Annals of Mathematics **180** (2014), 403–405
http://dx.doi.org/10.4007/annals.2014.180.1.9

Corrigendum to: Operator monotone functions and Löwner functions of several variables

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Abstract

We fix a gap in the proof of Theorem 7.24 in *Ann. of Math.* **176** (2012), 1783–1826.

There is a gap in the proof of Theorem 7.24 in [1], though the statement of the theorem is correct.

In the proof of necessity, we argue that Λ is in $\mathcal{G}$ by contradiction. If it were not, invoking the Hahn-Banach separation theorem would yield a real skew-symmetric matrix K and a constant $\delta \geq 0$ such that $\operatorname{tr}(\Gamma K) \geq -\delta$ for all Γ in $\mathcal{G}$, and $\operatorname{tr}(\Lambda K) < -\delta$. In the proof we assumed that $\delta = 0$, but this assumption is unjustified.

Instead, we argue as follows. Define Δ by

$$\Delta^r_{ij} \;=\; (x^r_j - x^r_i) K_{ji}, \quad i \neq j,$$

and with the diagonal entries Δ^r_{ii} chosen so that each $\Delta^r \geq 0$ and so that

$$\mu^r \;:=\; \sum_{i=1}^n f_{r,i} \Delta^r_{ii} \tag{0.1}$$

is minimal over all choices of $\Delta^r_{11}, \ldots, \Delta^r_{nn}$ such that $\Delta \geq 0$. (A minimal choice exists, since all the $f_{r,i}$ are strictly positive by assumption.) Then Δ is in SAM^d_n, and

$$[\Delta^s, S^r]_{ij} \;=\; (x^s_j - x^s_i) K_{ji} (x^r_j - x^r_i) \;=\; [\Delta^r, S^s]_{ij}.$$

J. A. was partially supported by NSF grant DMS-1068830. J. E. McC. was partially supported by NSF grant DMS-1300280. N. J. Y. was partially supported by LMS grant 41219 and EPSRC grant EP/K50340X/1.

As f is locally M_n monotone, we must have then that $D_\Delta f(S) \geq 0$ by Lemma 7.3. As

$$-\delta \;>\; \operatorname{tr}(\Lambda K) \;=\; \sum_{1\leq i,j\leq n} [D_\Delta f(S)]_{ij} - \sum_{r=1}^{d}\sum_{i=1}^{n} \Delta_{ii}^r f_{r,i},$$

we get that

$$\sum_{r=1}^{d} \mu^r - \delta \;>\; \sum_{1\leq i,j\leq n} [D_\Delta f(S)]_{ij} \geq 0. \tag{0.2}$$

By Duffin's strong duality theorem [2], the minimum μ^r in (0.1) satisfies

$$-\mu^r \;=\; \min \sum_{i\neq j} \Delta_{ij} A^r(i,j), \tag{0.3}$$

where A^r range over the set of real positive matrices such that the diagonal entries of A^r are $f_{r1}, \ldots, f_{rn}$ for each r.

For each such $A = (A^1, \ldots, A^d)$, let Γ be the corresponding element of $\mathcal{G}$: $\Gamma_{ii} = 0$ and

$$\Gamma_{ij} = \sum_{r=1}^{d} (x_j^r - x_i^r) A^r(i,j) \qquad \text{for } i \neq j.$$

We have

$$\begin{aligned} -\delta &\leq \operatorname{tr} \Gamma K \\ &= \sum_{i\neq j}\sum_{r=1}^{d} (x_j^r - x_i^r) A^r(i,j) K_{ji} \\ &= \sum_{r=1}^{d}\sum_{i\neq j} \Delta_{ij}^r A^r(i,j). \end{aligned}$$

Hence, by equation (0.3), $-\delta \leq \sum_{r=1}^{d}(-\mu^r)$, so $\sum_{r=1}^{d} \mu^r \;\leq\; \delta$. This contradicts (0.2), so it follows that $\Lambda \in \mathcal{G}$, and necessity is proved.

References

[1] J. Agler, J. E. McCarthy, and N. J. Young, Operator monotone functions and Löwner functions of several variables, *Ann. of Math.* **176** (2012), 1784–1826. MR 2979860. Zbl 1268.47025. http://dx.doi.org/10.4007/annals.2012.176.3.7.

[2] R. J. Duffin, Infinite programs, in *Linear Inequalities and Related Systems*, *Ann. of Math. Stud.* **38**, Princeton Univ. Press, Princeton, N.J., 1956, pp. 157–170. MR 0087573. Zbl 0072.37603.

(Received: September 3, 2013)

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