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Galois orbits and equidistribution of special subvarieties: towards the André-Oort conjecture

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Abstract

In this paper we develop a strategy and some technical tools for proving the André-Oort conjecture. We give lower bounds for the degrees of Galois orbits of geometric components of special subvarieties of Shimura varieties, assuming the Generalised Riemann Hypothesis. We proceed to show that sequences of special subvarieties whose Galois orbits have bounded degrees are equidistributed in a suitable sense.

Contents

1. Introduction

The main motivation for this paper is the André-Oort conjecture stated below.

Conjecture 1.1 (André-Oort). *Let S be a Shimura variety, and let Σ be a set of special points in S. Every irreducible component of the Zariski closure of Σ is a special subvariety of S.*

Some authors use the terminology "subvarieties of Hodge type" instead of "special subvarieties." The two terms refer to the same notion. There are two main approaches to this conjecture that proved fruitful in some cases. One, due to Edixhoven and Yafaev (see [7] and [21]), relies on the Galois properties of special points and geometric properties of images of subvarieties of Shimura varieties by Hecke correspondences. The other, due to Clozel and Ullmo (see [4]), aims at proving that certain sequences of special subvarieties are equidistributed in a certain sense. This approach uses some deep theorems from ergodic theory. The purpose of this paper is to explain how to combine these two approaches in order to obtain a strategy for proving the André-Oort conjecture in full generality and to provide essential ingredients to apply this strategy. The strategy and the results of this paper are subsequently used in [11] by Klingler and Yafaev to prove the André-Oort conjecture assuming the Generalised Riemann Hypothesis (GRH).

To explain the alternative, we need to introduce some terminology. Let S be a connected component of a Shimura variety. There are a Shimura datum (G, X) and a compact open subgroup K of $G(\mathbb{A}_f)$ such that S is a connected component of

$$\mathrm{Sh}_K(G, X) := G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K.$$

For the purpose of proving Conjecture 1.1, we may and do assume that S is the image of $X^+ \times \{1\}$ in $\mathrm{Sh}_K(G, X)$ (where X^+ is a fixed connected component of X). A special subvariety Z of S is associated to a Shimura subdatum (H, X_H) of (G, X). More precisely, Z is an irreducible component of the image of $\mathrm{Sh}_{K\cap H(\mathbb{A}_f)}(H, X_H)$ in $\mathrm{Sh}_K(G, X)$ contained in S. We are assuming that H is the generic Mumford-Tate group on X_H.

Let E be some number field over which S admits a canonical model. Let Z be a special subvariety of S associated to (H, X_H) as above.

By the degree of the Galois orbit of Z, denoted $\deg(\mathrm{Gal}(\overline{\mathbb{Q}}/E)\cdot Z)$, we mean the degree of the subvariety $\mathrm{Gal}(\overline{\mathbb{Q}}/E)\cdot Z$ calculated with respect to the natural ample line bundle on the Baily-Borel compactification of $\mathrm{Sh}_K(G, X)$. If Z is a special point, then $\deg(\mathrm{Gal}(\overline{\mathbb{Q}}/E)\cdot Z)$ is simply the number of $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$-conjugates of Z.

The “philosophy” of this paper is the following alternative. Let $(Z_n)_{n\in\mathbb{N}}$ be a sequence of special subvarieties of S. After possibly replacing (Z_n) by a subsequence and assuming the GRH for CM-fields, at least one of the following cases occurs:

(1) The sequence $\deg(\mathrm{Gal}(\overline{\mathbb{Q}}/E)\cdot Z_n)$ tends to infinity as $n\to\infty$ (and therefore Galois-theoretic and geometric techniques can be used).
(2) The sequence of probability measures (μ_n) canonically attached to (Z_n) weakly converges to some μ_Z, the probability measure canonically attached to a special subvariety Z of S. Moreover for every n large enough, Z_n is contained in Z. In other words, the sequence (Z_n) is equidistributed with respect to (Z, μ_Z).

Which of the two cases occurs depends on the geometric nature of the subvarieties Z_n. Let us explain this in more detail.

A special subvariety Z associated to a Shimura datum (H, X_H) as before (in particular, H is the generic Mumford-Tate on X_H) is called strongly special (see [4]) if the image of the group H in the adjoint group G^{ad} is semisimple. Note that condition (b) in the definition of "strongly special" ([4, 4.1]) is in fact implied by the first. (See [18, Rem. 3.9] or the proof of Theorem 3.8 of this paper.) Clozel and Ullmo proved in [4] that if the subvarieties Z_n are strongly special, then the second case of the alternative occurs. This result is unconditional.

On the other extreme, if H is a torus, then Z is a special point. If (Z_n) is a sequence of special points, then the first case of the alternative occurs (and the second in general does not: a sequence of special points is usually not equidistributed). This uses the GRH, but we believe that one might be able to get rid of this assumption. We also prove the equidistribution result unconditionally in the case where the subvarieties Z_n satisfy an additional assumption. In the paper [21], lower bounds for Galois orbits of special points are given and used to prove the André-Oort conjecture for curves. However, these bounds are not strong enough to prove that they are unbounded for a general infinite sequence of special points.

The first thing we do in this paper is to give lower bounds for the degree of Galois orbits of special subvarieties that are not *strongly special* (Theorem 2.19). In the special case where H is a torus, we can show that given an infinite set Σ of special points, the size of the Galois orbit of the point x is unbounded as x ranges through Σ. This result is explained in Conjecture 3.12. Lower bounds obtained in [21] do not allow us to prove such a statement.

We now explain our lower bounds in detail. Let N be an integer. Let H be the generic Mumford-Tate group on X_H, and let T be its connected centre. Suppose that T is a nontrivial torus. Let L_T be the splitting field of T. Let K_T^m be the maximal compact open subgroup of $T(\mathbb{A}_f)$. Note that K_T^m is a product of maximal compact open subgroups $K_{T,p}^m$ of $T(\mathbb{Q}_p)$ for all primes p. Let K_T be the compact open subgroup $T(\mathbb{A}_f) \cap K$ of $T(\mathbb{A}_f)$. We assume that K is a product of compact open subgroups K_p of $G(\mathbb{Q}_p)$ in which case K_T is also a product of compact open subgroups $K_{T,p}$ of $T(\mathbb{Q}_p)$. Let Z be a component of the image of $\mathrm{Sh}_{H(\mathbb{A}_f)\cap K}(H, X_H)$ in S.

We show (Theorem 2.19), assuming the GRH, that there is a positive constant B (independent of Z and N):

$$\deg(\mathrm{Gal}(\overline{\mathbb{Q}}/E) \cdot Z) \gg \prod_{\{p : K_{T,p}^m \neq K_{T,p}\}} \max(1, B|K_{T,p}^m / K_{T,p}|) \log(|\mathrm{disc}(L_T)|)^N.$$

We also obtain similar lower bounds for the degree of the Galois orbit of a Hodge generic irreducible subvariety Y of Z defined over $\overline{\mathbb{Q}}$ when Y moreover satisfies a technical property (see Theorem 2.19). This result will play no role in this paper but will be useful in the forthcoming paper by Klingler and Yafaev [11].

The next task we carry out is the analysis of the conditions, under which a given sequence of special subvarieties Z_n is such that $\deg(\mathrm{Gal}(\overline{\mathbb{Q}}/E) \cdot Z_n)$ is bounded. We translate this condition into explicit conditions on the Shimura data defining the subvarieties Z_n. We introduce the notion of a T-special subvariety. Suppose that G is semisimple of adjoint type, and fix a subtorus T of G such that $T(\mathbb{R})$ is compact. A T-Shimura subdatum (H, X_H) of (G, X) is a Shimura subdatum such that $T = Z(H)^0$ is the connected centre of H. A T-special subvariety is a special subvariety defined by a T-special Shimura subdatum. Fix an integer M. We show (Theorem 3.10), assuming the GRH, that there is a finite set $\{T_1, \ldots, T_r\}$ of subtori of G such that any special subvariety Z with $\deg(\mathrm{Gal}(\overline{\mathbb{Q}}/E) \cdot Z) \leq M$ is T_i-special for some $i = 1, \ldots, r$. This result crucially relies on a result of Gille and Moret-Bailly [10].

Finally, using the ergodic methods of [4], we prove that if the degree of $\mathrm{Gal}(\overline{\mathbb{Q}}/E) \cdot Z_n$ is bounded (when n varies), then the second case of the alternative occurs. We actually show (Theorem 3.8) that, for a fixed T, a sequence of T-special subvarieties is equidistributed in the sense explained above.

The alternative explained above is used in the forthcoming paper by Klingler and the second author [11] to prove the following theorem, which is the most general result on the André-Oort conjecture obtained so far.

THEOREM 1.2. *Let (G, X) be a Shimura datum and K a compact open subgroup of $G(\mathbb{A}_f)$. Let Σ be a set of special points in $\mathrm{Sh}_K(G, X)$. We make one of the two following assumptions*:

(1) *Assume the Generalised Riemann Hypothesis* (GRH) *for* CM *fields.*
(2) *Assume that there exists a faithful representation $G \hookrightarrow \mathrm{GL}_n$ such that with respect to this representation*, *the Mumford-Tate groups* $\mathrm{MT}(s)$ *lie in one* $\mathrm{GL}_n(\mathbb{Q})$*-conjugacy class as s ranges through Σ.*

Then every irreducible component of the Zariski closure of Σ in $\mathrm{Sh}_K(G, X)$ is a special subvariety.

Klingler and Yafaev started working together on this conjecture in 2003, trying to generalise the Edixhoven-Yafaev strategy to the general case of the André-Oort conjecture. In the process two main difficulties occurred. One is the question of irreducibility of transforms of subvarieties under Hecke correspondences. This problem is dealt with in the forthcoming paper by Klingler and Yafaev. This allows us to treat the cases where the first case of the alternative explained above occurs.

The other difficulty was dealing with sets of special subvarieties that are defined over number fields of bounded degree. We overcome this difficulty in the present paper. In fact, we show that this is precisely when the second case of the alternative occurs. This strategy — a combination of Galois theoretic and ergodic techniques — was discovered by the authors of this paper while the second author was visiting the University of Paris-Sud in January–February 2005. We tested our strategy on the case of subvarieties of a product of modular curves (see [19]).

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2. **Degrees of Galois orbits of special subvarieties**

In this section we give lower bounds for the degrees of the Galois orbits of special subvarieties that are not strongly special. (Actually we prove a more general statement as explained in the introduction.)

2.1. *Preliminaries on special subvarieties and reciprocity morphisms.* We start by recalling some facts about special subvarieties, reciprocity morphisms and the Galois action on the geometric components of Shimura varieties. If Z is a topological space, we denote by $\pi_0(Z)$ the set of connected components of Z.

Let (G, X) be a Shimura datum. We fix a faithful representation of G that allows us to view G as a closed subgroup of some GL_n. Let K be a compact open subgroup of $G(\mathbb{A}_f)$ that is contained in $\mathrm{GL}_n(\widehat{\mathbb{Z}})$. We also assume that K is a product of compact open subgroups K_p of $G(\mathbb{Q}_p)$.

Let (H, X_H) be a Shimura subdatum of (G, X). We suppose that H is not semisimple. (Its connected centre is nontrivial.) Let T be the connected centre of H, so that T is a nontrivial torus and H is an almost direct product TH^{der}.

Let K_H be the compact open subgroup $H(\mathbb{A}_f) \cap K$ of $H(\mathbb{A}_f)$. We first describe the Galois action on the set of components of $\mathrm{Sh}_{K_H}(H, X_H)$. We refer to Sections 2.4–2.6 of [6] for details and proofs.

Let N be a reductive group over $\mathbb{Q}$ and $\lambda\colon N \longrightarrow N^{\mathrm{ad}}$ the quotient of N by its centre. We denote

$$N(\mathbb{R})_+ := \lambda^{-1}(N^{\mathrm{ad}}(\mathbb{R})^+)$$

and $N(\mathbb{Q})_+ := N(\mathbb{R})_+ \cap N(\mathbb{Q})$.

Let $\pi_0(H, K_H)$ be the set of geometric components of $\mathrm{Sh}_{K_H}(H, X_H)$. Recall ([6, 2.1.3.1]) that

$$\pi_0(H, K_H) = H(\mathbb{Q})_+ \backslash H(\mathbb{A}_f)/K_H = H(\mathbb{A}_f)/H(\mathbb{Q})_+ K_H.$$

Let $E_H := E(H, X_H)$ be the reflex field of (H, X_H) and $T_{E_H} := \mathrm{Res}_{E_H/\mathbb{Q}} \mathbb{G}_{mE_H}$.

Following Deligne ([6, 2.0.15.1]) we define for any reductive $\mathbb{Q}$-group N,

$$\pi(N) := N(\mathbb{A})/N(\mathbb{Q})\rho(\widetilde{N}(\mathbb{A})).$$

Here $\rho\colon \widetilde{N} \longrightarrow N^{\mathrm{der}}$ denotes the universal covering of the derived group N^{der} of N. The group $\pi_0(\pi(N))$ is an abelian group (in fact $\pi(N)$ is an abelian group; see 1.6.6 of [14]) with a natural action of the abelian group $\pi_0(N(\mathbb{R})_+)$. Let

$$\overline{\pi_0}(\pi(N)) := \pi_0(\pi(N))/\pi_0(N(\mathbb{R})_+).$$

Then by [6, 2.1.3.2], we have

$$\pi_0(H, K_H) = \overline{\pi_0}(\pi(H))/K_H.$$

The action of $\mathrm{Gal}(\overline{\mathbb{Q}}/E_H)$ on $\pi_0(H, K_H)$ is given by the reciprocity morphism ([6, 2.6.1.1])

$$r_{(H,X_H)}\colon \mathrm{Gal}(\overline{\mathbb{Q}}/E_H) \longrightarrow \overline{\pi_0}(\pi(H)).$$

The morphism $r_{(H,X_H)}$ factors through $\mathrm{Gal}(\overline{\mathbb{Q}}/E_H)^{\mathrm{ab}}$, which is identified via the global class field theory with $\pi_0(\pi(T_{E_H}))$.

Let C be the torus H/H^{der}. To (H, X_H) one associates two Shimura data $(C, \{x\})$ and $(H^{\mathrm{ad}}, X_{H^{\mathrm{ad}}})$. The field E_H is the composite of $E(C, \{x\})$ and $E(H^{\mathrm{ad}}, X_{H^{\mathrm{ad}}})$ by Proposition 3.8 of [5]. There are morphisms of Shimura data,

$$\theta^{\mathrm{ab}}\colon (H, X_H) \longrightarrow (C, \{x\}) \text{ and } \theta^{\mathrm{ad}}\colon (H, X_H) \longrightarrow (H^{\mathrm{ad}}, X_{H^{\mathrm{ad}}}). \tag{1}$$

Let $r_{(C,\{x\})}$ be the reciprocity morphism associated with $(C, \{x\})$. Then $r_{(C,\{x\})}$ is induced from a morphism of algebraic tori $r_C : T_{E(C,\{x\})} \longrightarrow C$. Let F be a finite extension of $E(C, \{x\})$. The composite of the norm from T_F to $T_{E(C,\{x\})}$ with r_C is a surjective morphism of algebraic tori that we still denote $r_C\colon T_F \longrightarrow C$. This morphism induces a reciprocity morphism from $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ to $\overline{\pi_0}\pi(C)$ that we still denote $r_{(C,\{x\})}$. Notice that as E_H contains $E(C, \{x\})$, we have a morphism $r_C\colon T_{E_H} \longrightarrow C$.

It is convenient and sometimes essential to make the assumption that H is the generic Mumford-Tate group on X_H. Below we prove a lemma that will allow us to make this assumption. Let X^+ be a connected component of X. Then $G(\mathbb{Q})_+$ is the stabiliser of X^+ in $G(\mathbb{Q})$ (see [13, Prop. 5.7.b]). Let $\Gamma := G(\mathbb{Q})_+ \cap K$ and S be the component $\Gamma\backslash X^+$ of $\mathrm{Sh}_K(G,X)$. Note that S is the image of $X^+ \times \{1\}$ in $\mathrm{Sh}_K(G,X)$.

LEMMA 2.1. *Let V be a special subvariety of S. There exists a Shimura subdatum (H,X_H) of (G,X) such that H is the generic Mumford-Tate group on X_H and V is the image of a connected component of* $\mathrm{Sh}_{K\cap H(\mathbb{A}_f)}(H,X_H)$ *in* $\mathrm{Sh}_K(G,X)$ *by the natural map induced by the inclusion $(H,X_H) \subset (G,X)$ (We emphasise here that no Hecke correspondence is involved.)*

There exists a connected component X_H^+ of X_H contained in X^+ such that V is the image of $X_H^+ \times \{1\}$ in $\mathrm{Sh}_K(G,X)$.

Proof. Let $v \in V \subset S$ be a Hodge generic point of V and $x \in X^+$ mapping to v. Let H be the Mumford-Tate group of x, $X_H := H(\mathbb{R}).x$ and $X_H^+ := H(\mathbb{R})^+.x$. Then (H,X_H) is a Shimura subdatum of (G,X). (See, for example, [18, Lemme 3.3].) Therefore the image V' of $X_H^+ \times \{1\}$ in $\mathrm{Sh}_K(G,X)$ is a special subvariety containing v. As v is Hodge generic in V, it follows that V is the smallest special subvariety of $\mathrm{Sh}_K(G,X)$ containing v. Therefore $V \subset V'$. As V and V' are irreducible and have the same dimension $\dim(X_H^+)$, we have $V = V'$. $\square$

In view of this lemma, for the rest of this section, we only consider Shimura subdata $(H,X_H) \subset (G,X)$ such that H is the generic Mumford-Tate group on X_H. In particular, we will assume that G is the generic Mumford-Tate group on X.

LEMMA 2.2. *Suppose that the centre $Z(G)(\mathbb{R})$ is compact. Let (H,X_H) and K_H be as above, with H being the generic Mumford-Tate group on X_H. Let $f\colon \mathrm{Sh}_{K_H}(H,X_H) \longrightarrow \mathrm{Sh}_K(G,X)$ be the morphism induced by the inclusion (H,X_H) into (G,X).*

The restrictions of

$$f\colon \mathrm{Sh}_{K_H}(H,X_H) \longrightarrow f(\mathrm{Sh}_{K_H}(H,X_H))$$

to the irreducible components of $\mathrm{Sh}_{K_H}(H,X_H)$ are generically finite. Moreover the degrees of the restrictions of f to the irreducible components of $\mathrm{Sh}_{K_H}(H,X_H)$ are uniformly bounded when (H,X_H) varies. Furthermore, if K is neat, then f is generically injective and the restriction of f to the Hodge generic locus is injective. See [3, §17.1] *for a definition of a neat subgroup of $G(\mathbb{Q})$ and* [15, §0.6] *for the definition of a neat compact open subgroup of $G(\mathbb{A}_f)$. In particular, the number of irreducible components of $\mathrm{Sh}_{K_H}(H,X_H)$ is, up to a constant independent of (H,X_H), bounded by the number of irreducible components of its image in $\mathrm{Sh}_K(G,X)$.*

Remark 2.3. The assumption that $Z(G)(\mathbb{R})$ is compact implies that the stabiliser in $G(\mathbb{R})$ of any point $x \in X$ is compact. As a consequence, for any Shimura subdatum (H, X_H) of (G, X), the centre $Z(H)(\mathbb{R})$ is compact. This is, in particular, the case when G is semisimple of adjoint type.

Indeed, let $x \in X$. By the general theory of symmetric spaces, the stabiliser of x in $G(\mathbb{R})$ is compact modulo $Z(G)(\mathbb{R})$. By assumption, $Z(G)(\mathbb{R})$ is compact, and therefore the stabiliser of x in $G(\mathbb{R})$ is compact.

Let (H, X_H) be a Shimura subdatum of (G, X). Let $x_H \in X_H$. Then $Z(H)(\mathbb{R})$ is contained in the stabiliser of x_H in $G(\mathbb{R})$. Hence $Z(H)(\mathbb{R})$ is compact.

Proof. First note that it suffices to prove that the morphism f is generically injective when K is neat. Indeed, any compact open subgroup K of $G(\mathbb{A}_f)$ contains a neat compact open subgroup K' of finite index (see [15, §0.6]). Using the generic injectivity of $\mathrm{Sh}_{K'_H}(H, X_H) \longrightarrow \mathrm{Sh}_{K'}(G, X)$, one easily sees that the degrees of the restrictions of f to the irreducible components of $\mathrm{Sh}_{K_H}(H, X_H)$ are bounded by the index of K' in K.

Suppose K is neat. Let $\overline{(x_1, h_1)}$ and $\overline{(x_2, h_2)}$ be two points of $\mathrm{Sh}_{K_H}(H, X_H)$ having the same image by f. As we are proving injectivity on the Hodge generic locus, we assume that $\mathrm{MT}(x_1) = \mathrm{MT}(x_2) = H$.

There exist an element q of $G(\mathbb{Q})$ and an element k of K such that $x_2 = qx_1$ and $h_2 = qh_1k$.

The fact that $\mathrm{MT}(x_1) = \mathrm{MT}(x_2) = H$ implies that q belongs to the normaliser $N_G(H)(\mathbb{Q})$ of H in G. Therefore k belongs to $N_G(H)(\mathbb{A}_f) \cap K$. Let us check that the group $N_G(H)^0$ is reductive. There is an element x of X that factors through $N_G(H)_{\mathbb{R}}$. Then $x(\mathbb{S})$ normalises the unipotent radical R_u of $N_G(H)$; hence $\mathrm{Lie}(R_u)$ is a rational polarisable Hodge structure and the Killing form is nondegenerate on $\mathrm{Lie}(R_u)$. It follows that R_u is reductive and therefore is trivial.

We claim that the group $G' := N_G(H)/H$ has the property that $G'(\mathbb{R})$ is compact. Indeed as $N_G(H)^0$ is reductive, $N_G(H)^0$ is an almost direct product in G of the form $N_G(H)^0 = HL$ with L reductive. We will show that $L(\mathbb{R})$ is compact. It is enough to prove that $L(\mathbb{R})^+$ is compact. We claim that $L(\mathbb{R})^+$ and $H(\mathbb{R})^+$ commute. Consider the commutator map from $L(\mathbb{R}) \times H(\mathbb{R})$ to $L(\mathbb{R}) \cap H(\mathbb{R})$. The image of $L(\mathbb{R})^+ \times H(\mathbb{R})^+$ is a connected subgroup of $L(\mathbb{R}) \cap H(\mathbb{R})$. Furthermore the intersection $L(\mathbb{R}) \cap H(\mathbb{R})$ is finite. It follows that the image of $L(\mathbb{R})^+ \cap H(\mathbb{R})^+$ by the commutator map is trivial and therefore $L(\mathbb{R})^+$ and $H(\mathbb{R})^+$ commute.

Let $x \in X_H$. We view x as a morphism from $\mathbb{S}$ to $H_{\mathbb{R}}$. Then $x(\mathbb{S}) \subset H(\mathbb{R})^+$, and hence $x(\mathbb{S})$ is fixed by conjugation by $L(\mathbb{R})^+$. It follows that $L(\mathbb{R})^+$ stabilises any point of X_H. By Remark 2.3, $L(\mathbb{R})^+$ and hence $L(\mathbb{R})$ is compact. As the image of $L(\mathbb{R})$ in $G'(\mathbb{R})$ is of finite index in $G'(\mathbb{R})$, the group $G'(\mathbb{R})$ is compact.

The equality $h_2 = qh_1k$ shows that q belongs to $H(\mathbb{A}_f)\cdot(N_G(H)(\mathbb{A}_f)\cap K)$. Indeed, as q is in $N_G(H)(\mathbb{A}_f)$, there is an $h' \in H(\mathbb{A}_f)$ such that

$$qh_1 = h'q,$$

and we get $q = h'^{-1}h_2k^{-1} \in H(\mathbb{A}_f)\cdot(N_G(H)(\mathbb{A}_f)\cap K)$.

It follows that the image $\overline{q}$ of q in $G'(\mathbb{Q})$ is contained in the image K' of $N_G(H)(\mathbb{A}_f)\cap K$, which is a compact subgroup of $G'(\mathbb{A}_f)$. Therefore $\overline{q} \in \Gamma' := G'(\mathbb{Q})\cap K'$ and as $G'(\mathbb{R})$ is compact, the group Γ' is finite. As K is neat, K' is neat by [3, Cor. 17.3, p. 118] and Γ' is trivial. It follows that q belongs to $H(\mathbb{Q})$ and k to $K_H := H(\mathbb{A}_f)\cap K$. We conclude that the points $\overline{(x_1,h_1)}$ and $\overline{(x_2,h_2)}$ of $\mathrm{Sh}_{K_H}(H,X_H)$ are equal. This finishes the proof. □

Recall that T is the connected centre of H and C is H/H^{der}. Note that there is an isogeny $\alpha:\ T \longrightarrow C$ with kernel $T\cap H^{\mathrm{der}}$, given by the restriction of the quotient map $H \longrightarrow H/H^{\mathrm{der}}$ to T. We will make use of the following lemma.

LEMMA 2.4. *The order of the group $T\cap H^{\mathrm{der}}$ is uniformly bounded as (H,X_H) ranges through the Shimura subdata of (G,X). Let $\rho : \tilde{H} \to H^{\mathrm{der}}$ be the universal covering map. Then the degree of ρ is uniformly bounded as well.*

Proof. As $T\cap H^{\mathrm{der}}$ is contained in the centre of H^{der}, we just need a uniform bound on orders of the centres of the universal coverings of connected semisimple subgroups of G. Let L be a connected semisimple subgroup of G, and let D_L be the Dynkin diagram of $L_{\mathbb{C}}$. As the rank of $L_{\mathbb{C}}$ is bounded by the rank of $G_{\mathbb{C}}$, there are only finitely many possibilities for D_L. For each of these possibilities, the order of the centre of the universal covering of $L_{\mathbb{C}}$ is bounded by the index of the lattice of roots in the lattice of weights. □

We recall that we have fixed a faithful representation V of G. Let $\rho_T : T \hookrightarrow \mathrm{GL}(V)$ be the restriction of the representation $G \subset \mathrm{GL}(V)$ to T. We now prove some uniformity results regarding the characters occurring in ρ_T and the reciprocity morphism $r_C : T_{E_H} \to C$.

LEMMA 2.5. *There is a constant R_0 such that for any sub-Shimura datum (H,X_H), the degree of the reflex field $E(H,X_H)$ over $E(G,X)$ is bounded by R_0.*

Proof. By Remark 12.3(a) of [13], $E(H,X_H)$ is contained in any splitting field of H. The degree of any such splitting field is bounded in terms of the dimension of G only. Indeed, let T be a maximal torus in H. The dimension d of T is bounded in terms of the dimension of G. The degree of the splitting field is the size of the image of the representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the character group $X^*(T)$ of T. This is a finite subgroup of $\mathrm{GL}_d(\mathbb{Z})$, and its size is bounded in terms of d only. (See, for example, [8].) $\square$

Fix a positive integer $R \geq R_0$. For any Shimura subdatum (H, X_H) of (G, X), we let F be a finite extension of $\mathbb{Q}$ of degree bounded by R containing the reflex field $E(H, X_H)$. We assume that such a choice of R is made in the rest of the text.

Moreover assume in this section only that F is a Galois extension of $\mathbb{Q}$. By our assumption, there are only finitely many possibilities for the isomorphism class of $\mathrm{Gal}(F/\mathbb{Q})$. For the purposes of the present paper we can take F to be equal to the Galois closure of $E(H, X_H)$. However, we introduce extra flexibility on the field F for some applications in [11].

We may thus assume that $\mathrm{Gal}(F/\mathbb{Q})$ is isomorphic to a fixed abstract group Δ. Let T_F be the torus $\mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_{m,F}$. We write $H = T \cdot H^{\mathrm{der}}$, and we let $\mu\colon \mathbb{G}_{m,\mathbb{C}} \longrightarrow H_{\mathbb{C}}$ be the cocharacter $h_{\mathbb{C}}(z, 1)$ where h is an element of X_H such that $\mathrm{MT}(h) = H$. We write

$$V_{\mathbb{C}} = \oplus V_{\mathbb{C}}^{p,q} \tag{2}$$

for the Hodge decomposition of $V_{\mathbb{C}}$ induced by h.

The composition of μ with $H_{\mathbb{C}} \longrightarrow C_{\mathbb{C}}$ gives a cocharacter $\mathbb{G}_{m,\mathbb{C}} \longrightarrow C_{\mathbb{C}}$, which we denote by μ_C. The cocharacter μ_C is defined over F. Each σ in Δ defines a character χ_σ and a cocharacter μ_σ of the torus T_F. Moreover $X^*(T_F) = \oplus_{\sigma\in\Delta}\mathbb{Z}\chi_\sigma$ and $X_*(T_F) = \oplus_{\sigma\in\Delta}\mathbb{Z}\mu_\sigma$. In this way, we get a "canonical" basis for the character (respectively cocharacter) group of the torus T_F. There is a natural pairing

$$\langle\, ,\rangle\colon X^*(T_F) \times X_*(T_F) \longrightarrow \mathbb{Z}$$

defined by $\langle \chi_\sigma, \mu_\tau\rangle = \delta_{\sigma,\tau}$ for all σ, τ in Δ. As F contains $E(C, \{x\})$, we have a reciprocity morphism $r_C\colon T_F \longrightarrow C$. The morphism $r_C\colon T_F \longrightarrow C$ induces the morphism $r_{C*}\colon X_*(T_F) \to X_*(C)$ that sends the cocharacter μ_σ to $\sigma(\mu_C)$ and an injection $X^*(C) \subset X^*(T_F)$. We identify $X^*(C)$ with its image in $X^*(T_F)$.

By Lemma 2.4, the isogeny $\alpha\colon T \longrightarrow C$ has uniformly bounded degree, say m. Therefore there is a unique surjective morphism of algebraic tori $r\colon T_F \longrightarrow T$ such that

$$\alpha \circ r = r_C^m.$$

The morphism r identifies $X^*(T)$ with a submodule of $X^*(T_F)$. We will consider the coordinates of the characters in $X^*(T)$ with respect to the basis of $X^*(T_F)$ described previously.

LEMMA 2.6. *With respect to the chosen basis of $X^*(T_F)$ and the identification of $X^*(C)$ with a submodule of $X^*(T_F)$, there is a finite subset of $X^*(C)$ generating $X^*(C) \otimes \mathbb{Q}$ whose coordinates are bounded uniformly on (H, X_H).*

The coordinates of the characters χ of T intervening in the representation $\rho_T : T \hookrightarrow \mathrm{GL}(V)$, with respect to the basis of $X^*(T_F)$ described above, are bounded uniformly on (H, X_H).

The size of the torsion of $X^*(T_F)/X^*(T)$ is bounded uniformly on (H, X_H).

Proof. As the isogeny $T \longrightarrow C$ has order m, the representation ρ_T^m induces a representation (V, ρ_C) of C. The Shimura datum $(C, \{x\})$ as before induces a Hodge structure $V(\rho_C)$ on V by composing x with ρ_C. Let

$$V_{\mathbb{C}}(\rho_C) = \oplus_{(p,q)} V_{\mathbb{C}}(\rho_C)^{p,q} \tag{3}$$

be the associated Hodge decomposition. Let $\{\chi'_i\} \in X^*(C)$ be the set of characters that intervene in the representation ρ_C. As ρ_T is faithful, the $\{\chi'_i\}$ generate $X^*(C) \otimes \mathbb{Q}$. We will show that the coordinates of the χ'_i in the chosen basis of $X^*(T_F)$ are uniformly bounded.

These coordinates are the

$$\langle \chi'_i, r_{C*}(\mu_\sigma)\rangle = \langle \chi'_i, \sigma(\mu_C)\rangle = \langle \sigma^{-1}(\chi'_i), \mu_C\rangle$$

(where $\langle\, , \,\rangle : X^*(C) \times X_*(C) \longrightarrow \mathbb{Z}$ is the canonical pairing). These quantities are the integers p appearing in the Hodge decomposition (3) given by composing x with ρ_C. We just need to show that these weights are uniformly bounded on $(H^{\mathrm{der}}, X_{H^{\mathrm{der}}})$. This will be deduced by comparing the p's appearing in the Hodge decomposition $V(\rho_C)$ with the ones of V given in equation (2).

As the characters $\{\chi_i\} \in X^*(T)$ occuring in ρ_T are such that $\chi'_i = \chi_i^m$ with m uniformly bounded, the result for the coordinates of χ_i in the chosen basis of $X^*(T_F)$ is a consequence of the corresponding result for the χ'_i. The statement concerning the size of the torsion of $X^*(T_F)/X^*(T)$ is a direct consequence of the result on the coordinates of the χ_i.

Let $T_{H^{\mathrm{der}}}$ be a maximal torus of $H^{\mathrm{der}}_{\mathbb{C}}$ such that μ factors through $T_{\mathbb{C}} \cdot T_{H^{\mathrm{der}}}$. Let $\widetilde{T_{\mathbb{C}}}$ be the almost direct product $T_{\mathbb{C}} \cdot T_{H^{\mathrm{der}}}$. The torus $\widetilde{T_{\mathbb{C}}}$ is a maximal torus of $H_{\mathbb{C}}$.

Let $\mathcal{R}$ be the root system associated to $(T_{H^{\mathrm{der}}}, H^{\mathrm{der}}_{\mathbb{C}})$. There is only a finite, uniformly bounded number of possibilities for $\mathcal{R}$. The representation of H on V induces a representation of H^{der}. The dimensions of the irreducible factors of this representation are uniformly bounded; hence there is only a finite (uniformly bounded) number of characters of $T_{H^{\mathrm{der}}}$ that intervene in the representation.

As $T \cap H^{\mathrm{der}}$ is finite, we have a direct sum decomposition

$$X^*(\widetilde{T_{\mathbb{C}}})_{\mathbb{Q}} = X^*(T_{\mathbb{C}})_{\mathbb{Q}} \oplus X^*(T_{H^{\mathrm{der}}})_{\mathbb{Q}}$$

and a similar decomposition for $X_*(\widetilde{T_{\mathbb{C}}})_{\mathbb{Q}}$.

Let χ be a character of $\widetilde{T}_{\mathbb{C}}$ that intervenes in the representation $V_{\mathbb{C}}$ of $\widetilde{T}_{\mathbb{C}}$. The direct sum decompositions above give the decompositions $\chi = \chi_T + \chi_{H^{\mathrm{der}}}$ and $\mu = \mu_T + \mu_{H^{\mathrm{der}}}$.

The values taken by the $\langle \chi, \mu \rangle$ are the p such that $V_{\mathbb{C}}^{p,q}$ is nonzero in the Hodge decomposition (2). Hence they are finite in number and uniformly bounded. On the other hand, we have

$$\langle \chi, \mu \rangle = \langle \chi_T, \mu_T \rangle + \langle \chi_{H^{\mathrm{der}}}, \mu_{H^{\mathrm{der}}} \rangle,$$

where χ_T and $\chi_{H^{\mathrm{der}}}$ are the restrictions of χ to T and $T_{H^{\mathrm{der}}}$ respectively. In the decomposition

$$\mu = \mu_T + \mu_{H^{\mathrm{der}}},$$

there is only a finite number of possibilities for $\mu_{H^{\mathrm{der}}}$. This is a consequence of the theory of symmetric spaces. To see this, we decompose the root system $\mathcal{R}$ into irreducible factors $\mathcal{R}_i$. The components of the μ on $\mathcal{R}_i$ are either trivial or correspond to minuscule weights of the dual root system $\mathcal{R}_i^{\vee}$.

It follows that $\langle \chi_{H^{\mathrm{der}}}, \mu_{H^{\mathrm{der}}} \rangle$ takes only finitely many values and so does $\langle \chi_T, \mu_T \rangle$. As m is uniformly bounded, the $\langle \chi_T^m, \mu_T \rangle$ are uniformly bounded. This finishes the proof as the $\langle \chi_T^m, \mu_T \rangle$ are the p's appearing in the Hodge decomposition (3). □

Finally, for later use we prove a certain number of uniformity results concerning the reciprocity morphism. We keep the previous notation: (H, X_H) is a Shimura subdatum of (G, X), $H = T \cdot H^{\mathrm{der}}$, $C = H/H^{\mathrm{der}}$ and F, as before, is a finite Galois extension of $\mathbb{Q}$ containing the Galois closure of the reflex field $E(H, X_H)$ of (H, X_H), of degree over $\mathbb{Q}$ bounded by some constant R depending on (G, X) only.

The reciprocity morphism

$$r_{(C,\{x\})} : \mathrm{Gal}(\overline{\mathbb{Q}}/F)^{\mathrm{ab}} \simeq \pi_0\pi(T_F) \to \overline{\pi_0}(\pi(C))$$

factors through $\pi_0(\pi(C))$, and

$$r_{(H,X_H)} : \mathrm{Gal}(\overline{\mathbb{Q}}/F)^{\mathrm{ab}} \simeq \pi_0\pi(T_F) \to \overline{\pi_0}(\pi(H))$$

factors through $\pi_0(\pi(H))$. We will also write $r_{(C,\{x\})}$ and $r_{(H,X_H)}$ for the induced maps from $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ or $\mathrm{Gal}(\overline{\mathbb{Q}}/F)^{\mathrm{ab}}$ to $\pi_0(\pi(C))$ and $\pi_0(\pi(H))$ respectively. The map $r_C : T_F \to C$ induces a map

$$r_{C,\mathbb{A}/\mathbb{Q}} : \pi(T_F) \to \pi(C),$$

and $r_{(C,\{x\})}$ is obtained from $r_{C,\mathbb{A}/\mathbb{Q}}$ by applying the functor π_0. In view of [6, 2.5.3], the map $r_{(H,X_H)}$ is also obtained by applying the functor π_0 to a map

$$r_{H,\mathbb{A}/\mathbb{Q}} : \pi(T_F) \to \pi(H).$$

The projection $H \to C$ will be denoted by p, so $p = \theta^{\mathrm{ab}}$ in the notation of Section 2.1, equation (1).

If $\alpha : G_1 \to G_2$ is a morphism of reductive $\mathbb{Q}$-groups, we write $\alpha_l : G_1(\mathbb{Q}_l) \to G_2(\mathbb{Q}_l), \alpha_\infty : G_1(\mathbb{R}) \to G_2(\mathbb{R})$ and $\alpha_{\mathbb{A}} : G_1(\mathbb{A}) \to G_2(\mathbb{A})$ for the associated morphisms at the level of $\mathbb{Q}_l$-points, real points and adelic points respectively. The map $p : H \to C$ induces maps

$$p_{\mathbb{A}/\mathbb{Q}} : \pi(H) \to \pi(C),$$
$$\pi_0(p_{\mathbb{A}/\mathbb{Q}}) : \pi_0(\pi(H)) \to \pi_0(\pi(C))$$

and

$$\overline{\pi_0}(p_{\mathbb{A}/\mathbb{Q}}) : \overline{\pi_0}(\pi(H)) \to \overline{\pi_0}(\pi(C)).$$

Finally, for any reductive $\mathbb{Q}$-group G_1 and any $g \in G_1(\mathbb{A})$, we write $\overline{g}$ for the image of g in $\pi(G_1)$ and $\pi_0(\overline{g})$ (resp. $\overline{\pi_0}(\overline{g})$) for the image of g in $\pi_0(\pi(G_1))$ (resp. $\overline{\pi_0}(\pi(G_1))$).

LEMMA 2.7. *There is an integer n_1 such that for any sub-Shimura datum (H, X) of (G, X), the following holds*:

(a) *For any prime number l and for any $m \in T(\mathbb{Q}_l)$, $m^{n_1} \in p_l^{-1}(r_{C,l}(T_F(\mathbb{Q}_l)))$. For any $m \in T(\mathbb{R})$, $m^{n_1} \in p_\infty^{-1}(r_{C,\infty}(T_F(\mathbb{R})))$.*

(b) *Let us fix some models of T_F, H, T and C over $\mathbb{Z}$. Then for any l big enough* (*depending on (H, X_H) and the choice of the models over $\mathbb{Z}$*) *and any $m \in T(\mathbb{Z}_l)$, $m^{n_1} \in p_l^{-1}(r_{C,l}(T_F(\mathbb{Z}_l)))$.*

(c) *For any $m \in T(\mathbb{A})$, $m^{n_1} \in p_{\mathbb{A}}^{-1}(r_{C,\mathbb{A}}(T_F(\mathbb{A})))$ and the class $\overline{m^{n_1}}$ of m^{n_1} in $\pi(H)$ is in $p_{\mathbb{A}/\mathbb{Q}}^{-1}(r_{C,\mathbb{A}/\mathbb{Q}}(\pi(T_F)))$.*

Proof. The element x gives a cocharacter $\mu_{\mathbb{C}} \colon \mathbb{G}_{m\mathbb{C}} \longrightarrow C_{\mathbb{C}}$ defined by $\mu_{\mathbb{C}}(z) = x_{\mathbb{C}}(z, 1)$. The morphism $r_C \colon T_F \longrightarrow C$ corresponds to the morphism on cocharacter groups $X_*(T_F) \longrightarrow X_*(C)$ that sends the cocharacter $\mu_\sigma \in X_*(T_F)$ (induced by $\sigma \in \mathrm{Hom}(F, \overline{\mathbb{Q}})$) to $\sigma(\mu_{\mathbb{C}})$. The Lemma 2.6 says that there is a basis (χ_i) of characters of C such that the $\langle \chi_i, \sigma(\mu_{\mathbb{C}}) \rangle$ are uniformly bounded. We first verify that there is an integer n bounded independently of (H, X_H) and a morphism $\Phi : C \to T_F$ such that $f := r_C \circ \Phi$ is the $n[F : \mathbb{Q}]$-th power homomorphism from C to C.

As before, we identify the character group $X^*(C)$ with a sub-$\mathbb{Z}$-module of $X^*(T_F)$ via $X^*(r_C)$. Using Lemma 2.6 we see there exist a basis $\psi_1, \ldots, \psi_t$ of $X^*(T_F)$ and integers $d_1, \ldots, d_u$ bounded independently of (H, X_H), such that $d_1\psi_1, \ldots, d_u\psi_u$ is a basis of $X^*(C) \subset X^*(T_F)$. Let $n := \prod d_i$. This is an integer bounded independently of (H, X_H). Then the morphism $X^*(T_F) \longrightarrow X^*(C)$ sending ψ_i to $n\psi_i$ for $1 \leq i \leq u$ and ψ_i to 0 for $i > u$ corresponds to a morphism $s \colon C \longrightarrow T_F$ defined over $\overline{\mathbb{Q}}$. The morphism $r_C \circ s$ sends $x \in C(\overline{\mathbb{Q}})$ to x^n. Let $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. The morphism $s^\sigma \colon C \longrightarrow T_F$ is defined at the level of character groups by

$$X^*(s^\sigma)(\psi_i) = X^*(s)^\sigma(\psi_i^{\sigma^{-1}}).$$

We claim that at the level of $\overline{\mathbb{Q}}$-points, $r_C \circ s^\sigma$ is again $x \mapsto x^n$. Let $x \in C(\overline{\mathbb{Q}})$. We have

$$\begin{aligned} r_C(s^\sigma(x)) &= r_C(\sigma(s(\sigma^{-1}x))) \\ &= \sigma(r_C \circ s(\sigma^{-1}(x))) \\ &= \sigma((\sigma^{-1}x)^n) = x^n. \end{aligned}$$

We now define,

$$\Phi = \prod_{\sigma \in \mathrm{Gal}(F/\mathbb{Q})} s^\sigma.$$

What precedes shows that $f(x) = (r_C \circ \Phi)(x) = x^{[F:\mathbb{Q}]n}$ for $x \in C(\overline{\mathbb{Q}})$, and furthermore Φ is a morphism of $\mathbb{Q}$-tori, therefore f is the morphism $x \mapsto x^{[F:\mathbb{Q}]n}$ on C.

To simplify notation we now replace n by $n[F:\mathbb{Q}]$. Note that n is uniformly bounded as $[F:\mathbb{Q}]$ is bounded by R. It follows that $r_{C,l}(T_F(\mathbb{Q}_l))$ contains $U_n := f_l(C(\mathbb{Q}_l))$. The kernel of $f_l \otimes \overline{\mathbb{Q}_l}$ is killed by n. Writing down the corresponding Galois cohomology sequence, we see that $C(\mathbb{Q}_l)/f_l(C(\mathbb{Q}_l))$ is killed by n. Therefore, $C(\mathbb{Q}_l)/r_{C,l}(T_F(\mathbb{Q}_l))$ is also killed by n.

At the level of real points, notice that the map f_∞ induces a surjective morphism $C(\mathbb{R})^+ \longrightarrow C(\mathbb{R})^+$, where $C(\mathbb{R})^+$ is the neutral component of $C(\mathbb{R})$. By [20, 10.1],

$$C(\mathbb{R}) = (\mathbb{R}^*)^a \times (\mathbb{C}^*)^b \times \mathrm{SO}(2)(\mathbb{R})^c, \tag{4}$$

where a, b, c are some integers. It follows that $\pi_0(C(\mathbb{R})) = C(\mathbb{R})/C(\mathbb{R})^+$ is killed by two. (Notice that $\mathbb{C}^*$ and $\mathrm{SO}(2)(\mathbb{R})$ are connected.) For later use, notice that $|\pi_0(C(\mathbb{R}))|$ is bounded by 2^a (hence uniformly). Let n' be the maximum of all the possible integers n as above. Let $n_1 = \max(2, n')!$. Then n_1 satisfies the conditions of (a).

For (b), let $\theta \in C(\mathbb{Z}_l)$. Then $\theta^{n_1} = r_{C,l} s_l(\theta)$. For any l large enough, $T_F(\mathbb{Z}_l)$ is the maximal compact open subgroup of $T_F(\mathbb{Q}_l)$. As $s_l(C(\mathbb{Z}_l)) \subset T_F(\mathbb{Q}_l)$ is compact, for any l large enough, $s_l(C(\mathbb{Z}_l)) \subset T_F(\mathbb{Z}_l)$. Therefore, for l large enough, $s_l(\theta) \in T_F(\mathbb{Z}_l)$ and $\theta^{n_1} \in r_{C,l}(T_F(\mathbb{Z}_l))$.

Part (c) is a direct consequence of (a) and (b). □

From this point and for the rest of the paper we make the assumption that $Z(G)(\mathbb{R})$ is compact which, in particular, implies that $C(\mathbb{R})$ is compact by Remark 2.3.

LEMMA 2.8. *There exists an integer n_0 such that for any Shimura subdatum (H, X_H) of (G, X), any element of the kernel of*

$$\overline{\pi_0}(p_{\mathbb{A}/\mathbb{Q}}) : \overline{\pi_0}(\pi(H)) \to \overline{\pi_0}(\pi(C))$$

is killed by n_0.

Proof. Let $y \in H(\mathbb{A})$ such that $\overline{\pi_0}(\overline{y})$ is in the kernel of $\overline{\pi_0}(p_{\mathbb{A}/\mathbb{Q}})$. As $\pi_0(C(\mathbb{R}))$ is of uniformly bounded order (by the description of $C(\mathbb{R})$ given by equation (4)), we may assume that $\pi_0(\overline{y})$ is in the kernel of

$$\pi_0(p_{\mathbb{A}/\mathbb{Q}}) : \pi_0(\pi(H)) \to \pi_0(\pi(C)).$$

Recall that $T \cap H^{\mathrm{der}}$ is finite of uniformly bounded order by Lemma 2.4. Let M be a uniform bound on this order, and let $n_2 := M!$.

There exist an element t in $T(\mathbb{A})$ and h in $H^{\mathrm{der}}(\mathbb{A})$ such that

$$y^{n_2} = t \cdot h.$$

By Lemma 2.4 the group $H^{\mathrm{der}}(\mathbb{A})/\rho\widetilde{H}(\mathbb{A})$ is killed by a uniformly bounded integer. Let M' a uniform bound for this integer and $n_3 := M'!$. Then $\overline{y}^{n_2 n_3}$ and $\overline{t}^{n_3}$ coincide as elements of $\pi(H)$ and $\pi_0(\overline{y}^{n_2 n_3})$ and $\pi_0(\overline{t}^{n_3})$ coincide as elements of $\pi_0(\pi(H))$.

By the result of Deligne [6, 2.2.3],

$$\pi_0(\pi(T)) = \pi_0(T(\mathbb{R})) \times T(\mathbb{A}_f)/T(\mathbb{Q})^-,$$

where $T(\mathbb{Q})^-$ is the closure of $T(\mathbb{Q})$ in $T(\mathbb{A}_f)$ for the adelic topology . As $T(\mathbb{R})$ is compact (by Remark 2.3), $T(\mathbb{Q})$ is discrete in $T(\mathbb{A}_f)$ (see [13, Th. 5.26]). Therefore

$$\pi_0(\pi(T)) = \pi_0(T(\mathbb{R})) \times T(\mathbb{A}_f)/T(\mathbb{Q}).$$

As $C(\mathbb{R})$ is also compact, in the same way we have

$$\pi_0(\pi(C)) = \pi_0(C(\mathbb{R})) \times C(\mathbb{A}_f)/C(\mathbb{Q}).$$

As a consequence we obtain $\overline{\pi_0}(\pi(C)) = C(\mathbb{A}_f)/C(\mathbb{Q})$.

Consider the exact sequence

$$1 \longrightarrow W \longrightarrow T \xrightarrow{\alpha} C \longrightarrow 1,$$

where $W = T \cap H^{\mathrm{der}}$. Notice that the order of W divides n_2. We recall that the restriction of p to T is denoted α.

As $\pi_0(\overline{y})$ (and hence $\pi_0(\overline{y}^{n_2 n_3})$) is in the kernel of $\pi_0(p_{\mathbb{A}/\mathbb{Q}})$, we have

$$p_{\mathbb{A}}(t^{n_3}) = \alpha_{\mathbb{A}}(t^{n_3}) = c\underline{c}_\infty$$

with $c \in C(\mathbb{Q})$, and $\underline{c}_\infty = (c_\infty, 1)$ is an element of $C(\mathbb{A})$ with all finite components trivial and with the component at infinity $c_\infty \in C(\mathbb{R})^+$.

As α_∞ induces a surjective map from $T(\mathbb{R})^+$ to $C(\mathbb{R})^+$, there exists $\theta_\infty \in T(\mathbb{R})^+$ such that $\alpha_\infty(\theta_\infty) = c_\infty$. Let $\underline{\theta}_\infty$ be the element $(\theta_\infty, 1)$ of $T(\mathbb{A})$. Then $\alpha_{\mathbb{A}}(\underline{\theta}_\infty) = \underline{c}_\infty$.

An n_2-th power of any element of $C(\mathbb{Q})$ is in the image of $T(\mathbb{Q})$. Hence there exists a q in $T(\mathbb{Q})$ such that

$$\alpha_{\mathbb{A}}(t^{n_3 n_2}) = \alpha_{\mathbb{A}}(q)\alpha_{\mathbb{A}}(\underline{\theta}_\infty^{n_2}).$$

It follows that

$$t^{n_3 n_2} = qw\underline{\theta}_\infty^{n_2},$$

where w is in $W(\mathbb{A})$. As $W(\mathbb{A})$ is killed by n_2, we see that $t^{n_3 n_2^2} = q^{n_2}\underline{\theta}_\infty^{n_2^2}$. We deduce that the class $\pi_0(\bar{t}^{n_3 n_2^2})$ of $t^{n_3 n_2^2}$ in $\pi_0(\pi(H))$ is trivial.

We have

$$\pi_0(\bar{t})^{n_3 n_2^2} = \pi_0(\overline{y})^{n_3 n_2^3},$$

and therefore a uniform power of y has trivial image in $\pi_0(\pi(H))$. $\square$

We can now prove the following.

Proposition 2.9. *There is an integer A such that for any (H, X_H) and F as above and for any $m \in T(\mathbb{A})$, the class $\overline{\pi_0}(\overline{m}^A)$ of m^A in $\overline{\pi_0}(\pi(H))$ is in $r_{(H,X_H)}(\pi_0(\pi(T_F))) = r_{(H,X_H)}(\mathrm{Gal}(\overline{\mathbb{Q}}/F))$.*

Proof. By Lemma 2.7(c), we have $p_{\mathbb{A}/\mathbb{Q}}(\overline{m}^{n_1}) \in r_{C,\mathbb{A}/\mathbb{Q}}(\pi(T_F))$. Hence, there is an element σ of $T_F(\mathbb{A})$ such that

$$p_{\mathbb{A}/\mathbb{Q}}(\overline{m}^{n_1}) = r_{C,\mathbb{A}/\mathbb{Q}}(\overline{\sigma}).$$

Applying the functor $\overline{\pi_0}$, we get

$$\begin{aligned}\overline{\pi_0}(p_{\mathbb{A}/\mathbb{Q}})(\overline{\pi_0}(\overline{m}^{n_1})) = \overline{\pi_0}(r_{C,\mathbb{A}/\mathbb{Q}}(\overline{\sigma})) &= r_{(C,\{x\})}(\pi_0(\overline{\sigma})) \\ &= \overline{\pi_0}(p_{\mathbb{A}/\mathbb{Q}})(r_{(H,X_H)}(\pi_0(\overline{\sigma}))).\end{aligned}$$

The last equality is the natural functoriality of the reciprocity morphisms. As we have not been able to find a reference for this statement, we briefly explain the proof. Note that $\pi_0(\mathrm{Sh}(H, X_H))$ is a principal homogeneous space under $\overline{\pi}_0(\pi(H))$ and $\pi_0(\mathrm{Sh}(C, \{x\}))$ is a principal homogeneous space under $\overline{\pi}_0(\pi(C))$ (see [5, 2.1.16]). Let

$$\mathrm{Sh}_p : \mathrm{Sh}(H, X_H) \to \mathrm{Sh}(C, \{x\})$$

be the morphism of Shimura varieties induced by p. Let x_0 (resp. y_0) be some base points of $\pi_0(\mathrm{Sh}(H, X_H))$ — (resp. $\pi_0(\mathrm{Sh}(C, \{x\}))$) such that $y_0 = \pi_0(\mathrm{Sh}_p)(x_0)$. Then for any $\alpha \in \overline{\pi}_0(\pi(H))$, we have

$$\pi_0(\mathrm{Sh}_p)(\alpha.x_0) = \overline{\pi}_0(p_{\mathbb{A}/\mathbb{Q}})(\alpha).y_0.$$

By the theory of canonical models of Shimura varieties, the morphism of Shimura varieties Sh_p is defined over F. Therefore for any $\theta \in \mathrm{Gal}(\overline{\mathbb{Q}}/F)$,

$$\pi_0(\mathrm{Sh}_p)(x_0^\theta) = y_0^\theta.$$

As $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ acts on $\pi_0(\mathrm{Sh}(H, X_H))$ (resp. $\pi_0(\mathrm{Sh}(C, \{x\}))$) via $r_{(H,X_H)}$ (resp. $r_{(C,\{x\})}$), we have $x_0^\theta = r_{(H,X_H)}(\theta).x_0$ and $y_0^\theta = r_{(C,\{x\})}(\theta).y_0$. Therefore

$$r_{(C,\{x\})}(\theta).y_0 = \overline{\pi}_0(p_{\mathbb{A}/\mathbb{Q}})(r_{(H,X_H)}(\theta)).y_0,$$

which proves the claim.

It follows that there exists an element $y \in H(\mathbb{A})$ such that $\overline{\pi_0}(\overline{y})$ is in the kernel of $\overline{\pi_0}(p_{\mathbb{A}/\mathbb{Q}}) : \overline{\pi_0}(\pi(H)) \to \overline{\pi_0}(\pi(C))$ and such that

$$\overline{\pi_0}(\overline{m}^{n_1}) = \overline{\pi_0}(\overline{y}) r_{(H,X_H)}(\pi_0(\overline{\sigma})).$$

Let $A = n_1 n_0$ with n_0 the integer given by Lemma 2.8. By Lemma 2.8,

$$\overline{\pi_0}(\overline{m}^A) = r_{(H,X_H)}(\pi_0(\overline{\sigma}^{n_0})). \qquad \square$$

2.2. *Lower bounds for degrees of Galois orbits.* In this section we consider a Shimura datum (G, X) with G semisimple of adjoint type and we let K be a compact open subgroup of $G(\mathbb{A}_f)$. We also fix a faithful rational representation of G. We deal with the problem of bounding (below) the degree of Galois orbits of geometric components of subvarieties of $\mathrm{Sh}_K(G, X)$ defined over $\overline{\mathbb{Q}}$. We assume that $K \subset G(\mathbb{A}_f)$ is of the form $K = \prod_p K_p$ for some compact open subgroups K_p of $G(\mathbb{Q}_p)$.

Recall that we have fixed a faithful representation of G that allows us to view G as a closed subgroup of some GL_n. We may and do assume that K is contained in $\mathrm{GL}_n(\widehat{\mathbb{Z}})$. Let $\mathbf{K}_3$ be the principal congruence subgroup of level 3 of $\mathrm{GL}_n(\mathbb{Z}_3)$. We assume that K_3 is contained in $\mathbf{K}_3$. Hence K_3 is neat and K is neat (see [11, §4.1.5] and [15, §0.6]). All subvarieties are assumed to be closed.

Let M be a projective variety over $\mathbb{C}$, Y be an irreducible subvariety of M and $\mathcal{L}$ be an ample line bundle on M. Then $\deg_{\mathcal{L}}(Y)$ is the degree of Y computed with respect to $\mathcal{L}$. Let $c_1(\mathcal{L})$ be the first Chern class of $\mathcal{L}$. If Y is irreducible of dimension d, then $\deg_{\mathcal{L}}(Y)$ is the intersection number $c_1(\mathcal{L})^d.Y$ (see [9, Ch. 12, p. 211]). When Y is reducible, the degree of Y is defined to be the sum of the degrees of its irreducible components.

The Baily-Borel compactification of $\mathrm{Sh}_K(G, X)$ is denoted $\overline{\mathrm{Sh}_K(G, X)}$. Let $\mathcal{L}_K = \mathcal{L}_K(G, X)$ be the ample line bundle on $\overline{\mathrm{Sh}_K(G, X)}$ extending the line bundle of holomorphic differential forms of maximal degree on $\mathrm{Sh}_K(G, X)$. We say that $\mathcal{L}_K$ is the Baily-Borel line bundle on $\overline{\mathrm{Sh}_K(G, X)}$. We will also use the notation $\mathcal{L}_K$ for the Baily-Borel line bundle on the Baily-Borel compactification of $\mathrm{Sh}_K(G, X)$ even in the case when G is not of adjoint type (for example, for a sub-Shimura datum (H, X_H) of (G, X)).

Let Y be a subvariety of $\overline{\mathrm{Sh}_K(G, X)}$. We write $\deg(Y) = \deg_{\mathcal{L}_K}(Y)$ for the degree of Y computed with respect to the Baily-Borel line bundle. Let Z be a subvariety of $\mathrm{Sh}_K(G, X)$ and $\overline{Z}$ be its Zariski closure in $\overline{\mathrm{Sh}_K(G, X)}$. We will write $\deg(Z)$ for $\deg(\overline{Z})$.

Definition 2.10. Let Y be a geometrically irreducible subvariety of the variety $\mathrm{Sh}_K(G, X)$ defined over $\overline{\mathbb{Q}}$. Let F be a number field containing $E(G, X)$. We define the degree of the Galois orbit of Y, denoted $\deg(\mathrm{Gal}(\overline{\mathbb{Q}}/F) \cdot Y)$, to be the degree of the subvariety $\mathrm{Gal}(\overline{\mathbb{Q}}/F)\cdot Y$ of $\mathrm{Sh}_K(G,X)$ calculated with respect to the line bundle $\mathcal{L}_K$.

Let (H, X_H) be a Shimura subdatum of (G,X) such that H is the generic Mumford-Tate group on X_H. Let $K_H = K\cap H(\mathbb{A}_f)$. Let Y be as above, and suppose that Y is the image in $\mathrm{Sh}_K(G,X)$ of a geometrically irreducible subvariety Y_1 of $\mathrm{Sh}_{K_H}(H,X_H)$. Suppose that F contains $E(H,X_H)$. We define the *internal degree* of the Galois orbit of Y to be the degree of $\mathrm{Gal}(\overline{\mathbb{Q}}/F)\cdot Y_1$ calculated with respect to $\mathcal{L}_{K_H}$.

Note that when H is a torus (and hence Y is a special point), the degree $\deg(\mathrm{Gal}(\overline{\mathbb{Q}}/F)\cdot Y)$ is simply the number of conjugates of Y under $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$.

Let V be a geometric component of $\mathrm{Sh}_{K_H}(H,X_H)$. We will use the same notation for V and its image in $\mathrm{Sh}_K(G,X)$. This is justified in view of Lemma 2.2. We recall that T denotes the connected centre of H. Let $K_T := K\cap T(\mathbb{A}_f)$, and let K_T^m be the maximal compact open subgroup of $T(\mathbb{A}_f)$. We consider the compact open subgroup $K_H^m := K_T^m K_H$ of $H(\mathbb{A}_f)$. The group K_H is a normal subgroup of K_H^m and $K_H^m/K_H = K_T^m/K_T$. Note that as both K_H and K_T^m are products of compact open subgroups of $H(\mathbb{Q}_p)$ and $T(\mathbb{Q}_p)$ respectively, the group K_H^m is a product of compact open subgroups $K_{H,p}^m$ of $H(\mathbb{Q}_p)$.

We can find a neat compact open normal subgroup of $K_{H,3}^m$ of uniformly bounded index. Indeed $K_{H,3}^m$ is contained in a maximal compact open subgroup of $\mathrm{GL}_n(\mathbb{Q}_3)$. The maximal compact open subgroups of $\mathrm{GL}_n(\mathbb{Q}_3)$ are of the form $\alpha\mathrm{GL}_n(\mathbb{Z}_3)\alpha^{-1}$ for some $\alpha\in\mathrm{GL}_n(\mathbb{Q}_3)$. Fix $\alpha\in\mathrm{GL}_n(\mathbb{Q}_3)$ such that $K_{H,3}^m\subset\alpha\mathrm{GL}_n(\mathbb{Z}_3)\alpha^{-1}$. Then $K_{H,3}^{m'} := K_{H,3}^m\cap\alpha\mathbf{K}_3\alpha^{-1}$ is a neat compact open subgroup of $K_{H,3}^m$ of uniformly bounded index. One can check that this index is in fact bounded by $|\mathrm{GL}_n(\mathbb{F}_3)|$.

Let $K_H^{m'} := K_{H,3}^{m'}\times\prod_{p\neq 3}K_{H,p}^m$. Then $K_H^{m'}$ is a neat compact open normal subgroup of $H(\mathbb{A}_f)$ of uniformly bounded index in K_H^m. Let $K_H' := K_H^{m'}\cap K_H$. Note that K_H' is normal in K_H and $|K_H/K_H'|\leq|K_H^m/K_H^{m'}|$, and therefore both $|K_H^m/K_H^{m'}|$ and $|K_H/K_H'|$ are bounded by a uniform constant that we call a.

Lemma 2.11. *The morphism*

$$\pi'\colon \mathrm{Sh}_{K_H'}(H,X_H)\longrightarrow \mathrm{Sh}_{K_H^{m'}}(H,X_H)$$

is finite étale of degree $|K_H^{m'}/K_H'|$.

Proof. Let $\overline{(x,g)}$ be a point of $\mathrm{Sh}_{K_H^{m'}}(H,X_H)$. The preimage of $\overline{(x,g)}$ is $\overline{(x,gK_H^{m'})}$ in $\mathrm{Sh}_{K_H'}(H,X_H)$. Suppose

$$\overline{(x,g)}=\overline{(x,gk)},$$

with $k \in K_H^{m'}$. There exist q in $H(\mathbb{Q})$ and $k' \in K'_H$ such that $qx = x$ and $g = qgkk'$. The first condition implies that q is in a compact subgroup of $H(\mathbb{R})$, and the second condition implies that q is in the neat compact open subgroup $gK_H^{m'}g^{-1}$ of $H(\mathbb{A}_f)$. These two conditions imply that q is trivial. (Recall that $K_H^{m'}$ is neat.) Therefore $k = (k')^{-1} \in K'_H$. The preimage of $\overline{(x,g)}$ in $\mathrm{Sh}_{K'_H}(H, X_H)$ has a simply transitive action by $K_H^{m'}/K'_H$. Therefore π' is finite étale of degree $|K_H^{m'}/K'_H|$. $\square$

Let f be the morphism $\mathrm{Sh}_{K'_H}(H, X_H) \longrightarrow \mathrm{Sh}_{K_H}(H, X_H)$. As K_H is neat, the same proof as the proof of the previous Lemma 2.11 shows that f is finite étale of degree $|K_H/K'_H| \leq a$. As f is an étale map, we have an isomorphism $f^*(\mathcal{L}_{K_H}) \cong \mathcal{L}_{K'_H}$. (See Proposition 5.3.2 of [11] for a more precise statement.)

Let Y be a geometrically irreducible subvariety of V defined over $\overline{\mathbb{Q}}$. Let Y' be a geometrically irreducible component of $f^{-1}(Y)$ and V' be the geometrically irreducible component of $f^{-1}(V)$ containing Y'. The projection formula implies

$$\deg_{\mathcal{L}_{K_H}}(Y) \geq \frac{1}{a}\deg_{\mathcal{L}_{K'_H}}(Y').$$

As f is defined over F, we see that $f(\mathrm{Gal}(\overline{\mathbb{Q}}/F) \cdot Y') = \mathrm{Gal}(\overline{\mathbb{Q}}/F) \cdot Y$. By the projection formula applied to the subvariety $\mathrm{Gal}(\overline{\mathbb{Q}}/F) \cdot Y$ of $\mathrm{Sh}_{K_H}(H, X_H)$, we get

$$\deg_{\mathcal{L}_{K_H}}(\mathrm{Gal}(\overline{\mathbb{Q}}/F) \cdot Y) \geq \frac{1}{a}\deg_{\mathcal{L}_{K'_H}}(\mathrm{Gal}(\overline{\mathbb{Q}}/F) \cdot Y'). \tag{5}$$

For the purposes of giving a lower bound for $\deg_{\mathcal{L}_{K_H}}(\mathrm{Gal}(\overline{\mathbb{Q}}/F) \cdot Y)$, it is thus enough to give a lower bound for $\deg_{\mathcal{L}_{K'_H}}(\mathrm{Gal}(\overline{\mathbb{Q}}/F) \cdot Y')$.

Let us consider the following compact open subgroup of $T(\mathbb{A}_f)$:

$$K_T^{m'} := K_T^m \cap K_H^{m'}.$$

We have $K_T^m/K_T^{m'} \subset K_H^m/K_H^{m'}$, and therefore $|K_T^m/K_T^{m'}|$ is bounded by a.

Let $K'_T := K_T^{m'} \cap K_H \subset T(\mathbb{A}_f) \cap K = K_T$. Notice that

$$K'_T = K_T^{m'} \cap K_H \cap K_T = K_T^{m'} \cap K_T.$$

Hence $K_T^{m'}/K'_T$ is a subgroup of K_T^m/K_T. An application of the snake lemma shows that the index of $K_T^{m'}/K'_T$ in K_T^m/K_T is bounded by $|K_T^m/K_T^{m'}|$, therefore by a.

The next lemma splits the degree of the Galois orbit of Y' into two pieces, which we will estimate separately.

LEMMA 2.12. *As previously, let Y' be a geometrically irreducible subvariety of V' defined over $\overline{\mathbb{Q}}$ such that $f(Y') = Y$. The degree of the Galois* *orbit* $\mathrm{Gal}(\overline{\mathbb{Q}}/F)\cdot Y'$ *calculated with respect to* $\mathcal{L}_{K'_H}$ *is at least the degree of* $\mathrm{Gal}(\overline{\mathbb{Q}}/F)\cdot Y' \cap {\pi'}^{-1}\pi'(Y')$ *times the number of* $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$*-conjugates of* $\pi'(Y')$.

Proof. We need to check that the degree of $\mathrm{Gal}(\overline{\mathbb{Q}}/F)\cdot Y' \cap {\pi'}^{-1}(\sigma(\pi'(Y'))$ with $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/F)$ is independent of σ. As the line bundle $\mathcal{L}_{K'_H}$ is defined over F, for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/F)$, we have $\sigma^*\mathcal{L}_{K'_H} = \mathcal{L}_{K'_H}$. It follows that for any subvariety Z of $\mathrm{Sh}_{K'_H}(H, X_H)$ defined over $\overline{\mathbb{Q}}$ and any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/F)$, we have $\deg_{\mathcal{L}_{K'_H}}(Z) = \deg_{\mathcal{L}_{K'_H}}(\sigma Z)$. Applying this to

$$Z = \mathrm{Gal}(\overline{\mathbb{Q}}/F)\cdot Y' \cap {\pi'}^{-1}\pi'(Y')$$

and noticing that

$$\mathrm{Gal}(\overline{\mathbb{Q}}/F)\cdot Y' \cap {\pi'}^{-1}(\sigma\pi'(Y')) = \sigma(\mathrm{Gal}(\overline{\mathbb{Q}}/F)\cdot Y' \cap {\pi'}^{-1}\pi'(Y')),$$

we get that for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/F)$,

$$\deg_{\mathcal{L}_{K'_H}}(\mathrm{Gal}(\overline{\mathbb{Q}}/F)\cdot Y' \cap {\pi'}^{-1}\pi'(Y')) = \deg_{\mathcal{L}_{K'_H}}(\mathrm{Gal}(\overline{\mathbb{Q}}/F)\cdot Y' \cap {\pi'}^{-1}(\sigma(\pi'(Y'))).$$

□

We first deal with the second piece. From now on we assume, as in the previous section, that F is a finite extension of $\mathbb{Q}$ containing $E(H, X_H)$ of degree over $\mathbb{Q}$ bounded by R. We assume, moreover, that F contains the Galois closure of $E(H, X_H)$. This will be a harmless assumption in view of the kind of lower bounds for the degrees of the Galois orbits we are aiming to prove.

Let K_C^m be the maximal open compact subgroup of $C(\mathbb{A}_f)$. The number of components of the Galois orbit of $\pi'(V')$ is at least the size of the image of $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ in $\overline{\pi_0}(\pi(H))/K_H^m$ by $r_{(H,X_H)}$. By the proof of Proposition 2.9, $r_{(C,\{x\})} = \overline{\pi_0}(p_{\mathbb{A}/\mathbb{Q}}) \circ r_{(H,X_H)}$. Therefore the size of this image is at least the size of the image of $r_{(C,x)}((F \otimes \mathbb{A}_f)^*)$ in $\overline{\pi_0}(\pi(C))/K_C^m = C(\mathbb{Q})\backslash C(\mathbb{A}_f)/K_C^m$.

By Lemma 2.6, $X^*(T)$ has a set of generators $(\chi_1, \dots, \chi_d)$ such that the coordinates of the χ_i in the canonical basis $(\chi_\sigma)_{\sigma: F \to \mathbb{C}}$ of $X^*(T_F)$ are uniformly bounded. By Lemma 2.6, $X^*(C)$ has a set of generators $(\chi'_1, \dots, \chi'_{d'})$ such that the coordinates of the χ'_i in the canonical basis of $X^*(T_F)$ are uniformly bounded. As $(C, \{x\})$ is a Shimura datum of CM type such that the weight homomorphism is trivial (as G is of adjoint type), we see that for all $i \in \{1, \dots, d'\}$, $\chi'_i\overline{\chi'_i}$ is the trivial character. We are therefore in the situation of Theorem 2.13 of [21]. We get the following.

Proposition 2.13. *Assume the* GRH *for* CM *fields. Let N be a positive integer. Let L_C be the splitting field of C. The size of the image of* $r_{(C,\{x\})}((\mathbb{A}_f \otimes L_C)^*)$ *in* $C(\mathbb{Q})\backslash C(\mathbb{A}_f)/K_C^m$ *is at least a constant depending on N only times* $(\log|\mathrm{disc}(L_C)|)^N$.

We claim that L_C is the Galois closure E^c of $E = E(C, \{x\})$. By definition of the reflex field, E is contained in L_C. As L_C is a Galois extension, E^c is contained in L_C. Conversely, notice that the reciprocity morphism $r_C \colon \mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_{m,E} \longrightarrow C$ is surjective. This is a consequence of the fact that H is the generic Mumford-Tate group on X_H. This implies that L_C is contained in the splitting field of $\mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_{m,E}$ that is E^c. As $E(H, X_H)$ is the composite of E and $E(H^{\mathrm{ad}}, X_{H^{\mathrm{ad}}})$, the Galois closure of $E(H, X_H)$ contains L_C. We obtain the following consequence of Proposition 2.13.

PROPOSITION 2.14. *Assume the* GRH *for* CM *fields. Let N be a positive integer. Let L_C be the splitting field of C. The number of geometrically irreducible components of* $\mathrm{Gal}(\overline{\mathbb{Q}}/F) \cdot \pi'(V')$ *is at least a positive constant c_N depending on N and the degree of F over $\mathbb{Q}$ only, times* $(\log|\mathrm{disc}(L_C)|)^N$.

If Y' is a geometrically irreducible $\overline{\mathbb{Q}}$-subvariety of V', then the same lower bound holds for the number of components of $\mathrm{Gal}(\overline{\mathbb{Q}}/F) \cdot \pi'(Y')$.

The assertion regarding the subvariety Y' is a consequence of the fact that, as the conjugates of $\pi'(V')$ are disjoint (they are components of $\mathrm{Sh}_{K_H^{m'}}(H, X_H)$), the subvariety $\pi'(Y')$ has at least as many conjugates as $\pi'(V')$.

Now we deal with the first "piece": estimating the degree of the Galois orbit in the fibre over $\pi'(V')$.

The compact open subgroup K'_T is a product of compact open subgroups $K'_{T,p}$ of $T(\mathbb{Q}_p)$. Similarly, $K_T^{m'}$ is a product of compact open subgroups $K_{T,p}^{m'}$ of $T(\mathbb{Q}_p)$.

For an integer $e \geq 1$, we define Θ'_e as the image of the morphism $x \mapsto x^e$ on $K_T^{m'}/K'_T$ and Θ_e as the image of the morphism $x \mapsto x^e$ on K_T^m/K_T. We let $\pi \colon \mathrm{Sh}_{K_H}(H, X_H) \longrightarrow \mathrm{Sh}_{K_H^m}(H, X_H)$ be the natural map of Shimura varieties. Note that K_T^m/K_T acts transitively on the fibres of π.

For a scheme Z over some base field, $\mathrm{Irr}(Z)$ will denote the set of geometrically irreducible components of Z. The cardinality of a finite set Θ will be written $|\Theta|$. Hence $|\mathrm{Irr}(Z)|$ stands for the number of geometrically irreducible components of Z.

We prove the following key proposition.

LEMMA 2.15. *Let A be the integer given by Proposition* 2.9 *and a be the constant as in the beginning of the section. We have*

$$\Theta_A \cdot V \subset \mathrm{Gal}(\overline{\mathbb{Q}}/F) \cdot V \cap \pi^{-1}\pi(V)$$

and

$$\Theta'_A \cdot V' \subset \mathrm{Gal}(\overline{\mathbb{Q}}/F) \cdot V' \cap {\pi'}^{-1}\pi'(V').$$

Furthermore, we have

$$|\mathrm{Irr}(\mathrm{Gal}(\overline{\mathbb{Q}}/F) \cdot V' \cap {\pi'}^{-1}\pi'(V'))| \geq \frac{|\Theta'_A|}{|K_H^{m'}/K'_H|} |\mathrm{Irr}({\pi'}^{-1}\pi'(V'))|.$$

Proof. Recall that by the discussion at the beginning of Section 2.1,

$$\pi_0(\mathrm{Sh}_{K'_H}(H, X_H)) = \overline{\pi_0}(\pi(H))/K'_H = H(\mathbb{Q})_+\backslash H(\mathbb{A}_f)/K'_H.$$

This is a finite abelian group.

A class $\overline{\alpha}$ of $\alpha \in H(\mathbb{A}_f)$ in $H(\mathbb{Q})_+\backslash H(\mathbb{A}_f)/K'_H$ corresponds to the component $\overline{X_H^+ \times \{\alpha\}}$ that is the image of $X_H^+ \times \{\alpha\}$ in $\mathrm{Sh}_{K'_H}(H, X_H)$. The action of $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ on $\pi_0(\mathrm{Sh}_{K'_H}(H, X_H))$ is as follows. By slight abuse of notation, we denote by $r_{(H,X_H)}$ the composite of $r_{(H,X_H)}\colon \mathrm{Gal}(\overline{\mathbb{Q}}/F) \longrightarrow \overline{\pi_0}(\pi(H))$ with quotient by K'_H. Let $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/F)$, and let $t \in H(\mathbb{A}_f)$ such that $\bar{t}$ is $r_{(H,X_H)}(\sigma)$. Then for any $\alpha \in H(\mathbb{A}_f)$,

$$\sigma(\overline{X_H^+ \times \{\alpha\}}) = (\overline{X_H^+ \times \{t\alpha\}}) = (\overline{X_H^+ \times \{\alpha t\}}).$$

Let $m \in K_T^{m'}$. Then the image of m^A in $\pi_0(\mathrm{Sh}_{K'_H}(H, X_H))$ is $r_{(H,X_H)}(\sigma)$ for some $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/F)$. It follows that the image of Θ'_A in $H(\mathbb{Q})_+\backslash H(\mathbb{A}_f)/K'_H$ is contained in the image of $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$. Moreover $K_H^{m'}/K'_H$ acts transitively on $\mathrm{Irr}(\pi'^{-1}\pi'(V'))$. For $\overline{X_H^+ \times \{\alpha\}} \in \mathrm{Irr}(\pi'^{-1}\pi'(V'))$ and $k \in K_H^{m'}/K'_H$, this action is given by

$$(\overline{X_H^+ \times \{\alpha\}}).k = (\overline{X_H^+ \times \{\alpha k\}}).$$

Recall that $K_T^{m'}/K'_T$ is a subgroup of $K_H^{m'}/K'_H$. Consequently

$$\Theta'_A \cdot V' \subset \mathrm{Gal}(\overline{\mathbb{Q}}/F) \cdot V' \cap \pi'^{-1}\pi'(V').$$

Exactly the same proof with K_T^m instead of $K_T^{m'}$ shows that

$$\Theta_A \cdot V \subset \mathrm{Gal}(\overline{\mathbb{Q}}/F) \cdot V \cap \pi^{-1}\pi(V).$$

We now prove the second claim. The fact that $\Theta'_A \cdot V' \subset \mathrm{Gal}(\overline{\mathbb{Q}}/F) \cdot V' \cap \pi'^{-1}\pi'(V')$ implies that the number of Galois conjugates of V' contained in one fibre is at least the size of the orbit of V under the action of Θ'_A.

We have

$$|\mathrm{Irr}(\pi'^{-1}\pi'(V'))| = |\mathrm{Irr}((K_H^{m'}/K'_H) \cdot V')| \leq \frac{|K_H^{m'}/K'_H|}{|\Theta'_A|}|\mathrm{Irr}(\Theta'_A \cdot V')|$$

and

$$|\mathrm{Irr}(\mathrm{Gal}(\overline{\mathbb{Q}}/F) \cdot V' \cap \pi'^{-1}\pi'(V'))| \geq |\mathrm{Irr}(\Theta'_A \cdot V')|.$$

These inequalities yield the desired inequality. □

Lemma 2.16. *Let $e \geq 1$ be an integer. Recall that Y' denotes a geometrically irreducible subvariety of V' defined over $\overline{\mathbb{Q}}$ such that $f(Y') = Y$. Suppose that $\Theta'_{Ae} \cdot Y'$ is contained in*

$$\mathrm{Gal}(\overline{\mathbb{Q}}/F) \cdot Y' \cap \pi'^{-1}\pi'(Y').$$

In this situation, we have

$$|\mathrm{Irr}(\mathrm{Gal}(\overline{\mathbb{Q}}/F)\cdot Y' \cap \pi'^{-1}\pi'(Y'))| \geq \frac{|\Theta'_{Ae}|}{|K_H^{m'}/K'_H|}|\mathrm{Irr}(\pi'^{-1}\pi'(Y'))|.$$

Let $\pi\colon \mathrm{Sh}_{K_H}(H,X_H) \longrightarrow \mathrm{Sh}_{K_H^m}(H,X_H)$ *be the natural morphism. Suppose that* $\Theta_A \cdot Y$ *is contained in*

$$\mathrm{Gal}(\overline{\mathbb{Q}}/F)\cdot Y \cap \pi^{-1}\pi(Y).$$

Then $\Theta'_{Aa!}\cdot Y'$ *is contained in*

$$\mathrm{Gal}(\overline{\mathbb{Q}}/F)\cdot Y' \cap \pi'^{-1}\pi'(Y').$$

Proof. The first statement follows from the proof of Lemma 2.15. As for the second statement, let $\theta \in K_T^{m'}$. There exists a $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/F)$ such that

$$\theta^A \cdot Y = \sigma Y.$$

The group $K'_H = K_H^{m'} \cap K_H$ is a normal subgroup of K_H^m as both $K_H^{m'}$ and K_H are normal subgroups of K_H^m. It follows that there is a natural action of K_H^m on $\mathrm{Sh}_{K'_H}(H,X_H)$.

It follows, as the map f is $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ and K_H^m-equivariant, that

$$f(\theta^A\cdot Y') = f(\sigma Y').$$

Hence there exists $\alpha \in K_H$ such that

$$\alpha\cdot\theta^A\cdot Y' = \sigma Y'.$$

As K_T^m and K_H commute and the action of K_H^m on $\mathrm{Sh}_{K'_H}(H,X_H)$ is Galois-equivariant, we have

$$\alpha^{a!}\cdot\theta^{Aa!}\cdot Y' = \sigma^{a!}Y'.$$

Note that $\alpha^{a!}\in K'_H$ as $|K_H/K'_H|\leq a$. Thus $\alpha^{a!}$ acts trivially on $\mathrm{Sh}_{K'_H}(H,X_H))$. The second claim follows. □

LEMMA 2.17. *Let* Y' *be a geometrically irreducible subvariety of* V'. *Then*

$$\deg_{\mathcal{L}_{K'_H}}(\pi'^{-1}\pi'(Y')) \geq |K_H^{m'}/K'_H|.$$

Proof. Let Z' be the fibre $\pi'^{-1}\pi'(Y')$. The morphism of Shimura varieties $\pi'\colon \mathrm{Sh}_{K'_H}(H,X_H) \longrightarrow \mathrm{Sh}_{K_H^{m'}}(H,X_H)$ extends to a proper morphism

$$\overline{\pi'}\colon \overline{\mathrm{Sh}_{K'_H}(H,X_H)} \longrightarrow \overline{\mathrm{Sh}_{K_H^{m'}}(H,X_H)},$$

which is generically finite of degree $|K_H^{m'}/K'_H|$ by Lemma 2.11. Furthermore $\pi'^*\mathcal{L}_{K_H^{m'}} \cong \mathcal{L}_{K'_H}$. The projection formula gives

$$\begin{aligned}\deg_{\mathcal{L}_{K'_H}}(Z') &= \deg_{\pi'^*\mathcal{L}_{K_H^{m'}}}(Z') = \deg_{\mathcal{L}_{K_H^{m'}}}(\pi'_*Z') \\ &= [K_H^{m'} : K'_H] \deg_{\mathcal{L}_{K_H^{m'}}}(\pi(Z')) \geq [K_H^{m'} : K'_H]. \qquad \square\end{aligned}$$

Lemma 2.18. *There is a uniform integer $r > 0$ such that for any integer $e \geq 1$,*

$$|\Theta'_{Ae}| \geq \frac{1}{a!^r} \prod_{\{p : K_{T,p}^m \neq K_{T,p}\}} \max(1, B|K_{T,p}^m/K_{T,p}|),$$

with $B = \frac{1}{(Ae)^r}$.

Proof. Let $A' = Ae$. Since K_T^m/K_T and $K_T^{m'}/K'_T$ are products of the $K_{T,p}^m/K_{T,p}$ and the $K_{T,p}^{m'}/K'_{T,p}$ respectively, the groups $\Theta_{A'}$ and $\Theta'_{A'}$ are products

$$\Theta_{A'} = \prod_{\{p : K_{T,p}^m \neq K_{T,p}\}} \Theta_{A',p} \quad \text{and} \quad \Theta'_{A'} = \prod_{\{p : K_{T,p}^{m'} \neq K'_{T,p}\}} \Theta'_{A',p}.$$

For all $p \neq 3$, $\Theta_{A',p} = \Theta'_{A',p}$. As $K_T^{m'}/K'_T$ is a subgroup of K_T^m/K_T of index at most a, we see that $\Theta'_{Ae,3}$ contains $\Theta_{Aea!,3}$. Hence $\Theta'_{A'}$ contains

$$\Theta_{A'a!,3} \cdot \prod_{\{p \neq 3 : K_{T,p}^m \neq K_{T,p}\}} \Theta_{A',p}.$$

Fix a $p \neq 3$ such that $K_{T,p}^m \neq K_{T,p}$. It is enough to prove that the order of the kernel of the A'-th power morphism on $K_{T,p}^m/K_{T,p}$ is bounded uniformly on T and p.

Let E be the splitting field of T. Notice that the degree of E over $\mathbb{Q}$ is bounded in terms of the dimension of T, hence uniformly on (H, X_H). (See the proof of Lemma 2.5.) Using a basis of the character group of T, one can embed T into a product of a finite and uniformly bounded number of tori $\mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_{m,E}$. Via this embedding, $K_{T,p}^m$ and $K_{T,p}$ are subgroups of the product of $(\mathbb{Z}_p \otimes O_E)^*$. The group $(\mathbb{Z}_p \otimes O_E)^*$ is the direct product of the groups of units of E_v, completion of E at the place v with $v|p$.

By the local unit theorem (cf. [12]), the group of units of such an E_v is a direct product of a cyclic group with $\mathbb{Z}_p^{[E_v : \mathbb{Q}_p]}$.

It follows that there exists a uniform constant r such that the group $K_{T,p}^m/K_{T,p}$ is a finite abelian group, product of at most r cyclic factors. It follows that the size of the kernel of the A'-th power map on $K_{T,p}^m/K_{T,p}$ is bounded by $D := A'^r$. We now take $B := \frac{1}{D}$.

The result is then obtained by using the same argument for $p = 3$ after having replaced A' by $A'a!$. $\square$

We now put the previous ingredients together to prove lower bounds for Galois degrees. Let Y be a geometrically irreducible subvariety of V defined over $\overline{\mathbb{Q}}$ such that $\Theta_A \cdot Y$ is contained in $\mathrm{Gal}(\overline{\mathbb{Q}}/F) \cdot Y \cap \pi^{-1}\pi(Y)$. Let Y' be as before. We use the notation used throughout this section: K'_H, $\mathcal{L}_{K'_H}$, π', L_T, L_C, A, a, etc.

Recall that we have inequality (5):

$$\deg_{\mathcal{L}_{K_H}}(\mathrm{Gal}(\overline{\mathbb{Q}}/F) \cdot Y) \geq \frac{1}{a} \deg_{\mathcal{L}_{K'_H}}(\mathrm{Gal}(\overline{\mathbb{Q}}/F) \cdot Y').$$

We will give a lower bound for the right-hand side. By Lemma 2.12, $\deg_{\mathcal{L}_{K'_H}}(\mathrm{Gal}(\overline{\mathbb{Q}}/F) \cdot Y')$ is at least $\deg_{\mathcal{L}_{K'_H}}(\mathrm{Gal}(\overline{\mathbb{Q}}/F) \cdot Y' \cap {\pi'}^{-1}\pi'(Y'))$ times the number of $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$-conjugates of $\pi'(Y')$.

Let N be a positive integer. By Proposition 2.14 (under the assumption of the GRH) there is a constant c_N depending on N and the degree of F only such that the number of $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$-conjugates of $\pi'(Y')$ is at least $c_N(\log|\mathrm{disc}(L_C)|)^N$.

We now give a lower bound for $\deg_{\mathcal{L}_{K'_H}}(\mathrm{Gal}(\overline{\mathbb{Q}}/F) \cdot Y' \cap {\pi'}^{-1}\pi'(Y'))$. In the proof of Lemma 2.12, we saw that

$$\deg_{\mathcal{L}_{K'_H}}(Y') = \deg_{\mathcal{L}_{K'_H}}(\sigma Y')$$

for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/F)$. It follows that

$$\begin{aligned}\deg_{\mathcal{L}_{K'_H}}&(\mathrm{Gal}(\overline{\mathbb{Q}}/F) \cdot Y' \cap {\pi'}^{-1}\pi'(Y')) \\ &= \deg_{\mathcal{L}_{K'_H}}(Y') \cdot |\mathrm{Irr}(\mathrm{Gal}(\overline{\mathbb{Q}}/F) \cdot Y' \cap {\pi'}^{-1}\pi'(Y'))|.\end{aligned}$$

By Lemma 2.16 applied with $e = a!$, we have

$$|\mathrm{Irr}(\mathrm{Gal}(\overline{\mathbb{Q}}/F) \cdot Y' \cap {\pi'}^{-1}\pi'(Y'))| \geq \frac{|\Theta'_{Aa!}|}{|K_H^{m'}/K'_H|} |\mathrm{Irr}({\pi'}^{-1}\pi'(Y'))|.$$

We have

$$\deg_{\mathcal{L}_{K'_H}}(Y')|\mathrm{Irr}({\pi'}^{-1}\pi'(Y'))| = \deg_{\mathcal{L}_{K'_H}}({\pi'}^{-1}\pi'(Y')).$$

Therefore, by Lemma 2.17, we have

$$\deg_{\mathcal{L}_{K'_H}}(Y')|\mathrm{Irr}({\pi'}^{-1}\pi'(Y'))| \geq |K_H^{m'}/K'_H|.$$

These inequalities show that

$$\deg_{\mathcal{L}_{K'_H}}(\mathrm{Gal}(\overline{\mathbb{Q}}/F) \cdot Y' \cap {\pi'}^{-1}\pi'(Y')) \geq |\Theta'_{Aa!}|.$$

Finally, Lemma 2.18 implies that

$$|\Theta'_{Aa!}| \geq \frac{1}{a!^r} \prod_{\{p : K_{T,p}^m \neq K_{T,p}\}} \max(1, B|K_{T,p}^m/K_{T,p}|),$$

with $B = \frac{1}{A^r(a!)^r}$. As a consequence we get

$$\deg_{\mathcal{L}_{K'_H}}(\mathrm{Gal}(\overline{\mathbb{Q}}/F)\cdot Y' \cap {\pi'}^{-1}\pi'(Y')) \geq \frac{1}{a!^r} \prod_{\{p : K^m_{T,p} \neq K_{T,p}\}} \max(1, B|K^m_{T,p}/K_{T,p}|).$$

We get

$$\begin{aligned}\deg_{\mathcal{L}_{K'_H}}&(\mathrm{Gal}(\overline{\mathbb{Q}}/F)\cdot Y')\\ &\geq c_N \frac{1}{a!^r} \prod_{\{p : K^m_{T,p} \neq K_{T,p}\}} \max(1, B|K^m_{T,p}/K_{T,p}|)(\log|\mathrm{disc}(L_C)|)^N.\end{aligned}$$

In view of Lemma 2.15, what precedes applies to $Y' = V'$. Therefore, we obtain

$$\begin{aligned}\deg_{\mathcal{L}_{K'_H}}&(\mathrm{Gal}(\overline{\mathbb{Q}}/F)\cdot V')\\ &\geq c_N \frac{1}{a!^r} \prod_{\{p : K^m_{T,p} \neq K_{T,p}\}} \max(1, B|K^m_{T,p}/K_{T,p}|)(\log|\mathrm{disc}(L_C)|)^N.\end{aligned}$$

Using inequality (5) we obtain

$$\begin{aligned}\deg_{\mathcal{L}_{K_H}}&(\mathrm{Gal}(\overline{\mathbb{Q}}/F)\cdot Y)\\ &\geq c_N \frac{1}{a\cdot a!^r} \prod_{\{p : K^m_{T,p} \neq K_{T,p}\}} \max(1, B|K^m_{T,p}/K_{T,p}|)(\log|\mathrm{disc}(L_C)|)^N\end{aligned}$$

and

$$\begin{aligned}\deg_{\mathcal{L}_{K_H}}&(\mathrm{Gal}(\overline{\mathbb{Q}}/F)\cdot V)\\ &\geq c_N \frac{1}{a\cdot a!^r} \prod_{\{p : K^m_{T,p} \neq K_{T,p}\}} \max(1, B|K^m_{T,p}/K_{T,p}|)(\log|\mathrm{disc}(L_C)|)^N.\end{aligned}$$

We obtain the following theorem.

THEOREM 2.19. *Assume the* GRH *for* CM *fields. Let (G, X) be a Shimura datum with G semisimple of adjoint type. Fix positive integers N and R. There exist a positive real number B depending only on G, X and R and a positive constant c_N depending only on G, X, R and N such that the following holds. Let K be a neat compact open subgroup of $G(\mathbb{A}_f)$ that is a product of compact open subgroups K_p of $G(\mathbb{Q}_p)$. Let (H, X_H) be a Shimura subdatum of (G, X) such that H is the generic Mumford-Tate group on X_H. Let F be a finite extension of $\mathbb{Q}$ containing the reflex field $E(H, X_H)$ of (H, X_H) of degree bounded by R. Let K_H be $H(\mathbb{A}_f) \cap K$.*

Let T be the connected centre of H. We suppose that T is nontrivial, and we define L_T as the splitting field of T . We recall that $K_T := T(\mathbb{A}_f) \cap K$, and K^m_T is the maximal open compact subgroup of $T(\mathbb{A}_f)$. Then $K_T = \prod K_{T,p}$ and

$K_T^m = \prod K_{T,p}^m$ with $K_{T,p} = T(\mathbb{Q}_p) \cap K_p$, and $K_{T,p}^m$ is the maximal open compact subgroup of $T(\mathbb{Q}_p)$.

Let V be a geometric component of $\mathrm{Sh}_{K_H}(H, X_H)$:

$$(6)\quad \deg_{\mathcal{L}_{K_H}}(\mathrm{Gal}(\overline{\mathbb{Q}}/F)\cdot V) \geq c_N \prod_{\{p : K_{T,p}^m \neq K_{T,p}\}} \max(1, B|K_{T,p}^m/K_{T,p}|)\cdot(\log(|\mathrm{disc}(L_T)|))^N.$$

There exists an integer A depending on G, X and R only such that the following holds. Let Θ_A be the image of the map $x \mapsto x^A$ on K_T^m/K_T. Then $\Theta_A \cdot V \subset \mathrm{Gal}(\overline{\mathbb{Q}}/F)\cdot V \cap \pi^{-1}\pi(V)$.

If Y is a geometrically irreducible subvariety of V defined over $\overline{\mathbb{Q}}$ such that $\Theta_A \cdot Y$ is contained in $\mathrm{Gal}(\overline{\mathbb{Q}}/F)\cdot Y \cap \pi^{-1}\pi(Y)$, the same holds for Y:

$$(7)\quad \deg_{\mathcal{L}_{K_H}}(\mathrm{Gal}(\overline{\mathbb{Q}}/F)\cdot Y) \geq c_N \prod_{\{p : K_{T,p}^m \neq K_{T,p}\}} \max(1, B|K_{T,p}^m/K_{T,p}|)\cdot(\log(|\mathrm{disc}(L_T)|))^N.$$

Remark 2.20. Note that the field L_T is equal to L_C because the tori C and T are isogeneous. In the proof of Theorem 2.19, for technical reasons, we assumed that F contains the Galois closure of $E(H, X_H)$. This assumption does not change the validity of the theorem. The constant c_N from Theorem 2.19 is a multiple of that of Proposition 2.14; namely, $\frac{1}{a\cdot a!^r} c_N$ with c_N as in 2.14.

It should be noted that an inequality such as (7) holds for the degree $\deg_{\mathcal{L}_K}(\mathrm{Gal}(\overline{\mathbb{Q}}/F)\cdot Y)$ for *Hodge generic* subvarieties Y of $\mathrm{Sh}_{K_H}(H, X_H)$. This is due to the fact that to obtain such an inequality, one applies Corollary 5.3.10 of [11], which only holds for Hodge generic subvarieties. In particular, this applies to a component V of $\mathrm{Sh}_{K_H}(H, X_H)$. In this paper we will use the following inequality:

$$(8)\quad \deg_{\mathcal{L}_K}(\mathrm{Gal}(\overline{\mathbb{Q}}/E(G,X))\cdot V) \geq c_N \prod_{\{p : K_{T,p}^m \neq K_{T,p}\}} \max(1, B|K_{T,p}^m/K_{T,p}|)\cdot(\log(|\mathrm{disc}(L_T)|))^N.$$

This formula is a consequence of (6) and Corollary 5.3.10 of [11].

Note also that if we take F to be the Galois closure of $E(H, X_H)$ in Theorem 2.19, the degree of F over $\mathbb{Q}$ is uniformly bounded when (H, X_H) varies by Lemma 2.5.

In the case where we consider subvarieties V such that the associated tori T lie in one $\mathrm{GL}_n(\mathbb{Q})$-conjugacy class with respect to some faithful representation $G \hookrightarrow \mathrm{GL}_n$, we do not need to assume the GRH. Indeed, in this case the field L_T is fixed and hence the term involving it is constant. We only used the GRH to obtain this term.

3. Special subvarieties whose degree of Galois orbits are bounded

3.1. *Equidistribution of T-special subvarieties.* Let (G, X) be a Shimura datum with G semisimple of adjoint type, and let K be an open compact subgroup of $G(\mathbb{A}_f)$. Let X^+ be a connected component of X. In this case (G is semisimple of adjoint type), the stabiliser of X^+ is the neutral component $G(\mathbb{R})^+$ of $G(\mathbb{R})$. We let $G(\mathbb{Q})^+ = G(\mathbb{R})^+ \cap G(\mathbb{Q})$. We remark that in this situation $G(\mathbb{Q})^+ = G(\mathbb{Q})_+$ with our previous notation. Let $\Gamma = G(\mathbb{Q})^+ \cap K$ and $S = \Gamma \backslash X^+$ be a fixed component of $\mathrm{Sh}_K(G, X)$. Note that S is the image of $X^+ \times \{1\}$ in $\mathrm{Sh}_K(G, X)$.

If $(H, X_H) \subset (G, X)$ is a Shimura subdatum, we denote by $\mathrm{MT}(X_H)$ the generic Mumford-Tate group on X_H. If $H' = \mathrm{MT}(X_H)$, then $H' \subset H$, $H'^{\mathrm{der}} = H^{\mathrm{der}}$ and $Z(H')^0 \subset Z(H)^0$. This is a consequence of the proof of [18, Lemma 4.1]. In the situation of loc. cit., Hodge groups are considered instead of Mumford-Tate groups, but the proof can be easily adapted. Fix $x \in X_H$. Let X_H^+ be the $H(\mathbb{R})^+$-conjugacy class of x and $X_{H'}^+$ be the $H'(\mathbb{R})^+$-conjugacy class of x. Note that as the centre of H (resp. H') acts trivially on x, X_H^+ (resp. $X_{H'}^+$) is also the $H^{\mathrm{der}}(\mathbb{R})^+$ (resp.$H'^{\mathrm{der}}(\mathbb{R})^+$) conjugacy class of x. As $H'^{\mathrm{der}}(\mathbb{R})^+ = H^{\mathrm{der}}(\mathbb{R})^+$, $X_{H'}^+ = X_H^+$ and $(H', X_{H'})$ is a Shimura subdatum of (H, X_H).

Definition 3.1. Let $T_{\mathbb{Q}}$ be a subtorus of G such that $T(\mathbb{R})$ is compact. A T-Shimura subdatum (H, X_H) of (G, X) is a Shimura subdatum such that T is the connected centre of the generic Mumford-Tate group $H' = T \cdot H^{\mathrm{der}}$ of X_H. Note that in this definition T may be trivial. In this case the generic Mumford-Tate group H' of X_H is semisimple.

Definition 3.2. A T-special subvariety of S is an irreducible component Z of the image of $\mathrm{Sh}_{K \cap H(\mathbb{A}_f)}(H, X_H)$ contained in S for a T-Shimura subdatum $(H, X_H) \subset (G, X)$. In this case, we say that Z is associated to (H, X_H). A T-special subvariety of S is standard if there exist a T-Shimura subdatum (H, X_H) of (G, X) and a connected component X_H^+ of X_H contained in X^+ such that Z is the image of $X_H^+ \times \{1\}$ in S. If Z is standard, then Z is the image of $(\Gamma \cap H(\mathbb{R})^+) \backslash X_H^+$ in S. We denote by Σ_T the set of T-special subvarieties of S.

LEMMA 3.3. *A standard T-special subvariety Z is associated to a Shimura subdatum (H, X_H) such that $H = \mathrm{MT}(X_H) = T \cdot H^{\mathrm{der}}$.*

Proof. If Z is associated to (H_1, X_{H_1}) and Z is standard, then Z is the image of $X_{H_1}^+ \times \{1\}$ in S for some connected component $X_{H_1}^+$ of X_{H_1} contained in X^+. Write $H = \mathrm{MT}(X_{H_1})$. Then $X_H = X_{H_1}$ and Z is also associated to (H, X_H) and is standard. $\square$

LEMMA 3.4. *Recall that Σ_T is the set of T-special subvarieties of S. Let $\alpha \in \Gamma$ and $T_\alpha = \alpha T \alpha^{-1}$. Then $\Sigma_{T_\alpha} = \Sigma_T$.*

Proof. Let (H, X_H) be a T-Shimura subdatum of (G, X). Fix $x \in X_H$. Let $H_\alpha = \alpha H \alpha^{-1}$ and X_{H_α} be the $H_\alpha(\mathbb{R})$-conjugacy class of $\alpha.x$. Then (H_α, X_{H_α}) is a T_α-Shimura subdatum and the images of $\mathrm{Sh}_{K\cap H(\mathbb{A}_f)}(H, X_H)$ and $\mathrm{Sh}_{K\cap H_\alpha(\mathbb{A}_f)}(H_\alpha, X_{H_\alpha})$ in $\mathrm{Sh}_K(G, X)$ coincide. □

LEMMA 3.5. *Let $\{q_1, \ldots, q_l\}$ be a system of representatives of $G(\mathbb{Q})^+ \backslash G(\mathbb{Q})$. Let $T_\mathbb{Q}$ be a subtorus of $G_\mathbb{Q}$ such that $T(\mathbb{R})$ is compact. There exists a finite subset $\{r_1, \ldots, r_k\}$ of $G(\mathbb{A}_f)$ such that any T-special subvariety of S is a component of the image by the Hecke operator $T_{q_j r_i}$ of a standard $(q_j T q_j^{-1})$-special subvariety of S.*

Proof. There exist $r_1, \ldots, r_k$ in $Z_G(T)(\mathbb{A}_f)$ such that we have a finite double coset decomposition

$$Z_G(T)(\mathbb{A}_f) = \cup_{i=1}^k Z_G(T)(\mathbb{Q})^+ \cdot r_i \cdot (Z_G(T)(\mathbb{A}_f) \cap K).$$

Let Z be a T-special subvariety of S associated to a T-Shimura subdatum (H, X_H) of (G, X). Then Z is the image in S of $X_H^+ \times \{h\}$ for some $h \in H(\mathbb{A}_f)$ and for some component X_H^+ of X_H.

By definition of a T-Shimura subdatum, $T \subset Z(H)$ (where $Z(H)$ is the centre of H) and therefore $H \subset Z_G(T)$.

We can find $z \in Z_G(T)(\mathbb{Q})^+$, $k \in Z_G(T)(\mathbb{A}_f) \cap K$ and $i \in \{1, \ldots, k\}$ such that $h = z r_i k$. Therefore Z is in the image of $z^{-1}.X_H^+ \times \{r_i\}$ in S.

Fix $x \in X_H^+$. As $G(\mathbb{Q})$ is Zariski dense in $G(\mathbb{R})$, there exists a $j \in \{1, \ldots, l\}$ such that $q_j z^{-1}.x \in X^+$.

Define $H_{z,j} := q_j z^{-1} H z q_j^{-1}$ and $X_{H_{z,j}} := H_{z,j}(\mathbb{R}) \cdot (q_j z^{-1}.x)$. Then $(H_{z,j}, X_{H_{z,j}})$ is a Shimura subdatum of (G, X). The generic Mumford-Tate group of $X_{H_{z,j}}$ is

$$\mathrm{MT}(X_{H_{z,j}}) = q_j z^{-1} \mathrm{MT}(X_H) z q_j^{-1} = q_j z^{-1} (T H^{\mathrm{der}}) z q_j^{-1} = q_j T q_j^{-1}.H_{z,j}^{\mathrm{der}}.$$

Therefore $(H_{z,j}, X_{H_{z,j}})$ is a $(q_j T q_j^{-1})$-Shimura subdatum.

Note that $q_j z^{-1} X_H^+$ is a connected component $X_{H_{z,j}}^+$ of $X_{H_{z,j}}$ such that $X_{H_{z,j}}^+ \subset X^+$. Let Z_0 be the image of $X_{H_{z,j}}^+ \times \{1\}$ in S. Then Z_0 is a standard $(q_j T q_j^{-1})$-special subvariety associated to $(H_{z,j}, X_{H_{z,j}})$. This finishes the proof as Z is a component of $T_{q_j r_i}.Z_0$. □

Let (H, X_H) be a T-Shimura subdatum of (G, X). Our next task will be to construct a T-special Shimura subdatum (L, X_L) of (G, X) maximal amongst T-Shimura subdata of (G, X) containing (H, X_H). Our construction will show that L depends only on T and not on (H, X_H).

The algebraic group $Z_G(T)$ is reductive and connected as the centraliser of a torus. Let

$$Z_G(T) = \widetilde{T} L_1 \cdots L_r$$

be the decomposition of $Z_G(T)$ as an almost direct product of the connected centre $\widetilde{T}$ of $Z_G(T)$ and a product of $\mathbb{Q}$-simple factors $Z_G(T)^{\mathrm{der}}$.

Let $L = \widetilde{T} L_1 \cdots L_s$ be the almost direct product in G of $\widetilde{T}$ and of the L_i's such that $L_i(\mathbb{R})$ is not compact. Then

$$H \subset Z_G(T) = Z_G(\widetilde{T}).$$

Let

$$(L')^c = Z_G(T)/L$$

and $p : Z_G(T) \to (L')^c$ be the associated projection. Then $(L')^c(\mathbb{R})$ is compact. As the almost $\mathbb{Q}$-simple factors H_k of H^{der} are such that $H_k(\mathbb{R})$ are not compact, their projections by p on$(L')^c$ are trivial. We deduce from this that $H \subset L$. Let X_L be the $L(\mathbb{R})$-conjugacy class of some $x \in X_H$.

Lemma 3.6. *The pair (L, X_L) is a T-Shimura subdatum such that*

$$(H, X_H) \subset (L, X_L).$$

Proof. The proof of [4, Prop. 3.2] shows that (L, X_L) is a Shimura datum. As H is contained in L, $(H, X_H) \subset (L, X_L)$. We write $H' = \mathrm{MT}(X_H)$ and $L' = \mathrm{MT}(X_L)$. We have an inclusion of Shimura subdata

$$(H', X_H) \subset (L', X_L).$$

By definition, $T = Z(H')^0 \subset L'$ and T commutes with L', therefore $T \subset Z(L')^0$. Fix $x \in X_H$. Then X_L is the $L^{\mathrm{der}}(\mathbb{R})$-conjugacy-class of x. By definition of the generic Mumford-Tate group of X_H we know that

$$x(\mathbb{S})(\mathbb{R}) \subset (T \cdot H^{\mathrm{der}})(\mathbb{R}) \subset (T \cdot L^{\mathrm{der}})(\mathbb{R}).$$

We then see that for any $y \in X_L$, we have

$$y(\mathbb{S})(\mathbb{R}) \subset (T \cdot L^{\mathrm{der}})(\mathbb{R}).$$

Therefore $L' = \mathrm{MT}(X_L) \subset T \cdot L^{\mathrm{der}}$ and $Z(L')^0 \subset T$. Finally $T = Z(L')^0$ and (L, X_L) is a T-Shimura subdatum. $\square$

The following lemma will be useful later.

Lemma 3.7. *Let (M, X_M) be a Shimura subdatum of (G, X). Then there exist at most finitely many Y such that (M, Y) is a Shimura subdatum of (G, X). Moreover as M varies among the reductive subgroups of G, the number of Y is uniformly bounded.*

Proof. Let $X_{1,M}$ and $X_{2,M}$ such that $(M, X_{1,M})$ and $(M, X_{2,M})$ are subdata of (G, X). Fix $x_i \in X_{i,M}$ and $\alpha \in G(\mathbb{R})$ such that

$$x_2 = \alpha.x_1 = \alpha x_1 \alpha^{-1}.$$

Let $K_i = Z_G(x_i(\sqrt{-1}))(\mathbb{R})$ the associated maximal compacts of $G(\mathbb{R})$. We have the Cartan decompositions

$$G(\mathbb{R}) = P_1K_1 = P_2K_2 \quad \text{and} \quad M(\mathbb{R}) = (P_1 \cap M)\cdot(K_1 \cap M) = (P_2 \cap M)\cdot(K_2 \cap M).$$

We then have $K_2 = \alpha K_1 \alpha^{-1}$ and $P_2 = \alpha P_1 \alpha^{-1}$. As the Cartan decompositions are conjugate in $M(\mathbb{R})$, there exists $h \in M(\mathbb{R})$ such that

$$K_2 \cap M = h(K_1 \cap M)h^{-1} \quad \text{and} \quad P_2 \cap M = h(P_1 \cap M)h^{-1}.$$

Let $\gamma = h^{-1}\alpha = p.k$ with $p \in P_1$ and $k \in K_1$. Then

$$(\star) \quad K_1 \cap M = pK_1p^{-1} \cap M \quad \text{and} \quad P_1 \cap M = pP_1p^{-1} \cap M.$$

By [18, Lemma 3.11] we have the following:

(1) Let p, q and r be elements of P_1 such that $pqp^{-1} = r$ then $p^2q = qp^2$.
(2) Let $p \in P_1$ and k_1 and k_2 be elements of K_1 such that $pk_1p^{-1} = k_2$ then $p^2k_1 = k_1p^2$.

Then $(\star)$ and (1) imply that $p^2 \in Z_G(P_1 \cap M)(\mathbb{R})$ and $(\star)$ and (2) imply that $p^2 \in Z_G(K_1 \cap M)(\mathbb{R})$. We then find that

$$p^2 \in Z_G(M)(\mathbb{R}) \subset Z_G(x_1(\sqrt{-1}))(\mathbb{R}) = K_1,$$

so $p^2 \in P_1 \cap K_1$ is trivial and $p = 1$.

We now know that $\alpha = h\gamma$ with $h \in M(\mathbb{R})$ and $\gamma \in K_1$. Fix a set of representatives $\{\gamma_1, \ldots, \gamma_r\}$ in K_1 of K_1/K_1^+. As K_1^+ fixes x_1, we obtain that $\gamma_i.x_1 \in X_{2,M}$ for some $i \in \{1, \ldots, r\}$. This finishes the proof of the lemma. ☐

THEOREM 3.8. *Fix a subtorus $T_{\mathbb{Q}}$ of G with $T(\mathbb{R})$ compact. Let (Z_n) be a sequence of T-special subvarieties of S. Let $(\mu_n) := (\mu_{Z_n})$ be the associated sequence of probability measures. There exist a T-special subvariety Z of S and a subsequence (Z_{n_k}) such that (μ_{n_k}) converges weakly to μ_Z. Moreover Z contains Z_{n_k} for all k large enough.*

Proof. We first give the proof assuming that Z_n is a sequence of standard T-special subvarieties of S associated to T-special Shimura subdata (H_n, X_n) of (G, X) with $H_n = \mathrm{MT}(X_n) = TH_n^{\mathrm{der}}$. Using Lemmas 3.7 and 3.6 we may assume that for all $n \in \mathbb{N}$, (H_n, X_n) is a Shimura subdatum of the T-special Shimura datum (L, X_L).

Therefore we may assume that (Z_n) is contained in a fixed component S_L of $\mathrm{Sh}_{L(\mathbb{A}_f)\cap K}(L, X_L)$. Then (Z_n) is a sequence of strongly special subvarieties of S_L in the sense of [4, 4.1]. Let $(L^{\mathrm{ad}}, X_{L^{\mathrm{ad}}})$ be the adjoint Shimura datum and K_L^{ad} a compact open subgroup containing the image of $L(\mathbb{A}_f) \cap K$ in $L^{\mathrm{ad}}(\mathbb{A}_f)$.

We recall that Z_n is a strongly special subvariety of S_L if and only if its image Z_n^{ad} in $\mathrm{Sh}_{K_L^{\mathrm{ad}}}(L^{\mathrm{ad}}, X_{L^{\mathrm{ad}}})$ is strongly special. As T is the connected centre of H_n and T is contained in the centre of L, we see that Z_n^{ad} is defined by a Shimura subdatum (H_n', X_n') of $(L^{\mathrm{ad}}, X_{L^{\mathrm{ad}}})$ with H_n' semisimple and that Z_n^{ad} is strongly special.

Note that condition (b) in the definition of "strongly special" ([4, 4.1]) is in fact implied by the first. Let (F, X_F) be a Shimura subdatum of an adjoint Shimura datum (G, X) with F semisimple. Let $\alpha : \mathbb{S} \to F_{\mathbb{R}}$ be a element of X_F and $K_\alpha = Z_G(\alpha(\sqrt{-1}))$ be the associated maximal compact subgroup of $G(\mathbb{R})$. Then $Z_G(F)(\mathbb{R}) \subset Z_G(\alpha(\sqrt{-1}))$ is compact. Therefore $Z_G(F)$ is $\mathbb{Q}$-anisotropic (even $\mathbb{R}$-anisotropic) and (F, X_F) satisfies condition (b$''$) of [4, 4.1], which is equivalent to condition (b).

Theorem 4.6 of [4] proves that, after possibly having replaced (Z_n) by a subsequence, there exists a special subvariety $Z \subset S_L$ such that (μ_{Z_n}) converges weakly to μ_Z and $Z_n \subset Z$ for all $n \gg 0$. We can find a Shimura subdatum (H, X_H) associated to Z such that for any n large enough, the following inclusions of Shimura data hold:

$$(H_n, X_n) \subset (H, X_H) \subset (L, X_L).$$

We once again write $L' = \mathrm{MT}(X_L)$ and $H' = \mathrm{MT}(X_H)$. Then

$$(H_n, X_n) \subset (H', X_H) \subset (L', X_L).$$

By following the proof of Lemma 3.6 we deduce that $Z(H')^0 = Z(H_n)^0 = Z(L')^0$ for every n large enough, and consequently Z is a T-special subvariety.

This finishes the proof assuming the Z_n are standard T-special subvarieties of S. Without this assumption, using Lemmas 3.5 and 3.3 we may assume that there exist $q \in G(\mathbb{Q})$, $\theta \in G(\mathbb{A}_f)$ and a sequence of (qTq^{-1})-special Shimura subdata (H_n', X_n') of (G, X) with $H_n' = \mathrm{MT}(X_n')$ with the following property. There exists a sequence of standard (qTq^{-1})-special subvarieties Z_n' (with Z_n' the image of $X_n'^{+} \times \{1\}$ in S for some component $X_n'^{+}$ of X_n') such that Z_n is the image of $X_n'^{+} \times \{\theta\}$ in S. Let $\mu_n' := \mu_{Z_n'}$ be the associated sequence of probability measures. Then the weak convergence of μ_n to μ_Z for some special subvariety containing Z_n for n big enough are deduced from the corresponding weak convergence of μ_n' to $\mu_{Z'}$ for some special subvariety Z' containing the Z_n' for $n \gg 0$. The reader may check that the proof given previously guarantees that Z is T-special. □

A formal consequence of Theorem 3.8 is the following result.

COROLLARY 3.9. *Let $(Z_n)_{n\in\mathbb{N}}$ be a sequence of T-special subvarieties of S and Z be a component of the Zariski closure $\overline{\cup_{n\in\mathbb{N}} Z_n}$ of $\cup_{n\in\mathbb{N}} Z_n$. Then Z is T-special.*

Proof. Let $I_Z := \{n \in \mathbb{N}, Z_n \subset Z\}$. Then formal properties of the Zariski topology show that $\cup_{n\in I_Z} Z_n$ is Zariski dense in Z. If there exists $n \in I_Z$ such that $Z_n = Z$, then Z is T-special; otherwise I_Z is infinite. Passing to a subsequence we may and do assume that for all $n \in \mathbb{N}$, $Z_n \subset Z$. As Z_n is defined over $\overline{\mathbb{Q}}$ for all n, we see that Z is defined over $\overline{\mathbb{Q}}$. As Z has only countably many subvarieties defined over $\overline{\mathbb{Q}}$, using a diagonal process and passing to a subsequence we may assume that $(Z_n)_{n\in\mathbb{N}}$ is a "generic sequence" of Z. For any subvariety Y of Z with $Y \neq Z$, the set $I_Y := \{n \in \mathbb{N}, Z_n \subset Y\}$ is finite. In particular, for any subsequence $(Z_{n_k})_{k\in\mathbb{N}}$ of $(Z_n)_{n\in\mathbb{N}}$, we have $\overline{\cup_{k\in\mathbb{N}} Z_{n_k}} = Z$.

Moreover using Theorem 3.8 and passing to a subsequence we may and do assume that there exists a T-special subvariety Z' of S such that μ_{Z_n} converges weakly to $\mu_{Z'}$ and for all $n \in \mathbb{N}$, $Z_n \subset Z'$. As $(Z_n)_{n\in\mathbb{N}}$ is generic in Z we get $Z = \overline{\cup_{n\in\mathbb{N}} Z_n} \subset Z'$. As Z is closed and as for all n, $\mathrm{Supp}(\mu_{Z_n}) \subset Z$, using the weak convergence of $(\mu_{Z_n})_{n\in\mathbb{N}}$ to $\mu_{Z'}$ we get that $Z' = \mathrm{Supp}(\mu_{Z'}) \subset Z$. Therefore $Z = Z'$ is a T-special subvariety of S. □

3.2. *Special subvarieties whose Galois orbits have bounded degrees.* Let (G, X), X^+, Γ and S be as in the previous section. We recall that we have fixed a faithful representation $G \subset \mathrm{GL}(V_{\mathbb{Q}})$ on an n-dimensional $\mathbb{Q}$-vector space $V_{\mathbb{Q}}$. We fix a $\mathbb{Z}$-lattice $V_{\mathbb{Z}}$ and an isomorphism $V_{\mathbb{Z}} \simeq \mathbb{Z}^n$ such that $K \subset \mathrm{GL}_n(\widehat{\mathbb{Z}})$. Moreover we assume that $K = \prod_p K_p$ and that K is neat. For any algebraic subgroup H of G, we let $H_{\mathbb{Z}}$ (resp. $H_{\mathbb{Z}_p}$) be the Zariski-closure of H in $\mathrm{GL}_{n,\mathbb{Z}} = \mathrm{GL}(V_{\mathbb{Z}})$ (resp. $\mathrm{GL}_{n,\mathbb{Z}_p} = \mathrm{GL}(V_{\mathbb{Z}_p})$).

The aim of this section is to prove the following theorem, which provides a justification for the seemingly unnatural definition of T-special subvarieties. This result is used crucially in [11] in the proof of the André-Oort conjecture under the GRH.

Theorem 3.10. *Assume the GRH for CM fields. Let M be an integer. There exists a finite set $\{T_1, \dots, T_r\}$ of $\mathbb{Q}$-tori of G such that each $T_i(\mathbb{R})$ is compact and having the following property. Let Z be a special subvariety of S defined by the Shimura subdatum (H, X_H) (with H being the generic Mumford-Tate group on X_H) such that, with notation of* 2.19,

$$\max(1, B^{i(T)}|K_T^m/K_T|) \log|\mathrm{disc}(L_T)| \leq M.$$

In this last formula we wrote $i(T)$ for the cardinality of the set of primes p such that $K_{T,p}^m \neq K_{T,p}$. Then Z is a T_i-special subvariety for some $i \in \{1, \dots, r\}$.

Corollary 3.11. *Assume the* GRH *for* CM *fields, and let M be an integer. There exists a finite set $\{T_1, \dots, T_r\}$ of $\mathbb{Q}$-tori of G with the following property. Let Z be a special subvariety of S. If the degree of* $\mathrm{Gal}(\overline{\mathbb{Q}}/E(G,X))\cdot Z$ *is at most M, then Z is a T_i-special subvariety for some $i \in \{1, \dots, r\}$.*

Proof. Let Z be a special subvariety of S such that the degree

$$\deg(\mathrm{Gal}(\overline{\mathbb{Q}}/E(G,X)).Z)$$

is bounded by M. By 2.19, both $\max(1, B^{i(T)}|K_T^m/K_T|)$ and $\log(|\mathrm{disc}(L_T)|)$ are bounded. The conclusion then follows from Theorem 3.10. □

COROLLARY 3.12. *Assume the* GRH *for* CM *fields. Let* $\Sigma = \{x_i\}$ *be an infinite sequence of special points. Then the size* $|\mathrm{Gal}(\overline{\mathbb{Q}}/E(G,X)) \cdot x_i|$ *of the Galois orbit of* x_i *is unbounded as* x_i *ranges through* Σ.

Proof. When x is a special point, the degree of its Galois orbit is just its size. Suppose that $|\mathrm{Gal}(\overline{\mathbb{Q}}/E(G,X)) \cdot x_i|$ was bounded by an integer M. Then, by 3.11, each x_i would be T_j-special for some $j \in \{1, \dots, r\}$. But, for a fixed torus T with $T(\mathbb{R})$ compact, there are only finitely many T-special points. Hence Σ is finite. This contradicts the definition of Σ thus proving the corollary. □

We now proceed to prove Theorem 3.10. Now let Σ_M be the set of special subvarieties Z such that $\max(1, B^{i(T)}|K_T^m/K_T|) \log|\mathrm{disc}(L_T)| \leq M$. Then both $\max(1, B^{i(T)}|K_T^m/K_T|)$ and $|\mathrm{disc}(L_T)|$ are bounded. Let $C \simeq H/H^{\mathrm{der}}$, and let L_C be the splitting field of C. The discriminant $|\mathrm{disc}(L_C)| = |\mathrm{disc}(L_T)|$ is bounded when Z varies in Σ_M. To prove Theorem 3.10, it suffices to consider the set of $Z \in \Sigma_M$ such that the corresponding L_C is fixed.

LEMMA 3.13. (i) *Let* E_0 *be a number field. Let* $\mathbb{T}_{E_0}$ *be the set of* $\mathbb{Q}$*-subtori* T *of* G *such that there exists a Shimura subdatum* (H, X_H) *of* (G, X) *with* $H = \mathrm{MT}(X_H)$ *such that* T *is the connected centre of* H *and such that the splitting field of* T *is* E_0. *Then* $\mathbb{T}_{E_0}$ *is contained in a finite union of* $\mathrm{GL}_n(\mathbb{Q})$*-conjugacy classes.*

(ii) *Let* M *be an integer. Let* $\mathbb{T}_M$ *be the set of* $\mathbb{Q}$*-subtori* T *of* G *such that there exists* $Z \in \Sigma_M$ *associated with a Shimura subdatum* (H, X_H) *of* (G, X) *such that* $T = Z(\mathrm{MT}(X_H))^0$. *Then* $\mathbb{T}_M$ *is contained in a finite union of* $\mathrm{GL}_n(\mathbb{Q})$*-conjugacy classes.*

Proof. The assumption of part (ii) of this lemma implies that the discriminant of L_T is bounded. For the purpose of proving part (ii) of the lemma we may assume that L_T is fixed. As L_T is the splitting field of T, we see that part (ii) is a consequence of part (i) of the lemma.

We now prove the part (i) of the lemma. Let L be the torus $\mathrm{Res}_{E_0/\mathbb{Q}}\mathbb{G}_m$. As before, we identify $X^*(T)$ with a submodule of $X^*(L)$ via a "lifting" r_T of the reciprocity r_C. By Lemma 2.6, there is only a finite number of possibilities for the set of characters of L occurring in the representations $r_T : L \to T \subset \mathrm{GL}_n$. Let us fix such a set $\mathcal{X}$ of characters of L and write

$$V_{\overline{\mathbb{Q}}} = \oplus_{\chi \in \mathcal{X}} V_{\overline{\mathbb{Q}},\chi}$$

for the corresponding decomposition of $V_{\overline{\mathbb{Q}}}$ such that for all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, $\sigma(V_{\overline{\mathbb{Q}},\chi}) = V_{\overline{\mathbb{Q}},\chi^\sigma}$. Here the V_χ's are $\overline{\mathbb{Q}}$-vector subspaces of $V_{\overline{\mathbb{Q}}}$ and we can assume that their dimensions are fixed when T varies in $\mathbb{T}_{E_0}$.

It follows that the isomorphism class of the representation of the $\mathbb{Q}$-torus L on V is fixed. Therefore the morphism r_T is contained in a $\mathrm{GL}_n(\mathbb{Q})$-conjugacy class. This finishes the proof of the lemma as $T = r_T(L)$. □

The part (i) of the previous lemma will not be used in this text but will play a role in [11]. For the proof of Theorem 3.10, we need in fact the following more precise result than part (ii) of Lemma 3.13:

PROPOSITION 3.14. *The set* $\mathbb{T}_M$ *is contained in a finite union of* $\mathrm{GL}_n(\mathbb{Z})$-*conjugacy classes.*

We will need a weak version of the following result for the proof of the Proposition 3.14, but its full strength will be used in [11].

PROPOSITION 3.15. *There exists a positive constant c with the following property. Let (H, X_H) be a Shimura subdatum of (G, X). Let T be the connected centre of H. Let $L_T = L_C$ be the splitting field of T. Let p be a prime that is unramified in L_T and such that $K_p = \mathrm{GL}_n(\mathbb{Z}_p) \cap G(\mathbb{Q}_p)$. Assume that*

$$K_{T,p} := T(\mathbb{Q}_p) \cap K \neq K^m_{T,p}.$$

Then

$$|K^m_{T,p}/K_{T,p}| \geq cp. \tag{9}$$

Proof. This statement is a variant of Proposition 4.3.9 of [7]. We need to check that the proof can be adapted in our situation.

LEMMA 3.16. *The set $\mathbb{T}(G)$ of tori T in G occurring as the connected centre of a reductive subgroup H of G such that there exists a Shimura subdatum (H, X_H) of (G, X) is contained in a finite union of $\mathrm{GL}_n(\overline{\mathbb{Q}})$-conjugacy classes.*

Proof. By the discussion before Lemma 2.6 we may assume that there exists a finite Galois extension F of $\mathbb{Q}$ such that the isomorphism class Δ of $\mathrm{Gal}(F/\mathbb{Q})$ is fixed as an abstract group and a surjective map of tori

$$r_T : T_F = \mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_{m,F} \to T$$

obtained as a lifting of a uniformly bounded power of the reciprocity morphism r_C. Then $X^*(T_F)$ has a canonical basis $\mathcal{B}$ indexed by the elements of Δ. Let r be the cardinality of Δ. We can therefore find a $\overline{\mathbb{Q}}$–isomorphism $\mathbb{G}^r_{m,\overline{\mathbb{Q}}} \simeq T_{F,\overline{\mathbb{Q}}}$ such that the induced map $X^*(T_{F,\overline{\mathbb{Q}}}) \to X^*(\mathbb{G}^r_{m,\overline{\mathbb{Q}}})$ transforms the canonical basis $\mathcal{B}$ of $X^*(T_{F,\overline{\mathbb{Q}}})$ into the canonical basis of $\mathbb{Z}^r = X^*(\mathbb{G}^r_{m,\overline{\mathbb{Q}}})$. We end up with a representation

$$r_{T,\overline{\mathbb{Q}}} : \mathbb{G}^r_{m,\overline{\mathbb{Q}}} \to T_{\overline{\mathbb{Q}}} \subset \mathrm{GL}_{n,\overline{\mathbb{Q}}}$$

of the torus $\mathbb{G}^r_{m,\overline{\mathbb{Q}}}$. Using Lemma 2.6 we see that we may assume that the set of characters of $\mathbb{G}^r_{m,\overline{\mathbb{Q}}}$ and their multiplicities occurring in the representation $r_{T,\overline{\mathbb{Q}}}$ are fixed. As a consequence the $\overline{\mathbb{Q}}$-isomorphism class of the representation $r_{T,\overline{\mathbb{Q}}}$ is fixed. As $T_{\overline{\mathbb{Q}}} = r_{T,\overline{\mathbb{Q}}}(\mathbb{G}^r_{m,\overline{\mathbb{Q}}})$, we see that the tori $T \in \mathbb{T}(G)$ are contained in a finite union of $\mathrm{GL}_n(\overline{\mathbb{Q}})$-conjugacy classes. □

LEMMA 3.17. *Let T be a torus in $\mathbb{T}(G)$. Let $r_T : T_F \to T$ be as previously. Let p be a prime which is unramified in F. There exists $\alpha \in \mathrm{GL}_n(\mathbb{Q}_p)$ such that the Zariski closure of $T_\alpha := \alpha T \alpha^{-1}$ in $\mathrm{GL}_{n,\mathbb{Z}_p}$ is a torus $T_{\alpha,\mathbb{Z}_p}$. In this situation $K_{T_\alpha,p} = T_\alpha(\mathbb{Q}_p) \cap \mathrm{GL}_n(\mathbb{Z}_p)$ is the maximal compact open subgroup $K^m_{T_\alpha,p}$ of $T_\alpha(\mathbb{Q}_p)$.*

Proof. We first recall the following facts about models of tori over $\mathbb{Z}_p$ mainly due to Tits in the general context of reductive groups. Let Λ be a torus in $\mathrm{GL}_{n,\mathbb{Q}_p}$ and $\Lambda_{\mathbb{Z}_p}$ be its Zariski closure in $\mathrm{GL}_{n,\mathbb{Z}_p}$. Then $K_{\Lambda,p} := \Lambda(\mathbb{Z}_p) = \Lambda(\mathbb{Q}_p) \cap \mathrm{GL}_n(\mathbb{Z}_p)$ is a compact open subgroup of $\Lambda(\mathbb{Q}_p)$. If $\Lambda_{\mathbb{Z}_p}$ is a torus, then $K_{\Lambda,p} = \Lambda(\mathbb{Z}_p)$ is the maximal hyperspecial subgroup $K^m_{\Lambda,p}$ of $\Lambda(\mathbb{Q}_p)$ ([17, 3.8.1]). Conversely if $K^m_{\Lambda,p} = K_{\Lambda,p}$ is a maximal hyperspecial subgroup of $\Lambda(\mathbb{Q}_p)$, then $\Lambda_{\mathbb{Z}_p}$ is a torus over $\mathbb{Z}_p$ ([17, 3.8.1]). Note also that if the splitting field of Λ is an unramified extension of $\mathbb{Q}_p$, then $K^m_{\Lambda,p}$ is a hyperspecial subgroup of $\Lambda(\mathbb{Q}_p)$ ([17, 3.8.2]).

As p is unramified in F and as $r_T : T_F \to T$ is surjective, p is unramified in the splitting field of T. The maximal open compact subgroups of $\mathrm{GL}_n(\mathbb{Q}_p)$ are conjugate under $\mathrm{GL}_n(\mathbb{Q}_p)$, and any compact subgroup of $\mathrm{GL}_n(\mathbb{Q}_p)$ is contained in a maximal open compact subgroup of $\mathrm{GL}_n(\mathbb{Q}_p)$ (see [16, 3.3, p. 134]). Therefore there exists a maximal compact open subgroup $\alpha^{-1}\mathrm{GL}_n(\mathbb{Z}_p)\alpha$ of $\mathrm{GL}_n(\mathbb{Q}_p)$ for some $\alpha \in \mathrm{GL}_n(\mathbb{Q}_p)$ such that $T(\mathbb{Q}_p) \cap \alpha^{-1}\mathrm{GL}_n(\mathbb{Z}_p)\alpha = K^m_{T,p}$. Let $T_\alpha = \alpha T \alpha^{-1}$, and let $T_{\alpha,\mathbb{Z}_p}$ be its Zariski closure. Then $T_\alpha(\mathbb{Z}_p)$ is the maximal compact subgroup of $T_\alpha(\mathbb{Q}_p)$ and is hyperspecial. The previous discussion shows that $T_{\alpha,\mathbb{Z}_p}$ is a torus. □

We may now prove Proposition 3.15. Let p be a prime that is unramified in F. Let

$$r_\alpha : T_{F,\mathbb{Q}_p} \longrightarrow T_\alpha$$

be the map $r_\alpha = \alpha r_T \alpha^{-1}$. The torus $T_{F,\mathbb{Q}_p}$ is the generic fibre of a torus $T_{F,\mathbb{Z}_p}$ over $\mathbb{Z}_p$ (see [20, 10.3, Th. 2]), and the map r_α extends uniquely over $\mathbb{Z}_p$ as a map of algebraic tori

$$r_{\alpha,\mathbb{Z}_p} : T_{F,\mathbb{Z}_p} \longrightarrow T_{\alpha,\mathbb{Z}_p} \subset \mathrm{GL}_{n,\mathbb{Z}_p} = \mathrm{GL}(V_{\mathbb{Z}_p}).$$

Taking the special fibres we get over the residue field $\mathbb{F}_p$ of $\mathbb{Z}_p$ a map

$$r_{\alpha,\mathbb{F}_p} : T_{F,\mathbb{F}_p} \longrightarrow T_{\alpha,\mathbb{F}_p} \subset \mathrm{GL}_{n,\mathbb{F}_p} = \mathrm{GL}(V_{\mathbb{F}_p}).$$

Passing to the algebraic closure $\overline{\mathbb{F}}_p$ of $\mathbb{F}_p$ we get a map

$$r_{\alpha,\overline{\mathbb{F}}_p} : T_{F,\overline{\mathbb{F}}_p} \longrightarrow T_{\alpha,\overline{\mathbb{F}}_p} \subset \mathrm{GL}_{n,\overline{\mathbb{F}}_p} = \mathrm{GL}(V_{\overline{\mathbb{F}}_p}).$$

Using Lemma 4.1 of Exposé X of [1], we see that there is a canonical isomorphism between $X^*(T_{F,\mathbb{F}_p})$ and $X^*(T_F)$, and by our previous discussion we get a canonical basis $\mathcal{B}$ of $X^*(T_{F,\mathbb{F}_p})$. As in the proof of Lemma 3.16, we have an isomorphism of tori over $\overline{\mathbb{F}}_p$ between $\mathbb{G}^r_{m,\overline{\mathbb{F}}_p} \simeq T_{F,\overline{\mathbb{F}}_p}$ such that the associate map on the character groups send the canonical basis $\mathcal{B}$ on the canonical basis of $\mathbb{Z}^n = X^*(\mathbb{G}^r_{m,\overline{\mathbb{F}}_p})$. Composing this isomorphism with $r_{\alpha,\overline{\mathbb{F}}_p}$ we end up with a representation

$$r_{\alpha,\overline{\mathbb{F}}_p} : \mathbb{G}^r_{m,\overline{\mathbb{F}}_p} \longrightarrow T_{\alpha,\overline{\mathbb{F}}_p} \subset \mathrm{GL}_{n,\overline{\mathbb{F}}_p} = \mathrm{GL}(V_{\overline{\mathbb{F}}_p}).$$

Using Lemma 2.6 as in the proof of Lemma 3.16 we may assume that the characters of $\mathbb{G}^r_{m,\overline{\mathbb{F}}_p}$ and their multiplicities in the representation $r_{\alpha,\overline{\mathbb{F}}_p}$ are fixed.

By Lemma 4.4.1 of [7] there is a positive integer C_1 independent of (H, X_H) and p such that for all subspaces W of $V_{\overline{\mathbb{F}}_p}$, the group of connected components of the stabiliser of W in $\mathbb{G}^r_{m,\overline{\mathbb{F}}_p}$ is of order bounded by C_1. As the map $r_{\alpha,\overline{\mathbb{F}}_p}$ is surjective, the group of connected components of the stabiliser of W in $T_{\alpha,\overline{\mathbb{F}}_p}$ is also of cardinality uniformly bounded by C_1.

Assume now that $K_p = G(\mathbb{Q}_p) \cap \mathrm{GL}_n(\mathbb{Z}_p)$. Then $K_{T,p} = T(\mathbb{Q}_p) \cap \mathrm{GL}_n(\mathbb{Z}_p)$. If $K_{T,p} \neq K^m_{T,p}$, the Zariski closure $T_{\mathbb{Z}_p}$ of $T_{\mathbb{Q}_p}$ in $\mathrm{GL}_{n,\mathbb{Z}_p}$ is not a torus.

The conjugation morphism $x \mapsto \alpha x \alpha^{-1}$ establishes a bijection between $K^m_{T,p}/K_{T,p}$ and $K^m_{T_\alpha,p}/T_\alpha(\mathbb{Q}_p) \cap \alpha\mathrm{GL}_n(\mathbb{Z}_p)\alpha^{-1}$, where $K^m_{T_\alpha,p}$ is the maximal compact open subgroup of $T_\alpha(\mathbb{Q}_p)$. This last index is the size of the orbit $T_\alpha(\mathbb{Z}_p) \cdot \alpha\mathbb{Z}_p^n$. The fact that the Zariski closure $T_{\mathbb{Z}_p}$ of $T_{\mathbb{Q}_p}$ in $\mathrm{GL}_{n,\mathbb{Z}_p}$ is not a torus implies that $T_{\alpha,\mathbb{Z}_p}$ does not fix the lattice $\alpha\mathbb{Z}_p^n$ in the sense of [7, §3.3].

In view of the previous result on the size of the group of connected components of stabilisers of subspaces of $V_{\overline{\mathbb{F}}_p}$, the proof of Proposition 4.3.9 of [7] implies that this index is at least a uniform constant times p. □

Fix a torus $T_0 \in \mathbb{T}_M$, and let $\mathcal{D}(T_0)$ be the set of tori in G contained in the $\mathrm{GL}_n(\mathbb{Q})$-conjugacy class of T_0. To prove Proposition 3.14, we will analyse the variation of $B^{i(T)} \cdot |K^m_T/K_T|$ as T ranges through $\mathcal{D}(T_0)$.

LEMMA 3.18. *For all $T \in \mathcal{D}(T_0)$, we have the lower bound*

$$\prod_{\{p:K^m_{T,p} \neq K_{T,p}\}} \max(1, B|K^m_{T,p}/K_{T,p}|) \gg \prod_{\{p:K^m_{T,p} \neq K_{T,p}\}} cp,$$

where c is a uniform constant. Let M be an integer. There exists an integer $C_0 > 0$ such that the following holds. Let S_0 be the set of primes $p < C_0$, and let $\mathbb{Z}_{S_0}$ be the ring of S_0-integers. The set $\mathcal{D}(T_0) \cap \mathbb{T}_M$ is contained in a finite union of $\mathrm{GL}_n(\mathbb{Z}_{S_0})$-conjugacy classes.

Proof. Let p be a prime such that p is unramified in L_T, such that K_p is $G(\mathbb{Z}_p)$ for the $\mathbb{Z}_p$-structure given by our fixed representation of G and such that $T_{0,\mathbb{Z}_p}$ is a torus. These conditions are verified for almost all p.

Let $T \in \mathcal{D}(T_0)$ be such that $K^m_{T,p} \neq K_{T,p}$. By Proposition 3.15, we have the lower bound

$$|K^m_{T,p}/K_{T,p}| \geq cp.$$

Therefore there exists an integer C_0 such that for all $T \in \mathcal{D}(T_0) \cap \mathbb{T}_M$ and all primes $p > C_0$, $K_{T,p} = K^m_{T,p}$ and $K^m_{T,p}$ is hyperspecial. Let $T \in \mathcal{D}(T_0) \cap \mathbb{T}_M$ and $p > C_0$. Then $T_{\mathbb{Z}_p}$ is a torus.

Let $g \in \mathrm{GL}_n(\mathbb{Q})$ such that $T = gT_0g^{-1}$. The previous discussion shows that $T_{\mathbb{Z}_p}$ fixes the lattice $g\mathbb{Z}_p^n$. By [7, Lemma 3.3.1], there exist $c \in Z_{\mathrm{GL}_n}(T)(\mathbb{Q}_p)$ and $\alpha_p \in \mathrm{GL}_n(\mathbb{Z}_p)$ such that $g = c\alpha_p$. Therefore $T_{\mathbb{Z}_p} = \alpha_p T_{0,\mathbb{Z}_p} \alpha_p^{-1}$ for some $\alpha_p \in \mathrm{GL}_n(\mathbb{Z}_p)$.

By Corollary 6.4 of [10], the set $\mathcal{D}(T_0) \cap \mathbb{T}_M$ is contained in finitely many $\mathrm{GL}_n(\mathbb{Z}_{S_0})$-conjugacy classes. □

Proposition 3.14 will follow from the following proposition, whose proof was communicated to us by Laurent Clozel.

Proposition 3.19 (Clozel). *Let G be a reductive group over $\mathbb{Q}_p$, $T \subset G$ a nontrivial torus, and let $H = Z_G(T)$. Let K be a fixed compact open subgroup of $G(\mathbb{Q}_p)$, and let $K_T = K^m_T$ be the maximal compact subgroup of $T(\mathbb{Q}_p)$. The function*

$$I(g) = |K_T/T(\mathbb{Q}_p) \cap g^{-1}Kg| \to \infty$$

as $g \to \infty$ in $G(\mathbb{Q}_p)/H(\mathbb{Q}_p)$ (where a basis of neighborhoods of ∞ is given by the complements of compact subsets of $G(\mathbb{Q}_p)/H(\mathbb{Q}_p)$).

Let W be a set of $g \in G(\mathbb{Q}_p)/H(\mathbb{Q}_p)$ such that $I(g)$ is bounded. The image of W in $G(\mathbb{Z}_p)\backslash G(\mathbb{Q}_p)/H(\mathbb{Q}_p)$ is finite.

Proof. As $T(\mathbb{Q}_p) \cap g^{-1}Kg$ is a compact open subgroup of $T(\mathbb{Q}_p)$, $T(\mathbb{Q}_p) \cap g^{-1}Kg$ is contained in K_T. For $g \in G(\mathbb{Q}_p)$ and $h \in H(\mathbb{Q}_p)$, we find that

$$\begin{aligned} T(\mathbb{Q}_p) \cap h^{-1}g^{-1}Kgh &= h^{-1}(hT(\mathbb{Q}_p)h^{-1} \cap g^{-1}Kg)h \\ &= h^{-1}(T(\mathbb{Q}_p) \cap g^{-1}Kg)h = T \cap g^{-1}Kg \end{aligned}$$

as h commutes with T. So $I(g)$ is well defined on $G(\mathbb{Q}_p)/H(\mathbb{Q}_p)$.

Let $\mathbf{1}_K$ be the characteristic function of K on $G(\mathbb{Q}_p)$. Let μ_T be the normalised measure on K_T. Then $I(g) \to \infty$ if and only if

$$\int_{K_T} \mathbf{1}_K(gtg^{-1})\, d\mu_T \longrightarrow 0.$$

We just have to prove that for t outside a subset of K_T of μ_T-measure 0,

$$\mathbf{1}_K(gtg^{-1}) \to 0.$$

Let $T^{\mathrm{reg}} \subset T(\mathbb{Q}_p)$ be the set

$$T^{\mathrm{reg}} = \{t \in T(\mathbb{Q}_p) \mid Z_G(t) = Z_G(T) = H\}.$$

For $t \in T^{\mathrm{reg}}$, we have a homeomorphism

$$\begin{aligned}\pi_t :\ G(\mathbb{Q}_p)/H(\mathbb{Q}_p) &\to O(t)\\ g &\mapsto gtg^{-1},\end{aligned}$$

where $O(t)$ denotes the orbit of t under $G(\mathbb{Q}_p)$. As t is semisimple, this orbit is closed and the map π_t is proper. In this way we get that for $g \to \infty$, $\mathbf{1}_K(gtg^{-1}) = 0$. So the following lemma finishes the proof of the proposition. $\square$

LEMMA 3.20. *The set of $t \in K_T$ such that $t \notin T^{\mathrm{reg}}$ is of μ_T-measure* 0.

Proof. This last lemma is a consequence of [16, 2.1.11]. $\square$

We can now finish the proof of Proposition 3.14. Let $T_0 \in \mathbb{T}_M$. Let $T_{0,\mathbb{Z}}$ be the Zariski closure of T_0 in $\mathrm{GL}_{n,\mathbb{Z}}$. By Lemma 3.13, we just need to prove that $\mathcal{D}(T_0) \cap \mathbb{T}_M$ is contained in a finite union of $\mathrm{GL}_n(\mathbb{Z})$-conjugacy classes. By the proof of Lemma 3.18, there exists $C_0 > 0$ such that for all $T \in \mathcal{D}(T_0) \cap \mathbb{T}_M$ and all prime numbers $p > C_0$, there exists $\alpha_p \in \mathrm{GL}_n(\mathbb{Z}_p)$ such that $T_{\mathbb{Z}_p} = \alpha_p T_{0\mathbb{Z}_p} \alpha_p^{-1}$.

Let $g \in \mathrm{GL}_n(\mathbb{Q})$ be such that $T := gT_0g^{-1} \in \mathcal{D}(T_0) \cap \mathbb{T}_M$. By Theorem 2.19,

$$|K^m_{T,p}/K_{T,p}| = |K^m_{T_0,p}/T_0(\mathbb{Q}_p) \cap g^{-1}K_p g|$$

is bounded when T varies in $\mathcal{D}(T_0) \cap \mathbb{T}_M$. Using Proposition 3.19, we see that for all prime numbers $p \leq C_0$, there exists a finite subset W_p of

$$\mathrm{GL}_n(\mathbb{Z}_p)\backslash \mathrm{GL}_n(\mathbb{Q}_p)/Z_{\mathrm{GL}_n}(T_0)(\mathbb{Q}_p)$$

such that the image of g in $\mathrm{GL}_n(\mathbb{Z}_p)\backslash \mathrm{GL}_n(\mathbb{Q}_p)/Z_{\mathrm{GL}_n}(T_0)(\mathbb{Q}_p)$ is contained in W_p.

We therefore just need to prove that the set of tori $T = gT_0g^{-1} \in \mathcal{D}(T_0) \cap \mathbb{T}_M$ such that the image g_p in W_p is fixed for all $p \leq C_0$ is contained in a finite union of $\mathrm{GL}_n(\mathbb{Z})$-conjugacy classes.

If this set is not empty, there exists $T_1 \in \mathcal{D}(T_0) \cap \mathbb{T}_M$ such that for all primes p and all the tori T in this set, there exists $\alpha_p \in \mathrm{GL}_n(\mathbb{Z}_p)$ such that $T_{\mathbb{Z}_p} = \alpha_p T_{1\mathbb{Z}_p} \alpha_p^{-1}$. By the results of Gille and Moret-Bailly ([10, Cor. 6.4]) the set of tori under consideration is contained in a finite union of $\mathrm{GL}_n(\mathbb{Z})$-conjugacy classes. This finishes the proof of Proposition 3.14.

PROPOSITION 3.21. *The set $\mathbb{T}_M$ is a finite union of Γ-conjugacy classes.*

This proposition finishes the proof of Theorem 3.10. Fix $T_1, \ldots, T_s$ a system of representatives of the Γ-conjugacy classes in $\mathbb{T}_M$. In view of Lemma 3.4, any $Z \in \Sigma_F$ is a T_i-special subvariety.

Proof. Before starting the proof the proposition, we need to define the "type" of a torus. Let $\mathcal{S}$ be a finite set of finite places of $\mathbb{Q}$, and let A be the ring of $\mathcal{S}$-integers. Let $\overline{A}$ be the integral closure of A inside $\overline{\mathbb{Q}}$. Suppose that G_A is a smooth reductive model of $G_{\mathbb{Q}}$ over $\mathrm{Spec}(A)$.

We recall ([2, Exp. XIV, Def. 1.3]) that a maximal torus T of G_A is a torus in G_A such that for any geometric point $\overline{s}$ of $\mathrm{Spec}(A)$, $T_{\overline{s}}$ is a maximal torus of $G_{A,\overline{s}}$. For any $s \in \mathrm{Spec}(A)$, there exists a neighbourhood U of s such that $G_{|U}$ has a maximal torus ([2, Exp. XIV, Cor. 3.20]). By enlarging $\mathcal{S}$ we may and do assume that G_A has a maximal torus.

Let T_A be a torus in G_A. Then $Z_{G_A}(T_A)$ is a connected reductive subgroup of G_A such that, for any geometric point $\overline{s}$ of $\mathrm{Spec}(A)$, $(Z_{G_A}(T_A))_{\overline{s}}$ is a reductive subgroup of $(G_A)_{\overline{s}}$ of maximal reductive rank ([2, Exp. XXII, Prop. 5.10.3]). Moreover $Z_{G_A}(T_A)$ contains a maximal torus $T_A^{\max}$ of G_A ([2, Exp. XII, Prop. 7.9(d)]). By [2, Exp. XXII, Prop. 2.2], $T_{\overline{A}}^{\max}$ is a split maximal torus of $G_{\overline{A}}$. One can describe $Z_{G_{\overline{A}}}(T_{\overline{A}})$ using roots of $(G_{\overline{A}}, T_{\overline{A}}^{\max})$ that are trivial on $\tilde{T}_{\overline{A}} = Z(Z_{G_{\overline{A}}}(T_{\overline{A}}))^0$ ([2, Exp. XXII, §5.4]). Note that $Z_{G_{\overline{A}}}(T_{\overline{A}})$ is of type (R) in the sense of [2, Exp. XXII, Def. 5.2.1]. Then $Z_{G_{\overline{A}}}(T_{\overline{A}})$ is determined by a subset R' of the set $R = R(G_{\overline{A}}, T_{\overline{A}}^{\max})$ of roots of $(G_{\overline{A}}, T_{\overline{A}}^{\max})$ ([2, Exp. XXII, §5.4]). The possible subsets R' of R occuring as the roots of a subgroup of $G_{\overline{A}}$ of the form $Z_{G_{\overline{A}}}(T_{\overline{A}})$ are described in [2, Exp. XXII, §5.10]; see Proposition 5.10.3, Corollary 5.10.5 and Proposition 5.10.6 of loc. cit.

For any root data $R_1 = R(G_{\overline{A}}, T_1^{\max})$ and $R_2 = R(G_{\overline{A}}, T_2^{\max})$, there exists an inner automorphism ϕ of $G_{\overline{A}}$ transforming R_1 into R_2 ([2, Exp. XXIV, Lemma 1.5]). The subsets of R_1 occuring as root data for the reductive subgroups of type (R) are sent by ϕ on the corresponding subsets of R_2. Hence, there exist at most finitely many $G(\overline{A})$-conjugacy classes of subgroups of this form. If T_A is an A-torus in G_A the type of T_A is the $G(\overline{A})$-conjugacy class of $Z_{G_{\overline{A}}}(T_{\overline{A}})$. (Compare with [2, Exp. XXII,§2].)

We only need to prove Proposition 3.21 for a subset $\mathbb{T}'_M$ of $\mathbb{T}_M$ such that the tori in $\mathbb{T}'_M$ belong to a fixed $\mathrm{GL}_n(\mathbb{Z})$-conjugacy class of a torus $T_0 \in \mathbb{T}'_M$. Assume that $\mathcal{S}$ contains the primes p such that either $T_{0\mathbb{Z}_p}$ is not a torus or the Zariski-closure of G in $\mathrm{GL}_{n,\mathbb{Z}_p}$ is not reductive and smooth. The Zariski closures G_A of G and $T_{0,A}$ of T_0 in $\mathrm{GL}_{n,A}$ are smooth. By enlarging $\mathcal{S}$ we may and do assume that G_A has a maximal torus. As we work in a fixed $\mathrm{GL}_n(\mathbb{Z})$-conjugacy class, all the tori in $\mathbb{T}'_M$ have a smooth Zariski closure in GL_{nA}. We therefore may assume that all the tori in $\mathbb{T}'_M$ have the same type. Let $\tilde{T}_0$ be the maximal torus of $Z(Z_G(T_0))$. Then $Z_G(T_0) = Z_G(\tilde{T}_0)$ also has a smooth Zariski-closure in GL_{nA}.

If $T \in \mathbb{T}'_F$, we write $\tilde{T}$ for the maximal torus of $Z(Z_G(T))$. Then $\tilde{T}_A$ and $\tilde{T}_{0,A}$ are some A-subtori of G_A locally conjugate in the fppf topology. Corollary 6.4 of the paper by Gille and Moret-Bailly [10] tells us that there are at most finitely many $G(A)$-conjugacy classes of such subtori. We may therefore assume that for any $T \in \mathbb{T}'_F$, the associated A-torus $\tilde{T}_A$ is conjugate to $\tilde{T}_{0,A}$ by an element of $G(A)$.

Let $\alpha \in G(A)$ such that $\tilde{T}_A = \alpha \tilde{T}_{0,A} \alpha^{-1}$. Then

$$Z_{G_A}(\tilde{T}_A) = Z_{G_A}(T_A) = \alpha Z_{G_A}(\tilde{T}_{0,A}) \alpha^{-1}.$$

Over $\mathbb{Q}$ we get $Z_G(T) = \alpha Z_G(T_0) \alpha^{-1}$. Let L and L_0 be the reductive subgroups of $Z_G(T)$ and $Z_G(T_0)$ obtained by removing the $\mathbb{R}$-compact $\mathbb{Q}$-factors of $Z_G(T)$ and $Z_G(T_0)$ as described before Lemma 3.6. Let (L, X_L) and (L_0, X_{L_0}) be the associated Shimura data (see 3.6). Using Lemma 3.7 we may assume that for any $T \in \mathbb{T}'_M$, α induces an isomorphism of Shimura data between (L_0, X_{L_0}) and (L, X_L). Therefore the generic Mumford-Tate group $\mathrm{MT}(X_L)$ of X_L equals $\alpha MT(X_{L_0})\alpha^{-1}$. As a consequence, we have

$$T = Z(\mathrm{MT}(X_L)) = \alpha T_0 \alpha^{-1}.$$

Proposition 3.19 of Clozel shows that for all primes $p \in \mathcal{S}$, the image α_p of α in $G(\mathbb{Q}_p)/Z_G(T_0)(\mathbb{Q}_p)$ is contained in a finite union of $G(\mathbb{Z}_p)$-orbits. We may therefore assume that for all $p \in \mathcal{S}$ any torus T in $\mathbb{T}'_M$ is conjugate to T_0 by an element of $G(\mathbb{Z}_p)$. As T and T_0 are also conjugate by an element of $G(\mathbb{Z}_p)$ for all $p \notin \mathcal{S}$, Corollary 6.4 of the paper by Gille and Moret-Bailly [10] tells us that T is contained in a finite union of $G(\mathbb{Z})$-orbits. As Γ is of finite index in $G(\mathbb{Z})$, T is contained in a finite union of Γ-orbits. $\square$

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