H.

LEMMA 2.2. *Assume that H and L satisfy the Hodge-Riemann bilinear relations. Let $V \subset H$ denote an L-stable graded subspace such that* $\dim V^i = \dim V^{-i}$. *Then V and L satisfy the Hodge-Riemann bilinear relations* (*with respect to the restriction of $\langle -,- \rangle$ to V*).

Proof (*Sketch*: *the reader should provide a proof*). By symmetry of Betti numbers and hard Lefschetz, V admits a primitive decomposition, and the result follows. $\square$

The following lemma will serve as a substitute for the weak Lefschetz theorem:

LEMMA 2.3. *Suppose we have a map of graded* $\mathbb{R}[L]$*-modules* ($\deg L = 2$)

$$\phi : V \to W(1)$$

such that

(1) ϕ *is injective in degrees* ≤ -1;
(2) V *and* W *are equipped with graded bilinear forms* $\langle -, - \rangle_V$ *and* $\langle -, - \rangle_W$ *such that* $\langle \phi(\alpha), \phi(\beta) \rangle_W = \langle \alpha, L\beta \rangle_V$ *for all* $\alpha, \beta \in V$;
(3) W *satisfies the Hodge-Riemann bilinear relations.*

Then $L^i : V^{-i} \to V^i$ *is injective for* $i \geq 0$.

Proof. For $i = 0$ the statement is vacuous. Choose $0 \neq \alpha \in V^{-i}$ with $i \geq 1$, and consider $0 \neq \phi(\alpha) \in W^{-i+1}$. If $0 \neq L^i\phi(\alpha) = \phi(L^i\alpha)$, then $L^i\alpha \neq 0$. Alternatively, if $L^i\phi(\alpha) = 0$, then $\phi(\alpha)$ is primitive. Hence

$$(\phi(\alpha), \phi(\alpha))_L^{-i+1} = \langle \phi(\alpha), L^{i-1}\phi(\alpha) \rangle_W = \langle \alpha, L^i\alpha \rangle_V$$

is either strictly negative or positive by the Hodge-Riemann bilinear relations. In any case, $L^i\alpha \neq 0$. Hence $L^i : V^{-i} \to V^i$ is injective as claimed. $\square$

When $\dim(V^{-i}) = \dim(V^i)$ for all i, this lemma implies the hard Lefschetz theorem for V.

Remark 2.4. Suppose we are in the situation of the above lemma; that $-\ell$ is the lowest degree of W and that $-(\ell+1)$ is the lowest degree of V. The above proof indicates that $(-, -)_L^{-(\ell+1)}$ is $\pm$ definite on $V^{-(\ell+1)}$, with the same sign as on $W^{-\ell}$. In particular, if V also satisfies the Hodge-Riemann bilinear relations, then V has the standard sign if and only if W has the standard sign.

Finally, we will need the following lemma in Section 6.6. Let H, $\langle -, - \rangle$ and L be as in the first two paragraphs of this section, except that we no longer assume that $\langle -, - \rangle$ is nondegenerate. Suppose that there exists $d \in \mathbb{Z}$ such that $L^i : H^{-d-i} \to H^{-d+i}$ is an isomorphism for all $i \geq 0$ (so L satisfies the hard Lefschetz theorem if and only if $d = 0$).

LEMMA 2.5. *If* $d > 0$, *then the Lefschetz form* $(h, h')_L^{-i} := \langle h, L^i h \rangle$ *on* H^{-i} *for* $i \geq 0$ *is zero.*

Proof. For $i \geq 0$ consider the "shifted primitive spaces"

$$Q_L^{-d-i} := \ker L^{i+1} \subset H^{-d-i},$$

and set $Q_L^{-d-i} := 0$ if $i < 0$. Then our assumptions on L guarantee that we have a "shifted primitive decomposition"

$$H^m = \bigoplus_{j \geq 0} L^j Q_L^{m-2j}.$$

Fix a degree $m \leq 0$, and fix $x \in Q_L^{m-2j}$ and $y \in Q_L^{m-2k}$ for some $j \geq k \geq 0$, so that $L^j x$ and $L^k y$ are in degree m. Then

$$(L^j x, L^k y)_L^m = \langle x, L^{j+k-m} y\rangle = 0.$$

This follows because $y \in \ker L^{2k-d-m+1}$ and $2k-d-m+1 \leq j+k-m$, thanks to the assumption $d > 0$. $\square$

3. The Hecke algebra and Soergel bimodules

3.1. *Coxeter systems.* Fix a Coxeter system (W, S), and for simple reflections $s, t \in S$ denote by $m_{st} \in \{2, 3, \dots, \infty\}$ the order of st. We denote the length function on W by ℓ and the Bruhat order by $\leq$.

An *expression* is a word $\underline{z} = s_1 s_2 \cdots s_m$ in S. An expression will always be denoted by an underlined roman letter. Omitting the underline will denote the product in the Coxeter group. An expression $\underline{z} = s_1 s_2 \cdots s_m$ is *reduced* if $m = \ell(z)$.

Let us fix a finite-dimensional real vector space $\mathfrak{h}$ together with linearly independent subsets $\{\alpha_s\}_{s\in S} \subset \mathfrak{h}^*$ and $\{\alpha_s^\vee\}_{s\in S} \subset \mathfrak{h}$ such that

$$\alpha_s(\alpha_t^\vee) = -2\cos(\pi/m_{st}) \quad \text{for all } s, t \in S.$$

In addition, we assume that $\mathfrak{h}$ is of minimal dimension with these properties. The group W acts on $\mathfrak{h}$ by $s \cdot v = v - \alpha_s(v)\alpha_s^\vee$. This action is *reflection faithful* in the sense of [Soe07, Def. 1.5] (see [Soe07, Prop. 2.1]).

Remark 3.1. We have assumed that the representation $\mathfrak{h}$ is reflection faithful so that the theory of [Soe07] is available. It was shown by Libedinsky [Lib08] that Soergel's conjecture for $\mathfrak{h}$ is equivalent to Soergel's conjecture for the geometric representation. We discuss the choice of representation in detail in [EWb], where we give alternative proofs of the results of [Soe07] which are valid when $\mathfrak{h}$ is any "realization" of W.

Let R be the coordinate ring of $\mathfrak{h}$, graded so that its linear terms $\mathfrak{h}^*$ have degree 2. We denote by R^+ the ideal of elements of positive degree. Clearly W acts on R. For $s \in S$ we write R^s for the subring of invariants under s.

Because the vectors $\{\alpha_s^\vee\}_{s\in S}$ are linearly independent, the intersection of the open half spaces

$$\bigcap_{s\in S} \{v \in \mathfrak{h}^* \mid v(\alpha_s^\vee) > 0\} \subset \mathfrak{h}^*$$

is nonempty. We fix once and for all an element $\rho \in \mathfrak{h}^*$ in this intersection. That is, we fix ρ such that $\rho(\alpha_s^\vee) > 0$ for all $s \in S$. The following positivity property of the representation $\mathfrak{h}$ plays an important role below (see [Bou81, V.4.3] or [Hum90, Lemma 5.13] and the proof of [Soe07, Prop. 2.1]):

$$(w\rho)(\alpha_s^\vee) > 0 \Leftrightarrow sw > w. \tag{3.1}$$

3.2. *The Hecke algebra.* References for this section are [KL79] and [Soe97]. Recall that the Hecke algebra $\mathcal{H}$ is the algebra with free $\mathbb{Z}[v^{\pm 1}]$-basis given by symbols $\{H_x \mid x \in W\}$ with multiplication determined by

$$H_x H_s := \begin{cases} H_{xs} & \text{if } xs > x, \\ (v^{-1} - v)H_x + H_{xs} & \text{if } xs < x. \end{cases}$$

Given $p \in \mathbb{Z}[v^{\pm 1}]$ we write $\overline{p}(v) := p(v^{-1})$. We can extend this to an involution of $\mathcal{H}$ by setting $\overline{H_x} = H_{x^{-1}}^{-1}$. Denote the Kazhdan-Lusztig basis of $\mathcal{H}$ by $\{\underline{H}_x \mid x \in W\}$. It is characterised by the two conditions

(i) $\overline{\underline{H}_x} = \underline{H}_x$,
(ii) $\underline{H}_x \in H_x + \sum_{y<x} v\mathbb{Z}[v] H_y$

for all $x \in W$. For example, if $s \in S$, then $\underline{H}_s = H_s + vH_{\mathrm{id}}$.

Remark 3.2. In the notation of [KL79] we have $v = q^{-1/2}$, $H_x = v^{\ell(x)} T_x$ and $\underline{H}_x = C'_x$. If we write $\underline{H}_x = \sum h_{y,x} H_y$, then $v^{\ell(x)-\ell(y)} P_{y,x}(v^{-2}) = h_{y,x}$.

Consider the $\mathbb{Z}[v, v^{-1}]$-linear trace $\varepsilon : \mathcal{H} \to \mathbb{Z}[v^{\pm 1}]$ given by $\varepsilon(H_w) = \delta_{\mathrm{id},w}$. Define a bilinear form

$$\begin{aligned} (-,-) : \mathcal{H} \times \mathcal{H} &\to \mathbb{Z}[v^{\pm 1}] \\ (h, h') &\mapsto \varepsilon(a(h)h'), \end{aligned}$$

where a is the anti-involution of $\mathcal{H}$ determined by $a(v) = v$ and $a(H_x) = H_{x^{-1}}$. One checks easily that

(i) $(ph, qh') = pq(h, h')$ for all $p, q \in \mathbb{Z}[v^{\pm 1}]$ and $h, h' \in \mathcal{H}$;
(ii) $(h\underline{H}_s, h') = (h, h'\underline{H}_s)$, $(\underline{H}_s h, h') = (h, \underline{H}_s h')$ for all $h, h' \in \mathcal{H}$ and $s \in S$.

A straightforward induction shows $(H_x, H_y) = \delta_{xy}$, which we could have used as the definition of $(-,-)$.

An important property of this pairing (used repeatedly below) is that $(\underline{H}_x, \underline{H}_y) \in v\mathbb{Z}[v]$ when $x \neq y$, and $(\underline{H}_x, \underline{H}_x) \in 1 + v\mathbb{Z}[v]$.

Remark 3.3. This is not the form used in [EWb], which is more natural when one only considers Soergel bimodules. In this paper we also consider Δ- and ∇-filtered bimodules, for which the above form is more convenient.

3.3. *Bimodules.* We work in the abelian category of finitely generated graded R-bimodules. All morphisms preserve the grading (i.e., are homogeneous of degree 0). Given a graded R-bimodule $B = \bigoplus_{i\in\mathbb{Z}} B^i$, we denote by $B(1)$ the shifted bimodule: $B(1)^i = B^{i+1}$. We write $\mathrm{Hom}(-,-)$ for degree zero morphisms between bimodules (the morphisms in our category). For any two bimodules M and N set

$$\mathrm{Hom}^\bullet(M,N) = \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}(M, N(i)).$$

Given a polynomial $p = \sum_{i\in\mathbb{Z}} a_i v^i \in \mathbb{Z}_{\geq 0}[v^{\pm 1}]$, we let $B^{\oplus p}$ denote the bimodule $\oplus_{i\in\mathbb{Z}} B(i)^{\oplus a_i}$. Given bimodules B and B', we write $B' \overset{\oplus}{\subset} B$ to mean that B' is a direct summand of B. Throughout, "embedding" means "embedding as a direct summand."

The category of R-bimodules is a monoidal category under tensor product. Given R-bimodules B and B', we denote their tensor product by juxtaposition: $BB' := B \otimes_R B'$.

Throughout this paper, we have arbitrarily chosen the right action to be special for many constructions. For instance, for a bimodule B we will often consider $\overline{B} = B \otimes_R \mathbb{R}$; here $\mathbb{R} = R/R^+$ is the R-module where all positive degree polynomials vanish.

We define the dual of an R-bimodule B by $\mathbb{D}B := \mathrm{Hom}^\bullet_{-R}(B,R)$. Here, $\mathrm{Hom}^\bullet_{-R}(-,-)$ denotes homomorphisms of all degrees between right R-modules. We make $\mathbb{D}B$ into an R-bimodule via $(r_1 f r_2)(b) = f(r_1 b r_2)$ (where $f \in \mathbb{D}B$, $r_1, r_2 \in R$ and $b \in B$). Suppose that B is finitely generated and graded free as a right R-module, so that $B \cong R^{\oplus p}$ as a right R-module (for some $p \in \mathbb{Z}_{\geq 0}[v^{\pm 1}]$). Then $\mathbb{D}B \cong R^{\oplus \overline{p}}$. In particular, if $B \cong \mathbb{D}B$, then $\overline{B} \cong \overline{B}^*$ as graded $\mathbb{R}$-vector spaces and $\dim(\overline{B})^{-i} = \dim(\overline{B})^{+i}$.

We say that an R-valued form $\langle -,-\rangle_B$ on a graded R-bimodule B is *graded* if $\deg\langle b,b'\rangle_B = \deg b + \deg b'$ for homogeneous b, b'. A form $\langle -,-\rangle_B$ is *invariant* if it is graded and $\langle rb,b'\rangle = \langle b, rb'\rangle$ and $\langle br,b'\rangle = \langle b,b'r\rangle = \langle b,b'\rangle r$ for all $b,b' \in B$ and $r \in R$. (Note the left/right asymmetry.) The space of invariant forms is isomorphic to the space of R-bimodule maps $B \to \mathbb{D}B$. We say that an invariant form $\langle -,-\rangle_B$ on a bimodule B is *nondegenerate* if it induces an isomorphism $B \to \mathbb{D}B$. This is stronger than assuming nondegeneracy in the usual sense ($\langle b,b'\rangle = 0$ for all $b' \in B$ implies $b = 0$). An invariant form $\langle -,-\rangle_B$ on B induces a form $\langle -,-\rangle_{\overline{B}}$ on $\overline{B}$ by defining $\langle f,g\rangle_{\overline{B}}$ to be the image of $\langle f,g\rangle_B$ in $\mathbb{R} = R/R^+$.

Suppose that B is free of finite rank as a right R-module (as will be the case for all bimodules considered below). Then an invariant form $\langle -,-\rangle_B$ is nondegenerate if and only if $\langle -,-\rangle_{\overline{B}}$ gives a graded (in the sense of Section 2) nondegenerate form on the graded vector space $\overline{B}$, as follows from the graded Nakayama lemma.

3.4. *Bott-Samelson bimodules.* For any simple reflection $s \in S$ set $B_s := R \otimes_{R^s} R(1)$. It is an R-bimodule with respect to left and right multiplication by R. Consider the elements

$$c_{\mathrm{id}} := 1 \otimes 1 \in B_s, \qquad c_s := \frac{1}{2}(\alpha_s \otimes 1 + 1 \otimes \alpha_s) \in B_s$$

of degrees -1 and 1 respectively. Then $\{c_{\mathrm{id}}, c_s\}$ gives a basis for B_s as a right (or left) R-module, and one has the relations

$$r \cdot c_s = c_s \cdot r, \tag{3.2}$$

$$r \cdot c_{\mathrm{id}} = c_{\mathrm{id}} \cdot sr + \partial_s(r) \cdot c_s \tag{3.3}$$

for all $r \in R$. Here, ∂_s is the *Demazure operator*, given by

$$\partial_s(r) = \frac{r - sr}{\alpha_s} \in R.$$

For any expression $\underline{x} = st \cdots u$ we denote by $\mathrm{BS}(\underline{x})$ the corresponding Bott-Samelson bimodule:

$$\mathrm{BS}(\underline{x}) := B_s B_t \cdots B_u = B_s \otimes_R B_t \otimes_R \cdots \otimes_R B_u.$$

Given elements $b_s \in B_s$, $b_t \in B_t$, $\ldots$, $b_u \in B_u$ we denote the corresponding tensor simply by juxtaposition $b_s b_t \cdots b_u := b_s \otimes b_t \otimes \cdots \otimes b_u$. For any subexpression $\underline{\varepsilon}$ of $\underline{x}$ (that is, $\underline{\varepsilon} = \varepsilon_s \varepsilon_t \cdots \varepsilon_u$ with $\varepsilon_v \in \{\mathrm{id}, v\}$ for all $v \in S$) we can consider the element

$$c_{\underline{\varepsilon}} := c_{\varepsilon_s} c_{\varepsilon_t} \cdots c_{\varepsilon_u} \in \mathrm{BS}(\underline{x}).$$

One may check that the set $\{c_{\underline{\varepsilon}}\}$ gives a basis for $\mathrm{BS}(\underline{x})$ as a right (or left) R-module, as $\underline{\varepsilon}$ runs over all subexpressions of $\underline{x}$.

In the following, the element $c_{\mathrm{top}} := c_s c_t \cdots c_u \in \mathrm{BS}(\underline{x})$ will play an important role. Given $b \in \mathrm{BS}(\underline{x})$ we define $\mathrm{Tr}(b) \in R$ to be the coefficient of c_{top} when b is expressed in the basis $\{c_{\underline{\varepsilon}}\}$ of $\mathrm{BS}(\underline{x})$ as a right R-module.

Clearly, $\mathrm{BS}(\underline{x}) \cong (R \otimes_{R^s} R \otimes_{R^t} \cdots \otimes_{R^u} R)(d)$, where d is the length of the expression $\underline{x}$. It follows that $\mathrm{BS}(\underline{x})(-d)$ is a commutative ring, with term-wise multiplication. For example, $(f \otimes g) \cdot (f' \otimes g') = ff' \otimes gg'$ gives the multiplication on $B_s(-1) = R \otimes_{R^s} R$. Let us observe the following multiplication rules in B_s:

$$c_{\mathrm{id}} \cdot c_{\mathrm{id}} = c_{\mathrm{id}}, \tag{3.4}$$

$$c_{\mathrm{id}} \cdot c_s = c_s, \tag{3.5}$$

$$c_s \cdot c_s = c_s \alpha_s. \tag{3.6}$$

We define an invariant symmetric form $\langle -, - \rangle_{\mathrm{BS}(\underline{x})}$ on $\mathrm{BS}(\underline{x})$ via

$$\langle b, b' \rangle := \mathrm{Tr}(b \cdot b').$$

We call $\langle -,-\rangle_{\mathrm{BS}(\underline{x})}$ the *intersection form* on $\mathrm{BS}(\underline{x})$. It induces a symmetric $\mathbb{R}$-valued form $\langle -,-\rangle_{\overline{\mathrm{BS}(\underline{x})}}$ on $\overline{\mathrm{BS}(\underline{x})}$. If we write $\mathrm{Tr}_{\mathbb{R}}$ for the composition of Tr with the quotient map $R \to \mathbb{R} = R/R^+$, then we have

$$\langle b, b'\rangle_{\overline{\mathrm{BS}(\underline{x})}} = \mathrm{Tr}_{\mathbb{R}}(b \cdot b').$$

for all $b, b' \in \overline{\mathrm{BS}(\underline{x})}$. Because $R \otimes_{R^s} R \otimes_{R^t} \cdots \otimes_{R^u} R$ is a commutative ring, multiplication by any degree 2 element z of this ring gives a Lefschetz operator on $\mathrm{BS}(\underline{x})$. In other words, $\langle z\alpha, \beta\rangle = \langle \alpha, z\beta\rangle$ for any $\alpha, \beta \in \mathrm{BS}(\underline{x})$.

For $s \in S$ let $\mu : B_s \to R(1) : f \otimes g \mapsto fg$ denote the multiplication map. For $1 \le i \le m$ set $\underline{x_{\widehat{i}}} = s_1 \cdots \widehat{s_i} \cdots s_m$ ($\widehat{s_i}$ denotes omission). Consider the canonical maps

$$\begin{aligned}
\mathrm{Br}_i &: \mathrm{BS}(\underline{x}) \to \mathrm{BS}(\underline{x})(2) : b_1 \cdots b_i \cdots b_m \mapsto b_1 \cdots (b_i \cdot c_{s_i}) \cdots b_m,\\
\phi_i &: \mathrm{BS}(\underline{x}) \to \mathrm{BS}(\underline{x_{\widehat{i}}})(1) : b_1 \cdots b_i \cdots b_m \mapsto b_1 \cdots \mu(b_i) \cdots b_m,\\
\chi_i &: \mathrm{BS}(\underline{x_{\widehat{i}}}) \to \mathrm{BS}(\underline{x})(1) : b_1 \cdots b_{i-1} b_{i+1} \cdots b_m \mapsto b_1 \cdots b_{i-1} c_{s_i} b_{i+1} \cdots b_m.
\end{aligned}$$

By (3.2) we have $\mathrm{Br}_i = \chi_i \circ \phi_i$.

LEMMA 3.4. *As endomorphisms of* $\mathrm{BS}(\underline{x})$ *we have*

$$\rho \cdot (-) = \sum_{i=1}^{m} (s_{i-1} \cdots s_1 \rho)(\alpha_{s_i}^\vee) \chi_i \circ \phi_i + (-) \cdot x^{-1}\rho.$$

Here $\rho \cdot (-)$ *denotes left multiplication by* ρ *and* $(-) \cdot x^{-1}\rho$ *denotes right multiplication by* $x^{-1}\rho$.

Proof. This is an immediate consequence of (3.3). □

3.5. *Soergel bimodules.* By definition, a *Soergel bimodule* is an object in the additive Karoubian subcategory $\mathcal{B}$ of graded R-bimodules generated by Bott-Samelson bimodules and their shifts. In other words, indecomposable Soergel bimodules are the indecomposable R-bimodule summands of Bott-Samelson bimodules (up to shift).

It is a theorem of Soergel [Soe07] that, given any reduced expression $\underline{x}$ for $x \in W$, there is a unique (up to isomorphism) indecomposable summand B_x of $\mathrm{BS}(\underline{x})$ which does not occur as a direct summand of $\mathrm{BS}(\underline{y})$ for any expression $\underline{y}$ of length less than $\ell(x)$. Moreover, B_x does not depend (up to isomorphism) on the choice of reduced expression $\underline{x}$. The bimodules B_x for $x \in W$ give representatives for the isomorphism classes of indecomposable Soergel bimodules, up to shift.

Denote by $[\mathcal{B}]$ the split Grothendieck group of $\mathcal{B}$. That is, $[\mathcal{B}]$ is the abelian group generated by symbols $[B]$ for all objects $B \in \mathcal{B}$ subject to the relations $[B] = [B'] + [B'']$ whenever $B \cong B' \oplus B''$ in $\mathcal{B}$. We make $[\mathcal{B}]$ into a $\mathbb{Z}[v^{\pm 1}]$-module via $p[M] := [M^{\oplus p}]$ for $p \in \mathbb{Z}_{\ge 0}[v^{\pm 1}]$ and $M \in \mathcal{B}$. Because $\mathcal{B}$ is monoidal, $[\mathcal{B}]$ is a $\mathbb{Z}[v^{\pm 1}]$-algebra. The above results imply that $[\mathcal{B}]$ is free as a $\mathbb{Z}[v^{\pm 1}]$-module, with basis $\{[B_x] \mid x \in W\}$. In fact one has [Soe07, Th. 1.10]

Theorem 3.5 (Soergel's categorification theorem). *There is an isomorphism of $\mathbb{Z}[v^{\pm 1}]$-algebras*

$$\mathcal{H} \xrightarrow{\sim} [\mathcal{B}]$$

fixed by $\underline{H}_s \mapsto [B_s]$.

We now describe Soergel's construction of an inverse to the isomorphism $\mathcal{H} \xrightarrow{\sim} [\mathcal{B}]$. To do this it is natural to consider certain filtrations "by support" (see [Soe07, §§3 and 5]). For $x \in W$ consider the linear subspace (or "twisted graph")

$$\mathrm{Gr}(x) = \{(xv, v) \,|\, v \in \mathfrak{h}\} \subseteq \mathfrak{h} \times \mathfrak{h},$$

which we view as a subvariety in $\mathfrak{h} \times \mathfrak{h}$. For any subset A of W consider the corresponding union

$$\mathrm{Gr}(A) = \bigcup_{x \in A} \mathrm{Gr}(x) \subseteq \mathfrak{h} \times \mathfrak{h}.$$

Let us identify $R \otimes_{\mathbb{R}} R$ with the regular functions on $\mathfrak{h} \times \mathfrak{h}$. Any R-bimodule can be viewed as an $R \otimes_{\mathbb{R}} R$-module (because R is commutative) and hence as a quasi-coherent sheaf on $\mathfrak{h} \times \mathfrak{h}$. For example, one may check that the bimodule R_x corresponding to the structure sheaf on $\mathrm{Gr}(x)$ has the following simple description: $R_x \cong R$ as a left module, and the right action is twisted by x: $m \cdot r = m(xr)$ for $m \in R_x$ and $r \in R$.

Given any subset $A \subseteq W$ and R-bimodule M we define

$$\Gamma_A M := \{m \in M \mid \operatorname{supp} m \subseteq \mathrm{Gr}(A)\}$$

to be the subbimodule consisting of elements whose support is contained in $\mathrm{Gr}(A)$. Given $x \in W$ we will abuse notation and write $\leq x$ for the set $\{y \in W \mid y \leq x\}$ and similarly for $< x$, $\geq x$ and $> x$. With this notation, we obtain functors $\Gamma_{\leq x}$, $\Gamma_{<x}$, $\Gamma_{\geq x}$ and $\Gamma_{>x}$. For example, $\Gamma_{\leq x} = \Gamma_{\{y \in W \mid y \leq x\}}$.

For any $x \in W$ define $\Delta_x := R_x(-\ell(x))$ and $\nabla_x := R_x(\ell(x))$. Given a finitely generated R-bimodule M we say that M *has a Δ-filtration* (resp. *has a ∇-filtration*) if M is supported on Gr_A for some finite subset $A \subset W$ and, for all $x \in W$, we have isomorphisms

$$\Gamma_{\geq x} M / \Gamma_{>x} M \cong \Delta_x^{\oplus h_x^{\Delta}(M)} \qquad (\text{resp.} \quad \Gamma_{\leq x} M / \Gamma_{<x} M \cong \nabla_x^{\oplus h_x^{\nabla}(M)})$$

for some polynomials $h_x^{\Delta}(M)$ (resp. $h_x^{\nabla}(M)$) in $\mathbb{Z}_{\geq 0}[v^{\pm 1}]$. If M has a Δ-filtration (resp. ∇-filtration), we define its *Δ-character* (resp. *∇-character*) in the Hecke algebra via

$$\mathrm{ch}_{\Delta} M := \sum_{x \in W} h_x^{\Delta}(M) H_x \qquad (\text{resp.} \quad \mathrm{ch}_{\nabla} M := \sum_{x \in W} \overline{h_x^{\nabla}(M)} H_x).$$

Note that $\mathrm{ch}_{\Delta} M(1) = v \, \mathrm{ch}_{\Delta} M$ whilst $\mathrm{ch}_{\nabla} M(1) = v^{-1} \, \mathrm{ch}_{\nabla} M$.

By [Soe07, Props. 5.7 and 5.9], Soergel bimodules have both Δ- and ∇-filtrations, and by [Soe07, Bemerkung 6.16] we have

$$\mathrm{ch}_\Delta B = \overline{\mathrm{ch}_\nabla B}$$

for any Soergel bimodule. For any Soergel bimodule B we set

$$\mathrm{ch}(B) := \mathrm{ch}_\Delta(B).$$

By [Soe07, Th. 5.3], $\mathrm{ch} : [\mathcal{B}] \to \mathcal{H}$ gives an inverse of the isomorphism $\mathcal{H} \xrightarrow{\sim} [\mathcal{B}]$ of Soergel's categorification theorem.

Finally, Soergel has given a beautiful formula for the graded rank of homomorphism spaces between Soergel bimodules in terms of Δ and ∇-characters. Given a finite-dimensional graded $\mathbb{R}$-vector space $V = \bigoplus V^i$ we define

$$\underline{\dim}\, V = \sum (\dim V^i) v^{-i} \in \mathbb{Z}_{\geq 0}[v^{\pm 1}].$$

Our notation is chosen so that $\underline{\dim}(V^{\oplus p}) = p\, \underline{\dim}\, V$ for $p \in \mathbb{Z}_{\geq 0}[v^{\pm 1}]$. Given a free finitely generated graded right R-module M we set

$$\underline{\mathrm{rk}}\, M := \underline{\dim}(M \otimes_R \mathbb{R}).$$

THEOREM 3.6 (Soergel's hom formula). *Suppose that B has a Δ-filtration and $B' \in \mathcal{B}$ or that $B \in \mathcal{B}$ and B' has a ∇-filtration. Then* $\mathrm{Hom}^\bullet(B, B')$ *is a graded free right R-module of rank*

$$\underline{\mathrm{rk}}\, \mathrm{Hom}^\bullet(B, B') = \overline{(\mathrm{ch}_\Delta B, \mathrm{ch}_\nabla B')}.$$

If Soergel's conjecture holds for B_x and B_y, then $\mathrm{ch}\, B_x = \underline{H}_x$ and $\mathrm{ch}\, B_y = \underline{H}_y$. Soergel's hom formula then implies that $\mathrm{Hom}^\bullet(B_x, B_y)$ is concentrated in degrees ≥ 0, and $\dim \mathrm{Hom}(B_x, B_y) = \delta_{xy}$.

3.6. *Invariant forms on Soergel bimodules.* Let B denote a self-dual Soergel bimodule. Equipping B with an invariant nondegenerate bilinear form $\langle -, - \rangle_B$ is the same as giving an isomorphism $B \xrightarrow{\sim} \mathbb{D}B$. It is known (see [Soe07, Satz 6.14]) that each indecomposable Soergel bimodule is self-dual and hence admits a nondegenerate invariant form. Moreover, if Soergel's conjecture holds for B_x, then $\mathrm{End}(B_x) = \mathbb{R}$ (as follows immediately from Soergel's hom formula). This implies the following, which plays an important role in this paper:

LEMMA 3.7. *Suppose that Soergel's conjecture holds for B_x. Then B_x admits an invariant form which is unique up to a scalar. Moreover, any nonzero invariant form is nondegenerate.*

Proof. Giving an invariant form on B_x is the same thing as giving a graded R-bimodule morphism $B_x \to \mathbb{D}B_x$. By the remarks preceding the lemma, the space of such maps is one dimensional and contains an isomorphism. Hence

B_x admits an invariant form $\langle -,-\rangle_{B_x}$, and all others are scalar multiples of $\langle -,-\rangle_{B_x}$. The lemma now follows. $\square$

We now explain how Soergel bimodules may be inductively equipped with invariant forms. Fix a Soergel bimodule B and consider the two maps $\alpha, \beta : B \to BB_s = B \otimes_R B_s$ given by

$$\alpha(b) := bc_{\mathrm{id}} \quad \text{and} \quad \beta(b) := bc_s.$$

Note that β is a morphism of bimodules, whilst α is only a morphism of left modules: by (3.3) one has

$$\alpha(br) = \alpha(b)(sr) + \beta(b)\partial_s(r) \tag{3.7}$$

for $b \in B$ and $r \in R$.

Suppose that B is equipped with an invariant form $\langle -,-\rangle_B$. Then there is a unique invariant form $\langle -,-\rangle_{BB_s}$ on BB_s, which we call the *induced form*, satisfying

$$\langle \alpha(b), \alpha(b')\rangle_{BB_s} = \partial_s(\langle b, b'\rangle_B), \tag{3.8}$$

$$\langle \alpha(b), \beta(b')\rangle_{BB_s} = \langle b, b'\rangle_B \text{ and } \langle \beta(b), \alpha(b')\rangle_{BB_s} = \langle b, b'\rangle_B, \tag{3.9}$$

$$\langle \beta(b), \beta(b')\rangle_{BB_s} = \langle b, b'\rangle_B \alpha_s \tag{3.10}$$

for all $b, b' \in B$. Indeed, if $e_1, \dots, e_m$ denotes a basis for B as a right R-module, then $\alpha(e_1), \dots, \alpha(e_m), \beta(e_1), \dots, \beta(e_m)$ is a basis for BB_s and the above formulas fix $\langle -,-\rangle_{BB_s}$ on this basis. It is straightforward to check that $\langle -,-\rangle_{BB_s}$ satisfies (3.8), (3.9) and (3.10) for all $b, b' \in B$ and that $\langle rb, b'\rangle_{BB_s} = \langle b, rb'\rangle$ for all $b, b' \in B$ and $r \in R$. Clearly $\langle -,-\rangle_{BB_s}$ is symmetric if $\langle -,-\rangle$ is.

Suppose that B is a summand of a Bott-Samelson bimodule $\mathrm{BS}(\underline{x})$. Then B is equipped with an invariant symmetric form $\langle -,-\rangle_B$, obtained by restriction from the intersection form on $\mathrm{BS}(\underline{x})$. There are now two ways to equip BB_s with an invariant form: either via the induced form as above, or by viewing BB_s as a summand of $\mathrm{BS}(\underline{x})B_s = \mathrm{BS}(\underline{x}s)$ and considering the restriction of the intersection form. It is an easy exercise to see that these two forms agree, which motivates the above formulas. If we apply this for $B = \mathrm{BS}(\underline{x})$, we conclude that the intersection form on $\mathrm{BS}(\underline{x})$ can also be obtained by starting with the canonical multiplication form on R and iterating the construction of the induced form.

LEMMA 3.8. *Suppose that B is an R-bimodule which is equipped with an invariant form $\langle -,-\rangle_B$. Assume that B is free as a right R-module and that $\langle -,-\rangle_B$ is nondegenerate. Then $\langle -,-\rangle_{BB_s}$ is nondegenerate.*

Proof. Because $\langle -,-\rangle_B$ is nondegenerate and B is free as a right R-module, we can fix a basis $e_1, \dots, e_m$ and dual basis $e_1^*, \dots, e_m^*$ for B as a right R-module.

Then

$$\alpha(e_1), \ldots, \alpha(e_m), \beta(e_1), \ldots, \beta(e_m)$$

and

$$\beta(e_1^*), \ldots, \beta(e_m^*), \alpha(e_1^*), \ldots, \alpha(e_m^*)$$

are bases for BB_s as a right R-module. Now (3.8), (3.9) and (3.10) show that the matrix of $\langle -, - \rangle_{BB_s}$ in this pair of bases has the form

$$\begin{pmatrix} I_m & \alpha_s I_m \\ 0 & I_m \end{pmatrix},$$

where I_m denotes the $m \times m$ identity matrix. The zero matrix in the lower left arises because $\partial_s(1) = 0$. Hence $\langle -, - \rangle_{BB_s}$ is nondegenerate as claimed. $\square$

COROLLARY 3.9. *The intersection form on a Bott-Samelson bimodule is nondegenerate.*

The following positivity calculation is not entirely necessary for the proofs below. However, it does give a simple explanation of why the global sign in the Hodge-Riemann bilinear relations is correct.

LEMMA 3.10. *The Lefschetz form $(-,-)_\rho^{-\ell(x)}$ on $\overline{\mathrm{BS}(\underline{x})}^{-\ell(x)} \cong \mathbb{R}$ is positive-definite when $\underline{x}$ is a reduced expression.*

Proof. Let $c_{\mathrm{bot}} := c_{\mathrm{id}} c_{\mathrm{id}} \cdots c_{\mathrm{id}}$, which spans $\mathrm{BS}(\underline{x})^{-\ell(x)}$. We claim that $\rho^{\ell(x)} c_{\mathrm{bot}} = N c_{\mathrm{top}} \in \overline{\mathrm{BS}(\underline{x})}$ for some $N > 0$, which will imply the result. We induct on $\ell(x)$. The result is clear when $\ell(x) = 0$.

By Lemma 3.4 we have

$$\rho \cdot c_{\mathrm{bot}} = \sum_i (s_{i-1} \cdots s_1 \rho)(\alpha_{s_i}^\vee) \chi_i(c_{\mathrm{bot}}) + c_{\mathrm{bot}} \cdot (x^{-1}\rho)$$

inside $\mathrm{BS}(\underline{x})$. Note that $(s_{i-1} \cdots s_1 \rho)(\alpha_{s_i}^\vee)$ is positive for all i, by our positivity assumption on ρ and the fact that $\underline{x}$ is a reduced expression. The final term clearly vanishes in $\overline{\mathrm{BS}(\underline{x})}$, so it remains to see what happens when $\rho^{\ell(x)-1}$ is applied to every other term.

Suppose that $\underline{x}_{\hat{i}}$ is a reduced expression. Then by induction, $\rho^{\ell(x)-1} c_{\mathrm{bot}} = N_i c_{\mathrm{top}} \in \overline{\mathrm{BS}(\underline{x}_{\hat{i}})}$ for some $N_i > 0$. Clearly $\chi_i(c_{\mathrm{top}}) = c_{\mathrm{top}}$, so $\rho^{\ell(x)-1} \chi_i(c_{\mathrm{bot}}) = N_i c_{\mathrm{top}} \in \overline{\mathrm{BS}(\underline{x})}$.

Suppose that $\underline{x}_{\hat{i}}$ is not a reduced expression. In this case,

$$\mathrm{BS}(\underline{x}_{\hat{i}}) \cong \bigoplus B_z^{\oplus p_z}$$

with all z appearing on the right-hand side satisfying $\ell(z) < \ell(x) - 1$ and $p_z \in \mathbb{Z}_{\geq 0}[v^{\pm 1}]$. For degree reasons, $\rho^{\ell(x)-1}$ vanishes on $\overline{B_z}$ for any such z and therefore vanishes identically on $\overline{\mathrm{BS}(\underline{x}_{\hat{i}})}$. Therefore, $\rho^{\ell(x)-1} \chi_i(c_{\mathrm{bot}}) = 0$ for such i.

Therefore, $\rho^{\ell(x)} c_{\mathrm{bot}} = \left(\sum_i (s_{i-1} \cdots s_1 \rho)(\alpha_{s_i}^\vee) N_i\right) c_{\mathrm{top}} \in \overline{\mathrm{BS}(\underline{x})}$, with

$$\sum_i (s_{i-1} \cdots s_1 \rho)(\alpha_{s_i}^\vee) N_i > 0. \qquad \square$$

The following simple observation was promised in the introduction:

LEMMA 3.11. *If* $\mathrm{S}(x)$ *holds, then* $\mathrm{HR}(x)$ *and* $\mathrm{HR}(\underline{x})$ *are equivalent for any reduced expression* $\underline{x}$.

Proof. Obviously $B_x^i = 0$ for $i < -\ell(x)$. By considering the ∇-character of B_x it is easy to see that $B_x^{-\ell(x)}$ is one dimensional. Hence any embedding $B_x \hookrightarrow \mathrm{BS}(\underline{x})$ induces an isomorphism $B_x^{-\ell(x)} \xrightarrow{\sim} \mathrm{BS}(\underline{x})^{-\ell(x)} = \mathbb{R}(c_{\mathrm{bot}})$.

Given $\mathrm{S}(x)$, Lemma 3.7 implies that the restriction of the intersection form on $\mathrm{BS}(\underline{x})$ to B_x must be a scalar multiple of the intersection form on B_x. The Lefschetz form on $\overline{\mathrm{BS}(\underline{x})}^{-\ell(x)}$ is positive definite, and hence this scalar must be positive. Now $\mathrm{HR}(x)$ and $\mathrm{HR}(\underline{x})$ are equivalent. $\square$

4. **The embedding theorem**

In this section we fix $x \in W$ and $s \in S$ with $xs > x$, and we assume $\mathrm{S}(y)$ and $\mathrm{HR}(\underline{y})$ for all $y < xs$. By $\mathrm{HR}(\underline{y})$, if we choose an embedding $B_y \overset{\oplus}{\subset} \mathrm{BS}(\underline{y})$, then the restriction of the intersection form on $\mathrm{BS}(\underline{y})$ to B_y yields a nondegenerate invariant form $\langle -, - \rangle_{B_y}$ on B_y which satisfies the Hodge-Riemann bilinear relations. Let us also fix a generator c_{bot} of the one-dimensional vector space $B_y^{-\ell(y)}$. Then $\mathrm{HR}(\underline{y})$ implies

$$\langle \rho^{\ell(y)} \cdot c_{\mathrm{bot}}, c_{\mathrm{bot}} \rangle_{B_y} = N \tag{4.1}$$

for some $0 < N \in \mathbb{R}$.

Similarly, we fix an embedding $B_x \overset{\oplus}{\subset} \mathrm{BS}(\underline{x})$ which induces a nondegenerate form $\langle -, - \rangle_{B_x}$ on B_x. As discussed in Section 3.6, this induces a nondegenerate invariant symmetric form $\langle -, - \rangle_{B_x B_s}$ on $B_x B_s$, compatible with the induced embedding $B_x B_s \overset{\oplus}{\subset} \mathrm{BS}(\underline{x}) B_s = \mathrm{BS}(\underline{x}s)$.

Having fixed these forms on B_y and $B_x B_s$ we obtain a canonical identification

$$\mathrm{Hom}(B_y, B_x B_s) \xrightarrow{\sim} \mathrm{Hom}(B_x B_s, B_y)$$

sending $f \in \mathrm{Hom}(B_y, B_x B_s)$ to its adjoint f^*. That is, f^* is uniquely determined by the identity $\langle f(b), b' \rangle_{B_x B_s} = \langle b, f^*(b') \rangle_{B_y}$ for all $b \in B_y$ and $b' \in B_x B_s$.

On $\mathrm{Hom}(B_y, B_x B_s)$ we can consider the *local intersection form*

$$(f, g)_y^{x,s} := g^* \circ f \in \mathrm{End}(B_y) = \mathbb{R}.$$

Theorem 4.1 (Embedding theorem). *The map*

$$\iota : \mathrm{Hom}(B_y, B_xB_s) \to (\overline{B_xB_s})^{-\ell(y)} : f \mapsto \overline{f(c_{\mathrm{bot}})}$$

is injective, with image contained in the primitive subspace

$$P_\rho^{-\ell(y)} \subset (\overline{B_xB_s})^{-\ell(y)}.$$

Moreover, ι is an isometry with respect to the Lefschetz form up to a factor of N: for all $f, g \in \mathrm{Hom}(B_y, B_xB_s)$ we have

$$N(f,g)_y^{x,s} = (\iota(f), \iota(g))_\rho^{-\ell(y)}. \tag{4.2}$$

Remark 4.2. The above constructions depend on the choices ($\mathbb{R}_{>0}$-torsors) of invariant forms on B_y and B_x and the choice (an $\mathbb{R}^\times$-torsor) of $c_{\mathrm{bot}} \in B_y^{-\ell(y)}$. The reader can confirm that both sides of (4.2) are affected equally by any rescaling, and the coefficient of isometry N is positive for any choice.

Proof. Consider the exact sequence of modules with Δ-flag

$$\Delta_y = \Gamma_{\geq y} B_y \hookrightarrow B_y \twoheadrightarrow B/\Gamma_{\geq y} B_y.$$

We know that $\mathrm{ch}_\Delta \Delta_y = H_y$, $\mathrm{ch}\, B_y = \underline{H}_y$, and that $\mathrm{ch}_\Delta(B_y/\Gamma_{\geq y}B_y) = \underline{H}_y - H_y$ because this is part of the Δ-flag on B_y. Therefore, the characters add up, and we can use Soergel's hom formula (Theorem 3.6) to conclude that we have an exact sequence

$$\mathrm{Hom}^\bullet(B_y/\Gamma_{\geq y}B_y, B_xB_s) \hookrightarrow \mathrm{Hom}^\bullet(B_y, B_xB_s) \twoheadrightarrow \mathrm{Hom}^\bullet(\Delta_y, B_xB_s).$$

Now $\underline{H}_y - H_y \in \bigoplus v\mathbb{Z}_{\geq 0}[v]H_z$ and $\mathrm{ch}(B_xB_s) = \underline{H}_x\underline{H}_s \in \bigoplus \mathbb{Z}_{\geq 0}[v]H_z$. Hence $\mathrm{Hom}^{\leq 0}(B_y/\Gamma_{\geq y}B_y, B_xB_s) = 0$ and we have an isomorphism

$$\mathrm{Hom}(B_y, B_xB_s) \xrightarrow{\sim} \mathrm{Hom}(\Delta_y, B_xB_s) = \Gamma_y(B_xB_s)(\ell(y)).$$

Using Soergel's hom formula again we see that $\mathrm{Hom}^\bullet(B_y, B_xB_s)$ is concentrated in degrees ≥ 0 and hence $\Gamma_y(B_xB_s)$ is concentrated in degrees $\geq \ell(y)$. Now B_xB_s is free as a right R-module and it is known that $\Gamma_y(B_xB_s)$ is a direct summand of B_xB_s as a right R-module. (See the proof of Proposition 6.4 in [Soe07].) It follows that if $m \in B_xB_s$ and $mr \in \Gamma_y(B_xB_s)$ for some $r \in R$, then $m \in \Gamma_y(B_xB_s)$. Hence the induced map

$$\Gamma_y(B_xB_s)^{\ell(y)} \to (\overline{B_xB_s})^{\ell(y)}$$

is injective.

Let c be the image of a generator of Δ_y under $\Delta_y \hookrightarrow \Gamma_y B_y \subset B_y$. It projects to a generator $\overline{c}$ of the one-dimensional space $(\overline{B_y})^{\ell(y)} \cong \mathbb{R}$. The isomorphisms of the previous paragraph imply that

$$\iota' : \mathrm{Hom}(B_y, B_xB_s) \to \overline{B_xB_s}^{\ell(y)} : f \mapsto \overline{f(c)}$$

is injective. In addition, $\mathrm{hL}(y)$ implies that $\rho^{\ell(y)} \cdot c_{\mathrm{bot}}$ also has nonzero image in $(\overline{B_y})^{\ell(y)}$ and therefore is equal to $\overline{c}$ up to a nonzero scalar. Hence

$$\iota : \mathrm{Hom}(B_y, B_x B_s) \to \overline{B_x B_s}^{-\ell(y)} : f \mapsto \overline{f(c_{\mathrm{bot}})}$$

is injective too. Finally, $\rho^{\ell(y)+1}$ annihilates $\overline{B_y}$ and hence the image of ι is contained in the primitive subspace $P_\rho^{-\ell(y)} \subset (\overline{B_x B_s})^{-\ell(y)}$. The first part of the theorem now follows.

Fix $f, g \in \mathrm{Hom}(B_y, B_x B_s)$. We have

$$\begin{aligned} N(f,g)_y^{x,s} &= \langle g^*(f(c_{\mathrm{bot}})), \rho^{\ell(y)} \cdot c_{\mathrm{bot}} \rangle_{\overline{B_y}} \\ &= \langle f(c_{\mathrm{bot}}), \rho^{\ell(y)} \cdot g(c_{\mathrm{bot}}) \rangle_{\overline{B_x B_s}} \\ &= (\iota(f), \iota(g))_\rho^{-\ell(y)}. \end{aligned}$$

(The first equality follows from (4.1), the second by adjointness, and the third by definition.) (4.2) now follows. □

Because the restriction of a definite form to a subspace stays nondegenerate, we have

Corollary 4.3. $\mathrm{HR}(x,s)$ *and* $\mathrm{S}(y)$ *for all* $y < xs$ *implies* $\mathrm{S}(xs)$.

5. Hodge-Riemann bilinear relations

In this section we prove (1.6) from the introduction. We actually prove a more general version. Let us fix a (not necessarily reduced) expression $\underline{x}$ and a summand $B \overset{\oplus}{\subset} \mathrm{BS}(\underline{x})$. On B we have an invariant form induced from the intersection form on $\mathrm{BS}(\underline{x})$ and a Lefschetz operator induced by left multiplication by ρ. Using the terminology of Section 2, for all $i \geq 0$ we get a Lefschetz form on $(\overline{B})^{-i}$ given by

$$(p,q)_\rho^{-i} = \mathrm{Tr}_{\mathbb{R}}(\rho^i(pq)).$$

For all $\zeta \geq 0$ we consider the Lefschetz operator

$$L_\zeta := (\rho \cdot -) + \mathrm{id}_B(\zeta\rho \cdot -)$$

on BB_s. Here $(\rho \cdot -)$ denotes the operator of left multiplication by ρ and $\mathrm{id}_B(\zeta\rho \cdot -)$ denotes the tensor product of the identity on B and the operator of left multiplication by $\zeta\rho$ on B_s. In this section $(-,-)_\rho^{-i}$ will always refer to the Lefschetz form on $\overline{B}$, while $(-,-)_{L_\zeta}^{-i}$ will refer to the Lefschetz form on $\overline{BB_s}$. Thus, $(-,-)_{L_0}^{-i}$ is the Lefschetz form on $\overline{BB_s}$ induced by left multiplication by ρ. We abusively write $\mathrm{Tr}_{\mathbb{R}}$ for the real valued trace on both $\mathrm{BS}(\underline{x})$ and $\mathrm{BS}(\underline{x}s)$.

Theorem 5.1. *Suppose that* $\overline{B}$ *satisfies hard Lefschetz and the Hodge-Riemann bilinear relations with the standard sign. Then for* $\zeta \gg 0$, *the induced action of* L_ζ *on* $\overline{BB_s}$ *satisfies the hard Lefschetz theorem and the Hodge-Riemann bilinear relations with the standard sign.*

The following lemma reduces this theorem to a statement relating the signatures of the forms on $\overline{B}$ and $\overline{BB_s}$:

LEMMA 5.2. *Let V and W be two finite-dimensional graded vector spaces, equipped with graded nondegenerate symmetric forms and Lefschetz operators satisfying the hard Lefschetz theorem. Assume that W is even or odd and that $\underline{\dim} V = (v + v^{-1}) \underline{\dim} W$, so that V is odd or even. Suppose that W satisfies the Hodge-Riemann bilinear relations with the standard sign. Then V satisfies the Hodge-Riemann bilinear relations with the standard sign if and only if for all $i \geq 0$ the signature of the Lefschetz form on the primitive subspace $P^{-i+1} \subset W^{-i+1}$ is equal to the signature of the Lefschetz form on all of V^{-i}. (By convention, $P^1 = 0$.)*

Proof. Let $\ell \geq 0$ be such that $W^{-\ell}$ is the lowest nonzero degree of W. For $j \in \mathbb{Z}$ write $v_j := \dim V^j$ and $w_j := \dim W^j$ for the Betti numbers of V and W. For $j \geq 0$ write $p_{-j} := v_{-j} - v_{-j-2}$ for the dimension of the primitive subspace $P^{-j} \subset V^{-j}$. Because $\underline{\dim} V = (v + v^{-1}) \underline{\dim} W$, the lowest nonzero degree of V is $-\ell - 1$ and we have $v_{-j} = w_{-j+1} + w_{-j-1}$. Hence, for all $j \geq 0$ we have

$$p_{-j} = w_{-j+1} - w_{-j-3}.$$

Now V satisfies the Hodge-Riemann bilinear relations with the standard sign if and only if for all $j \geq -1$ the signature of the Lefschetz form on V^{-j-1} is equal to

$$\begin{aligned}
&(-1)^{(j+1-(\ell+1))/2}(p_{-j-1} - p_{-j-3} + p_{-j-5} - p_{-j-7} + \cdots) \\
&\quad = (-1)^{(j-\ell)/2}((w_{-j} - w_{-j-4}) - (w_{-j-2} - w_{-j-6}) + (w_{-j-4} - w_{-j-8}) - \cdots) \\
&\quad = (-1)^{(j-\ell)/2}(w_{-j} - w_{-j-2}).
\end{aligned}$$

The last term is the signature of the Lefschetz form on the primitive subspace $P^{-j} \subset W^{-j}$ by the Hodge-Riemann bilinear relations. The lemma now follows. $\square$

Clearly, the lemma will apply to $W = \overline{B}$ and $V = \overline{BB_s}$, so long as V satisfies hard Lefschetz. The proof below establishes a statement about signatures. The essential argument is to show that, as $\zeta \to \infty$, the form on $\overline{BB_s}$ tends to the "product" of the forms on $\overline{B}$ and on $\overline{B_s}$.

Proof of Theorem 5.1. Recall from Section 3.6 the maps α and β from B to BB_s, and the formulae (3.8), (3.9) and (3.10) which control the invariant form on BB_s. As a reminder, for $x \in B^{-i+1}$ and $y \in B^{-i-1}$ we have

$$\alpha(x) := xc_{\mathrm{id}} \quad \text{and} \quad \beta(y) := yc_s$$

in $(BB_s)^{-i}$.

We are interested in the $\mathbb{R}$-valued form on $\overline{BB_s}$. It is immediate from (3.10) that two elements in the image of β are orthogonal with respect to $\langle -,-\rangle_{\overline{BB_s}}$, because the positive degree polynomial α_s appears on the right. For similar reasons, $L_\zeta = L_0$ when applied to an element in the image of β, because left and right multiplication by $\zeta\rho$ agree on $c_s \in B_s$. Therefore,

$$(\beta(y), \beta(y'))^{-i}_{L_\zeta} = 0 \tag{5.1}$$

and

$$\begin{aligned} (\alpha(x), \beta(y))^{-i}_{L_\zeta} &= (\alpha(x), \beta(y))^{-i}_{L_0} \\ &= \mathrm{Tr}_{\mathbb{R}}(\rho^i(xy)c_s) \\ &= (x, \rho y)^{-i+1}_\rho. \end{aligned} \tag{5.2}$$

This second equation relates the form on $(\overline{BB_s})^{-i}$ to the form on $(\overline{B})^{-i+1}$. The only "difficult" pairings are of the form $(\alpha(x), \alpha(x'))^{-i}_{L_\zeta}$. We will have more to say about these below.

Now fix $i \geq 0$, and choose elements $e_1, \ldots, e_n \in B^{-i-1}$ which project to an orthogonal basis of $(\overline{B})^{-i-1}$. Choose elements $p_1, \ldots, p_m \in B^{-i+1}$ which project to an orthogonal basis of the primitive subspace $P^{-i+1}_\rho \subset (\overline{B})^{-i+1}$. Then

$$\rho e_1, \ldots, \rho e_n, p_1, \ldots, p_m$$

project to an orthogonal basis for $(\overline{B})^{-i+1}$. It follows that

$$\alpha(\rho e_1), \ldots, \alpha(\rho e_n), \beta(e_1) \ldots, \beta(e_n), \alpha(p_1), \ldots, \alpha(p_m)$$

project to a basis of $(\overline{BB_s})^{-i}$.

With this choice of basis, equations (5.1) and (5.2) imply that the Gram matrix of the form $(-,-)^{-i}_{L_\zeta}$ has the form

$$M^{-i}_\zeta := \begin{pmatrix} * & J & * \\ J & 0 & 0 \\ * & 0 & Q_\zeta \end{pmatrix},$$

where J is a nondegenerate diagonal matrix. We have not yet computed Q_ζ or the $*$'s. The determinant of M^{-i}_ζ only depends on the entries of J and Q_ζ. Hence M^{-i}_ζ is nondegenerate if and only if Q_ζ is, in which case we can find a path in the space of real nondegenerate symmetric matrices to the matrix

$$M := \begin{pmatrix} 0 & J & 0 \\ J & 0 & 0 \\ 0 & 0 & Q_\zeta \end{pmatrix}$$

and we can conclude that the signature of M^{-i}_ζ is equal to that of Q_ζ.[2]

[2]More formally, let $\mathrm{Sym}_n^{\det\neq 0}$ denote the space of real nondegenerate symmetric matrices, with its Euclidean topology. We can find a path $t : [0,1] \to \mathrm{Sym}_n^{\det\neq 0}$ such that $t(1) = M^{-i}_\zeta$ and $t(0) = M$. Using that the signature is constant on connected components of $\mathrm{Sym}_n^{\det\neq 0}$ we conclude that the signatures of M and M^{-i}_ζ coincide. Finally, the signatures of M and Q_ζ are easily seen to agree.

We claim that, for $\zeta \gg 0$, Q_ζ is nondegenerate and has signature equal to the signature of $(-,-)_\rho^{-i+1}$ on $P_\rho^{-i+1} \subset \overline{B}^{-i+1}$. If this is true, then L_ζ satisfies hard Lefschetz, and Lemma 5.2 will conclude the proof.

Firstly, if $i = 0$, then $m = 0$ and the result follows. Hence we may assume $i > 0$. Let $p, q \in B^{-i+1}$. We have

$$(\alpha(p), \alpha(q))_{L_\zeta}^{-i} = \mathrm{Tr}_{\mathbb{R}}(L_\zeta^i((pq)c_{\mathrm{id}})) = \mathrm{Tr}_{\mathbb{R}}\left(\sum_{j=0}^{i}\binom{i}{j}\rho^{i-j}(pq)(\zeta\rho)^j c_{\mathrm{id}}\right).$$

By (3.3) we have for $j \geq 1$

$$\rho^{i-j}(pq)(\zeta\rho)^j c_{\mathrm{id}} = \rho^{i-j}(pq)c_s \cdot \partial_s((\zeta\rho)^j) + \rho^{i-j}(pq)c_{\mathrm{id}} \cdot s(\zeta\rho)^j.$$

Applying $\mathrm{Tr}_{\mathbb{R}}$ we obtain (again for $j \geq 1$)

$$\mathrm{Tr}_{\mathbb{R}}(\rho^{i-j}(pq)(\zeta\rho)^j c_{\mathrm{id}}) = \begin{cases} \zeta\rho(\alpha_s^\vee)\,\mathrm{Tr}_{\mathbb{R}}(\rho^{i-1}(pq)) & \text{if } j = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$(\alpha(p), \alpha(q))_{L_\zeta}^{-i} = \mathrm{Tr}_{\mathbb{R}}(\rho^i(pq)c_{\mathrm{id}}) + \zeta i\rho(\alpha_s^\vee)(p,q)_\rho^{-i+1}.$$

Note that the first term is independent of ζ. It follows that

$$\lim_{\zeta\to\infty}\frac{1}{\zeta}Q_\zeta = i\rho(\alpha_s^\vee)\cdot Q,$$

where Q is the matrix $((p_i,p_j)_\rho^{-i+1})_{1\leq i,j\leq n}$. Now, $\overline{B}$ satisfies the Hodge-Riemann bilinear relations, and hence Q is definite. It follows that Q_ζ is too, for $\zeta \gg 0$, and has the same signature as Q (i, ζ and $\rho(\alpha_s^\vee)$ are all strictly positive). The theorem now follows. ☐

The upshot of Theorem 5.1 is the following corollary:

Corollary 5.3. *If* $\mathrm{HR}(\underline{x})$ *holds, then* $\mathrm{hL}(\underline{x},s)_\zeta$ *for all* $\zeta \geq 0$ *implies* $\mathrm{HR}(\underline{x},s)_\zeta$ *for all* $\zeta \geq 0$.

Proof. By Theorem 5.1 we have $\mathrm{HR}(\underline{x},s)_\zeta$ for some $\zeta \gg 0$. By Lemma 2.1 we have $\mathrm{HR}(\underline{x},s)_\zeta$ for all $\zeta \geq 0$. ☐

All that remains is to prove hard Lefschetz for the family L_ζ of Lefschetz operators. This task occupies the rest of the paper.

6. **Hard Lefschetz for Soergel bimodules**

In this section we establish the hard Lefschetz theorem for Soergel bimodules using Rouquier complexes. Although the basic idea is simple, the details are somewhat complicated. Before giving the details we give a brief motivational sketch.

Let us first recall a key fact from Hodge theory: the weak Lefschetz theorem together with the Hodge-Riemann bilinear relations in dimension

$n-1$ imply the hard Lefschetz theorem in dimension n. Let X denote a smooth projective variety, and let X_H denote a general hyperplane section and $r : X_H \hookrightarrow X$ the inclusion. A key point in the proof is the observation that one can factor the Lefschetz operator as the composition of the restriction map $r^* : H^*(X) \to H^*(X_H)$ and its dual $r_* : H^*(X_H) \to H^{*+2}(X)$. The weak Lefschetz theorem implies that r^* is injective in degrees $\leq \dim_{\mathbb{C}} X - 1$, and one can then use Lemma 2.3 to deduce the hard Lefschetz theorem for $H^*(X)$ from the Hodge-Riemann bilinear relations for $H^*(X_H)$.

The weak Lefschetz theorem actually gives a situation stronger than that of Lemma 2.3 because r^* (resp. r_*) is an isomorphism in degrees $\leq \dim_{\mathbb{C}} X - 2$ (resp. $\geq \dim_{\mathbb{C}} X$), which can be used to deduce the Hodge-Riemann bilinear relations for $H^*(X)$ in all degrees except $\dim_{\mathbb{C}} X$. This aspect of the proof is not replicated in this paper.

A major initial hurdle in the setting of Soergel bimodules is the apparent absence of the weak Lefschetz theorem. Indeed, even if geometric tools are available, taking a general hyperplane section in a Bott-Samelson resolution or flag variety leaves the world of varieties whose cohomology admits a simple combinatorial description.

The first key observation is that for any expression $\underline{x}$, left multiplication by ρ on $\overline{\mathrm{BS}(\underline{x})}$ (our substitute for a Lefschetz operator) still admits a factorization (see Section 6.7)

$$\overline{\mathrm{BS}(\underline{x})} \xrightarrow{\phi} \bigoplus \overline{\mathrm{BS}(\underline{x}_{\hat{i}})}(1) \xrightarrow{\chi} \overline{\mathrm{BS}(\underline{x})}(2).$$

However, the modules appearing above will generally not satisfy hard Lefschetz, because one has no control over the shifts of indecomposable Soergel bimodules that may occur.

The second key observation is that ϕ (and χ) are (up to some positive scalars) differentials on Rouquier complexes. One can then use homological algebra to replace $\overline{\mathrm{BS}(\underline{x})} \xrightarrow{\phi} \bigoplus \overline{\mathrm{BS}(\underline{x}_{\hat{i}})}(1) \to \cdots$ by a minimal subcomplex without affecting exactness properties. It is this subcomplex that serves as a replacement for the weak Lefschetz theorem and allows us to deduce hard Lefschetz.

6.1. *Complexes and their minimal complexes.* Let $C^b(\mathcal{B})$ denote the category of bounded complexes of Soergel bimodules (all differentials are required to be of degree zero), and let $K^b(\mathcal{B})$ denote its homotopy category. Because we already use right indices to indicate the degree in the grading, we use left indices to indicate the cohomological degree. In other words, an object $F \in C^b(\mathcal{B})$ looks like

$$\cdots \to {}^{i}F \xrightarrow{d} {}^{i+1}F \to \cdots,$$

with ${}^{i}F \in \mathcal{B}$ and d a morphism in $\mathcal{B}$. We regard $\mathcal{B}$ as a full subcategory of $C^b(\mathcal{B})$ and $K^b(\mathcal{B})$ consisting of complexes concentrated in degree 0. As with bimodules we write $F' \overset{\oplus}{\subset} F$ to mean that F' is a direct summand (as complexes) of the complex F.

Let $\mathrm{rad}(\mathcal{B}) \subset \mathcal{B}$ denote the radical of $\mathcal{B}$ (see, e.g., [Kra, §1.8]). It is an ideal of the category $\mathcal{B}$, and we write $\mathrm{rad}(\mathcal{B})(B, B') \subset \mathrm{Hom}(B, B')$ for the corresponding subspace for any $B, B' \in \mathcal{B}$. On may show [Kra, Prop. 1.8.1] that $\mathrm{rad}(\mathcal{B})(B, B) \subset \mathrm{End}(B)$ coincides with the Jacobson radical $J\,\mathrm{End}(B) \subset \mathrm{End}(B)$ for any $B \in \mathcal{B}$. Because the endomorphism ring of B is a finite dimensional $\mathbb{R}$-algebra (remember that morphisms in $\mathcal{B}$ are assumed to be of degree zero), $\mathrm{End}\,B/J\,\mathrm{End}\,B$ is a semi-simple $\mathbb{R}$-algebra. We conclude that $\mathcal{B}^{\mathrm{ss}} := \mathcal{B}/\mathrm{rad}(\mathcal{B})$ is semi-simple.

Given any indecomposable Soergel bimodule B_x, $\mathrm{End}\,B_x/J\,\mathrm{End}\,B_x = \mathbb{R}$. (For general reasons $\mathrm{End}\,B_x/J\,\mathrm{End}\,B_x$ is a division algebra over $\mathbb{R}$. However, one always has a surjection $\mathrm{End}\,B_x \twoheadrightarrow \mathbb{R}$ (see the proof of [Soe07, Satz 6.14]), and hence $\mathrm{End}\,B_x/J\,\mathrm{End}\,B_x = \mathbb{R}$.) We conclude that the images of $B_x(i)$ for $x \in W$ and $i \in \mathbb{Z}$ in $\mathcal{B}^{\mathrm{ss}}$ give a complete set of pairwise nonisomorphic simple objects, all of whose endomorphism rings are isomorphic to $\mathbb{R}$. We denote by $q : \mathcal{B} \to \mathcal{B}^{\mathrm{ss}}$ the quotient functor. One may check that $f : B \to B'$ is an isomorphism if and only if $q(f)$ is.

Now consider a complex $F \in C^b(\mathcal{B})$. We say that F is *minimal* if all differentials on $q(F) \in C^b(\mathcal{B}^{\mathrm{ss}})$ are zero. This is equivalent to requiring that $q(F)$ contain no contractible direct summands, which by the isomorphism lifting statement, is equivalent to requiring that F itself has no contractible direct summands.[3]

Given any complex $F \in C^b(\mathcal{B})$ there exists a direct summand $F_{\min} \overset{\oplus}{\subset} F$ such that $F_{\min}$ is minimal and such that the inclusion $F_{\min} \to F$ is an isomorphism in $K^b(\mathcal{B})$. We call such a summand a *minimal subcomplex*. Any two minimal subcomplexes are isomorphic as complexes. (If $f : F \to G$ is a homotopy isomorphism between minimal complexes, then $q(f)$ is a homotopy isomorphism between complexes with trivial differential. It follows that $q(f)$, and hence f, is an isomorphism of complexes.)

6.2. *The perverse filtration on bimodules.* A Soergel bimodule B is *perverse* if $\mathrm{ch}(B) = \sum a_x \underline{H}_x$ with $a_x \in \mathbb{Z}_{\geq 0}$. A Soergel bimodule B is *p-split* if each indecomposable summand of B is isomorphic to $B'(m)$ for some $m \in \mathbb{Z}$

[3] If $F \in C^b(\mathcal{B})$ is a complex, and a differential $d : {}^iF = M \oplus B \to {}^{i+1}F = M' \oplus B'$ has the form

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \mathrm{iso} \end{pmatrix}.$$

for some isomorphism $\mathrm{iso} : B \to B'$, then one can choose new decompositions ${}^iF = M \oplus B$ and ${}^{i+1}F = M' \oplus B'$ such that d is a diagonal matrix, with entries $\alpha' : M \to M'$ and $\mathrm{iso} : B \to B'$. Hence F is homotopic to a complex F' with the contractible summand $B \xrightarrow{\sim} B'$ removed.

and perverse Soergel bimodule B'. Any summand of a perverse (resp. p-split) Soergel bimodule is perverse (resp. p-split). (Use that $\overline{\mathrm{ch}(B_x)} = \mathrm{ch}(B_x)$ for any $x \in W$ and that the character of any Soergel bimodule is positive in the standard basis.)

If B_1 and B_2 are perverse, then Soergel's hom formula (Theorem 3.6) implies that

$$\mathrm{Hom}(B_1, B_2(-i)) = 0 \quad \text{for } i > 0. \tag{6.1}$$

Let B be a p-split Soergel bimodule, and choose a decomposition

$$B \xrightarrow{\sim} \bigoplus B_x^{\oplus m_{x,i}}(i) \tag{6.2}$$

of B as a direct sum of indecomposable bimodules. Because B is assumed p-split, we know that if $m_{x,i} \neq 0$, then B_x is perverse. We define the *perverse filtration* to be the filtration

$$\tau_{\leq j} B := \bigoplus_{i \geq -j} B_x^{\oplus m_{x,i}}(i).$$

(The more geometrically-minded reader might prefer ${}^p\tau_{\leq j}$.) Using (6.1) one can show that this filtration does not depend on the choice of decomposition (6.2) and is preserved (possibly nonstrictly) by all maps between Soergel bimodules. Of course this filtration always splits; however, the splitting is not canonical in general.

We set $\tau_{<j} := \tau_{\leq j-1}$ and define

$$\tau_{\geq j} B := B / \tau_{<j} B$$

and

$$\mathcal{H}^j(B) := \tau_{\leq j}(B)/\tau_{<j}(B)(j).$$

One can check $\tau_{\leq j}(-)$, $\tau_{\geq j}(-)$ and $\mathcal{H}^j(-)$ define endofunctors on the full subcategory of p-split Soergel bimodules.

6.3. *The perverse filtration on complexes.* Let ${}^pK^b(\mathcal{B})^{\geq 0}$ denote the full subcategory of $K^b(\mathcal{B})$ with objects those complexes which are isomorphic to complexes F such that

(1) each term of F is p-split,
(2) $\tau_{<-i}{}^iF = 0$ for all $i \in \mathbb{Z}$.

Similarly, we define ${}^pK^b(\mathcal{B})^{\leq 0}$ to be the full subcategory of complexes which are isomorphic to complexes F such that

(1) each term of F is p-split,
(2) ${}^iF = \tau_{\leq -i}{}^iF$ for all $i \in \mathbb{Z}$.

Alternatively, F belongs to ${}^pK^b(\mathcal{B})^{\leq 0}$ (resp. ${}^pK^b(\mathcal{B})^{\geq 0}$) if and only if its minimal complex satisfies the conditions above.

In other words, if a minimal complex is in ${}^pK^b(\mathcal{B})^{\geq 0}$, then an indecomposable summand in cohomological degree 0 has the form $B_x(k)$ for $k \leq 0$, an indecomposable summand in cohomological degree 1 has the form $B_x(k)$ for $k \leq 1$, etc.

LEMMA 6.1. *Let $F' \to F \to F'' \xrightarrow{[1]}$ be a distinguished triangle in $K^b(\mathcal{B})$. If $F', F'' \in {}^pK^b(\mathcal{B})^{\geq 0}$, then $F \in {}^pK^b(\mathcal{B})^{\geq 0}$. Similarly, if $F', F'' \in {}^pK^b(\mathcal{B})^{\leq 0}$, then $F \in {}^pK^b(\mathcal{B})^{\leq 0}$.*

Proof. We prove the first statement; the second statement follows by an identical argument. We may assume that ${}^iF'$ and ${}^iF''$ are p-split and that $\tau_{<-i}{}^iF' = \tau_{<-i}{}^iF'' = 0$ for all $i \in \mathbb{Z}$. Turning the triangle we see that F is isomorphic to the cone over a map $F''[-1] \to F'$. This cone has i^{th} term ${}^iF'' \oplus {}^iF$. The result follows because $\tau_{<-i}({}^iF'' \oplus {}^iF') = 0$ for all $i \in \mathbb{Z}$. □

Remark 6.2. Once one has proven Soergel's conjecture one may show that $({}^pK^b(\mathcal{B})^{\leq 0}, {}^pK^b(\mathcal{B})^{\geq 0})$ gives a nondegenerate t-structure on $K^b(\mathcal{B})$. Its heart can be thought of as a category of mixed equivariant perverse sheaves on the (possibly nonexistent) flag variety associated to (W, S).

6.4. *Rouquier complexes.* The monoidal structure on $\mathcal{B}$ induces a monoidal structure on $K^b(\mathcal{B})$ (total complex of tensor product of complexes) which we denote by juxtaposition. Given a distinguished triangle $F' \to F \to F'' \xrightarrow{[1]}$ and $G \in K^b(\mathcal{B})$, the triangle

$$F'G \to FG \to F''G \xrightarrow{[1]}$$

is also distinguished.

For $s \in S$ consider the complex

$$F_s := 0 \to B_s \to R(1) \to 0,$$

where B_s occurs in cohomological degree 0 and the only nonzero differential is given by the multiplication map $f \otimes g \mapsto fg$. It is known and easily checked that F_s is invertible in $K^b(\mathcal{B})$; hence tensoring on the left or right by F_s gives an equivalence of $K^b(\mathcal{B})$.

Fix $x \in W$ and a reduced expression $\underline{x} = s_1 s_2 \cdots s_m$. As an object in the homotopy category $K^b(\mathcal{B})$, the object $F_{s_1} \cdots F_{s_m}$ depends only on x up to canonical isomorphism (see [Rou06a]). In this paper, a *Rouquier complex* is any choice $F_x \overset{\oplus}{\subset} F_{s_1} \cdots F_{s_m}$ of minimal subcomplex, which again does not depend on the choice of reduced expression.

Remark 6.3. Braid group actions on categories appearing in highest weight representation theory have been around for decades (see, e.g., [Car86], [Ric94]). One obtains the above complexes by translating these actions into Soergel bimodules. The term "Rouquier complex" seems to have been introduced by Khovanov. We feel it is justified in our setting because it was Rouquier who first emphasised that concrete algebraic properties of these complexes should have applications for arbitrary Coxeter systems [Rou06b, 4.2.1]. This is a key idea in the present article.

A straightforward induction shows that F_x is homotopic to $R(-\ell(x))$ when viewed as a complex of right R-modules. This implies the following lemma:

Lemma 6.4. *We have*

$$H^i(\overline{F_x}) = \begin{cases} \mathbb{R}(-\ell(x)) & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

For the rest of this section and the next we examine the perverse filtration on Rouquier complexes.

Lemma 6.5. *Let $x \in W$, $s \in S$, and assume* $\mathrm{S}(x)$. *Regard $B_x \in K^b(\mathcal{B})$ as a complex concentrated in degree* 0.

(1) *If $xs < x$, then $B_xF_s \cong B_x(-1)$ in $K^b(\mathcal{B})$.*
(2) *If $xs > x$, then $B_xF_s \in {}^pK^b(\mathcal{B})^{\geq 0}$.*

Proof. (1) Under our assumptions, $B_xB_s \cong B_x(1) \oplus B_x(-1)$. Hence B_xF_s has the form

$$0 \to B_x(1) \oplus B_x(-1) \to B_x(1) \to 0.$$

Now B_x is indecomposable and tensoring with F_s gives an equivalence of $K^b(\mathcal{B})$. Hence the above complex is also indecomposable. It follows that the map $B_x(1) \to B_x(1)$ induced by the differential is nonzero and is an isomorphism because $\mathrm{End}(B_x) = \mathbb{R}$. It follows that the subcomplex $B_x(1) \to B_x(1)$ is contractible, yielding the result.

(2) If Soergel's conjecture holds for B_x, then $\mathrm{ch}(B_xB_s) = \underline{H}_x\underline{H}_s \in \bigoplus \mathbb{Z}_{\geq 0}\underline{H}_z$ and B_xB_s is perverse. The result is now immediate from the definitions. ☐

Lemma 6.6. *Suppose that $F \in {}^pK^b(\mathcal{B})^{\geq 0}$ and that Soergel's conjecture holds for all indecomposable summands of all ${}^iF$. Then $FF_s \in {}^pK^b(\mathcal{B})^{\geq 0}$.*

Proof. We can assume that F is a minimal complex. Consider the stupid filtration of F:

$$w_{\geq k}F := \cdots \to 0 \to {}^kF \to {}^{k+1}F \to \cdots .$$

Then for all k we have distinguished triangles

$$w_{\geq k+1}F \to w_{\geq k}F \to {}^kF[-k] \xrightarrow{[1]} .$$

By Lemma 6.1, if $(w_{\geq k+1}F)F_s$ and ${}^kF[-k]F_s$ are in ${}^pK^b(\mathcal{B})^{\geq 0}$, then so is $(w_{\geq k}F)F_s$. By Lemma 6.5, ${}^kF[-k]F_s \in {}^pK^b(\mathcal{B})^{\geq 0}$. The result now follows by induction. ☐

Corollary 6.7. *Assume* $\mathrm{S}(y)$ *for all $y < x$. Then $F_x \in {}^pK^b(\mathcal{B})^{\geq 0}$.*

Proof. Choose a reduced expression $\underline{x}$ for x, ending in some $s \in S$. Let $y = xs < x$. By an inductive application of the previous lemma, $F_y \in {}^pK^b(\mathcal{B})^{\geq 0}$. Then $F_x \cong F_y F_s$, and F_y satisfies the conditions of the previous lemma, so $F_x \in {}^pK^b(\mathcal{B})^{\geq 0}$. □

In particular, in the setting of the above corollary, we know Soergel's conjecture for every summand of every ${}^iF_x$ except possibly B_x itself, which occurs only in degree zero.

6.5. *Rouquier complexes are linear.* In the present section we establish that Rouquier complexes are "linear" under the assumption of Soergel's conjecture. We will need the following result of Libedinsky and the second author:

Proposition 6.8 (Rouquier complexes are Δ-split). *Fix $x \in W$, and let F_x denote a Rouquier complex. Then for any $y \in W$ we have an isomorphism in the homotopy category of graded R-bimodules*

$$\Gamma_{\geq y/>y} F_x = \begin{cases} \Delta_x & \text{if } x = y, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. This is [LW, Prop. 3.7]. □

The precise statement of "linearity" is the following:

Theorem 6.9 (Rouquier complexes are linear). *Assume* $\mathrm{S}(y)$ *for all* $y \leq x$. *Then*

(1) ${}^0F_x = B_x$;
(2) *for* $i \geq 1$, ${}^iF_x = \bigoplus B_z(i)^{\oplus m_{z,i}}$ *for* $z < x$ *and* $m_{z,i} \in \mathbb{Z}_{\geq 0}$.

In particular, $F_x \in {}^pK^b(\mathcal{B})^{\leq 0} \cap {}^pK^b(\mathcal{B})^{\geq 0}$.

Remark 6.10 (Positivity of inverse Kazhdan-Lusztig polynomials). One can show that

$$H_x = \mathrm{ch}(F_x) := \sum (-1)^i \,\mathrm{ch}({}^iF_x).$$

Therefore, defining $g_{z,x}$ by $H_x = \sum g_{z,x} \underline{H}_z$, we have $g_{x,x} = 1$ and $g_{z,x} = \sum (-1)^i m_{z,i} v^i$ for $z \leq x$. Hence one can determine all multiplicities $m_{z,i}$ using only Kazhdan-Lusztig combinatorics. Furthermore, a straightforward inductive argument gives that $m_{z,i} = 0$ if i and $\ell(x) - \ell(z)$ have different parity. Hence $(-1)^{\ell(x)-\ell(z)} g_{z,x}$ has positive coefficients for all $z \leq x$.

The theorem will be deduced from the following:

Lemma 6.11. *Assume* $\mathrm{S}(\leq x)$. *If* ${}^iF_x$ *contains a summand isomorphic to* $B_z(j)$ *with* $z < x$, *then* ${}^{i-1}F_x$ *contains a summand isomorphic to* $B_{z'}(j')$ *with* $z' > z$ *and* $j' < j$.

In the proof we use the following facts, which are immediate from the definition of the Δ-character: if $\mathrm{S}(y)$ holds, then $\Gamma_{\geq z/>z}(B_y)$ is zero unless $y \geq z$; it is Δ_z when $y = z$; and it is $\oplus\Delta_z(k)^{\oplus m_k}$ for $y > z$ with all k strictly positive.

Proof. Choose a summand $B_z(j) \Subset {}^iF_x$, and consider its image in ${}^{i+1}F_x$ under the differential. By (6.1) this image must project trivially to any summand of the form $B_y(k)$ for $k < j$. If $\mathrm{S}(z)$ and $\mathrm{S}(y)$ hold, then Soergel's hom formula (Theorem 3.6) implies that any nonzero map $B_z(j) \to B_y(j)$ is an isomorphism (and so $z = y$). Such an isomorphism cannot appear as the projection of the differential in a minimal complex because it would yield a contractible summand. Therefore, $B_z(j)$ maps to $\tau_{<-j}{}^{i+1}F_x$, the sum of terms $B_y(k)$ for $k > j$. Similarly, if some summand $B_y(k)$ of ${}^{i-1}F_x$ is sent nontrivially to $B_z(j)$ by the differential and then projection, we must have $k < j$.

Now apply $\Gamma_{\geq z/>z}$ to F_x. The result is split by Proposition 6.8 and has a summand in $\Gamma_{\geq z/>z}{}^iF_x$ isomorphic to $\Delta_z(j)$ coming from our chosen summand $B_z(j)$. This summand cannot survive in the cohomology of the complex, and thus it must map isomorphically to some $\Delta_z(j)$ in $\Gamma_{\geq z/>z}({}^{i+1}F_x)$ or be mapped to isomorphically from some $\Delta_z(j)$ in $\Gamma_{\geq z/>z}({}^{i-1}F_x)$. The former is impossible, because this summand maps to $\Gamma_{\geq z/>z}\tau_{<-j}({}^{i+1}F_x)$ which can only contain $\Delta_z(k)$ for $k > j$. Thus some summand $B_y(k)$ of ${}^{i-1}F_x$ contributes $\Delta_z(j)$ to $\Gamma_{\geq z/>z}$ (in particular, $y \geq z$), and this maps to $B_z(j)$. As mentioned above we must have $k < j$, which means that $y > z$. This proves the lemma. □

Proof of Theorem 6.9. Lemma 6.11 implies that the only summands of ${}^0F_x$ are of the form B_x, because ${}^{-1}F_x = 0$. In fact, ${}^0F_x \cong B_x$, as can be seen by applying $\Gamma_{\geq x/>x}$. Induction using Lemma 6.11 then implies that $\tau_{>-i}{}^iF_x = 0$. The theorem now follows because $F \in {}^pK^b(\mathcal{B})^{\leq 0}$ by Lemma 6.7. □

6.6. *Rouquier complexes are Hodge-Riemann.* We will use the following proposition repeatedly in what follows:

PROPOSITION 6.12. *Fix $\zeta \geq 0$, $s \in S$ and a Soergel bimodule $B = \bigoplus_{z \in W} B_z^{\oplus m_z}$ (for $m_z \in \mathbb{Z}_{\geq 0}$) such that if $m_z \neq 0$, then $\mathrm{S}(z)$ and $\mathrm{HR}(z,s)_\zeta$ hold. If $\zeta = 0$, we assume, in addition, that $m_z = 0$ if $zs < z$.*

Assume that $\overline{B}$ is even or odd and that B is equipped with an invariant nondegenerate form $\langle -,- \rangle_B$ such that $\overline{B}$ satisfies the Hodge-Riemann bilinear relations with the standard sign (with respect to left multiplication by ρ and $\langle -,- \rangle_{\overline{B}}$).

Then $\overline{BB_s}$ satisfies the Hodge-Riemann bilinear relations with the standard sign (with respect to L_ζ and the induced form $\langle -,- \rangle_{\overline{BB_s}}$).

Proof. We claim that we can choose our isomorphism $B \cong \bigoplus B_z^{\oplus m_z}$ such that each indecomposable summand is orthogonal under $\langle -,- \rangle_B$. Because Soergel's conjecture holds for each summand, the decomposition into isotypic components must be orthogonal, as $\mathrm{Hom}(B_z, \mathbb{D}B_y) = 0$ for $y \neq z$. Applying Soergel's conjecture again, we know that $\mathrm{End}(B_z^{\oplus m_z})$ is a matrix algebra, and choosing a decomposition of $B_z^{\oplus m_z}$ is the same as choosing a basis for $(B_z^{\oplus m_z})^{-\ell(z)}$. It is not difficult to check that if one chooses an orthogonal basis of $(B_z^{\oplus m_z})^{-\ell(z)}$ with respect to the (definite) Lefschetz form, then one obtains an orthogonal decomposition of $B_z^{\oplus m_z}$.

Hence we may assume that the decomposition $B = \bigoplus B_z^{\oplus m_z}$ is orthogonal with respect to $\langle -, - \rangle$. It follows that the induced form is orthogonal with respect to the decomposition $BB_s = \bigoplus (B_z B_s)^{\oplus m_z}$. By the Hodge-Riemann bilinear relations for $\overline{B}$, the Lefschetz form on the primitive subspace in degree $m + 2i$ is $(-1)^i$-definite, where m denotes the minimal nonzero degree in $\overline{B}$. Hence the restriction of $\langle -, - \rangle_B$ to any summand isomorphic to B_z is $(-1)^{(\ell(z)-m)/2}$ times a positive multiple of the intersection form on B_z. It follows from $\mathrm{HR}(z,s)_\zeta$ that the Lefschetz form on the primitive subspace of each summand $\overline{B_z B_s} \subset \overline{BB_s}$ is $(-1)^i$-definite in degree $-m - 1 + 2i$. Hence the same is true of $\overline{BB_s}$ (being the orthogonal direct sum of such spaces). Hence $\overline{BB_s}$ satisfies the Hodge-Riemann bilinear relations with the standard sign as claimed. $\square$

For the rest of this section, fix $x \in W$ and assume $\mathrm{S}(\leq x)$. By Theorem 6.9 we know that ${}^jF_x$ is concentrated in perverse degree $-j$. By definition, F_x is a direct summand of $F_{s_1} \cdots F_{s_m}$ for any choice of reduced expression $\underline{x} = s_1 \cdots s_m$. Hence ${}^jF_x$ is a direct summand of ${}^j(F_{s_1} \cdots F_{s_m})$. In other words, for all $j \geq 0$,

$$ {}^jF_x \overset{\oplus}{\subset} \bigoplus_{\underline{x}' \in \pi(\underline{x}, j)} \mathrm{BS}(\underline{x}')(j), \tag{6.3} $$

where $\pi(\underline{x}, j)$ denotes the set of all subexpressions of $\underline{x}$ obtained by omitting j simple reflections. Shifting, we deduce that ${}^jF_x(-j)$ is a summand of $\bigoplus \mathrm{BS}(\underline{x}')$.

Fix a tuple $\lambda = (\lambda_{\underline{x}'})_{\underline{x}' \in \pi(\underline{x}, j)}$ of strictly positive real numbers. We use these scalars to rescale the direct sum of the intersection form on $\bigoplus \mathrm{BS}(\underline{x}')$: if $b = (b_{\underline{x}'})$ and $b' = (b_{\underline{x}'})$ are elements of $\bigoplus \mathrm{BS}(\underline{x}')$, we set

$$ \langle b, b' \rangle^\lambda := \sum_{\underline{x}' \in \pi(\underline{x}, j)} \lambda_{\underline{x}'} \langle b_{\underline{x}'}, b'_{\underline{x}'} \rangle_{\mathrm{BS}(\underline{x}')}. $$

We say that F_x *satisfies the Hodge-Riemann bilinear relations* if for all reduced expressions $\underline{x} = s_1 \ldots s_m$ one can choose an embedding

$$ F_x \overset{\oplus}{\subset} F_{s_1} \cdots F_{s_m} $$

such that, for all tuples of strictly positive real numbers $\lambda = (\lambda_{\underline{x}'})$, each $\overline{{}^jF_x(-j)}$ satisfies the Hodge-Riemann bilinear relations with respect to the form induced by $\langle -, - \rangle^\lambda$ and the Lefschetz operator given by left multiplication by ρ, and with global sign determined as follows. The Lefschetz form should be positive definite on primitive subspaces in degrees congruent to $-m + j$ modulo 4.

(One can show that this is equivalent to satisfying the Hodge-Riemann bilinear relations with the standard sign. We will not need this.)

PROPOSITION 6.13. *Assume* $\mathrm{S}(\leq x)$. *Also assume that* $\mathrm{HR}(y,s)$ *holds for all* $y < x$ *and* $s \in S$ *with* $ys > y$. *Then* F_x *satisfies the Hodge-Riemann bilinear relations.*

Proof. We prove the proposition by induction over the Bruhat order, with the case $x = \mathrm{id}$ being obvious. Fix a reduced expression $\underline{x} = s_1 s_2 \cdots s_m$ for x as above and let $\underline{y} = s_1 \cdots s_{m-1}$ and $s = s_m$ so that $x = ys$. By induction we may assume that F_y satisfies the Hodge-Riemann bilinear relations. Hence we may choose an embedding $F_y \overset{\oplus}{\subset} F_{s_1} \cdots F_{s_{m-1}}$ such that for all j and any choice of scalars $(\mu_{\underline{y}'})_{\underline{y}' \in \pi(\underline{y},j)}$ the form on $\overline{{}^jF_y(-j)}$ induced by the pullback of the form $\langle -,- \rangle^\mu$ under the embedding

$$ {}^jF_y(-j) \overset{\oplus}{\subset} \bigoplus_{\underline{y}' \in \pi(\underline{y},j)} \mathrm{BS}(\underline{y}') $$

satisfies the Hodge-Riemann bilinear relations. Now F_x is a summand of $F_y F_s$ and hence we have natural embeddings

$$ \begin{aligned} {}^jF_x(-j) &\overset{\oplus}{\subset} {}^jF_y B_s(-j) \oplus {}^{j-1}F_y(-j+1) \\ &\overset{\oplus}{\subset} \bigoplus_{\underline{y}' \in \pi(\underline{y},j)} \mathrm{BS}(\underline{y}')B_s \oplus \bigoplus_{\underline{y}'' \in \pi(\underline{y},j-1)} \mathrm{BS}(\underline{y}'') = \bigoplus_{\underline{x}' \in \pi(\underline{x},j)} \mathrm{BS}(\underline{x}'). \end{aligned} $$

We claim that $\overline{{}^jF_x(-j)}$ satisfies the Hodge-Riemann bilinear relations with respect to this embedding for any tuple $\lambda = (\lambda_{\underline{x}'})_{\underline{x}' \in \pi(\underline{x},j)}$ of strictly positive real numbers (or equivalently any pair $(\mu'_{\underline{y}'})_{\underline{y}' \in \pi(\underline{y},j)}$ and $(\mu''_{\underline{y}'})_{\underline{y}'' \in \pi(\underline{y},j-1)}$ of tuples of strictly positive real numbers).

Soergel's conjecture holds for all indecomposable summands of F_y, and hence we have a canonical decomposition

$$ {}^jF_y(-j) = \bigoplus_{z \in W} V_z \otimes_{\mathbb{R}} B_z $$

for some (degree zero) multiplicity spaces V_z. Set

$$ B^\uparrow := \bigoplus_{\substack{z \in W \\ zs > z}} V_z \otimes_{\mathbb{R}} B_z \quad \text{and} \quad B^\downarrow := \bigoplus_{\substack{z \in W \\ zs < z}} V_z \otimes_{\mathbb{R}} B_z $$

so that

$$ {}^jF_y(-j) = B^\uparrow \oplus B^\downarrow. \tag{6.4} $$

This decomposition is orthogonal with respect to the induced forms because $\mathrm{Hom}(B^\uparrow, \mathbb{D}B^\downarrow) = \mathrm{Hom}(B^\downarrow, \mathbb{D}B^\uparrow) = 0$. Character calculations yield that $B^\uparrow B_s$ is perverse and that $B^\downarrow B_s \cong B^\downarrow(-1) \oplus B^\downarrow(1)$. (See also the proof of Theorem 6.19.)

Now, as we have already remarked above, F_x is a summand of F_yF_s and so ${}^jF_x$ is a summand of ${}^jF_yB_s \oplus {}^{j-1}F_y(1)$. We rewrite this using (6.4):

$$ {}^jF_x(-j) \Subset B^{\downarrow}B_s \oplus B^{\uparrow}B_s \oplus {}^{j-1}F_y(-j+1). $$

This decomposition is orthogonal with respect to the induced forms, and the inclusion of ${}^jF_x(-j)$ is an isometry.

The decomposition $B^{\downarrow}B_s \cong B^{\downarrow}(-1) \oplus B^{\downarrow}(1)$ is not orthogonal with respect to the induced form. In fact, the induced form is nondegenerate and hence induces a nondegenerate pairing of $B^{\downarrow}(-1)$ and $B^{\downarrow}(1)$. Nonetheless, we claim that in the decomposition

$$ \overline{B^{\downarrow}B_s} \cong \overline{B^{\downarrow}}(1) \oplus \overline{B^{\downarrow}}(-1) $$

the restriction of the Lefschetz form to $\overline{B^{\downarrow}}(1)$ is zero. Indeed, our assumptions imply that left multiplication by ρ satisfies the hard Lefschetz theorem on $\overline{B^{\downarrow}}$, and the fact that the Lefschetz form is zero follows from Lemma 2.5.

Because ${}^jF_x(-j)$ lives in perverse degree 0 by Theorem 6.9, Hom vanishing (6.1) implies that the projection ${}^jF_x(-j) \to B^{\downarrow}B_s$ will land entirely within $B^{\downarrow}(1)$, and therefore the image of $\overline{{}^jF_x(-j)}$ in $\overline{B^{\downarrow}B_s}$ will not contribute to the Lefschetz form.

Hence the projection to the second two factors above gives a map

$$ \iota : \overline{{}^jF_x(-j)} \to \overline{B^{\uparrow}B_s} \oplus \overline{{}^{j-1}F_y(-j+1)} $$

which is an isometry for the Lefschetz forms. We claim that ι is injective. Recall the functor $q : \mathcal{B} \to \mathcal{B}^{\mathrm{ss}}$ from Section 6.1. Any map ${}^jF_x(-j) \to B^{\downarrow}(1)$ cannot be an isomorphism because it is a map between objects in perverse degrees 0 and -1 respectively, and hence vanishes after applying q. On the other hand, if we apply q to the original inclusion ${}^jF_x \hookrightarrow {}^j(F_yF_s)$, then we obtain an injection, because this map is the inclusion of a direct summand. We conclude that if we apply q to $\iota' : {}^jF_x(-j) \to B^{\uparrow}B_s \oplus {}^{j-1}F_y(-j+1)$, then we obtain a split inclusion. Hence ι' is a split inclusion and the claim follows.

By assumption the induced intersection form on $\overline{{}^{j-1}F_y(-j+1)}$ satisfies the Hodge-Riemann bilinear relations and is positive definite on primitive subspaces in degrees congruent to $-(m-1)+j-1 = -m+j$ modulo 4. By the same inductive assumption, $\overline{B^{\uparrow}}$ satisfies the Hodge-Riemann bilinear relations with global sign given as follows. The Lefschetz form is > 0 on primitives in degrees congruent to $-(m-1)+j$ modulo 4. By Proposition 6.12 (essentially applying $\mathrm{HR}(z,s)$ to each summand) it follows that $\overline{B^{\uparrow}B_s}$ satisfies the Hodge-Riemann bilinear relations, with the Lefschetz forms > 0 on primitive subspaces in degrees congruent to $-(m+1)+j-1 = -m+j$ modulo 4.

We conclude that the codomain of ι satisfies the Hodge-Riemann bilinear relations. Hence the same is true for $\overline{{}^jF_x(-j)}$, being a ρ-stable summand with symmetric Betti numbers (Lemma 2.2). $\square$

6.7. *Factoring the Lefschetz operator.* Fix an expression $\underline{x} = s_1 s_2 \cdots s_m$. Recall the morphisms Br_i, ϕ_i and χ_i introduced in Section 3.4. Let us denote by Tr and $\langle -,- \rangle$ the trace map and intersection form on $\mathrm{BS}(\underline{x})$. To avoid confusion, we denote the trace map and intersection form on $\mathrm{BS}(\underline{x}_{\widehat{i}})$ by Tr_i and $\langle -,- \rangle_i$.

LEMMA 6.14. *For $b, b' \in \mathrm{BS}(\underline{x})$ we have $\langle b, \mathrm{Br}_i\, b' \rangle = \langle \phi_i b, \phi_i b' \rangle_i$.*

Proof. We may assume $b = b_1 b_2 \cdots b_m$ and $b' = b'_1 b'_2 \cdots b'_m$ with $b_i, b'_i \in B_{s_i}$. We calculate

$$\begin{aligned}\langle b, \mathrm{Br}_i\, b' \rangle &= \mathrm{Tr}((b_1 b'_1) \cdots (b_i b'_i c_s) \cdots (b_m b'_m))\\ &= \mathrm{Tr}((b_1 b'_1) \cdots \mu(b_i)\mu(b'_i) c_s \cdots (b_m b'_m))\\ &= \mathrm{Tr}(\chi_i((b_1 b'_1) \cdots \mu(b_i)\mu(b'_i) \cdots (b_m b'_m)))\\ &= \mathrm{Tr}_i((b_1 b'_1) \cdots \mu(b_i)\mu(b'_i) \cdots (b_m b'_m))\\ &= \langle \phi_i(b), \phi_i(b') \rangle_i.\end{aligned}$$

The second to last equality follows from the identity $\mathrm{Tr}(\chi_i(\gamma)) = \mathrm{Tr}_i(\gamma)$ valid for all $\gamma \in \mathrm{BS}(\underline{x}_{\widehat{i}})$. $\square$

Let us rescale the forms on each $\mathrm{BS}(\underline{x}_{\widehat{i}})$ by defining

$$\langle -,- \rangle'_i := (s_{i-1} \cdots s_1 \rho)(\alpha^\vee_{s_i}) \langle -,- \rangle_i.$$

Let $\langle -,- \rangle'$ denote the direct sum of the forms $\langle -,- \rangle'_i$ on $\bigoplus \mathrm{BS}(\underline{x}_{\widehat{i}})$.

If we set $\phi := \sum \phi_i$, then for $b, b' \in \mathrm{BS}(\underline{x})$ we have

$$\begin{aligned}\langle \phi(b), \phi(b') \rangle' &= \sum_{1 \le i \le m} (s_{i-1} \cdots s_1 \rho)(\alpha^\vee_{s_i}) \langle \phi_i(b), \phi_i(b') \rangle_i\\ &= \sum_{1 \le i \le m} (s_{i-1} \cdots s_1 \rho)(\alpha^\vee_{s_i}) \langle b, \mathrm{Br}_i(b') \rangle_i\\ &= \langle b, \rho b' \rangle - \langle b, b' \rangle \cdot w^{-1}\rho\end{aligned}$$

by Lemmas 3.4 and 6.14 respectively. We conclude:

LEMMA 6.15. *Consider the induced map*

$$\overline{\mathrm{BS}(\underline{x})} \xrightarrow{\phi} \bigoplus \overline{\mathrm{BS}(\underline{x}_{\widehat{i}})}(1).$$

For all $b, b' \in \overline{\mathrm{BS}(\underline{x})}$ we have

$$\langle b, \rho b' \rangle = \langle \phi b, \phi b' \rangle' \in \mathbb{R}.$$

Remark 6.16. Lemma 6.15 will be a key tool in our proof of the hard Lefschetz theorem for Soergel bimodules. It serves as a partial replacement for the weak Lefschetz theorem.

Remark 6.17. When we apply the above lemma, $\underline{x}$ will be a reduced expression. Because ρ is assumed dominant regular, it follows that all the scaling factors $(s_{i-1}\cdots s_1\rho)(\alpha_{s_i}^\vee)$ are positive, by (3.1). Hence, although we rescale the forms on each Bott-Samelson bimodule, this does not affect the signs appearing in the Hodge-Riemann bilinear relations.

6.8. *Proof of hard Lefschetz.* Fix $x \in W$ and $s \in S$. Let $\underline{x}$ denote a reduced expression for x. Recall the operator L_ζ on $B_{\underline{x}}B_s$ from Section 5. The goal of this section is to prove three incarnations of the hard Lefschetz theorem for the induced action of L_ζ on $\overline{B_x B_s}$ under certain inductive assumptions. The three cases are

(1) $\zeta > 0$ and $xs < x$ (Theorem 6.19),
(2) $\zeta > 0$ and $xs > x$ (Theorem 6.20),
(3) $\zeta = 0$ and $xs > x$ (Theorem 6.21).

(It will also be clear in the proof of (1) that hard Lefschetz fails in the missing case $\zeta = 0$ and $xs < x$.)

Remark 6.18. We warn the reader that the proof in case (1) is comparatively straightforward and has little in common with the proofs of cases (2) and (3). On the other hand, the proofs of cases (2) and (3) (which use positivity considerations in a crucial way) are similar, with (3) being more involved. The reader is encouraged to view the proof in case (2) as a warm-up for (3).

Theorem 6.19 (Hard Lefschetz for $\zeta > 0$, $xs < x$). *Suppose* $\zeta > 0$ *and* $xs < x$. *If* $\mathrm{hL}(x)$ *holds, then so does* $\mathrm{hL}(x,s)_\zeta$.

Proof. The basic idea is as follows. Because $xs < x$, we have $B_xB_s \cong B_x(1) \oplus B_x(-1)$. We will fix such an isomorphism and see that the operator L_ζ on $\overline{B_xB_s} = \overline{B_x(1)} \oplus \overline{B_x(-1)}$ has the form

$$L_\zeta = \begin{pmatrix} \rho\cdot(-) & 0 \\ \zeta\rho(\alpha_s^\vee) & \rho\cdot(-) \end{pmatrix}, \tag{6.5}$$

where $\rho\cdot(-)$ is the Lefschetz operator on $\overline{B_x}$ given by left multiplication by ρ, and $\zeta\rho(\alpha_s^\vee)$ denotes a scalar multiple of the identity, viewed as a degree two map $\overline{B_x(1)} \to \overline{B_x(-1)}$. Because $\rho\cdot(-)$ satisfies the hard Lefschetz theorem on $\overline{B_x}$, we can complete the ρ-action to an action of $\mathfrak{sl}_2(\mathbb{R}) = \mathbb{R}f \oplus \mathbb{R}h \oplus \mathbb{R}e$ such that $e = \rho\cdot(-)$ and $hb = kb$ for all $b \in (\overline{B_x})^k$. In this case, after rescaling (under the assumption that $\zeta \neq 0$), the above matrix describes the action of e on the tensor product of $\overline{B_x}$ with the standard two-dimensional representation of $\mathfrak{sl}_2(\mathbb{R})$. Hence $e = L_\zeta$ satisfies the hard Lefschetz theorem as claimed.

It remains to show that L_ζ has the form given in (6.5). By assumption $xs < x$ and hence, by [Wil11, Th. 1.4], we can find an (R, R^s)-bimodule $B_{\overline{x}}$ such that $B_{\overline{x}} \otimes_{R^s} R \cong B_x$. We conclude that any choice of isomorphism

$R \cong R^s \oplus R^s(-2)$ of graded R^s-modules yields an isomorphism

$$B_x B_s \cong B_{\overline{x}} \otimes_{R^s} R \otimes_{R^s} R(1) \cong B_x(1) \oplus B_x(-1). \tag{6.6}$$

Now we fix such an isomorphism. Consider the maps $\iota_1, \iota_2 : R^s \to R$, where ι_1 is the inclusion and $\iota_2(r) = \frac{1}{2}\alpha_s \iota_1(r)$. Let $\pi_1, \pi_2 : R \to R^s$ be given by

$$\pi_1(r) = \frac{1}{2}(r + sr) \quad \text{and} \quad \pi_2(r) = \partial_s(r).$$

Then $\pi_a \circ \iota_b = \delta_{ab}$ for $a, b \in \{1, 2\}$, and so these maps give the inclusions and projections in an R^s-bimodule isomorphism $R \cong R^s \oplus R^s(-2)$. Tensoring these isomorphisms with the identity on both sides yields the inclusion and projection maps fixing an isomorphism as in (6.6).

With respect to this fixed isomorphism, a straightforward calculation yields that L_ζ is given by the matrix

$$\begin{pmatrix} \rho \cdot (-) + \zeta(-) \cdot \pi_1(\rho) & \frac{1}{2}\zeta(-) \cdot \pi_1(\alpha\rho) \\ \zeta\rho(\alpha_s^\vee)(-) & \rho \cdot (-) + \frac{1}{2}\zeta(-) \cdot \partial_s(\alpha\rho) \end{pmatrix}.$$

Passing to $\overline{B_x B_s}$ the operator of right multiplication by a polynomial of positive degree becomes zero, and the above matrix reduces to (6.5). This completes the proof. $\square$

Theorem 6.20 (Hard Lefschetz for $\zeta > 0$, $xs > x$). *Suppose $\zeta > 0$ and $xs > x$. Assume the following*:

(1) $\mathrm{S}(\leq x)$ *holds*;
(2) $\mathrm{HR}(z, t)$ *holds for all* $(z, t) \in W \times S$ *such that* $z < x$ *and* $zt > t$;
(3) $\mathrm{HR}(<x, s)_\zeta$ *holds*;
(4) $\mathrm{HR}(x)$ *holds.*

Then $\mathrm{hL}(x, s)_\zeta$ *holds.*

Proof. Write $\underline{x} = s_1 s_2 \cdots s_m$, and set

$$\gamma_i := (s_{i-1} \cdots s_1 \rho)(\alpha_{s_i}^\vee) \quad \text{for } 1 \leq i \leq m \text{ and } \quad \gamma_{m+1} := (x^{-1}\rho)(\alpha_s^\vee) + \zeta\rho(\alpha_s^\vee).$$

The scalars $\gamma_1, \ldots, \gamma_m$ are all positive because $\underline{x}$ is reduced, and γ_{m+1} is positive because $xs > x$; see (3.1). As in Section 6.7 we use the tuple $\gamma_{\leq m} = (\gamma_i)_{i=1}^m$ (resp. $\gamma = (\gamma_i)_{i=1}^{m+1}$) to define a rescaled intersection forms $\langle -, - \rangle^{\gamma_{\leq m}}$ (resp. $\langle -, - \rangle^\gamma$) on $\bigoplus \mathrm{BS}(\underline{x}_{\hat{i}})$ (resp. $\bigoplus \mathrm{BS}((\underline{x}s)_{\hat{i}})$). By a slight variant of Section 6.7 we have the relation

$$\langle b, L_\zeta b' \rangle_{\overline{\mathrm{BS}(\underline{x}s)}} = \langle \phi(b), \phi(b') \rangle_{\mathbb{R}}^\gamma \quad \text{for all } b, b' \in \overline{\mathrm{BS}(\underline{x}s)}, \tag{6.7}$$

where ϕ is the first differential in the complex $F_{s_1} \cdots F_{s_m} F_s$ and $\langle -, - \rangle_{\mathbb{R}}^\gamma$ denotes the form on $\bigoplus \overline{\mathrm{BS}((\underline{x}s)_{\hat{i}})}$ induced by $\langle -, - \rangle^\gamma$.

Now fix a minimal complex $F_x \Subset F_{s_1} F_{s_2} \cdots F_{s_m}$. Because we assume $\mathrm{S}(\leq x)$, Theorem 6.9 allows us to conclude that ${}^0F_x = B_x$ and that ${}^kF_x$ is concentrated in perverse degree $-k$. Because we assume $\mathrm{S}(\leq x)$ and $\mathrm{HR}(z,t)$ for all $z < x$ with $zt > z$, we may apply Proposition 6.13 to find an embedding $F_x \Subset F_{s_1} \cdots F_{s_m}$ so that ${}^1F_x(-1) \Subset \bigoplus \overline{\mathrm{BS}(\underline{x}_{\hat{i}})}$ satisfies the Hodge-Riemann bilinear relations with respect to the form induced by $\langle -,-\rangle^{\gamma_{\leq m}}$.

The first two terms of F_xF_s have the form

$$B_xB_s \xrightarrow{\phi} {}^1F_xB_s \oplus B_x(1).$$

We use this decomposition to write $\phi = (d_1, d_2)$ for maps $d_1 : B_xB_s \to {}^1F_xB_s(-1)$ and $d_2 : B_xB_s \to B_x(1)$. It is straightforward to verify that d_1 commutes with L_ζ and that for d_2 we have

$$d_2(L_\zeta b) = \rho \cdot d_2(b) + d_2(b) \cdot \zeta\rho$$

for all $b \in B_xB_s$. Hence, if we denote by L the operator on $\overline{{}^1F_xB_s} \oplus \overline{B_x}(1)$ given by L_ζ on the first summand and $\rho \cdot (-)$ on the second, then we have

$$\overline{\phi}(L_\zeta b) = L\overline{\phi}(b) \quad \text{for all } b \in \overline{B_xB_s}, \tag{6.8}$$

where $\overline{\phi}$ denotes the induced map $\overline{\phi} : \overline{B_xB_s} \to \overline{{}^1F_xB_s} \oplus \overline{B_x}(1)$. Moreover,

(1) $\overline{\phi}$ is injective in degrees $\leq \ell(x)$ (by Lemma 6.4).
(2) $\langle b, L_\zeta b'\rangle_{\overline{B_xB_s}} = \langle \overline{\phi}(b), \overline{\phi}(b')\rangle^\gamma_{\mathbb{R}}$ for all $b, b' \in \overline{B_xB_s}$ (by (6.7)).
(3) $\overline{{}^1F_xB_s(-1)} \oplus \overline{B_x}$ satisfies the Hodge-Riemann bilinear relations with respect to the Lefschetz operator L and the form $\langle -,-\rangle^\gamma_{\mathbb{R}}$. (The decomposition $\overline{{}^1F_xB_s(-1)} \oplus \overline{B_x}$ is orthogonal. For $\overline{B_x}$ the Hodge-Riemann bilinear relations hold by assumption. For $\overline{{}^1F_xB_s(-1)}$ the Hodge-Riemann relations hold by Proposition 6.12 and our assumption $\mathrm{HR}(y,s)_\zeta$ for all $y < x$.)

Now we can apply Lemma 2.3 to conclude that $L_\zeta^k : (\overline{B_xB_s})^{-k} \to (\overline{B_xB_s})^k$ is injective for all $k \geq 0$. Finally, $\overline{B_xB_s}$ is self-dual as a graded vector space and hence has symmetric Betti numbers. Hence L_ζ satisfies the hard Lefschetz theorem on $\overline{B_xB_s}$ as claimed. □

THEOREM 6.21 (Hard Lefschetz for $\zeta = 0$, $xs > x$). *Assume that*

(1) $\mathrm{S}(\leq x)$ *holds*;
(2) $\mathrm{HR}(y,t)$ *holds for all* $(y,t) \in W \times S$ *such that* $y < x$ *and* $yt > y$;
(3) $\mathrm{HR}(x)$ *holds*;
(4) $\mathrm{hL}(z)$ *holds for all* $z < xs$.

Then $\mathrm{hL}(x,s)$ *holds.*

Proof. Write $\underline{x} = s_1s_2\cdots s_m$ for x, and set

$$\gamma_i := (s_{i-1}\cdots s_1\rho)(\alpha_{s_i}^\vee) \quad \text{for } 1 \leq i \leq m \text{ and} \quad \gamma_{m+1} := (x^{-1}\rho)(\alpha_s^\vee).$$

By (3.1), $\gamma_1, \ldots, \gamma_{m+1}$ are positive. As in Section 6.7 we use the tuple $\gamma_{\leq m} = (\gamma_i)_{i=1}^m$ (resp. $\gamma = (\gamma_i)_{i=1}^{m+1}$) to define a rescaled intersection forms $\langle -,-\rangle^{\gamma_{\leq m}}$

(resp. $\langle -,-\rangle^{\gamma}$) on $\bigoplus \mathrm{BS}(\underline{x}_{\hat{i}})$ (resp. $\bigoplus \mathrm{BS}((\underline{x}s)_{\hat{i}})$). By Section 6.7 we have the relation

$$\langle b, \rho \cdot b' \rangle_{\overline{\mathrm{BS}(\underline{x}s)}} = \langle \phi(b), \phi(b') \rangle^{\gamma}_{\mathbb{R}} \quad \text{for all } b, b' \in \overline{\mathrm{BS}(\underline{x}s)}, \tag{6.9}$$

where ϕ is the first differential in the complex $F_{s_1} \cdots F_{s_m} F_s$ and $\langle -,-\rangle^{\gamma}_{\mathbb{R}}$ denotes the form on $\bigoplus \overline{\mathrm{BS}((\underline{x}s)_{\hat{i}})}$ induced by $\langle -,-\rangle^{\gamma}$.

We now choose a minimal subcomplex $F_x \Subset F_{s_1} F_{s_2} \cdots F_{s_m}$. We know that ${}^0F_x = B_x$, that ${}^kF_x$ is concentrated in perverse degree $-k$ by Theorem 6.9, and that we can choose our embedding such that ${}^kF_x(-k) \Subset \bigoplus \mathrm{BS}(\underline{x})$ satisfies the Hodge-Riemann bilinear relations by Proposition 6.13. As in the proof of Proposition 6.13 let us decompose

$${}^1F_x(-1) = B^{\uparrow} \oplus B^{\downarrow}$$

so that $B^{\uparrow}B_s$ is perverse and $\mathcal{H}^0(B^{\downarrow}B_s) = 0$. This decomposition is orthogonal with respect to $\langle -,-\rangle^{\gamma_{\le m}}$ because

$$\mathrm{Hom}(B^{\uparrow}, \mathbb{D}B^{\downarrow}) = \mathrm{Hom}(B^{\downarrow}, \mathbb{D}B^{\uparrow}) = 0.$$

The first two terms of F_xF_s have the form

$$B_xB_s \to B_x(1) \oplus B^{\uparrow}B_s(1) \oplus B^{\downarrow}B_s(1).$$

We claim that this decomposition of ${}^1(F_xF_s)$ is orthogonal with respect to $\langle -,-\rangle^{\gamma}$. Indeed, under the inclusion of ${}^1(F_xF_s) \Subset \bigoplus \mathrm{BS}((\underline{x}s)_{\hat{i}})$ we have $B_x(1) \Subset \mathrm{BS}(\underline{x})$ and $B^{\uparrow}B_s(1) \oplus B^{\downarrow}B_s(1) \Subset \bigoplus \mathrm{BS}(\underline{x}_{\hat{i}}s)$. Hence $B_x(1)$ is orthogonal to $B^{\uparrow}B_s(1) \oplus B^{\downarrow}B_s(1)$. The form on $B^{\uparrow}B_s(1) \oplus B^{\downarrow}B_s(1) = (B^{\uparrow} \oplus B^{\downarrow})B_s$ coincides with the induced form from $\langle -,-\rangle^{\gamma_{\le m}}$ on $B^{\uparrow} \oplus B^{\downarrow}$ (see §3.6). The claimed orthogonality for the decomposition of ${}^1(F_xF_s)$ now follows from the orthogonality of $B^{\uparrow}$ and $B^{\downarrow}$ under $\langle -,-\rangle^{\gamma_{\le m}}$.

We also know that $F_xF_s \in {}^pK^b(\mathcal{B})^{\ge 0}$ by Corollary 6.7, and hence the restriction of the second differential to $\tau_{\le -2}({}^1(F_xF_s)) = \tau_{\le -2}(B^{\downarrow}B_s(1))$ is a split injection. Canceling this contractible direct summand we obtain a summand of F_xF_s such that the inclusion is a homotopy equivalence. Observing that $\tau_{\ge -1}(B^{\downarrow}B_s(1)) = \tau_{\ge 0}(B^{\downarrow}B_s(1)) \cong B^{\downarrow}$ we see that the first two terms of this summand have the form

$$B_xB_s \xrightarrow{d} B_x(1) \oplus B^{\uparrow}B_s(1) \oplus B^{\downarrow}.$$

We use this decomposition to write $d = (d_1, d_2, d_3)$ for maps $d_1 : B_xB_s \to B_x(1)$, $d_2 : B_xB_s \to B^{\uparrow}B_s(1)$ and $d_3 : B_xB_s \to B^{\downarrow}$. Consider the induced map

$$\overline{B_xB_s} \xrightarrow{\overline{d}} \overline{B_x}(1) \oplus \overline{B^{\uparrow}B_s}(1) \oplus \overline{B^{\downarrow}}$$

with components $\overline{d_1}$, $\overline{d_2}$ and $\overline{d_3}$. By Lemma 6.4, $\overline{d}$ is injective in degrees $\le \ell(x)$.

Now fix $0 \ne b \in (\overline{B_xB_s})^{-k}$ for some $k \ge 0$. Because $\overline{B_xB_s}$ has symmetric Betti numbers, to prove the theorem it is enough to show that $\rho^k \cdot b \ne 0$. Because $\overline{d}(b) \ne 0$, the theorem follows from the following two claims:

Claim 1. *If* $\overline{d_3}(b) \neq 0$, *then* $\rho^k(b) \neq 0$.

Each indecomposable summand of $B^{\downarrow}$ is of the form B_z with $z < xs$ and $zs < z$. For such z, left multiplication by ρ on $\overline{B_z}$ satisfies the hard Lefschetz theorem by assumption. Hence left multiplication by ρ satisfies the hard Lefschetz theorem on $\overline{B^{\downarrow}}$. Now $\overline{d_3}$ commutes with left multiplication by ρ. Hence $0 \neq \rho^k(\overline{d_3}(b)) = \overline{d_3}(\rho^k(b))$ and the claim follows.

Claim 2. *If* $\overline{d_3}(b) = 0$, *then* $\rho^k(b) \neq 0$.

Consider $V := \mathrm{Ker}(\overline{d_3}) \subset \overline{B_x B_s}$ and $W = \overline{B_x} \oplus \overline{B^{\uparrow} B_s}$. By restricting $\langle -, - \rangle_{\overline{B_x B_s}}$ to V and $\langle -, - \rangle_{\mathbb{R}}^{\gamma}$ to W, we obtain graded forms on these spaces. The operator given by left multiplication by ρ is a Lefschetz operator on both spaces. Write ϕ_V for the restriction of $\overline{d}$ to V, viewed as a map $V \to W(1)$. Then

(1) $\phi_V(\rho b) = \rho(\phi_V(b))$ for all $b \in V$.
(2) ϕ_V is injective in degrees ≤ -1 (or even $\leq \ell(x)$).
(3) $\langle b, L_\zeta b' \rangle_V = \langle \phi_V(b), \phi_V(b') \rangle_W$ for all $b, b' \in V$ (by (6.7)).
(4) W satisfies the Hodge-Riemann bilinear relations. (For $\overline{B_x}$ this holds by assumption. For $\overline{B^{\uparrow} B_s}$ this holds because every indecomposable summand of $B^{\uparrow}$ is of the form B_z with $zs > z$. Hence the Hodge-Riemann bilinear relations hold for $\overline{B^{\uparrow} B_s}$ by our assumption (2) in the statement of the theorem, combined with Proposition 6.12 and the fact that $\overline{B^{\uparrow}}$ satisfies the Hodge-Riemann bilinear relations.)

We now apply Lemma 2.3 to conclude that $\rho^k : V^{-k} \to V^k$ is injective. $\square$

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