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Stable logarithmic maps to Deligne–Faltings pairs I

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Abstract

We introduce a new compactification of the space of relative stable maps. This approach uses logarithmic geoemetry in the sense of Kato-Fontaine-Illusie without taking expansions of the target. The underlying structures of the stable logarithmic maps are stable in the usual sense.

Contents

1. **Introduction**

1.1. *Background on relative Gromov-Witten theory.* Gromov-Witten theory relative to a smooth divisor was introduced during the past decade, for the purpose of proving a degeneration formula, in the symplectic setting by A.M. Li and Y. Ruan [LR01] and at about the same time by E.N. Ionel and T. Parker [IP03]. On the algebraic side, this was worked out by Jun Li [Li01], [Li02]. This approach uses the idea of expanded degenerations, which was introduced by Ziv Ran [Ran87]. A related idea of admissible covers was introduced even earlier by Harris and Mumford [HM82].

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Recently, the idea of expanded degeneration was systematically studied using orbifold techniques by D. Abramovich and B. Fantechi [AF11], and an elegant proof of degeneration formula was given there.

On the other hand, the idea of admissible covers was revisited by Mochizuki [Moc95] using logarithmic geometry. Following a similar idea, B. Kim defined the logarithmic stable maps [Kim10], by putting certain log structures on Jun Li's predeformable maps. Then, using the work of M. Olsson [Ols03a], [Ols05], he proved that the stack parametrizing such maps is a proper DM stack and has an explicit perfect obstruction theory. A degeneration formula under Kim's setting is proved in [Che].

Another approach using logarithmic structures without expansions was first proposed by Bernd Siebert in 2001 [Sie]. The goal here is also to obtain the degeneration formula, but in a more general situation, such as simple normal crossings divisors. However, the program was on hold for a while since Mark Gross and Bernd Siebert were working on other projects in mirror symmetry. Only recently they have taken up the unfinished project of Siebert jointly. In particular, they succeeded in finding a definition of basic log maps, a crucial ingredient for a good moduli theory of stable log maps to a fixed target with Zariski log structures [GS13]. Their definition builds on insights from tropical geometry, obtained by probing the stack of log maps using the standard log point and is compatible with the minimality introduced in this paper.

A different approach using exploded manifolds to studying holomorphic curves was recently introduced by Brett Parker in [Par12], [Par09a], and [Par09b]. It also aimed at defining and computing relative and degenerated Gromov-Witten theories in a general situation. The theory of exploded manifolds uses a generalized version of tropical curves, and is closely related to logarithmic geometries — the explosion functors, which is central to this theory, can be phrased in terms of certain kind of base change in log geometry; see [Par12, §5]. It was pointed out by Mark Gross that this approach is parallel and possibly equivalent to the logarithmic approach.

1.2. *The approach and main results of this paper.* The goal of this paper is to develop the relative Gromov-Witten theory along the logarithmic approach proposed by Bernd Siebert. However, we use somewhat different methods. Instead of using tropical geometry and probing the stack using standard log point, we associate to each log map a marked graph as in Section 3.3, which allows us to define the right base log structure. We now describe our methods as follows.

The target we will consider in this paper is a projective variety X equipped with a rank-one Deligne-Faltings log structure $\mathcal{M}_X$ on X as in Definition 3.1.1, which comes from a line bundle L on X, with a morphism of sheaves $s : L \to \mathcal{O}_X$. In particular, if L is the ideal sheaf of a smooth divisor $D \subset X$ and s is the natural inclusion, then this will recover the relative case. See Section A.2 for more details on DF log structures. Denote by $X^{\log} = (X, \mathcal{M}_X)$ the corresponding log scheme. Instead of considering usual stable maps to the expansions of X, we investigate the usual stable maps to X, with fixed log structure $\mathcal{M}_X$ on the target, and suitable log structures on the source curves.

In the subsequent paper [AC11], we will consider target X with generalized Deligne-Faltings log structure $\mathcal{M}_X$, namely where there exists a global map $P \to \overline{\mathcal{M}}_X$ from a toric monoid P, which locally lifts to a chart. In particular, this covers many interesting cases, such as a variety with a simple normal crossings divisor, or a simple normal crossings degeneration of a variety with simple normal crossings singularities. It does not cover the case of a normal crossings divisor that is not simple. We hope one can also cover this using the descent argument along this approach.

A key point of this paper is the observation made in Section 3.2, which describes the log map on the level of characteristic monoids. This leads us to the notions of marked graphs 3.3.2 and minimality 3.5.1. Such a minimality condition can be explained as the "minimal requirements" that a log map needs to satisfy. Then our minimal stable log maps are defined to be usual stable maps with the minimal log structures. Denote by $\mathcal{K}_\Gamma(X^{\log})$ the category fibered over the category of schemes, which for any scheme T associates the groupoid of minimal stable log maps over T with numerical data Γ. We refer to Section 3.6 for the precise meaning of $\mathcal{K}_\Gamma(X^{\log})$. The main result of this paper is the following:

THEOREM 1.2.1. *The fibered category $\mathcal{K}_\Gamma(X^{\log})$ is a proper Deligne-Mumford stack. Furthermore, the natural map $\mathcal{K}_\Gamma(X^{\log}) \to \mathcal{K}_{g,n}(X,\beta)$ by removing the log structures from minimal stable log maps is representable and finite.*

Remark 1.2.2. In fact, the stack $\mathcal{K}_\Gamma(X^{\log})$ carries a universal minimal log structure, which will be denoted by $\mathcal{M}_{\mathcal{K}_\Gamma(X^{\log})}$. Thus the pair

$$(\mathcal{K}_\Gamma(X^{\log}), \mathcal{M}_{\mathcal{K}_\Gamma(X^{\log})})$$

can be viewed as a log stack in the sense of Olsson; see Section A.3. By applying the standard technique in [BF97] and replacing the usual cotangent complex by logarithmic cotangent complex [Ols05], one can produce a perfect obstruction theory of $\mathcal{K}_\Gamma(X^{\log})$ relative to $\mathfrak{M}^{\mathrm{pre}}_{g,n}$, the stack of log prestable curves defined in Section B.3. We will discuss this perfect obstruction theory and the corresponding virtual fundamental class in another paper.

Up to now, we only introduce $\mathcal{K}_\Gamma(X^{\log})$ as category fibered over $\mathfrak{Sch}$, the category of schemes. Denote by $\mathfrak{LogSch}^{\mathrm{fs}}$ the category of fine and saturated log schemes as introduced in Section A.1. The following result exhibits another important aspect of our construction:

Theorem 1.2.3. *The pair* $(\mathcal{K}_\Gamma(X^{\log}), \mathcal{M}_{\mathcal{K}_\Gamma(X^{\log})})$ *defines a category fibered over* $\mathfrak{LogSch}^{\mathrm{fs}}$, *which for any* fs *log scheme* $(S, \mathcal{M}_S)$ *associates the category of stable log maps over* $(S, \mathcal{M}_S)$.

The above categorical interpretation allows us to work systematically with fs log schemes rather than usual schemes. This point of view will be a useful tool in [AC11], where we reduce the case with generalized DF-log structure on the target to the case of this paper.

1.3. *Outline of the paper.* In Section 2, we fix a morphism of fs log schemes $X^{\log} \to B^{\log}$, define an auxiliary category $\mathcal{LM}_{g,n}(X^{\log}/B^{\log})$ of all log maps with target X, fibered over schemes, and show that it is an algebraic stack in the sense of Artin. This stack is unbounded and serves mainly as a construction technique. This will be achieved by verifying Artin's criteria [Art74, 5.1]. Here the deformation theory of our log maps will be given by the log cotangent complex developed in [Ols05].

Section 3 is aimed at the construction of minimal log maps. In fact, for each log map over a geometric point with fs log structure, we can associate a marked graph; see Construction 3.4.1. These graphs will be used to describe the minimality condition. Then we show that minimality is an open condition; see Proposition 3.5.2. This implies the algebricity of the stack of minimal log maps using the results of Section 2. In Section 3.7 we show that there are at most finitely many minimal stable log maps over a fixed underlying stable map with a fixed marked graph. The finiteness of automorphisms of minimal stable log maps over a geometric point is proved in Proposition 3.8.1.

Section 4 is devoted to proving Theorem 1.2.3. This will follow naturally from the universal property of minimal log maps in Proposition 4.1.1.

In Section 5, we will show that $\mathcal{K}_\Gamma(X^{\log})$ is of finite type by stratifying the stack and bounding the stratum associated to each marked graph. Indeed, we will prove the boundedness of $\mathcal{K}_\Gamma(X^{\log})$ relative to the stack of usual stable maps. One issue here is the construction of all maps of log structures for a given graph. We will turn this into constructing isomorphisms of corresponding line bundles.

The weak valuative criterion of $\mathcal{K}_\Gamma(X^{\log})$ is proved in Section 6. In fact, the universal property of minimality produces an extension of *minimal* stable log map, once we can find *any* extension of stable log maps, not necessarily minimal. For separateness, we first show that the marked graph is uniquely determined by the generic fiber. Then we introduce a new map $\bar{\beta}$ as in (6.4.3), which helps us compare any two possible extensions; see Lemma 6.5.1. This leads us to the uniqueness of the extension. In the end, we give a proof of Theorem 1.2.1 and show that the stack of minimal stable log maps is representable and finite over the stack of usual stable maps.

Finally, we have two appendices collecting various results of logarithmic geometry and logarithmic curves, as we need in this paper. The notion of log pre-stable curve is introduced in Definition B.2.3.

1.4. *Conventions.* Throughout this paper, we will work over an algebraically closed field of characteristic 0, denoted by $\mathbb{C}$.

The word locally always means étale locally, and neighborhood always means étale neighborhood, unless otherwise specified.

Given a scheme or algebraic stack X, a point $p \in X$ means a closed point unless otherwise specified. We denote by $\bar{p}$ an algebraic closure of p.

We usually use $X^{\log}$ to denote a log scheme $(X, \mathcal{M}_X)$ if no confusion could arise. The map $\exp : \mathcal{M}_X \to \mathcal{O}_X$ is reserved for the structure map of $\mathcal{M}_X$. Given a section $u \in \mathcal{O}_X$, we sometimes use $\log u$ to denote the corresponding section in $\mathcal{M}_X$ if no confusion could arise.

The category of schemes, fine log schemes, and fs log schemes are denoted by $\mathfrak{Sch}$, $\mathfrak{LogSch}$, and $\mathfrak{LogSch}^{\mathrm{fs}}$ respectively. See Section A.1 for the precise definitions.

The letter ξ (respectively $\xi^{\log}$) is reserved for log maps over a usual scheme (respectively log scheme). Given a log map $\xi = (C \to S, \mathcal{M}_S, f)$ over S as in Remark 3.1.5, we will denote by $\mathcal{M}_C$ the corresponding log structure on C if no confusion could arise.

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2. **Algebricity of the stack of log maps**

In this section, we prove that the stack $\mathcal{LM}_{g,n}(X^{\log}/B^{\log})$ parametrizing log maps as in Definition 2.1.5 is algebraic by checking Artin's criteria [Art74, 5.1]. The result in this section is only used to prove that the stack of minimal stable log maps $\mathcal{K}_{g,n}(X^{\log}, \beta)$, as in Definition 3.6.5, is algebraic. The reader may wish to assume the result of Theorem 2.1.10, and skip to the next section.

2.1. *Log maps over* $\mathfrak{LogSch}$ *and over* $\mathfrak{Sch}$.

Conventions 2.1.1. In this section, we fix a separated, finite type, and log flat morphism of log schemes $\pi : X^{\log} \to B^{\log}$. See [Ols03a, Def. 4.1, Th. 4.6] for the equivalent definitions of log flat morphisms. Denote by B and X the underlying schemes of $B^{\log}$ and $X^{\log}$ respectively. Let $\mathcal{M}_B$ and $\mathcal{M}_X$ be the log structure on $B^{\log}$ and $X^{\log}$ respectively. Given any B-scheme S, denote by $(X_S, \mathcal{M}_{X_S}^{X_S/S}) \to (S, \mathcal{M}_S^{X_S/S})$ the pull-back of $X^{\log} \to B^{\log}$ over S.

As an analogue of usual pre-stable maps, we introduce our log maps over log schemes as follows.

Definition 2.1.2. A log map $\xi^{\log}$ over a fine $B^{\log}$-log scheme $(S, \mathcal{M}_S)$ is a commutative diagram of log schemes

$$\begin{tikzcd}(C, \mathcal{M}_C) \arrow[rr, "f"] \arrow[dr] & & (X_S, \mathcal{M}_{X_S}) \arrow[dl] \\ & (S, \mathcal{M}_S), & \end{tikzcd} \tag{2.1.1}$$

where

(1) the arrow $(C, \mathcal{M}_C) \to (S, \mathcal{M}_S)$ is given by the log curve $(C \to S, \mathcal{M}_S^{C/S} \to \mathcal{M}_S)$ as in Definition B.2.2;
(2) the arrow $(X_S, \mathcal{M}_{X_S}) \to (S, \mathcal{M}_S)$ is obtained from the following cartesian diagram of log schemes:

$$\begin{tikzcd}(X_S, \mathcal{M}_X) \arrow[r] \arrow[d] & X^{\log} \arrow[d, "\pi"] \\ (S, \mathcal{M}_S) \arrow[r] & B^{\log}.\end{tikzcd}$$

Given a log map of fine log schemes $g : (T, \mathcal{M}_T) \to (S, \mathcal{M}_S)$, we define the pull-back $g^*\xi^{\log}$ to be the log map $\xi_T^{\log}$ over $(T, \mathcal{M}_T)$ obtained by pulling back (2.1.1) via the log map g.

The above definition gives a category of log maps fibered over $\mathfrak{LogSch}_{B^{\log}}$, the category of fine log schemes over $B^{\log}$. This category allows pull-back via arbitrary log maps, hence changes the base log structures. In another word, it only parametrizes the "log maps," without the information of log structures on the base. This is the category of most interest to us.

However, algebraic stacks are built over the category of schemes, rather than the category of log schemes. In order to have the algebraic structure, we need to introduce another fibered category over $\mathfrak{Sch}_B$, the category of B-schemes. This leads to the following definition:

Definition 2.1.3. A log map ξ over a B-scheme S consists of a fine log scheme $(S, \mathcal{M}_S)$ over $B^{\log}$ and a log map $\xi^{\log}$ over $(S, \mathcal{M}_S)$ as in Definition 2.1.2. Usually we denote it by

$$\xi = (C \to S, X_S \to S, \mathcal{M}_S^{X_S/S} \to \mathcal{M}_S, \mathcal{M}_S^{C/S} \to \mathcal{M}_S, f),$$

where $\mathcal{M}_S^{X_S/S}$ is the pull-back of $\mathcal{M}_B$ via the structure map $S \to B$.

Consider another B-scheme T and a B-scheme morphism $T \to S$. Then we have an induced strict arrow $g : (T, \mathcal{M}_T) \to (S, \mathcal{M}_S)$, where $\mathcal{M}_T := g^*(\mathcal{M}_S)$. The pull-back ξ_T of ξ via $T \to S$ is given by the log scheme $(T, \mathcal{M}_T)$, and the log map $\xi_T^{\log} = g^* \xi^{\log}$ over $(T, \mathcal{M}_T)$. In the rest of this section, if no confusion could arise, we will use $(C \to S, X_S \to S, \mathcal{M}_S, f)$ to denote the log map ξ over S.

Since isomorphisms are central to the structure of stacks, we spell out the resulting notion of an isomorphism of log maps over schemes.

Definition 2.1.4. Consider two log maps $\xi_1 = (C_1 \to S, X_S \to S, \mathcal{M}_1, f_1)$ and $\xi_2 = (C_2 \to S, X_S \to S, \mathcal{M}_2, f_2)$ over a scheme S. An isomorphism $\xi_1 \to \xi_2$ over S is given by a triple (ρ, θ, γ) fitting in the following commutative diagram of log schemes:

(2.1.2)
$$\begin{tikzcd}
(C_1, \mathcal{M}_{C,1}) \arrow[rr] \arrow[dr] \arrow[dd, "\rho"'] & & (X_S, \mathcal{M}_{X,1}) \arrow[dl] \arrow[dd, "\gamma"] \\
& (S, \mathcal{M}_1) \arrow[dd, "\theta", near start] & \\
(C, \mathcal{M}_{C,2}) \arrow[rr, crossing over] \arrow[dr] & & (X_S, \mathcal{M}_{X,2}) \arrow[dl] \\
& (S, \mathcal{M}_2), &
\end{tikzcd}$$

where

(1) The pair (ρ, θ) is an arrow of log curves $(C_1 \to S, \mathcal{M}_1) \to (C_2 \to S, \mathcal{M}_2)$ as in Definition B.3.1.

(2) The arrow θ is an isomorphism of log schemes over $B^{\log}$ fitting in the following commutative diagram:
$$\begin{tikzcd}
(S, \mathcal{M}_1) \arrow[rr, "\theta"] \arrow[dr] & & (S, \mathcal{M}_2) \arrow[dl] \\
& B^{\log}. &
\end{tikzcd}$$

(3) The arrow γ is obtained from the following cartesian diagram of log schemes:
$$\begin{tikzcd}
(X_S, \mathcal{M}_{X_S,1}) \arrow[r, "\gamma"] \arrow[d] & (X_S, \mathcal{M}_{X_S,2}) \arrow[d] \\
(S, \mathcal{M}_1) \arrow[r, "\theta"] & (S, \mathcal{M}_2).
\end{tikzcd}$$

Note that the underlying maps $\underline{\theta}$ and $\underline{\gamma}$ are both identities of the corresponding underlying schemes.

Denote by $\mathcal{I}som_S(\xi_1, \xi_2)$ the functor over S that for any S-scheme $T \to S$ associates the set of isomorphisms of $\xi_{T,1}$ and $\xi_{T,2}$ over T, where $\xi_{T,1}$ and $\xi_{T,2}$ are the pull-back of ξ_1 and ξ_2 via $T \to S$ respectively. Denote by $\mathcal{A}ut_S(\xi)$ the functor of automorphisms of ξ over S.

Definition 2.1.5. Denote by $\mathcal{LM}_{g,n}(X^{\log}/B^{\log})$ the fibered category over $\mathfrak{Sch}_B$ that for any $S \to B$ associates the groupoid of log maps ξ over S, with the underlying pre-stable curves of genus g and n-markings. For simplicity of notation, in this section we will use $\mathcal{LM}$ to denote $\mathcal{LM}_{g,n}(X^{\log}/B^{\log})$.

Remark 2.1.6. By Definition 2.1.3, we only allow pull-backs of log maps via strict log maps in $\mathcal{LM}$, hence we do not change the log structures. Thus, given a scheme S, the groupoid $\mathcal{LM}(S)$ contains all possible log structures $\mathcal{M}_S$ on S with log maps over $(S, \mathcal{M}_S)$. This is a huge stack as it parametrizes, in particular, all possible log structures on the base. One would like to consider a smaller stack parametrizing only log maps without the information of the base log structures. It will be shown in Section 4 that if we work over fs log schemes rather than the usual category of schemes for the base, then the stack we want is $\mathcal{K}^{\mathrm{pre}}_{g,n}(X^{\log})$ as introduced in Section 3.

Denote by $\mathfrak{M}_{g,n}$ the algebraic stack of genus g, n-marked pre-stable curves with the canonical log structure as in Section B.1. Consider the new algebraic stack

$$\mathfrak{B} = \mathcal{L}og_{\mathfrak{M}_{g,n}\times B^{\log}},$$

where the fibered product is in the log sense, and $\mathcal{L}og_{\bullet}$ is the log stack introduced in Section A.3. By Theorem A.3.2, the stack $\mathfrak{B}$ is algebraic over B.

Remark 2.1.7. We give the moduli interpretation of $\mathfrak{B}$ as follows. For any B-scheme S, an object $\zeta \in \mathfrak{B}(S)$ is a diagram

$$\begin{tikzcd}
(C, \mathcal{M}_C) \arrow[dr] & & (X_S, \mathcal{M}_{X_S}) \arrow[dl] \\
& (S, \mathcal{M}), &
\end{tikzcd} \tag{2.1.3}$$

where the left arrow is a family of genus g, n-marked log curves given by the induced map $(S, \mathcal{M}_S) \to \mathfrak{M}_{g,n}$ and the right arrow is given by the induced map $(S, \mathcal{M}_S) \to B^{\log}$. Consider two objects ζ_1 and ζ_2 in $\mathfrak{B}(S)$. An arrow $\zeta_1 \to \zeta_2$ over the scheme S is a triple (ρ, θ, γ) given by the following diagram

$$\begin{tikzcd}
(C_1, \mathcal{M}_{C,1}) \arrow[dr] \arrow[dd, "\rho"'] & & (X_S, \mathcal{M}_{X_S,1}) \arrow[dl] \arrow[dd, "\gamma"] \\
& (S, \mathcal{M}_1) \arrow[dd, "\theta"] & \\
(C, \mathcal{M}_{C,2}) \arrow[dr] & & (X_S, \mathcal{M}_{X_S,2}) \arrow[dl] \\
& (S, \mathcal{M}_2), &
\end{tikzcd} \tag{2.1.4}$$

where the square on the left is an isomorphism of log curves and the square on the right satisfies the condition in Definition 2.1.4(2) and (3).

Observation 2.1.8. Note that there is a natural morphism of fibered categories $\mathcal{LM} \to \mathfrak{B}$ by removing the log map f as in Definition 2.1.3. Note that any nontrivial isomorphism of a log map is a nontrivial isomorphism of the corresponding log source and target. This implies that $\mathcal{LM}$ is a pre-sheaf over $\mathfrak{B}$.

We denote by $\mathcal{K}^{\mathrm{pre}}_{g,n}(X/B)$ the stack of usual pre-stable maps with the source given by genus g, n-marked pre-stable curves, and family of targets given by $X \to B$. This is an algebraic stack over B. For simplicity of notation, we will denote this stack by $\mathcal{K}$.

Observation 2.1.9. Note that we have a natural arrow $\mathcal{LM} \to \mathcal{K}$ by removing all log structures. Given a log map ξ, denote by $\underline{\xi}$ the corresponding object in $\mathcal{K}$.

Our main result of this section is the following:

Theorem 2.1.10. *The fibered category $\mathcal{LM}$ is an algebraic stack.*

Proof. The rest of this section is devote to the proof of this theorem. The representability of the diagonal $\mathcal{LM} \to \mathcal{LM} \times \mathcal{LM}$ is proved in Section 2.2. By Observation 2.1.8, we have a natural map $\mathcal{LM} \to \mathfrak{B}$ to the algebraic stack $\mathfrak{B}$. Thus, to produce a smooth cover for $\mathcal{LM}$, it is enough to check Artin's criteria [Art74, 5.1] relative to $\mathfrak{B}$. This will be done from Section 2.3 to 2.7. ☐

2.2. *Representability of the isomorphism functors of log maps.*

Proposition 2.2.1. *Consider two log maps ξ_1 and ξ_2 over a B-scheme S as in Definition* 2.1.4. *The functor $\mathcal{I}som_S(\xi_1, \xi_2)$ is represented by an algebraic space of finite type over S.*

Proof. Using the notation as in Definition 2.1.4, Remark 2.1.7, and Observation 2.1.9, we form the following commutative diagram:

(2.2.1)
$$\begin{tikzcd}
\mathcal{I}som_S(\xi_1,\xi_2) \arrow[dr, "\phi_3"] \arrow[drr, "\phi_2"] \arrow[ddr, "\phi_1"'] & & \\
& I \arrow[r] \arrow[d] & \mathcal{I}som_S(\underline{\xi_1},\underline{\xi_2}) \arrow[d, "\psi_2"] \\
& \mathcal{I}som_S(\zeta_1,\zeta_2) \arrow[r, "\psi_1"] & \mathcal{I}som_S(\underline{\zeta_1},\underline{\zeta_2}),
\end{tikzcd}$$

where the square is cartesian and ϕ_3 is given by the universal property of fiber product. Here ζ_i is the corresponding log source and target of ξ_i given by the natural map in Observation 2.1.8 and $\underline{\xi_i}$ is the underline map of ξ_i given by the natural map in Observation 2.1.9. The object $\underline{\zeta_i}$ can be obtained by removing log structures on ζ_i, or given by the source and target of $\underline{\xi_i}$.

Note that any isomorphism of ξ_1 and ξ_2 induces trivial isomorphism of the underlying structure of the target $X_S \to S$. Thus, the sheaf $\mathcal{I}som_S(\underline{\zeta_1}, \underline{\zeta_2})$ is the sheaf of isomorphisms of the underlying curves. Since $\mathcal{I}som_S(\underline{\xi_1}, \underline{\xi_2})$, $\mathcal{I}som_S(\underline{\zeta_1}, \underline{\zeta_2})$, and $\mathcal{I}som_S(\zeta_1, \zeta_2)$ are represented by algebraic spaces of finite type over S, it is enough to show that ϕ_3 is representable and of finite type.

Consider an S-scheme U and an arrow $U \to I$ given by a pair (τ, λ), where

$$\tau \in \mathcal{I}som_S(\zeta_1, \zeta_2)(U) \quad \text{and} \quad \lambda \in \mathcal{I}som_S(\underline{\xi_1}, \underline{\xi_2})(U),$$

such that their induced elements in $\mathcal{I}som_S(\underline{\zeta_1}, \underline{\zeta_2})(U)$ coincide. Now we have a cartesian diagram

$$\begin{tikzcd} I' \arrow[r] \arrow[d] & \mathcal{I}som_S(\xi_1, \xi_2) \arrow[d] \\ U \arrow[r, "(\tau,\lambda)"'] & I. \end{tikzcd}$$

Here I' is the sheaf over U that for any $V \to U$, associated a unital set $\{*\}$ if $(\tau, \lambda)_V$ induces an isomorphism between $\xi_{1,V}$ and $\xi_{2,V}$, and the empty set otherwise. Next we will show that $I' \to U$ is a locally closed immersion of finite type.

For simplicity, we assume $U = S$, and we let $\tau = (\rho, \theta, \gamma)$ as in Remark 2.1.7. We need to show that the commutativity of the following diagram of log schemes is represented by a locally closed immersion of finite type:

$$\begin{tikzcd} (C_1, \mathcal{M}_{C_1}) \arrow[r, "f_1"] \arrow[d, "\rho"'] & (X, \mathcal{M}_{X,1}) \arrow[d, "\gamma"] \\ (C_2, \mathcal{M}_{C_2}) \arrow[r, "f_2"] & (X, \mathcal{M}_{X,2}). \end{tikzcd}$$

Since the map τ already gives an isomorphism of the underlying structure, we only need to consider the commutativity of

(2.2.2)
$$\begin{tikzcd} \mathcal{M}_{C_1} & f_1^*\mathcal{M}_{X,1} \arrow[l, "f_1^\flat"'] \\ \rho^*\mathcal{M}_{C_2} \arrow[u, "\rho^\flat"] & \rho^* \circ f_2^*\mathcal{M}_{X,2}. \arrow[l, "\rho^*\circ f_2^\flat"] \arrow[u, "\gamma^\flat"'] \end{tikzcd}$$

Our statement follows from the following lemma. ☐

Lemma 2.2.2. *The condition that* (2.2.2) *commutes is represented by a quasi-compact locally closed immersion* $Z \to S$.

Proof. The commutativity of (2.2.2) is equivalent to the equality

$$\rho^{\flat} \circ (\rho^{*} \circ f_2^{\flat}) = f_1^{\flat} \circ \gamma^{\flat}. \tag{2.2.3}$$

By [Ols03a, 3.6], the condition that (2.2.3) holds on the level of characteristic is represented by a quasi-compact open immersion $j : C^{\circ} \to C_1$. Since the question is local on S, we may further shrink S, and we assume that C_1 is a neighborhood of some fiber $C_{1,s}$ for some point $s \in S$. Since the family $C_1 \to S$ is proper, we may choose a finite set of étale maps $\{U_i \to C_1\}$ that covers the fiber $C_{1,s}$, and each open set $V_i = C^{\circ} \times_{C_1} U_i$ is connected. Note that the projection $\pi_1 : C_1 \to S$ is flat, hence is open. Thus, we obtain an open sub-set $S^{\circ} = \cap_i \pi_1(V_i) \subset S$. Replacing S by S°, we may assume that the equality (2.2.3) on the level of characteristic holds.

With this assumption on the characteristic, the proof in [Ols03a, 3.6] shows that the (2.2.3) is represented by a closed subscheme $T \subset C_1$ on the fiber. Note that the statement is locally on S. Further shrinking S, we can assume that the family $C \to S$ is projective. Now what we want is the maximal closed subscheme $Z \subset S$ parametrizing fibers $C_1 \times_S Z \subset T$ as in [Abr94, Defs. 3, 4]. Then the lemma can be deduced from the "essential free" case of [SGA3, VIII, Th. 6.5]. See [Abr94, Th. 6(3)] for the reduction argument. $\square$

Next, we check the Artin's criteria [Art74, 5.1].

2.3. *$\mathcal{LM}$ is a stack in the étale topology.* By [Art74, 1.1], or [LMB00, Def. 3.1], we need to prove the following:

(1) the isomorphism functor is a sheaf in the étale topology,
(2) any étale descent datum for an object of $\mathcal{LM}$ is effective.

Since the isomorphism functor is shown to be representable, it is a sheaf in the étale topology. For the second condition, let $\{S_i \to S\}_i$ be an étale covering of S, and let $\xi_i \in \mathcal{LM}(S_i)$ for each i. Assume that we have isomorphisms $\phi_{ij} : \xi_i|_{S_i \times_S S_j} \to \xi_j|_{S_i \times_S S_j}$ for each pair (i, j) that satisfy the cocycle condition.

For any i, let ζ_i be the corresponding log curve and target as in Remark 2.1.7 for ξ_i. Since such ζ_i is parametrized by the algebraic stack $\mathfrak{B}$, we can glue them together to obtain ζ over S, whose restriction to each S_i is ζ_i. Then étale locally we have a log map from ζ given by ξ_i. Since log maps are defined in terms of homomorphisms of étale sheaves, they can be glued from étale local data. Therefore we can glue ξ_i to obtain the log map ξ with the source curve given by ζ.

2.4. *$\mathcal{LM}$ is limit preserving.* Consider

$$R = \varinjlim R_i,$$

where $\{R_i\}$ is a filtering inductive system of neotherian rings. Set $S = \operatorname{Spec} R$ and $S_i = \operatorname{Spec} R_i$. By [Art74, §1], we need to show that the following map of groupoids is an equivalence of categories:

$$\varprojlim \mathcal{LM}(S_i) \to \mathcal{LM}(S).$$

Consider a log map $\xi = (C \to S, X_S \to S, \mathcal{M}_S, f)$ in $\mathcal{LM}(S)$. Since the stack $\mathfrak{B}$ is locally of finite type, we have the family $\zeta = (C \to S, X_S \to S, \mathcal{M}_S)$ coming from $\zeta_i = (C_i \to S_i, X_{S_i} \to S_i, \mathcal{M}_{S_i})$ over S_i for some i. Also notice that we have an induced map $S \to \mathcal{K}$ given by the underlying map. Since $\mathcal{K}$ is locally of finite type, the underlying map $\underline{f}$ comes from $\underline{f}_{i'}$ over some $S_{i'}$. We choose an index i_0 such that $i_0 > i$ and $i_0 > i'$.

It remains to consider the map of log structures $f^\flat : f^*\mathcal{M}_X \to \mathcal{M}_C$. We first introduce two stacks $\mathcal{L}^\Delta$ and $\mathcal{L}^\Lambda$ as in [Ols05, §2].

Remark 2.4.1. Consider a scheme U over $\mathbb{Z}$. Objects in $\mathcal{L}^\Delta(U)$ are commutative diagrams of log structures on U of the following form:

(2.4.1)
$$\begin{tikzcd} & \mathcal{M}_1 \arrow[dl] \arrow[dr] & \\ \mathcal{M}_2 \arrow[rr] & & \mathcal{M}_3. \end{tikzcd}$$

Objects in $\mathcal{L}^\Lambda$ are diagrams of log structures on U of the following form:

(2.4.2)
$$\begin{tikzcd} & \mathcal{M}_1 \arrow[dl] \arrow[dr] & \\ \mathcal{M}_2 & & \mathcal{M}_3. \end{tikzcd}$$

It was shown in [Ols05, 2.4] that those two stacks $\mathcal{L}^\Delta$ and $\mathcal{L}^\Lambda$ are algebraic stacks locally of finite type. Note that there is a natural morphism $\mathcal{L}^\Delta \to \mathcal{L}^\Lambda$ by dropping the bottom arrow in (2.4.1) to obtain (2.4.2).

Observation 2.4.2. Consider $\zeta = (\pi_C : C \to S, X_S \to S, \mathcal{M}_S)$, which is the family of log sources and targets constructed above. There is a natural diagram of log structures on C as follows:

(2.4.3)
$$\begin{tikzcd} & \pi_C^*\mathcal{M}_S \arrow[dl] \arrow[dr] & \\ f^*\mathcal{M}_X & & \mathcal{M}_C. \end{tikzcd}$$

This induces a natural map $C \to \mathcal{L}^\Lambda$. Consider the fiber product $\mathcal{L}^\Delta \times_{\mathcal{L}^\Lambda} C$. This gives an algebraic stack parametrizing the bottom arrows $f^\flat$ that fits in the above commutative diagram.

The map $f^\flat$ is equivalent to a map $C \to \mathcal{L}^\Delta \times_{\mathcal{L}^\Lambda} C$. Note that the algebraic stack $\mathcal{L}^\Delta \times_{\mathcal{L}^\Lambda} C$ is locally of finite presentation. By [LMB00, Prop. 4.18(i)], we have the map $f^\flat$ coming from some $f^\flat_{i_1}$ over S_{i_1} for some $i_1 > i_0$. This map is compatible with all the log structures coming from base and target. Indeed, consider the composition

$$p_j : C_j \to \mathcal{L}^\Delta \times_{\mathcal{L}^\Lambda} C_j \to C_j.$$

Applying [LMB00, Prop. 4.18(i)] again, we see that the identity $p = \mathrm{id}_C : C \to C$ comes from p_j for some $i_2 > i_1$. Thus, the map f_{i_2} also compatible with the underlying map $\underline{f}$. This proves the essential surjectivity.

The full faithfulness follows from [LMB00, Prop. 4.15(i)] and the fact that the diagonal $\mathcal{LM} \to \mathcal{LM} \times_B \mathcal{LM}$ is representable and locally of finite type.

2.5. *Deformations and obstructions.* By [Art74, Def. 5.1], it remains to find a smooth cover of $\mathcal{LM}$. As in Observation 2.1.8, we have a representable map of stack $\mathcal{LM} \to \mathfrak{B}$. Since $\mathfrak{B}$ is an algebraic stack, it would be enough to produce a smooth cover for $\mathcal{LM}_U := \mathcal{LM} \times_{\mathfrak{B}} U$, where $U \to \mathfrak{B}$ is an arbitrary smooth map. This can be done by checking Artin's criteria [Art74, 5.2] for $\mathcal{LM}_U$ relative to U. First we consider the deformations and obstructions.

Let A_0 be a reduced neotherian ring over U, and let $A' \to A \to A_0$ be an infinitesimal extension of A_0, where $A' \to A$ is surjective whose kernel I is a finite A_0-module and hence a square-zero ideal. Set $S = \operatorname{Spec} A$ and $S' = \operatorname{Spec} A'$. Consider a log map $\xi_A = (C \to S, X_S \to S, \mathcal{M}_S, f) \in \mathcal{LM}_U$. Let $\xi_0 = (C_0 \to S_0, X_{S_0} \to S_0, \mathcal{M}_{S_0}, f_0)$ be the restriction of ξ_A over A_0. Since we are over U, the log sources and targets $(C \to S, X_S \to S, \mathcal{M}_S)$ comes from the structure morphism $S \to U$. Note that we have another family of log sources and targets $(C' \to S', X_{S'} \to S', \mathcal{M}_{S'})$, which are also from the structure map $S' \to U$. To obtain a deformation of ξ_A over S', it is equivalent to producing a dotted arrow f' that fits in the following commutative diagram:

(2.5.1)
$$\begin{tikzcd}
(C,\mathcal{M}_C) \arrow[rr, "k"] \arrow[dr, "f"] \arrow[ddr, bend right] & & (C',\mathcal{M}_{C'}) \arrow[dr, dotted, "f'"] \arrow[ddr, bend right] & \\
& (X_S,\mathcal{M}_{X_S}) \arrow[rr, "j"] \arrow[d] & & (X_{S'},\mathcal{M}_{X_{S'}}) \arrow[d] \\
& (S,\mathcal{M}_S) \arrow[rr, "i"] & & (S',\mathcal{M}_{S'}).
\end{tikzcd}$$

Note that the front and back squares in (2.5.1) are cartesian squares. Let $\mathbf{L}^{\log}_{X_S/S}$ be the logarithmic cotangent complex of the log map $(X_S, \mathcal{M}_{X_S}) \to (S, \mathcal{M}_S)$ as in [Ols05]. By [Ols05, 5.9], we have the following results:

(1) there is a canonical class $o \in \operatorname{Ext}^1(f^*\mathbf{L}^{\log}_{X_S/S}, I \otimes_{A_0} \mathcal{O}_{C_0})$, whose vanishing is necessary and sufficient for the existence of a morphism f' fitting in the above diagram;

(2) if $o = 0$, then the set of such maps f' is a torsor under $\operatorname{Ext}^0(f^*\mathbf{L}^{\log}_{X_S/S}, I \otimes_{A_0} \mathcal{O}_{C_0})$.

Since the family of targets $X^{\log} \to B^{\log}$ is log flat, by [Ols05, 1.1(iv)] we have

$$\operatorname{Ext}^1(f^*\mathbf{L}^{\log}_{X_S/S}, I \otimes_{A_0} \mathcal{O}_{C_0}) \cong \operatorname{Ext}^1(f_0^*\mathbf{L}^{\log}_{X_{S_0}/S_0}, I \otimes_{A_0} \mathcal{O}_{C_0}).$$

Thus we define

$$\mathcal{D}_{\xi_A}(I) = \operatorname{Ext}^0(f^*\mathbf{L}^{\log}_{X_S/S}, I \otimes_{A_0} \mathcal{O}_{C_0}) \text{ and } \mathcal{O}_{\xi_0}(I) = \operatorname{Ext}^1(f_0^*\mathbf{L}^{\log}_{X_{S_0}/S_0}, I \otimes_{A_0} \mathcal{O}_{C_0})$$

to be the modules of deformations and obstructions respectively. Note that the log cotangent complex $\mathbf{L}^{\log}_{X_S/S}$ is bounded above with coherent cohomologies. The conditions of deformation and obstruction modules in [Art74, 5.2(4)] follows from the standard property of cohomology; see, e.g., [AV02, 5.3.4].

2.6. *Schlessinger's conditions.* By [Art74, 5.2(2)], we need to verify Schlessinger's conditions (S1) and (S2) as in [Art74, §2]. The condition (S2) follows from the cohomological description of the module of deformation $\mathcal{D}$. Next we check the condition (S1$'$) [Art74, 2.3], which is a stronger version of (S1).

Indeed, consider an infinitesimal extension $A' \to A \to A_0$ as in Section 2.5, and consider a U-algebra homomorphism $B \to A$ such that the composition $B \to A_0$ is surjective. Consider $\xi_A \in \mathcal{LM}_U(A)$. For any surjection $R \to A$, denote by $\mathcal{LM}_{\xi_A}(R)$ the category of log maps over $\operatorname{Spec} R$ whose restriction to $\operatorname{Spec} A$ is ξ_A. Then we need to show that

$$\mathcal{LM}_{\xi_A}(A' \times_A B) \to \mathcal{LM}_{\xi_A}(A') \times \mathcal{LM}_{\xi_A}(B)$$

is an equivalence of categories.

First, consider the essential surjectivity. Consider objects $\xi_{A'} \in \mathcal{LM}_{\xi_A}(A')$ and $\xi_B \in \mathcal{LM}_{\xi_A}(B)$. Set $\xi_{A'} = (\zeta_{A'}, f_{A'})$ and $\xi_B = (\zeta_B, f_B)$, where $\zeta_{A'}$ and ζ_B are the corresponding log sources and targets as in Remark 2.1.7. Note that the two families $\zeta_{A'}$ and ζ_B correspond to maps $\operatorname{Spec} A' \to U$ and $\operatorname{Spec} B \to U$, which induce the same map $\operatorname{Spec} A \to U$ by restricting to $\operatorname{Spec} A$. Then we can glue them together to obtain a map $\operatorname{Spec} B \times_A A' \to U$. This induces a family $\zeta_{B\times_A A'}$ over $\operatorname{Spec} B \times_A A'$, whose restrictions to $\operatorname{Spec} A'$ and $\operatorname{Spec} B$ are $\zeta_{A'}$ and ζ_B respectively. Since the stack $\mathcal{K}$ parametrizing the underlying maps is algebraic, the same argument as above produces a gluing $\underline{f}_{A'\times_A B}$ of $\underline{f}_{A'}$ and $\underline{f}_B$.

It remains to produce a compatible morphism of log structures $f^\flat_{A'\times_A B}$. Next we choose an affine open cover $V_{B\times_A A'} = \bigcup_i V_i$ of the log source curve in $\zeta_{B\times_A A'}$; its restrictions to A' and B give the affine open covers V_B and V_A for curves of $\zeta_{A'}$ and ζ_B respectively. Consider the stack

$$\mathcal{L}^{\Delta} \times_{\mathcal{L}^{\Lambda}} C_{A'}$$

induced by $\zeta_{A'}$ and the map $\underline{f}_A$ as in Observation 2.4.2. Similarly, we have stack

$$\mathcal{L}^{\Delta} \times_{\mathcal{L}^{\Lambda}} C_B$$

induced by ζ_B and $\underline{f}_B$. They can be glued to give $\mathcal{L}^{\Delta} \times_{\mathcal{L}^{\Lambda}} C_{A'\times_A B}$, which corresponds to $\zeta_{A'\times_A B}$. Consider the maps $V_{A'} \to \mathcal{L}^{\Delta} \times_{\mathcal{L}^{\Lambda}} C_{A'}$ and $V_B \to \mathcal{L}^{\Delta} \times_{\mathcal{L}^{\Lambda}} C_B$ induced by $f_{A'}$ and f_B respectively. Note that these maps can be glued together and descend to a map

$$C_{A'\times_A B} \to \mathcal{L}^{\Delta} \times_{\mathcal{L}^{\Lambda}} C_{A'\times_A B}.$$

This induce a map of log structures

$$f^{\flat}_{A'\times_A B} : \underline{f}^{*}_{A'\times_A B}\mathcal{M}_{X_{A'\times_A B}} \to \mathcal{M}_{C_{A'\times_A B}}.$$

By construction, $f^{\flat}_{A'\times_A B}$ is compatible with $\zeta_{A'} \times_A B$ and the underlying map $\underline{f}_{A'\times_A B}$.

The full faithfulness follows from the representability of isomorphism functor of log maps.

2.7. *Compatibility with formal completions.* Let $\hat{A}$ be a complete local ring, and let m be the maximal ideal of $\hat{A}$. Set $A_n = \hat{A}/m^n$, $S = \operatorname{Spec}\hat{A}$, and $S_n = \operatorname{Spec} A_n$. Since we work over a fixed chart $U \to \mathcal{LM}$, it is enough to consider a family of log maps $\{\xi_n = (C_n \to S_n, X_{S_n} \to S_n, \mathcal{M}_S, f_n)\}_n$ such that $\xi_n \in \mathcal{LM}_U(S_n)$, and $\xi_n|_{S_k} = \xi_k$ for any $n \geq k$. According to [Art74, 5.2(3)], we need to show that there exists an element $\xi \in \mathcal{LM}_U(S)$ such that $\xi|_{S_n} = \xi_n$ for any n.

Denote by $\zeta_n = (C_n \to S_n, X_{S_n} \to S_n, \mathcal{M}_{S_n})$ the family of log sources and targets of ξ_n. For each n, there is a map $S_n \to U$ induced by ζ_n such that they fit in the following commutative diagrams for any $k \leq n$:

$$\begin{tikzcd} S_n \arrow[r] \arrow[d] & U. \\ S_k \arrow[ur] & \end{tikzcd}$$

Thus the above diagram induces a map $S \to U$, whose restriction to S_n is the map given by ζ_n as above. By pulling back the universal family over U, we obtain a family of log sources and targets $\zeta = (C \to S, X_S \to S, \mathcal{M}_S)$. Note that $\zeta|_{S_n} = \zeta_n$ for any n.

Denote by $\underline{\xi}_n$ the usual pre-stable map over S_n. Consider the family of compatible underlying maps $\{\underline{\xi}_n\}$. By [Gro61, 5.4.1], there exists a unique underlying map (up to a unique isomorphism) $\underline{f} : C \to X_S$ such that $\underline{f}|_{S_n} = \underline{f}_n$.

Now to construct ξ, we need to construct a log map $f : (C, \mathcal{M}_C) \to (X_S, \mathcal{M}_{X_S})$, which is compatible with the underlying map $\underline{f}$ and f_n for all n. By definition of log maps, this is equivalent to constructing a map of log structures $f^\flat : \underline{f}^*\mathcal{M}_{X_S} \to \mathcal{M}_C$, which is compatible with $f_n^\flat$ and the log structure $\mathcal{M}_S$ on the base. For simplicity, set $\mathcal{M} = \underline{f}^*\mathcal{M}_{X_S}$.

Choose an affine étale cover of C such that over each affine chart $V \to C$, the log structures $\mathcal{M}|_V$, $\mathcal{M}_C|_V$, and $\pi_1^*\mathcal{M}_S$ can be obtained by taking the log structures associated to $\Gamma(\mathcal{M}, V) \to \mathcal{O}_V$, $\Gamma(\mathcal{M}_C, V) \to \mathcal{O}_V$, and $\pi_1^*\mathcal{M}_S \to \mathcal{O}_V$ respectively. Since the charts of fine log structures always exist étale locally, the above choice of cover exists. We first construct $f^\flat$ on such chart V.

Set $V_n = V \times_S S_n$. Then the canonical map $V_n \to C_n$ gives an affine étale chart. Consider the compatible families of monoids $\{\Gamma(\mathcal{M}_n, V_n)\}_n$ and $\{\Gamma(\mathcal{M}_{C_n}, V_n)\}_n$. For simplicity, let $\mathcal{N}$ be one of the monoids $\Gamma(\mathcal{M}, V)$, $\Gamma(\mathcal{M}_C, V)$, or $\Gamma(\pi_1^*\mathcal{M}_S, V)$, and let $\mathcal{N}_n$ be one of the corresponding reductions $\Gamma(\mathcal{M}_n, V_n)$, $\Gamma(\mathcal{M}_{C_n}, V_n)$, or $\Gamma(\pi_1^*\mathcal{M}_{S_n}, V_n)$. Denote by $q_n : \mathcal{N} \to \mathcal{N}_n$ the restriction map and by $p_n : \varprojlim \mathcal{N}_n \to \mathcal{N}_n$ the canonical map. Let p_{nk} : $\mathcal{N}_n \to \mathcal{N}_k$ be the restriction map for all $k \leq n$. Assume that $V_n = \operatorname{Spec} R_n$ and $R = \varprojlim R_n$. Thus, we write $V = \operatorname{Spec} R$.

Note that inverse limit exists in the category of monoids and their formation commutes with the forgetful functor to the category of sets ([Ogu06, Ch. I, 1.1]). Furthermore, the inverse limit of a family of integral monoids is again integral ([Ogu06, Ch. I, 1.2]). Consider an element $e \in \mathcal{N}$. This induces a family of compatible elements $\{q_n(e)\}_n \in \varprojlim \mathcal{N}_n$. In this way, we obtain a canonical map of integral monoids:

$$p : \mathcal{N} \to \varprojlim \mathcal{N}_n.$$

LEMMA 2.7.1.

(1) *Consider an element $e \in \mathcal{N}_n$. Then $p_{nk}(e) \in R_k^*$ for some $k \leq n$ if and only if $e \in R_n^*$. Furthermore, the map p_{nk} induces a natural isomorphism $\bar{p}_{nk} : \mathcal{N}_n/R_n^* \to \mathcal{N}_k/R_k^*$.*

(2) *Consider an element $e \in \mathcal{N}$. Then $q_n(e) \in R_n^*$ for some n if and only if $e \in R^*$. Furthermore, the map q_n induces a natural isomorphism of monoids $\bar{q}_n : \mathcal{N}/R^* \to \mathcal{N}_n/R_n^*$.*

(3) *There is a natural inclusion $R^* \hookrightarrow \varprojlim \mathcal{N}_n$ that fits in the following commutative diagram:*

$$\begin{tikzcd} & R^* \arrow[dl] \arrow[dr] & \\ \mathcal{N} \arrow[rr, "p"] & & \varprojlim \mathcal{N}_n, \end{tikzcd} \tag{2.7.1}$$

where the left side arrow is the natural inclusion of units given by the corresponding log structures.

(4) *The canonical projection* $p_n : \varprojlim \mathcal{N}_n \to \mathcal{N}_n$ *induces an isomorphism of monoids*
$$\bar{p}_n : (\varprojlim \mathcal{N}_n)/R^* \to \mathcal{N}_n/R_n^*.$$
(5) *The canonical map* $p : \mathcal{N} \to \varprojlim \mathcal{N}_n$ *induces an isomorphism of monoids*
$$\bar{p} : \mathcal{N}/R^* \to (\varprojlim \mathcal{N}_n)/R^*.$$

Proof. The first part of Statement (1) follows from the following commutative diagram:

$$\begin{tikzcd} \mathcal{N}_n \arrow[r, "p_{nk}"] \arrow[d] & \mathcal{N}_k \arrow[d] \\ R_n \arrow[r] & R_k, \end{tikzcd}$$

where the two vertical maps are given by the structure morphism of the corresponding log structures. This immediately implies the existence of $\bar{p}_{nk}$. The surjectivity of p_{nk} for any $k \le n$ implies that $\bar{p}_{nk}$ is also surjective. To see the injectivity, consider two elements $a, b \in \mathcal{N}_n$ such that $p_{nk}(a) = p_{nk}(b) + \log u$ for some $u \in R_k^*$. Without loss of generality, we can assume that $a + c' = b + c$ in $\mathcal{N}_n$. Thus $p_{nk}(c') = p_{nk}(c) + \log u$, which implies $p_{nk}(c' - c) \in R_k^*$, hence $c' - c \in R_n^*$. This proves the second part of Statement (1).

Statement (2) can be proved similarly as the first one.

To prove (3), consider $e \in \varprojlim \mathcal{N}_n$, which can be represented by a family of compatible elements $\{e_n \in \mathcal{N}_n\}_n$. Assume that $e_{n'} \in R_{n'}^*$ for some n'. Then the first statement implies that $e_n \in R_n^*$ for all n. Thus we have a unique element $e \in R^* \subset R$ such that $e|_{V_n} = e_n$. This induces a canonical map $R^* \hookrightarrow \varprojlim \mathcal{N}_n$. Now the commutativity of (2.7.1) can be checked directly.

In fact the above argument proves that $p_n(e) \in R_n^*$ for some $e \in \varprojlim \mathcal{N}_n$ if and only if $e \in R^*$. Thus we obtain a canonical map $\bar{p}_n : (\varprojlim \mathcal{N}_n)/R^* \to \mathcal{N}_n/R_n^*$. Note that we have the following commutative diagram:

(2.7.2)
$$\begin{tikzcd} \mathcal{N} \arrow[d, "p"'] \arrow[dr, "q_n"] & \\ \varprojlim \mathcal{N}_n \arrow[r, "p_n"] & \mathcal{N}_n. \end{tikzcd}$$

The surjectivity of q_n implies that $\bar{p}_n$ is also surjective. The injectivity of $\bar{p}_n$ can be proved similarly as for the first statement. This proves (4).

Finally, note that (2.7.2) induces a commutative diagram

(2.7.3)
$$\begin{tikzcd} \mathcal{N}/R^* \arrow[d, "\bar{p}"'] \arrow[dr, "\bar{q}_n"] & \\ (\varprojlim \mathcal{N}_n)/R^* \arrow[r, "\bar{p}_n"] & \mathcal{N}_n/R_n^*. \end{tikzcd}$$

Since both $\bar{q}_n$ and $\bar{p}_n$ are isomorphisms of monoids, we conclude that $\bar{p}$ is also an isomorphism. This proves (5). $\square$

PROPOSITION 2.7.2. *The map of monoids* $p : \mathcal{N} \to \varprojlim \mathcal{N}_n$ *is an isomorphism.*

Proof. By Lemma 2.7.1(3) and (5), we have a commutative diagram

$$\begin{tikzcd} \mathcal{N} \arrow[r, "p"] \arrow[d] & \varprojlim \mathcal{N}_n \arrow[d] \\ \mathcal{N}/R^* \arrow[r, "\bar{p}"] & \varprojlim \mathcal{N}_n/R^*. \end{tikzcd}$$

Pick two sections $e, e' \in \mathcal{N}$ such that $p(e) = p(e')$. Denote by $\bar{e}$ and $\bar{e}'$ the corresponding images in $\mathcal{N}/R^*$. It follows from Lemma 2.7.1(5) that $\bar{e} = \bar{e}'$. Thus, we have $e = e' + \log u$ for some $u \in R^*$. But the assumption $p(e) = p(e')$ implies that $p(u) = 1 \in \varprojlim \mathcal{N}_n$. By Lemma 2.7.1(3), we have $u = 1$. This proves that p is also injective.

To prove the surjectivity, consider an element $a \in \varprojlim \mathcal{N}_n$. Since $\bar{p}$ is an isomorphism, denote by $\bar{a}$ the image of a in $\mathcal{N}/R^*$. Let a' be a lifting of $\bar{a}$ in $\mathcal{N}$. Then there exists an element $u \in R^*$ such that $a = p(a') + \log u$. Thus $p(a' + \log u) = a$. $\square$

Pick an element $\{e_n \in \Gamma(\mathcal{M}_n, V_n)\}_n \in \varprojlim \Gamma(\mathcal{M}_n, V_n)$. We obtain a compatible family $\{f_n(e_n)\}_n \in \varprojlim \Gamma(\mathcal{M}_{C_n}, V_n)$. Thus the compatible morphism of log structures $\{f_n\}$ induces a natural map of monoids

$$\varprojlim \Gamma(\mathcal{M}_n, V_n) \to \varprojlim \Gamma(\mathcal{M}_{C_n}, V_n).$$

By Proposition 2.7.2, we have a natural map of monoids

$$\Gamma(f^\flat, V) : \Gamma(\mathcal{M}, V) \to \Gamma(\mathcal{M}_C, V).$$

Next we show that $\Gamma(f^\flat, V)$ induces a map of log structures $f^\flat_V : \mathcal{M}|_V \to \mathcal{M}_C|_V$. Since the two log structures $\mathcal{M}|_V$ and $\mathcal{M}_C|_V$ can be obtained from $\Gamma(\mathcal{M}, V)$ and $\Gamma(\mathcal{M}_C, V)$ respectively, it is enough to show that the following diagram is commutative:

$$\begin{tikzcd} \Gamma(\mathcal{M}, V) \arrow[rr, "\Gamma(f^\flat, V)"] \arrow[dr, "\exp_1"'] & & \Gamma(\mathcal{M}_C, V) \arrow[dl, "\exp_2"] \\ & R, & \end{tikzcd}$$

where $\exp_1$ and $\exp_2$ are the structure morphism of the corresponding log structures. To see this, consider any section $s \in \Gamma(\mathcal{M}, V)$. Since $\exp_1(s)|_{S_n} = \exp_1 \circ \Gamma(f^\flat, V)(s)|_{S_n}$ for any n, we have $\exp_1(s) = \exp_1 \circ \Gamma(f^\flat, V)(s)$. This proves the commutativity.

We claim that $f_V^\flat$ is compatible with the log structure on the base. This is equivalent to showing the commutativity of the following diagram of log structures on V:

$$\begin{tikzcd} & \pi_1^*\mathcal{M}_S|_V \arrow[dl, "\pi_2^\flat"'] \arrow[dr, "\pi_1^\flat"] & \\ \mathcal{M}|_V \arrow[rr, "f_V^\flat"] & & \mathcal{M}_C|_V, \end{tikzcd} \tag{2.7.4}$$

where $\pi_1 : C \to S$ and $\pi_2 : X_S \to S$ are the projections. Note that $\pi_1^*\mathcal{M}_S|_V$ can be obtained by taking the log structure associated to $\Gamma(\pi_1^*\mathcal{M}_S, V) \to \mathcal{O}_V$. Hence to verify the commutativity of (2.7.4), it is enough to show that the following diagram is commutative:

$$\begin{tikzcd} & \Gamma(\pi_1^*\mathcal{M}_S, V) \arrow[dl, "\pi_2^\flat"'] \arrow[dr, "\pi_1^\flat"] & \\ \Gamma(\mathcal{M}, V) \arrow[rr, "f_V^\flat"] & & \Gamma(\mathcal{M}_C, V). \end{tikzcd}$$

This follows from the definition of $f_V^\flat$, and we have the following commutative diagram for each n:

$$\begin{tikzcd} & \pi_{1,n}^*\mathcal{M}_{S_n} \arrow[dl] \arrow[dr] & \\ f^*\mathcal{M}_n \arrow[rr, "f_n^\flat"] & & \mathcal{M}_{C_n}. \end{tikzcd}$$

Thus we obtain the desired map $f_V^\flat$ over each affine chart V.

Finally, notice that the construction of $f_V^\flat$ is functorial. Hence, we are able to obtain a global map $f^\flat$ by gluing $f_V^\flat$ on each affine chart. This finishes the proof of compatibility with formal completions.

3. **Minimal logarithmic maps to rank-one Deligne-Faltings log pairs**

3.1. *Basic definitions and notation.*

Definition 3.1.1. We call the log scheme $X^{\log} = (X, \mathcal{M}_X)$ a *rank-one Deligne-Faltings pair* if

(1) X is a projective variety over $\mathbb{C}$;
(2) $\mathcal{M}_X$ is a DF log structure on X as in Definition A.2.1, with a global presentation $\mathbb{N} \to \overline{\mathcal{M}}_X$.

Remark 3.1.2. The results in Section 3 and 4 still hold if we assume X to be only separated of finite type over $\mathbb{C}$. However, the projectivity is essential for the properness of the stack $\mathcal{K}_\Gamma(X)$ as in Definition 3.6.5.

Conventions 3.1.3. In the rest of this paper, we fix a Deligne-Faltings pair $(X, \mathcal{M}_X)$ as our target of log maps, with a global presentation $\mathbb{N} \to \overline{\mathcal{M}}_X$. Denote by (L, s) the pair consisting of a line bundle L and a morphism of sheaves $s : L \to \mathcal{O}_X$ corresponding to $\mathcal{M}_X$. Let D be the vanishing locus of the section $s \in H^0(L^\vee)$. Denote by δ the standard generator of $\mathbb{N}$. For convenience, locally we identify δ with its image in $\overline{\mathcal{M}}_X$.

Remark 3.1.4. Note that if $s = 0$, then $D = X$. If s is not a zero section, then D is a divisor in X. Thus, we have $L = \mathcal{O}_X(-D)$, with the natural inclusion $s : \mathcal{O}_X(-D) \hookrightarrow \mathcal{O}_X$. The section δ locally lifts to a section in $\mathcal{O}_X$, whose vanishing locus gives the divisor D.

Remark 3.1.5. The target $X^{\log}$ should be viewed as a log scheme over a point with trivial log structures. Thus, we can simplify the notation in Section 2.1 as follows. A log map over a usual scheme S is given by the triple $(C \to S, \mathcal{M}_S, f)$, where $(C \to S, \mathcal{M}_S)$ is a log curve in Definition B.2.2 and $f : (C, \mathcal{M}_C) \to (X, \mathcal{M}_X)$ is a log map.

Consider two log maps $\xi = (C \to S, \mathcal{M}_S, f)$ and $\xi' = (C' \to S, \mathcal{M}'_S, f')$ over a scheme S. An arrow $\xi \to \xi'$ over S is a pair (ρ, θ) as in Definition B.3.1 such that the following diagram commutes:

$$\begin{tikzcd}
 & & (X, \mathcal{M}_X) \\
(C, \mathcal{M}_C) \arrow[r, "\rho"] \arrow[d] \arrow[urr, bend left, "f"] & (C', \mathcal{M}_{C'}) \arrow[d] \arrow[ur, bend right, "f'"'] & \\
(S, \mathcal{M}_S) \arrow[r, "\theta"] & (S, \mathcal{M}'_S), &
\end{tikzcd}$$

where the square is a cartesian square of log schemes. This is compatible with Definitions 2.1.3 and 2.1.4.

3.2. *Log maps on the level of characteristics.* Consider a log map $\xi = (\pi : C \to S, \mathcal{M}_S, f)$ as in Remark 3.1.5, where $S = \operatorname{Spec} k$ is a geometric point and $(C \to S, \mathcal{M}_S)$ is a log pre-stable curve. Pick a point $p \in C$ that sits in an irreducible component Z. We have a map of characteristic monoids:

$$\bar{f}^\flat : f^*(\overline{\mathcal{M}}_X)_p \to \overline{\mathcal{M}}_{C,p}. \tag{3.2.1}$$

First consider the case p is a smooth nonmarked point. By the description in Definition B.2.1, we have $\bar{f}^\flat(\delta) = e \in \overline{\mathcal{M}}_S$ at p. By [Ols03a, 3.5(i),(iii)], the equality $\bar{f}^\flat(\delta) = e$ lifts to an étale neighborhood of p.

Definition 3.2.1. We call e *the degeneracy of* Z. Note that if $p \notin D$ for some $p \in Z$, then the image e vanishes in $\overline{\mathcal{M}}_S$. A component Z is called *degenerate* if its degeneracy is not zero. This is equivalent to saying that Z maps to D via f.

Next, we consider the case where p is a marked point. Locally at p, we have $\mathcal{M}_C \cong \pi^*\mathcal{M}_S \oplus_{\mathcal{O}^*_C} \mathcal{N}$, where $\mathcal{N}$ is the canonical log structure associated to the marked point p. Then on the level of characteristic, we have

$$\bar{f}^\flat(\delta) = e + c_p \cdot \sigma_p, \tag{3.2.2}$$

where $e \in \overline{\mathcal{M}}_S$, the element σ_p is the generator of $\overline{\mathcal{N}}_p$, and c_p is a nonnegative integer.

Observation 3.2.2. When we generalize (3.2.2) to nearby smooth points, any lifting of σ_p in the structure sheaf becomes invertible. Thus, the element e is the degeneracy of the component Z containing p.

Definition 3.2.3. We call c_p *the contact order of f at p.*

LEMMA 3.2.4. *Consider a log map $\xi = (C' \to S', \mathcal{M}_{S'}, g)$ over a scheme S' and a marking Σ_i on C'. There is an open subset in S' such that the contact order along the fixed marking Σ_i is constant.*

Proof. Consider the relative characteristic $\overline{\mathcal{M}}_{C'/S'}$. This is a locally constant sheaf along Σ_i, with stalks given by $\mathbb{N}$. Thus along Σ_i there is a map of locally constant sheaves $g^*\overline{\mathcal{M}}_X \to \overline{\mathcal{M}}_{C'/S'}$, which locally at $p \in \Sigma_i$ is given by $\mathbb{N} \to \mathbb{N}$ by $1 \mapsto c$, for some positive integer c. Note that the correspondence $1 \mapsto c$ can be generalize to the nearby points of p. Therefore it forms an open condition on the base. □

Remark 3.2.5. When D is a divisor, the contact order of a marked point Σ in a nondegenerate component can be identified with the local intersection multiplicity $(C \cdot D)_\Sigma$.

Finally, let us consider the case where p is a node joining two irreducible components Z and Z'. Let e_p be the element in $\overline{\mathcal{M}}_S$ smoothing the node p. Denote by $\log x_p$ and $\log y_p$ the elements in $\overline{\mathcal{M}}_C$ given by the local coordinates of the two components Z and Z' at p respectively as in Section B.4. Then locally at p we have the equation in $\overline{\mathcal{M}}_C$:

$$e_p = \log x_p + \log y_p. \tag{3.2.3}$$

Thus, without loss of generality we can assume that

$$\bar{f}^\flat(\delta) = e + c_p \cdot \log x_p, \tag{3.2.4}$$

where c_p is a positive integer.

Definition 3.2.6. The integer c_p is called *the contact order of f at the node p*. If $c_p \neq 0$, then p is called a *distinguished node*. A point $p \in C$ is called a *distinguished point* if it is a marked point or node with nontrivial contact order. Otherwise, it is called *nondistinguished point*.

Lemma 3.2.7. *Using the notation as above, the degeneracy of Z is e and the degeneracy of Z' is $e + c_p \cdot e_p$.*

Proof. When we generalize (3.2.4) to a smooth point in Z', the section y becomes invertible. Then the statement for Z' follows from the definition of degeneracy of a component. For Z, the proof is similar. □

Lemma 3.2.7 gives a way to put a partial order on the set of irreducible components as follows.

Definition 3.2.8. Using the notation as above, we call Z the *lower component* of p and Z' the *upper component* of p.

Lemma 3.2.9. *Consider a log map $\xi = (C' \to S', \mathcal{M}_{S'}, g)$ and a connected singularity $p \subset C'$. There is an open subset in S', such that over each fiber we have that either the node p is smoothed out, or its contact order remains the same.*

Proof. The proof is similar to the one for Lemma 3.2.4. □

3.3. *Marked graph.* We next introduce the marked graph that will be used later to describe the combinatorial data associated to log maps.

Definition 3.3.1. A *weighted graph G* is a connected graph with the following data:

(1) a subset $V_n(G) \subset V(G)$ of the set of vertices of G, which is called the *set of nondegenerate vertices*;
(2) for each edge $l \in E(G)$, we associate a nonnegative integer weight c_l called the *contact order* of l.

Note that the set $V_n(G)$ can be empty. If the contact order of an edge l is zero, then l is called the *nondistinguished edge*; otherwise it is called a *distinguished edge*. Two vertices are called *adjacent* if they are connected by an edge. Denote by $\underline{G}$ the underlying graph of G, obtained by removing all weights.

Definition 3.3.2. Consider a weighted graph G as in the above definition. An *orientation* on G is an orientation on the underlying graph $\underline{G}$, except that we allow some edges to be *nonoriented*, i.e., an edge with two directions. Consider an edge l from v_1 to v_2 under the orientation. Then v_1 is called the *initial vertex of l*, and v_2 is called the *end vertex of l*. We denote this by $v_1 \leq v_2$. If l is orientated, then we write $v_1 < v_2$.

An orientation on G is called *compatible* if

(1) an edge $l \in E(G)$ is nonoriented if and only if $c_l = 0$;
(2) if $v \in V_n(G)$, then for any other adjacent vertex v' of v we have $v \leq v'$.

Note that if $v, v' \in V_n(G)$, then any edges between them is nonoriented. The graph G is called a *marked graph* if it is a weighted graph with a compatible orientation.

A *path* is a nonrepeated sequence of edges $(l_1, l_2, \cdots, l_m)$ such that the end vertex of l_j is the initial vertex of l_{j+1}. Such a path is called a *cycle* if the initial vertex of l_1 is the end vertex of l_m. A cycle is called *strict* if it contains at least one oriented edges. A vertex $v \in V(G)$ is called *minimal* (respectively *maximal*) if it is not the end (respectively initial) vertex of any oriented edge. Thus by condition (2) above, any vertex $v \in V_n(G)$ is minimal.

Construction 3.3.3. Consider a marked graph G as in Definition 3.3.2. For each edge $l \in E(G)$ (respectively each vertex $v \in V(G)$), we introduce a variable e_l (respectively e_v), which is called the *element associated to* l (respectively v). For any $v \in V_n(G)$, we set

$$h_v : e_v = 0. \tag{3.3.1}$$

Consider an edge $l \in E(G)$ with initial vertex v_1 and end vertex v_2. We associate an equation

$$h_l : e_{v_2} = e_{v_1} + c_l \cdot e_l. \tag{3.3.2}$$

Consider the monoid

$$M(G) = \Big\langle e_v, e_l \;\Big|\; v \in V(G),\ l \in E(G) \Big\rangle \Big/ \Big\langle h_l, h_v \;\Big|\; l \in E(G),\ v \in V_n(G) \Big\rangle. \tag{3.3.3}$$

Denote by $T(G)$ the torsion part of $M(G)^{\mathrm{gp}}$. Then we have the following composition:

$$M(G) \to M(G)^{\mathrm{gp}} \to M(G)^{\mathrm{gp}}/T(G).$$

Denote by $N(G)$ the image of $M(G)$ in $M(G)^{\mathrm{gp}}/T(G)$ and by $\overline{\mathcal{M}}(G)$ the saturation of $N(G)$ in $M(G)^{\mathrm{gp}}/T(G)$.

Definition 3.3.4. The monoid $\overline{\mathcal{M}}(G)$ constructed above is called the *associated monoid* of the marked graph G.

Note that $N(G)$ is the image of $M(G)$ in $\overline{\mathcal{M}}(G)$. By the definition of $\overline{\mathcal{M}}(G)$ and Proposition A.1.1, we have the following:

LEMMA 3.3.5. *By viewing $N(G)$ and $\overline{\mathcal{M}}(G)$ as sub-monoids of $\overline{\mathcal{M}}(G)^{\mathrm{gp}} = M(G)^{\mathrm{gp}}/T(G)$, we have that for any $a \in \overline{\mathcal{M}}(G)$, there exist $b \in N(G)$ and a positive integer m such that $b = m \cdot a$.*

Definition 3.3.6. The marked graph G is called *admissible* if $\overline{\mathcal{M}}(G)$ is a sharp monoid, and the image of e_l in $\overline{\mathcal{M}}(G)$ is nontrivial for all $l \in E(G)$.

For example, consider a loop l attached to some vertex v. If $c_l \neq 0$, then (3.3.2) would imply that $e_l = 0$. In general, we have the following result:

COROLLARY 3.3.7. *If G is admissible, then there is no strict cycle in G.*

Proof. If there is a strict cycle $(l_1, \cdots, l_k)$, then we have $\sum_{i=1}^{k} c_{l_i} e_{l_i} = 0$. The strictness implies that at least one of the c_{l_i} is nonzero. Thus, the monoid $\overline{\mathcal{M}}(G)$ fails to be sharp, which contradicts the admissibility assumption. $\square$

Note that when G is admissible, the monoid $\overline{\mathcal{M}}(G)$ generates a strongly convex rational cone $C(\overline{\mathcal{M}}(G))$ in the vector space $\overline{\mathcal{M}}(G)^{\mathrm{gp}} \otimes \mathbb{Q}$ (see [Ful93, p. 4]).

LEMMA 3.3.8. *Consider an irreducible element $e \in \overline{\mathcal{M}}(G)$, where G is admissible. Assume that e lies on an extremal ray of $C(\overline{\mathcal{M}}(G))$. Then at least one of the following holds*:

(1) *there is a positive integer n and a minimal vertex v such that $n \cdot e = e_v$,*
(2) *there is a positive integer n and an edge l such that $n \cdot e = e_l$.*

Proof. Let n be the minimal positive integer such that $n \cdot e \in N(G)$. Assume that $n \cdot e = b + c$ with $b, c \in N(G)$. Note that e generates an extremal ray of the strongly convex rational cone $C(\overline{\mathcal{M}}(G))$. Thus we have positive numbers n_1 and n_2 such that $b = n_1 \cdot e$ and $c = n_2 \cdot e$. The minimality of n implies that either $b = 0$, or $c = 0$. Since b and c in $N(G)$ are elements associated to edges or vertices, the element $n \cdot e$ must satisfy one of the two possibilities above. $\square$

3.4. *Marked graphs associated to log maps.* Consider a log map $\xi = (C \to S, \mathcal{M}_S, f)$ over a geometric point S such that the log structure $\mathcal{M}_S$ is fs.

Construction 3.4.1. We construct a weighted graph G_ξ of ξ with an orientation as in Definition 3.3.2:

(1) The underlying graph $\underline{G_\xi}$ is given by the dual graph of the curve C.
(2) The subset $V_n(G)$ consists of the vertices corresponding to nondegenerate components.
(3) For each edge $l \in E(G_\xi)$, we associate a nonnegative integers c_l, where c_l is the contact order of the node l as in Definition 3.2.6.
(4) Let $l \in E(G_\xi)$ be a node joining two irreducible components $v_1, v_2 \in V(G_\xi)$. Then we define an orientation by putting $v_1 \leq v_2$ if v_1 is the lower component and v_2 is the upper component of l as in Definition 3.2.8.

Consider a node $l \in E(G_\xi)$. Denote by e_l' the element in $\overline{\mathcal{M}}_S$ smoothing l and by e_l the element associated to l in $\overline{\mathcal{M}}(G_\xi)$. Then consider an irreducible component $v \in V(G_\xi)$. Denote by e_v' the degeneracy of v in ξ and by $e_v \in \overline{\mathcal{M}}(G_\xi)$ the element associated to v. We define a correspondence

$$e_l \mapsto e_l' \quad \text{and} \quad e_v \mapsto e_v'. \tag{3.4.1}$$

PROPOSITION 3.4.2. *Assuming that $\mathcal{M}_S$ is* fs, *the correspondence* (3.4.1) *induces a canonical morphism of monoids*:

$$\phi : \overline{\mathcal{M}}(G_\xi) \to \overline{\mathcal{M}}_S.$$

Proof. Note that (3.4.1) induces a map $M(G_\xi) \to \overline{\mathcal{M}}_S$. By Proposition A.1.1, this induces a unique map $\phi' : M(G_\xi)^{\mathrm{Sat}} \to \overline{\mathcal{M}}_S$. Since the monoid $\overline{\mathcal{M}}_S$ is sharp, if $e \in M(G_\xi)^{\mathrm{Sat}}$ is torsion, then $\phi'(e) = 0$. Thus, there is a unique map $\phi : \overline{\mathcal{M}}(G_\xi) \to \overline{\mathcal{M}}_S$ induced by ϕ'. □

COROLLARY 3.4.3. *The graph G_ξ is an admissible marked graph.*

Proof. The compatibility of the orientation follows from Lemma 3.2.7. Let us consider the admissibility. First note that e_l is nontrivial for any $l \in E(G)$, since its image in $\overline{\mathcal{M}}_S$ is the element smoothing the node l, which is nontrivial. For any element $a \in \overline{\mathcal{M}}(G_\xi)$, if a is invertible, then by Lemma 3.3.5, there exists some positive integer m such that $m \cdot a = \sum_i d_i e_i$, where e_i are elements associated to some edges or vertices and d_i are nonnegative integers. Since the monoid $\overline{\mathcal{M}}_S$ is sharp, we have $\phi(a) = \sum_i d_i \phi(e_i) = 0$ in $\overline{\mathcal{M}}_S$. If $d_i \neq 0$, then $\phi(e_i) = 0$, which implies that e_i is the element associated to a nondegenerate component. Thus we have $e_i = 0$ in $\overline{\mathcal{M}}(G_\xi)$. This implies that $a = 0$ in $\overline{\mathcal{M}}(G_\xi)$, which proves the statement. □

Definition 3.4.4. We call G_ξ the *marked graph of* ξ.

3.5. *Minimal logarithmic maps.* We still consider a log map $\xi = (C \to S, \mathcal{M}_S, f)$ over a geometric point S with fs log structure $\mathcal{M}_S$.

Definition 3.5.1. The log map ξ over S is called *minimal* if the induced canonical map ϕ in Proposition 3.4.2 is an isomorphism of monoids. A family of log maps ξ_T over a scheme T is called *minimal* if each geometric fiber is minimal.

PROPOSITION 3.5.2 (Openness of minimal log maps). *Let $\xi = (C \to S, \mathcal{M}_S, f)$ be a family of log maps over a scheme S, and assume that $\xi_{\bar{s}}$ is minimal for some point $s \in S$. Then there exists an étale neighborhood of s with all geometric fibers minimal.*

Proof. Shrinking S, we may assume that S is connected, and we have a chart $\beta : \overline{\mathcal{M}}_{S,\bar{s}} \to \mathcal{M}_S$ by Proposition A.1.3. We next show that for any $\bar{t} \in S$, the fiber $\xi_{\bar{t}}$ is minimal.

We have

$$K_{\bar{t}} = \{a \in \overline{\mathcal{M}}_{S,\bar{s}} \mid \beta(a) \text{ is invertible at } \bar{t}\}.$$

Note that $K_{\bar{t}}$ is a submonoid of $\overline{\mathcal{M}}_{S,\bar{s}}$. Consider the following composition

$$\overline{\mathcal{M}}_{S,\bar{s}} \to \overline{\mathcal{M}}^{\mathrm{gp}}_{S,\bar{s}} \to \overline{\mathcal{M}}^{\mathrm{gp}}_{S,\bar{s}} / K^{\mathrm{gp}}_{\bar{t}}.$$

By [Ols03a, 3.5], we have $\overline{\mathcal{M}}^{\mathrm{gp}}_{S,\bar{s}}/K^{\mathrm{gp}}_{\bar{t}} \cong \overline{\mathcal{M}}^{\mathrm{gp}}_{S,\bar{t}}$. The above composition induces a map $q : \overline{\mathcal{M}}_{S,\bar{s}} \to \overline{\mathcal{M}}_{S,\bar{t}}$, which is exactly the specialization map as in [Ols03a, 3.5(iii)]. We construct a new graph from the marked graph $G_{\xi_{\bar{s}}}$ as follows:

(1) for an edge $l \in E(G_{\xi,\bar{s}})$, if $q(e_l) = 0$, then we contract l and identify the two end vertices of l and the corresponding associated elements;
(2) for a vertex $v \in V(G_{\xi,\bar{s}})$, if $q(e_v) = 0$, then we put $e_v = 0$ in G'.

Other vertices and edges in $G_{\xi_{\bar{s}}}$ and their associated elements and contact orders remain the same. We denote by G' the resulting graph.

First note that the underlying graph $\underline{G'}$ is the dual graph of $C_{\bar{t}}$ since an edge $l \in E(G_{\xi,\bar{s}})$ gets contracted if and only if the corresponding node is smoothed out over $\bar{t}$. Furthermore, the orientation on $G_{\xi_{\bar{s}}}$ induces a natural orientation on G'. Since all contact orders remain the same, the graph G' is in fact the marked graph $G_{\xi_{\bar{t}}}$ of $\xi_{\bar{t}}$.

The construction of G' gives a map of monoids:

$$M(G_{\xi_{\bar{s}}}) \to M(G_{\xi_{\bar{t}}}) \to \overline{\mathcal{M}}_{G_{\xi_{\bar{t}}}}. \tag{3.5.1}$$

By the same argument in Proposition 3.4.2, we obtain a canonical map of monoids:

$$q' : \overline{\mathcal{M}}_{S,\bar{s}} \cong \overline{\mathcal{M}}(G_{\xi_{\bar{s}}}) \to \overline{\mathcal{M}}(G_{\xi_{\bar{t}}}), \tag{3.5.2}$$

which gives the following commutative diagram

$$\begin{tikzcd} & \overline{\mathcal{M}}_{S,\bar{s}} \arrow[dl, "q'"'] \arrow[dr, "q"] & \\ \overline{\mathcal{M}}(G_{\xi_{\bar{t}}}) \arrow[rr] & & \overline{\mathcal{M}}_{S,\bar{t}}, \end{tikzcd} \tag{3.5.3}$$

where the bottom map is the canonical map as in Proposition 3.4.2. Consider the induced commutative diagram:

$$\begin{tikzcd} & \overline{\mathcal{M}}^{\mathrm{gp}}_{S,\bar{s}} \arrow[dl, "(q')^{\mathrm{gp}}"'] \arrow[dr, "q^{\mathrm{gp}}"] & \\ \overline{\mathcal{M}}(G_{\xi_{\bar{t}}})^{\mathrm{gp}} \arrow[rr] & & \overline{\mathcal{M}}^{\mathrm{gp}}_{S,\bar{t}}. \end{tikzcd} \tag{3.5.4}$$

Note that both q^{gp} and $(q')^{\mathrm{gp}}$ are surjective maps. By the construction of q, the group $K^{\mathrm{gp}}_{\bar{t}}$ is the kernel of q^{gp}. On the other hand, the construction of G' and the fact that $G' = G_{\xi_{\bar{t}}}$ implies that $K^{\mathrm{gp}}_{\bar{t}}$ is also the kernel of $(q')^{\mathrm{gp}}$. Since the monoids in (3.5.3) are fine and saturated, the map $\overline{\mathcal{M}}(G_{\xi_{\bar{t}}}) \to \overline{\mathcal{M}}_{S,\bar{t}}$ is an isomorphism. This proves the statement. $\square$

Definition 3.5.3. Denote by $\mathcal{K}^{\mathrm{pre}}_{g,n}(X^{\log})$ the stack parametrizing minimal log maps to $X^{\log}$, with the fixed genus g, and n-markings.

Corollary 3.5.4. *$\mathcal{K}^{\mathrm{pre}}_{g,n}(X^{\log})$ is an open substack of t $\mathcal{LM}_{g,n}(X^{\log})$ and hence is an algebraic stack.*

Proof. This follows from Theorem 2.1.10 and Proposition 3.5.2. □

3.6. *Stable log maps.*

Definition 3.6.1. A log map $\xi = (C \to S, \mathcal{M}_S, f)$ over a geometric point S is called *stable* if its underlying map is stable in the usual sense, and $\mathcal{M}_S$ is fs. A family of log maps ξ_T over a scheme T is called stable if its geometric fibers are stable. A stable log map is called *minimal stable* if it satisfies the minimality condition as in Definition 3.5.1.

Similarly, we can work over log schemes rather than the usual schemes. Then we have the following:

Definition 3.6.2. A log map $\xi^{\log}$ over an fs log scheme $(S, \mathcal{M}_S)$ as in Definition 2.1.2 is called stable if its underlying map is stable in the usual sense.

Conventions 3.6.3. In this paper, we fix the discrete data $\Gamma = (\beta, g, n, \mathbf{c})$, where

(1) $\beta \in H^2(X, \mathbb{Z})$ is a curve class in X;
(2) n and g are two nonnegative integers;
(3) $\mathbf{c} = (c_i)_{i=1}^n$ is a set of nonnegative integers such that

$$\sum_{i=1}^{n} c_i = c_1(L^\vee) \cap \beta, \tag{3.6.1}$$

where $c_1(L^\vee)$ is the first Chern class of the line bundle $L^\vee$ as in Conventions 3.1.3.

Definition 3.6.4. A minimal stable log map $\xi = (C \to S, \mathcal{M}_S, f)$ over a geometric point S is called a Γ-*minimal stable log* if

(1) the source curve $(C \to S, \mathcal{M}_S)$ is a log pre-stable curve of genus g with n marked points;
(2) $f_*[C] = \beta$;
(3) for any $i \in \{1, 2, \cdots, n\}$, the contact order along section Σ_i is given by c_i.

A log map ξ' over an arbitrary scheme T is called Γ-*minimal stable log* if its geometric fibers are all Γ-minimal stable log. The arrows between stable log maps are the same as the arrow of log maps in Definition 2.1.4.

Definition 3.6.5. Denote by $\mathcal{K}_{g,n}(X^{\log}, \beta)$ the stack parametrizing minimal stable log maps with genus g, n marked points, and curve class β. Let $\mathcal{K}_\Gamma(X^{\log})$ be the stack parametrizing Γ-minimal stable log maps. These are substacks of $\mathcal{LM}_{g,n}(X^{\log})$ as in Theorem 2.1.10.

THEOREM 3.6.6. *The stack* $\mathcal{K}_{g,n}(X^{\log}, \beta)$ *is an open substack of* $\mathcal{K}^{\text{pre}}_{g,n}(X^{\log})$, *hence it is algebraic.*

Proof. Note that the stability condition is a condition on the underlying map that is well known to be open. $\square$

Remark 3.6.7. Denote by Λ the set of discrete data Γ as in Convention 3.6.3 with fixed g, n, and β. Note that Λ is a finite set. By Lemma 3.2.4, we have the disjoint union of open and closed substacks

$$\mathcal{K}_{g,n}(X^{\log}, \beta) = \coprod_{\Gamma \in \Lambda} \mathcal{K}_{\Gamma}(X^{\log}).$$

3.7. *A quasi-finiteness result.* We fix a minimal stable log map $\xi_1 = (\pi : C \to S, \mathcal{M}_1, f_1)$ over a geometric point S. Denote by G the marked graph of ξ_1. Choose a chart $\beta : \overline{\mathcal{M}}(G) \to \mathcal{M}_S$. Let $\mathcal{N}_1 \subset \mathcal{M}_S$ be the sub-log structure generated by $\beta(N(G))$. Since different choices of β only differ by invertible elements, the log structure $\mathcal{N}_1$ does not depend on the choice of β.

Consider the fine log scheme $(S, \mathcal{N}_1)$ with the sub-log structure $\mathcal{N}_1 \subset \mathcal{M}_1$ induced by $N(G)$. Since the map of characteristics $\overline{\mathcal{M}}^{C/S}_S \to \overline{\mathcal{M}}(G)$ factors through $N(G)$, the structure map $\mathcal{M}^{C/S}_S \to \mathcal{M}_S$ factors through $\mathcal{N}_1$. This induces a log curve $(C \to S, \mathcal{N}_1)$. Denote by $\mathcal{N}_{C,1}$ the log structure of $(C \to S, \mathcal{N}_1)$ on C and by $\mathcal{M}_{C,1}$ the log structure of ξ_1 on C. Then $\mathcal{N}_{C,1}$ is a sub-log structure of $\mathcal{M}_{C,1}$. Again by considering the map of characteristics, we see that the log map $f^{\flat}_1 : f^*_1\mathcal{M}_X \to \mathcal{M}_{C,1}$ factors through $\mathcal{N}_{C,1}$. Then we obtain a log map $g_1 : (C, \mathcal{N}_{C,1}) \to X^{\log}$. Denote by $\xi'_1 = (C \to S, \mathcal{N}_1, g_1)$ the log map over S.

Definition 3.7.1. The log map ξ'_1 is called the *coarse log map* of ξ_1.

Remark 3.7.2. The log structures of coarse log maps are in general not saturated. The above construction yields a natural arrow $\xi_1 \to \xi'_1$.

COROLLARY 3.7.3. *The pair* $(\xi_1, \xi_1 \to \xi'_1)$ *is unique up to a unique isomorphism*

Proof. This follows from the uniqueness of the log structure $\mathcal{N}_1$. $\square$

The following result reveals the importance of the notion of coarse log maps:

LEMMA 3.7.4. *Consider another minimal stable log map* $\xi_2 = (C \to S, \mathcal{M}_2, f_2)$ *whose underlying structure and marked graph are identical to those of* ξ_1. *Then there exists (up to a unique isomorphism) a unique isomorphism of coarse log maps* $\xi'_2 \cong \xi'_1$.

Proof. The above statement means that the coarse log maps are up to a unique isomorphism, uniquely determined by their underlying structures and the combinatorial structures. In fact, the sections in the log structures of the two coarse log maps associated to edges are canonically identified by the canonical log structures of the curves. On the other hand, roughly speaking, the sections of the two log structures associated to vertices can be identified first using the combinatorial structures, which determines the characteristics, and then the underlying structure, which determines the "continuous part" of the log structure. We now make this construction precise.

Let $\xi_2' = (C \to S, \mathcal{N}_2, g_2)$ be the coarse log map of ξ_2. Denote by $\mathcal{N}_{C,i}$ the log structures on C corresponding to ξ_i' for $i = 1, 2$. Consider the solid diagram of log structures on C:

(3.7.1)
$$\begin{tikzcd}
& \mathcal{N}_{C,1} \arrow[dd, dashed, "\psi_{\mathcal{N}}"] & \\
\underline{f}^*\mathcal{M}_X \arrow[ur, "g_1^\flat"] \arrow[dr, "g_2^\flat"'] & & \mathcal{M}_C^{C/S} \arrow[ul, "\phi_1"'] \arrow[dl, "\phi_2"] \\
& \mathcal{N}_{C,2}, &
\end{tikzcd}$$

where $\mathcal{N}_{C,i}$ is the associated log structure on C with respect to $\mathcal{N}_i$. We will first construct the dashed arrow $\psi_{\mathcal{N}}$, which makes (3.7.1) commutative.

Since the underlying structures of ξ_1 and ξ_2 are identical, it would be enough to construct $\psi_{\mathcal{N}} : \pi^*\mathcal{N}_1 \to \pi^*\mathcal{N}_2$. We fix a chart $\beta_i : N(G) \to \mathcal{N}_i$. Consider a section $e \in \pi^*\mathcal{N}_i$ for $i = 1, 2$. We want to define the image $\psi_{\mathcal{N}}(e)$. For this, it is enough to assume that $e = \pi^*(\beta_1(\bar{e}))$ for some element $\bar{e} \in N(G)$ associated to a vertex or an edge.

We first assume that $\bar{e}$ is an element associated to an edge $l \in E(G)$. Then there exists a section $e_l \in \mathcal{M}_S^{C/S}$ such that $\phi_1(e_l) = e$. Thus, to make (3.7.1) commutative, one has to define

$$\psi_{\mathcal{N}}(e) = \phi_2(e_l). \tag{3.7.2}$$

We then consider the case that $\bar{e}$ is an element associated to a vertex $v \in V(G)$. We can assume that v is degenerate, otherwise $\bar{e}$ is trivial in $N(G)$. We may restrict (3.7.1) to a small neighborhood of a nondistinguished point as in Definition 3.2.6 on the component corresponding to v. Denote by $\delta \in \underline{f}^*\mathcal{M}_X$ a local generator. Since on the level of characteristic we have $\bar{\phi}_1(\delta) = \bar{e}$ in $\overline{\mathcal{N}}_{C,1}$, we may assume $f_1^\flat(\delta) = e$. Thus, one has to define

$$\psi_{\mathcal{N}}(e) = g_2^\flat(\delta). \tag{3.7.3}$$

It is clear that (3.7.2) and (3.7.3) make (3.7.1) commutative. Now we need to show that the map $\psi_{\mathcal{N}}$ is well defined. The only issue here is to check the left triangle of (3.7.1) at a distinguished node.

We may assume that p is a distinguished node joining two components Z_1 and Z_2 with contact order c. We need to check that the map $\psi_{\mathcal{N}}$ defined at the generic points of the two components can be extended to p. Let x_j be the local coordinate of Z_j at p. Denote by $\log x_j$ the corresponding section in both $\mathcal{N}_{C,1}$ and $\mathcal{N}_{C,2}$. Then we automatically have

$$\psi_{\mathcal{N}}(\log x_i) = \log x_i.$$

Without loss of generality, assume that the orientation of the node is given by $Z_1 > Z_2$. Then locally at p, we have

$$f_1^{\flat}(\delta) = e_{Z_2} + c \cdot \log x_2 \tag{3.7.4}$$

and

$$f_2^{\flat}(\delta) = \psi_{\mathcal{N}}(e_{Z_2}) + c \cdot \log x_2, \tag{3.7.5}$$

where $\psi_{\mathcal{N}}(e_{Z_2})$ is defined by (3.7.3) at some smooth nonmarked point of Z_2.

On the other hand, we have a section $e_l \in \mathcal{M}_S^{C/S}$ such that

$$e_l = \log x_1 + \log x_2.$$

We identify e_l with the corresponding sections in $\mathcal{N}_{S,i}$ and $\mathcal{N}_{C,i}$ via ϕ_i. Now combining this with (3.7.4) and (3.7.5), and generalizing to a smooth nonmarked point of Z_1, we get

$$f_1^{\flat}(\delta) + c \cdot \log x_1 = e_{Z_2} + c \cdot e_l$$

and

$$f_2^{\flat}(\delta) + c \cdot \log x_1 = \psi_{\mathcal{N}}(e_{Z_2}) + c \cdot e_l.$$

Hence

$$\psi_{\mathcal{N}}(e_{Z_2} + c \cdot e_l) = \psi_{\mathcal{N}}(e_{Z_2}) + c \cdot e_l.$$

This proves that the definitions of $\psi_{\mathcal{N}}$ on the two components meeting at p are compatible. Therefore the map $\psi_{\mathcal{N}}$ is well defined. In particular, the above construction gives a canonical isomorphism $\pi^*\mathcal{N}_1 \cong \pi^*\mathcal{N}_2$.

Note that $\pi : C \to S$ forms a flat cover. Since log structures can be glued under fppf topology [Ols03a], the map $\psi_{\mathcal{N}}$ descends to a well-defined isomorphism of log structures $\mathcal{N}_1 \to \mathcal{N}_2$ that induces an isomorphism $\xi_1' \cong \xi_2'$. The uniqueness follows from that of $\psi_{\mathcal{N}}$ in the above construction. □

PROPOSITION 3.7.5. *There are at most finitely many minimal stable log maps over a geometric point with fixed underlying map and marked graph.*

Proof. Fixing a discrete data Γ, the number of possible choices of contact orders along marked points is finite. It is enough to show that the number of

Γ-minimal stable log maps with fixed underlying structure and marked graph is at most finite.

Consider a Γ-minimal stable log map $\xi = (C \to S, \mathcal{M}_S, f)$ over a geometric point S with the fixed underlying structure $\underline{\xi}$ and marked graph G. Denote by $\xi' = (C \to S, \mathcal{N}_S, g)$ the coarse log map of ξ over S. Then we have the natural arrow $\xi \to \xi'$.

On the other hand, consider the saturation map $\mathcal{S} : (S^{\mathcal{S}}, \mathcal{M}) = (S, \mathcal{N}_S)^{\mathcal{S}at} \to (S, \mathcal{N}_S)$. Denote by $\xi^{\mathcal{S}}$ the stable log map over $S^{\mathcal{S}}$ obtained by pulling back ξ' via $\mathcal{S}$. It is easy to check that $\xi^{\mathcal{S}}$ is minimal. By [Ogu06, Ch. II,2.4.5], we have a canonical map $h' : (S, \mathcal{M}_S) \to (S^{\mathcal{S}}, \mathcal{M})$ such that $h = \mathcal{S} \circ h'$. This induces an arrow of minimal stable log maps $\xi \to \xi^{\mathcal{S}}$. By comparing the characteristic, it is easy to see that h' is a strict closed immersion.

Since the underlying map of $\mathcal{S}$ is finite, the statement follows from Lemma 3.7.4. ☐

3.8. *Finiteness of automorphisms.* Let $\xi = (C \to S, \mathcal{M}_S, f)$ be a minimal stable log map over a geometric point S. We fix a chart $\overline{\mathcal{M}}_S \to \mathcal{M}_S$, and we identify the elements e_v and e_l for $v \in V(G_\xi)$ and $l \in E(G_\xi)$ with their images in $\mathcal{M}_S$.

PROPOSITION 3.8.1. *Using the notation as above, the set $\mathcal{A}ut_S(\xi)(S)$ is finite.*

Proof. Note that the set of underlying automorphisms of $\underline{f}$ is finite. Fixing a underlying automorphism $(\underline{\rho}, \mathrm{id}_S)$, it is enough to show that there are finitely many automorphisms of ξ whose underlying structure are given by $(\underline{\rho}, \mathrm{id}_S)$. For simplicity, we assume that $\underline{\rho} = \mathrm{id}_C$, and other cases can be proved similarly.

Let (ρ, θ) be an automorphism with the underlying structure given by $(\mathrm{id}_C, \mathrm{id}_S)$. First we consider a node $l \in E(G_\xi)$. Denote by x and y the local coordinates of l. We can choose x and y so that $e_l = \log x + \log y$. Note that we have

$$\rho^\flat(e_l) = \rho^\flat(\log x) + \rho^\flat(\log y) = \log \rho^*(x) + \log \rho^*(y) = \log x + \log y. \tag{3.8.1}$$

Since $\underline{\rho} = \mathrm{id}_C$, the element e_l is fixed by ρ for any l. The same argument shows that the log structure from the marked points is also fixed by ρ.

Now consider a minimal vertex $v \in V(G_\xi)$. Locally on the component of v, we have

$$f^\flat(\delta) = e_v + \log h,$$

where h is a locally invertible section. Note that we have

$$\rho^\flat(e_v + \log h) = \rho^\flat(e_v) + \log \rho^*(h) = \rho^\flat(e_v) + \log h. \tag{3.8.2}$$

Since ρ fixes the section $f^\flat(\delta)$, the map $\rho^\flat$ also fixes the element e_v. Thus, the automorphism (ρ, θ) acts trivially on all elements associated to vertices and edges of G_ξ. By Lemmas 3.3.5 and 3.3.8, the number of choices of (ρ, θ) is finite. $\square$

Denote by $\xi' = (C \to S, \mathcal{N}_S, g)$ the coarse log map of ξ. Then any automorphism $(\rho, \theta) \in \mathcal{A}ut_S(\xi)(S)$ induces a unique automorphism of ξ'. Indeed, the above proofs of Proposition 3.8.1 and Lemma 3.7.4 imply the following:

Corollary 3.8.2. *Consider an automorphism $(\rho, \theta) \in \mathcal{A}ut_S(\xi)(S)$. The induced map of sub-log structures $\theta^\flat : \mathcal{N}_S \to \mathcal{N}_S$ is uniquely determined by the underlying automorphism $(\underline{\rho}, \mathrm{id}_S)$ and the automorphism of the marked graph. In particular, the automorphism of ξ' induced by (ρ, θ) is uniquely determined by the underlying automorphism $(\underline{\rho}, \mathrm{id}_S)$.*

Denote by $\underline{\xi}$ the usual stable maps obtained by removing log structures on ξ. We can strengthen the result of Proposition 3.8.1 as follows.

Lemma 3.8.3. *The map of groups $\mathcal{A}ut_S(\xi)(S) \to \mathcal{A}ut_S(\underline{\xi})(S)$ is injective.*

Proof. Consider an element $(\underline{\rho}, \mathrm{id}_S) \in \mathcal{A}ut_S(\underline{\xi})(S)$. It is enough to show that there is at most one element $(\rho, \theta) \in \mathcal{A}ut_S(\xi)(S)$ that is the pre-image of $(\underline{\rho}, \mathrm{id}_S)$. Consider the following diagram:

$$\begin{tikzcd} \mathcal{N}_S \arrow[r, hook] \arrow[d] & \mathcal{M}_S \arrow[d, dashed, "\theta^\flat"] \\ \mathcal{N}_S \arrow[r, hook] & \mathcal{M}_S, \end{tikzcd} \tag{3.8.3}$$

where the left vertical arrow can be constructed similarly by (3.8.1) and (3.8.2), which is uniquely determined by the underlying map $(\underline{\rho}, \mathrm{id}_S)$ and the automorphism of the marked graph. Corollary 3.8.2 implies that any (ρ, θ) over $(\underline{\rho}, \mathrm{id}_S)$ induces a unique map $\mathcal{N}_S \to \mathcal{N}_S$ as in (3.8.3). Hence to find (ρ, θ), it is equivalent to find the dashed arrow $\theta^\flat$, which makes the above diagram (3.8.3) commutative. By the adjointness of saturation and inclusion functors of log structures as in [Ogu06, Ch. II,2.4.5], we have the following commutative diagram:

$$\begin{tikzcd} (S, \mathcal{M}_S) \arrow[r, hook, "i"] \arrow[dr, "\theta'"'] & (S, \mathcal{N}_S)^{\mathcal{S}at} \arrow[r] \arrow[d, "\exists!\cong"] & (S, \mathcal{N}_S) \arrow[d] \\ & (S, \mathcal{N}_S)^{\mathcal{S}at} \arrow[r] & (S, \mathcal{N}_S), \end{tikzcd} \tag{3.8.4}$$

where $(S, \mathcal{N}_S)^{\mathcal{S}at}$ is the saturation of $(S, \mathcal{N}_S)$.

Denote by ξ' the coarse log curve of ξ over S. Then by taking the saturation, we obtain a minimal stable log map $(\xi')^{\mathcal{S}}$ over $(S, \mathcal{N}_S)^{\mathcal{S}at}$. Note that the left triangle of (3.8.4) induces a commutative diagram of minimal stable log maps

$$
\begin{tikzcd}
\xi \arrow[r] \arrow[dr] & (\xi')^{\mathcal{S}} \arrow[d] \\
 & (\xi')^{\mathcal{S}}.
\end{tikzcd}
\tag{3.8.5}
$$

This gives a unique $\theta^\flat$ as in (3.8.3) and hence a unique isomorphism of ξ. □

PROPOSITION 3.8.4. *The natural map $\mathcal{K}_\Gamma(X^{\log}) \to \mathcal{K}_{g,n}(X,\beta)$ is representable by removing log structures from minimal stable log maps.*

Proof. This follows from Lemma 3.8.3 and [LMB00, 8.1.1]. □

4. The stack of minimal log maps as category fibered over $\mathfrak{LogSch}^{\mathrm{fs}}$

By the construction in last section, the stacks $\mathcal{K}^{\mathrm{pre}}_{g,n}(X^{\log})$, $\mathcal{K}_{g,n}(X^{\log},\beta)$, and $\mathcal{K}_\Gamma(X^{\log})$ as open substacks of $\mathcal{LM}_{g,n}(X^{\log})$ are fibered categories over $\mathfrak{Sch}$, parametrizing minimal log maps over usual schemes with various numerical conditions. In this section, we give a different categorical explanation as categories fibered over $\mathfrak{LogSch}^{\mathrm{fs}}$.

4.1. *The universal property of minimal log maps.* In this subsection, we fix a log map $\xi = (C \to S, \mathcal{M}_S, f : (C, \mathcal{M}_C) \to (X, \mathcal{M}_X))$ such that the log structure $\mathcal{M}_S$ is fs.

PROPOSITION 4.1.1. *There exist a minimal log map over S*

$$\xi_{\min} = (C \to S, \mathcal{M}_S^{\min}, f_{\min} : (C, \mathcal{M}_C^{\min}) \to (X, \mathcal{M}_X))$$

and a map of fs *log schemes $\Phi : (S, \mathcal{M}_S) \to (S, \mathcal{M}_S^{\min})$ that fit in the following commutative diagram*

$$
\begin{tikzcd}
 & & (X,\mathcal{M}_X) \\
(C,\mathcal{M}_C) \arrow[r, "\Phi_C"'] \arrow[d] \arrow[urr, bend left, "f"] & (C,\mathcal{M}_C^{\min}) \arrow[d] \arrow[ur, bend right, "f_{\min}"'] & \\
(S,\mathcal{M}_S) \arrow[r, "\Phi"] & (S,\mathcal{M}_S^{\min}), &
\end{tikzcd}
\tag{4.1.1}
$$

where the square is a cartesian square of log schemes. Furthermore, the datum $(\Phi, \xi_{\min})$ is unique up to a unique isomorphism.

Proof. Note that the statement is local on S. Then the proposition follows from Lemmas 4.1.2, 4.1.3, 4.1.4, and 4.1.5. □

By Construction 3.4.1, for each geometric point $\bar{t} \in S$, we can associate a marked graph $G_{\xi_{\bar{t}}}$ to the fiber $\xi_{\bar{t}}$. It was shown in Lemma 3.4.3 that $G_{\xi_{\bar{t}}}$ is admissible. By Proposition 3.4.2, we have a canonical morphism of monoids

$$\phi_{\bar{t}} : \overline{\mathcal{M}}(G_{\xi_{\bar{t}}}) \to \overline{\mathcal{M}}_{S,\bar{t}}. \tag{4.1.2}$$

LEMMA 4.1.2. *Assume that we have a log pre-stable curve* $(C \to S, \mathcal{M}_S^{\min})$ *and a morphism* $\Phi : (S, \mathcal{M}_S) \to (S, \mathcal{M}_S^{\min})$ *such that*

(1) *for each point* $s \in S$, *we have a fixed isomorphism* $\overline{\mathcal{M}}_{S,\bar{s}}^{\min} \cong \overline{\mathcal{M}}(G_{\xi_{\bar{s}}})$;
(2) *the induced map* $\bar{\Phi}_{\bar{s}}^{\flat} : \overline{\mathcal{M}}(G_{\xi_{\bar{s}}}) \cong \overline{\mathcal{M}}_{S,\bar{s}}^{\min} \to \overline{\mathcal{M}}_{S,\bar{s}}$ *on the level of characteristic is identical to the canonical map* $\phi_{\bar{s}}$ *as* (4.1.2);
(3) *the log pre-stable curve* $(C \to S, \mathcal{M}_S)$ *is the pull-back of* $(C \to S, \mathcal{M}_S^{\min})$ *via* Φ.

Then we have a unique log map $f_{\min} : (C, \mathcal{M}_C^{\min}) \to (X, \mathcal{M}_X)$ *that fits in diagram* 4.1.1. *Note that* $(C \to S, \mathcal{M}_S^{\min}, f_{\min})$ *forms a minimal log map over the scheme* S.

Proof. Since all the underlying maps are fixed, it is enough to construct a map of log structures $f_{\min}^{\flat} : f^*(\mathcal{M}_X) \to \mathcal{M}_C^{\min}$ that fits in the following commutative diagram:

$$\begin{tikzcd} & f^*(\mathcal{M}_X) \arrow[dl, "f^{\flat}_{\min}"'] \arrow[dr, "f^{\flat}"] & \\ \mathcal{M}_C^{\min} \arrow[rr, "\Phi_C^{\flat}"] & & \mathcal{M}_C. \end{tikzcd}$$

Consider an arbitrary closed point $p \in C$ that lies in an irreducible component corresponding to the vertex $v \in V(G_{\xi_{\bar{s}}})$. Then locally at p, we have

$$f^{\flat}(\delta) = e_{v,0} + \log h, \tag{4.1.3}$$

where $e_{v,0} \in \mathcal{M}_S$ near $\bar{s}$ and h is a nonzero regular section locally near p. Note that there are two possible cases: if p is a smooth nonmarked point, then h is a locally invertible section; if p is a special point with contact order c, then $h = u \cdot \sigma^c$, where u is a locally invertible section and σ is a local coordinate function vanishing at p. Note that the underlying map $\underline{\Phi}_C$ is an identity. Thus, to define $f_{\min}^{\flat}(\delta)$ locally at p, it is enough to find a lifting $\tilde{e}_v \in \mathcal{M}_S^{\min}$ of $e_{v,0}$ such that the image of $\tilde{e}_v$ in $\overline{\mathcal{M}}_S^{\min}$ is the element associated to the vertex v.

We first consider the uniqueness. Assume that we have two liftings $\tilde{e}_v$ and $\tilde{e}_v'$ such that their images in $\overline{\mathcal{M}}_S^{\min}$ are given by the element associated to v. Then, we have $\tilde{e}_v = \log u + \tilde{e}_v'$ for some locally invertible function u. This implies that

$$\Phi_C^{\flat}(\tilde{e}_v) = \Phi_C^{\flat}(\log u) + \Phi_C^{\flat}(\tilde{e}_v').$$

Since $\tilde{e}_v$ and $\tilde{e}'_v$ are two liftings of $e_{v,0}$, we have $\Phi^\flat_C(\log u) = 1$. Note that the underlying map $\underline{\Phi}_C = \mathrm{id}_C$. It follows that $u = 1$. This shows that the lifting is unique.

Now we consider the existence of the lifting. Denote by $\bar{e}_{v,0}$ the image of $e_{v,0}$ in the characteristic $\overline{\mathcal{M}}_{S,\bar{s}}$. By (2) the map of monoids $\bar{\Phi}^\flat_{\bar{s}}$ is identical to $\phi_{\bar{s}}$. Then we have a unique element $\bar{e}_v \in \overline{\mathcal{M}}^{\min}_{S,\bar{s}}$ that corresponds to the element associated to v in the graph G, and $\bar{\Phi}^\flat_{\bar{s}}(\bar{e}_v) = \bar{e}_{v,0}$. Thus, locally we can lift $\bar{e}_v$ to an element $\tilde{e}_v \in \mathcal{M}^{\min}_S$ such that $\Phi^\flat_{\bar{s}}(\tilde{e}_v) = e_v$. Then we define

$$f^\flat_{\min}(\delta) = \tilde{e}_v + \log h. \tag{4.1.4}$$

The uniqueness of the lifting shows that the construction in (4.1.4) can be glued globally to obtain a unique map $f^\flat_{\min}$. We can check locally that the map of monoids $f^\flat_{\min}$ is compatible with the structure morphisms of the corresponding log structures. This finishes the proof of the statement. □

In fact, in the above proof we constructed a log map $f_{\min}$ that is minimal at $\bar{s}$, hence minimal in a neighborhood of $\bar{s}$, by the openness of minimality. We next construct a unique log prestable curve $(C \to S, \mathcal{M}^{\min}_S)$ satisfying the three conditions in the above lemma. Note that the question is local on S. Pick a point $\bar{s} \in S$. Shrinking S, we can assume that there is a global chart $\beta : \overline{\mathcal{M}}_{S,\bar{s}} \to \mathcal{M}_S$. We have the canonical map $\phi_{\bar{s}} : \overline{\mathcal{M}}(G_{\xi_{\bar{s}}}) \to \overline{\mathcal{M}}_{S,\bar{s}}$. Consider the pre-log structure given by the following composition:

$$\overline{\mathcal{M}}(G_{\xi_{\bar{s}}}) \xrightarrow{\phi_{\bar{s}}} \overline{\mathcal{M}}_{S,\bar{s}} \xrightarrow{\beta} \mathcal{M}_S \xrightarrow{\exp} \mathcal{O}_S.$$

Denote by $\mathcal{M}^{\min}_S$ the log structure associated to the above pre-log structure. Thus, the construction above gives a global chart $\beta_{\min} : \overline{\mathcal{M}}(G_{\xi_{\bar{s}}}) \to \mathcal{M}^{\min}_S$ and a natural map $\Phi^\flat : \mathcal{M}^{\min}_S \to \mathcal{M}_S$.

Note that the construction of $\mathcal{M}^{\min}_S$ depends on the choice of the chart β. Assume that we have another log structure $\mathcal{M}^{\min}_1$ and a map $\Phi^\flat_1 : \mathcal{M}^{\min}_1 \to \mathcal{M}_S$ over S that comes from another chart $\beta_1 : \overline{\mathcal{M}}_{S,\bar{s}} \to \mathcal{M}_S$. Then we have

Lemma 4.1.3. *There is a unique isomorphism of log structures $\mathcal{M}^{\min}_1 \to \mathcal{M}^{\min}_S$ fitting in the following commutative diagram*:

$$\begin{tikzcd} \mathcal{M}^{\min}_1 \arrow[rr] \arrow[dr, "\Phi^\flat_1"'] & & \mathcal{M}^{\min}_S \arrow[dl, "\Phi^\flat"] \\ & \mathcal{M}_S. & \end{tikzcd}$$

Proof. Consider an irreducible element $a \in \overline{\mathcal{M}}(G)$. Then the construction of $\mathcal{M}^{\min}_1$ and $\mathcal{M}^{\min}_S$ implies that

$$\Phi^\flat_1 \circ \beta_1(a) + \log u = \Phi^\flat \circ \beta_{\min}(a),$$

where u is a unique invertible section. We define

$$\beta_1(a) \mapsto \beta_{\min}(a) + \log u^{-1}.$$

This induces a unique map $\mathcal{M}_1^{\min} \to \mathcal{M}_S^{\min}$ that satisfies the condition of the lemma. $\square$

Lemma 4.1.4. *There exists a unique dashed arrow that fits in the following commutative diagram*:

$$\begin{tikzcd} \mathcal{M}_S^{\min} \arrow[rr, "\Phi^\flat"] & & \mathcal{M}_S \\ & \mathcal{M}_S^{C/S}, \arrow[ul, dashed, "\phi_{\min}"] \arrow[ur, "\phi"'] & \end{tikzcd} \tag{4.1.5}$$

where ϕ is the structure arrow defining the log pre-stable curve $(C \to S, \mathcal{M}_S)$, see Definition B.2.2.

Proof. Further shrinking S, we can choose a global chart $\mathbb{N}^m \to \mathcal{M}_S^{C/S}$. Let e be a generator of $\mathbb{N}^m$ that corresponds to an edge $l \in V(G_{\xi_{\bar{s}}})$. For convenience, we will identify e with its image in $\mathcal{M}_S^{C/S}$. Consider $\phi(e) \in \mathcal{M}_S$ and its image $\bar{\phi}(e) \in \overline{\mathcal{M}}_S$. Now on the level of characteristic, there is a unique element $\bar{e}' \in \overline{\mathcal{M}}_S^{\min}$, which corresponds to the element associated to l, such that $\bar{\Phi}^\flat(\bar{e}') = \bar{\phi}(e)$. A similar argument as in the proof of Lemma 4.1.2 shows that there is a unique section $e' \in \mathcal{M}_S^{\min}$ such that $\Phi^\flat(e') = \phi(e)$. Then we can define $\phi_{\min}(e) = e'$ for every generator e. This gives the map $\phi_{\min} : \mathcal{M}_S^{\min} \to \mathcal{M}_S$.

Note that our construction depends on a fixed chart $\mathbb{N}^m \to \mathcal{M}_S^{C/S}$. However, a similar argument as in the proof of Lemma 4.1.3 shows that a different choice of the global chart will induces the same map $\phi_{\min}$. This finishes the proof. $\square$

Note that the arrow $\phi_{\min}$ induces a log pre-stable curve $(C \to S, \mathcal{M}_S^{\min})$. Denote by $\mathcal{M}_C^{\min}$ the corresponding log structure on C associated to the log curve. By Lemma 3.2.9, we can further shrink S and assume that the contact order of the nodes on each geometric fiber is given by the weights of the edges of $G_{\xi_{\bar{s}}}$. Now we have

Lemma 4.1.5. *The log pre-stable curve $(C \to S, \mathcal{M}_S^{\min})$ and the log map $(S, \mathcal{M}_S)^{\min} \to (S, \mathcal{M}_S)$ induced by $\Phi^\flat$ satisfy conditions* (1), (2), *and* (3) *in Lemma* 4.1.2.

Proof. Note that (3) follows from the commutativity of (4.1.5) and Definition B.2.2. For (1) and (2), we can repeat the argument in Lemma 4.1.2. Indeed, the same construction there yields a log map $(C, \mathcal{M}_C^{\min}) \to (X, \mathcal{M}_X)$, which is minimal at $\bar{s}$. Now the openness of minimality implies that all points in a neighborhood of $\bar{s}$ are minimal. Therefore, properties (1) and (2) in Lemma 4.1.2 follow. $\square$

4.2. *Proof of Theorem* 1.2.3. By the definition of log stack in Section A.3, the stack $\mathcal{K}_\Gamma(X^{\log})$ carries a natural log structure $\mathcal{M}_{\mathcal{K}_\Gamma(X^{\log})}$ as follows. For any $g : S \to \mathcal{K}_\Gamma(X^{\log})$, the log structure $g^*\mathcal{M}_{\mathcal{K}_\Gamma(X^{\log})}$ is exactly the log structure on S given by the minimal log map ξ over S induced by g. Now we have a universal diagram of log stacks

$$\begin{tikzcd} (\mathfrak{C}_\Gamma, \mathcal{M}_{\mathfrak{C}_\Gamma}) \arrow[r] \arrow[d] & X^{\log} \\ (\mathcal{K}_\Gamma(X^{\log}), \mathcal{M}_{\mathcal{K}_\Gamma(X^{\log})}), & \end{tikzcd}$$

where the pair $(\mathfrak{C}_\Gamma, \mathcal{M}_{\mathfrak{C}_\Gamma})$ is the universal curve over $\mathcal{K}_\Gamma(X^{\log})$ with universal log structures $\mathcal{M}_{\mathfrak{C}_\Gamma}$. This diagram gives a stable log map $\xi_{\mathcal{K}_\Gamma(X^{\log})}$ over $(\mathcal{K}_\Gamma(X^{\log}), \mathcal{M}_{\mathcal{K}_\Gamma(X^{\log})})$.

Now consider a stable log map $\xi^{\log}$ over $(S, \mathcal{M}_S)$. The tuple $(\xi^{\log}, S, \mathcal{M}_S)$ then gives a stable log map over S. The universal property of minimal log map implies that there is a unique minimal log map $\xi_{\min} = (\xi^{\log}_{\min}, S, \mathcal{M}_S^{\min})$ over S, and a map of log schemes $g : (S, \mathcal{M}_S) \to (S, \mathcal{M}_S^{\min})$, such that $\xi^{\log} = g^*\xi^{\log}_{\min}$ as in Definition 2.1.2. This induces a unique log map $f : (S, \mathcal{M}_S) \to (\mathcal{K}_\Gamma(X^{\log}), \mathcal{M}_{\mathcal{K}_\Gamma(X^{\log})})$ such that $\xi^{\log} = f^*\xi_{\mathcal{K}_\Gamma(X^{\log})}$.

Theorem 1.2.3 follows.

Remark 4.2.1. Using the same argument as above, we can shows that the two stacks $\mathcal{K}_{g,n}(X^{\log}, \beta)$ and $\mathcal{K}^{\mathrm{pre}}_{g,n}(X^{\log})$ with their universal minimal log structures can be viewed as categories fibered over $\mathfrak{LogSch}^{\mathrm{fs}}$, parametrizing log maps over fs log schemes with corresponding numerical conditions.

Remark 4.2.2. If the log structure $\mathcal{M}_X$ on the target X is trivial, the stack $\mathcal{K}_{g,n}(X^{\log}, \beta)$ is isomorphic to the stack $\mathcal{K}_{g,n}(X, \beta)$ of usual stable maps with the minimal log structure coming from the canonical log structure of its universal curve.

5. The boundedness theorem for minimal stable log maps

5.1. *Statement of the boundedness theorem.* In this section, we fix the target $X^{\log} = (X, \mathcal{M}_X)$ as in Convention 3.1.3. The main result of this section is the following:

THEOREM 5.1.1. *There exist a scheme T of finite type and a map $g : T \to \mathcal{K}_\Gamma(X^{\log})$ that exhausts all geometric point of $\mathcal{K}_\Gamma(X^{\log})$. Namely, for any point $\xi \in \mathcal{K}_\Gamma(X^{\log})(\mathbb{C})$, there exists a lifting* $\operatorname{Spec}\mathbb{C} \to T$ *such that its composition with g gives ξ.*

Proof. The proof of this theorem will occupy the whole section. Indeed, we will prove that the map $\mathcal{K}_\Gamma(X^{\log}) \to \mathcal{K}_{g,n}(X,\beta)$, obtained by removing all log structures, is of finite type. In Section 5.3, we will bound the choices of marked graph by stratifying $\mathcal{K}_{g,n}(X,\beta)$. In Section 5.4, we will construct a family of minimal stable log maps that exhausts all possible minimal log structures with fixed underlying map and marked graph. This will be achieved by considering isomorphisms of corresponding line bundles. The result from Section 5.2 will be used in the above argument. □

5.2. *Isomorphisms of line bundles induced by stable log maps.* Consider a stable log map (not necessarily minimal) $\xi = (C \to S, \mathcal{M}_S, f)$ over a scheme S. In this subsection, we put the following assumption:

(5.2.1) The characteristic $\overline{\mathcal{M}}_S$ is a constant sheaf of monoids on S.

Lemma 5.2.1. *With the assumptions as above, the marked graphs of all geometric fibers of* ξ *are isomorphic.*

Proof. Note that the elements smoothing the distinguished nodes are in $\overline{\mathcal{M}}_S$. Then the statement follows from assumption (5.2.1). □

Given the stable log map ξ over S as above, let us consider the following commutative diagram:

$$\begin{tikzcd} & f^*(\mathcal{M}_X) \arrow[r, "f^\flat"] \arrow[d, "p_1"] & \mathcal{M}_C \arrow[d, "p_2"] \\ \mathbb{N} \arrow[r] & f^*(\overline{\mathcal{M}}_X) \arrow[r, "\bar{f}^\flat"] & \overline{\mathcal{M}}_C. \end{tikzcd}$$

The composition of the bottom arrow $\mathbb{N} \to \overline{\mathcal{M}}_C$ locally lifts to a chart of a sub-log structure of $\mathcal{M}_C$. Denote by $\mathcal{M}$ the resulting sub-log structure. Note that this is also a DF-log structure. The map $f^\flat$ induces an isomorphism of log structures $f^*(\mathcal{M}_X) \to \mathcal{M}$. By the argument in Section A.2, this gives an isomorphism of the corresponding line bundles and sections. Next, we will describe this isomorphism on each irreducible component of C.

Pick a point $\bar{s} \in S$. Shrinking S, we may choose a lifting of global chart $\beta : \overline{\mathcal{M}}_{S,\bar{s}} \to \mathcal{M}_S$. Consider the induced map $\hat{\beta} : \overline{\mathcal{M}}_S \to \mathcal{M}_C$. Denote by $\widehat{\mathcal{M}}_C = \mathcal{M}_C^{\mathrm{gp}}/(\overline{\mathcal{M}}_S)^{\mathrm{gp}}$ the quotient given by the map $\hat{\beta}$. Consider the following commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & (\overline{\mathcal{M}}_S)^{\mathrm{gp}} \arrow[r, "\hat{\beta}^{\mathrm{gp}}"] & \mathcal{M}_C^{\mathrm{gp}} \arrow[r] & \widehat{\mathcal{M}}_C, \\ & & f^*(\mathcal{M}_X) \arrow[u] \arrow[ur, "\hat{f}^\flat"'] & \end{tikzcd} \tag{5.2.2}$$

where the map $\hat{f}^\flat$ is given by the composition $f^*(\mathcal{M}_X) \to \mathcal{M}_C \to \widehat{\mathcal{M}}_C$.

Remark 5.2.2. Note that the morphism $\hat{f}^\flat$ depends on the choice of a lifting $\beta : \overline{\mathcal{M}}_{S,\bar{s}} \to \mathcal{M}_S$. This will be important when we discuss the valuative criterion.

Conventions 5.2.3. Consider the irreducible component C_v of C corresponding to a vertex $v \in G_\xi$. Note that C_v is connected. Denote by $\{p_l\}_{l\in\Lambda_v^{\mathrm{low}}}$ the set of splitting nodes, joining v with v' for some $v' \leq v$. Let $\{p_l\}_{l\in\Lambda_v^{\mathrm{up}}}$ be the set consisting of the following special points in C_v:

(1) the set of splitting nodes, joining v with v'' for some $v \leq v''$;
(2) the marked points with nontrivial contact orders.

Denote by c_l the contact order at p_l for $l \in \Lambda_v^{\mathrm{low}} \cup \Lambda_v^{\mathrm{up}}$. Consider the line bundle

$$L_v = \prod_{l\in\Lambda_v^{\mathrm{low}}} \mathcal{O}_{C_v}(c_l \cdot p_l) \otimes \prod_{l\in\Lambda_v^{\mathrm{up}}} \mathcal{O}_{C_v}(-c_l \cdot p_l).$$

Note that the line bundle L_v only depends on the graph G_ξ.

Proposition 5.2.4. *Assume that the element associated to a vertex $v \in G_\xi$ is not zero. Then the map $\hat{f}^\flat$ induces a natural isomorphism of line bundles*

$$\hat{f}_v^\flat : f^*(L)_v \to L_v.$$

Proof. We first construct $\hat{f}_v^\flat$ locally. There are three cases.

Case 1. Consider a closed point p of p_l for $l \in \Lambda_v^{\mathrm{up}}$. Locally at p we have

$$f^\flat(\delta) = e_v + c_l \log \sigma_l,$$

where σ_l is the local coordinate of p in C_v defining the marking p_l and e_v is contained in the image of $\hat{\beta}$. Thus, we have $\hat{f}^\flat(\delta) = c_l \log \sigma_l$. Then locally near p, we define

$$\hat{f}_v^\flat(\delta) = \sigma_l^{c_l}. \tag{5.2.3}$$

Note that $\sigma_l^{c_l}$ is the local section of L_v at p.

Case 2. Consider a closed point p of the splitting node p_l for $l \in \Lambda_v^{\mathrm{low}}$. Assume that p_l joins vertices v' and v such that $v' \leq v$. Locally at p, we have

$$f^\flat(\delta) = e_{v'} + c_l \log \sigma_l', \tag{5.2.4}$$

where $e_{v'}$ is in the image of $\hat{\beta}$. By a nice choice of coordinates, we have

$$c_l \cdot e_l = c_l \log \sigma_l + c_l \log \sigma_l' \quad \text{in } \mathcal{M}_C, \tag{5.2.5}$$

where σ_l' is the local coordinate of p_l in C_v' and e_l is the element smoothing node, contained in the image of $\hat{\beta}$. Then we have

$$1 = c_l \log \sigma_l + c_l \log \sigma_l' \quad \text{in } \hat{\mathcal{M}}_C.$$

This induces

$$\hat{f}^\flat(\delta) = c_l \log \sigma_l = 1 - c_l \log \sigma_l.$$

Then locally at the node p we define

$$\hat{f}_v^\flat(\delta) = \left(\frac{1}{\sigma_l}\right)^{c_l}. \tag{5.2.6}$$

Note that this is a local generator of L_v at p.

Case 3. Locally at a point p that is not contained in one of the p_l for $l \in \Lambda_v^{\mathrm{up}} \cup \Lambda_v^{\mathrm{low}}$, we have

$$f^\flat(\delta) = e_v + \log h,$$

where h is an invertible function at p and e_v is contained in the image of $\hat{\beta}$. Then the map $\hat{f}^\flat(\delta) = \log h$ induces

$$\hat{f}_v^\flat(\delta_\lambda) = h. \tag{5.2.7}$$

Note that the local construction of $\hat{f}_v^\flat$ is uniquely determined by $\hat{f}^\flat$, which is a map of sheaves of monoids. Thus these local definitions can be glued to obtain a global map. We also notice that δ lifts to a the local generator of L. Therefore we construct an isomorphism of line bundles $\hat{f}_v^\flat$ as required. □

Remark 5.2.5. The local calculation shows that the isomorphism $\hat{f}_v^\flat$ in Proposition 5.2.4 depends on the choice of the chart β.

5.3. *Finiteness of the discrete data.*

PROPOSITION 5.3.1. *The following set is finite*:

$$\{G \mid G \textit{ is the marked graph of some } \xi \in \mathcal{K}_\Gamma(X^{\log})(\mathbb{C})\}.$$

Proof. Step 1: *Bounding the choices of underlying dual graph.* Denote by $\mathcal{K}_{g,n}(X,\beta)$ the Kontsevich moduli space of stable maps, with n-marked points, genus g, and curve class β in X. Note that we have a morphism

$$\mathcal{K}_\Gamma(X^{\log}) \to \mathcal{K}_{g,n}(X,\beta),$$

by removing all log structures. Let $U \to \mathcal{K}_{g,n}(X,\beta)$ be an affine étale chart. Consider the following cartesian diagram without log structures:

$$\begin{tikzcd} \mathcal{K}_U \arrow[r] \arrow[d] & \mathcal{K}_\Gamma(X^{\log}) \arrow[d] \\ U \arrow[r] & \mathcal{K}_{g,n}(X,\beta). \end{tikzcd}$$

Since the stack $\mathcal{K}_{g,n}(X,\beta)$ is of finite type, it is enough to prove that the set of dual graphs corresponding to the geometric point of $\mathcal{K}_U$ is finite. Denote by $C_U \to U$ the universal curve and by $\underline{f}_U : C_U \to X$ the universal map over U.

Since U is of finite type, it is covered by finitely many strata, where the family of curves over each stratum have a fixed dual graph. We put the reduced scheme structure on each stratum. Let S be the stratum corresponding to a graph G. Denote by $\underline{f}: C \to X$ the universal map over S.

Step 2: *Bounding the choices of distinguished nodes and orientations.* Since G is a finite graph, the number of choices of distinguished nodes is finite. We first fix a choice of distinguished nodes on G. So we fix an orientation on G such that

(1) if C_v does not degenerate to D, then $v \in V_n(G)$;
(2) the nonoriented edges are in one-to-one correspondence to the nondistinguished nodes;
(3) no cycles contain distinguished edges.

Step 3: *Bounding the choices of contact orders.* Since we fixed the orientation and distinguished edges on G, we can use the notation $\{p_l\}_{l\in\Lambda_v^{\mathrm{low}}}$ and $\{p_l\}_{l\in\Lambda_v^{\mathrm{up}}}$ for the two sets of distinguished points on the subcurve C_v as in Conventions 5.2.3. Denote by c_l the possible contact order at the distinguished point p_l. Since the dual graph of the underlying curve is fixed, the multi-degree of $\underline{f}^*(L)$ on C_v is fixed for any $v \in V(G)$. By Proposition 5.2.4, we have

$$\deg \underline{f}^*(L)|_{C_v} = \sum_{l\in\Lambda_v^{\mathrm{low}}} c_l - \sum_{l'\in\Lambda_v^{\mathrm{up}}} c_{l'}. \tag{5.3.1}$$

First, consider a maximal vertex $v \in V(G)$. Then the set $\{p_l\}_{l\in\Lambda_v^{\mathrm{up}}}$ is given by the discrete data Γ. Since the contact orders are all positive, the choices of c_l for $l \in \Lambda_v^{\mathrm{low}}$ is finite by (5.3.1).

Consider an arbitrary vertex $v' \in V(G)$. We assume that for any adjacent vertex v of v' such that $v' \leq v$, the number of choices of the contact orders along the splitting nodes joining v' and v is finite. Then by taking into account all contact orders from adjacent vertices and those from marked points of C, a similar argument shows that the possible choices of contact order c_l for $l \in \Lambda_{v'}^{\mathrm{low}}$ are also finite in number. Since G is a finite graph, this proves that the choice of contact orders on G is finite.

This finishes the proof of the proposition. $\square$

5.4. *Proof of Theorem* 5.1.1. Consider the family of usual stable maps $\underline{f}: C \to X$ over S as in Step 1 of the above proof. Fix a possible marked graph G_0 with $\underline{G_0} = G$ the dual graph of C. We use the notation as in Step 3 of the above proof, and we assume that (5.3.1) holds for any $v \in V(G_0)$. Since the stack $\mathcal{K}_{g,n}(X,\beta)$ is of finite type, to prove Theorem 5.1.1, it is enough to prove the following:

Proposition 5.4.1. *Using the notation and assumptions as above, there exist a scheme T of finite type over S and a family of minimal stable log maps*

ξ over T that satisfy the following conditions: for any minimal stable log map ξ' over $\bar{s}$, with marked graph given by G_0, and underlying map $\underline{\xi}'$ given by the pull-back of $\underline{f}$ via $\bar{s} \to S$, there exists a lifting $\bar{s} \to T$ such that ξ' is isomorphic to the pull-back $\xi_{\bar{s}}$.

Proof. By shrinking S, we can assume that S is affine, and the canonical log structure $\mathcal{M}_S^{C/S}$ on S coming from the family $C \to S$ has a global chart $\mathbb{N}^n \to \overline{\mathcal{M}}_{S,\bar{s}}^{C/S}$ for some geometric point $\bar{s} \in S$. Consider the pre-log structure $\overline{\mathcal{M}}(G_0) \to \mathcal{O}_S$, given by $e \mapsto 0$ for any nontrivial element $e \in \overline{\mathcal{M}}(G_0)$. Denote by $\mathcal{M}_S$ the new log structure associated to the pre-log structure. Note that there is a map $\mathbb{N}^n \to \overline{\mathcal{M}}(G_0)$ given by the corresponding nodes. This induces a map $\mathcal{M}_S^{C/S} \to \mathcal{M}_S$, hence a log pre-stable curve $\zeta = (C \to S, \mathcal{M}_S)$ over S. Note that any minimal log map ξ' over $\bar{s} \in S$ as in the statement has the source log curve isomorphic to $\zeta_{\bar{s}}$.

Denote by $\mathcal{M}_C$ the log structure on C corresponding to the log pre-stable curve ζ. Note that over C we have another log structure $\underline{f}^*(\mathcal{M}_X)$. Since the dual graph G_0 is fixed, we have a morphism of sheaves of monoids on C,

$$\bar{f}^\flat : \underline{f}^*(\overline{\mathcal{M}}_X) \to \overline{\mathcal{M}}_C,$$

which is locally described as in Section 3.2. To define a log map $f : (C, \mathcal{M}_C) \to X^{\log}$, it is enough to define a map of log structures $f^\flat : \underline{f}^*(\mathcal{M}_X) \to \mathcal{M}_C$ fitting in the following commutative diagram:

(5.4.1)
$$\begin{tikzcd}
 & & \mathcal{O}_C \\
 & \underline{f}^*(\mathcal{M}_X) \arrow[ur, bend left] \arrow[r, dashed, "f^\flat"] \arrow[d, "p_1"] & \mathcal{M}_C \arrow[u, bend right] \arrow[d, "p_2"] \\
\mathbb{N} \arrow[r] & \underline{f}^*(\overline{\mathcal{M}}_X) \arrow[r, "\bar{f}^\flat"] & \overline{\mathcal{M}}_C,
\end{tikzcd}$$

where the two vertical arrows are the canonical projection and the arrow $\mathbb{N} \to \underline{f}^*(\overline{\mathcal{M}}_X)$ is the pull-back of the global presentation. Note that the arrow $\bar{f}^\flat$ is injective. Denote by δ_X and δ_C the image of δ in $\underline{f}^*(\overline{\mathcal{M}}_X)$ and $\overline{\mathcal{M}}_C$ respectively. The inverse images $p_1^{-1}(\delta_X)$ and $p_2^{-1}(\delta_C)$ form two $\mathcal{O}_C^*$-torsors. Denote by $\mathcal{I}som_C(p_1^{-1}(\delta_X), p_2^{-1}(\delta_C))$ the presheaf of isomorphisms of the two torsors over C. To find a dashed arrow as in (5.4.1) is equivalent to have a global section of $\mathcal{I}som_C(p_1^{-1}(\delta_X), p_2^{-1}(\delta_C))$. Note that the torsor $p_1^{-1}(\delta_X)$ corresponds to the line bundle $\underline{f}^*L$. Denote by L_C the corresponding line bundle of $p_2^{-1}(\delta_C)$. Then we have

$$\begin{aligned}
\mathcal{I}som_C(p_1^{-1}(\delta_X), p_2^{-1}(\delta_C)) &\cong \mathcal{I}som_C(\underline{f}^*L, L_C) \\
&\cong \mathcal{I}som_C(\underline{f}^*L \otimes L_C^{-1}, \mathcal{O}_C).
\end{aligned}$$

Denote by I the above presheaf. It is well known that line bundles are parametrized by the algebraic stack $\mathcal{B}\mathbb{G}_m$. Thus, I is represented by a $\mathbb{G}_m$-torsor that is a separated algebraic space of finite type. Let $\pi : C \to S$ be the projection. By [Ols06, Th. 1.5], there is an algebraic space $\pi_* I$ locally of finite type over S that for any $Y \to S$ associates the groupoid of isomorphisms $(\underline{f}^* L \otimes L_C^{-1})_Y \to \mathcal{O}_{C_Y}$. We have the following lemma for the boundedness of $\pi_* I$: $\square$

LEMMA 5.4.2. *The algebraic space $\pi_* I$ is of finite type over S.*

Proof. By our assumption on G_0, the two line bundles L_C and f^*L have the same degree when restricted to each irreducible component over $\bar{s} \in S$. Since S is affine, by [FP97, Prop. 1], there is a unique closed subscheme $T \subset S$ that represents the condition that the line bundle $f^*L \otimes L_{C_T}^{-1}$ is trivial. The same proof shows that over that locus, the line bundle is pulled back from the base. Its sheaf of trivializations is again represented by a $\mathbb{G}_m$-torsor $U \to \pi_* I$ that is of finite type. $\square$

By pulling back via $\pi_* I \to S$, we have a family of log pre-stable curves $\zeta_{\pi_* I} = (C_I \to \pi_* I, \mathcal{M}_{\pi_* I})$, a usual stable map $\underline{f}_{\pi_* I} : C_I \to X$, and a morphism of sheaves of monoids $f^\flat_{\pi_* I} : \underline{f}^*_{\pi_* I}\mathcal{M}_X \to \mathcal{M}_{C_I}$, where $\mathcal{M}_{C_I}$ is the log structure on C_I given by the log curve $\zeta_{\pi_* I}$.

LEMMA 5.4.3. *The set of points $t \in \pi_* I$, whose fiber $f^\flat_{\pi_* I, \bar{t}}$ gives a morphism of log structures, forms a closed subset of $\pi_* I$.*

Proof. The condition that $f^\flat_{\pi_* I}$ is a morphism of log structures is equivalent to having the following commutative diagram:

$$\begin{tikzcd} \underline{f}^*_{\pi_* I}\mathcal{M}_X \arrow[rr, "f^\flat_{\pi_* I}"] \arrow[dr, "\exp_X"'] & & \mathcal{M}_{C_I} \arrow[dl, "\exp_C"] \\ & \mathcal{O}_{C_I}, & \end{tikzcd} \tag{5.4.2}$$

where the two arrows $\exp_X$ and $\exp_C$ are the structure maps of the corresponding log structures $\underline{f}^*_{\pi_* I}\mathcal{M}_X$ and $\mathcal{M}_{C_I}$. Locally on C_I, we choose a generator $\delta \in \underline{f}^*_{\pi_* I}\mathcal{M}_X$; then the commutativity of the diagram is equivalent to the following equality of sections of $\mathcal{O}_{C_I}$:

$$\exp_X(\delta) = \exp_C \circ f^\flat_{\pi_* I}(\delta),$$

which is a closed condition. Let $V \subset C_I$ be the closed sub-scheme representing the commutativity of (5.4.2) over C_I, and let V^c be the complement of V in C_I. Denote by W the image of V^c in $\pi_* I$ via the projection $C_I \to \pi_* I$. Since the family of curves is flat, the image W is open in $\pi_* I$. Thus, the complement W^c of W is closed in $\pi_* I$. This proves the lemma. $\square$

We take $T = W^c$ as in the above proof with the reduced scheme structure. Then T is a closed subspace of $\pi_* I$. By pulling back families from $\pi_* I$, we obtain a family of minimal stable log maps ξ over T. According to our construction, this family ξ over T satisfies the lifting property in Proposition 5.4.1.

Theorem 5.1.1 follows from the above arguments.

6. The weak valuative criterion for minimal stable log maps

6.1. *Statement of the weak valuative criterion.* Let R be a discrete valuation ring, and let K be the fraction field of R. Denote by π the uniformizer of R, and write $S = \operatorname{Spec} R$. Let s and η be the closed and generic point of S respectively. If R' is another discrete valuation ring, we will write π' for its uniformizer. Denote by s' and η' the closed and generic point of $S' = \operatorname{Spec} R'$ respectively.

Theorem 6.1.1. *With the notation above, consider a minimal stable log map ξ_η over η. Possibly after a base change given by an injection $R \hookrightarrow R'$ of DVR, which induces a finite extension of fraction fields, we have an extension of minimal stable log maps given by the following cartesian diagram:*

$$\begin{tikzcd} \xi_{\eta'} \arrow[r] \arrow[d] & \xi_{S'} \arrow[d] \\ \eta' \arrow[r] & S', \end{tikzcd}$$

where $\xi_{\eta'}$ is the pull-back of ξ_η via $\eta' \to \eta$ and $\xi_{S'}$ is a minimal stable log map over S'. Furthermore, the extension $\xi_{S'}$ is unique up to a unique isomorphism and its formation commutes with further injections of discrete valuation rings.

Proof. We first assume that ξ_η is a minimal log map over η that is not necessarily stable. Possibly after base change, we fix an extension of the underlying pre-stable map $\underline{f} : C \to S$ such that its restriction to the generic fiber is given by the pull-back of $\underline{\xi}_\eta$. Denote by $\underline{\xi}$ the extended underlying map. Here for simplicity, we still use S to denote the new base. The existence of compatible minimal log structures on $\underline{\xi}$ will be proved in Section 6.3. This will be achieved by constructing an extension of certain simplified log maps and using the universal property of minimal log maps. The uniqueness of the extended minimal log structure on $\underline{\xi}$ will be proved in Section 6.5.

In case of stable maps, the extended underlying map $\underline{\xi}$ is unique up to a unique isomorphism. Hence the theorem will be proved by the above argument. □

Remark 6.1.2. By Observation 2.1.9, there is a map

$$\mathcal{K}^{\mathrm{pre}}_{g,n}(X^{\log}, \beta) \to \mathcal{K}^{\mathrm{pre}}_{g,n}(X, \beta),$$

where $\mathcal{K}^{\mathrm{pre}}_{g,n}(X,\beta)$ is the stack of usual pre-stable maps and $\mathcal{K}^{\mathrm{pre}}_{g,n}(X^{\log},\beta)$ is as in Definition 3.5.3. Then our proof implies that this map of stacks satisfies the weak valuative criterion.

6.2. *Local analysis of the extended underlying map.* Let $\xi_\eta = (C_\eta \to \eta, \mathcal{M}_\eta, f_\eta)$. We first consider the case where ξ_η is a log map (not necessarily minimal and stable). We still use $\underline{f} : C \to X$ to denote the extended underlying map over S. Possibly after a base change, we fix a chart $\beta_\eta : \overline{\mathcal{M}}_\eta \to \mathcal{M}_\eta$. Denote by G_η the marked graph of ξ_η. If a node of C is smoothed out over η, then we call it a *special node*; otherwise we call it a *generic node*.

Consider a point $p \in C_{\bar{s}}$, and choose an étale neighborhood U of p. Write $U_\eta := U \times_S \eta$. Shrinking U, we have

$$f^\flat_\eta(\delta) = \beta_\eta(e_v) + \log u_p \text{ over } U_\eta, \tag{6.2.1}$$

where $e_v \in \overline{\mathcal{M}}_\eta$ is the degeneracy of some vertex $v \in G_\eta$ and $u_p \in \mathcal{O}_{U_\eta}$ is some nonzero section. Note that for any section $e \in \overline{\mathcal{M}}_\eta$, we have the corresponding section $\alpha \circ \beta_\eta(e_v) = 0 \in \mathcal{O}_{C_\eta}$. Since we require $u_p \in \mathcal{O}_{U_\eta}$ to be a nonzero section, with the choice of a chart β_η, such section u_p can be unique determined by the formation of log structures on curves; see Definition B.2.1(2).

Assume that p is not a generic node. Shrinking U further, we can assume that U is connected and does not contain a generic node. We also assume that U does not contain points on other components that do not contain p. Note that in this case U is normal and u_p extends to a rational function on U. Denote by ν_π the valuation of the divisor given by the uniformizer π. Let $n_p = \nu_\pi(u_p)$; then we have the following result:

Lemma 6.2.1. *With the above assumption, the point p satisfies one of the following possibilities*:

(1) *if p is a smooth nonmarked point, then there is a neighborhood of p that contains only nondistinguished points over η, and we have $u_p = \pi^{n_p} \cdot h_p$, where $h_p \in \mathcal{O}^*_U$*;
(2) *if p is a marked point with contact order c over η, then $u_p = \pi^{n_p} \cdot x^c \cdot h_p$, where $h_p \in \mathcal{O}^*_U$, and the section containing p is given by the vanishing of $x \in \mathcal{O}_U$*;
(3) *if p is a special node, then $u_p = \pi^{n_p} \cdot x^c \cdot h_p$, where $h_p \in \mathcal{O}^*_U$, the section $x \in \mathcal{O}_U$ is a local coordinate of one component at p, and c is a nonnegative integer.*

Note that if in (3) *we have $c = 0$, then this is compatible with the case described in* (1).

Proof. Since $n_p = \nu_\pi(u_p)$, and u_p is well defined over the generic fiber, we have $u_p = \pi^{n_p} \cdot h'_p$ for some $h'_p \in \mathcal{O}_U$. Since p is a smooth nonmarked point, there is a neighborhood of p that contains only smooth nonmarked points over η. It follows that $h'_p \in \mathcal{O}^*_{U_\eta}$. Note that $\nu_p(h'_p|_{C_{\bar{s}}}) = 0$ by restricting to the central fiber, where ν_p is the valuation map given by the divisor p of the central fiber $C_{\bar{s}}$. Thus $h'_p \in \mathcal{O}^*_U$. This proves (1).

For (2), we have $\nu_x(h'_p) = c$, where ν_x is the valuation map given by the divisor corresponding to the vanishing of x. Then we have $h'_p = x^c \cdot h_p$ such that the restriction $h_p|_{U_\eta}$ is invertible. The same argument as for (1) shows that $h_p \in \mathcal{O}^*_U$.

Consider the case where p is a special node. Let x and y be local coordinates of the two components meeting at p. Choosing the coordinates appropriately, we may assume that $x \cdot y = \pi^n$ for some positive integer n. Without loss of generality, we can assume that $\nu_y(h'_p) = 0$ and $\nu_x(h'_p) = c$ for some nonnegative integer c. Here ν_x (respectively ν_y) is the valuation map corresponding to the divisor defined by the ideal generated by x (respectively y). Thus, as in (2), we have $h'_p = x^c \cdot h_p$ for some $h_p \in \mathcal{O}^*_U$. This proves (3). □

Observation 6.2.2. For a smooth point p, the integer n_p and the rational section u_p in (6.2.1) depend on the choice of the chart β_η. We call the integer n_p *the special degeneracy of p with respect to the chart β_η*. Let Z be the irreducible component of the fiber containing p. Then it is not hard to see that generic points on Z also have n_p as the special degeneracy under β_η. Thus, we call n_p *the special degeneracy of Z under β_η*.

Remark 6.2.3. Consider a node p joining two irreducible components Z_1 and Z_2 over $\bar{s}$. First we assume that p is a special node. Let x and y be local coordinates on Z_1 and Z_2 at p respectively such that $x \cdot y = \pi^n$. By Lemma 6.2.1(3), we can assume that $u_p = \pi^{n_p} \cdot x^c \cdot h_p$. Thus, we can check that the special degeneracy of Z_1 is n_p and the special degeneracy of Z_2 is $n_p + c \cdot n$. In this case, we write $Z_1 \leq Z_2$. Note that if $c = 0$, we have both $Z_1 \leq Z_2$ and $Z_2 \leq Z_1$.

Consider the case where p is a generic node. We take the normalization of C along all the generic node. Then we obtain a set of usual pre-stable curves $\{C_v\}_{v \in V(G_\eta)}$ over S. If $Z_1 \subset C_{v_1}$ and $Z_2 \subset C_{v_2}$, and $v_1 \leq v_2$, then we define $Z_1 \leq Z_2$. We thus define an orientation on the dual graph G of the curve $C_{\bar{s}}$.

The following result, which gives a way of comparing sections in the base log structure, is crucial in the proof of the uniqueness of the extension:

Lemma 6.2.4. *Using the notation as above, consider another chart $\beta'_\eta : \overline{\mathcal{M}}_\eta \to \mathcal{M}_\eta$ and a generic point $p \in C_{\bar{s}}$ lies in the component corresponding to $v \in V(G_\eta)$. The two special degeneracies of p given by β_η and β'_η are the same if and only if $\beta'_\eta(e_v) = \log u + \beta_\eta(e_v)$ for some $u \in R^*$.*

Proof. The "if" part is obvious. Consider the other direction. As in (6.2.1), locally at a nonmarked point p we have

$$f_\eta^\flat(\delta) = \beta'_\eta(e_v) + \log u'_p = \beta_\eta(e_v) + \log u_p.$$

Then the assumption implies that $u \cdot u'_p = u_p$ for some $u \in R^*$. Thus, we have

$$\beta'_\eta(e_v) = \beta_\eta(e_v) + \log u,$$

which proves the statement. □

LEMMA 6.2.5. *With the notation as above, the integer c as in* (2) *and* (3) *of Lemma* 6.2.1 *does not depend on the choice of chart β_η. Therefore the orientation on G defined in Remark* 6.2.3 *does not depend on the choice of the chart β_η.*

Proof. Consider another chart $\beta_\eta : \overline{\mathcal{M}}_\eta \to \mathcal{M}_\eta$ and

$$f^\flat(\delta) = \beta'_\eta(e_v) + \log u'_p$$

for some $u'_p \in \mathcal{O}^*_{U_\eta}$. Then we have

$$\beta'_\eta(e_v) = \beta_\eta(e_v) + \log a$$

for some element $a \in K$. Comparing with (6.2.1), we have

$$u_p = a \cdot u'_p.$$

Since c is given by the valuation ν_x, this implies the statement of the lemma. □

We take the normalization of C along all generic nodes. For each $v \in V(G_\eta)$, denote by C_v the corresponding connected component. Now we consider the case where p is a generic node. Let $l \in E(G)$ be the edge corresponding to the generic node p, and assume that l connects two vertices v_1 and v_2. Denote by p_η the corresponding node over the generic point. Again we have the section u_p over U_η as in (6.2.1). By shrinking U, we can choose two regular sections x and y on U_η, which correspond to the coordinates of the two components meeting at p_η in C_η. Choosing the coordinates appropriately, we may assume

$$\beta_\eta(e_l) = \log x + \log y \quad \text{in} \quad U_\eta. \tag{6.2.2}$$

Without loss of generality, we can assume that $u_p = x^c$, where c is the contact order at p_η. Then u_p vanishes along the component with coordinate y.

By taking the normalization of U along the generic node given by l, we obtain two sub-schemes U_1 and U_2 of U. By shrinking U, we can assume that $U_i \subset C_{v_i}$ for $i = 1, 2$. We still use x and y to denote restriction of x and y to U_1 and U_2 respectively, and p_i the pre-image of p in U_i for $i = 1, 2$. Then x and y can be viewed as a rational function on U_1 and U_2 respectively. Let

Σ_1 and Σ_2 be the two sections in U_1 and U_2 respectively, coming from the splitting node l. Let σ_i be the regular functions on U_i, whose vanishing gives the section Σ_i for $i = 1, 2$.

LEMMA 6.2.6. *With the notation as above, locally at p_i, we have*

(1) $x = \pi^{n_1} \cdot \sigma_1 \cdot h_1$, *where* $n_1 = \nu_\pi(x)$ *and* $h_1 \in \mathcal{O}^*_{U_1}$;
(2) $y = \pi^{n_2} \cdot \sigma_2 \cdot h_2$, *where* $n_2 = \nu_\pi(y)$ *and* $h_2 \in \mathcal{O}^*_{U_2}$.

Proof. The proof of this is similar to that for Lemma 6.2.1. □

Remark 6.2.7. Note that σ_1 and σ_2 are the local coordinates of the two components joining at p. Choosing those coordinates appropriately, we may assume that $h_1 = h_2 = 1$ in Lemma 6.2.6. Thus, we have $u_p = \pi^{c \cdot n_1} \cdot \sigma_1^c$.

6.3. *Existence of the extension.* Now we consider the minimal log map ξ_η and the extended underlying map $\underline{\xi}$. Denote by $C(\overline{\mathcal{M}}_\eta)$ the convex rational polyhedral cone of $\overline{\mathcal{M}}_\eta$ in $\overline{\mathcal{M}}^{\mathrm{gp}}_\eta \otimes_{\mathbb{Z}} \mathbb{Q}$. Since $\overline{\mathcal{M}}_\eta$ is sharp, the cone $C(\overline{\mathcal{M}}_\eta)$ is strongly convex.

LEMMA 6.3.1. *There is a lattice point $\tilde{v} \in \overline{\mathcal{M}}^{\mathrm{gp}}_\eta$ such that $(u, \tilde{v}) > 0$ for any nonzero element $u \in C(\overline{\mathcal{M}}_\eta)$, where $(\cdot, \cdot)$ is the standard pairing in the Euclidean space $\overline{\mathcal{M}}^{\mathrm{gp}}_\eta \otimes_{\mathbb{Z}} \mathbb{Q}$.*

Proof. This follows from [Ful93, §1.2(iv)]. □

We fix a lattice point $\tilde{v}$ satisfying the condition in the above lemma. The set

$$\{(u, \tilde{v}) \mid u \in C(\overline{\mathcal{M}}_\eta)\} \subset \mathbb{Q}$$

forms a monoid, whose saturation is the rank-one free monoid $\mathbb{N}$. Thus, we have a map of saturated monoids $l_{\tilde{v}} : \overline{\mathcal{M}}_\eta \to \mathbb{N}$. Consider the log structure $\mathcal{M}'_\eta$ associated to the pre-log structure $\mathbb{N} \to K, \ e \mapsto 0$ over η. We fix two charts $\beta_\eta : \overline{\mathcal{M}}_\eta \to \mathcal{M}_\eta$ and $\beta'_\eta : \overline{\mathcal{M}}'_\eta \cong \mathbb{N} \to \mathcal{M}'_\eta$. Then we have a morphism of log structures $\mathcal{M}_\eta \to \mathcal{M}'_\eta$ given by

$$\beta_\eta(e) \mapsto \beta'_\eta \circ l_{\tilde{v}}(e).$$

Denote by $\xi'_\eta = (C \to S, \mathcal{M}'_\eta, f'_\eta)$ the log map obtained by pulling back ξ_η via the map $(\eta, \mathcal{M}_\eta) \to (\eta, \mathcal{M}'_\eta)$. By Proposition 4.1.1, it is enough to construct a log map (not necessarily minimal) ξ' such that its generic fiber is given by ξ'_η as above.

LEMMA 6.3.2. *Using the notation as above, there exists a chart $\beta'_\eta : \overline{\mathcal{M}}'_\eta \to \mathcal{M}'_\eta$ such that no components of C over $\bar{s}$ have negative special degeneracy under β'_η as in Observation* 6.2.2.

Proof. We fix an arbitrary chart β'_η as above. Consider an irreducible component Z over the closed point $\bar{s}$. Let $p \in Z$ be a smooth nondistinguished point $p \in Z$. Consider the nearby points of p over η. By Lemma 6.2.1, we have

$$(f'_\eta)^\flat(\delta) = \beta'_\eta(e) + \log \pi^n \cdot u, \tag{6.3.1}$$

where u is a locally invertible section near p and $e \in \overline{\mathcal{M}}'_\eta$. If $n \geq 0$, then there is nothing to prove. Consider the case $n < 0$. Since the number of irreducible components over $\bar{s}$ is finite, we can assume that n is minimal among the special degeneracy of all irreducible components of the closed fiber under β'_η. Consider the new chart given by

$$\beta''_\eta : \overline{\mathcal{M}}'_\eta \to \mathcal{M}_\eta, \quad e \mapsto \beta'_\eta(e) - n \cdot \log \pi. \tag{6.3.2}$$

It is not hard to check that (6.3.1) becomes

$$(f'_\eta)^\flat(\delta) = \beta''_\eta(e) + \log u.$$

Since n is minimal, by applying (6.3.1) and (6.3.2) to other components, it follows that no irreducible component of C over $\bar{s}$ has negative special degeneracy under β''_η. $\square$

We fix a chart $\beta'_\eta : \overline{\mathcal{M}}'_\eta \to \mathcal{M}_\eta$, which satisfies the condition in Lemma 6.3.2. Consider the log structure $\mathcal{M}'_S$ associated to the following pre-log structure on S:

$$\mathbb{N}^2 \to R, \quad e_\eta \mapsto 0, \text{ and } e_s \mapsto \pi,$$

where e_η and e_s form the basis of $\mathbb{N}^2$. Now we identify $\mathcal{M}'_{S,\eta}$ with $\mathcal{M}'_\eta$, and the element e_η corresponds to the chart $\beta'_\eta : \overline{\mathcal{M}}'_\eta \to \mathcal{M}_\eta$.

LEMMA 6.3.3. *With the notation as above, there is a morphism of log structures $\mathcal{M}^{C/S}_S \to \mathcal{M}'_S$, whose restriction to the generic point η is identical to the morphism of log structures $\mathcal{M}^{C_\eta/\eta}_\eta \to \mathcal{M}'_\eta$ given by ξ'_η.*

Proof. Possibly after a base change, we can choose a global chart $\beta^{C/S} : \overline{\mathcal{M}}^{C/S}_{S,\bar{s}} \to \mathcal{M}^{C/S}_S$. Denote by G the dual graph of $C_{\bar{s}}$ and by $\{e_l\}_{l \in E(G)}$ the set of generators of $\overline{\mathcal{M}}^{C/S}_{S,\bar{s}}$ such that $\beta^{C/S}(e_l)$ is an element in $\mathcal{M}^{C/S}_S$ smoothing the node corresponding to l in the closed fiber. Assume that l is smoothed out over η; then $\exp \circ \beta^{C/S}(e_l) = \pi^n \cdot h$, where n is an positive integer and h is an invertible element in R. Thus, we define

$$e_l \mapsto n \cdot e_s + \log h.$$

If the node corresponding to l persists over η, then we have

$$e_l \mapsto n_\eta \cdot e_\eta + \log \pi^{n_s} + \log h \quad \text{over } \eta,$$

where n_η and n_s are two integers and h is an invertible element in R. Note that n_η is positive, and we may assume that n_s is also positive by choosing a sufficiently large n in (6.3.2). Thus, we define

$$e_l \mapsto n_\eta \cdot e_\eta + n_s \cdot e_s + \log h.$$

This induces a map $\mathcal{M}_S^{C/S} \to \mathcal{M}'_S$, whose restriction to the generic point coincides with $\mathcal{M}_\eta^{C_\eta/\eta} \to \mathcal{M}'_\eta$. □

Note that the map $\mathcal{M}_S^{C/S} \to \mathcal{M}'_S$ in the above lemma gives a log pre-stable curves $(C \to S, \mathcal{M}'_S)$, whose restriction to η is given by the log-prestable curve $(C_\eta \to \eta, \mathcal{M}'_\eta)$ of ξ'_η.

PROPOSITION 6.3.4. *There is a log map ξ' over $(S, \mathcal{M}'_S)$ with the log curve $(C \to S, \mathcal{M}'_S)$ and underlying map $\underline{\xi}$, whose restriction to η is identical to ξ'_η.*

Proof. It is enough to define the morphism of log structures $f^\flat : \underline{f}^*\mathcal{M}_X \to \mathcal{M}'_C$, where $\mathcal{M}'_C$ is the log structure on C corresponding to the log curve $(C \to S, \mathcal{M}'_S)$. Pick a point $p \in C$ over $\bar{s}$ and an étale neighborhood U of p. By shrinking U as in (6.2.1), we can assume that over the generic point, we have

$$f_\eta^\flat(\delta) = n \cdot e_\eta + \log u_p \ \text{ in } U_\eta,$$

where $u_p \in \mathcal{O}_{U_\eta}$.

We first assume that p is not a generic node. By Lemma 6.2.1, further shrinking U if necessary, the section u_p extend to U of the following form:

$$u_p = \pi^{n_1} \cdot h',$$

where $n_1 = \nu_\pi(u_p)$ and $h' \in \mathcal{O}_U$. Note that Lemma 6.3.2 implies that the integer n_1 is nonnegative. Thus, the only possible way to define $f^\flat$ near p is given by

$$f^\flat(\delta) = n \cdot e_\eta + n_1 \cdot e_s + \log h'.$$

Next we consider the case p is a generic node. With the notation in Remark 6.2.7, we have

$$u_p = \pi^{c \cdot n_1} \cdot \sigma_1^c.$$

Thus, we define

$$f^\flat(\delta) = n \cdot e_\eta + c_1 \cdot n_1 \cdot e_s + c \cdot \log \sigma_1.$$

Note that our local construction is obtained by specializing the section u_p to the closed fiber. Since the underlying structure is fixed, such specialization is unique. Thus, the above construction can be glued together to obtain a global map $f^\flat$ as we want. □

6.4. *Specializing the dual graph.* Consider the dual graph $\underline{G}$ of the underlying curve $C_{\bar{s}}$ of the fixed extension $\underline{\xi}$. For each edge $l \in E(\underline{G})$, if l corresponds to a special node, then we can associate to l a nonnegative integer c given by Lemma 6.2.1(3); if l corresponds to a generic node, then we associate to l the contact order given by ξ_η. Denote by $V_n(\underline{G})$ the set of nondegenerate components of $C_{\bar{s}}$. Note that Remark 6.2.3 gives an orientation on $\underline{G}$ that is compatible with the contact orders defined on each edge and the subset $V_n(\underline{G}) \subset V(\underline{G})$. Thus, we obtain a marked graph. We use G to denote this graph with the discrete data.

PROPOSITION 6.4.1. *Consider any minimal log map ξ over S with the fixed underlying map $\underline{\xi}$ that is an extension of ξ_η. Then the marked graph $G_{\xi_{\bar{s}}}$ is identical to the graph G with the orientation and contact orders as above.*

Proof. First note that the underlying graph of $G_{\xi_{\bar{s}}}$ and G are both given by the dual graph $\underline{G}$ of the underlying curve, and their sets of nondegenerate vertices are the same. It is enough to check that the two graphs have the same contact orders and orientations. We denote the underlying graph to be $\underline{G}$.

Consider an edge $l \in E(\underline{G})$. If l corresponds to a generic node, then by Lemma 3.2.9, the orientation and contact order of l is uniquely determined by the generic fiber ξ_η. Hence the two graphs $G_{\xi_{\bar{s}}}$ and G have the same orientation and contact orders along l.

Next, consider the case where l corresponds to a special node p. Assume that the contact order of ξ at p is c. Note that the log structure is compatible with the underlying structure. Hence the two graphs have the same weight c in Lemma 6.2.1(3) associated to l that only depends on f_η. By Lemma 6.2.5, the orientation of l in $G_{\xi_{\bar{s}}}$ is given by the one described in Remark 6.2.3. This implies that the two graphs $G_{\xi_{\bar{s}}}$ and G have the same orientation and contact order along l.

This finishes the proof of the statement. □

COROLLARY 6.4.2. *The graph G is admissible.*

Proof. This follows from the existence of the extension of ξ_η and the above proposition. □

Consider a minimal log map $\xi = (C \to S, \mathcal{M}_S, f)$, which is an extension of ξ_η over S with underlying map $\underline{\xi}$. Consider the natural map $q^{\mathrm{gen}} : \overline{\mathcal{M}}(G)^{\mathrm{gp}} \cong \overline{\mathcal{M}}^{\mathrm{gp}}_{S,\bar{s}} \to \overline{\mathcal{M}}^{\mathrm{gp}}_{\eta}$. This is a surjection. Denote by K_{sp} the kernel of q^{gen}. Then we have the following exact sequence:

$$0 \longrightarrow K_{\mathrm{sp}} \longrightarrow \overline{\mathcal{M}}(G)^{\mathrm{gp}} \xrightarrow{q^{\mathrm{gen}}} \overline{\mathcal{M}}^{\mathrm{gp}}_{\eta} \longrightarrow 0. \tag{6.4.1}$$

Note that all groups involved in the exact sequence (6.4.1) are free abelian groups. We fix a noncanonical decomposition that is compatible with (6.4.1):

$$\overline{\mathcal{M}}(G)^{\mathrm{gp}} = K_{\mathrm{sp}} \oplus \overline{\mathcal{M}}^{\mathrm{gp}}_{\eta}. \tag{6.4.2}$$

Denote by $q^{\mathrm{sp}} : \overline{\mathcal{M}}(G)^{\mathrm{gp}} \to K_{\mathrm{sp}}$ the natural projection. Then for any element $e \in \overline{\mathcal{M}}(G)^{\mathrm{gp}}$, we write $e = q^{\mathrm{gen}}(e) + q^{\mathrm{sp}}(e)$.

Possibly after a base change, we fix a global chart $\beta : \overline{\mathcal{M}}(G) \cong \overline{\mathcal{M}}_{S,\bar{s}} \to \mathcal{M}_S$. Thus, we have a map of groups $\beta^{\mathrm{gp}} : \overline{\mathcal{M}}^{\mathrm{gp}}_{S,\bar{s}} \to \mathcal{M}^{\mathrm{gp}}_S$. By [Ols03a, 3.5(i)], the group K_{sp} is generated by elements in $\overline{\mathcal{M}}(G)$, whose images in R is not zero. Consider the composition

$$\bar{\beta} := \nu_\pi \circ \exp \circ \beta^{\mathrm{gp}} : K_{\mathrm{sp}} \to \mathbb{Z}, \tag{6.4.3}$$

where ν_π is the valuation of of the fraction field K.

LEMMA 6.4.3. *The map $\bar{\beta}$ only depends on the base S and the fixed underlying extension $\underline{\xi}$.*

Proof. Note that all other irreducible elements in $\overline{\mathcal{M}}(G)$ can be expressed as some nonnegative rational linear combinations of the irreducible elements lying on some extremal rays of $C(\overline{\mathcal{M}}(G))$. It is enough to consider an irreducible element $e \in \overline{\mathcal{M}}(G)$ that lies on an extremal ray of the cone $C(\overline{\mathcal{M}}(G))$ in $\overline{\mathcal{M}}(G) \otimes_{\mathbb{Z}} \mathbb{Q}$ such that its image in R is nonzero. Without loss of generality, we can assume that $a = \pi^n$. By Lemma 3.3.8, there is a minimal positive integer n' such that $n' \cdot e$ is the element associated to some vertex or some special node l in G. In the first case, the the minimal vertex is specialized from a nondegenerate component over η. Hence, Lemma 6.2.1(1) implies that the degeneracy $n \cdot n'$ is uniquely determined by the generic fiber and the base S. If $n' \cdot e$ is the element associated to a special node, then this is also determined by the generic fiber and the base S. This proves the statement. □

Consider the map β'_η given by the composition

$$\overline{\mathcal{M}}_\eta \longrightarrow \overline{\mathcal{M}}^{\mathrm{gp}}_\eta \longrightarrow \overline{\mathcal{M}}^{\mathrm{gp}}_{S,\bar{s}} \xrightarrow{\beta^{\mathrm{gp}}} \mathcal{M}^{\mathrm{gp}}_S,$$

where the middle arrow is the natural inclusion given by (6.4.2). Note that for any $e \in \overline{\mathcal{M}}_\eta$, the element $\beta'_\eta(e)$ generalize to a unique element in $\mathcal{M}_\eta$. Thus we obtain a chart for $\mathcal{M}_\eta$.

Definition 6.4.4. A chart $\beta'_\eta : \overline{\mathcal{M}}_\eta \to \mathcal{M}_\eta$ is called *specializable* if it comes from a global chart $\beta : \overline{\mathcal{M}}_{S,\bar{s}} \to \mathcal{M}_S$ as above.

Remark 6.4.5. The specializable chart can be viewed as a restriction of the chart $\beta : \overline{\mathcal{M}}(G) \to \mathcal{M}_S$ to the generic point. However, it depends on the choice of the noncanonical splitting (6.4.2).

For any element $e \in \overline{\mathcal{M}}_{S,\bar{s}}$, consider the decomposition given by (6.4.2):

$$e = q^{\mathrm{sp}}(e) + q^{\mathrm{gen}}(e) = e^{\mathrm{sp}} + e^{\mathrm{gen}}.$$

By the construction of β'_η, we have

$$\beta(e)_\eta = \beta^{\mathrm{gp}}(e^{\mathrm{sp}})_\eta + \beta'_\eta(e^{\mathrm{gen}}) \text{ in } \mathcal{M}_\eta. \tag{6.4.4}$$

Note that $\exp\circ\beta^{\mathrm{gp}}(e^{\mathrm{sp}}) \in K$, hence it is an invertible element in $\mathcal{M}_\eta$.

LEMMA 6.4.6. *For any $v \in V(G)$, the special degeneracy of v with respect to β'_η as in Observation 6.2.2 only depends on $\bar\beta$ and q^{sp}.*

Proof. It follows from (6.4.4) that the special degeneracy of v with respect to β'_η is given by $\bar\beta\circ q^{\mathrm{sp}}(e_v)$, where $e_v \in \overline{\mathcal{M}}(G)$ is the element associated to v. □

6.5. *Uniqueness of the extension.* Assume that we have two minimal extensions $\xi_1 = (C \to S, \mathcal{M}_1, f_1)$ and $\xi_2 = (C \to S, \mathcal{M}_2, f_2)$ of ξ_η, with the same underlying $\underline{\xi}$. After a base change, we can assume that we have two global charts

$$\beta_1 : \overline{\mathcal{M}}(G) \to \mathcal{M}_1 \quad \text{and} \quad \beta_2 : \overline{\mathcal{M}}(G) \to \mathcal{M}_2 \tag{6.5.1}$$

for ξ_1 and ξ_2 respectively.

LEMMA 6.5.1. *For any $e \in \overline{\mathcal{M}}(G)$, we have a unique element $u \in R^*$ such that*

$$\beta_1(e)_\eta = \log u + \beta_2(e)_\eta \quad \textit{in } \mathcal{M}_\eta.$$

Thus, we have a canonical isomorphism of log structures $\mathcal{M}_1 \cong \mathcal{M}_2$.

Proof. We only need to consider the irreducible elements of $\overline{\mathcal{M}}(G)$. Let e be an irreducible element of $\overline{\mathcal{M}}(G)$. By Proposition 6.4.1, we have

$$\overline{\beta_1(e)_\eta} = \overline{\beta_2(e)_\eta} \quad \text{in } \overline{\mathcal{M}}_\eta.$$

Hence we have a unique element $u \in K$ such that

$$\beta_1(e)_\eta = u \cdot \beta_2(e)_\eta \quad \text{in } \mathcal{M}_\eta,$$

where K is the fraction field of R. It remains to prove that u is an invertible element in R.

First assume that e lies on an extremal ray of $C(\overline{\mathcal{M}}(G))$. By Lemma 3.3.8, we have a minimal positive integer n such that $n\cdot e \in N(G)$ is either the element associated to some edge, or the element associated to some minimal vertex.

Consider the case where $n \cdot e$ is the element associated to some edge l. We identify the element $e_l \in \mathcal{M}_S^{C/S}$ smoothing l with its image in $\mathcal{M}_1$ or $\mathcal{M}_2$. Then we have

$$n \cdot \beta_1(e) + \log u_1 = e_l \text{ in } \mathcal{M}_1 \text{ and } n \cdot \beta_2(e) + \log u_2 = e_l \text{ in } \mathcal{M}_2,$$

where $u_1, u_2 \in R^*$. By restricting to the generic point η, we have

$$e_{l,\eta} = n \cdot \beta_1(e)_\eta + \log u_1 = n \cdot \beta_2(e)_\eta + \log u_2 \text{ in } \mathcal{M}_\eta.$$

This implies that $u^n = u_2/u_1$, hence $u \in R^*$.

Next, we consider the case where $n \cdot e$ is the element associated to some minimal vertex $v' \in V(G)$, and we assume that v' is specialized from $v \in V(G_\eta)$. Set $e^{\mathrm{sp}} = q^{\mathrm{sp}}(e)$ and $e^{\mathrm{gen}} = q^{\mathrm{gen}}(e)$. Let $\beta_{i,\eta}$ be the specializable chart as in Definition 6.4.4 induced by β_i for $i = 1, 2$. By (6.4.4), we may assume that

$$n \cdot \beta_1(e)_\eta = n_1 \cdot \log \pi + n \cdot \beta'_{1,\eta}(e^{\mathrm{gen}})$$

and

$$n \cdot \beta_2(e)_\eta = n_2 \cdot \log \pi + n \cdot \beta'_{2,\eta}(e^{\mathrm{gen}}).$$

Note that the special degeneracy of v' with respect to $\beta'_{i,\eta}$ is given by n_i for $i = 1, 2$. By Lemmas 6.4.3 and 6.4.6, the special degeneracy of v' does not depend on the choice of β_i. Thus we have $n_1 = n_2$. By Lemma 6.2.4, we obtain a unique element $u \in R^*$ such that

$$\beta'_{2,\eta}(e^{\mathrm{gen}}) = \log u + \beta'_{1,\eta}(e^{\mathrm{gen}}).$$

Finally assume that e does not lie on any extremal ray. Then for some sufficiently divisible positive integer n, we have

$$n \cdot e = \sum_i n_i e_i,$$

where n_i is a positive integer and e_i is an irreducible element lying on some extremal ray for each i. Then the above argument implies that there exists a unique $u_i \in R^*$ such that $\beta_1(e_i)_\eta = \beta_2(e_i)_\eta + \log u_i$ for each i. Thus, we have

$$n \cdot \beta_1(e)_\eta = n \cdot \beta_2(e)_\eta + \log h,$$

where $h = \prod_i u_i^{n \cdot n_i} \in R^*$. This implies that $u^n = h$, hence $u \in R^*$. $\square$

PROPOSITION 6.5.2. *Possibly after a base change, the isomorphism* $\xi_{1,\eta} \cong \xi_{2,\eta}$ *can be extended uniquely to an isomorphism of* ξ_1 *and* ξ_2.

Proof. For simplicity, we assume that $\xi_{1,\eta} = \xi_{2,\eta} = \xi_\eta$. We fix two global chart β_1 and β_2 as in (6.5.1). Denote by $\beta_{i,\eta} : \overline{\mathcal{M}}_\eta \to \mathcal{M}_\eta$ the specializable chart induced by β_i for $i = 1, 2$. By Lemma 6.5.1, we can identify $\mathcal{M}_1$ and $\mathcal{M}_2$. Thus, the two chart $\beta_{1,\eta}$ and $\beta_{2,\eta}$ are identical.

We first show that the following diagram of log structures commutes:

$$\begin{tikzcd} & \mathcal{M}_S^{C/S} \arrow[dl, "\psi_1"'] \arrow[dr, "\psi_2"] & \\ \mathcal{M}_1 \arrow[rr, equal] & & \mathcal{M}_2, \end{tikzcd}$$

where ψ_i is the structure map defining the corresponding log curve of ξ_i. Since we put the standard log structure along nondistinguished nodes, we only need to consider a special distinguished node p over the closed point. Let $e_p \in \mathcal{M}_S$ be a section smoothing p. Then we have

$$\psi_1(e_p) = \psi_2(e_p) + \log u,$$

where u is a unit in R. Since $\xi_{1,\eta} = \xi_{2,\eta} = \xi_\eta$, by restricting the above equation to the generic point η, we obtain $u = 1$. This proves the commutativity. Thus, we can identify the two log curves of ξ_1 and ξ_2.

It remains to show that the two morphisms of log structures $f^\flat_{\xi_i}$ for $i = 1, 2$ are identical. Pick a point $p \in C$ over $\bar{s}$. Then we need to prove that locally at p we have

$$f^\flat_{\xi_1}(\delta) = f^\flat_{\xi_2}(\delta). \tag{6.5.2}$$

Since the two log maps ξ_1 and ξ_2 are minimal, locally at p we have

$$\bar{f}^\flat_{\xi_1}(\delta) = \bar{f}^\flat_{\xi_2}(\delta) \text{ in } \overline{\mathcal{M}}_S.$$

Thus, locally at p, there exists an invertible function u such that

$$f^\flat_{\xi_1}(\delta) = f^\flat_{\xi_2}(\delta) + \log u.$$

Since $\xi_{1,\eta} = \xi_{2,\eta}$, by restricting to the generic fiber, we obtain $u = 1$. This proves (6.5.2) at p. Therefore, the statement of the proposition holds. □

6.6. *Proof of Theorem* 1.2.1 *and finiteness.* Now we can give the proof of the main Theorem 1.2.1.

Proof. The boundedness is proved in Section 5, and the weak valuative criterion is proved in Section 6. Since the stack has finite diagonal, it was shown in [EHKV01, Th. 2.7] that $\mathcal{K}_\Gamma(X^{\log})$ admits a finite surjective morphism from a scheme. With this property and the weak valuative criterion, by [LMB00, Prop. 7.12] the stack is proper. The Deligne-Mumford property follows from Proposition 3.8.1.

The representability and finiteness of the map $\mathcal{K}_\Gamma(X^{\log}) \to \mathcal{K}_{g,n}(X, \beta)$ follow from Propositions 3.8.4 and 3.7.5. □

Denote by $K_\Gamma(X^{\log})$ and $K_{g,n}(X, \beta)$ the coarse moduli spaces of $\mathcal{K}_\Gamma(X^{\log})$ and $\mathcal{K}_{g,n}(X, \beta)$ respectively. It follows from [KM97, 1.3 Cor.] that $K_\Gamma(X^{\log})$ exists and is proper. By the universal property of coarse moduli spaces, we have a natural map

$$K_\Gamma(X^{\log}) \to K_{g,n}(X, \beta). \tag{6.6.1}$$

Again, since this arrow is quasi-finite, we have

Corollary 6.6.1. *The natural map* (6.6.1) *is finite.*

Appendix A. Prerequisites on logarithmic geometry

A.1. *Basic definitions and properties.* Following [Kat89] and [Ogu06], we recall some basic terminology from logarithmic geometry.

Monoids. A *monoid* is a commutative semi-group with a unit. We usually use " + " and "0" to denote the binary operation and the unit of a monoid. A *morphism between two monoids* is required to preserve the unit.

Let P be a monoid. We can associate a group

$$P^{\mathrm{gp}} := \{(a,b)|(a,b) \sim (c,d) \text{ if } \exists s \in P \text{ such that } s+a+d = s+b+c\}.$$

We have the following terminology:

(1) P is called *integral* if the natural map $P \to P^{\mathrm{gp}}$ is injective.
(2) P is called *saturated* if it is integral and satisfies that for any $p \in P^{\mathrm{gp}}$, if $n \cdot p \in P$ for some positive integer n, then $p \in P$.
(3) P is *coherent* if it is finitely generated.
(4) P is *fine* if it is integral and coherent.
(5) P is *fs* if it is fine and saturated.
(6) P is *sharp* if there are no other units except 0. A nonzero element p in a sharp monoid P is called *irreducible* if $p = a + b$ implies either $a = 0$ or $b = 0$. Denote by $\mathrm{Irr}(P)$ the set of irreducible elements in a sharp monoid P.
(7) A fine monoid P is called *free* if $P \cong \mathbb{N}^n$ for some positive integer n.
(8) A monoid P is called torsion free if the associated group P^{gp} is torsion free.
(9) The monoid P is called toric if P is fine, saturated, and sharp. Note that in this case P is automatically torsion free.
(10) A morphism $h : Q \to P$ of two integral monoids is called *integral* if for any $a_1, a_2 \in Q$, and $b_1, b_2 \in P$ that satisfy $h(a_1)b_1 = h(a_2)b_2$, there exist $a_2, a_4 \in Q$ and $b \in P$ such that $b_1 = h(a_3)b$ and $a_1a_3 = a_2a_4$.

Denote by $\mathrm{Mon}^{\mathrm{int}}$ and $\mathrm{Mon}^{\mathrm{sat}}$ the categories of integral and saturated monoids respectively. Then there is a natural inclusion

$$\iota : \mathrm{Mon}^{\mathrm{sat}} \to \mathrm{Mon}^{\mathrm{int}}.$$

On the other hand, given an integral monoid M, the elements $a \in M^{\mathrm{gp}}$, such that $m \cdot a \in M$ for some positive integer m, form a saturated submonoid $M^{\mathrm{sat}} \subset M^{\mathrm{gp}}$. This induces another functor

$$\mathcal{S}at : \mathrm{Mon}^{\mathrm{int}} \to \mathrm{Mon}^{\mathrm{sat}}.$$

Proposition A.1.1. [Ogu06, Ch. I, 1.2.3(3)] *The functor $\mathcal{S}at$ is left adjoint to the functor ι.*

Logarithmic structures. Let X be a scheme. A *pre-log structure* on X is a pair $(\mathcal{M}, \exp)$, which consists of a sheaf of monoids $\mathcal{M}$ on the étale site $X_{\text{ét}}$ of X and a morphism of sheaves of monoids $\exp : \mathcal{M} \to \mathcal{O}_X$, called the *structure morphism of* $\mathcal{M}$. Here we view $\mathcal{O}_X$ as a sheaf of monoid under multiplication.

A pre-log structure $\mathcal{M}$ on X is called a *log structure* if $\exp^{-1}(\mathcal{O}_X^*) \cong \mathcal{O}_X^*$ via exp. We sometimes omit the morphism exp, and we only use $\mathcal{M}$ to denote the log structure if no confusion could arise. We call the pair $(X, \mathcal{M})$ a *log scheme*.

Given two log structures $\mathcal{M}$ and $\mathcal{N}$ on X, a *morphism of the log structures* $h : \mathcal{M} \to \mathcal{N}$ is a morphism of sheaves of monoids that is compatible with the structure morphisms of $\mathcal{M}$ and $\mathcal{N}$.

Given a pre-log structure $\mathcal{M}$ on X, we can associate a log structure $\mathcal{M}^a$ given by

$$\mathcal{M}^a := \mathcal{M} \oplus_{\exp^{-1}(\mathcal{O}_X^*)} \mathcal{O}_X^*.$$

Consider a morphism of schemes $f : X \to Y$ and a log structure $\mathcal{M}_Y$ on Y. We can define the *pull-back log structure* $f^*(\mathcal{M}_Y)$ to be the log structure associated to the pre-log structure

$$f^{-1}(\mathcal{M}_Y) \to f^{-1}(\mathcal{O}_Y) \to \mathcal{O}_X.$$

Consider two log schemes $(X, \mathcal{M}_X)$ and $(Y, \mathcal{M}_Y)$. A morphism of log schemes $(X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ is a pair $(f, f^\flat)$, where $f : X \to Y$ is a morphism of the underlying schemes and $f^\flat : f^*(\mathcal{M}_Y) \to \mathcal{M}_X$ is a morphism of log structures on X. The morphism $(f, f^\flat)$ is called *strict* if $f^\flat$ is an isomorphism of log structures. It is called *vertical* if $\mathcal{M}_X/f^*(\mathcal{M}_Y)$ is a sheaf of groups under the induced monoidal operation. A standard example of log structures is the following:

Example A.1.2. Let D be a normal crossing divisor on a smooth scheme X. Then

$$\mathcal{M}_X = \{g \in \mathcal{O}_X \mid g \text{ is invertible outside } D\}$$

with the natural injection $\mathcal{M}_X \to \mathcal{O}_X$ forms a log structure on X.

A.1.1. *Charts of log structures.* Let $(X, \mathcal{M})$ be a log scheme, and let P be a monoid. Denote by P_X the constant sheaf of monoid P on X. A chart of $\mathcal{M}$ is a morphism $P_X \to \mathcal{M}$ such that the associated log structure of the composition $P_X \to \mathcal{M} \to \mathcal{O}_X$ is $\mathcal{M}$. The log structure $\mathcal{M}$ is called a *fine log structure* on X if P is fine. If the monoid P is fs, then $\mathcal{M}$ is called a *fs log structure*. We denote by $\mathfrak{LogSch}$ the category of fine log schemes and $\mathfrak{LogSch}^{\mathrm{fs}}$ the category of fs log schemes.

Let $\overline{\mathcal{M}} = \mathcal{M}/\mathcal{O}_X^*$ be the quotient sheaf. We call it the *characteristic of the log structure* $\mathcal{M}$. It is useful to notice that $\overline{f^*(\mathcal{M})} = f^{-1}(\overline{\mathcal{M}})$ for any morphism of schemes $f : Y \to X$. A fine log structure $\mathcal{M}$ is called locally free if for any $\bar{x} \in X$, we have $\overline{\mathcal{M}}_{\bar{x}} \cong \mathbb{N}^n$ for some positive integer n. Let $\overline{\mathcal{M}}_{\bar{x}}^{\mathrm{gp,tor}}$ be the torsion part of $\overline{\mathcal{M}}_{\bar{x}}^{\mathrm{gp}}$. The following result is very useful for creating charts:

PROPOSITION A.1.3. [Ols03a, 2.1] *Using the notation as above, there exist an* fppf *neighborhood $f : X' \to X$ of x and a chart $\beta : P \to f^*(\mathcal{M})$ such that for some geometric point $\bar{x}' \to X'$ lying over x, the natural map $P \to f^{-1}\overline{\mathcal{M}}_{\bar{x}'}$ is bijective. If $\overline{\mathcal{M}}_{\bar{x}}^{\mathrm{gp,tor}} \otimes k(x) = 0$, then such a chart exists in an étale neighborhood of x.*

Remark A.1.4. If $\mathcal{M}$ is an fs log structure on X, then the above proposition implies that there exists a section $\overline{\mathcal{M}}_{\bar{x}} \to \mathcal{M}_{\bar{x}}$ that can be lifted to a chart étale locally near x.

Consider a morphism $f : (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ of fine log schemes. A *chart of f* is a triple $(P_X \to \mathcal{M}_X, Q_Y \to \mathcal{M}_Y, Q \to P)$, where $P_X \to \mathcal{M}_X$ and $Q_Y \to \mathcal{M}_Y$ are charts of $\mathcal{M}_X$ and $\mathcal{M}_Y$ respectively, and $Q \to P$ is a morphism of monoids such that the following diagram is commutative:

$$\begin{array}{ccc} Q_X & \longrightarrow & P_X \\ \downarrow & & \downarrow \\ f^*(\mathcal{M}_Y) & \longrightarrow & \mathcal{M}_X. \end{array}$$

Note that the charts of morphism of fine log schemes exist étale locally.

Consider a morphism of log schemes $f : (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$. With the help of charts, we can describe the log smoothness properties of f due to K. Kato [Kat89, Th. 3.5]. The log map f is called *log smooth* if étale locally, there is a chart $(P_X \to \mathcal{M}_X, Q_Y \to \mathcal{M}_Y, Q \to P)$ of f such that

(1) $\mathrm{Ker}(Q^{\mathrm{gp}} \to P^{\mathrm{gp}})$ and the torsion part of $\mathrm{Coker}(Q^{\mathrm{gp}} \to P^{\mathrm{gp}})$ are finite groups;
(2) the induced map $X \to Y \times_{\mathrm{Spec}(\mathbb{Z}[Q])} \mathrm{Spec}\, \mathbb{Z}[P]$ is smooth in the usual sense.

The map f is called *integral* if for every $p \in X$, the induced map $\overline{\mathcal{M}}_{f(\bar{p})} \to \overline{\mathcal{M}}_{\bar{p}}$ is integral. In general, the underlying structure map of a log smooth morphism need not be flat. However, it is shown in [Kat89, 4.5] that the underlying map of a log smooth and integral morphism is flat.

A.2. *Deligne-Faltings log structures.*

Definition A.2.1. Consider a scheme X. A locally free log structure $\mathcal{M}_X$ on X is called a *Deligne-Faltings* (DF) *log structure* if there is a morphism of locally constant sheaves of monoids $\beta : \mathbb{N}^k \to \overline{\mathcal{M}}_X$ that locally lifts to a chart. We call the map β a *global presentation of* $\mathcal{M}_X$.

Remark A.2.2. Consider a DF log structure $\mathcal{M}_X$ with a global presentation $\mathbb{N}^k \to \overline{\mathcal{M}}_X$. Denote by $\{\delta_i\}_{i=1}^k$ the standard generators of $\mathbb{N}^k$. Then locally we have a lifting $\tilde{\beta} : \mathbb{N}^k \to \mathcal{M}_X$. Note that the section $\beta(\delta_i)$ with its inverse image under the canonical map $\pi : \mathcal{M}_X \to \overline{\mathcal{M}}_X$ is a $\mathcal{O}_X^*$-torsor, which corresponds to a line bundle L_i. The composition

$$\pi^{-1}\big(\beta(\delta_i)\big) \subset \mathcal{M}_X \to \mathcal{O}_X$$

gives a morphism of line bundles $s_i : L_i \to \mathcal{O}_X$. In fact, it was shown in [Kat89, Complement 1] that a locally free DF log structure as above is equivalent to the data consisting of a k-tuple of line bundles $(L_i)_{i=1}^k$ and morphisms of line bundles $s_i : L_i \to \mathcal{O}_X$ for each i.

Note that $s_i \in H^0(L_i^\vee)$. Denote by $D_i \subset X$ the vanishing locus of s_i. Note that D_i consists of the points where the image of δ_i in $\overline{\mathcal{M}}_X$ is nontrivial. If s_i is not a zero section, then D_i is a Cartier divisor in X. If s_i is a zero section, then $D_i = X$. Consider the sub-log structure $\mathcal{M}_X^g \subset \mathcal{M}_X$ that is given by the set of zero sections and the corresponding line bundles. We call $\mathcal{M}_X^g$ the *generic part of* $\mathcal{M}_X$. Note that if $D_i = \emptyset$, then the sub-log structure generated by δ_i is trivial.

Example A.2.3. Consider a smooth Cartier divisor $D \subset X$ and the log structure $\mathcal{M}_X$ associated to D defined in Example A.1.2. Then $\mathcal{M}_X$ forms a DF log structure on X that corresponds to the line bundle $\mathcal{O}_X(-D)$ and the natural inclusion $\mathcal{O}_X(-D) \hookrightarrow \mathcal{O}_X$.

A.3. *Olsson's Log Stacks.* We follow [Ols03a] to introduce the algebraic stack parametrizing log structures. Let us fix a base scheme S and consider an algebraic stack $\mathcal{X}$ in the sense of [Art74], which means that

(1) the diagonal $\mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is representable and of finite type,
(2) there exists a surjective smooth morphism $X \to \mathcal{X}$ from a scheme.

Now we can define a fine log structure $\mathcal{M}_{\mathcal{X}}$ on $\mathcal{X}$ by repeating the definition of log structure on schemes in A.1, but using a lisse-étale site instead of the étale site. We refer to [Ols03a, §5] for details of log structure on Artin stacks.

For any S-scheme T and an arrow $g : T \to \mathcal{X}$, we obtain a fine log structure $g^*(\mathcal{M}_{\mathcal{X}})$ on the lisse-étale site $T_{\text{lis-ét}}$ of T. It is shown in [Ols03a, 5.3] that such $g^*(\mathcal{M}_{\mathcal{X}})$ is isomorphic to a unique fine log structure on the étale site $T_{\text{ét}}$ of T. By abuse of notation, we denote by $g^*(\mathcal{M}_{\mathcal{X}})$ the corresponding log structure on T. Thus, we define a functor from $\mathcal{X}$ to $\mathfrak{LogSch}_S$ by pulling back the log structure $\mathcal{M}_{\mathcal{X}}$. The stack $\mathcal{X}$ associated with this functor is called a *log stack* in [Kat00]. A fine log scheme $(X, \mathcal{M}_X)$ can be naturally viewed as a log algebraic stack.

Consider the fibered category $\mathcal{L}og_{(\mathcal{X},\mathcal{M}_{\mathcal{X}})}$ over $\mathcal{X}$. Its objects are pairs

$$(g : X \to \mathcal{X}, g^*(\mathcal{M}_{\mathcal{X}}) \to \mathcal{M}_X),$$

where g is a map from scheme X to $\mathcal{X}$ and $g^*(\mathcal{M}_{\mathcal{X}}) \to \mathcal{M}_X$ is a morphism of fine log structures on X. An arrow

$$\big(g : X \to \mathcal{X}, g^*(\mathcal{M}_{\mathcal{X}}) \to \mathcal{M}_X\big) \longrightarrow \big(h : Y \to \mathcal{X}, h^*(\mathcal{M}_{\mathcal{X}}) \to \mathcal{M}_Y\big)$$

is a strict morphism of log schemes $(X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$, such that

(1) the underlying map $X \to Y$ is a morphism over $\mathcal{X}$;
(2) the following diagram of log schemes commutes:

$$\begin{tikzcd}
(X, \mathcal{M}_X) \arrow[r] \arrow[d] & (Y, \mathcal{M}_Y) \arrow[d] \\
\big(X, g^*(\mathcal{M}_{\mathcal{X}})\big) \arrow[r] & \big(Y, h^*(\mathcal{M}_{\mathcal{X}})\big).
\end{tikzcd}$$

Remark A.3.1. In fact, the stack $\mathcal{L}og_{(\mathcal{X},\mathcal{M}_{\mathcal{X}})}$ parametrizes log structures over $(\mathcal{X}, \mathcal{M}_{\mathcal{X}})$. An object $\big(g : X \to \mathcal{X}, g^*(\mathcal{M}_{\mathcal{X}}) \to \mathcal{M}_X\big)$ as above can be viewed as a morphism of log stacks $(X, \mathcal{M}_X) \to (\mathcal{X}, \mathcal{M}_{\mathcal{X}})$.

THEOREM A.3.2. [Ols03a, 5.9] *The fibered category $\mathcal{L}og_{(\mathcal{X},\mathcal{M}_{\mathcal{X}})}$ is an algebraic stack locally of finite presentation over $\mathcal{X}$.*

Appendix B. Logarithmic curves and their stacks

In this section, we introduce the notion of log pre-stable curves. We will prove that the stack $\mathfrak{M}^{\mathrm{pre}}_{g,n}$ parametrizing log pre-stable curves of genus g and n marked points is an open substack of some Olsson's log stack as above, and hence is algebraic in the sense of [Art74, 5.1]. We refer to [Kat00], [Moc95], and [Ols07] for more details of log structures on curves.

B.1. *The canonical log structure on pre-stable curves.* Consider the stack $\mathfrak{M}_{g,n}$ parametrizing genus g pre-stable curves with n marked points, and let $\mathfrak{C}_{g,n}$ be the universal family over $\mathfrak{M}_{g,n}$. Denote by $\{\Sigma_i : \mathfrak{M}_{g,n} \to \mathfrak{C}_{g,n}\}_{i=1}^n$ the n sections. The boundary $\mathfrak{M}^{\mathrm{sing}}_{g,n} \subset \mathfrak{M}_{g,n}$ that parametrizes singular curves is a divisor with normal crossings on $\mathfrak{M}_{g,n}$. Hence the boundary divisor induces a canonical log structure $\mathcal{M}_{\mathfrak{M}_{g,n}}$ on $\mathfrak{M}_{g,n}$, which is defined using the smooth topology as in [Ols03a].

Note that each section Σ_i corresponds to a smooth divisor on $\mathfrak{C}_{g,n}$. By Example A.1.2, we have a log structure $\mathcal{M}^{\Sigma_i}$ associated to this section Σ_i. The pre-image of $\mathfrak{M}^{\mathrm{sing}}_{g,n}$ also gives a normal crossing divisor in $\mathfrak{C}_{g,n}$, hence a log structure $\mathcal{M}^{\sharp}_{\mathfrak{C}_{g,n}}$ on $\mathfrak{C}_{g,n}$. Consider the log structure

$$\mathcal{M}_{\mathfrak{C}_{g,n}} := \mathcal{M}^{\sharp}_{\mathfrak{C}_{g,n}} \oplus_{\mathcal{O}^*_{\mathfrak{C}_{g,n}}} \sum_i \mathcal{M}^{\Sigma_i}.$$

We call it the canonical log structure on $\mathfrak{C}_{g,n}$. There is a natural log smooth, integral, vertical map

$$(\mathfrak{C}_{g,n}, \mathcal{M}^{\sharp}_{\mathfrak{C}_{g,n}}) \to (\mathfrak{M}_{g,n}, \mathcal{M}_{\mathfrak{M}_{g,n}}), \tag{B.1.1}$$

whose underlying map is given by the family $\mathfrak{C}_{g,n} \to \mathfrak{M}_{g,n}$. By adding the log structure from the marked points, we have an induced log smooth, integral map

$$(\mathfrak{C}_{g,n}, \mathcal{M}_{\mathfrak{C}_{g,n}}) \to (\mathfrak{M}_{g,n}, \mathcal{M}_{\mathfrak{M}_{g,n}}). \tag{B.1.2}$$

Given any family $C \to S$ of usual pre-stable curves of genus g, with n marked points, we have the following cartesian diagram:

$$\begin{tikzcd} C \arrow[r] \arrow[d, "\pi"'] & \mathfrak{C}_{g,n} \arrow[d] \\ S \arrow[r] & \mathfrak{M}_{g,n}. \end{tikzcd}$$

Denote by $\mathcal{M}^{\sharp C/S}_{C}$, $\mathcal{M}^{C/S}_{C}$ and $\mathcal{M}^{\Sigma_i}_{C}$ the log structures on C, obtained by pulling back $\mathcal{M}^{\sharp}_{\mathfrak{C}_{g,n}}$, $\mathcal{M}_{\mathfrak{C}_{g,n}}$ and $\mathcal{M}^{\Sigma_i}$ respectively. Let $\mathcal{M}^{C/S}_{S}$ be the log structure on S obtained by pulling back $\mathcal{M}_{\mathfrak{M}_{g,n}}$. Note that $\mathcal{M}^{\Sigma_i}_{C}$ is the log structure given by the section Σ_i. Now we have two canonical log maps obtained by pulling back (B.1.1) and (B.1.2) respectively:

$$(C, \mathcal{M}^{\sharp C/S}_{C}) \to (S, \mathcal{M}^{C/S}_{S}) \tag{B.1.3}$$

and

$$(C, \mathcal{M}^{C/S}_{C}) \to (S, \mathcal{M}^{C/S}_{S}). \tag{B.1.4}$$

Lemma B.1.1. *For any pair of fine log structures $(\mathcal{M}'_C, \mathcal{M}_S)$ over the family of prestable curves $C \to S$, such that the log map $(C, \mathcal{M}'_C) \to (S, \mathcal{M}_S)$ is log smooth, proper, integral and vertical, we have a unique pair of maps $\mathcal{M}^{\sharp C/S}_{C} \to \mathcal{M}'_C$ and $\mathcal{M}^{C/S}_{S} \to \mathcal{M}_S$ fitting in the following cartesian diagram of fine log schemes:*

$$\begin{tikzcd} (C, \mathcal{M}'_C) \arrow[r] \arrow[d] & (C, \mathcal{M}^{\sharp C/S}_{C}) \arrow[d] \\ (S, \mathcal{M}_S) \arrow[r] & (S, \mathcal{M}^{C/S}_{S}). \end{tikzcd}$$

Proof. See [Ols07], and [Ols03b, 2.7] for a proof. ∎

B.2. *Log curves.* With the description above, we are able to introduce the log structure on curves that we are interested in.

Definition B.2.1. A map of fine log schemes $(C, \mathcal{M}_C) \to (S, \mathcal{M}_S)$ with sections $\{\Sigma_i\}_{i=1}^{n}$ is called a genus g log curve with n-markings if

(1) the family $C \to S$ with $\{\Sigma_i\}$ is the usual prestable curve of genus g and n-markings;
(2) the log structure $\mathcal{M}_C$ is of the form $\mathcal{M}_C = \mathcal{M}'_C \oplus_{\mathcal{O}^*_C} (\sum_j \mathcal{M}_C^{\Sigma_j})$;
(3) the log map $(C, \mathcal{M}_C) \to (S, \mathcal{M}_S)$ comes from a log smooth, integral vertical map $(C, \mathcal{M}'_C) \to (S, \mathcal{M}_S)$ plus the log structure $\mathcal{M}^{\Sigma_i}$ given by the markings.

By Lemma B.1.1, we have an equivalent definition of log curves using the canonical log structure.

Definition B.2.2. A genus g, log curve with n-marked points over a scheme S is given by the following data $(C \to S, \{\Sigma\}_{i=1}^n, \mathcal{M}_S^{C/S} \to \mathcal{M}_S)$, where
(1) $(C \to S, \{\Sigma\}_{i=1}^n)$ is a usual family of pre-stable curves of genus g with n-markings;
(2) $\mathcal{M}_S^{C/S} \to \mathcal{M}_S$ is a morphism of fine log structures.

If no confusion could arise, we will use $(C \to S, \mathcal{M}_S)$ for the log curves in the definition for short, and we denote by $\mathcal{M}_C$ the log structure on the curves in the above Definition B.2.1.

Definition B.2.3. A log curve $(C \to S, \mathcal{M}_S)$ is called log pre-stable if the log structure $\mathcal{M}_S$ is fine and saturated.

Remark B.2.4. By [Ols03a, 5.26], the condition that the base log structure $\mathcal{M}_S$ is fine and saturated is an open condition on S.

B.3. *The stack of log curves.*

Definition B.3.1. Consider two log curves $(C \to S, \mathcal{M}_S)$ and $(C' \to S, \mathcal{M}'_S)$ over S. Denote by $\mathcal{M}_C$ and $\mathcal{M}_{C'}$ the log structure on C and C' associated to the two log curves respectively. An isomorphism between the above two log curves is a pair (ρ, θ) such that
(1) $\theta : (S, \mathcal{M}_S) \to (S, \mathcal{M}'_S)$ and $\rho : (C, \mathcal{M}_C) \to (C', \mathcal{M}_{C'})$ are isomorphisms of log schemes;
(2) the underlying map $\underline{\theta} : S \to S$ is the identity, and $\underline{\rho} : C \to C'$ is an isomorphism of usual prestable curves over S;
(3) the pair (ρ, θ) fit in the following commutative diagram:

$$\begin{tikzcd} (C, \mathcal{M}_C) \arrow[r, "\rho"] \arrow[d] & (C', \mathcal{M}_{C'}) \arrow[d] \\ (S, \mathcal{M}_S) \arrow[r, "\theta"] & (S, \mathcal{M}'_S). \end{tikzcd}$$

Denote by $\mathfrak{M}_{g,n}^{\log}$ the fibered category over $\mathbb{C}$ parametrizing log curves with the arrow defined above. In fact, we have

$$\mathfrak{M}_{g,n}^{\log} \cong \mathcal{L}og_{(\mathfrak{M}_{g,n}, \mathcal{M}_{\mathfrak{M}_{g,n}})}.$$

Thus, the fibered category $\mathfrak{M}^{\log}_{g,n}$ forms an algebraic stack in the sense of [Art74]. Denote by $\mathfrak{M}^{\mathrm{pre}}_{g,n}$ the substack of $\mathfrak{M}^{\log}_{g,n}$ parametrizing log prestable curves. Then by Remark B.2.4, we have the following:

COROLLARY B.3.2. *The fibered category $\mathfrak{M}^{\mathrm{pre}}_{g,n}$ is an open substack in $\mathfrak{M}^{\log}_{g,n}$, hence it is algebraic.*

B.4. *The canonical log structure at nodes.* Note that the log structure $\mathcal{M}_{\mathfrak{M}_{g,n}}$ is locally free. Consider a usual prestable curve $C \to S$. Then the canonical log structure $\mathcal{M}_S^{C/S}$ is also locally free. For any closed point $s \in S$, we have

$$\overline{\mathcal{M}}^{C/S}_{S,\bar{s}} \cong \mathbb{N}^m,$$

where m is a nonnegative integer.

Shrinking S if necessary, by Proposition A.1.3 we can choose a global chart $\overline{\mathcal{M}}^{C/S}_{S,\bar{s}} \cong \mathbb{N}^m \to \mathcal{M}_S^{C/S}$. Denote by $\{e_i\}_{i=1}^m$ the standard generators of $\mathbb{N}^m$.

Consider a node point $p \in C_{\bar{s}}$ in the fiber. Then there is an étale neighborhood U of p that contains no other nodes and marked points. We have a special element $e_j \in \{e_i\}_{i=1}^m$, with the following chart:

$$\begin{tikzcd}
\mathbb{N}^{m-1} \oplus \mathbb{N}^2 \arrow[r] & \mathcal{M}_C^{C/S}|_U \\
\mathbb{N}^{m-1} \oplus \mathbb{N} \arrow[r] \arrow[u, "{(\mathrm{id},\Delta)}"] & \pi^*(\mathcal{M}_S^{C/S})|_U. \arrow[u, "\pi^\flat"']
\end{tikzcd}$$

Here on the bottom, the monoids $\mathbb{N}^{m-1}$ and $\mathbb{N}$ are generated by $\{e_i\}_{i\neq j}$ and e_j respectively, and on the top we assume that a and b are the standard generators of the monoid $\mathbb{N}^2$. The map (id, Δ) is given by the identity on $\mathbb{N}^{n-1}$ and the diagonal map $\Delta : e_j \mapsto a + b$.

Conventions B.4.1. Consider a log curve $(C \to S, \mathcal{M}_S)$. For convenience, we identify e_j with its image in $\overline{\mathcal{M}}_S$ and call it *the element in $\overline{\mathcal{M}}_S$ smoothing the node p*, or simply *the element smoothing p*.

For each node p_i over s, we fix an element $e_i \in \overline{\mathcal{M}}^{C/S}_{S,\bar{s}}$ smoothing it. Let $\mathrm{Irr}(\overline{\mathcal{M}}^{C/S}_{S,\bar{s}})$ be the set of irreducible elements in the monoid $\overline{\mathcal{M}}^{C/S}_{S,\bar{s}}$. In fact we have $\{e_i\}_{i=1}^m = \mathrm{Irr}(\overline{\mathcal{M}}^{C/S}_{S,\bar{s}})$ and a natural map

$$s_{C_{\bar{s}}} : \{\text{nodes in } C_{\bar{s}}\} \to \mathrm{Irr}\left(\overline{\mathcal{M}}^{C/S}_{S,\bar{s}}\right)$$

given by $p_i \mapsto$ (the element e_i smoothing p_i). It was shown in [Kat00] that this map is a one-to-one correspondence. This means that all nodes in the fiber are smoothed independently.

Remark B.4.2. The bijection $s_{C_{\bar{s}}}$ implies that the canonical log structures $(\mathcal{M}_S^{C/S}, \mathcal{M}_C^{C/S})$ is special in the sense of [Ols03b, 2.6].

We give a local description of the relation between canonical log structure and the underlying structure at the nodes as in [Kat00, §3]. Let A be a local neotherian henselian ring, and let s be an element in the maximal ideal m_A of A. Let R be the henselization of $A[x,y]/(xy-s)$ at the ideal generated by x, y and m_A. We still use x, y to denote the corresponding elements in R.

LEMMA B.4.3. [Kat00, 2.1] *Given* $x', y' \in R$ *such that* $x'y' \in A$ *and* $(x', y', m_A) = (x, y, m_A)$ (*equality of ideals in* R), *then there exist units* $u_x, u_y \in R^*$ *with* $u_x u_u \in A$ *such that* $x' = u_x x$ *and* $y' = u_y y$ (*or* $y' = u_x x$ *and* $x' = u_y y$).

Consider the local family $\operatorname{Spec} R \to \operatorname{Spec} A$. The canonical log structure $(\mathcal{M}_R, \mathcal{M}_A)$ is given by the following commutative diagram of pre-log structures:

$$\begin{array}{ccc} \mathbb{N}^2 & \xrightarrow{(e_1,e_2)\mapsto(x,y)} & R \\ \Delta\uparrow & & \uparrow \\ \mathbb{N} & \xrightarrow{e\mapsto s} & A, \end{array}$$

where e_1, e_2 (resp. e) are the standard generators of $\mathbb{N}^2$ (resp. $\mathbb{N}$), and $\Delta : e \mapsto e_1 + e_2$ is the diagonal map. For convenience, we sometimes use $\log x, \log y$ and $\log s$ denote the image of e_1, e_2 and e in the corresponding log structures.

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