Annals of Mathematics **180** (2014), 1–85
http://dx.doi.org/10.4007/annals.2014.180.1.1

On the coherence conjecture of Pappas and Rapoport

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Abstract

We prove the (generalized) coherence conjecture proposed by Pappas and Rapoport. As a corollary, one of their theorems, which describes the geometry of the special fibers of the local models for ramified unitary groups, holds unconditionally. Our proof is based on the study of the geometry (in particular, certain line bundles and ℓ-adic sheaves) of the global Schubert varieties, which are the equal characteristic counterparts of the local models.

Contents

1. **Introduction**

The goal of this paper is to prove the coherence conjecture of Pappas and Rapoport as proposed in [PR08]. The precise formulation of the conjecture is a little bit technical and will be given in Section 2.3. In this introduction, we would like to describe a vague form of this conjecture, to convey the ideas behind it and to outline the proofs.

The coherence conjecture was proposed by Pappas and Rapoport in order to understand the special fibers of local models. Local models were systematically introduced by Rapoport and Zink in [RZ96] (special cases were constructed earlier by Deligne-Pappas [DP94] and independently by de Jong [dJ93]) as a tool to analyze the étale local structure of certain integral models of (PEL-type) Shimura varieties with parahoric level structures over p-adic fields. Unlike the Shimura varieties themselves, which are usually moduli spaces of abelian varieties, local models are defined in terms of linear algebra and therefore are much easier to study. For example, using local models, Görtz (see [Gör01], [Gör03]) proved the flatness of certain PEL-type Shimura varieties associated to unramified unitary groups and symplectic groups. (Some special cases were obtained in earlier works [CN92], [dJ93], [DP94].) On the other hand, a discovery of G. Pappas (cf. [Pap00]) showed that the originally defined integral models in [RZ96] are usually not flat when the groups are ramified. Therefore, nowadays the (local) models defined in [RZ96] are usually called the naive models. In a series of papers ([PR03], [PR05], [PR09]), Pappas and Rapoport investigated the corrected definition of flat local models. The easiest

definition of these local models is by taking the flat closures of the generic fibers in the naive local models. Usually, an integral model defined in this way is not useful since the moduli interpretation is lost and therefore it is very difficult to study the special fiber, etc. (In fact, a considerable part [PR03], [PR05], [PR09] is devoted in an attempt to cutting out the correct closed subschemes inside the naive models by strengthening the original moduli problem of [RZ96].) Indeed, most investigations of local models so far used these strengthened moduli problems in a way or another. (For a survey of most progress in this area, we refer to the recent paper [PRS13].)

However, as observed by Pappas and Rapoport in [PR08], the brute force definition of the local models by taking the flat closure is not totally out of control as one might think. Namely, it is known after Görtz' work that the special fibers of the naive models always embed in the affine flag varieties and that their reduced subschemes are a union of Schubert varieties. Therefore, two questions arise: which Schubert varieties appear in the special fibers (of the flat models) and whether the special fibers are reduced. These questions are reduced to the coherence conjecture (see [PR08], [PR09], at least in the case the group splits over a tamely ramified extension), which characterizes the dimension of the spaces of global sections of certain ample line bundles on certain union of Schubert varieties. Therefore, we will have a fairly good understanding of the local models even if we do not know the moduli problem they represent, provided we can prove the coherence conjecture.

Let us be a little bit more precise. To this goal, we first need to recall the theory of affine flag varieties. (We refer to Section 2.2 for unexplained notation and more details.) Let k be a field and $\mathcal{G}$ be a flat affine group scheme of finite type over $k[[t]]$. Let G be fiber of $\mathcal{G}$ over the generic point $F = k(\!(t)\!) = k[[t]][t^{-1}]$. Then one can define the affine flag variety $\mathcal{F}\ell_{\mathcal{G}} = LG/L^+\mathcal{G}$, which is an ind-scheme, of ind-finite type (cf. [BL94], [Fal03], [PR08] and Section 2.2). When G is an almost simple, simply-connected algebraic group over $k(\!(t)\!)$ and $\mathcal{G}$ is a parahoric group scheme of G, $\mathcal{F}\ell_{\mathcal{G}}$ is ind-projective and coincides with the affine flag varieties arising from the theory of affine Kac-Moody groups as developed in [Kum02], [Mat88] (at least when G splits over a tamely ramified extension of $k(\!(t)\!)$). The jet group $L^+\mathcal{G}$ acts on $\mathcal{F}\ell_{\mathcal{G}}$ by left translations and the orbits are finite dimensional; their closures are called (affine) Schubert varieties. When $\mathcal{G}$ is an Iwahori group scheme of G, Schubert varieties are parametrized by elements in the affine Weyl group W_{aff} of G. (More generally, if G is not simply-connected, they are parametrized by elements in the Iwahori-Weyl group $\widetilde{W}$.) For $w \in \widetilde{W}$, we denote the corresponding Schubert variety by $\mathcal{F}\ell_w$.

Let us come back to local models. Let $(G, K, \{\mu\})$ be a triple, where G is a reductive group over a p-adic field F, with finite residue field k_F, K is a parahoric subgroup of G and $\{\mu\}$ is a geometric conjugacy class of one-parameter

subgroups of G. Let E/F be the reflex field (i.e., the field of definition for $\{\mu\}$), with ring of integers $\mathcal{O}_E$ and residue field k_E. Then for most such triples (at least when μ is minuscule; cf. [PRS13] for a complete list), one can define the so-called naive model $\mathcal{M}^{\mathrm{naive}}_{K,\{\mu\}}$), which is an $\mathcal{O}_E$-scheme, whose generic fiber is the flag variety $X(\mu)$ of parabolic subgroups of G_E of type μ. Inside $\mathcal{M}^{\mathrm{naive}}_{K,\{\mu\}}$, one defines $\mathcal{M}^{\mathrm{loc}}_{K,\{\mu\}}$ as the flat closure of the generic fiber. (For an example of the definitions of such schemes, see Section 8.) In all known cases, one can find a reductive group G' defined over $k(\!(t)\!)$ and a parahoric group scheme $\mathcal{G}$ over $k[\![t]\!]$ such that the special fiber

$$\overline{\mathcal{M}}^{\mathrm{naive}}_{K,\{\mu\}} := \mathcal{M}^{\mathrm{naive}}_{K,\{\mu\}} \otimes k_E$$

embeds into the affine flag variety $\mathcal{F}\ell_{\mathcal{G}} = LG'/L^+\mathcal{G}$ as a closed subscheme, which is, in addition, invariant under the action of $L^+\mathcal{G}$. In particular, the reduced subscheme of $\overline{\mathcal{M}}^{\mathrm{naive}}_{K,\{\mu\}}$ is a union of Schubert varieties inside $\mathcal{F}\ell_{\mathcal{G}}$. Which Schubert variety will appear in $\overline{\mathcal{M}}^{\mathrm{naive}}_{K,\{\mu\}}$ usually can be read from the moduli definition of $\mathcal{M}^{\mathrm{naive}}_{K,\{\mu\}}$. However, the special fiber of $\mathcal{M}^{\mathrm{loc}}_{K,\{\mu\}}$ is more mysterious, and a lot of work has been done in order to understand it. (We refer to [PRS13] (in particular, its Section 4) and references therein for a detailed survey of the current progress.)

Here we review two strategies to study $\mathcal{M}^{\mathrm{loc}}_{K,\{\mu\}}$. For simplicity, we assume that the derived group of G is simply-connected and K is an Iwahori subgroup of G at this moment. In this case, $\mathcal{G}$ will be an Iwahori group scheme of G'. One can attach to $\{\mu\}$ a subset $\mathrm{Adm}(\mu)$ in the Iwahori-Weyl group $\widetilde{W}$, usually called the μ-admissible set; cf. [Rap05] and Section 2.1 for the definitions. In all known cases, it is not hard to see that the Schubert varieties $\mathcal{F}\ell_w$ for $w \in \mathrm{Adm}(\mu)$ indeed appear in $\overline{\mathcal{M}}^{\mathrm{loc}}_{K,\{\mu\}}$; i.e.,

$$\mathcal{A}(\mu) := \bigcup_{w \in \mathrm{Adm}(\mu)} \mathcal{F}\ell_w \subset \overline{\mathcal{M}}^{\mathrm{loc}}_{K,\{\mu\}}.$$

Now, the first strategy to determine the (underlying reduced closed subscheme of) the special fiber $\overline{\mathcal{M}}^{\mathrm{loc}}_{K,\{\mu\}}$ goes as follows. Write down a moduli functor $\mathcal{M}'_{K,\{\mu\}}$ that is a closed subscheme of $\mathcal{M}^{\mathrm{naive}}_{K,\{\mu\}}$ such that

$$\mathcal{M}'_{K,\{\mu\}} \otimes E = \mathcal{M}^{\mathrm{naive}}_{K,\{\mu\}} \otimes E, \qquad \overline{\mathcal{M}}'_{K,\{\mu\}}(\bar{k}) = \mathcal{A}(\mu)(\bar{k}),$$

where $\bar{k}$ is an algebraic closure of k_E. Clearly, this will imply that the reduced subscheme

$$(\overline{\mathcal{M}}^{\mathrm{loc}}_{K,\{\mu\}})_{\mathrm{red}} = \mathcal{A}(\mu). \tag{1.0.1}$$

In fact, much of the previous works about $\mathcal{M}^{\mathrm{loc}}_{K,\{\mu\}}$ followed this strategy. However, let us mention that (so far) the definition of $\mathcal{M}'_{K,\{\mu\}}$ itself is not group

theoretical (i.e., it depends on choosing some representations of the group G). In particular, when G is ramified, its definition can be complicated. In addition, except a few cases, it is not known whether $\mathcal{M}'_{K,\{\mu\}} = \mathcal{M}^{\mathrm{loc}}_{K,\{\mu\}}$ in general.

There is another strategy to determine $\overline{\mathcal{M}}^{\mathrm{loc}}_{K,\{\mu\}}$, as proposed in [PR08]. Namely, let us choose an ample line bundle $\mathcal{L}$ over $\mathcal{M}^{\mathrm{naive}}_{K,\{\mu\}}$. Then since by definition $\mathcal{M}^{\mathrm{loc}}_{K,\{\mu\}}$ is flat over $\mathcal{O}_E$ with generic fiber $X(\mu)$, for $n \gg 0$,

$$\dim_{k_E} \Gamma(\mathcal{A}(\mu), \mathcal{L}^n) \leq \dim_{k_E} \Gamma(\overline{\mathcal{M}}^{\mathrm{loc}}_{K,\{\mu\}}, \mathcal{L}^n) = \dim_E \Gamma(X(\mu), \mathcal{L}^n).$$

The general expectation (which has been verified in all known cases) that

$$\overline{\mathcal{M}}^{\mathrm{loc}}_{K,\{\mu\}} = \mathcal{A}(\mu)$$

led Pappas and Rapoport to conjecture the following equivalent statement:

$$\dim_{k_E} \Gamma(\mathcal{A}(\mu), \mathcal{L}^n) = \dim_E \Gamma(X(\mu), \mathcal{L}^n).$$

Apparently, this conjecture would not be very useful unless one could say something about the line bundle $\mathcal{L}$. In fact, the statement of the conjecture in [PR08] is different and more precise. Namely, in loc. cit., they constructed some line bundle $\mathcal{L}_1$ on the affine flag variety $\mathcal{F}\ell_{\mathcal{G}}$ and some line bundle $\mathcal{L}_2$ on $X(\mu)$, both of which are explicit and are given purely in terms of group theory. (See Section 2.3 for the precise construction.) Then they conjecture

The Coherence Conjecture. *For $n \gg 0$,*

$$\dim_{\bar{k}} \Gamma(\mathcal{A}(\mu), \mathcal{L}_1^n) = \dim_E \Gamma(X(\mu), \mathcal{L}_2^n).$$

In addition, in loc. cit., for certain groups, they constructed natural ample line bundles $\mathcal{L}$ on the corresponding local models, whose restrictions give $\mathcal{L}_1$ and $\mathcal{L}_2$.

What makes the coherence conjecture useful? First of all, the conjecture is group theoretic; i.e., the statement is uniform for all groups. The nongroup theoretic part then is absorbed into the construction of natural line bundles on local models and the identification of their restrictions with the group theoretically constructed line bundles. This is a much simpler problem. An example is illustrated in Section 8. More importantly, the right-hand side in the coherence conjecture is defined over $\mathcal{O}_E$ and therefore, it is equivalent to prove that

$$\dim_{\bar{k}} \Gamma(\mathcal{A}(\mu), \mathcal{L}_1^n) = \dim_{\bar{k}} \Gamma(X(\mu), \mathcal{L}_2^n).$$

Observe that in the above formulation, everything is over the field k rather than over a mixed characteristic ring.

How can we prove this conjecture? Suppose that we can find a scheme $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}$ (the reason we choose this notation will be clear soon), which is flat over $\bar{k}[t]$, together with a line bundle $\mathcal{L}$ such that its fiber over $0 \in \mathbb{A}^1$ is $(\mathcal{A}(\mu), \mathcal{L}_1)$ and its fiber over $y \neq 0$ is $(X(\mu), \mathcal{L}_2)$. Then the coherence conjecture will

follow. In fact, such $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}$ does exist and can be constructed purely group theoretically. They are the (generalized) equal characteristic counterparts of local models, which we will call the global Schubert varieties. Let us briefly indicate the construction of $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}$ here. (The construction of the line bundle $\mathcal{L}$, which we ignore here, is also purely group theoretical, see Section 4.) For simplicity, let us assume that G' is split over k. (The nonsplit case will also be considered in the paper.) Let B be a Borel subgroup of G'. Then in [Gai01], Gaitsgory (following ideas of Kottwitz and Beilinson) constructed a family of ind-schemes $\mathrm{Gr}_{\mathcal{G}}$ over $\mathbb{A}^1$, which is a deformation from the affine Grassmannian $\mathrm{Gr}_{G'}$ of G' to the affine flag variety $\mathcal{F}\ell_{G'}$ of G'. By its construction,

$$\mathrm{Gr}_{\mathcal{G}}|_{\mathbb{G}_m} \cong (\mathrm{Gr}_{G'} \times G'/B) \times \mathbb{G}_m,^1 \quad \mathrm{Gr}_{\mathcal{G}}|_0 \cong \mathcal{F}\ell_{G'},$$

where $\mathrm{Gr}_{\mathcal{G}}|_0$ denotes the scheme theoretic fiber of $\mathrm{Gr}_{\mathcal{G}}$ over $0 \in \mathbb{A}^1$. When μ is minuscule, the Schubert variety $\overline{\mathrm{Gr}}_{\mu}$ corresponds to μ in $\mathrm{Gr}_{G'}$ is in fact isomorphic to $X(\mu)$. In addition, we can "spread it out" over $\mathbb{G}_m$ as $(\overline{\mathrm{Gr}}_{\mu} \times *) \times \mathbb{G}_m$ to get a closed subscheme of $\mathrm{Gr}_{\mathcal{G}}|_{\mathbb{G}_m}$, where $*$ is the base point in G'/B. Now define $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}$ as the closure of $(\overline{\mathrm{Gr}}_{\mu} \times *) \times \mathbb{G}_m$ inside $\mathrm{Gr}_{\mathcal{G}}$. By definition, its fiber over $y \neq 0$ is isomorphic to $X(\mu)$. On the other hand, it is not hard to see that $\mathcal{A}(\mu) \subset \overline{\mathrm{Gr}}_{\mathcal{G},\mu}|_0$ (cf. Lemma 3.7). Therefore, the coherence conjecture will follow if we can show that $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}|_0 = \mathcal{A}(\mu)$ (and if we can construct the corresponding line bundle).

At the first sight, it seems the idea is circular. However, it is not the case. The reason, as we mentioned before, is that $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}$ now is a scheme over k and we have many more tools to attack the problem. Observe that to prove that $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}|_0 = \mathcal{A}(\mu)$, we need to show that

(1) $(\overline{\mathrm{Gr}}_{\mathcal{G},\mu}|_0)_{\mathrm{red}} = \mathcal{A}(\mu)$ (Theorem 3.8);
(2) $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}|_0$ is reduced (Theorem 3.9).

Part (1) can be achieved by the calculation of the nearby cycle $\mathcal{Z}_{\mu} = \Psi_{\overline{\mathrm{Gr}}_{\mathcal{G},\mu}}(\mathbb{Q}_\ell)$ of the family $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}$ (see Lemma 7.1). Usually, such a calculation is a hard problem. The miracle here is that if $\mathcal{Z}_{\mu}$ is regarded as an object in the category of Iwahori equivariant perverse sheaves on $\mathcal{F}\ell_{G'}$, it has very nice properties. Namely, by the main result of [Gai01] (in the case when G' is split), $\mathcal{Z}_{\mu}$ is a central sheaf; i.e., for any other Iwahori equivariant perverse sheaf $\mathcal{F}$ on $\mathcal{F}\ell_{G'}$, the convolution product $\mathcal{Z}_{\mu} \star \mathcal{F}$ (see (7.2.3)–(7.2.4) for the definition) is perverse and

$$\mathcal{Z}_{\mu} \star \mathcal{F} \cong \mathcal{F} \star \mathcal{Z}_{\mu}.$$

Then by a result of Arhkipov-Bezrukavnikov [AB09, Th. 4], the above properties put a strong restriction of the support of $\mathcal{Z}_{\mu}$, which will imply Part (1). We

[1] In the main body of this paper, we will work with a different family so that this extra G'/B factor does not appear.

shall mention that although we assume here that G' is split, the same strategy can be applied to the nonsplit groups. This is done in Section 7, where we generalize the results of [Gai01] and [AB09] to ramified groups as well. Our arguments are simpler than the originally arguments in [Gai01], [AB09] and will provide the following technical advantage. As we mentioned above, $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}$ should be regarded as the equal characteristic counterparts of local models. Therefore, it is natural (and indeed important) to determine the nearby cycles $\Psi_{\mathcal{M}^{\mathrm{loc}}_{K,\{\mu\}}}(\mathbb{Q}_\ell)$ for the local models. For example, if one could prove that these sheaves are also central (the *Kottwitz conjecture*[2]), then one could conclude (1.0.1) directly. It turns out the arguments in Section 7 have a direct generalization to the mixed characteristic situation, and in joint work with Pappas [PZ13], we use this to show the Kottwitz conjecture. (Some previous cases are proved by Haines and Ngô [HN02].)

Now we turn to Part (2), which is more difficult. The idea is that we can assume $\operatorname{char} k > 0$ and use the powerful technique of Frobenius splitting (cf. [MR85], [BK05]). To prove that $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}|_0$ is reduced, it is enough to prove that it is Frobenius split. To achieve this goal, we embed $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}$ into a larger scheme $\overline{\mathrm{Gr}}^{\mathrm{BD}}_{\mathcal{G},\mu,\lambda}$ over $\mathbb{A}^1$, which is a closed subscheme of a version of the Beilinson-Drinfeld Grassmannian. The scheme $\overline{\mathrm{Gr}}^{\mathrm{BD}}_{\mathcal{G},\mu,\lambda}$ is normal, and its fiber over 0 is reduced. Then to prove that

$$\overline{\mathrm{Gr}}_{\mathcal{G},\mu}|_0 = \overline{\mathrm{Gr}}_{\mathcal{G},\mu} \cap \overline{\mathrm{Gr}}^{\mathrm{BD}}_{\mathcal{G},\mu,\lambda}|_0$$

is Frobenius split, it is enough to construct a Frobenius splitting of $\overline{\mathrm{Gr}}^{\mathrm{BD}}_{\mathcal{G},\mu,\lambda}$, compatible with $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}$ and $\overline{\mathrm{Gr}}^{\mathrm{BD}}_{\mathcal{G},\mu,\lambda}|_0$. Since $\overline{\mathrm{Gr}}^{\mathrm{BD}}_{\mathcal{G},\mu,\lambda}$ is normal, it is enough to prove this for some nice open subscheme $U \subset \overline{\mathrm{Gr}}^{\mathrm{BD}}_{\mathcal{G},\mu,\lambda}$ such that $\overline{\mathrm{Gr}}^{\mathrm{BD}}_{\mathcal{G},\mu,\lambda} - U$ has codimension two. In particular, the open subscheme U will not intersect with $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}|_0$, which is our primary interest. Section 6 is devoted to realizing this idea.

Now let us describe the organization of the paper and some other results proved in it.

In Section 2, we review the coherence conjecture of Pappas and Rapoport. In Section 2.1, we review the basic theory of reductive groups over local fields and introduce various notation used in the rest of the paper. In Section 2.2, we rapidly recall the main results of [PR08] (and [Fal03]) concerning loop groups and the geometry of their flag varieties. In Section 2.3, we state the main theorems (Theorems 1 and 2) of our paper, which give a modified version of

[2]In fact, the Kottwitz conjecture is weaker than this statement, and its significance lies in its use in the Langlands-Kottwitz method for calculating the Zeta functions of Shimura varieties.

original coherence conjecture of Pappas and Rapoport. (See Remark 2.1 for the reason of the modification.)

In Section 3, we introduce the main geometric object we are going to study in the paper, namely, the global Schubert varieties. They are varieties projective over the affine line $\mathbb{A}^1$, which are the counterparts of local models in the equal characteristic situation. In Section 3.1, we define the global affine Grassmannian over a curve for general (nonconstant) group schemes. After the work of [PR08], [PR10], [Hei10], this construction is now standard. In Section 3.2, we construct a special Bruhat-Tits group scheme over $C = \mathbb{A}^1$, i.e., a group scheme that is only ramified at the origin. Let us remark that similar constructions are also considered in [HNY13], [Ric13]. In Section 3.3, we apply the construction of the global affine Grassmannian to the group scheme we consider in the paper. We introduce the global Schubert variety $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}$, which is associated to a geometric conjugacy class of one-parameter subgroup $\{\mu\}$ of G, over a ramified cover $\tilde{C}$ of C. We then state another main theorem (Theorem 3), which asserts that the special fiber of $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}$ is $\mathcal{A}^Y(\mu)$, and first show that the variety $\mathcal{A}^Y(\mu)$ is contained in this special fiber (Lemma 3.7). In Section 4, we explain that our assertion about the special fiber of $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}$ is equivalent to the coherence conjecture. The key ingredient is a certain line bundle on the global affine Grassmannian, namely, the pullback of the determinant line bundle along the closed embedding

$$\mathrm{Gr}_{\mathcal{G}} \to \mathrm{Gr}_{\mathrm{Lie}(\mathcal{G})}.$$

We calculate its central charges at each fiber (which turn out to be twice of the dual Coxeter number) and find the remarkable fact that the central charge of line bundles on the global affine Grassmannians are constant along the curve (Proposition 4.1).

In Section 5, we make some preparations towards the proof of our main theorem. We study two basic geometrical structures of $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}$: (i) in Section 5.2, we will construct certain affine charts of $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}$, which turn out to be isomorphic to affine spaces over $\tilde{C}$; and (ii) in Section 5.3, we will construct a $\mathbb{G}_m$-action on $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}$, so that the map $\overline{\mathrm{Gr}}_{\mathcal{G},\mu} \to \tilde{C}$ is $\mathbb{G}_m$-equivariant, where $\mathbb{G}_m$ acts on $\tilde{C} = \mathbb{A}^1$ by natural dilatation. To establish (i), we will need to first construct the global root subgroups of $\mathcal{LG}$ as in Section 5.1.

The next two sections are then devoted to the proof of the theorem concerning the special fiber of $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}$, as has been already outlined above. The first part of the proof, presented in Section 6, concerns the scheme theoretic structure of the special fiber. Namely, we prove that it is reduced. This is achieved by the technique of Frobenius splitting. As a warm up, we prove in Section 6.1 that Theorem 1 is a special case of Theorem 2, which should be well known to experts. Then we introduce the Beilinson-Drinfeld Grassmannian and the

convolution Grassmannian and reduce Theorem 3.9 to Theorem 6.8. In Section 6.3, we prove a special case of Theorem 6.8 by studying the affine flag variety associated to a special parahoric group scheme. Recall that a result of Beilinson-Drinfeld (cf. [BD, 4.6]) asserts that the Schubert varieties in the affine Grassmannian are Gorenstein. In Section 6.3 we examine to what extend this result holds for ramified groups (i.e., reductive groups split over a ramified extension). It turns out this result extends to all affine flag varieties associated to special parahorics except in the case the special parahoric is a parahoric of the ramified odd unitary group SU_{2n+1}, whose special fiber has reductive quotient SO_{2n+1} (Theorem 6.11). In this exceptional case, no Schubert variety of positive dimension in the corresponding affine flag variety is Gorenstein (Remark 6.1). In Section 7, we give the second part of the proof, which asserts that topologically, the special fiber of $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}$ coincides with $\mathcal{A}^Y(\mu)$. This is achieved by the description of the support of the nearby cycle (for the intersection cohomology sheaf) of this family. In the case when the group is split, this follows the earlier works of [Gai01] and [AB09]. In Sections 7.2 and Section 7.3, we generalize their results to ramified groups, with certain simplifications of the original arguments.

The paper has two appendices. The first one, Section 8, calculates the line bundles on the local models for the ramified unitary groups. The study of these local models was the main motivation for Pappas and Rapoport to make the coherence conjecture. Since their original conjecture is not as stated in our main theorem, we explain in this appendix why our main theorem is correct for the applications to local models. The second appendix (Section 9) collects and strengthens some results, which already exist in literature, in a form needed in the main body of the paper.

Notation. Let k be a field, and fix $\bar{k}$ to be an algebraic closure of k. We will denote by $k^s \subset \bar{k}$ the separable closure of k in $\bar{k}$.

If X be a Y-scheme and $V \to Y$ is a morphism, the base change $X \times_Y V$ is denoted by X_V or $X|_V$. If $V = \mathrm{Spec}R$, it is sometimes also denoted by X_R. If $V = x = \mathrm{Spec}k$ is a point, then it is sometimes also denoted by $(X)_x$.

For a vector bundle $\mathcal{V}$ on a scheme V, we denote by $\det(\mathcal{V})$ the top exterior power of $\mathcal{V}$, which is a line bundle.

If A is an affine algebraic group (not necessarily a torus) over a field k, we denote by $\mathbb{X}^\bullet(A)$ (resp. $\mathbb{X}_\bullet(A)$) its character group (resp. cocharacter group) over k^s. The Galois group $\Gamma = \mathrm{Gal}(k^s/k)$ acts on $\mathbb{X}^\bullet(A)$ (resp. $\mathbb{X}_\bullet(A)$) and the invariants (resp. coinvariants) are denoted by $\mathbb{X}^\bullet(A)^\Gamma$ (resp. $\mathbb{X}^\bullet(A)_\Gamma, \mathbb{X}_\bullet(A)^\Gamma, \mathbb{X}_\bullet(A)_\Gamma$).

If $\mathcal{G}$ is a flat group scheme over V, the trivial $\mathcal{G}$-torsor (i.e., $\mathcal{G}$ itself regarded as a $\mathcal{G}$-torsor by right multiplication) is denoted by $\mathcal{E}^0$. For a $\mathcal{G}$-torsor $\mathcal{E}$, we

use ad $\mathcal{E}$ to denote the associated adjoint bundle. If $\mathcal{P}$ is a $\mathcal{G}$ torsor and X is a scheme over V with an action of $\mathcal{G}$, we denote the twisted product by $\mathcal{P} \times^{\mathcal{G}} X$, which is the quotient of $\mathcal{P} \times_V X$ by the diagonal action of $\mathcal{G}$.

If G is a reductive group over a field, we denote by G_{der} its derived group, G_{sc} the simply-connected cover of G_{der} and G_{ad} its the adjoint group.

Acknowledgements. The author is grateful to D. Gaitsgory, G. Pappas, M. Rapoport, J.-K. Yu for useful discussions, and G. Pappas and M. Rapoport for reading an early draft of this paper. He is also extremely grateful to the referee for pointing out an enormous number of mistakes and typos in an early draft. The work of the author is supported by the NSF grant DMS-1001280/1313894.

2. **Review of the local picture, formulation of the conjecture**

2.1. *Group theoretical data.* Let k be an algebraically closed field. Let $\mathcal{O} = k[[t]]$ and $F = k(\!(t)\!)$. Let $\Gamma = \mathrm{Gal}(F^s/F)$ be the inertial (Galois) group, where F^s is the separable closure of F. Let us emphasize that we choose a uniformizer t. Let G be a connected reductive group over F. In this paper, unless otherwise stated, G is assumed to split over a *tamely* ramified extension $\tilde{F}/F$. It is called a ramified group if it is nonsplit over F.

Let S be a maximal F-split torus of G. Let $T = \mathcal{Z}_G(S)$ be the centralizer of S in G, which is a maximal torus of G since G is quasi-split over F. (As F is a field of cohomological dimension one, this follows from [Ste65, Th. 1.9].) Let us choose a rational Borel subgroup $B \supset T$. Let H be a split Chevalley group over $\mathbb{Z}$ such that $H \otimes F^s \cong G \otimes F^s$. We need to choose this isomorphism carefully. Let us fix a pinning (H, B_H, T_H, X) of H over $\mathbb{Z}$. Let us recall that this means that B_H is a Borel subgroup of H, T_H is a split maximal torus contained in B_H, and $X = \Sigma_{\tilde{a} \in \Delta} X_{\tilde{a}} \in \mathrm{Lie}\, B$, where Δ is the corresponding set of simple roots, $\tilde{U}_{\tilde{a}}$ is the root subgroup corresponding to $\tilde{a}$ and $X_{\tilde{a}}$ is a generator in the rank one free $\mathbb{Z}$-module $\mathrm{Lie}\, \tilde{U}_{\tilde{a}}$. Let us choose an isomorphism $(G, B, T) \otimes_F \tilde{F} \cong (H, B_H, T_H) \otimes_{\mathbb{Z}} \tilde{F}$, where $\tilde{F}/F$ is a cyclic extension such that $G \otimes \tilde{F}$ splits. This induces an isomorphism of the root data $(\mathbb{X}^\bullet(T_H), \Delta, \mathbb{X}_\bullet(T_H), \Delta^\vee) \cong (\mathbb{X}^\bullet(T), \Delta, \mathbb{X}_\bullet(T), \Delta^\vee)$. Let Ξ be the group of pinned automorphisms of (H, B_H, T_H, X). The natural map from Ξ to the group of the automorphisms of the root datum $(\mathbb{X}^\bullet(T_H), \Delta, \mathbb{X}_\bullet(T_H), \Delta^\vee)$ is an isomorphism ([Con, Prop. 7.1.6]).

Now the action of $\Gamma = \mathrm{Gal}(\tilde{F}/F)$ on $G \otimes_F \tilde{F}$ induces a homomorphism $\psi : \Gamma \to \Xi$. Then we can always choose an isomorphism

$$(G, B, T) \otimes_F \tilde{F} \cong (H, B_H, T_H) \otimes_{\mathbb{Z}} \tilde{F} \tag{2.1.1}$$

such that the action of $\gamma \in \Gamma$ on the left-hand side corresponds to $\psi(\gamma) \otimes \gamma$. In the rest of the paper, we fix such an isomorphism. This determines a point

v_0 in $A(G,S)$, the apartment associated to (G,S) ([BT72]).[3] This is a special point of $A(G,S)$, which in turn gives a parahoric group scheme $\mathcal{G}_{v_0}$ over $\mathcal{O}$, namely

$$\mathcal{G}_{v_0} := ((\mathrm{Res}_{\mathcal{O}_{\tilde{F}}/\mathcal{O}}(H \otimes \mathcal{O}_{\tilde{F}}))^{\Gamma})^0. \tag{2.1.2}$$

Let us explain the notation. Here Res stands for the Weil restriction, so that $\mathrm{Res}_{\mathcal{O}_{\tilde{F}}/\mathcal{O}}(H \otimes \mathcal{O}_{\tilde{F}})$ is a smooth group scheme over $\mathcal{O}$ (cf. [Edi92, 2.2]), with an action of Γ. The notation $(-)^{\Gamma}$ stands for taking the Γ-fixed point subscheme. Under our tameness assumption, $\tilde{\mathcal{G}}_{v_0} := (\mathrm{Res}_{\mathcal{O}_{\tilde{F}}/\mathcal{O}}(H \otimes \mathcal{O}_{\tilde{F}}))^{\Gamma}$ is smooth by [Edi92, 3.4]. Finally, $(-)^0$ stands for taking the neutral connected component. Therefore, $\mathcal{G}_{v_0}$ and $\tilde{\mathcal{G}}_{v_0}$ have the same generic fiber and the special fiber of $\mathcal{G}_{v_0}$ is the neutral connected component of the special fiber of $\tilde{\mathcal{G}}_{v_0}$.

Recall that $A(G,S)$ is an affine space under $\mathbb{X}_\bullet(S)_{\mathbb{R}}$. For every facet $\sigma \subset A(G,S)$, let $\mathcal{G}_\sigma$ be the parahoric group scheme over $\mathcal{O}$. (In particular, the special fiber of $\mathcal{G}_\sigma$ is connected.) Let us choose a special vertex $v \in A(G,S)$ (e.g., v_0), and identify $A(G,S)$ with $\mathbb{X}_\bullet(S)_{\mathbb{R}}$ via this choice. Let $\mathbf{a}$ be the unique alcove in $A(G,S)$, whose closure contains the point v and is contained in the finite Weyl chamber determined by our chosen Borel. This determines a set of simple affine roots $\alpha_i, i \in \mathbf{S}$, where $\mathbf{S}$ is the set of vertices of the affine Dynkin diagram associated to G.

Let $\widetilde{W}$ be the Iwahori-Weyl group of G (cf. [HR]) that acts on $A(G,S)$. This is defined to be $\mathcal{N}_G(S)(F)/\ker\kappa$, where $\mathcal{N}_G(S)$ is the normalizer of S in G, and

$$\kappa : T(F) \to \mathbb{X}_\bullet(T)_\Gamma \tag{2.1.3}$$

is the Kottwitz homomorphism (cf. [Kot97, §7]). One has the following exact sequence:

$$1 \to \mathbb{X}_\bullet(T)_\Gamma \to \widetilde{W} \to W_0 \to 1, \tag{2.1.4}$$

where W_0 is the relative Weyl group of G over F. In what follows, we use t_λ to denote the translation element in $\widetilde{W}$ given by $\lambda \in \mathbb{X}_\bullet(T)_\Gamma$ from the above map (2.1.4).[4] But occasionally, we also use λ itself to denote this translation element if no confusion is likely to arise. The pinned isomorphism (2.1.1) determines a set of positive roots $\Phi^+ = \Phi(G,S)^+$ for G. There is a natural map $\mathbb{X}_\bullet(T)_\Gamma \to \mathbb{X}_\bullet(S)_{\mathbb{R}}$. We define

$$\mathbb{X}_\bullet(T)^+_\Gamma = \{\lambda \mid (\lambda, a) \geq 0 \text{ for } a \in \Phi^+\}. \tag{2.1.5}$$

[3]More precisely, v_0 is a point in the apartment associated to the adjoint group $(G_{\mathrm{ad}}, S_{\mathrm{ad}})$. But since in the paper we only use the combinatorial structures of $A(G,S)$, we will not distinguish it from the one associated to the adjoint group.

[4]Note that under the sign convention of the Kottwitz homomorphism in [Kot97], t_λ acts on $A(G,S)$ by $v \mapsto v - \lambda$.

Our choice of the special vertex v of $A(G,S)$ gives a splitting of the exact sequence, and therefore we can write $w = t_\lambda w_f$ for $\lambda \in \mathbb{X}_\bullet(T)_\Gamma$ and $w_f \in W_0$.

Let W_{aff} be the affine Weyl group of G, i.e., the Iwahori-Weyl group of G_{sc}, that is a Coxeter group. One has

$$1 \to \mathbb{X}_\bullet(T_{\mathrm{sc}})_\Gamma \to \ W_{\mathrm{aff}} \to W_0 \to 1,$$

where T_{sc} is the inverse image of T in G_{sc}. One can write $\widetilde{W} = W_{\mathrm{aff}} \rtimes \Omega$, where Ω is the subgroup of $\widetilde{W}$ that fixes the chosen alcove $\mathbf{a}$. This gives $\widetilde{W}$ a quasi-Coxeter group structure. Hence it makes sense to talk about the length of an element $w \in \widetilde{W}$ and there is a Bruhat order on $\widetilde{W}$. Namely, if we write $w_1 = w_1'\tau_1, w_2 = w_2'\tau_2$ with $w_i' \in W_{\mathrm{aff}}, \tau_i \in \Omega$, then $\ell(w_i) = \ell(w_i')$ and $w_1 \leq w_2$ if and only if $\tau_1 = \tau_2$ and $w_1' \leq w_2'$. A lot of the combinatorics of the Iwahori-Weyl group arises from the study of the restriction of the length function and the Bruhat order to $\mathbb{X}_\bullet(T)_\Gamma \subset \widetilde{W}$. Some of them will be reviewed in Section 9.1.

Now let us recall the definition of the *admissible set* in the Iwahori-Weyl group. Let $\bar{W}$ be the absolute Weyl group of G, i.e., the Weyl group for (H, T_H). Suppose that $\mu : (\mathbb{G}_m)_{\tilde{F}} \to G \otimes \tilde{F}$ gives a geometric conjugacy class of one-parameter subgroups. It determines a $\bar{W}$-orbit in $\mathbb{X}_\bullet(T)$. One can associate $\{\mu\}$ a W_0-orbit Λ in $\mathbb{X}_\bullet(T)_\Gamma$ as follows. Choose a Borel subgroup of G containing T and defined over F. This gives a unique element in this $\bar{W}$-orbit, still denoted by μ, which is dominant with respect to this Borel subgroup. Let $\bar{\mu}$ be its image in $\mathbb{X}_\bullet(T)_\Gamma$, and let $\Lambda = W_0\bar{\mu}$. It turns out Λ does not depend on the choice of the rational Borel subgroup of G since any two such F-rational Borels that contain T will be conjugate to each other by an element in W_0. For $\mu \in \mathbb{X}_\bullet(T)$, define the admissible set

$$\mathrm{Adm}(\mu) = \{w \in \widetilde{W} \mid w \leq t_\lambda \text{ for some } \lambda \in \Lambda\}. \tag{2.1.6}$$

Under the map $\mathbb{X}_\bullet(T)_\Gamma \to \widetilde{W} \to \widetilde{W}/W_{\mathrm{aff}} \cong \Omega$, the set Λ maps to a single element (cf. [Rap05, Lemma 3.1]), denoted by τ_μ. Define

$$\mathrm{Adm}(\mu)^\circ = \tau_\mu^{-1}\mathrm{Adm}(\mu).$$

For $Y \subset \mathbf{S}$ any subset, let W^Y denote the subgroup of W_{aff} generated by $\{r_i, i \in \mathbf{S} - Y\}$, where r_i is the simple reflection corresponding to i. Then set

$$\mathrm{Adm}^Y(\mu) = W^Y\mathrm{Adm}(\mu)W^Y \subset \widetilde{W}$$

and

$$\mathrm{Adm}^Y(\mu)^\circ = \tau_\mu^{-1}\mathrm{Adm}^Y(\mu).$$

Note that $\mathrm{Adm}^Y(\mu)^\circ \subset W_{\mathrm{aff}}$, and this subset only depends on the image of μ under $\mathbb{X}_\bullet(T) \to \mathbb{X}_\bullet(T_{\mathrm{ad}})$, where T_{ad} is the image of T in G_{ad}.

2.2. *Loop groups and their flag varieties.* Let $\sigma \subset A(G,S)$ be a facet. Let

$$\mathcal{F}\ell_\sigma = LG/L^+\mathcal{G}_\sigma$$

be the (partial) flag variety of LG. Let us recall that LG is the loop group of G, which represents the functor which associates to every k-algebra R the group $G(R(\!(t)\!))$, $L^+\mathcal{G}_\sigma$ is the jet group of $\mathcal{G}_\sigma$, which represents the functor that associates to every k-algebra R the group $\mathcal{G}_\sigma(R[[t]])$, and $\mathcal{F}\ell_\sigma = LG/L^+\mathcal{G}_\sigma$ is the fpqc quotient. Let us also recall that LG is represented by an ind-affine ind-scheme, $L^+\mathcal{G}_\sigma$ is represented by an affine scheme, which is a closed subscheme of LG, and $\mathcal{F}\ell_\sigma$ is represented by an ind-scheme, ind-projective over k. Denote by $I = L^+\mathcal{G}_{\mathbf{a}}$ the Iwahori subgroup of LG, and denote $\mathcal{F}\ell_{\mathbf{a}}$ by $\mathcal{F}\ell$, which we call the affine flag variety of G. If G splits over F, so that $G = H \otimes F$ and (2.1.1) is the natural isomorphism, then the special vertex v_0 is hyperspecial, and corresponds to the parahoric group scheme $H \otimes k[[t]]$. Then we denote $\mathcal{F}\ell_{v_0}$ by Gr_H and call it the affine Grassmannian of H. Let $Y \subset \mathbf{S}$ be a subset, and let $\sigma_Y \subset A(G,S)$ be the facet such that $\alpha_i(\sigma_Y) = 0$ for $i \in \mathbf{S} - Y$. Observe that $\sigma_{\mathbf{S}} = \mathbf{a}$ is the chosen alcove. We also denote $\mathcal{F}\ell_{\sigma_Y}$ by $\mathcal{F}\ell^Y$ for simplicity.

Let us recall that the I-orbits of $\mathcal{F}\ell$ are parametrized by $\widetilde{W}$. In general, the $L^+\mathcal{G}_{\sigma_Y}$-orbits of $\mathcal{F}\ell^{Y'}$ are parametrized by $W^Y \setminus \widetilde{W}/W^{Y'}$, where W^Y is the Weyl group of $\mathcal{G}_{\sigma_Y} \otimes k$. For $w \in \widetilde{W}$, let ${}^Y\mathcal{F}\ell^{Y'}_w \subset \mathcal{F}\ell^{Y'}$ denote the corresponding Schubert variety, i.e., the closure of the $L^+\mathcal{G}_{\sigma_Y}$-orbit through w. If $Y = Y'$, then we simply denote it by $\mathcal{F}\ell^Y_w$. If G is split and $\mathcal{G} = H \otimes k[[t]]$ is a hyperspecial model, recall that $L^+\mathcal{G}$-orbits of Gr_H are parametrized by $\bar{W} \setminus \widetilde{W}/\bar{W} \cong \mathbb{X}_\bullet(T)^+$, the set of dominant coweights of G. For $\mu \in \mathbb{X}_\bullet(T)^+$, let $\overline{\mathrm{Gr}}_\mu$ be the corresponding Schubert variety in Gr_H.

Let us recall the following result of [Fal03], [PR08].

THEOREM 2.1. *Let $p = \mathrm{char}\, k$. Assume that $p \nmid |\pi_1(G_{\mathrm{der}})|$, where G_{der} is the derived group of G. Then the Schubert variety $\mathcal{F}\ell^Y_w$ is normal, has rational singularities, and is Frobenius-split if $p > 0$.*

For $\mu \in \mathbb{X}_\bullet(T)$, let

$$\mathcal{A}^Y(\mu)^\circ = \bigcup_{w \in \mathrm{Adm}^Y(\mu)^\circ} {}^{Y^\circ}\mathcal{F}\ell^Y_{\mathrm{sc},w}, \tag{2.2.1}$$

where $\sigma_{Y^\circ} = \tau_\mu^{-1}(\sigma_Y)$ and where ${}^{Y^\circ}\mathcal{F}\ell^Y_{\mathrm{sc},w}$ is the union of Schubert varieties (more precisely, the closure of $L^+\mathcal{G}_{\sigma_{Y^\circ}}$-orbits) in the partial affine flag variety $\mathcal{F}\ell^Y_{\mathrm{sc}} = LG_{\mathrm{sc}}/L^+\mathcal{G}_{\sigma_Y}$. Then $\mathcal{A}^Y(\mu)^\circ$ is a reducible subvariety of $\mathcal{F}\ell^Y_{\mathrm{sc}}$, with irreducible components

$${}^{Y^\circ}\mathcal{F}\ell^Y_{\mathrm{sc},\tau_\mu^{-1}t_\lambda}, \quad \lambda \in \Lambda \subset \mathbb{X}_\bullet(T)_\Gamma \subset \widetilde{W}.$$

Observe that $\mathcal{A}^Y(\mu)^\circ$ only depends on the image of μ under $\mathbb{X}_\bullet(T) \to \mathbb{X}_\bullet(T_{\mathrm{ad}})$.

When $p \nmid |\pi_1(G_{\mathrm{der}})|$, it is also convenient to consider

$$\mathcal{A}^Y(\mu) = \bigcup_{w \in \mathrm{Adm}^Y(\mu)} \mathcal{F}\ell_w^Y. \tag{2.2.2}$$

Choosing a lift $g \in G(F)$ of $\tau_\mu \in \widetilde{W}$ and identifying $\mathcal{F}\ell_{\mathrm{sc}}^Y$ with the reduced part of the neutral connected component of $\mathcal{F}\ell^Y$ (see [PR08, §6]), we can define a map $\mathcal{F}\ell_{\mathrm{sc}}^Y \to \mathcal{F}\ell^Y$, $x \mapsto gx$. Clearly, this map induces an isomorphism $\mathcal{A}^Y(\mu)^\circ \cong \mathcal{A}^Y(\mu)$.

In particular, if $G = H \otimes F$ is split and $\sigma_Y = v_0$ is the hyperspecial vertex corresponding to $H \otimes \mathcal{O}$, then $\mathcal{A}^Y(\mu)^\circ$ is denoted by $\mathrm{Gr}_{\leq \mu}$, so that if $p \nmid |\pi_1(G_{\mathrm{der}})|$, then we have the isomorphism $\mathrm{Gr}_{\leq \mu} \cong \overline{\mathrm{Gr}}_\mu$.

We also need to review the Picard group of $\mathcal{F}\ell$. For simplicity, we assume that G is simple, simply-connected, absolutely simple. In this case $\mathcal{F}\ell$ is connected. For each $i \in \mathbf{S}$, let P_i be the corresponding parahoric subgroup scheme such that $L^+P_i \supset I$ so that $L^+P_i/I \cong \mathbb{P}^1$. This $\mathbb{P}^1$ maps naturally to $\mathcal{F}\ell$ via $L^+P_i \to LG$, and the image will be denoted as $\mathbb{P}_i^1$. Then it is known ([PR08, §10]) that there is a unique line bundle $\mathcal{L}(\varepsilon_i)$ on $\mathcal{F}\ell$, whose restriction to the $\mathbb{P}_i^1$ is $\mathcal{O}_{\mathbb{P}^1}(1)$, and whose restrictions to other $\mathbb{P}_j^1$s with $j \neq i$ is trivial. Then there is an isomorphism

$$\mathrm{Pic}(\mathcal{F}\ell) \cong \bigoplus_{i \in \mathbf{S}} \mathbb{Z}\mathcal{L}(\varepsilon_i).$$

Let us write $\otimes_i \mathcal{L}(\varepsilon_i)^{n_i}$ as $\mathcal{L}(\sum_i n_i \varepsilon_i)$. As explained in loc. cit., the ε_i can be thought of as the fundamental weights of the Kac-Moody group associated to LG, and therefore, $\mathrm{Pic}(\mathcal{F}\ell)$ is identified with the weight lattice of the corresponding Kac-Moody group.

There is also a morphism

$$c : \mathrm{Pic}(\mathcal{F}\ell) \to \mathbb{Z} \tag{2.2.3}$$

called the central charge. If we identify $\mathcal{L} \in \mathrm{Pic}(\mathcal{F}\ell)$ with a weight of the corresponding Kac-Moody group, then $c(\mathcal{L})$ is just the restriction of this weight to the central $\mathbb{G}_m$ in the Kac-Moody group. Explicitly,

$$c(\mathcal{L}(\varepsilon_i)) = a_i^\vee, \tag{2.2.4}$$

where $a_i^\vee (i \in \mathbf{S})$ are defined as in [Kac90, 6.1]. The kernel of c can be described as follows. Let s denote the closed point of $\mathrm{Spec}\mathcal{O}$, and let $(\mathcal{G}_{\mathbf{a}})_s$ denote the special fiber of $\mathcal{G}_{\mathbf{a}}$. Recall that for any k-algebra R, $\mathcal{F}\ell(R)$ is the set of $\mathcal{G}_{\mathbf{a}}$-torsors on $\mathrm{Spec}R[[t]]$ together with a trivialization over $\mathrm{Spec}R((t))$. Therefore, by restriction of the $\mathcal{G}_{\mathbf{a}}$-torsors by $t \mapsto 0$ to $\mathrm{Spec}R \subset \mathrm{Spec}R[[t]]$, we obtain a natural morphism $\mathcal{F}\ell \to \mathbb{B}(\mathcal{G}_{\mathbf{a}})_s$ (here $\mathbb{B}(\mathcal{G}_{\mathbf{a}})_s$ is the classifying stack

of $(\mathcal{G}_{\mathbf{a}})_s)$, which induces $\mathbb{X}^\bullet((\mathcal{G}_{\mathbf{a}})_s) \cong \mathrm{Pic}(\mathbb{B}(\mathcal{G}_{\mathbf{a}})_s) \to \mathrm{Pic}(\mathcal{F}\ell)$. We have the short exact sequence

$$0 \to \mathbb{X}^\bullet((\mathcal{G}_{\mathbf{a}})_s) \to \mathrm{Pic}(\mathcal{F}\ell) \xrightarrow{c} \mathbb{Z} \to 0. \tag{2.2.5}$$

Now let $Y \subset \mathbf{S}$ be a nonempty subset. Observe that if $\mathcal{L}(\sum n_i\varepsilon_i)$ is a line bundle on $\mathcal{F}\ell$, with $n_i = 0$ for $i \in \mathbf{S} - Y$, then this line bundle is the pullback of a unique line bundle along $\mathcal{F}\ell \to \mathcal{F}\ell^Y$, denoted by $\mathcal{L}^Y(\sum_{i\in Y} n_i\varepsilon_i)$. In this way, we have

$$\mathrm{Pic}(\mathcal{F}\ell^Y) \cong \bigoplus_{i\in Y} \mathbb{Z}\mathcal{L}(\varepsilon_i). \tag{2.2.6}$$

The central charge of a line bundle $\mathcal{L}$ on $\mathcal{F}\ell^Y$ is defined to be the central charge of its pullback to $\mathcal{F}\ell$, i.e., the image of $\mathcal{L}$ under $\mathrm{Pic}(\mathcal{F}\ell^Y) \to \mathrm{Pic}(\mathcal{F}\ell) \xrightarrow{c} \mathbb{Z}$. Observe that $\mathcal{L}^Y(\sum_{i\in Y} n_i\varepsilon_i)$ is ample on $\mathcal{F}\ell^Y$ if and only if $n_i > 0$ for all $i \in Y$.

In the case $G = H \otimes F$ is split, the central charge map induces an isomorphism $c : \mathrm{Pic}(\mathrm{Gr}_H) \cong \mathbb{Z}$. We will denote by $\mathcal{L}_b$ the ample generator of the Picard group of Gr_H. Observe that, for $Y = \{i\}$ not special, the ample generator of $\mathrm{Pic}(\mathcal{F}\ell^Y)$ has central charge $a_i^\vee$, which is in general greater than one. That is, the composition $\mathrm{Pic}(\mathcal{F}\ell^Y) \to \mathrm{Pic}(\mathcal{F}\ell) \xrightarrow{c} \mathbb{Z}$ is injective but not surjective in general.

2.3. *The coherence conjecture.* Now we formulate the coherence conjecture of Pappas and Rapoport. However, the original conjecture, as stated in loc. cit., needs to be modified (see Remark 2.1).

Assume that G is simple, absolutely simple, simply-connected and splits over a tamely ramified extension $\tilde{F}/F$. Let $\{\mu\}$ be a geometric conjugacy class of one-parameter subgroups $(\mathbb{G}_m)_{\tilde{F}} \to G_{\mathrm{ad}} \otimes \tilde{F}$. First assume that μ is minuscule. Let $P(\mu)$ be the corresponding maximal parabolic subgroup of H, and let $X(\mu) = H/P(\mu)$ be the corresponding partial flag variety of H. Let $\mathcal{L}(\mu)$ be the ample generator of the Picard group of $X(\mu)$. Then define

$$h_\mu(a) = \dim H^0(X(\mu), \mathcal{L}(\mu)^a).$$

If $\mu = \mu_1 + \cdots + \mu_n$ is a sum of minuscule coweights, let $h_\mu = h_{\mu_1} \cdots h_{\mu_n}$. The following is the main theorem of this paper, which is a modified version of the original coherence conjecture of Pappas and Rapoport in [PR08].

Theorem 1. *Let $\mu = \mu_1 + \cdots + \mu_n$ be a sum of minuscule coweights. Then for any $Y \subset \mathbf{S}$ and ample line bundle $\mathcal{L}$ on $\mathcal{F}\ell^Y$, we have*

$$\dim H^0(\mathcal{A}^Y(\mu)^\circ, \mathcal{L}^a) = h_\mu(c(\mathcal{L})a),$$

where $c(\mathcal{L})$ is the central charge of $\mathcal{L}$.

This theorem is a consequence of the following more general theorem.

Theorem 2. *Let $\mu \in \mathbb{X}_\bullet(T_{\mathrm{ad}})$. Then for any $Y \subset \mathbf{S}$ and ample line bundle $\mathcal{L}$ on $\mathcal{F}\ell^Y$, we have*

$$\dim H^0(\mathcal{A}^Y(\mu)^\circ, \mathcal{L}) = \dim H^0(\mathrm{Gr}_{\leq \mu}, \mathcal{L}_b^{c(\mathcal{L})}).$$

Since Theorem 1 is not the same as what Pappas and Rapoport originally conjectured and their conjecture is aimed at studying the local models of Shimura varieties, we will explain why this is the correct theorem for applications to local models in Section 8. Let us remark that if G is split of type A or C, Theorem 1 is proved in [PR08], using the previous results on the local models of Shimura varieties (cf. [Gör01], [Gör03], [PR05]). However, it seems that Theorem 2 is new even for symplectic groups.

One consequence of our main theorem (see Section 8) is that

Corollary 2.2. *The statement of Theorem* 0.1 *in* [PR09] *holds unconditionally.*

Our main theorem can be also applied to local models of other types (for example for the (even) orthogonal groups) to deduce some geometrical properties of the special fibers. This will be done in [PZ13].

Remark 2.1. The original coherence conjecture in [PR08] needs to be modified. This is due to a miscalculation in [PR08, 10.a.1]. Namely, when G is simply-connected, the affine flag variety of G (denoted by $\mathcal{F}_G$ temporarily) embeds into the affine flag variety of H (denoted by $\mathcal{F}_H$ temporarily). Therefore, there is a restriction map $\mathrm{Pic}(\mathcal{F}_H) \to \mathrm{Pic}(\mathcal{F}_G)$, which was described explicitly in loc. cit. This description is wrong in the case when the group is a nonsplit unitary group in an even number of variables. Adjusting the work in loc. cit. to account for this produces the modified coherence conjecture that we show in this paper. Let us remark that the same miscalculation led to an incorrect example in [Hei10, Rem. 19(4) and an incorrect statement in the last sentence of the first paragraph in p. 502] (see Proposition 4.1).

3. **The global Schubert varieties**

Theorem 2 will be a consequence of the geometry of the global Schubert varieties, which will be introduced in what follows. Global Schubert varieties are the function field counterparts of the local models.

3.1. *The global affine Grassmannian.* Let C be a smooth curve over k, and let $\mathcal{G}$ be a smooth *affine* group scheme over C. Let $\mathrm{Gr}_{\mathcal{G}}$ be the global affine Grassmannian over C. Let us recall the functor it represents. For every k-algebra R,

$$\mathrm{Gr}_{\mathcal{G}}(R) = \left\{ (y, \mathcal{E}, \beta) \;\middle|\; \begin{array}{l} y : \mathrm{Spec} R \to C,\ \mathcal{E} \text{ is a } \mathcal{G}\text{-torsor on } C_R, \\ \beta : \mathcal{E}|_{C_R - \Gamma_y} \cong \mathcal{E}^0|_{C_R - \Gamma_y} \text{ is a trivialization} \end{array} \right\}, \tag{3.1.1}$$

where Γ_y denotes the graph of y. This is a formally smooth ind-scheme over C ([PZ13, Prop. 5.5]).

We also have the jet group $\mathcal{L}^+\mathcal{G}$ and the loop group $\mathcal{L}\mathcal{G}$ of $\mathcal{G}$. To define it, we need some notation. Let $y : \mathrm{Spec}R \to C$. We will denote by $\Gamma_y \subset C_R$ the closed subscheme given by the graph of y and consider the formal completion of C_R along Γ_y, which is an affine formal scheme. Following [BD, 2.12], we can also consider the affine scheme $\hat{\Gamma}_y$ given by the relative spectrum of the ring of regular functions on that completion. There is a natural closed immersion $\Gamma_y \to \hat{\Gamma}_y$, and we will denote by $\hat{\Gamma}_y^\circ := \hat{\Gamma}_y - \Gamma_y$ the complement of the image. In our paper, we will soon specialize to the case $C = \mathbb{A}^1 = \mathrm{Spec}k[v]$ so that $y : \mathrm{Spec}R \to C$ is given by $v \mapsto y \in R$ and therefore $\Gamma_y = \mathrm{Spec}R[v]/(v-y)$ and $\hat{\Gamma}_y \simeq \mathrm{Spec}R[[w]]$ and the map $p : \hat{\Gamma}'_y \to C_R$ is given by $v \mapsto w + y$. We will often write $\hat{\Gamma}_y = \mathrm{Spec}R[[v-y]]$. Then $\hat{\Gamma}_y^\circ = \mathrm{Spec}R[[v-y]][(v-y)^{-1}]$.

Now, we define $\mathcal{L}^+\mathcal{G}$ and $\mathcal{L}\mathcal{G}$. For any k-algebra R,

$$\mathcal{L}^+\mathcal{G}(R) = \left\{ (y, \beta) \;\middle|\; y : \mathrm{Spec}R \to C, \beta \in \mathcal{G}(\hat{\Gamma}_y) \right\} \tag{3.1.2}$$

and

$$\mathcal{L}\mathcal{G}(R) = \left\{ (y, \beta) \;\middle|\; y : \mathrm{Spec}R \to C, \beta \in \mathcal{G}(\hat{\Gamma}_y^\circ) \right\}. \tag{3.1.3}$$

The former is a scheme formally smooth (but not of finite type) over C, and the latter is a formally smooth ind-scheme over C.

Let us describe the fibers of $\mathcal{L}\mathcal{G}, \mathcal{L}^+\mathcal{G}, \mathrm{Gr}_\mathcal{G}$ over C. Let $x \in C$ be a closed point. Let $\mathcal{O}_x$ denote the completion of the local ring of C at x and F_x be the fractional field of $\mathcal{O}_x$. Then

$$(\mathcal{L}\mathcal{G})_x \cong L(\mathcal{G}_{F_x}), \quad (\mathcal{L}^+\mathcal{G})_x \cong L^+(\mathcal{G}_{\mathcal{O}_x}), \quad (\mathrm{Gr}_\mathcal{G})_x \cong \mathrm{Gr}_{\mathcal{G}_{\mathcal{O}_x}} := L(\mathcal{G}_{F_x})/L^+(\mathcal{G}_{\mathcal{O}_x}).$$

The groups $\mathcal{L}\mathcal{G}$ and $\mathcal{L}^+\mathcal{G}$ naturally act on $\mathrm{Gr}_\mathcal{G}$. To see this, we can use the descent lemma of Beauville-Laszlo (see [BL95], or rather a general form of this lemma given in [BD, Th. 2.12.1] and [PZ13, Lemma 5.1]) to show

Lemma 3.1. *The natural map*

$$\mathrm{Gr}_\mathcal{G}(R) \to \left\{ (y, \mathcal{E}, \beta) \;\middle|\; \begin{array}{l} y : \mathrm{Spec}R \to C, \mathcal{E} \text{ is a } \mathcal{G}\text{-torsor on} \\ \hat{\Gamma}_y,\ \beta : \mathcal{E}|_{\hat{\Gamma}_y^\circ} \cong \mathcal{E}^0|_{\hat{\Gamma}_y^\circ} \text{ is a trivialization} \end{array} \right\}$$

is a bijection for each R.

Then $\mathcal{L}\mathcal{G}$ and $\mathcal{L}^+\mathcal{G}$ act on $\mathrm{Gr}_\mathcal{G}$ by changing the trivialization β. The trivial $\mathcal{G}$-torsor gives $\mathrm{Gr}_\mathcal{G} \to C$ a section e. Then we have the projection

$$\mathrm{pr} : \mathcal{L}\mathcal{G} \to \mathcal{L}\mathcal{G} \cdot e = \mathrm{Gr}_\mathcal{G}. \tag{3.1.4}$$

We need the following lemma in the sequel.

LEMMA 3.2. *The formation of* $\mathrm{Gr}_{\mathcal{G}}, \mathcal{LG}, \mathcal{L}^+\mathcal{G}$ *commute with any étale base change, i.e., if* $f : C' \to C$ *is étale, then* $\mathrm{Gr}_{\mathcal{G}} \times_C C' \cong \mathrm{Gr}_{\mathcal{G}\times_C C'}$, *etc. In addition, the action of* $\mathcal{LG}$ *on* $\mathrm{Gr}_{\mathcal{G}}$ *also commutes with any étale base change.*

Proof. We have the following observation. Let $y' : \mathrm{Spec}R \to C'$ be an R-point of C' and $f(y) : \mathrm{Spec}R \to C$ be the corresponding R-point of C. Since f is étale, the morphism obtained from f by completing along y' and y gives an isomorphism of affine formal schemes that induces an isomorphism $\hat{\Gamma}_y \simeq \hat{\Gamma}_{y'}$ between the affine spectra of their coordinate rings. In addition, this isomorphism restricts to an isomorphism $\hat{\Gamma}^\circ_{y'} \simeq \hat{\Gamma}^\circ_y$. The lemma now follows. □

3.2. *The group scheme.* We will be mostly interested in the case that $\mathcal{G}$ is a Bruhat-Tits group scheme over C. Let us specify the meaning of this term. Let η denote the generic point of C. Then a smooth group scheme $\mathcal{G}$ over C is called a Bruhat-Tits group scheme if $\mathcal{G}_\eta$ is (connected) reductive, and for any closed point y of C, $\mathcal{G}_{\mathcal{O}_y}$ is a parahoric group scheme of $\mathcal{G}_{F_y}$.

Now let us specify the Bruhat-Tits group scheme that will be relevant to us. Let G_1 be an almost simple, absolutely simple and simply-connected, and split over a tamely ramified extension $\tilde{F}/F$, as in the coherent conjecture. Then we can assume that $\tilde{F}/F$ is cyclic of order $e = 1, 2, 3$. Let γ be a generator of $\Gamma = \mathrm{Gal}(\tilde{F}/F)$. For technical reasons, which are apparent from the statement of Theorem 2.1, we need the following well-known result.

LEMMA 3.3. *There is a connected reductive group* G *over* F, *which splits over* $\tilde{F}/F$, *such that* $G_{\mathrm{der}} \cong G_1$ *and* $\mathbb{X}_\bullet(T) \to \mathbb{X}_\bullet(T_{\mathrm{ad}})$ *is surjective. Here* T *is a maximal torus of* G *as in Section* 2.1.

For example, if $G_1 = \mathrm{SL}_n$ or Sp_{2n}, then G can be chosen as GL_n and GSp_{2n} respectively.

We let (H, B_H, T_H, X) be a split pinned group over $\mathbb{Z}$, together with an isomorphism $(G, B, T) \otimes_F \tilde{F} \cong (H, B_H, T_H) \otimes \tilde{F}$ as in Section 2.1. Let us choose the special vertex v_0 to identify $A(G, S)$ with $\mathbb{X}_\bullet(S)_\mathbb{R}$, and let $\mathbf{a}$ be the chosen alcove in $A(G, S)$ as in Section 2.1. Let $Y \subset \mathbf{S}$ as before.

Let $[e] : \mathbb{A}^1 \to \mathbb{A}^1$ be the ramified cover given by $y \to y^e$. To distinguish these two $\mathbb{A}^1$s, let us denote it as $[e] : \tilde{C} \to C$. The origin of C is denoted by 0, and the origin of $\tilde{C}$ is denoted by $\tilde{0}$. Write $C^\circ = C - \{0\}$ and $\tilde{C}^\circ = \tilde{C} - \{\tilde{0}\}$. Observe that Γ acts on $H \times \tilde{C}$ naturally. Namely, it acts on the first factor by pinned automorphisms and the second by transport of structures. Let

$$\mathcal{G}|_{C^\circ} = (\mathrm{Res}_{\tilde{C}^\circ/C^\circ}(H \times \tilde{C}))^\Gamma.$$

Then $\mathcal{G}_{F_0} \cong G$ after choosing some $F_0 \cong F$. Now, gluing $\mathcal{G}|_{C^\circ}$ and $\mathcal{G}_{\sigma_Y}$ along the the fpqc cover $C = C^\circ \cup \mathrm{Spec}\mathcal{O}_0$ (see [Hei10, Lemma 5] for the detailed

discussion of the descent theory in this case), we get a group scheme $\mathcal{G}$ over C, satisfying

(1) $\mathcal{G}_\eta$ is connected reductive with connected center, splits over a tamely ramified extension, such that $(\mathcal{G}_\eta)_{\mathrm{der}}$ is simple, absolutely simple, and simply-connected;
(2) for some choice of isomorphism $F_0 \cong F$, $\mathcal{G}_{F_0} \cong G$;
(3) for any $y \neq 0$, $\mathcal{G}_{\mathcal{O}_y}$ is hyperspecial, (noncanonically) isomorphic to $H \otimes \mathcal{O}_y$;
(4) $\mathcal{G}_{\mathcal{O}_0} = \mathcal{G}_{\sigma_Y}$ under the isomorphism $\mathcal{G}_{F_0} \cong G$.

A more detailed account of the construction of this group scheme is given in Section 5.1. Let us mention that similar group schemes have been constructed in [HNY13], [Ric13]. For this group scheme $\mathcal{G}$, we know that the fiber of $\mathrm{Gr}_{\mathcal{G}}$ over $y \neq 0$ is isomorphic to the affine Grassmannian Gr_H of H, and the fiber over 0 is isomorphic to the affine flag variety $\mathcal{F}\ell^Y$ of G. Likewise, the fiber of $\mathcal{L}^+\mathcal{G}$ over $y \neq 0$ is isomorphic to L^+H and the fiber over 0 is isomorphic to $L^+\mathcal{G}_{\sigma_Y}$.

Let $\mathcal{T}$ be the subgroup scheme of $\mathcal{G}$ such that

(1) $\mathcal{T}_\eta$ is a maximal torus of $\mathcal{G}_\eta$;
(2) for any $y \neq 0$, $\mathcal{T}_{\mathcal{O}_y}$ is a split torus;
(3) $\mathcal{T}_{F_0}$ is the torus T and $\mathcal{T}_{\mathcal{O}_0}$ is the connected Néron model of $\mathcal{T}_{F_0}$.

We can construct $\mathcal{T}$ as the neutral connected component of

$$\tilde{\mathcal{T}} = (\mathrm{Res}_{\tilde{C}/C}(T_H \times \tilde{C}))^\Gamma. \tag{3.2.1}$$

Note that $\mathcal{T}$ embeds into $\mathcal{G}$ naturally. Indeed, under our tameness assumption, $\mathcal{T}$ is the connected Néron model of $(\mathrm{Res}_{\tilde{C}^\circ/C^\circ}(T_H \times \tilde{C}))^\Gamma$, and $\mathcal{T}(\mathcal{O}_0) \subset \mathcal{G}(\mathcal{O}_0)$. Then the claim follows by the construction of parahoric group schemes as in [BT84, 5.2].

3.3. *The global Schubert variety.* It turns out that it is more convenient to base change everything over C to $\tilde{C}$. Let u (resp. v) denote a global coordinate of $\tilde{C}$ (resp. C) such that the map $[e] : \tilde{C} \to C$ is given by $v \mapsto u^e$. Recall that $0 \in C(k)$ (resp. $\tilde{0} \in \tilde{C}(k)$) is given by $v = 0$ (resp. $u = 0$). The crucial step toward the construction of the global Schubert varieties is the following proposition.

Proposition 3.4. *For each $\mu \in \mathbb{X}_\bullet(\mathcal{T}_\eta) \cong \mathbb{X}_\bullet(T_H)$, there is a section*

$$s_\mu : \tilde{C} \to \mathcal{L}\mathcal{T} \times_C \tilde{C}$$

such that for any $\tilde{y} \in \tilde{C}(k)$, the element

$$s_\mu(\tilde{y}) \in (\mathcal{L}\mathcal{T})_y(k) = \mathcal{T}_{F_y}(F_y), \qquad y = [e](\tilde{y})$$

maps under the Kottwitz homomorphism $\kappa : \mathcal{T}_{F_y}(F_y) \to \mathbb{X}_\bullet(\mathcal{T}_\eta)_{\mathrm{Gal}(F_y^s/F_y)}$ to the image of μ under the natural projection $\mathbb{X}_\bullet(\mathcal{T}_\eta) \to \mathbb{X}_\bullet(\mathcal{T}_\eta)_{\mathrm{Gal}(F_y^s/F_y)}$.

The proposition is obvious for split groups. But for the ramified groups, the proof is a little bit complicated, and only the statement of the proposition will be used in the main body of the paper. Therefore, those who are only interested in split groups can skip the proof.

Proof. Let us first review how to construct an element in $t_\mu \in T(k(\!(t)\!))$ whose image under the Kottwitz homomorphism (2.1.3) is μ under the map $\mathbb{X}_\bullet(T) \to \mathbb{X}_\bullet(T)_\Gamma$. Let $k(\!(s)\!)/k(\!(t)\!)$ be a finite separable extension of degree n so that $T_{k(\!(s)\!)}$ splits, where $s^n = t$. Then $\lambda(s) \in T(k(\!(s)\!))$. By the construction of the Kottwitz homomorphism (cf. [Kot97, §7]), we can take t_λ to be the image of $\lambda(s)$ under the norm map $T(k(\!(s)\!)) \to T(k(\!(t)\!))$.

Now we construct s_μ. Let $\tilde{\mathcal{T}}$ is as in (3.2.1). We will first construct a section $s_\mu : \tilde{C} \to \mathcal{L}\tilde{\mathcal{T}}$ and then prove it indeed factors as $s_\mu : \tilde{C} \to \mathcal{L}\mathcal{T} \to \mathcal{L}\tilde{\mathcal{T}}$.

Let $\Gamma_{[e]}$ denote the graph of $[e] : \tilde{C} \to C$. By definition,

$$\mathrm{Hom}_C(\tilde{C}, \mathcal{L}\tilde{\mathcal{T}}) = \mathrm{Hom}_C(\hat{\Gamma}^\circ_{[e]}, \tilde{\mathcal{T}}) = \mathrm{Hom}(\hat{\Gamma}^\circ_{[e]} \times_C \tilde{C}, T_H)^\Gamma,$$

where Γ acts on $\hat{\Gamma}^\circ_{[e]} \times_C \tilde{C}$ via the action on the second factor.

Recall that we have the global coordinates u, v and the map $[e] : \tilde{C} \to C$ is given by $v \mapsto u^e$. Then $\mathcal{O}_{\hat{\Gamma}^\circ_{[e]}} \cong k[u](\!(v - u^e)\!)$. Therefore, the ring of functions on $\hat{\Gamma}^\circ_{[e]} \times_C \tilde{C}$ can be written as

$$A = k[u_1](\!(v - u_1^e)\!) \otimes_{k[v]} k[u_2],$$

where the map $k[v] \to k[u_2]$ is given by $v \mapsto u_2^e$. Let γ be a generator of $\Gamma = \mathrm{Aut}(\tilde{C}/C)$ acting on u_2 as $u_2 \mapsto \xi u_2$, where ξ is a primitive e-th root of unit. For $i = 1, \ldots, e$, the element $(\xi^i \otimes u_2 - u_1 \otimes 1)$ is invertible in A and therefore gives a morphism

$$x_i : \hat{\Gamma}^\circ_{[e]} \times_C \tilde{C} \to \mathbb{G}_m.$$

Clearly $x_i \circ \gamma = x_{i+1}$. (As usual, $x_{i+e} = x_i$.)

Now choose a basis $\omega_1, \ldots, \omega_\ell$ of $\mathbb{X}^\bullet(T_H)$. Let us define

$$s_\mu : \hat{\Gamma}^\circ_{[e]} \times_C \tilde{C} \to T_H$$

as

$$\omega_j(s_\mu) = x_1^{\langle \mu, \gamma\omega_j \rangle} x_2^{\langle \mu, \gamma^2\omega_j \rangle} \cdots x_e^{\langle \mu, \gamma^e\omega_j \rangle}.$$

Clearly, s_μ is independent of the choice of $\omega_1, \ldots, \omega_\ell$. (However, it depends on the global coordinate u on $\tilde{C}$.) Furthermore, s_μ is Γ-equivariant. Therefore, we constructed a section $s_\mu : \tilde{C} \to \mathcal{L}\tilde{\mathcal{T}}$.

Now we prove that this section indeed factors as $s_\mu : \tilde{C} \to \mathcal{L}\mathcal{T} \to \mathcal{L}\tilde{\mathcal{T}}$. In other words, the morphism $\hat{\Gamma}^\circ_{[e]} \to \tilde{\mathcal{T}}$ factors as $\hat{\Gamma}^\circ_{[e]} \to \mathcal{T} \to \tilde{\mathcal{T}}$. By definition, $\mathcal{T}$ is the neutral connected component of $\tilde{\mathcal{T}}$. Therefore, it is enough to prove that the image of $\hat{\Gamma}^\circ_{[e]}|_0 \to \tilde{\mathcal{T}}|_0$ lands in the neutral connected component

of $\tilde{\mathcal{T}}|_0$. Observe that $\hat{\Gamma}^\circ_{[e]}|_0 \cong \mathrm{Spec} k(\!(u_1)\!)$. Let $\tilde{C}_0$ be the fiber of $\tilde{C} \to C$ over 0 so that $\tilde{C}_0 \cong k[u]/u^e$ with a Γ-action. It has a unique closed point $\tilde{0}$. Recall that $\tilde{T}|_0 = (\mathrm{Res}_{\tilde{C}_0/k}(T_H \times \tilde{C}_0))^\Gamma$ and therefore, there is a canonical map $\varepsilon : \tilde{\mathcal{T}}|_0 \to T_H^\Gamma$ given by adjunction, making the following diagram commute:

$$\begin{array}{ccccc}
\mathrm{Hom}_C(\hat{\Gamma}^\circ_{[e]}, \tilde{\mathcal{T}}) & \longrightarrow & \mathrm{Hom}(\hat{\Gamma}^\circ_{[e]}|_0, \tilde{\mathcal{T}}|_0) & \xrightarrow{\varepsilon} & \mathrm{Hom}(\hat{\Gamma}^\circ_{[e]}|_0, T_H^\Gamma) \\
\| & & \| & & \| \\
\mathrm{Hom}(\hat{\Gamma}^\circ_{[e]} \times_C \tilde{C}, T_H)^\Gamma & \longrightarrow & \mathrm{Hom}(\hat{\Gamma}^\circ_{[e]}|_0 \times \tilde{C}_0, T_H)^\Gamma & \longrightarrow & \mathrm{Hom}(\hat{\Gamma}^\circ_{[e]}|_0 \times \{\tilde{0}\}, T_H^\Gamma)
\end{array}$$

In our case, $\varepsilon(s_\mu) : \hat{\Gamma}^\circ_{[e]}|_0 \to T_H$ is given by

$$\omega_j(\varepsilon(s_\mu)) = (-u_1)^{(\sum_{\gamma\in\Gamma} \gamma\mu, \omega_j)}.$$

In other words, $\varepsilon(s_\mu)$ is the composition

$$\hat{\Gamma}^\circ_{[e]}|_0 \xrightarrow{-u_1} \mathbb{G}_m \xrightarrow{\sum_{\gamma\in\Gamma} \gamma\mu} T_H.$$

Since for any Γ-invariant coweight μ, the image $\mu : \mathbb{G}_m \to T_H^\Gamma$ lands in the neutral connected component of T_H^Γ (the torus part), $s_\mu : \tilde{C} \to \mathcal{L}\tilde{\mathcal{T}}$ factors through $\tilde{C} \to \mathcal{L}\mathcal{T} \to \mathcal{L}\tilde{\mathcal{T}}$.

Finally, let us check that $s_\mu : \tilde{C} \to \mathcal{L}\mathcal{T} \times_C \tilde{C}$ satisfies the desired properties as claimed in the proposition.

Let $\tilde{y} \in \tilde{C}(k)$ be a closed point given by $u \mapsto \tilde{y} \in k$. Then $s_\mu(\tilde{y})$ corresponds to $s_\mu(\tilde{y}) : \mathrm{Spec} k(\!(v - \tilde{y}^e)\!) \otimes_{k[v]} k[u_2] \to T_H$ given by

$$\omega_j(s_\mu(\tilde{y})) = \prod_{i=1}^{e} (\xi^i 1 \otimes u_2 - y)^{(\mu, \gamma^i\omega_j)}.$$

If $\tilde{y} = 0$, the assertion of the proposition follows directly from the review of the construction of t_μ at the beginning. If $\tilde{y} \neq 0$, let $w = 1 \otimes u_2 - y$. Then

$$\prod_{i=1}^{e} (\xi^i 1 \otimes u_2 - y)^{(\mu, \gamma^i\omega_j)} = w^{(\mu,\omega_j)} f(w),$$

where

$$f(w) = \prod_{i=1}^{e-1} (\xi^i 1 \otimes u_2 - y)^{(\mu, \gamma^i\omega_j)} \in k[[w]]^\times.$$

Therefore, as an element in $T_H(k(\!(w)\!))$, which is canonically isomorphic to $\mathcal{L}\mathcal{T}_y$, $s_\mu(\tilde{y})$ maps to μ under the Kottwitz homomorphism. □

Remark 3.1. Note that the natural map $\mathcal{L}\mathcal{T} \to \mathcal{L}\tilde{\mathcal{T}}$ induces isomorphisms $\mathcal{L}\mathcal{T}|_{C^\circ} \simeq \mathcal{L}\tilde{\mathcal{T}}|_{C^\circ}$ and $\mathcal{L}\mathcal{T}|_0 \to \mathcal{L}\tilde{\mathcal{T}}|_0$, but itself is not an isomorphism.

Remark 3.2. For a general μ, there is no such section $C \to \mathcal{L}\mathcal{T}$ satisfying the property of the proposition. This is why we want to base change everything over C to $\tilde{C}$. However, if $\mu \in \mathbb{X}_\bullet(T)$ is defined over F, then s_μ indeed descents

to a section $C \to \mathcal{LT}$. This means that in this case the variety $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}$ defined below, which *a priori* is a variety over $\tilde{C}$, descends to a variety over C. One can summarize this by saying that $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}$ is defined over the "reflex field" of the geometric conjugacy class $\{\mu\}$ (which is the same as the field of definition of μ as G is quasi-split over F ([Kot84, Lemma 1.1.3])). The same phenomenon appears in the theory of Shimura varieties.

The composition of s_μ and the natural morphism (see (3.1.4)) $\mathrm{pr} : \mathcal{LT} \to \mathrm{Gr}_{\mathcal{T}}$ (resp. $\mathcal{LT} \to \mathcal{LG}$) gives a section $\tilde{C} \to \mathrm{Gr}_{\mathcal{T}} \times_C \tilde{C}$ (resp. $\tilde{C} \to \mathcal{LG} \times_C \tilde{C}$), which is still denoted by s_μ.

The construction of $\tilde{C} \to \mathcal{LT} \times_C \tilde{C}$ will depend on the choice of the global coordinate u of $\tilde{C}$, but the section $s_\mu : \tilde{C} \to \mathrm{Gr}_{\mathcal{T}} \times_C \tilde{C}$ does not. Indeed, there is the following moduli interpretation of such section. Recall that $\mathrm{Gr}_{\mathcal{T}}$ is ind-proper over C ([Hei10]), and therefore, s_μ is uniquely determined by a section $\tilde{C}^\circ \to \mathrm{Gr}_{\mathcal{T}} \times_C \tilde{C}^\circ \cong \mathrm{Gr}_{T_H \times \tilde{C}^\circ}$ (by Lemma 3.2). Then this section, under the moduli interpretation of $\mathrm{Gr}_{T_H \times \tilde{C}^\circ}$, is given as follows: let Δ be the diagonal of $\tilde{C}^\circ \times \tilde{C}^\circ$ and $\mathcal{O}_{(\tilde{C}^\circ)^2}(\mu\Delta)$ be the T_H-torsor on $(\tilde{C}^\circ)^2$ such that for any weight ν of T_H, the associated line bundle is $\mathcal{O}_{\tilde{C}^\circ}((\mu,\nu)\Delta)$. This T_H-torsor has a canonical trivialization away from Δ.

LEMMA 3.5. *The map $s_\mu : \tilde{C}^\circ \to \mathrm{Gr}_{\mathcal{T}}$ corresponds to $(\mathcal{E}, \beta)$, where $\mathcal{E}$ is the T_H-torsor $\mathcal{O}_{(\tilde{C}^\circ)^2}(\mu\Delta)$ and β is its canonical trivialization over $(\tilde{C}^\circ)^2 - \Delta$.*

Proof. The Kottwitz homomorphism $\kappa : LT_H(k) \to \mathbb{X}_\bullet(T_H)$ induces an isomorphism $\mathrm{Gr}_H(k) \cong LT_H(k)/L^+T_H(k)$. On the other hand, recall that if we fix a point x on the curve $\tilde{C}$, we can interpret Gr_{T_H} as the set of $(\mathcal{E}, \beta)$, where $\mathcal{E}$ is a T_H-torsor and β is a trivialization of $\mathcal{E}$ away from x. Under this interpretation, any $t_\mu \in \mathbb{X}_\bullet(T_H)$ is interpreted as the T_H-torsor $\mathcal{O}_{\tilde{C}}(\mu x)$,[5] with its canonical trivialization away from x. Then the lemma is clear. □

By composing with the natural morphism $\mathrm{Gr}_{\mathcal{T}} \to \mathrm{Gr}_{\mathcal{G}}$, we obtain a section of $\mathrm{Gr}_{\mathcal{G}} \times_C \tilde{C}$, still denoted by s_μ.

Notation. In what follows, we denote $\mathrm{Gr}_{\mathcal{G}} \times_C \tilde{C}$ (resp. $\mathcal{L}^+\mathcal{G} \times_C \tilde{C}$, resp. $\mathcal{LG} \times_C \tilde{C}$) by $\widetilde{\mathrm{Gr}}_{\mathcal{G}}$ (resp. $\widetilde{\mathcal{L}^+\mathcal{G}}$, resp. $\widetilde{\mathcal{LG}}$).

Definition 3.1. For each $\mu \in \mathbb{X}_\bullet(\mathcal{T}_\eta) \cong \mathbb{X}_\bullet(T_H)$, the global Schubert variety $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}$ is the minimal $\widetilde{\mathcal{L}^+\mathcal{G}}$-stable irreducible closed subvariety of $\widetilde{\mathrm{Gr}}_{\mathcal{G}}$ that contains s_μ.

Let us emphasize that $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}$ is not a subvariety of $\mathrm{Gr}_{\mathcal{G}}$. Rather, it lies in $\mathrm{Gr}_{\mathcal{G}} \times_C \tilde{C}$. Recall that for any $\mu \in \mathbb{X}_\bullet(T)$, one defines a subset $\mathrm{Adm}^Y(\mu) \subset \widetilde{W}$

[5]The reason that t_μ represents $\mathcal{O}_{\tilde{C}}(\mu x)$ rather than $\mathcal{O}_{\tilde{C}}(-\mu x)$ is due to the original sign convention of the Kottwitz homomorphism in [Kot97].

as in (2.1.6). The main geometric property of $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}$ that we will prove in this paper is as follows.

Theorem 3. *Assume that the group G is as in Lemma* 3.3 *and the group scheme $\mathcal{G}$ is constructed from G as in Section* 3.2. *Let y be a closed point of $\tilde{C}$. Then*

$$(\overline{\mathrm{Gr}}_{\mathcal{G},\mu})_y \cong \begin{cases} \bigcup_{w\in \mathrm{Adm}^Y(\mu)} \mathcal{F}\ell^Y_w & y=\tilde{0}, \\ \overline{\mathrm{Gr}}_\mu & y\neq \tilde{0}. \end{cases}$$

In particular, all the fibers are reduced.

We first prove the easy part of the theorem.

LEMMA 3.6. $(\overline{\mathrm{Gr}}_{\mathcal{G},\mu})_y \cong \overline{\mathrm{Gr}}_\mu$ *for* $y\neq\tilde{0}$.

Proof. Write $\tilde{C}^\circ = \tilde{C} - \tilde{0}$. We want to show that $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}|_{\tilde{C}^\circ}$ is isomorphic to $\overline{\mathrm{Gr}}_\mu \times \tilde{C}^\circ$. First we have a canonical isomorphism

$$\mathcal{G}\times_C \tilde{C}^\circ \cong H\times \tilde{C}^\circ. \tag{3.3.1}$$

Therefore, by Lemma 3.2,

$$\mathrm{Gr}_{\mathcal{G}}\times_C \tilde{C}^\circ \cong \mathrm{Gr}_{H\times\tilde{C}^\circ}, \quad \mathcal{L}\mathcal{G}\times_C \tilde{C}^\circ \cong \mathcal{L}(H\times\tilde{C}). \tag{3.3.2}$$

Secondly, $\tilde{C}^\circ \cong \mathbb{G}_m$, which admits a global coordinate u so that $\mathcal{L}(H\times\tilde{C}^\circ)\cong LH\times\tilde{C}^\circ$ and $\mathrm{Gr}_{H\times\tilde{C}^\circ}\cong \mathrm{Gr}_H\times\tilde{C}^\circ$. Finally, by Lemma 3.5, the section $s_\mu : \tilde{C}^\circ \to \mathrm{Gr}_{\mathcal{G}}\times_C\tilde{C}^\circ \cong \mathrm{Gr}_H\times\tilde{C}^\circ$ satisfies $s_\mu(\tilde{C}^\circ)\subset \overline{\mathrm{Gr}}_\mu\times\tilde{C}^\circ$. □

Using this lemma, we see that it is enough to make the following convention.

Convention. When we discuss $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}$, we will assume that $\mu\in\mathbb{X}_\bullet(T_H)$ is dominant with respect to the chosen Borel B_H as in Sections 2.1 and 3.2.

At this moment, we can also see that

LEMMA 3.7. *The scheme $(\overline{\mathrm{Gr}}_{\mathcal{G},\mu})_{\tilde{0}}\subset(\mathrm{Gr}_{\mathcal{G}})_0\cong\mathcal{F}\ell^Y$ contains $\mathcal{F}\ell^Y_w$ for $w\in\mathrm{Adm}^Y(\mu)$.*

Proof. Clearly, it is enough to show that $\mathcal{F}\ell^Y_\lambda\subset(\overline{\mathrm{Gr}}_{\mathcal{G},\mu})_{\tilde{0}}$ for any λ in Λ, where Λ is the W_0-orbit in $\mathbb{X}_\bullet(T)_\Gamma$ containing μ as constructed in Section 2.1. Observe that $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}$ is the flat closure of $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}|_{\tilde{C}^\circ}$ in $\widetilde{\mathrm{Gr}}_{\mathcal{G}}$, since the later is clearly $\widetilde{\mathcal{L}^+\mathcal{G}}$-stable. Then the claim follows from the fact that for any $\lambda\in\mathbb{X}_\bullet(T_H)$ in the $\bar{W}$-orbit of μ, $s_\lambda(\tilde{0})\in\mathcal{F}\ell^Y_\lambda$ and $s_\lambda(\tilde{C}^\circ)\subset\overline{\mathrm{Gr}}_{\mathcal{G},\mu}|_{\tilde{C}^\circ}\cong\overline{\mathrm{Gr}}_\mu\times\tilde{C}^\circ$. □

To prove the theorem, it is remains to show that

THEOREM 3.8. *The underlying reduced subscheme of $(\overline{\mathrm{Gr}}_{\mathcal{G},\mu})_{\tilde{0}}$ is the union $\bigcup_{w\in\mathrm{Adm}^Y(\mu)}\mathcal{F}\ell^Y_w$.*

THEOREM 3.9. *$(\overline{\mathrm{Gr}}_{\mathcal{G},\mu})_{\tilde{0}}$ is reduced.*

By the same argument as in [PZ13, 9.2.1], $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}$ is normal. We conjecture that it is Cohen-Macaulay as well.

4. **Line bundles on $\mathrm{Gr}_{\mathcal{G}}$ and $\mathrm{Bun}_{\mathcal{G}}$**

This section explains why Theorems 3 and 2 are equivalent to each other. The key ingredients are the line bundles on the global affine Grassmannian $\mathrm{Gr}_{\mathcal{G}}$. Observe that $\mathrm{Gr}_{\mathcal{G}}$ can be disconnected. This will create some complications in trying to determine the line bundle on $\mathrm{Gr}_{\mathcal{G}}$ directly. Instead, we will pass to its group scheme $\mathcal{G}_{\mathrm{der}}$ (defined below), whose generic fiber then is simply-connected so that we can apply the results of Heinloth [Hei10] directly.

4.1. *Line bundles on $\mathrm{Gr}_{\mathcal{G}}$ and $\mathrm{Bun}_{\mathcal{G}}$.* In this subsection, we temporary assume that C is a smooth curve over k and $\mathcal{G}$ is a Bruhat-Tits group scheme over C such that $\mathcal{G}_\eta$ is almost simple, absolutely simple, and simply-connected.

PROPOSITION 4.1. *Let $\mathcal{L}$ be a line bundle on $\mathrm{Gr}_{\mathcal{G}}$. Then the function $c_{\mathcal{L}}$ that associates to every $y \in C(k)$ the central charge of the restriction of $\mathcal{L}$ to $(\mathrm{Gr}_{\mathcal{G}})_y$ is constant.*

This proposition implies that the statement in the last sentence of the first paragraph in page 502 of [Hei10] is not correct.

Proof. Let $\mathrm{Pic}(\mathrm{Gr}_{\mathcal{G}}/C)$ denote the relative sheaf of Picard groups over C. As explained in [Hei10], this is an étale sheaf over C. Let $D = \mathrm{Ram}(\mathcal{G})$ be the set of points of C such that for every $y \in \mathrm{Ram}(\mathcal{G})$, the fiber $\mathcal{G}_y$ is not semi-simple. This is a finite set. Then there is a short exact sequence

$$1 \to \prod_{y \in D} \mathbb{X}^\bullet(\mathcal{G}_y) \to \mathrm{Pic}(\mathrm{Gr}_{\mathcal{G}}/C) \to \mathfrak{c} \to 1, \tag{4.1.1}$$

where $\mathfrak{c}$ is a constructible sheaf, with all fibers isomorphic to $\mathbb{Z}$, and is constant on $C - D$.

According to the description of the sheaf $\mathfrak{c}$ in Remark 19 (3) of loc. cit., if $\mathcal{L}$ is a line bundle on $\mathrm{Gr}_{\mathcal{G}}$ such that $c_{\mathcal{L}}(y) = 0$ for some $y \in C(k)$, then $c_{\mathcal{L}} = 0$. Therefore, to prove the proposition, it is enough to construct one line bundle $\mathcal{L}_{2c}$ on $\mathrm{Gr}_{\mathcal{G}}$ such that $c_{\mathcal{L}_{2c}}$ is constant on C.

Let $\mathcal{V}_0 = \mathrm{Lie}\,\mathcal{G}$ be the Lie algebra of $\mathcal{G}$. This is a locally free $\mathcal{O}_C$-module on C of rank $\dim_\eta \mathcal{G}_\eta$, on which $\mathcal{G}$ acts by the adjoint representation. This induces a morphism $\mathcal{G} \to \mathrm{GL}(\mathcal{V}_0)$ and therefore a morphism $i : \mathrm{Gr}_{\mathcal{G}} \to \mathrm{Gr}_{\mathrm{GL}(\mathcal{V}_0)}$. Let $\mathcal{L}_{\det}$ denote the determinant line bundle on $\mathrm{Gr}_{\mathrm{GL}(\mathcal{V}_0)}$. Let us recall its construction. We want to associate to every $\mathrm{Spec}R \to \mathrm{Gr}_{\mathrm{GL}(\mathcal{V}_0)}$ a line bundle on $\mathrm{Spec}R$ in a compatible way. Recall that a morphism $\mathrm{Spec}R \to \mathrm{Gr}_{\mathrm{GL}(\mathcal{V}_0)}$

represents a morphism $y \in C(R)$, a vector bundle $\mathcal{V}$ on C_R and an isomorphism $\mathcal{V}|_{C_R-\Gamma_y} \cong \mathcal{V}_0|_{C_R-\Gamma_y}$. There exists some N large enough such that

$$\mathcal{V}_0(-N\Gamma_y) \subset \mathcal{V} \subset \mathcal{V}_0(N\Gamma_y)$$

and $\mathcal{V}_0(N\Gamma_y)/\mathcal{V}$ is R-flat. Then the line bundle on $\mathrm{Spec}R$ is

$$\det(\mathcal{V}_0(N\Gamma_y)/\mathcal{V}) \otimes \det(\mathcal{V}_0(N\Gamma_y)/\mathcal{V}_0)^{-1},$$

which is independent of the choice of N up to a canonical isomorphism.

The pullback $i^*\mathcal{L}_{\det}$ is a line bundle on $\mathrm{Gr}_{\mathcal{G}}$, which will be our $\mathcal{L}_{2c}$. To see this is the desired line bundle, we need to calculate its central charge when restricted to each $y \in C(k)$. Let $D = \mathrm{Ram}(\mathcal{G})$. First consider $y \in C - D$. Then the map $i_y : (\mathrm{Gr}_{\mathcal{G}})_y \to (\mathrm{Gr}_{\mathrm{GL}(\mathcal{V}_0)})_y$ is just

$$\mathrm{Gr}_H \to \mathrm{Gr}_{\mathrm{GL}(\mathrm{Lie}\, H)},$$

where H is the split Chevalley group over $\mathbb{Z}$ such that $G \otimes k(\eta)^s \cong H \otimes k(\eta)^s$. It is well known that in this case $i_y^*\mathcal{L}_{\det}$ over y has central charge $2h^\vee$, where $h^\vee$ is the dual Coxeter number of H. (In fact, this statement is a consequence of the following argument.)

It remains to calculate the central charge of $\mathcal{L}_{2c}$ over $y \in D$. Without loss of generality, we can assume that D consists of one point, denoted by 0. So let $y = 0$ and $G = \mathcal{G}_{F_0}$. Then the closed embedding $i_0 : (\mathrm{Gr}_{\mathcal{G}})_0 \to (\mathrm{Gr}_{\mathrm{GL}(\mathcal{V}_0)})_0$ is just

$$LG/L^+\mathcal{G}_{\mathcal{O}_0} \to \mathrm{Gr}_{\mathrm{GL}(\mathrm{Lie}\, \mathcal{G}_{\mathcal{O}_0})}.$$

Let us first assume that $\mathcal{G}_{\mathcal{O}_0}$ is an Iwahori group scheme of $\mathcal{G}_{F_0}$. Write $I = L^+\mathcal{G}_{\mathcal{O}_0}$ and $\mathcal{F}\ell = LG/I$ as usual. We claim that in this case

LEMMA 4.2. *We have an isomorphism* $i_0^*\mathcal{L}_{\det} \cong \mathcal{L}(2\sum_{i\in\mathbf{S}} \varepsilon_i)$.

Assuming this fact, we find the central charge of $i_0^*\mathcal{L}_{\det}$ is $2\sum_{i\in\mathbf{S}} a_i^\vee$. By checking all the affine Dynkin diagrams, we find that

$$\sum_{i\in\mathbf{S}} a_i^\vee = h^\vee.$$

In fact, we find that for affine Dynkin diagrams $X_N^{(r)}$, where $X = A, B, C, D, E, F, G$ and $r = 1, 2, 3$, the sum $\sum a_i^\vee$ is independent of r (see [Kac90, Rem. 6.1]), and it is well known (or by definition) that for $r = 1$, $\sum a_i^\vee = h^\vee$. Therefore, the proposition follows in this case.

Now we prove Lemma 4.2. This is equivalent to proving that the restriction of $i_0^*\mathcal{L}_{\det}$ to each $\mathbb{P}^1_j$ (whose definition is given in Section 2.2) is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(2)$. Recall that a k-point $gI \in \mathcal{F}\ell$ corresponds to a pro-algebraic subgroup of $L\mathcal{G}_{F_0}$ given by $I' := gIg^{-1}$, which is the jet group of an Iwahori group scheme of $\mathcal{G}_{F_0}$. By abuse of notation, we still denote this Iwahori group scheme by I'. Then $\mathcal{F}\ell \to \mathrm{Gr}_{\mathrm{GL}(\mathrm{Lie}\, \mathcal{G}_{\mathcal{O}_0})}$ maps an Iwahori group scheme I' of G

to its Lie algebra $\operatorname{Lie} I'$, which is a free $\mathcal{O}_0$-module, together with the canonical isomorphism $\operatorname{Lie} I' \otimes F_0 \cong \operatorname{Lie} G \cong \operatorname{Lie} I \otimes F_0$.

For $j \in \mathbf{S}$, let P_j be the minimal parahoric (but not Iwahori) group scheme corresponding to j. Then the subscheme $\mathbb{P}^1_j \subset \mathcal{F}\ell$ classifies the Iwahori group schemes of G that map to P_j. Let $P_j^u \to P_j$ be the "unipotent radical" of P_j. More precisely, P_j^u is smooth over $\mathcal{O}_0$ with $P_j^u \otimes F_0 = G$ and the special fiber of P_j^u maps onto the unipotent radical of the special fiber of P_j. If I' is an Iwahori group scheme of G that maps to P_j, then

$$\operatorname{Lie} P_j^u \subset \operatorname{Lie} I' \subset \operatorname{Lie} P_j.$$

Let $\bar{P}_j^{\mathrm{red}}$ be the reductive quotient of the special fiber of P_j. Then $\bar{P}_j^{\mathrm{red}}$ is isomorphic to $\mathrm{GL}_2, \mathrm{SL}_2$ or SO_3 over k. Let $\mathrm{Gr}(2, \operatorname{Lie} \bar{P}_j^{\mathrm{red}}) \cong \mathbb{P}^2$ denote the Grassmannian (over k) of 2-planes in the three-dimensional vector space $\operatorname{Lie} \bar{P}_j^{\mathrm{red}}$. We have the following commutative diagram:

$$\begin{array}{ccc} \mathbb{P}^1_j & \longrightarrow & \mathrm{Gr}(2, \operatorname{Lie} \bar{P}_j^{\mathrm{red}}) \\ \downarrow & & \downarrow \\ \mathcal{F}\ell & \longrightarrow & \mathrm{Gr}_{\mathrm{GL}(\operatorname{Lie} \mathcal{G}_{\mathcal{O}_0})}, \end{array}$$

where $\mathbb{P}^1_j \to \mathbb{G}(2, \operatorname{Lie} \bar{P}_j^{\mathrm{red}})$ is given by

$$I' \mapsto (\operatorname{Lie} I' / \operatorname{Lie} P_j^u \subset \operatorname{Lie} P_j / \operatorname{Lie} P_j^u) \cong \operatorname{Lie} \bar{P}_j^{\mathrm{red}})$$

and $\mathrm{Gr}(2, \operatorname{Lie} \bar{P}_j^{\mathrm{red}}) \to \mathrm{Gr}_{\mathrm{GL}(\operatorname{Lie} I)}$ is given by realizing that $\mathrm{Gr}(2, \operatorname{Lie} \bar{P}_j^{\mathrm{red}})$ represents the free $\mathcal{O}_0$-modules that are in between $\operatorname{Lie} P_j^u$ and $\operatorname{Lie} P_j$. Observe that the degree of the map $\mathbb{P}^1_j \to \mathrm{Gr}(2, \operatorname{Lie} \bar{P}_j^{\mathrm{red}}) \simeq \mathbb{P}^2$ is two as it is just the map that sends a Borel subgroup of SL_2 to the two-dimensional vector subspace of $\mathfrak{sl}_2$ given by the Lie algebra of the Borel subgroup.

By construction, the restriction of $\mathcal{L}_{\det}$ to $\mathrm{Gr}(2, \operatorname{Lie} \bar{P}_j^{\mathrm{red}})$ is the (positive) determinant line bundle on $\mathbb{G}(2, \operatorname{Lie} \bar{P}_j^{\mathrm{red}})$, or $\mathcal{O}_{\mathbb{P}^2}(1)$. Therefore, the restriction of $\mathcal{L}_{\det}$ to $\mathbb{P}^1_j$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(2)$. This finishes the proof of Lemma 4.2, and therefore the proposition in the case $\mathcal{G}_{\mathcal{O}_0}$ is Iwahori.

Now let $\mathcal{G}_{\mathcal{O}_0}$ be a general parahoric group scheme. Let $\mathcal{G}'$ be the group scheme over C together with $\mathcal{G}' \to \mathcal{G}$, such that $\mathcal{G}'_{\mathcal{O}_0}$ is Iwahori and the restriction of $\mathcal{G}' \to \mathcal{G}$ to $C - \{0\}$ is an isomorphism. Let $\mathcal{V}_0 = \operatorname{Lie} \mathcal{G}$ and $\mathcal{V}'_0 = \operatorname{Lie} \mathcal{G}'$. We have the natural map

$$p : (\mathrm{Gr}_{\mathcal{G}'})_0 \to (\mathrm{Gr}_{\mathcal{G}})_0$$

induced from $\mathcal{G}' \to \mathcal{G}$ and the maps

$$i : \mathrm{Gr}_{\mathcal{G}} \to \mathrm{Gr}_{\mathrm{GL}(\mathcal{V}_0)}, \quad i' : \mathrm{Gr}_{\mathcal{G}'} \to \mathrm{Gr}_{\mathrm{GL}(\mathcal{V}'_0)}.$$

Let $\mathcal{L}_{\det}$ (resp. $\mathcal{L}'_{\det}$) be the determinant line bundle on $\mathrm{Gr}_{\mathrm{GL}(\mathcal{V}_0)}$ (resp. $\mathrm{Gr}_{\mathrm{GL}(\mathcal{V}'_0)}$). We need to show that $p^* i_0^* \mathcal{L}_{\det}$ and $i_0'^* \mathcal{L}'_{\det}$ have the same central

charge. (Observe that these two line bundles are not isomorphic.) From this, we conclude that the central charge of $i^*\mathcal{L}_{\det}$ is also constant along C.

Let us extend $\mathcal{G}$ and $\mathcal{G}'$ to group schemes over the complete curve $\bar{C}$ such that $\mathcal{G}|_{\bar{C}-\{0\}} = \mathcal{G}'|_{\bar{C}-\{0\}}$. Let $\mathrm{Bun}_{\mathcal{G}}$ (resp. $\mathrm{Bun}_{\mathcal{G}'}$) be the moduli stack of $\mathcal{G}$-torsors ($\mathcal{G}'$-torsors) on $\bar{C}$. Let $\mathcal{G}_0, \mathcal{G}'_0$ be the restriction of the two group schemes over $0 \in C$, and let P be the image of $\mathcal{G}'_0 \to \mathcal{G}_0$. This is indeed a Borel subgroup of $\mathcal{G}'_0$. Recall that by restricting a $\mathcal{G}'$-torsor to $0 \in C$, we obtain a map $(\mathrm{Gr}_{\mathcal{G}'})_0 \xrightarrow{r} \mathbb{B}\mathcal{G}'_0$, and we have the similar map for $\mathcal{G}$. Then we have the following diagram with both squares Cartesian:

$$\begin{array}{ccccc} (\mathrm{Gr}_{\mathcal{G}'})_0 & \longrightarrow & \mathrm{Bun}_{\mathcal{G}'} & \xrightarrow{r} & \mathbb{B}\mathcal{G}'_0 \longrightarrow \mathbb{B}P \\ \downarrow & & \downarrow & & \downarrow \\ (\mathrm{Gr}_{\mathcal{G}})_0 & \longrightarrow & \mathrm{Bun}_{\mathcal{G}} & \xrightarrow{\quad r \quad} & \mathbb{B}\mathcal{G}_0. \end{array} \tag{4.1.2}$$

Indeed, it is clear that the left square is Cartesian because $\mathcal{G}|_{\bar{C}-\{0\}} = \mathcal{G}'|_{\bar{C}-\{0\}}$. The fact that the second square is Cartesian is established in Proposition 9.7.

Let $y : \mathrm{Spec}R \to (\mathrm{Gr}_{\mathcal{G}'})_0$ be a morphism given by $(\mathcal{E}, \beta)$, where $\mathcal{E}$ is a $\mathcal{G}'$-torsor on C_R. Then we have the natural short exact sequence

$$0 \to \mathrm{ad}\mathcal{E} \to \mathrm{ad}(\mathcal{E} \times^{\mathcal{G}'} \mathcal{G}) \to \mathcal{E} \times^{\mathcal{G}'} (\mathrm{Lie}\,\mathcal{G}/\,\mathrm{Lie}\,\mathcal{G}') \to 0.$$

On the other hand, $p : (\mathrm{Gr}_{\mathcal{G}'})_0 \to (\mathrm{Gr}_{\mathcal{G}})_0$ is a relatively smooth morphism since $\mathbb{B}P \to \mathbb{B}\mathcal{G}'_0$ is smooth. Let $\mathcal{T}_p$ denote the relative tangent sheaf. We claim that $\mathcal{E} \times^{\mathcal{G}'} (\mathrm{Lie}\,\mathcal{G}/\,\mathrm{Lie}\,\mathcal{G}') \cong y^*\mathcal{T}_p$, where $y^*\mathcal{T}_p$ is the sheaf on $\mathrm{Spec}R$, regarded as a sheaf on C_R via the closed embedding $\{0\} \times \mathrm{Spec}R =: \{0\}_R \to C_R$. But this follows from (4.1.2) and

$$\mathcal{E} \times^{\mathcal{G}'} (\mathrm{Lie}\,\mathcal{G}/\,\mathrm{Lie}\,\mathcal{G}') \cong (\mathcal{E}|_{\{0\}_R} \times^{\mathcal{G}'_0} P) \times^P (\mathrm{Lie}\,\mathcal{G}_0/\,\mathrm{Lie}\,P).$$

Therefore, we have

$$0 \to \mathrm{ad}\ \mathcal{E} \to \mathrm{ad}(\mathcal{E} \times^{\mathcal{G}''} \mathcal{G}') \to y^*\mathcal{T}_p \to 0. \tag{4.1.3}$$

Let us finish the proof that $p^*i_0^*\mathcal{L}_{\det}$ and $i_0'^*\mathcal{L}'_{\det}$ have the same central charge and therefore the proof of the proposition. From the above lemma,

$$p^*i_0^*\mathcal{L}_{\det} \cong i_0'^*\mathcal{L}'_{\det} \otimes \det(\mathcal{T}_p). \tag{4.1.4}$$

So it is enough to prove that $\det \mathcal{T}_p$ as a line bundle on $(\mathrm{Gr}_{\mathcal{G}'})_0$ has central charge zero. But from (4.1.2), $\det \mathcal{T}_p$ is a pullback of some line bundle from $\mathbb{B}P$, and hence from $\mathbb{B}\mathcal{G}'_0$, which has zero central charge by (2.2.5). □

Now, we assume that C is a complete curve and let $\mathrm{Bun}_{\mathcal{G}}$ be the moduli stack of $\mathcal{G}$-torsors on C. Let $\mathrm{Pic}(\mathrm{Bun}_{\mathcal{G}})$ be the Picard group of rigidified line bundles (trivialized over the trivial $\mathcal{G}$-torsor) on $\mathrm{Bun}_{\mathcal{G}}$. Let $D = \mathrm{Ram}(\mathcal{G})$. Observe that $\prod_{y \in C(k)} \mathbb{X}^\bullet(\mathcal{G}_y) = \prod_{y \in D} \mathbb{X}^\bullet(\mathcal{G}_y)$. Fix $0 \in C(k)$. Let $\mathcal{F}\ell^Y =$

$L\mathcal{G}_{F_0}/L^+\mathcal{G}_{\mathcal{O}_0}$, which is a partial affine flag variety of $\mathcal{G}_{F_0}$. According to [Hei10, §7], we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \prod_{y\in C(k)}\mathbb{X}^\bullet(\mathcal{G}_y) & \longrightarrow & \mathrm{Pic}(\mathrm{Bun}_\mathcal{G}) & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{X}^\bullet(\mathcal{G}_0) & \longrightarrow & \mathrm{Pic}(\mathcal{F}\ell^Y) & \xrightarrow{c} & \mathbb{Z} & \longrightarrow & 0. \end{array}$$

The left vertical arrow is the projection to the factor corresponding to 0, and the right vertical arrow is injective (but not necessarily surjective). Presumably, the map $\mathrm{Pic}(\mathrm{Bun}_\mathcal{G})\to\mathbb{Z}$ is in fact given by $\mathrm{Pic}(\mathrm{Bun}_\mathcal{G})\to\Gamma(C,\mathrm{Pic}(\mathrm{Gr}_\mathcal{G}/C))$ $\to\Gamma(C,\mathfrak{c})\cong\mathbb{Z}$ and the right vertical arrow is the natural restriction map $\Gamma(C,\mathfrak{c})\to\mathfrak{c}|_0$. Here we will not need to show this. We can, however, use the above diagram to show that, for any $\mathcal{L}\in\mathrm{Pic}(\mathcal{F}\ell^Y)$, a certain tensor power of it will descend to a line bundle on $\mathrm{Bun}_\mathcal{G}$. Therefore, we conclude

Corollary 4.3. *Let C be a smooth but (not necessarily complete) curve, and let $\mathcal{G}$ be a Bruhat-Tits group scheme over C such that $\mathcal{G}_\eta$ is almost simple, absolutely simple and simply-connected. Let H be the split Chevalley group over $\mathbb{Z}$ such that $\mathcal{G}\otimes k(\eta)^s\cong H\otimes k(\eta)^s$. Let $0\in C(k)$, and let $\mathcal{L}$ be a line bundle on $\mathcal{F}\ell^Y=L\mathcal{G}_{F_0}/L^+\mathcal{G}_{\mathcal{O}_0}$. Then there is a line bundle on $\mathrm{Gr}_\mathcal{G}$ whose restriction to $(\mathrm{Gr}_\mathcal{G})_0\cong\mathcal{F}\ell^Y$ is isomorphic to $\mathcal{L}^n$ for some $n\geq 1$ and whose restriction to $(\mathrm{Gr}_\mathcal{G})_y\cong\mathrm{Gr}_H$(for $y\notin\mathrm{Ram}(\mathcal{G})$) is isomorphic to $\mathcal{L}_b^{nc(\mathcal{L})}$, where $\mathcal{L}_b$ is the ample generator of* $\mathrm{Pic}(\mathrm{Gr}_H)\cong\mathbb{Z}$.

Proof. Let $\bar{C}$ be a complete curve containing C. We extend $\mathcal{G}$ to a Bruhat-Tits group scheme over $\bar{C}$. Then some tensor power $\mathcal{L}^n$ of $\mathcal{L}$ descends to a line bundle $\mathcal{L}'$ on $\mathrm{Bun}_\mathcal{G}$. Let $h_{\mathrm{glob}}:\mathrm{Gr}_\mathcal{G}\to\mathrm{Bun}_\mathcal{G}$ be the natural projection. Then $h_{\mathrm{glob}}^*\mathcal{L}'$ is a line bundle on $\mathrm{Gr}_\mathcal{G}$ whose restriction to $(\mathrm{Gr}_\mathcal{G})_0$ is isomorphic to $\mathcal{L}^n$, and whose restriction to $(\mathrm{Gr}_\mathcal{G})_y\cong\mathrm{Gr}_H$ ($y\notin\mathrm{Ram}(\mathcal{G})$) has central charge $c(\mathcal{L}^n)$, and therefore is isomorphic to $\mathcal{L}_b^{nc(\mathcal{L})}$. □

4.2. *Theorem* 3 *is equivalent to Theorem* 2. Let us begin with a general construction. Let $\mathcal{G}$ be a Bruhat-Tits group scheme over a curve C. Then away from a finite subset $D\subset C$, $\mathcal{G}|_{C-D}$ is reductive. Let $\mathcal{G}_{\mathrm{der}}|_{C-D}$ be the derived group of $\mathcal{G}|_{C-D}$ so that for $y\in C(k)$, $(\mathcal{G}_{\mathrm{der}})_{F_y}$ is the derived group of $\mathcal{G}_{F_y}$ ([DG70, Exposé XXII 6.2]). It is known that there is a canonical bijection between the facets in the building of $(\mathcal{G}_{\mathrm{der}})_{F_y}$ and those in the building of $\mathcal{G}_{F_y}$, and under this bijection, the corresponding parahoric group scheme for $(\mathcal{G}_{\mathrm{der}})_{F_y}$ maps to the corresponding parahoric group scheme for $\mathcal{G}_{F_y}$. For example, see [HR, Prop. 3] and its proof for the last statement. Therefore, we can extend $\mathcal{G}_{\mathrm{der}}|_{C-D}$ to a Bruhat-Tits group scheme over C together with a morphism $\mathcal{G}_{\mathrm{der}}\to\mathcal{G}$ such that for all $y\in D$, $(\mathcal{G}_{\mathrm{der}})_{\mathcal{O}_y}\to\mathcal{G}_{\mathcal{O}_y}$ is the morphism of parahoric group schemes given by the facet determined by $\mathcal{G}_{\mathcal{O}_y}$.

Definition 4.1. The group scheme $\mathcal{G}_{\mathrm{der}}$ together with the morphism $\mathcal{G}_{\mathrm{der}} \to \mathcal{G}$ is called the derived group scheme of $\mathcal{G}$.

Now let G be the group as in Lemma 3.3, and assume that the group scheme $\mathcal{G}$ over $C = \mathbb{A}^1$ is constructed from G as in Section 3.2. Let us denote $\mathcal{G}_1 = \mathcal{G}_{\mathrm{der}}$ for simplicity. Let $C^\circ = C - \{0\}$. Observe that $\mathcal{G}_1|_{C^\circ}$ is reductive, $(\mathcal{G}_1)_{F_0} \cong G_1 = G_{\mathrm{der}}$, and for $y \neq 0$, $(\mathcal{G}_1)_{\mathcal{O}_y}$ is hyperspecial for $H_{\mathrm{der}} \otimes \mathcal{O}_y$. In addition, $(\mathcal{G}_1)_\eta$ is simply-connected.

Let us explain why Theorems 3 and 2 are equivalent. The natural morphism $\mathcal{G}_1 \to \mathcal{G}$ induces a morphism $\mathrm{Gr}_{\mathcal{G}_1} \to \mathrm{Gr}_{\mathcal{G}}$. One can show that this is a closed embedding. (We will not use this fact; it follows however, a posteriori, from the argument below.) But at least it follows directly from [PR08, §6] that both $(\mathrm{Gr}_{\mathcal{G}_1})_0 \to (\mathrm{Gr}_{\mathcal{G}})_0$ and $\mathrm{Gr}_{\mathcal{G}_1}|_{C^\circ} \to \mathrm{Gr}_{\mathcal{G}}|_{C^\circ}$ are closed immersions. These induce isomorphisms from $(\mathrm{Gr}_{\mathcal{G}_1})_0$ and $(\mathrm{Gr}_{\mathcal{G}_1})_{C^\circ}$ to the reduced subschemes of the neutral connected component of $(\mathrm{Gr}_{\mathcal{G}})_0$ and of $\mathrm{Gr}_{\mathcal{G}}|_{C^\circ}$ respectively. Let $\mu \in \mathbb{X}_\bullet(T)$, and let $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}$ be the corresponding global Schubert variety as in Section 3.3. Recall the section s_μ from Proposition 3.4. Regard it as a section of $\widetilde{\mathcal{L}\mathcal{G}}$ that acts on $\widetilde{\mathrm{Gr}}_{\mathcal{G}}$. Then

$$s_\mu^{-1}\overline{\mathrm{Gr}}_{\mathcal{G},\mu}|_{\tilde{C}^\circ} \subset \widetilde{\mathrm{Gr}}_{\mathcal{G}_1}|_{\tilde{C}^\circ}.$$

This follows from $t_\mu^{-1}\overline{\mathrm{Gr}}_\mu \subset \mathrm{Gr}_{H_{\mathrm{der}}}$ for any $\mu \in \mathbb{X}_\bullet(T_H)$, where t_μ is considered as any lifting of $t_\mu \in \widetilde{W}$ to $T_H(F)$. Let $\mathrm{Gr}_{\mathcal{G}_1,\leq\mu}$ be the flat closure of $s_\mu^{-1}\overline{\mathrm{Gr}}_{\mathcal{G},\mu}|_{\tilde{C}^\circ}$ in $\widetilde{\mathrm{Gr}}_{\mathcal{G}_1}$. We claim that the natural map

$$\mathrm{Gr}_{\mathcal{G}_1,\leq\mu} \to s_\mu^{-1}\overline{\mathrm{Gr}}_{\mathcal{G},\mu} \tag{4.2.1}$$

is an isomorphism. To see this, first note that it is proper and bijective at the level of points. Therefore, it is a finite morphism by Zariski's main theorem. Now, let $x \in (\mathrm{Gr}_{\mathcal{G}_1,\leq\mu})_{\tilde{0}}$ and y be its image in $(s_\mu^{-1}\overline{\mathrm{Gr}}_{\mathcal{G},\mu})_{\tilde{0}}$. Let A and B be their local rings respectively. Then A is a finite B-module. Let u be a local coordinate around $\tilde{0}$. Note that since $B[u^{-1}] \to A[u^{-1}]$ is an isomorphism and B has no u-torsion, $B \to A$ is injective. On the other hand, $(\mathrm{Gr}_{\mathcal{G}_1,\leq\mu})_{\tilde{0}} \to (s_\mu^{-1}\overline{\mathrm{Gr}}_{\mathcal{G},\mu})_{\tilde{0}}$ is a closed embedding since $\mathcal{F}\ell^Y_{\mathrm{sc}} \to \mathcal{F}\ell^Y$ is a closed embedding. Therefore, $B/uB \to A/uA$ is surjective. By Nakayama's lemma, $B = A$, and therefore (4.2.1) is an isomorphism.

Let τ_μ be the image of μ in $\Omega \cong \mathbb{X}_\bullet(T)_\Gamma/\mathbb{X}_\bullet(T_{\mathrm{sc}})_\Gamma$, and let $Y^\circ \subset \mathbf{S}$ so that $\sigma_{Y^\circ} = \tau_\mu^{-1}(\sigma_Y)$ as before. Let $g \in G_1(F)$ be a lifting of $t_{-\mu}\tau_\mu \in W_{\mathrm{aff}}$. Then since $\mathcal{F}\ell^Y_w \in (\overline{\mathrm{Gr}}_{\mathcal{G},\mu})_{\tilde{0}}$ for $w \in \mathrm{Adm}^Y(\mu)$ (see Lemma 3.7), $g({}^{Y^\circ}\mathcal{F}\ell^Y_{\mathrm{sc},w}) \subset (\mathrm{Gr}_{\mathcal{G}_1,\leq\mu})_{\tilde{0}}$ for $w \in \mathrm{Adm}^Y(\mu)^\circ$. In other words, $\mathcal{A}^Y(\mu)^\circ \subset (\mathrm{Gr}_{\mathcal{G}_1,\leq\mu})_{\tilde{0}}$.

Let $\mathcal{L}$ be an ample line bundle on $\mathcal{F}\ell^Y_{\mathrm{sc}}$. Suppose that its certain tensor power $\mathcal{L}^n$ extends to a line bundle on $\mathrm{Gr}_{\mathcal{G}_1}$ by Corollary 4.3. Then we have

$$\dim \Gamma((\mathrm{Gr}_{\mathcal{G}_1,\leq\mu})_y, \mathcal{L}_b^{nc(\mathcal{L})}) = \dim \Gamma((\mathrm{Gr}_{\mathcal{G}_1,\leq\mu})_{\tilde{0}}, \mathcal{L}^n) \geq \dim \Gamma(\mathcal{A}^Y(\mu)^\circ, \mathcal{L}^n)$$

by the flatness and the fact that $H^1({}^Y\mathcal{F}\ell_w^{Y'}, \mathcal{L}) = 0$ for any Schubert variety ${}^Y\mathcal{F}\ell_w^{Y'}$ and any ample line bundle $\mathcal{L}$. In addition, the equality holds if and only if $\mathcal{A}^Y(\mu)^\circ = (\mathrm{Gr}_{\mathcal{G}_1,\leq\mu})_{\tilde{0}}$. Clearly, for $y \neq \tilde{0}$, $(\mathrm{Gr}_{\mathcal{G}_1,\leq\mu})_y = g\mathrm{Gr}_{\leq\mu}$. Therefore,

$$\Gamma((\mathrm{Gr}_{\mathcal{G}_1,\leq\mu})_y, \mathcal{L}_b^{nc(\mathcal{L})}) \cong \Gamma(\mathrm{Gr}_{\leq\mu}, \mathcal{L}_b^{nc(\mathcal{L})})$$

by Corollary 4.3. Therefore, Theorem 2 implies Theorem 3. Conversely, Theorem 3 implies that the statement of Theorem 2 holds for $\mathcal{L}^n, \mathcal{L}^{2n}, \ldots$. Therefore, we have the equality of Euler characteristic $\chi(\mathrm{Gr}_{\leq\mu}, \mathcal{L}_b^{mc(\mathcal{L})}) = \chi(\mathcal{A}^Y(\mu)^\circ, \mathcal{L}^m)$ for any m as both are polynomial in m. But it is well known that both $\mathcal{L}_b^{mc(\mathcal{L})}$ and $\mathcal{L}^m$ have no higher cohomology. (The charateristic $p > 0$ case follows from Frobenius splitting, and the characteristic zero case follows from the semicontinuity; see [Mat88] for details.) Therefore, the statement of Theorem 2 also holds for $\mathcal{L}$.

To finish this section, let us mention the following observation.

Corollary 4.4. *If Theorem* 2 (*equivalently*, *Theorem* 3) *holds for one prime* $p \nmid e$, *then it holds for all* $p \nmid e$ *as well as in the case* $\mathrm{char}\, k = 0$.

Proof. Recall that the affine flag varieties and Schubert varieties are defined over $W(k)$, the ring of Witt vectors of k, and the formation commutes with base change ([Fal03], [PR08]). In addition, line bundles are also defined over $W(k)$. (After identifying the affine flag varieties with those arising from Kac-Moody theory ([PR08]), this follows from [Mat88, XVIII]. In fact, they are even defined over $\mathbb{Z}'$, where $\mathbb{Z}$ is obtained from $\mathbb{Z}$ by adding e-th roots of unity and inverting e.) By the vanishing of corresponding H^1 (by the reason mentioned above), both sides are free $W(k)$-modules and the formation of cohomology commutes with base change. The corollary follows. □

5. Some properties of $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}$

In this section, we study two basic geometrical structures of $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}$: (i) in Section 5.2, we will construct certain affine charts of $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}$, which turn out to be isomorphic to affine spaces over $\tilde{C}$; and (ii) in Section 5.3, we will construct a $\mathbb{G}_m$-action on $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}$, so that the map $\overline{\mathrm{Gr}}_{\mathcal{G},\mu} \to \tilde{C}$ is $\mathbb{G}_m$-equivariant, where $\mathbb{G}_m$ acts on $\tilde{C} = \mathbb{A}^1$ by natural dilatation. To establish (ii), we will need to first construct the global root subgroups of $\mathcal{LG}$ as in Section 5.1. We shall remark that the proofs of these results for G split are very straightforward. It is only when G is not split that some complicated discussion is needed. Those who are only interested in split groups can skip this section.

5.1. *Global root groups.* We will introduce certain "root subgroups" of $\mathcal{LG}$ (more precisely, of $\mathcal{L}^+\mathcal{G}$; see Remark 5.1(i)), whose fibers over $0 \in C$ is the usual root subgroups of the loop group LG as constructed in [PR08, 9.a,9.b].

Let us first review the shape of root groups of G. Let (H, B_H, T_H, X) be a pinned Chevalley group over $\mathbb{Z}$ as in Section 2.1. In particular, H_{der} is simply-connected. Let Ξ be the group of pinned automorphisms of H_{der} that is simple, almost simple, simply-connected by our assumption. So Ξ is a cyclic group of order $1, 2$ or 3. Let $\tilde{\Phi} = \Phi(H, T_H)$ be the set of roots of H with respect to T_H. For each $\tilde{a} \in \Phi(H, T_H)$, let $\tilde{U}_{\tilde{a}}$ denote the corresponding root group. Then for each $\gamma \in \Xi$, one has an isomorphism $\gamma : \tilde{U}_{\tilde{a}} \cong \tilde{U}_{\gamma\tilde{a}}$. The stabilizer of $\tilde{a}$ in Ξ is either trivial or the whole group. Let us choose a Chevalley-Steinberg system of H; i.e., for each $\tilde{a} \in \Phi(H, T_H)$, an isomorphism $x_{\tilde{a}} : \mathbb{G}_a \cong \tilde{U}_{\tilde{a}}$ over $\mathbb{Z}$. In addition, we require that

(1) if $\tilde{a} \in \Delta$ is a simple root, then $X_{\tilde{a}} = dx_{\tilde{a}}(1)$, where $X = \sum_{\tilde{a}\in\Delta} X_{\tilde{a}}$;
(2) if the stabilizer of $\tilde{a}$ in Ξ is trivial, then $\gamma \circ x_{\tilde{a}} = x_{\gamma\tilde{a}}$ for any $\gamma \in \Xi$.

Note that if γ stabilizes $\tilde{a}$, it is not necessarily always the case that $\gamma \circ x_{\tilde{a}} = x_{\tilde{a}}$, as can be seen for SL_3. In this case, one obtains a quadratic character

$$\chi_{\tilde{a}} : \Xi \to \mathrm{Aut}_{\mathbb{Z}}(\mathbb{G}_a) = \{\pm 1\} \tag{5.1.1}$$

such that $\gamma \circ x_{\tilde{a}} = x_{\tilde{a}} \circ \chi_{\tilde{a}}(\gamma)$. Of course, this can happen only if the order of Ξ is 2.

Recall that $\Gamma = \mathrm{Aut}(\tilde{C}/C)$ is a group of order $e = 1, 2, 3$, which acts on H via pinned automorphisms and the corresponding map $\Gamma \to \Xi$ is injective.

Let $j : \Phi(H, T_H) \to \Phi(G, S)$ be the restriction of the root systems. For $a \in \Phi(G, S)$, let

$$\eta(a) = \{\tilde{a} \in \Phi(H, T_H) | j(\tilde{a}) = ma, m \geq 0\}.$$

This is a subset of $\Phi(H, T_H)$ satisfying the condition of [Con, 5.1.16]. Let $\tilde{U}_{\eta(a)}$ be the closed subgroup scheme of H as defined in loc. cit. As a scheme, $\tilde{U}_{\eta(a)} \cong \prod_{\tilde{a}\in\eta(a)} \tilde{U}_{\tilde{a}}$, where the product is taken over any given order (which we fix from now on) on $\eta(a)$. Informally, this is the subgroup of H generated by $\tilde{U}_{\tilde{a}}, \tilde{a} \in \eta(a)$. This subgroup is invariant under Ξ. Then $(\mathrm{Res}_{\tilde{F}/F}\tilde{U}_{\tilde{a}})^{\Gamma}$ is the root group of G corresponding to a.

For an integer n, let us denote by $\mathbb{G}_{a,n,\tilde{C}}$ the group scheme over $\tilde{C}$ that is the n-th congruence group scheme of $\mathbb{G}_{a,\tilde{C}}$. In other words, $\mathbb{G}_{a,n+1,\tilde{C}}$ is the dilatation of $\mathbb{G}_{a,n,\tilde{C}}$ along the trivial subgroup in the fiber over $\tilde{0}$ (see [BLR90, §3.2] or Section 9.2). More concretely, $\mathbb{G}_{a,n,\tilde{C}} = \mathrm{Spec}k[u, t_n] \simeq \mathbb{G}_{a,\tilde{C}}$ and the map $\mathbb{G}_{a,n+1,\tilde{C}} \to \mathbb{G}_{a,n,\tilde{C}}$ is given by $t_n \mapsto ut_{n+1}$. We also have the congruence group schemes $\tilde{U}_{\tilde{a},n,\tilde{C}}$ of $\tilde{U}_{\tilde{a},\tilde{C}}$. The Chevalley-Steinberg isomorphism $x_{\tilde{a}} : \mathbb{G}_a \to \tilde{U}_{\tilde{a}}$ induces the isomorphism

$$x_{\tilde{a},n} : \mathbb{G}_{a,\tilde{C}} \simeq \mathbb{G}_{a,n,\tilde{C}} \to \tilde{U}_{\tilde{a},n,\tilde{C}}$$

making the following diagram commutative:

$$\begin{array}{ccc} \mathbb{G}_{a,\tilde{C}} & \xrightarrow{x_{\tilde{a},n+1}} & \tilde{U}_{\tilde{a},n+1,\tilde{C}} \\ {\scriptstyle t_n \mapsto u t_{n+1}}\downarrow & & \downarrow \\ \mathbb{G}_{a,\tilde{C}} & \xrightarrow{x_{\tilde{a},n}} & \tilde{U}_{\tilde{a},n,\tilde{C}}. \end{array} \tag{5.1.2}$$

Our goal is to construct some global root groups for $\mathcal{LG}$. For the purpose, we describe a construction of $\mathcal{G}$.

Let us normalize the valuation so that u has value $1/e$. Then we embed $A(G,S)$ into $A(H,T_H)$. Let $x \in \sigma_Y$ be a point. It determines a parahoric group scheme $\tilde{\mathcal{G}}_x$ of $H \otimes \tilde{F}$, and $\mathcal{G}_{\sigma_Y}$ is the neutral connected component of $(\mathrm{Res}_{\tilde{F}/F}\tilde{\mathcal{G}}_x)^\Gamma$. (One can see the claim as follows: Lifting x to a point in the extended building of G, then $(\tilde{\mathcal{G}}_x(\tilde{\mathcal{O}}))^\Gamma \subset G(F)$ is the stabilizer of this point. On the other hand, by [Edi92, 2.2, 3.4], $(\mathrm{Res}_{\tilde{F}/F}\tilde{\mathcal{G}}_x)^\Gamma$ is smooth. Therefore, its neutral connected component is the parahoric group scheme of G given x.)

We extend $\tilde{\mathcal{G}}_x$ to a group scheme $\tilde{\mathcal{G}}$ over $\tilde{C}$ as in Section 3.2, so that $\tilde{\mathcal{G}}|_{\tilde{C}^\circ} = H \times \tilde{C}^\circ$ and $\tilde{\mathcal{G}}|_{\mathcal{O}_0} = \tilde{\mathcal{G}}_x$ (under the identification $\tilde{F} \cong \tilde{F}_0$). From the construction, $\tilde{\mathcal{G}}$ contains

$$\prod_{\tilde{a}\in\Phi(H,T_H)^-} \tilde{U}_{\tilde{a},\lceil e\tilde{a}(v_0-x)\rceil,\tilde{C}} \times T_{H,\tilde{C}} \times \prod_{\tilde{a}\in\Phi(H,T_H)^+} \tilde{U}_{\tilde{a},\lceil e\tilde{a}(v_0-x)\rceil,\tilde{C}}$$

as a fiberwise dense open subscheme ([BT84, 2.2.10, 3.9.4]), where $\lceil y \rceil$ denotes the smallest integer that is $\geq y$. Observe that since x is fixed under the action of Γ, for $a \in \Phi$, the closed subgroup scheme $\prod_{\tilde{a}\in\eta(a)} \tilde{U}_{\tilde{a},\lceil e\tilde{a}(x-v_0)\rceil,\tilde{C}}$ of $\tilde{\mathcal{G}}$ is invariant under the action of Γ. Let

$$U_{a,\sigma_Y,C} = \left(\mathrm{Res}_{\tilde{C}/C} \prod_{\tilde{a}\in\eta(a)} \tilde{U}_{\tilde{a},\lceil e\tilde{a}(v_0-x)\rceil,\tilde{C}}\right)^\Gamma,$$

which does not depend on x. By [Edi92, 2.2, 3.4], $U_{a,\sigma_Y,C}$ is smooth. In addition, a check for SL_2 and SU_3 cases shows that $U_{a,\sigma_Y,C}$ is connected. Then $(U_{a,\sigma_Y,C})_{F_0}$ is the root group of $\mathcal{G}_{F_0} \cong G$ corresponding to a, and for $y \neq 0$, $(U_{a,\sigma_Y,C})_y \cong \tilde{U}_{\eta(a)}$ noncanonically. In addition,

$$\prod_{a\in\Phi^{nd,-}} U_{a,\sigma_Y,C} \times (\mathrm{Res}_{\tilde{C}/C} T_{H,\tilde{C}})^{\Gamma,0} \times \prod_{a\in\Phi^{nd,+}} U_{a,\sigma_Y,C}$$

is a fiberwise dense open subscheme of $\mathcal{G}$, where $\Phi^{nd} \subset \Phi = \Phi(G,S)$ denote the set of nondivisible roots; i.e., $a \in \Phi^{nd}$ if $a/2 \notin \Phi$. Given an affine root α of G with vector part a, the corresponding root subgroup of $\mathcal{LG}$ will be constructed as a closed subgroup scheme of $\mathcal{L}U_{a,\sigma_Y,C}$.

Recall that we constructed the special vertex v_0 in Section 2.1, where we used this vertex to identify $A(G,S)$ with $\mathbb{X}_\bullet(S)_\mathbb{R}$. Then we can write affine

roots as $a+m$, where $a \in \Phi(G,S)$ and $m \in \frac{1}{e}\mathbb{Z}$. Let $a+m$ be an affine root such that $em \geq \lceil ea(v_0 - x)\rceil$. Let us construct a closed immersion

$$x_{a+m} : \mathbb{G}_{a,C} \to \mathcal{L}U_{a,\sigma_Y,C}. \tag{5.1.3}$$

Let us describe x_{a+m} at the level of R-points, where R is a k-algebra. Recall that we write $C = \mathrm{Spec}k[v], \tilde{C} = \mathrm{Spec}k[u]$ such that $[e] : \tilde{C} \to C$ is given by $v \mapsto u^e$. Let $y : \mathrm{Spec}R \to C$ be an R-point of C. We identify $\mathrm{Hom}_C(\mathrm{Spec}R, \mathbb{G}_{a,C})$ with R in an obvious manner. We thus need to construct a map (functorial in R)

$$x_{a+m} : R \to \mathrm{Hom}_C(\mathrm{Spec}R, \mathcal{L}U_{a,\sigma_Y,C}).$$

The graph of $y : \mathrm{Spec}R \to C$ is $\Gamma_y = \mathrm{Spec}R[v]/(v-y)$, and therefore

$$\hat{\Gamma}^\circ_y = \mathrm{Spec}R((v-y)).$$

Now, by definition,

$$\mathrm{Hom}_C\left(\mathrm{Spec}R, \mathcal{L}U_{a,\sigma_Y,C}\right) = \mathrm{Hom}\left(\mathrm{Spec}R((v-y)) \times_C \tilde{C}, \tilde{U}_{a,\sigma_Y,\tilde{C}}\right)^\Gamma, \tag{5.1.4}$$

where Γ acts on $\mathrm{Spec}R((v-y)) \times_C \tilde{C}$ via the action on $\tilde{C}$ and acts on $\tilde{U}_{a,\sigma_Y,\tilde{C}} := \prod_{\tilde{a}\in\eta(a)} \tilde{U}_{\tilde{a},\lceil e\tilde{a}(v_0-x)\rceil,\tilde{C}}$ as above.

Let us introduce the following notation. Each element $s \in R((v-y)) \otimes_{k[v]} k[u]$ determines a morphism $\mathrm{Spec}R((v-y)) \times_C \tilde{C} \to \mathbb{G}_{a,\tilde{C}}$. Let

$$x_{\tilde{a},n}(s) : \mathrm{Spec}R((v-y)) \times_C \tilde{C} \to \tilde{U}_{\tilde{a},n,\tilde{C}}$$

denote the composition of this morphism with $x_{\tilde{a},n} : \mathbb{G}_{a,\tilde{C}} \to \tilde{U}_{\tilde{a},n,\tilde{C}}$.

Now we construct x_{a+m}. There are two cases.

(i) $2a \notin \Phi(G,S)$. In this case, Γ acts transitively on $\eta(a)$. There are two subcases.

(ia) $\eta(a) = \tilde{a}$, so that Γ fixes $\tilde{a}$ and $\tilde{U}_{\eta(a)} = \tilde{U}_{\tilde{a}}$. Define

$$x_{a+m}(r) = x_{\tilde{a},\lceil ea(v_0-x)\rceil}\left(r \otimes u^{em-\lceil ea(v_0-x)\rceil}\right).$$

Since $a+m$ is an affine root, Γ acts on $u^{em-\lceil ea(v_0-x)\rceil}$ exactly via the quadratic character $\chi_{\tilde{a}}$ as defined in (5.1.1), and $x_{a+m}(r)$ is an element in (5.1.4).

(ib) Γ acts simply transitively on $\eta(a)$. Choose $\tilde{a} \in \eta(a)$ and $\gamma \in \Gamma$ a generator. Using the isomorphism $\prod_{i=1}^e \tilde{U}_{\gamma^i(\tilde{a})} \cong \tilde{U}_{\eta(a)}$, define

$$x_{a+m}(r) = \prod_{i=1}^{e} x_{\gamma^i(\tilde{a}),\lceil ea(v_0-x)\rceil}\left(r \otimes \gamma^i(u)^{em-\lceil ea(v_0-x)\rceil}\right).$$

Since for $\tilde{a}, \tilde{a}' \in \eta(a)$, the groups $\tilde{U}_{\tilde{a}}$ and $\tilde{U}_{\tilde{a}'}$ commute, and therefore $x_{a+m}(r)$ is an element in (5.1.4).

(ii) $2a \in \Phi(G,S)$, so that $\eta(a) = \{\tilde{a}, \tilde{a}', \tilde{a}+\tilde{a}'\}$. In this case, $\operatorname{char} k \neq 2$, $e = 2$, and the group is the odd unitary group. In addition, the quadratic character $\chi_{\tilde{a}+\tilde{a}'}$ is nontrivial. Recall that for any s, s',

$$x_{\tilde{a}}(s)x_{\tilde{a}'}(s') = x_{\tilde{a}'}(s')x_{\tilde{a}}(s)x_{\tilde{a}+\tilde{a}'}(\pm ss'), \tag{5.1.5}$$

where $\pm$ depends on $x_{\tilde{a}}, x_{\tilde{a}'}, x_{\tilde{a}+\tilde{a}'}$, but not on s, s'. Define

$$\begin{aligned} x_{a+m}(r) = {} & x_{\tilde{a},\lceil e\tilde{a}(v_0-x)\rceil}\left(r \otimes u^{em-\lceil e\tilde{a}(v_0-x)\rceil}\right) \\ & \times x_{\tilde{a}',\lceil e\tilde{a}(v_0-x)\rceil}\left((-1)^{em-\lceil e\tilde{a}(v_0-x)\rceil} r \otimes u^{em-\lceil e\tilde{a}(v_0-x)\rceil}\right) \\ & \times x_{\tilde{a}+\tilde{a}',\lceil 2e\tilde{a}(v_0-x)\rceil}\left(\mp(-1)^{em-\lceil e\tilde{a}(v_0-x)\rceil}\frac{1}{2}r^2 \otimes u^{2em-\lceil 2e\tilde{a}(v_0-x)\rceil)}\right), \end{aligned}$$

where $\mp$ is the sign *opposite* to the sign $\pm$ in (5.1.5). Using (5.1.5), it is clear that $x_{a+m}(r)$ is again an element in (5.1.4).

We have completed the construction of (5.1.3). Note that they are independent of the choice of $x \in \sigma_Y$ by (5.1.2). In addition, over $0 \in C$ (i.e., by setting $y = 0$), the map (5.1.3) reduces to an isomorphism of $\mathbb{G}_a$ and the root subgroup of LG corresponding to $a + m$, as constructed in [PR08, 9.a,9.b]. This motivates us to define

Definition 5.1. Let $a + m$ be an affine root of G that satisfies $em \geq \lceil ea(v_0-x)\rceil$. The subgroup scheme $\mathcal{U}_{a+m} = x_{a+m}(\mathbb{G}_{a,C})$ is called root subgroup of $\mathcal{LG}$ corresponding to $a + m$.

Remark 5.1. (i) Note that in the above definition, the requirement $em \geq \lceil ea(v_0 - x)\rceil$ is necessary, as we need $r \otimes u^{em-\lceil ea(v_0-x)\rceil}$ to be an element in $R(\!(v-y)\!) \otimes_{k[v]} k[u]$. Note that in fact $\mathcal{U}_{a+m} \subset \mathcal{L}^+U_{a,\sigma_Y,C}$. If $f : \mathcal{G}' \to \mathcal{G}$ is a map of Bruhat-Tits group schemes, then $\mathcal{L}f x_{a+m} = x_{a+m}$ if x_{a+m} is defined for $\mathcal{G}'$ (and therefore for $\mathcal{G}$).

(ii) By taking the fibers $U_{a+m} = (\mathcal{U}_{a+m})_0 \subset (\mathcal{LG})_0 \cong LG$, we obtain the root subgroups of LG. Note that, however, as $R(\!(v)\!) \otimes_{k[v]} k[u] = R(\!(u)\!)$, we could drop the requirement $em \geq \lceil e\tilde{a}(v_0 - x)\rceil$, and $U_{a+m} \subset LG$ is defined for *all* affine roots of G . If we do not identify $A(G,S)$ with $\mathbb{X}_\bullet(S)_\mathbb{R}$ via v_0, we write them as U_α, where α is an affine root.

The following lemma about the root subgroups for (global) loop groups is the counterpart of a well-known fact about the root subgroups of Kac-Moody groups. To describe it, let us use the following notation. For a group (ind)-scheme $\mathcal{U}$ over C and $y \in C(R)$ and R-point, $\mathcal{U}(R)$ will denote the group of R-points of $\mathcal{U}$ over y.

Lemma 5.1. *Let R be a k-algebra and $y \in C(R)$. Let $a+m, b+n$ ($a \notin \mathbb{R}b$) be two affine roots of G such that $\mathcal{U}_{a+m}, \mathcal{U}_{b+n}$ are defined. Then the commutator $[\mathcal{U}_{a+m}(R), \mathcal{U}_{b+n}(R)]$ is contained in the group generated by $\mathcal{U}_{(pa+qb)+(pm+qn)}(R)$,*

where $p, q \in \mathbb{Z}_{>0}$ such that $(pa+qb)+(pm+qn)$ is also an affine root of G. (The groups $\mathcal{U}_{(pa+qb)+(pm+qn)}$ are clearly defined for $\mathcal{G}$.)

Proof. Let us define a subset $\Psi_{a,b} \subset \tilde{\Phi} = \Phi(H, T_H)$ as

$$\Psi_{a,b} = \{\tilde{a} \in \tilde{\Phi} \mid j(\tilde{a}) = pa+qb \text{ for } p, q \in \mathbb{Z}_{>0}\} = \bigcup_{pa+qb \in \Phi^{nd}, p,q>0} \eta(pa+qb),$$

where $j : \tilde{\Phi} \to \Phi$. For $\tilde{a} \in \Psi_{a,b}$ such that $j(\tilde{a}) = pa+qb$, let $k(\tilde{a}) = pm+qn$. Using the same notation as above, let us define

$$\tilde{\mathcal{U}}_{\tilde{a}+k(\tilde{a})} \subset \mathcal{L}\mathrm{Res}_{\tilde{C}/C}\tilde{U}_{a,\sigma_Y,\tilde{C}},$$

where $\tilde{U}_{a,\sigma_Y,\tilde{C}} = \tilde{U}_{\tilde{a},\lceil e\tilde{a}(v_0-x)\rceil,\tilde{C}}$, to be the group over C whose R-points over $y : \mathrm{Spec}R \to C$ are given by

$$\left\{x_{\tilde{a},\lceil e\tilde{a}(v_0-x)\rceil}\left(r \otimes u^{ek(\tilde{a})-\lceil e\tilde{a}(v_0-x)\rceil}\right), r \in R\right\} \subset \mathrm{Hom}_C(\mathrm{Spec}R, \mathcal{L}\left(\mathrm{Res}_{\tilde{C}/C}\tilde{U}_{a,\sigma_Y,\tilde{C}}\right)).$$

Let $p, q \in \mathbb{Z}_{>0}$, and let $\tilde{\mathcal{U}}_{\eta(pa+qb),pm+qn}(R)$ be the subgroup of $\mathcal{L}\mathrm{Res}_{\tilde{C}/C}(\tilde{\mathcal{G}})(R)$ generated by $\tilde{\mathcal{U}}_{\tilde{a}+k(\tilde{a})}(R)$ for those $\tilde{a} \in \eta(pa+qb) \subset \Psi_{a,b}$. Although one can show that $\tilde{\mathcal{U}}_{\eta(pa+qb),pm+qn}(R)$ is the group of R-points of some group scheme over C, it is enough for us to consider it as an abstract group here. Then

$$\mathcal{U}_{(pa+qb)+(pm+qn)}(R) = \tilde{\mathcal{U}}_{\eta(pa+qb),pm+qn}(R) \cap \mathcal{L}U_{pa+qb,\sigma_Y,C}(R).$$

Likewise, let $\tilde{\mathcal{U}}_{\Psi_{a,b},m,n}(R)$ be the group generated by $\tilde{\mathcal{U}}_{\tilde{a}+k(\tilde{a})}(R), \tilde{a} \in \Psi_{a,b}$. Recall that for the fixed Chevalley-Steinberg system $\{x_{\tilde{a}}, \tilde{a} \in \tilde{\Phi}\}$, and for two roots $\tilde{a}, \tilde{b} \in \tilde{\Phi}$, there exists $c(p,q) \in \mathbb{Z}$ for $p, q \in \mathbb{Z}_{>0}$ such that for any ring R and $r, s \in R$, the commutator $[x_{\tilde{a}}(r), x_{\tilde{b}}(s)]$ can be written as

$$[x_{\tilde{a}}(r), x_{\tilde{b}}(s)] = \prod_{p\tilde{a}+q\tilde{b} \in \tilde{\Phi}, p,q>0} x_{p\tilde{a}+q\tilde{b}}(c(p,q)r^p s^q)$$

([Con, Prop. 5.1.14]). Therefore, the commutator of $[\tilde{\mathcal{U}}_{\tilde{a}+k(\tilde{a})}(R), \tilde{\mathcal{U}}_{\tilde{b}+k(\tilde{b})}(R)]$ is contained in the group generated by $\tilde{\mathcal{U}}_{p\tilde{a}+q\tilde{b}+k(p\tilde{a}+q\tilde{b})}(R)$, where $p, q \in \mathbb{Z}_{>0}$ and $p\tilde{a}+q\tilde{b} \in \tilde{\Phi}$. Now we can apply [BT72, Prop. 6.1.6], with the pair $\tilde{\mathcal{U}}_{\tilde{a}+k(\tilde{a})}(R) \subset \tilde{U}_{\tilde{a}}$ playing the role $Y_a \subset U_a$ in loc. cit. Then we have

$$\tilde{\mathcal{U}}_{\Psi_{a,b},m,n}(R) \xleftarrow{\sim} \prod_{\tilde{a} \in \Psi_{a,b}} \tilde{\mathcal{U}}_{\tilde{a}+k(\tilde{a})}(R) \xrightarrow{\sim} \prod_{pa+qb \in \Phi^{nd}, p,q>0} \tilde{\mathcal{U}}_{\eta(pa+qb),pm+qn}(R).$$

Here the first isomorphism is obtained by setting Ψ^{red} in loc. cit. as $\Psi_{a,b}$, and the second isomorphism is obtained by setting $\Psi^{\mathrm{red}} = \eta(pa+qb)$ for all $pa+qb \in \Phi^{nd}, p, q > 0$.

Next, let $\mathcal{L}U_{(a,b)}(R)$ be the group generated by $\mathcal{L}U_{pa+qb,\sigma_Y,C}(R)$, $pa+qb \in \Phi, p, q \in \mathbb{Z}_{>0}$. Again by loc. cit., there is a bijection

$$\prod_{pa+qb\in\Phi^{nd},p,q>0} \mathcal{L}U_{pa+qb,\sigma_Y,C}(R) \xrightarrow{\sim} \mathcal{L}U_{(a,b)}(R).$$

Combining the above two isomorphisms, we thus obtain that

$$\begin{aligned}(\mathcal{U}_{a+m}(R),\mathcal{U}_{b+n}(R)) &\subset \tilde{\mathcal{U}}_{\Psi_{a,b},m,n}(R)\cap \mathcal{L}U_{(a,b)}(R)\\ &= \prod_{pa+qb\in\Phi^{nd},p,q>0} (\tilde{\mathcal{U}}_{\eta(pa+qb),pm+qn}(R)\cap \mathcal{L}U_{pa+qb,\sigma_Y,C}(R))\\ &= \prod_{pa+qb\in\Phi^{nd},p,q>0} \mathcal{U}_{(pa+qb)+(pm+qn)}(R). \qquad \square\end{aligned}$$

5.2. *Some affine charts of* $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}$. We introduce certain affine charts of $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}$ that turn out to be isomorphic to affine spaces. Let $\Lambda = W_0\mu \subset \mathbb{X}_\bullet(T)_\Gamma$ as before, and let $\lambda \in \Lambda$. Denote $\Phi_\lambda \subset \Phi(G,S)$ to be the subset

(5.2.1)

$$\Phi_\lambda = \left\{ a+m \mid (a,\lambda)>0, a+m \text{ affine root }, 0 \le m - \frac{\lceil ea(v_0-x)\rceil}{e} < (a,\lambda)\right\}.$$

By Lemma 9.1, this is a set with $(2\rho,\mu)$ elements. (Recall that $\mu \in \mathbb{X}_\bullet(T)^+$.) For each $a+m$, $\mathcal{U}_{a+m}$ is defined and is a subgroup of $\mathcal{L}^+\mathcal{G}$.

Let us endow a total order on the set Φ_λ as follows: First fix an order on $\{a \mid (a,\lambda)>0\}\cap\Phi^{nd}$. Then we can extend it to an order on $\{a \mid (a,\lambda)>0\}$ by requiring if $a, 2a \in \{a \mid (a,\lambda)>0\}$, then $a<2a<b$ for any $b \in \{a \mid (a,\lambda)>0\} \cap \Phi^{nd}$ such that $a<b$. Finally, we can give an order on Φ_λ by requiring $a+m<b+n$ if either $a<b$ or $a=b, m<n$.

Now, consider $\prod_{\Phi_\lambda}\mathcal{U}_{a+m} \to \mathcal{L}^+\mathcal{G}$ given by multiplication. Here and everywhere else the fiber products are over C. This is a closed immersion. In fact, let $\Psi \subset \Phi(G,S)$ be the image of the map $\Phi_\lambda \to \Phi(G,S)$ by taking the vector part of an affine root. Then $\Psi\cap(-\Psi)=\emptyset$. Therefore, $\prod_{a\in\Psi} U_{a,\sigma_Y,C} \to \mathcal{G}$ is a closed embedding. On the other hand, for $a \in \Psi$, it is not hard to see that the morphism $\prod_m \mathcal{U}_{a+m}\prod_{m'}\mathcal{U}_{2a+m'} \to \mathcal{L}^+U_{a,\sigma_Y,C}$ is a closed immersion. The claim follows.

Let us denote by $\mathcal{U}_{\Phi_\lambda} \subset \mathcal{L}^+\mathcal{G}$ the image of the above map. This is a closed subscheme of $\mathcal{L}^+\mathcal{G}$. We claim that $\mathcal{U}_{\Phi_\lambda}$ is indeed a closed subgroup scheme of $\mathcal{L}^+\mathcal{G}$.

Lemma 5.2. *Let R be a k-algebra and $y \in C(R)$. Then the R-points of $\mathcal{U}_{\Phi_\lambda}$ over y form the subgroup of $\mathcal{L}^+\mathcal{G}(R)$ generated by $\mathcal{U}_{a+m}(R), a+m\in\Phi_\lambda$.*

Proof. Let us denote the subgroup generated by $\mathcal{U}_{a+m}(R), a+m\in\Phi_\lambda$ by $\langle\mathcal{U}_{a+m}(R)\rangle$. By Lemma 5.1, the collection of groups $\{\mathcal{U}_{a+m}(R), a+m\in\Phi_\lambda\}$ satisfies the condition as required by [BT72, Lemma 6.1.7]. Then by loc. cit.,

we have

$$\mathcal{U}_{\Phi_\lambda}(R) = \prod_{a+m\in\Phi_\lambda} \mathcal{U}_{a+m}(R) \cong \langle \mathcal{U}_{a+m}(R)\rangle.$$

The lemma follows. □

Recall the section $s_\lambda : \tilde{C} \to \widetilde{\mathcal{LG}}$ as constructed in the paragraph after Proposition 3.4.[6] Consider

$$c_\lambda : \mathcal{U}_{\Phi_\lambda} \times_C \tilde{C} \to \overline{\mathrm{Gr}}_{\mathcal{G},\mu}, \quad g \mapsto g s_\lambda.$$

PROPOSITION 5.3. *The morphism c_λ is an open immersion.*

Proof. We first show that the stabilizer of s_λ in $\mathcal{U}_{\Phi_\lambda}$ is trivial. Recall that $\mathcal{LG}$ acts on $\mathrm{Gr}_{\mathcal{G}}$, and the stabilizer of the section $e : C \to \mathrm{Gr}_{\mathcal{G}}$ (defined by the trivial $\mathcal{G}$-torsor) is $\mathcal{L}^+\mathcal{G}$. Therefore, the stabilizer in $\widetilde{\mathcal{LG}}$ of the section s_λ is $s_\lambda(\widetilde{\mathcal{L}^+\mathcal{G}})s_\lambda^{-1}$. Therefore, it is enough to prove that $\widetilde{\mathcal{L}^+\mathcal{G}} \cap s_\lambda^{-1}(\mathcal{U}_{\Phi_\lambda} \times_C \tilde{C})s_\lambda$ is trivial, or equivalently,

$$(\mathcal{L}^+ U_{a,\sigma_Y,C} \times_C \tilde{C}) \cap s_\lambda^{-1}(\mathcal{U}_{a+m} \times_C \tilde{C})s_\lambda$$

is trivial for all $a + m \in \Phi_\lambda$.

Let us analyze the R-points of $s_\lambda^{-1}(\mathcal{U}_{a+m} \times_C \tilde{C})s_\lambda$ over $y : \mathrm{Spec}R \to \tilde{C}$. Recall that $s_\lambda(y)$ is given by the Γ-equivariant map

$$s_\lambda(y) : \mathrm{Spec}R(\!(v - y^e)\!) \otimes_{k[v]} k[u] \to T_H$$

such that for any weight ω of T_H, the composition $\omega s_\lambda(y)$ (which is determined by an invertible element in $R(\!(v-y^e)\!)\otimes_{k[v]} k[u]$) is $\prod_{i=1}^e (1\otimes\gamma^i(u) - y\otimes 1)^{(\lambda,\gamma^i\omega)}$. Note that for any $\tilde{a} \in \Phi(H, T_H)$ such that $j(\tilde{a}) = a \in \{a \mid (a,\lambda) > 0\}$,

$$\prod_{i=1}^{e} (1\otimes\gamma^i(u) - y\otimes 1)^{(-\lambda,\gamma^i(\tilde{a}))} u^{em - \lceil e\tilde{a}(v_0 - x)\rceil} \notin R[[v - y^e]] \otimes_{k[v]} k[u],$$

as $em - \lceil e\tilde{a}(v_0 - x)\rceil < e(\lambda, a)$, which implies

$$\mathcal{L}^+ U_{a,\sigma_Y,C} \times_C \tilde{C} \cap s_\lambda^{-1}(\mathcal{U}_{a+m} \times_C \tilde{C})s_\lambda$$

is the trivial subgroup for all $a + m \in \Phi_\lambda$.

Let us remark that using the construction in Section 5.3 there is a less computational way to show that the stabilizer of s_λ in $\mathcal{U}_{\Phi_\lambda}$ is trivial. Namely, note that the $\mathbb{G}_m$-action on $\mathcal{G}$ constructed in Section 5.3 also induces a $\mathbb{G}_m$-action on $\mathcal{U}_{\Phi_\lambda}$ and the map c_λ is equivariant. Therefore, it is enough to show that the stabilizer of $s_\lambda(\tilde{0})$ in $(\mathcal{U}_{\Phi_\lambda})_0$ is trivial. For this purpose, it is enough to do the above calculation for $y = 0$, which is easier. We will use this argument again in the proof of Lemma 6.3.

[6] More precisely, we need to choose a lifting of $\tilde{\lambda} \in \mathbb{X}_\bullet(T)$ of λ, but to simply the notation, we denote this lifting by λ.

Therefore, the stabilizer of s_λ in $\mathcal{U}_{\Phi_\lambda}$ is trivial. Then, c_λ is a monomorphism of irreducible varieties over k of the same dimension. We show that $U = c_\lambda(\mathcal{U}_{\Phi_\lambda} \times_C \tilde{C})$ is open. For simplicity, we write $\overline{\mathrm{Gr}}_{\mathcal{G},\mu} \to \tilde{C}$ as $f : X \to \tilde{C}$, and we write the group scheme $\mathcal{U}_{\Phi_\lambda} \times_C \tilde{C}$ by $\mathcal{U}$. As $\dim \mathcal{U} = \dim X = (2\rho, \mu)+1$, and U is constructible, it contains a nonempty open subset of X. Let $W \subset U$ be the maximal open subset of X contained in U. Then W is $\mathcal{U}$-stable. In particular, $W = \mathcal{U}(W \cap s_\lambda(\tilde{C}))$. We claim that $s_\lambda(\tilde{C}) \subset W$, which implies that $W = U$. Otherwise, $\tilde{C} - f(W \cap s_\lambda(\tilde{C}))$ consist of finitely many points $x_1, \ldots, x_n$. Then $W \cap f^{-1}(x_i) = \emptyset$. Note that $U_{x_i} := f^{-1}(x_i) \cap U$ is just the orbit of $s_\lambda(x_i))$ under $\mathcal{U}_{x_i}$ in $f^{-1}(x_i)$. As it contains a nonempty open subset of $f^{-1}(x_i)$, U_{x_i} is open in $f^{-1}(x_i)$. Let $Z_{x_i} = f^{-1}(x_i) - U_{x_i}$. This is a closed subset of $f^{-1}(x_i)$.

Let $Y = f^{-1}(f(W \cap s_\lambda(\tilde{C})))$. Then W is open dense in Y. Let $D = Y - W$, which is a proper closed subset of Y, and let $\bar{D}$ be its closure in X. Then $\bar{D}$ flat over $\tilde{C}$. Therefore, $\bar{D}_{x_i} := f^{-1}(x_i) \cap \bar{D}$ is a proper closed subset of $f^{-1}(x_i)$, of dimension strictly smaller than $(2\rho, \mu)$. Therefore, $U_{x_i} \not\subset \bar{D}_{x_i}$. Now, $X - \bar{D} - \bigcup_i Z_{x_i}$ is open, contained in U, and is strictly larger than W. This is a contradiction.

We therefore have proved that c_λ is a monomorphism that maps onto an open subset U of $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}$. Finally we show that $c_\lambda : \mathcal{U} \to U$ is an isomorphism, where U is regarded as an open subvariety of $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}$. We can check this étale locally on U. Note that the $\mathcal{L}^+\mathcal{G}$-torsor $\mathcal{L}\mathcal{G} \to \mathrm{Gr}_{\mathcal{G}}$ given by (3.1.4) can be trivialized in étale topology. (See the proof of [PZ13, Prop. 5.2] with obvious modifications.) Therefore, we can find a section $U \to \mathcal{L}\mathcal{G} \times_C \tilde{C}$ and can assume that the closed embedding $\mathcal{U} \to \widetilde{\mathcal{L}\mathcal{G}},\ g \mapsto g s_\lambda$ factors through $\mathcal{U} \to U \to \widetilde{\mathcal{L}\mathcal{G}}$. Therefore, $\mathcal{U} \to U$ is a closed embedding. But this map is surjective and U is integral, so it must be an isomorphism. □

In what follows, we denote the image of c_λ ($\lambda \in \Lambda$) by U_λ, so that U_λ is affine open in $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}$ that is smooth over $\tilde{C}$ (indeed an affine space over $\tilde{C}$). Note that

$$(U_\lambda)_{\tilde{0}} = (\mathcal{U}_{\Phi_\lambda})_{\tilde{0}} s_\lambda(0) = L^+\mathcal{G}_{\mathbf{a}} t_\lambda$$

is exactly the $L^+\mathcal{G}_{\mathbf{a}}$-orbit through t_λ in $\mathcal{F}\ell^Y$.

5.3. *A $\mathbb{G}_m$-action on $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}$.* Let $\mathcal{G}$ be a group scheme over C as in Section 3.2. Let $\underline{f} : \widetilde{\mathrm{Gr}}_{\mathcal{G}} \to \tilde{C}$ be the structural map. We construct a natural $\mathbb{G}_m$-action on $\widetilde{\mathrm{Gr}}_{\mathcal{G}}$ that lifts the natural action of $\mathbb{G}_m$ on $\tilde{C}$ via dilatations. In addition, each $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}$ is stable under this $\mathbb{G}_m$-action.

The construction of the $\mathbb{G}_m$-action on $\widetilde{\mathrm{Gr}}_{\mathcal{G}}$ is straightforward. Recall that the global coordinate on $\tilde{C}$ is u and on C is v and that the map $[e] : \tilde{C} \to C$ is given by $v \mapsto u^e$. Recall that an R-point of $\widetilde{\mathrm{Gr}}_{\mathcal{G}}$ is given by $u \mapsto y$ and a $\mathcal{G}$-torsor $\mathcal{E}$ on C_R that is trivialized over $C_R - \Gamma_{[e](y)}$. Let $r \in R^\times$ be an

R-point of $\mathbb{G}_m$. We need to construct a new $\mathcal{G}$-torsor on C_R, together with a trivialization over $C_R - \Gamma_{[e](ry)}$. Indeed, let $r^e : C_R \to C_R$ be given by $v \mapsto r^e v$. It maps $\Gamma_{[e](y)}$ to $\Gamma_{[e](ry)}$. Then the pullback of $\mathcal{E}$ along r^{-e} is an $(r^{-e})^*\mathcal{G}$-torsor on C_R, together with a trivialization on $C_R - \Gamma_{[e](ry)}$. Therefore, to complete the construction, it is enough to show that $(r^{-e})^*\mathcal{G}$ is canonically isomorphic to $\mathcal{G}$ as group schemes over C_R. Let us remark that the same construction will give an action of $\mathbb{G}_m$ on $\widetilde{\mathcal{L}\mathcal{G}}$ (resp. $\widetilde{\mathcal{L}^+\mathcal{G}}$), compatible with the dilatations on $\tilde{C}$. Furthermore, the action of $\widetilde{\mathcal{L}\mathcal{G}}$ (resp. $\widetilde{\mathcal{L}^+\mathcal{G}}$) on $\widetilde{\mathrm{Gr}}_{\mathcal{G}}$ is $\mathbb{G}_m$-equivariant.

Let us define the action of $\mathbb{G}_m$ on $C = \mathrm{Spec} k[v]$ via $(r, v) \mapsto r^e v$. Observe that $\mu_e \subset \mathbb{G}_m$ acts trivially on C via this action.

LEMMA 5.4. *Given the action of $\mathbb{G}_m$ on C as above, there is a natural action of $\mathbb{G}_m$ on $\mathcal{G}$ such that $\mathcal{G} \to C$ is $\mathbb{G}_m$-equivariant.*

Remark 5.2. However, the natural dilatation on C does not lift to $\mathcal{G}$.

Proof. As has been explained in Section 5.1, there is a group scheme $\tilde{G}$ over $\tilde{C}$ satisfying $\tilde{G}|_{\tilde{C}^\circ} = H \times \tilde{C}^\circ$, and $\tilde{\mathcal{G}}_{\tilde{\mathcal{O}}_0}$ is a parahoric group scheme of $H \otimes \tilde{F}_0$, given by a point $x \in A(H, T_H)^\Gamma$, such that $\mathcal{G}$ is the neutral connected component of $(\mathrm{Res}_{\tilde{C}/C}\tilde{\mathcal{G}})^\Gamma$. To prove the proposition, it is enough to prove that there is a natural $\mathbb{G}_m$ action on $\tilde{\mathcal{G}}$, compatible with the rotation on $\tilde{C}$. In addition, this $\mathbb{G}_m$-action should be compatible with the action of Γ on $\tilde{\mathcal{G}}$.

Let $m, p : \mathbb{G}_m \times \tilde{C} \to \tilde{C}$ be the action map and the projection map respectively. We need show that there is an isomorphism of group schemes $p^*\tilde{\mathcal{G}} \cong m^*\tilde{\mathcal{G}}$ over $\mathbb{G}_m \times \tilde{C}$ satisfying the usual compatibility conditions. Since $\mathbb{G}_m$ naturally acts on $\tilde{\mathcal{G}}|_{\tilde{C}^\circ} = H \times \tilde{C}^\circ$ by acting via rotation on the second factor, there is a natural isomorphism

$$c : p^*\tilde{\mathcal{G}}|_{\mathbb{G}_m \times \tilde{C}^\circ} \cong m^*\tilde{\mathcal{G}}|_{\mathbb{G}_m \times \tilde{C}^\circ}$$

that is compatible with the Γ-actions. We need to show that this uniquely extends to an isomorphism over $\mathbb{G}_m \times \tilde{C}$. Then it will be automatically compatible with the Γ-actions. Indeed, the uniqueness is clear since $p^*\tilde{\mathcal{G}}$ (resp. $m^*\tilde{\mathcal{G}}$) is flat over $\mathbb{G}_m \times \tilde{C}$, so that $\mathcal{O}_{p^*\tilde{\mathcal{G}}} \subset \mathcal{O}_{p^*\tilde{\mathcal{G}}}[u^{-1}]$ (resp. $\mathcal{O}_{m^*\tilde{\mathcal{G}}} \subset \mathcal{O}_{m^*\tilde{\mathcal{G}}}[u^{-1}]$). We need to prove that the map $c : \mathcal{O}_{m^*\tilde{\mathcal{G}}}[u^{-1}] \to \mathcal{O}_{p^*\tilde{\mathcal{G}}}[u^{-1}]$ indeed sends $\mathcal{O}_{m^*\tilde{\mathcal{G}}}$ to $\mathcal{O}_{p^*\tilde{\mathcal{G}}}$. But this can be checked over each closed point of $\mathbb{G}_m$. Therefore, it remains to prove that for every $r \in \mathbb{G}_m(k)$, the isomorphism of $r^*\tilde{\mathcal{G}}|_{\tilde{C}^\circ} \cong \tilde{\mathcal{G}}|_{\tilde{C}^\circ}$ extends to an isomorphism $r^*\tilde{\mathcal{G}} \cong \tilde{\mathcal{G}}$. We can replace $\tilde{C}$ by $\mathcal{O}_{\tilde{0}}$. By [BT84, Prop. 1.7.6], it is enough to prove that the isomorphism $r : \tilde{\mathcal{G}}(F_{\tilde{0}}) \to \tilde{\mathcal{G}}(F_{\tilde{0}})$ induces an isomorphism $\tilde{\mathcal{G}}(\mathcal{O}_{\tilde{0}}) \to \tilde{\mathcal{G}}(\mathcal{O}_{\tilde{0}})$. But it is clear that each root subgroup of $L(H \otimes \tilde{F}_0)$ with respect to $(H \otimes \tilde{F}_0, T_H \otimes \tilde{F}_0)$ as constructed in Section 5.1 (see Remark 5.1(ii)) is invariant under this $\mathbb{G}_m$-action. Therefore, for any $x \in A(H, T_H)$, the corresponding parahoric group of $H \otimes \tilde{F}_0$ is invariant under this $\mathbb{G}_m$-action. $\square$

It remains to show that each $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}$ is invariant under this $\mathbb{G}_m$-action. It is enough to show that the section $s_\mu : \tilde{C}^\circ \to \widetilde{\mathrm{Gr}}_{\mathcal{T}} \subset \widetilde{\mathrm{Gr}}_{\mathcal{G}}$ is invariant under this $\mathbb{G}_m$-action. Recall that $s_\mu : \tilde{C}^\circ \to \mathrm{Gr}_{\mathcal{T}} \times_C \tilde{C}^\circ \cong \mathrm{Gr}_{T_H \times \tilde{C}^\circ}$ is given by the T_H-bundle $\mathcal{O}_{\tilde{C}^\circ}(\mu\Delta)$ with its canonical trivialization away from Δ (see Lemma 3.5). From this moduli interpretation, it is clear that s_μ is $\mathbb{G}_m$-invariant.

By restriction to $(\widetilde{\mathrm{Gr}}_{\mathcal{G}})_{\tilde{0}} \cong \mathcal{F}\ell^Y$, we obtain an action of $\mathbb{G}_m$ on $\mathcal{F}\ell^Y$ (and therefore on $\mathcal{F}\ell^Y_{\mathrm{sc}}$). As is shown in [PR08], the affine flag variety $\mathcal{F}\ell^Y_{\mathrm{sc}}$ coincides with the affine flag variety in the Kac-Moody setting. Under this identification, the above $\mathbb{G}_m$-action on $\mathcal{F}\ell^Y_{\mathrm{sc}}$ should correspond to the action of the extra one-dimensional torus (usually called the rotation torus) in the maximal torus of the affine Kac-Moody group. We do not make the statement precise. Instead, we mention

LEMMA 5.5. *Each Schubert variety in $\mathcal{F}\ell^Y$ is invariant under this action of $\mathbb{G}_m$ on $\mathcal{F}\ell^Y$.*

Proof. Since $\mathbb{G}_m$ acts on $\mathcal{G}$, it acts on $L^+\mathcal{G}_{\mathcal{O}_0}$. Clearly, it also acts on $L^+\mathcal{T}_{\mathcal{O}_0}$, and therefore it acts on the normalizer $N_{G(F_0)}(\mathcal{T}(\mathcal{O}_0))$ of $\mathcal{T}(\mathcal{O}_0)$ in $\mathcal{G}(F_0)$. Since $N_{G(F_0)}(\mathcal{T}(\mathcal{O}_0))/\mathcal{T}(\mathcal{O}_0) \cong \widetilde{W}$ is discrete, the induced $\mathbb{G}_m$-action fixes every element. The lemma follows. □

6. **Proofs I: Frobenius splitting of global Schubert varieties**

In this section, we prove Theorem 3.9 assuming Theorem 3.8. We also deduce Theorem 1 from Theorem 2.

6.1. *Factorization of affine Demazure modules.* In this subsection, we show that Theorem 2 implies Theorem 1. This proof is essentially contained in [Zhu09]. However, we repeat some arguments since they serve as a prototype for the following subsections.

Let H be a split Chevalley group over k such that H_{der} is almost simple, simply-connected, as assumed in Section 2.1. Let Gr_H be the affine Grassmannian of H and $\mathcal{L}_b$ be the line bundle on $(\mathrm{Gr}_H)_{\mathrm{red}}$ (the reduced ind-subscheme of Gr_H) that restricts to the ample generator of the Picard group of each of connected component (which is isomorphic to $\mathrm{Gr}_{H_{\mathrm{der}}}$). We have the following two assertions.

LEMMA 6.1. *Let $\mu \in \mathbb{X}_\bullet(T_H)$ be a minuscule coweight, so that $\overline{\mathrm{Gr}}_\mu \cong X(\mu) = H/P(\mu)$, where $P(\mu)$ is the maximal parabolic subgroup corresponding to μ. Then the restriction of $\mathcal{L}_b$ to $\overline{\mathrm{Gr}}_\mu$ is the ample generator of the Picard group of $X(\mu)$.*

Proof. Let us use the following notation. For ν a dominant weight of $P(\mu)$, let $\mathcal{L}(\nu)$ be the line bundle on $H/P(\mu)$ such that $\Gamma(H/P(\mu), \mathcal{L}(\nu))^*$ is the Weyl module of H of highest weight ν.

First assume that $\operatorname{char} k = 0$. Let us fix a normalized invariant form $(\cdot,\cdot)_{\mathrm{norm}}$ on $\mathbb{X}_\bullet(T_H)$ so that the square of the length of short coroots is two. Note that this invariant form may not be unique if H is not semi-simple. For a coweight $\mu \in \mathbb{X}_\bullet(T_H)$ of T_H, let μ^* be the image of μ under $\mathbb{X}_\bullet(T_H) \to \mathbb{X}^\bullet(T_H)$ induced by this form. In other words, $(\mu^*, \lambda) = (\mu, \lambda)_{\mathrm{norm}}$. Let $t_\mu \in \mathrm{Gr}_{T_H}(k) \subset \mathrm{Gr}_H(k)$ be the corresponding point as in the proof of Lemma 3.5. Now assume that μ is dominant. For every positive root $\tilde{a}$ of H, corresponding to a copy of $\mathrm{SL}_2 \subset H$, let $\mathrm{SL}_2 t_\mu \subset H t_\mu \cong H/P(\mu)$ be the corresponding rational curve. Then according to [Zhu09, Lemma 2.2.2], the restriction of $\mathcal{L}_b$ to this rational curve has degree $\frac{2(\mu,\tilde{a})}{(\tilde{a},\tilde{a})}$. Therefore, the restriction of $\mathcal{L}_b$ to $Ht_\mu \cong H/P(\mu)$ is isomorphic to $\mathcal{L}(\mu^*)$. Note that for μ minuscule, $(\mu,\mu)_{\mathrm{norm}} = 2$, and therefore μ^* is the corresponding minuscule weight. The lemma follows in this case.

To prove the lemma in the case $\operatorname{char} k > 0$, observe that everything is defined over $\mathbb{Z}$. (See [Fal03], where it is proven that the Schubert varieties are defined over $\mathbb{Z}$ and commute with base change.) It is well known that $\mathrm{Pic}(H/P(\mu)_{\mathbb{Z}}) \cong \mathrm{Pic}(H/P(\mu)_k) \cong \mathbb{Z}$. The lemma follows. □

The following proposition is essentially equivalent to [FL06, Th. 1], whose proof is of combinatorial nature. Here we reproduce a proof given in [Zhu09, Th. 1.2.2].

PROPOSITION 6.2. *Let $\mathcal{L}$ be a line bundle on $(\mathrm{Gr}_H)_{\mathrm{red}}$ whose restriction to each connected component of Gr_H has the same central charge. Then*

$$H^0(\overline{\mathrm{Gr}}_{\mu+\lambda}, \mathcal{L}) \cong H^0(\overline{\mathrm{Gr}}_\mu, \mathcal{L}) \otimes H^0(\overline{\mathrm{Gr}}_\lambda, \mathcal{L}).$$

Proof. Recall that $H^1(\overline{\mathrm{Gr}}_\mu, \mathcal{L}) = 0$ since $\overline{\mathrm{Gr}}_\mu$ is Frobenius split and $\mathcal{L}$ is ample. Therefore, it is enough to prove the proposition for $\mathcal{L}^n, \mathcal{L}^{2n}, \dots$ and some $n \geq 1$. Therefore, by replacing $\mathcal{L}$ by $\mathcal{L}^n$, we can assume that the central charge of $\mathcal{L}$ is $2h^\vee$; i.e., $\mathcal{L} = \mathcal{L}_b^{2h^\vee}$. Then $\mathcal{L}^n$ is the pullback of the n-tensor of the determinant line bundle $\mathcal{L}^n_{\det}$ of $\mathrm{Gr}_{\mathrm{GL}(\mathrm{Lie}\, H)}$ along $i : \mathrm{Gr}_H \to \mathrm{Gr}_{\mathrm{GL}(\mathrm{Lie}\, H)}$, as has been discussed in the proof of Theorem 4.3. Let us choose a complete curve (e.g., $\bar{C} = C \cup \{\infty\}$) and let Bun_H be the moduli stack of H-bundles on the curve. Then we know that $\mathcal{L}$ is the pullback along $\mathrm{Gr}_H \to \mathrm{Bun}_H$ of a line bundle on Bun_H (which in turn is the pullback along $\mathrm{Bun}_H \to \mathrm{Bun}_{\mathrm{GL}(\mathrm{Lie}\, H)}$ of the determinant line bundle on $\mathrm{Bun}_{\mathrm{GL}(\mathrm{Lie}\, H)}$). Denote this line bundle on Bun_H as ω^{-1}. (In fact, this is the anti-canonical bundle of Bun_H.)

Consider the convolution affine Grassmannian $\mathrm{Gr}^{\mathrm{Conv}}_{H\times C}$ over C, defined as

$$\mathrm{Gr}^{\mathrm{Conv}}_{H\times C}(R) = \left\{ (y, \mathcal{E}, \mathcal{E}', \beta, \beta') \;\middle|\; \begin{array}{l} y \in C(R),\ \mathcal{E}, \mathcal{E}' \text{ are two } H\text{-torsors on } C_R, \\ \beta : \mathcal{E}|_{C_R - \Gamma_y} \cong \mathcal{E}^0|_{C_R - \Gamma_y} \text{ is a trivialization,} \\ \text{and } \beta' : \mathcal{E}'|_{(C-\{0\})_R} \cong \mathcal{E}|_{(C-\{0\})_R}. \end{array} \right\}.$$

This is an ind-scheme formally smooth over C, and by the same argument as in [Gai01, Prop. 5], we have

$$\mathrm{Gr}^{\mathrm{Conv}}_{H\times C}|_{C^\circ} \cong \mathrm{Gr}_{H\times C^\circ} \times \mathrm{Gr}_H, \quad (\mathrm{Gr}^{\mathrm{Conv}}_{H\times C})_0 \cong \mathrm{Gr}_H \tilde{\times} \mathrm{Gr}_H,$$

where $\mathrm{Gr}_H \tilde{\times} \mathrm{Gr}_H := LH \times^{L^+H} \mathrm{Gr}_H$ is the local convolution Grassmannian. In addition, $\mathrm{Gr}^{\mathrm{Conv}}_{H\times C}$ is a fibration over $\mathrm{Gr}_{H\times C}$ by sending $(y, \mathcal{E}, \mathcal{E}', \beta, \beta')$ to $(y, \mathcal{E}, \beta)$, with the fibers isomorphic to Gr_H.

Now $\overline{\mathrm{Gr}}_\mu \times \overline{\mathrm{Gr}}_\lambda$ extends naturally to a closed variety of $\mathrm{Gr}_{H\times C^\circ} \times \mathrm{Gr}_H$. The closure of this variety in $\mathrm{Gr}^{\mathrm{Conv}}_{H\times C}$ is denoted as $\overline{\mathrm{Gr}}^{\mathrm{Conv}}_{H\times C,\mu,\lambda}$. As is proven in [Zhu09, 1.2.2], for $y \neq 0$, $(\overline{\mathrm{Gr}}^{\mathrm{Conv}}_{H\times C,\mu,\lambda})_y \cong \overline{\mathrm{Gr}}_\mu \times \overline{\mathrm{Gr}}_\lambda$ and $(\overline{\mathrm{Gr}}^{\mathrm{Conv}}_{H\times C,\mu,\lambda})_0 \cong \overline{\mathrm{Gr}}_\mu \tilde{\times} \overline{\mathrm{Gr}}_\lambda$, where $\overline{\mathrm{Gr}}_\mu \tilde{\times} \overline{\mathrm{Gr}}_\lambda$ is the "twisted product" of $\overline{\mathrm{Gr}}_\mu$ and $\overline{\mathrm{Gr}}_\lambda$. (See loc. cit. or (6.2.6) below for the precise definition.)

Let $h : \mathrm{Gr}^{\mathrm{Conv}}_{H\times C} \to \mathrm{Bun}_H$ be the map sending $(y, \mathcal{E}, \mathcal{E}', \beta, \beta')$ to $\mathcal{E}'$. Then as explained in [Zhu09, 1.2.2], $h^*(\omega^{-1})^n$, when restricted to $\mathrm{Gr}^{\mathrm{Conv}}_{H\times C}|_{C^\circ}$, is isomorphic to $\mathcal{L}^n \boxtimes \mathcal{L}^n$, whereas over $(\mathrm{Gr}^{\mathrm{Conv}}_{H\times C})_0$, it is isomorphic to $m^*\mathcal{L}^n$, where $m : \mathrm{Gr}_H \tilde{\times} \mathrm{Gr}_H \to \mathrm{Gr}_H$ is the natural convolution map (which is obtained from multiplication in the loop group). Therefore,

$$H^0(\overline{\mathrm{Gr}}_\mu, \mathcal{L}^n) \otimes H^0(\overline{\mathrm{Gr}}_\lambda, \mathcal{L}^n) \cong H^0(\overline{\mathrm{Gr}}_\mu \tilde{\times} \overline{\mathrm{Gr}}_\lambda, m^*\mathcal{L}^n) \cong H^0(\overline{\mathrm{Gr}}_{\mu+\lambda}, \mathcal{L}^n).$$

The last isomorphism is due to the fact that $\mathcal{O}_{\overline{\mathrm{Gr}}_{\mu+\lambda}} \cong m_*\mathcal{O}_{\overline{\mathrm{Gr}}_\mu \tilde{\times} \overline{\mathrm{Gr}}_\lambda}$. □

Clearly, Lemma 6.1 and Proposition 6.2 together with Theorem 2 will imply Theorem 1.

6.2. *Reduction of Theorem* 3.9 *to Theorem* 6.8. In this subsection, we prove Theorem 3.9, assuming Theorem 3.8. The key ingredient is the Frobenius splitting of varieties in characteristic p.

We begin with introducing more ind-schemes. Let $\mathcal{G}$ be the group scheme over C as in Section 3.2. In particular, $\mathcal{G}_{\mathcal{O}_0} = \mathcal{G}_{\sigma_Y}$. Let $\mathrm{Gr}^{\mathrm{BD}}_{\mathcal{G}}$ be the Beilinson-Drinfeld affine Grassmannian for $\mathcal{G}$ over C. That is, for every k-algebra R,

$$\mathrm{Gr}^{\mathrm{BD}}_{\mathcal{G}}(R) = \left\{ (y, \mathcal{E}, \beta) \;\middle|\; \begin{array}{l} y \in C(R), \mathcal{E} \text{ is a } \mathcal{G}\text{-torsor on } C_R, \text{ and} \\ \beta : \mathcal{E}|_{C^\circ_R - \Gamma_y} \cong \mathcal{E}^0|_{C^\circ_R - \Gamma_y} \text{ is a trivialization} \end{array} \right\}. \tag{6.2.1}$$

This is a formally smooth ind-scheme ind-proper over C. (The ind-representability of $\mathrm{Gr}^{\mathrm{BD}}_{\mathcal{G}}$ is explained in the proof of Theorem 10.5 of [PZ13].) Again, by the same argument as in [Gai01, Prop. 5], we have

$$\mathrm{Gr}^{\mathrm{BD}}_{\mathcal{G}}|_{C^\circ} \cong \mathrm{Gr}_{\mathcal{G}}|_{C^\circ} \times (\mathrm{Gr}_{\mathcal{G}})_0, \quad (\mathrm{Gr}^{\mathrm{BD}}_{\mathcal{G}})_0 \cong (\mathrm{Gr}_{\mathcal{G}})_0 \cong \mathcal{F}\ell^Y.$$

We also need the convolution affine Grassmannian $\mathrm{Gr}_{\mathcal{G}}^{\mathrm{Conv}}$. The functor it represents is as follows. Let R be a k-algebra; then

(6.2.2)

$$\mathrm{Gr}_{\mathcal{G}}^{\mathrm{Conv}}(R)=\left\{(y,\mathcal{E},\mathcal{E}',\beta,\beta')\;\middle|\;\begin{array}{l} y\in C(R),\ \mathcal{E},\mathcal{E}' \text{ are two } \mathcal{G}\text{-torsors on } C_R,\\ \beta:\mathcal{E}|_{C_R-\Gamma_y}\cong\mathcal{E}^0|_{C_R-\Gamma_y} \text{ is a trivialization,}\\ \text{and } \beta':\mathcal{E}'|_{C_R^\circ}\cong\mathcal{E}|_{C_R^\circ}\end{array}\right\}.$$

The ind-representibility of $\mathrm{Gr}_{\mathcal{G}}^{\mathrm{Conv}}$ can be seen from another construction of $\mathrm{Gr}_{\mathcal{G}}^{\mathrm{Conv}}$. Namely, there is a $L^+\mathcal{G}_{\mathcal{O}_0}$-torsor $\mathrm{Gr}_{\mathcal{G},\underline{0}}$ over $\mathrm{Gr}_{\mathcal{G}}$ whose R-points classify

$$\mathrm{Gr}_{\mathcal{G},\underline{0}}(R)=\left\{(y,\mathcal{E},\beta,\gamma)\;\middle|\;\begin{array}{l}(y,\mathcal{E},\beta)\in\mathrm{Gr}_{\mathcal{G}}(R), \text{ and a trivialization}\\ \gamma:\mathcal{E}|_{\hat{\Gamma}_0}\cong\mathcal{E}^0|_{\hat{\Gamma}_0}\end{array}\right\},\tag{6.2.3}$$

where $\Gamma_0=\{0\}\times\mathrm{Spec}R$ is the graph of the constant map $\mathrm{Spec}R\to\{0\}$. Then

$$\mathrm{Gr}_{\mathcal{G}}^{\mathrm{Conv}}\cong\mathrm{Gr}_{\mathcal{G},\underline{0}}\times^{L^+\mathcal{G}_{\mathcal{O}_0}}\mathcal{F}\ell^Y.$$

The projection

$$\pi:\mathrm{Gr}_{\mathcal{G}}^{\mathrm{Conv}}\to\mathrm{Gr}_{\mathcal{G}}$$

sends $(y,\mathcal{E},\mathcal{E}',\beta,\beta')$ to $(y,\mathcal{E},\beta)$.

There is a natural map

$$m:\mathrm{Gr}_{\mathcal{G}}^{\mathrm{Conv}}\to\mathrm{Gr}_{\mathcal{G}}^{\mathrm{BD}}\tag{6.2.4}$$

sending $(y,\mathcal{E},\mathcal{E}',\beta,\beta')$ to $(y,\mathcal{E}',\beta\circ\beta')$. This is a morphism over C, which is an isomorphism over $C-\{0\}$. Over 0, this morphism is the local convolution morphism

$$m:\mathcal{F}\ell^Y\tilde{\times}\mathcal{F}\ell^Y:=LG\times^{L^+\mathcal{G}_{\mathcal{O}_0}}\mathcal{F}\ell^Y\to\mathcal{F}\ell^Y,\tag{6.2.5}$$

given by the natural multiplication of the loop group.

In addition, there is a section

$$z:\mathrm{Gr}_{\mathcal{G}}\to\mathrm{Gr}_{\mathcal{G}}^{\mathrm{Conv}}$$

given by sending $(y,\mathcal{E},\beta)$ to $(y,\mathcal{E},\mathcal{E},\beta,\mathrm{id})$. Therefore, via z (resp. $m\circ z$), $\mathrm{Gr}_{\mathcal{G}}$ is realized as a closed subscheme of $\mathrm{Gr}_{\mathcal{G}}^{\mathrm{Conv}}$ (resp. $\mathrm{Gr}_{\mathcal{G}}^{\mathrm{BD}}$).

Let $w\in W^Y\backslash\widetilde{W}/W^Y$ be an element in the extended affine Weyl group and let $\mathcal{F}\ell_w^Y$ denote the corresponding Schubert variety in $\mathcal{F}\ell^Y$. Then $\overline{\mathrm{Gr}}_\mu\times\mathcal{F}\ell_w^Y\subset\mathrm{Gr}_H\times\mathcal{F}\ell^Y$ extends to a variety

$$\overline{\mathrm{Gr}}_{\mathcal{G},\mu}|_{\tilde{C}^\circ}\times\mathcal{F}\ell_w^Y\subset(\mathrm{Gr}_{\mathcal{G}}\times_C\tilde{C}^\circ)\times\mathcal{F}\ell^Y\cong\mathrm{Gr}_{\mathcal{G}}^{\mathrm{BD}}\times_C\tilde{C}^\circ\cong\mathrm{Gr}_{\mathcal{G}}^{\mathrm{Conv}}\times_C\tilde{C}^\circ.$$

Let $\overline{\mathrm{Gr}}_{\mathcal{G},\mu,w}^{\mathrm{BD}}$ denote its flat closure in $\mathrm{Gr}_{\mathcal{G}}^{\mathrm{BD}}\times_C\tilde{C}$ and $\overline{\mathrm{Gr}}_{\mathcal{G},\mu,w}^{\mathrm{Conv}}$ denote its flat closure in $\mathrm{Gr}_{\mathcal{G}}^{\mathrm{Conv}}\times_C\tilde{C}$. Then $\overline{\mathrm{Gr}}_{\mathcal{G},\mu,w}^{\mathrm{Conv}}$ maps to $\overline{\mathrm{Gr}}_{\mathcal{G},\mu,w}^{\mathrm{BD}}$ via m.

We have the following key observation.

LEMMA 6.3. *Suppose that ν is sufficiently large. Then the fiber $(\overline{\mathrm{Gr}}^{\mathrm{BD}}_{\mathcal{G},\mu,\nu})_{\tilde{0}}$ is irreducible and generically reduced.*

Proof. We first show the irreducibility. Clearly, the map $m_{\tilde{0}} : (\overline{\mathrm{Gr}}^{\mathrm{Conv}}_{\mathcal{G},\mu,\nu})_{\tilde{0}} \to (\overline{\mathrm{Gr}}^{\mathrm{BD}}_{\mathcal{G},\mu,\nu})_{\tilde{0}}$ is dominant. From the second definition of $\mathrm{Gr}^{\mathrm{Conv}}_{\mathcal{G}}$, $(\overline{\mathrm{Gr}}^{\mathrm{Conv}}_{\mathcal{G},\mu,\nu})_{\tilde{0}}$ is a fibration over $(\overline{\mathrm{Gr}}_{\mathcal{G},\mu})_{\tilde{0}}$, with fibers isomorphic to $\mathcal{F}\ell^Y_\nu$. Therefore, by Theorem 3.8, the underlying reduced subschemes of irreducible components of $(\overline{\mathrm{Gr}}^{\mathrm{Conv}}_{\mathcal{G},\mu,\nu})_{\tilde{0}}$ are just

$$\mathcal{F}\ell^Y_\lambda \tilde{\times} \mathcal{F}\ell^Y_\nu, \quad \lambda \in \Lambda.$$

Here and in the sequel we use the following notation. Let S_1, S_2 be two subschemes of $\mathcal{F}\ell^Y$, and assume that S_2 is $L^+\mathcal{G}_{\mathcal{O}_0}$ stable. Then we denote

$$S_1 \tilde{\times} S_2 := \widetilde{S_1} \times^{L^+\mathcal{G}_{\mathcal{O}_0}} S_2, \tag{6.2.6}$$

where $\widetilde{S_1}$ is the preimage of S_1 under $L\mathcal{G}_{\mathcal{O}_0} \to \mathcal{F}\ell^Y$.

Therefore, the underlying reduced subscheme of each irreducible component of

$$(\overline{\mathrm{Gr}}^{\mathrm{BD}}_{\mathcal{G},\mu,\nu})_{\tilde{0}} \subset \mathcal{F}\ell^Y$$

is contained in one of $m_{\tilde{0}}(\mathcal{F}\ell^Y_\lambda \tilde{\times} \mathcal{F}\ell^Y_\nu), \lambda \in \Lambda$. Observe that if $\lambda \in \Lambda$ is not dominant, for ν sufficiently dominant so that $\lambda+\nu$ is dominant, by Lemma 9.1, we have

$$\ell(t_{\nu+\lambda}) = (2\rho, \nu+\lambda) < (2\rho,\nu) + (2\rho,\mu) = \ell(t_\nu) + \ell(t_\lambda).$$

However, by flatness, all the irreducible components of $(\overline{\mathrm{Gr}}^{\mathrm{BD}}_{\mathcal{G},\mu,\nu})_{\tilde{0}}$ have dimension $\ell(t_\mu) + \ell(t_\nu)$. This implies that $(\overline{\mathrm{Gr}}^{\mathrm{BD}}_{\mathcal{G},\mu,\nu})_{\tilde{0}}$ has only one irreducible component whose underlying reduced subscheme is $m_{\tilde{0}}(\mathcal{F}\ell^Y_\mu \tilde{\times} \mathcal{F}\ell^Y_\nu) = \mathcal{F}\ell^Y_{\mu+\nu}$.

Next, we show that $(\overline{\mathrm{Gr}}^{\mathrm{BD}}_{\mathcal{G},\mu,\nu})_{\tilde{0}}$ is generically reduced. Let $t_\nu = s_\nu(\tilde{0}) \in \mathcal{F}\ell^Y(k)$ be the point corresponding to ν. Let $s_{\mu,\nu}$ denote the section of $\mathrm{Gr}^{\mathrm{BD}}_{\mathcal{G}} \times_C \tilde{C}$ over $\tilde{C}$ given by the closure of $s_\mu|_{\tilde{C}^\circ} \times \{t_\nu\} : \tilde{C}^\circ \to \widetilde{\mathrm{Gr}}_{\mathcal{G}}|_{\tilde{C}^\circ} \times \mathcal{F}\ell^Y$. Using a similar construction as in Section 5.2, we will exhibit a smooth affine chart around $t_{\mu+\nu} = s_{\mu,\nu}(\tilde{0})$ in $\overline{\mathrm{Gr}}^{\mathrm{BD}}_{\mathcal{G},\mu,\nu}$. We consider the group ind-scheme $\mathcal{L}^{\mathrm{pol}}\mathcal{G}$ over C whose R-points are pairs (y,γ), where $y \in C(R)$ and $\gamma : C_R \to \mathcal{G}$. This is a constant group ind-scheme over C whose fibers are just the group of (polynomial) maps from $C \to \mathcal{G}$. Note $\mathcal{L}^{\mathrm{pol}}\mathcal{G}$ acts on $\mathrm{Gr}^{\mathrm{BD}}_{\mathcal{G}}$ by changing the trivialization β.

Note that the root group $\mathcal{U}_{a+m}$ constructed in Section 5.1 can be regarded as subgroups of $\mathcal{L}^{\mathrm{pol}}\mathcal{G}$. In addition, for λ, $\mathcal{U}_{\Phi_\lambda}$ as constructed in Section 5.2 is a natural closed subgroup scheme of $\mathcal{L}^{\mathrm{pol}}\mathcal{G}$. Now we claim that the morphism

$$\mathcal{U}_{\Phi_{\mu+\nu}} \times_C \tilde{C} \to \mathrm{Gr}^{\mathrm{BD}}_{\mathcal{G}} \times_C \tilde{C}, \quad g \mapsto g s_{\mu,\nu} \tag{6.2.7}$$

induces an open immersion $\mathcal{U}_{\Phi_{\mu+\nu}} \times_C \tilde{C} \to \overline{\mathrm{Gr}}^{\mathrm{BD}}_{\mathcal{G},\mu,\nu}$. This will complete the proof of the lemma.

For this purpose, we consider the Beilinson-Drinfeld version of the jet (resp. loop) group $\mathcal{L}^+\mathcal{G}^{\mathrm{BD}}$ (resp. $\mathcal{L}\mathcal{G}^{\mathrm{BD}}$) whose R-points are pairs (y,γ), where $y \in C(R)$, and γ is a section of $\mathcal{G}$ over $\widehat{\Gamma_y \cup \Gamma_0}$ (resp. over $(\widehat{\Gamma_y \cup \Gamma_0})^\circ$). They are represented by ind-schemes formally smooth over C. We have

$$\mathcal{L}^+\mathcal{G}^{\mathrm{BD}}|_{C^\circ} \cong \mathcal{L}^+\mathcal{G}|_{C^\circ} \times L^+\mathcal{G}_{\sigma_Y}, \quad (\mathcal{L}^+\mathcal{G}^{\mathrm{BD}})_0 \cong L^+\mathcal{G}_{\sigma_Y},$$

and a similar description for $\mathcal{L}\mathcal{G}^{\mathrm{BD}}$. In addition, there is a canonical map $\mathcal{L}\mathcal{G}^{\mathrm{BD}} \to \mathrm{Gr}^{\mathrm{BD}}_{\mathcal{G}}$ (similar to (3.1.4)) that is an $\mathcal{L}^+\mathcal{G}^{\mathrm{BD}}$-torsor and admits sections étale locally on $\mathrm{Gr}^{\mathrm{BD}}_{\mathcal{G}}$. The natural map $\widehat{\Gamma_y \cup \gamma_0} \to C_R$ induces $\mathcal{L}^{pol}\mathcal{G} \subset \mathcal{L}^+\mathcal{G}^{\mathrm{BD}}$, and $\mathcal{U}_{\Phi_{\mu+\nu}}$ can be regarded as a closed subgroup scheme of $\mathcal{L}^+\mathcal{G}^{\mathrm{BD}}$. Clearly, $\overline{\mathrm{Gr}}^{\mathrm{BD}}_{\mathcal{G},\mu,\nu}$ is stable under the action of $\mathcal{L}^+\mathcal{G}^{\mathrm{BD}} \times_C \tilde{C}$. Therefore, the morphism (6.2.7) factors as $\mathcal{U}_{\Phi_{\mu+\nu}} \to \overline{\mathrm{Gr}}^{\mathrm{BD}}_{\mathcal{G},\mu,\nu}$. Using the similar argument as in the proof of Proposition 5.3, one shows that (6.2.7) is a monomorphism and in fact is an open immersion to $\overline{\mathrm{Gr}}^{\mathrm{BD}}_{\mathcal{G},\mu,\nu}$. □

Proposition 6.4. *Let $\nu \in \mathbb{X}_\bullet(T)$ be a sufficiently dominant coweight. Then the variety $\overline{\mathrm{Gr}}^{\mathrm{BD}}_{\mathcal{G},\mu,\nu}$ is normal and the fiber $(\overline{\mathrm{Gr}}^{\mathrm{BD}}_{\mathcal{G},\mu,\nu})_{\tilde{0}}$ over $\tilde{0} \in \tilde{C}$ is reduced.*

Proof. The proposition follows from Lemma 6.3 and Hironaka's lemma (cf. [Gro65, 5.12.8]). Namely, let V denote the underlying reduced subscheme of $(\overline{\mathrm{Gr}}^{\mathrm{BD}}_{\mathcal{G},\mu,\nu})_{\tilde{0}}$. Then V is irreducible and, therefore, is a Schubert variety of $\mathcal{F}\ell^Y$ that is normal by Theorem 2.1. Therefore, the proposition follows. □

In fact, we proved that the fiber $(\overline{\mathrm{Gr}}^{\mathrm{BD}}_{\mathcal{G},\mu,\nu})_{\tilde{0}}$ is isomorphic to $\mathcal{F}\ell^Y_{\mu+\nu}$.

If $\nu \in \mathbb{X}_\bullet(T_{\mathrm{sc}})_\Gamma \subset \mathbb{X}_\bullet(T)_\Gamma$ so that $\nu \in W_{\mathrm{aff}}$, then $z(\overline{\mathrm{Gr}}_{\mathcal{G},\mu}) \subset \overline{\mathrm{Gr}}^{\mathrm{Conv}}_{\mathcal{G},\mu,\nu}$ (resp. $m \circ z(\overline{\mathrm{Gr}}_{\mathcal{G},\mu}) \subset \overline{\mathrm{Gr}}^{\mathrm{BD}}_{\mathcal{G},\mu,\nu}$) is naturally a closed subscheme.

By Corollary 4.4, we just need to prove Theorem 3.9 for one prime. Therefore, we will assume $\mathrm{char}\, k = p > 2$. Recall the notation of Frobenius splitting (cf. [MR85], [BK05]) for varieties in characterisic $p > 0$.

Theorem 6.5. *Assume that $\nu \in \mathbb{X}_\bullet(T_{\mathrm{sc}})$ is a coweight dominant enough so that Proposition 6.4 holds. Then $\overline{\mathrm{Gr}}^{\mathrm{BD}}_{\mathcal{G},\mu,\nu}$ is Frobenius split, compatibly with $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}$ and $(\overline{\mathrm{Gr}}^{\mathrm{BD}}_{\mathcal{G},\mu,\nu})_{\tilde{0}}$.*

Corollary 6.6. *Theorem 3.9 holds. That is, the scheme $(\overline{\mathrm{Gr}}_{\mathcal{G},\mu})_{\tilde{0}}$ is reduced.*

Proof. This is because

$$(\overline{\mathrm{Gr}}_{\mathcal{G},\mu})_{\tilde{0}} = \overline{\mathrm{Gr}}_{\mathcal{G},\mu} \cap (\overline{\mathrm{Gr}}^{\mathrm{BD}}_{\mathcal{G},\mu,\nu})_{\tilde{0}},$$

and therefore it is Frobenius split. In particular, it is reduced. □

The remaining goal of this subsection is to reduce Theorem 6.5 to the following Theorem 6.8 via Proposition 6.7. Theorem 6.8 itself will be proven in later subsections. First, it is enough to prove Theorem 6.5 for the case where $\mathcal{G}_{\mathcal{O}_0} \cong \mathcal{G}_{\mathbf{a}}$ is the Iwahori group scheme. To see this, assume that we have $\mathcal{G}_1 \to \mathcal{G}_2$, where $(\mathcal{G}_1)_{\mathcal{O}_0}$ is Iwahori and $(\mathcal{G}_2)_{\mathcal{O}_0}$ is a general parahoric group scheme. Then the natural projection $\overline{\mathrm{Gr}}^{\mathrm{BD}}_{\mathcal{G}_1,\mu,\nu} \to \overline{\mathrm{Gr}}^{\mathrm{BD}}_{\mathcal{G}_2,\mu,\nu}$ is proper birational and therefore the push-forward of the structure sheaf is the structure sheaf by the normality. Furthermore, under the projection, the scheme-theoretical image of $\overline{\mathrm{Gr}}_{\mathcal{G}_1,\mu}$ (resp. $(\overline{\mathrm{Gr}}^{\mathrm{BD}}_{\mathcal{G}_1,\mu,\nu})_{\tilde{0}}$) is $\overline{\mathrm{Gr}}_{\mathcal{G}_2,\mu}$ (resp. $(\overline{\mathrm{Gr}}^{\mathrm{BD}}_{\mathcal{G}_2,\mu,\nu})_{\tilde{0}}$). Therefore, from now on we assume that $\mathcal{G}_{\mathcal{O}_0} = \mathcal{G}_{\mathbf{a}}$ and write $I = L^+\mathcal{G}_{\mathbf{a}}$.

Since $\overline{\mathrm{Gr}}^{\mathrm{BD}}_{\mathcal{G},\mu,\nu}$ is normal, we just need to find an open subscheme of $U \subset \overline{\mathrm{Gr}}^{\mathrm{BD}}_{\mathcal{G},\mu,\nu}$, whose complement has codimension at least two, such that U is Frobenius split, compatibly with $U \cap (\overline{\mathrm{Gr}}^{\mathrm{BD}}_{\mathcal{G},\mu,\nu})_{\tilde{0}}$ and $U \cap \overline{\mathrm{Gr}}_{\mathcal{G},\mu}$ ([BK05, Lemma 1.1.7(iii)]). Therefore, we can throw away some bad loci of $\overline{\mathrm{Gr}}^{\mathrm{BD}}_{\mathcal{G},\mu,\nu}$ that are hard to control. In particular, we can throw away $(\overline{\mathrm{Gr}}_{\mathcal{G},\mu})_{\tilde{0}} \subset \overline{\mathrm{Gr}}_{\mathcal{G},\mu} \subset \overline{\mathrm{Gr}}^{\mathrm{BD}}_{\mathcal{G},\mu,\nu}$, which is of our main interest.

More precisely, we have

Proposition 6.7. *There is an open subscheme U of $\overline{\mathrm{Gr}}^{\mathrm{Conv}}_{\mathcal{G},\mu,\nu}$ such that*

1. *$m : \overline{\mathrm{Gr}}^{\mathrm{Conv}}_{\mathcal{G},\mu,\nu} \to \overline{\mathrm{Gr}}^{\mathrm{BD}}_{\mathcal{G},\mu,\nu}$ maps U isomorphically onto an open subscheme $m(U)$ of $\overline{\mathrm{Gr}}^{\mathrm{BD}}_{\mathcal{G},\mu,\nu}$, and the complement of $m(U)$ in $\overline{\mathrm{Gr}}^{\mathrm{BD}}_{\mathcal{G},\mu,\nu}$ has codimension two*;
2. *U is Frobenius split, compatible with $U \cap (\overline{\mathrm{Gr}}^{\mathrm{Conv}}_{\mathcal{G},\mu,\nu})_{\tilde{0}}$ and $U \cap z(\overline{\mathrm{Gr}}_{\mathcal{G},\mu})$.*

It is clear that Theorem 6.5 will follow from this proposition.

Proof. Let us first construct this open subscheme U. Recall that in Section 5.2, we have constructed the affine open chart $c_\mu : U_\mu \subset \overline{\mathrm{Gr}}_{\mathcal{G},\mu}$ that satisfies

1. $s_\mu(\tilde{C}) \subset U_\mu$;
2. U_μ is an affine space over $\tilde{C}$ and therefore smooth over $\tilde{C}$;
3. $(U_\mu)_{\tilde{0}} = C(\mu) \subset \mathcal{F}\ell^Y$ is the Schubert cell containing t_μ, i.e., the $L^+\mathcal{G}_{\mathbf{a}}$-orbit containing t_μ.

Recall that we constructed the section $s_\mu : \tilde{C} \to \widetilde{\mathrm{Gr}}_{\mathcal{G}}$ and that $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}$ is the minimal irreducible closed subvariety of $\widetilde{\mathrm{Gr}}_{\mathcal{G}}$ that is invariant under $\widetilde{\mathcal{L}^+\mathcal{G}}$ and contains $s_\mu(\tilde{C})$. Let $\mathrm{Gr}_{\mathcal{G},\mu}$ denote the $\widetilde{\mathcal{L}^+\mathcal{G}}$-orbit through s_μ. Then $\mathrm{Gr}_{\mathcal{G},\mu}$ is an open subscheme of $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}$ that is smooth over $\tilde{C}$. In fact, $\mathrm{Gr}_{\mathcal{G},\mu}$ is open in $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}$ since $U_\mu \subset \mathrm{Gr}_{\mathcal{G},\mu}$ and the transformations of U_μ by $\widetilde{\mathcal{L}^+\mathcal{G}}$ form a cover of $\mathrm{Gr}_{\mathcal{G},\mu}$. Therefore, $\mathrm{Gr}_{\mathcal{G},\mu}$ is flat over $\tilde{C}$. Observe that under the isomorphism $\mathrm{Gr}_{\mathcal{G}} \times_C \tilde{C}^\circ \cong \mathrm{Gr}_H \times \tilde{C}^\circ$,

$$\mathrm{Gr}_{\mathcal{G},\mu}|_{\tilde{C}^\circ} \cong \mathrm{Gr}_\mu \times \tilde{C}^\circ,$$

where Gr_μ denotes the L^+H-orbit in Gr_H through t_μ that is smooth. On the other hand $(\mathrm{Gr}_{\mathcal{G},\mu})_{\tilde{0}} = (U_\mu)_{\tilde{0}}$ is the Schubert cell $C(\mu)$ in $\mathcal{F}\ell$ containing μ that is irreducible and smooth. Therefore, $\mathrm{Gr}_{\mathcal{G},\mu}$ is smooth over $\tilde{C}$.

Let U_1 be the preimage of $\mathrm{Gr}_{\mathcal{G},\mu}$ under $\pi : \overline{\mathrm{Gr}}^{\mathrm{Conv}}_{\mathcal{G},\mu,\nu} \to \overline{\mathrm{Gr}}_{\mathcal{G},\mu}$. Then U_1 is a fibration over $\mathrm{Gr}_{\mathcal{G},\mu}$ with fibers $\mathcal{F}\ell_\nu$. As a scheme over $\tilde{C}$, the fiber of U_1 over $\tilde{0}$ is

$$C(\mu)\tilde{\times}\mathcal{F}\ell_\nu.$$

We define U to be the open subscheme of U_1 that coincides with U_1 over $\tilde{C}^\circ$ and that is given by

$$C(\mu)\tilde{\times}C(\nu) \subset C(\mu)\tilde{\times}\mathcal{F}\ell_\nu$$

over $\tilde{0}$.

We claim that $m : U \to m(U)$ is an isomorphism and the complement of $m(U)$ in $\overline{\mathrm{Gr}}^{\mathrm{BD}}_{\mathcal{G},\mu,\nu}$ has codimension two. Over $\tilde{C}^\circ$, m is an isomorphism. Over $\tilde{0}$, the morphism

$$m : U_{\tilde{0}} \to (\overline{\mathrm{Gr}}^{\mathrm{BD}}_{\mathcal{G},\mu,\nu})_{\tilde{0}}$$

is the same as

$$m : C(\mu)\tilde{\times}C(\nu) \to \mathcal{F}\ell_{\mu+\nu}.$$

For λ, recall the group scheme $\mathcal{U}_{\Phi_\lambda}$ as constructed in Section 5.2. Denote $(\mathcal{U}_{\Phi_\lambda})_0 \subset I$ by U_{Φ_λ}. Note that $U_{\Phi_\mu} t_\mu U_{\Phi_\nu} t_\nu = U_{\Phi_{\mu+\nu}} t_{\mu+\nu}$. Therefore, m induces an isomorphism from $C(\mu)\tilde{\times}C(\nu)$ onto $C(\mu+\nu)$. In addition, the preimage of $C(\mu+\nu)$ is $C(\mu)\tilde{\times}C(\mu)$, as $m(\mathcal{F}\ell_w\tilde{\times}\mathcal{F}\ell_v)$ for $wv < \mu+\nu$ is contained in a proper Schubert subvariety of $\mathcal{F}\ell_{\mu+\nu}$. Therefore, $m : U \to m(U)$ is a homeomorphism and $m^{-1}m(U) = U$. Therefore, $m : U \to m(U)$ is a proper, birational homeomorphism with $m(U)$ normal, which must be an isomorphism. Note that $(\overline{\mathrm{Gr}}_{\mathcal{G},\mu})_{\tilde{0}} \subset \overline{\mathrm{Gr}}_{\mathcal{G},\mu} \subset \overline{\mathrm{Gr}}^{\mathrm{BD}}_{\mathcal{G},\mu,\nu}$ is not contained in $m(U)$.

To see that the complement of $m(U)$ has codimension two, first observe that over $\tilde{C}^\circ$,

$$\overline{\mathrm{Gr}}^{\mathrm{BD}}_{\mathcal{G},\mu,\nu}|_{\tilde{C}^\circ} - m(U)|_{\tilde{C}^\circ} \cong (\overline{\mathrm{Gr}}_\mu - \mathrm{Gr}_\mu) \times \mathcal{F}\ell_\nu \times \tilde{C}^\circ,$$

which has codimension two, since $\overline{\mathrm{Gr}}_\mu - \mathrm{Gr}_\mu$ has codimension two in $\overline{\mathrm{Gr}}_\mu$. Over $\tilde{0}$,

$$(\overline{\mathrm{Gr}}^{\mathrm{BD}}_{\mathcal{G},\mu,\nu})_{\tilde{0}} - m(U)_{\tilde{0}} \cong \mathcal{F}\ell_{\mu+\nu} - C(\mu+\nu),$$

which has codimension at least one. This proves that the complement of $m(U)$ in $\overline{\mathrm{Gr}}^{\mathrm{BD}}_{\mathcal{G},\mu,\nu}$ has codimension two.

Next we turn to the second part of the proposition. Recall that U_1 is the preimage of $\mathrm{Gr}_{\mathcal{G},\mu}$ under $\pi : \overline{\mathrm{Gr}}^{\mathrm{Conv}}_{\mathcal{G},\mu,\nu} \to \overline{\mathrm{Gr}}_{\mathcal{G},\mu}$. From the construction of U, we know that $U \subset U_1 \subset \overline{\mathrm{Gr}}^{\mathrm{Conv}}_{\mathcal{G},\mu,\nu}$. Therefore, it is enough to show that the same statement of Proposition 6.7 (2) holds for U_1. Recall that

$$U_1 \cong (\mathrm{Gr}_{\mathcal{G},\mu} \times_{\mathrm{Gr}_\mathcal{G}} \mathrm{Gr}_{\mathcal{G},\underline{0}}) \times^I \mathcal{F}\ell_\nu,$$

where $\mathrm{Gr}_{\mathcal{G},\underline{0}}$ is the I-torsor over $\mathrm{Gr}_{\mathcal{G}}$ as in (6.2.3). To simplify the notation, for any I-variety V, we denote

$$\mathrm{Gr}_{\mathcal{G},\mu}\tilde{\times}V := (\mathrm{Gr}_{\mathcal{G},\mu} \times_{\mathrm{Gr}_{\mathcal{G}}} \mathrm{Gr}_{\mathcal{G},\underline{0}}) \times^I V.$$

Now, let $* \in \mathcal{F}\ell_\nu$ be the base point. (Recall that $\nu \in \mathbb{X}^\bullet(T_{\mathrm{sc}})$, so that $*$, the Schubert variety corresponding to the identity element in the affine Weyl group, is contained in $\mathcal{F}\ell_\nu$.) Then the closed embedding $z : \mathrm{Gr}_{\mathcal{G},\mu} \to U_1$ corresponds to

$$\mathrm{Gr}_{\mathcal{G},\mu}\tilde{\times}* \to \mathrm{Gr}_{\mathcal{G},\mu}\tilde{\times}\mathcal{F}\ell_\nu.$$

Now the assertion follows from the following more general statement. □

Theorem 6.8. *For any $w \in \widetilde{W}$, there is a Frobenius splitting of the scheme $\mathrm{Gr}_{\mathcal{G},\mu}\tilde{\times}\mathcal{F}\ell_w$ that is compatible with*

$$(\mathrm{Gr}_{\mathcal{G},\mu}\tilde{\times}\mathcal{F}\ell_w)_{\tilde{0}} \cong (\mathrm{Gr}_{\mathcal{G},\mu})_{\tilde{0}}\tilde{\times}\mathcal{F}\ell_w \cong C(\mu)\tilde{\times}\mathcal{F}\ell_w.$$

In addition, for any $v \leq w$ in $\widetilde{W}$, $\mathrm{Gr}_{\mathcal{G},\mu}\tilde{\times}\mathcal{F}\ell_v \subset \mathrm{Gr}_{\mathcal{G},\mu}\tilde{\times}\mathcal{F}\ell_w$ is also compatible with this splitting.

The remaining goal of this section is to prove this theorem.

6.3. *Special parahorics.* We continue by assuming that G and $\mathcal{G}$ are as given in Section 3.2, but we are particularly interested in the case when $\mathcal{G} = \mathcal{G}^s$ is the group scheme over C such that $\mathcal{G}^s_{\mathcal{O}_0}$ is a special parahoric group scheme of G. In this case, we can easily deduce Theorem 3.9 (assuming Theorem 3.8) directly from Hironaka's lemma (without going into the argument presented in the previous subsection). This will in turn help us prove a special case of Theorem 6.8; namely, the case when $w = 1$ (see Corollary 6.17). Let us remark that if G is split, Proposition 6.15 directly follows from Frobenius splitting of Schubert varieties. Those who are only interested in split groups can go directly to the paragraph after this proposition.

So let $v \in A(G,S)$ be a special point in the apartment associated to (G,S), and let $\mathcal{G}_v$ be the corresponding special parahoric group scheme over $\mathcal{O}$. Let $\mathcal{F}\ell_v = LG/L^+\mathcal{G}_v$ be the partial affine flag variety. To emphasize that it is the affine flag variety associated to a special parahoric, we sometimes also denote it by $\mathcal{F}\ell^s$. As before, for each $\mu \in \mathbb{X}_\bullet(T)_\Gamma$, let us use t_μ to denote its lifting to $T(F)$ under the Kottwitz homomorphism $T(F) \to \mathbb{X}_\bullet(T)_\Gamma$. It gives a point in $\mathcal{F}\ell^s$, still denoted by t_μ. Then the Schubert variety $\mathcal{F}\ell^s_\mu$ is the closure of the $L^+\mathcal{G}_v$-orbit in $\mathcal{F}\ell_v$ passing through t_μ. We have the following results special for Schubert varieties in $\mathcal{F}\ell^s$, which generalize the corresponding results for Gr_H. (See also [Ric13, Cor. 2.10] for more detailed discussion.)

Lemma 6.9. *The Schubert varieties are parametrized by $\mathbb{X}_\bullet(T)^+_\Gamma$. For $\mu \in \mathbb{X}_\bullet(T)^+_\Gamma$, the dimension of $\mathcal{F}\ell^s_\mu$ is $(\mu, 2\rho)$. Let $\mathring{\mathcal{F}}\ell^s_\mu \subset \mathcal{F}\ell^s_\mu$ be the unique open $L^+\mathcal{G}_v$-orbit in $\mathcal{F}\ell^s_\mu$. Then $\mathcal{F}\ell^s_\mu - \mathring{\mathcal{F}}\ell^s_\mu$ has codimension at least two.*

Proof. Observe that the natural map $\mathbb{X}_\bullet(T)^+_\Gamma \subset \mathbb{X}_\bullet(T)_\Gamma \to W_0\backslash\widetilde{W}/W_0$ is a bijection. The first claim follows. Let $I \subset L^+\mathcal{G}_v$ be the Iwahori subgroup of LG corresponding to the alcove $\mathbf{a}$. (Recall that v is contained in the closure of C.) Then the I-orbits in $\mathcal{F}\ell^s$ are parametrized by minimal length representatives in $\widetilde{W}/W_0$. Let $\lambda \in \Lambda = W_0\mu \subset \mathbb{X}_\bullet(T)_\Gamma$. By Lemmas 9.1 and 9.3, if $w \in \widetilde{W}$ is a minimal length representative for the coset $t_\lambda W_0$, then

$$\dim IwL^+\mathcal{G}_v/L^+\mathcal{G}_v \leq (\mu, 2\rho),$$

and if $\lambda \in \mathbb{X}_\bullet(T)^+_\Gamma$, the equality holds. Therefore, $\dim \mathcal{F}\ell^s_\mu = (\mu, 2\rho)$. To prove the last claim, observe that if $\mathcal{F}\ell^s_\lambda \subsetneq \mathcal{F}\ell^s_\mu$, then $\mu - \lambda \in \mathbb{X}_\bullet(T_{\mathrm{sc}})_\Gamma$ and therefore $(\mu - \lambda, 2\rho)$ is an even integer. □

Recall that in [BD, §4.6], Beilinson and Drinfeld proved that $\overline{\mathrm{Gr}}_\mu$ is Gorenstein; i.e., the dualizing sheaf $\omega_{\overline{\mathrm{Gr}}_\mu}$ is indeed a line bundle. (See Equation (241) in loc. cit.) It is natural to ask whether the same result hold for $\mathcal{F}\ell^s_\mu$. However, the situation is more complicated in the ramified case due to the fact that not all special points in the building of G are conjugate under $G_{\mathrm{ad}}(F)$. More precisely, if G_{der} is the odd ramified special unitary group SU_{2n+1} (see Section 8 for the definition), then there are two types of special parahoric group schemes (see Remark 8.1(ii)).

Let us begin with the following lemma. Let v be any point in the apartment $A(G,S)$, and let $\mathcal{G}_v$ be the corresponding parahoric group scheme for G. For simplicity, we write $K = L^+\mathcal{G}_v$. Then K acts on $\mathrm{Lie}\, G$ by the adjoint representation. Let $\mu \in \mathbb{X}_\bullet(T)_\Gamma$. Let

$$P = K \cap \mathrm{Ad}_{t_\mu} K,$$

considered as a proalgebraic group over k. Then $\mathrm{Lie}\, K$ and $\mathrm{Ad}_{t_\mu}\mathrm{Lie}\, K$ are P-modules.

Lemma 6.10. *As P-modules,*

$$\det \frac{\mathrm{Lie}\, K}{\mathrm{Lie}\, K \cap \mathrm{Ad}_{t_\mu}\mathrm{Lie}\, K} \cong \left(\det \frac{\mathrm{Ad}_{t_\mu}\mathrm{Lie}\, K}{\mathrm{Lie}\, K \cap \mathrm{Ad}_{t_\mu}\mathrm{Lie}\, K}\right)^{-1}. \tag{6.3.1}$$

Proof. Recall that we denote by S the chosen maximal split F-torus of G. Its (connected) Néron model $\mathcal{S}$ maps naturally into $\mathcal{G}_v$ since $v \in A(G,S)$ ([BT84, §5.2]), and $L^+\mathcal{S}$ maps to P. The special fiber S_k of $\mathcal{S}$ can be regarded as the "constant" maps from $\mathcal{O}$ to $\mathcal{S}$ and can therefore be regarded a subgroup of $L^+\mathcal{S}$. Then $S_k \subset P$ is a maximal torus of P. Thus, $\mathbb{X}^\bullet(P) \subset \mathbb{X}^\bullet(S_k)$. Therefore, it is enough to prove (6.3.1) as an isomorphism of S_k-modules.

In Section 5.1.3, in particular Remark 5.1 (see also [PR08, 9.a,9.b]), we have attached to each affine root α of (G,S) a one-dimensional unipotent

subgroup $U_\alpha \cong \mathbb{G}_a \subset LG$. Let $\mathfrak{u}_\alpha$ be the Lie algebra of U_α. By definition,

$$\operatorname{Lie} K = \operatorname{Lie} \mathcal{T}^{\flat,0} \oplus \prod_{\alpha(v)\geq 0} \mathfrak{u}_\alpha,$$

where $\mathcal{T}^{\flat,0}$ is the connected Néron model of T. Then clearly, as S_k-modules (we fix an embedding $S_k \to L^+\mathcal{S}$),

$$\frac{\operatorname{Lie} K}{\operatorname{Lie} K \cap \operatorname{Ad}_{t_\mu} \operatorname{Lie} K} \cong \bigoplus_{\alpha(v)\geq 0, \alpha(v-\mu)<0} \mathfrak{u}_\alpha,$$

$$\frac{\operatorname{Ad}_{t_\mu} \operatorname{Lie} K}{\operatorname{Lie} K \cap \operatorname{Ad}_{t_\mu} \operatorname{Lie} K} \cong \bigoplus_{\alpha(v)<0, \alpha(v-\mu)\geq 0} \mathfrak{u}_\alpha.$$

By identifying $A(G,S)$ with $\mathbb{X}_\bullet(S)_\mathbb{R}$ using the point v, we can write affine roots of G by $\alpha = a + m, a \in \Phi(G,S)$, where a is the vector part of α and $m = v(\alpha)$. Therefore,

$$\{\alpha(v) \geq 0, \alpha(v-\mu) < 0\} = \{a + m | a \in \Phi(G,S)^+, 0 \leq m < (\mu, a)\}$$

and

$$\{\alpha(v) < 0, \alpha(v-\lambda) \geq 0\} = \{a + m | a \in \Phi(G,S)^-, (\mu, a) \leq m < 0\}.$$

Since S_k acts on $\mathfrak{u}_{a+m}$ via the weight a, the lemma follows. □

Now we should specify the special vertex. Recall that we assume that G_{der} is simple and simply-connected. If $G_{\mathrm{der}} \neq \mathrm{SU}_{2n+1}$, we can choose an arbitrary special vertex in the building of G since they are all conjugate under $G_{\mathrm{ad}}(F)$. If $G_{\mathrm{der}} = \mathrm{SU}_{2n+1}$, we choose the special vertex so that the corresponding parahoric group has reductive quotient Sp_{2n}. (See Remark 8.1).

THEOREM 6.11. *Let G be as in Section 3.2. With the choice of the special vertex v as above, the Schubert variety $\mathcal{F}\ell^s_\mu$ is Gorenstein for all μ.*

Proof. As above, we denote by $\mathcal{G}_v$ the parahoric group of G corresponding to v and $K = L^+\mathcal{G}_v$. Recall that since $\mathcal{F}\ell^s_\mu$ is Cohen-Macaulay, the dualizing sheaf $\omega_{\mathcal{F}\ell^s_\mu}$ exists. We need to show that it is indeed a line bundle. Let $j : \mathring{\mathcal{F}}\ell^s_\mu \to \mathcal{F}\ell^s_\mu$ be the open K-orbit in $\mathcal{F}\ell^s_\mu$. Then we have shown that $\mathcal{F}\ell^s_\mu - \mathring{\mathcal{F}}\ell^s_\mu$ has codimension at least two. As $\mathcal{F}\ell^s_\mu$ is normal, $\omega_{\mathcal{F}\ell^s_\mu} = j_*(\omega_{\mathring{\mathcal{F}}\ell^s_\mu})$. Let $\mathcal{L}_{2c}$ be the pullback to $\mathcal{F}\ell^s$ of the determinant line bundle $\mathcal{L}_{\det}$ of $\mathrm{Gr}_{\mathrm{GL}(\operatorname{Lie}\mathcal{G}_v)}$ along $i : \mathcal{F}\ell^s \to \mathrm{Gr}_{\mathrm{GL}(\operatorname{Lie}\mathcal{G}_v)}$. We first prove that there is an isomorphism of line bundles $\omega^{-2}_{\mathring{\mathcal{F}}\ell^s_\mu} \cong \mathcal{L}_{2c}|_{\mathring{\mathcal{F}}\ell^s_\mu}$ on $\mathring{\mathcal{F}}\ell^s_\mu$.

Indeed, observe that both sheaves are K-equivariant. The K-equivariant structure of $\omega^{-2}_{\mathring{\mathcal{F}}\ell^s_\mu}$ is induced from the action of K on $\mathring{\mathcal{F}}\ell^s_\mu$. On the other hand, a central extension of LG acts on $\mathcal{L}_{2c}$, and a splitting of this central extension over K defines a K-equivariant structure on $\mathcal{L}_{2c}$. To fix this K-equivariant

structure uniquely, we will require that the action of K on the fiber of $\mathcal{L}_{2c}$ over $* \in \mathcal{F}\ell^s$ is trivial. Then the K-equivariant structure on $\mathcal{L}_{2c}$ is given as follows. (For simplicity, we only describe it at the level of k-points, but the generalization to R-points is clear; for example, see cf. [FZ12, §§2.2.2–2.2.3].) Recall that for $x \in \mathcal{F}\ell^s$, $i(x)$ is a lattice in $\operatorname{Lie} G$ and $\mathcal{L}_{2c}|_x$ is the k-line

$$\mathcal{L}_{2c}|_x = \det(i(x)|\operatorname{Lie} K) := \det \frac{\operatorname{Lie} K}{\operatorname{Lie} K \cap i(x)} \otimes \det \left(\frac{i(x)}{\operatorname{Lie} K \cap i(x)} \right)^{-1}.$$

Then for $g \in K$, $\mathcal{L}_{2c}|_x \to \mathcal{L}_{2c}|_{gx}$ is given by

$$\det(g) : \det(i(x)|\operatorname{Lie} K) \cong \det(i(gx)|g \operatorname{Lie} K) = \det(i(gx)|\operatorname{Lie} K).$$

Now it is enough to prove that there is an isomorphism $\mathcal{L}_{2c}|_{t_\mu} \cong \omega^{-2}_{\mathring{\mathcal{F}}\ell^s_\mu}|_{t_\mu}$ as one-dimensional representations of $P = t_\mu K t_\mu^{-1} \cap K$, the stabilizer of $t_\mu \in \mathcal{F}\ell^s$ in K. As the tangent space of $\mathring{\mathcal{F}}\ell^s_\mu$ at t_μ as a P-module is isomorphic to $\frac{\operatorname{Lie} K}{\operatorname{Lie} K \cap \operatorname{Ad}_{t_\mu} \operatorname{Lie} K}$,

$$\omega^{-2}_{\mathring{\mathcal{F}}\ell^s_\mu}|_{t_\mu} \cong \left(\det \frac{\operatorname{Lie} K}{\operatorname{Lie} K \cap \operatorname{Ad}_{t_\mu} \operatorname{Lie} K} \right)^2$$

as P-modules. On the other hand, it follows from the construction of the determinant line bundle that

$$\mathcal{L}_{2c}|_{t_\mu} \cong \det \frac{\operatorname{Lie} K}{\operatorname{Lie} K \cap \operatorname{Ad}_{t_\mu} \operatorname{Lie} K} \otimes \left(\det \frac{\operatorname{Ad}_{t_\mu} \operatorname{Lie} K}{\operatorname{Lie} K \cap \operatorname{Ad}_{t_\mu} \operatorname{Lie} K} \right)^{-1}$$

as P-modules. Therefore, the assertion follows from Lemma 6.10.

Next, we prove that there is a K-equivariant line bundle $\mathcal{L}_c$ on $(\mathcal{F}\ell^s)_{\mathrm{red}}$ such that $\mathcal{L}_c^2 \cong \mathcal{L}_{2c}$. Indeed, for any $g \in G(F)$ acting on $\mathcal{F}\ell^s$ by left translation, we have $g^*\mathcal{L}_{2c} \cong \mathcal{L}_{2c}$. Therefore, it is enough to construct $\mathcal{L}_c$ in the neutral connected component of $(\mathcal{F}\ell^s)_{\mathrm{red}}$, which is isomorphic to $\mathcal{F}\ell^s_{\mathrm{sc}}$, the corresponding affine flag variety for G_{der} by [PR08, §6]. Since v is a special vertex, $\operatorname{Pic}(\mathcal{F}\ell^s_{\mathrm{sc}}) \cong \mathbb{Z}\mathcal{L}(\varepsilon_i)$, where $i \in \mathbf{S}$ is a special vertex in the local Dynkin diagram of G corresponding to v. By checking [Kac90, §§4, 6], we see that for our choice of v, we have$a_i^\vee = 1$. (For SU_{2n+1}, there is another special vertex $i' \in \mathbf{S}$ such that $a_{i'}^\vee = 2$, and the reductive quotient of the corresponding parahoric group is SO_{2n+1}; see the following remark and Remark 8.1.) Therefore, the central charge of $\mathcal{L}(\varepsilon_i)$ is 1, whereas the central charge of $\mathcal{L}_{2c}$ is $2h^\vee$ by (2.2.4) and Lemma 4.2. Therefore, $\mathcal{L}_c = \mathcal{L}(h^\vee \varepsilon_i)$.

As $\mathbb{X}^\bullet(P) \subset \mathbb{X}^\bullet(S_k)$ is torsion free, $\mathcal{L}_c|_{t_\mu} \cong \det \frac{\operatorname{Lie} K}{\operatorname{Lie} K \cap \operatorname{Ad}_{t_\mu} \operatorname{Lie} K}$ as P-modules. Therefore, we have $\omega^{-1}_{\mathring{\mathcal{F}}\ell^s_\mu} \cong \mathcal{L}_c|_{\mathring{\mathcal{F}}\ell^s_\mu}$, which in turn implies that $\omega^{-1}_{\mathcal{F}\ell^s_\mu} = j_*(\omega_{\mathring{\mathcal{F}}\ell^s_\mu})$ $\cong j_*(\mathcal{L}_c|_{\mathring{\mathcal{F}}\ell^s_\mu}) = \mathcal{L}_c$. □

Observe that the above proof implies that no matter what special vertex we choose, $\omega_{\mathcal{F}\ell^s_\mu}^{-2}$ is always a line bundle, where following Section 9.3, we denote $j_*((\omega_{\mathcal{F}\ell^s_\mu}|_{\mathring{\mathcal{F}}\ell^s_\mu})^n)$ by $\omega^n_{\mathcal{F}\ell^s_\mu}$.

The following corollary is what we need in the sequel.

COROLLARY 6.12. *For any special vertex v of G, $H^1(\mathcal{F}\ell^s_\mu, \omega^{-n}_{\mathcal{F}\ell^s_\mu}) = 0$ for all positive even integers n.*

Remark 6.1. In the case $G_{\mathrm{der}} = \mathrm{SU}_{2n+1}$, if we take the special vertex to be v_0, the one defined by the pinning (2.1.1) so that $\mathcal{G}_{v_0}$ is of the form (2.1.2), then the reductive quotient is SO_{2n+1} and the corresponding $a_i^\vee = 2$. Since the dual Coxeter number of SL_{2n+1} is $2n+1$, this means that on the partial flag variety $\mathcal{F}\ell^s$ corresponding to this special vertex, $\mathcal{L}_{2c}$ does *not* have a square root. Let I be the Iwahori group of G_{der} corresponding to the chosen alcove $\mathbf{a}$, $i \in \mathbf{S}$ given by v_0. Let $\mathbb{P}^1_i = P_i/I$ be the rational line in $\mathcal{F}\ell_{\mathrm{sc}} = LG_{\mathrm{der}}/I$ as constructed in Section 2.2. It projects to a rational curve in $\mathcal{F}\ell^s$ under $LG_{\mathrm{der}}/I \to LG_{\mathrm{der}}/L^+\mathcal{G}_{v_0}$. (An explicit description of this rational line is given in (8.0.1).) Then the restriction of $\mathcal{L}_{2c}$ to this rational line has degree $2n+1$. Since this line is contained in any Schubert variety $\mathcal{F}\ell^s_\mu$, this means that $\omega^{-1}_{\mathcal{F}\ell^s_\mu}$ is *not* a line bundle; i.e., $\mathcal{F}\ell^s_\mu$ is not Gorenstein.

Now we turn to the global Schubert varieties. Let

$$\mathcal{G}^s = ((\mathrm{Res}_{\tilde{C}/C}(H \times \tilde{C}))^\Gamma)^0$$

be the Bruhat-Tits group scheme over C as constructed in Section 3.2. Therefore, $\mathcal{G}^s_{\mathcal{O}_0} \cong \mathcal{G}_{v_0}$ is the special parahoric group scheme for $\mathcal{G}_{F_0}$ as in (2.1.2).

PROPOSITION 6.13. *Assume Theorem* 3.8. *Then Theorem* 3.9 *holds for $\mathcal{G}^s$.*

Proof. By Theorem 3.8, the support of $(\overline{\mathrm{Gr}}_{\mathcal{G}^s,\mu})_{\tilde{0}}$ is a single Schubert variety. This is because, when $\mathcal{G}^s_{\mathcal{O}_0} = \mathcal{G}_{v_0}$ is a special parahoric group scheme, $W^Y = W_0$ and $W_0 \backslash \mathrm{Adm}^Y(\mu)/W_0$ consists of only one extremal element in the Bruhat order, namely t_μ under the projection $\mathrm{Adm}^Y(\mu) \to W_0 \backslash \mathrm{Adm}^Y(\mu)/W_0$. This proves that the special fiber of $\overline{\mathrm{Gr}}_{\mathcal{G}^s,\mu}$ is irreducible. On the other hand, we have the affine chart U_μ that is an affine space over $\tilde{C}$ (see Section 5.2) of $\overline{\mathrm{Gr}}_{\mathcal{G}^s,\mu}$ and $(U_\mu)_{\tilde{0}}$ is open in $(\overline{\mathrm{Gr}}_{\mathcal{G}^s,\mu})_{\tilde{0}}$. Therefore, the special fiber of $\overline{\mathrm{Gr}}_{\mathcal{G}^s,\mu}$ is generically reduced. By Hironaka's lemma again, $\overline{\mathrm{Gr}}_{\mathcal{G}^s,\mu}$ is normal over $\tilde{C}$, with special fiber reduced, indeed isomorphic to $\mathcal{F}\ell^s_\mu$. □

COROLLARY 6.14. *The global Schubert variety $\overline{\mathrm{Gr}}_{\mathcal{G}^s,\mu}$ is normal and Cohen-Macaulay.*

Proof. The normality follows from Hironaka's lemma. Since $\mathcal{F}\ell^s_\mu$ is Cohen-Macaulay, the assertion follows. □

We refer to Section 9.3 for a brief discussion of some facts about Frobenius splittings.

PROPOSITION 6.15. *The variety* $\overline{\mathrm{Gr}}_{\mathcal{G}^s,\mu}$ *is Frobenius split, compatibly with* $(\overline{\mathrm{Gr}}_{\mathcal{G}^s,\mu})_{\tilde{0}}$.

Proof. For simplicity, let us denote $\overline{\mathrm{Gr}}_{\mathcal{G}^s,\mu}$ by X. Then $f : X \to \tilde{C}$ is flat and is fiberwise normal and Cohen-Macaulay (since each X_y is a Schubert variety). Let $\omega_{X/\tilde{C}}$ be the relative dualizing sheaf on X. We know that $f_*\omega_{X/\tilde{C}}^{1-p}$ is a vector bundle on $\tilde{C}$ by Corollary 6.12.

By the construction of Section 5.3, the sheaf $f_*\omega_{X/\tilde{C}}^{1-p}$ is $\mathbb{G}_m$-equivariant, and therefore, we can choose a $\mathbb{G}_m$-equivariant isomorphism

$$f_*\omega_{X/\tilde{C}}^{1-p} \cong H^0(X_{\tilde{0}}, \omega_{X_{\tilde{0}}}^{1-p}) \otimes \mathcal{O}_{\tilde{C}}, \tag{6.3.2}$$

where the $\mathbb{G}_m$ action on $H^0(X_{\tilde{0}}, \omega_{X_{\tilde{0}}}^{1-p})$ comes from the $\mathbb{G}_m$-equivariant structure on $\omega_{X_{\tilde{0}}}^{1-p}$. Let $\sigma \in H^0(X_{\tilde{0}}, \omega_{X_{\tilde{0}}}^{1-p})$ be a $\mathbb{G}_m$-invariant section that splits X_0 (i.e., σ is a splitting of the natural map $\mathcal{O}_{X_{\tilde{0}}} \to F_*\mathcal{O}_{X_{\tilde{0}}}$, when regarded as a morphism from $F_*\mathcal{O}_{X_{\tilde{0}}} \to \mathcal{O}_{X_{\tilde{0}}}$ via (9.3.1)). Such a section always exists by Lemma 6.16 below. Let $\sigma \otimes 1$ be a section of $f_*\omega_{X/\tilde{C}}^{1-p}$ via the isomorphism (6.3.2). We claim that $\sigma \otimes 1$, regarded as a morphism $(F_{X/\tilde{C}})_*\mathcal{O}_X \to \mathcal{O}_{X^{(p)}}$ via (9.3.3), will map 1 to 1. In fact, $(\sigma \otimes 1)(1)$ is a $\mathbb{G}_m$-invariant nonzero function on $\overline{\mathrm{Gr}}_{\mathcal{G}^s,\mu}$ since its restriction to $X_{\tilde{0}}$ is nonzero by (9.3.7). But since all regular functions on $\overline{\mathrm{Gr}}_{\mathcal{G}^s,\mu}$ come from $\tilde{C}$, $(\sigma \otimes 1)(1)$ is a $\mathbb{G}_m$-invariant nonzero function on $\tilde{C}$, which must be a constant. But its restriction to $X_{\tilde{0}}$ is 1; the claim follows.

Now, let $(\sigma \otimes 1) \otimes (\frac{u}{du})^{p-1} \in f_*\omega_{X/\tilde{C}}^{1-p} \otimes \omega_{\tilde{C}}^{1-p} \cong f_*\omega_X^{1-p}$. By the formula (9.3.2) (applied to $\tilde{C}$) and the commutative diagram (9.3.6), the proposition follows. □

LEMMA 6.16. *Let X be an algebraic variety over an algebraically closed field k of positive characteristic with a $\mathbb{G}_m$-action. Let $\tau : F_*\mathcal{O}_X \to \mathcal{O}_X$ be a splitting map of the inclusion $\mathcal{O}_X \to F_*\mathcal{O}_X$. Decompose $\tau = \sum_j \tau_j$ according to the weights of the natural action of $\mathbb{G}_m$. Then τ_0 is also a splitting map.*

Proof. By definition, $1 = \tau(1) = \sum_j \tau_j(1)$, where $\tau_j(1)$ is a function on X of weight j under the action of $\mathbb{G}_m$. Comparing the weights of both sides, we find that $\tau_0(1) = 1$ and $\tau_j(1) = 0$ for $j \neq 0$. □

Let $\mathcal{G}$ be the group scheme with $\mathcal{G}_{\mathcal{O}_0} = \mathcal{G}_{\mathbf{a}}$. Let $I = L^+\mathcal{G}_{\mathbf{a}}$. Observe that the natural projection $\mathrm{Gr}_{\mathcal{G}} \to \mathrm{Gr}_{\mathcal{G}^s}$ induces an isomorphism from $\mathrm{Gr}_{\mathcal{G},\mu}$ to its image in $\overline{\mathrm{Gr}}_{\mathcal{G}^s,\mu}$. To see this, observe that $\mathrm{Gr}_{\mathcal{G},\mu}$ is covered by U_μ and $\mathrm{Gr}_{\mathcal{G},\mu}|_{\tilde{C}^\circ}$, both of which map isomorphically to their images in $\overline{\mathrm{Gr}}_{\mathcal{G}^s,\mu}$. We thus

regard $\mathrm{Gr}_{\mathcal{G},\mu}$ as an open subscheme of $\overline{\mathrm{Gr}}_{\mathcal{G}^s,\mu}$ under this map. The boundary $\overline{\mathrm{Gr}}_{\mathcal{G}^s,\mu} - \mathrm{Gr}_{\mathcal{G},\mu}$ has codimension at least two. Therefore, we have proven

COROLLARY 6.17. *$\mathrm{Gr}_{\mathcal{G},\mu}$ is Frobenius split, compatibly with $(\mathrm{Gr}_{\mathcal{G},\mu})_{\tilde{0}}$.*

COROLLARY 6.18. *The pullback along $f : \mathrm{Gr}_{\mathcal{G},\mu} \to \tilde{C}$ gives an isomorphism $f^* : H^0(\tilde{C}, \mathcal{O}_{\tilde{C}}) \xrightarrow{\sim} H^0(\mathrm{Gr}_{\mathcal{G},\mu}, \mathcal{O}_{\mathrm{Gr}_{\mathcal{G},\mu}})$.*

6.4. *Proof of Theorem* 6.8. The goal of this subsection is to prove Theorem 6.8. Without loss of generality, we can assume that $w \in W_{\mathrm{aff}}$. Let $s_i (i \in \mathbf{S})$ be the simple reflections (determined by the alcove $\mathbf{a}$). Let us recall that for $\tilde{w} = (s_{i_1}, s_{i_2}, \ldots, s_{i_m})$ a sequence of simple reflections corresponding to affine simple roots with $w = s_{i_1} \cdots s_{i_m}$, the Bott-Samelson-Demazure-Hasen (BSDH) variety is defined as

$$D_{\tilde{w}} = L^+P_{i_1} \times^I L^+P_{i_2} \times^I \cdots \times^I L^+P_{i_n}/I,$$

where P_i is the parahoric group corresponding to i (so that $L^+P_i/I \cong \mathbb{P}^1$). This is a smooth variety that is an iterated fibration by $\mathbb{P}^1$. For any subset $\{j_1, \ldots, j_n\} \subset \{1, \ldots, m\}$, let $\tilde{v} = (s_{i_{j_1}}, \ldots, s_{i_{j_n}})$ be the corresponding subsequence of $\tilde{w}$. Let $H_{i_p}, p = 1, 2, \ldots, m$, be defined as

$$H_{i_p} = \begin{cases} I & \text{if } p \notin \{j_1, \ldots, j_n\}, \\ L^+P_{i_p} & \text{if } p \in \{j_1, \ldots, j_n\}. \end{cases}$$

Then there is a closed embedding $\sigma_{\tilde{v},\tilde{w}} : D_{\tilde{v}} \to D_{\tilde{w}}$ given by

$$\begin{aligned} D_{\tilde{v}} = L^+P_{j_1} \times^I L^+P_{j_2} \times^I \cdots \times^I L^+P_{j_n}/I &\cong H_{i_1} \times^I \cdots \times^I H_{i_m}/I \\ &\hookrightarrow L^+P_{i_1} \times^I L^+P_{i_2} \times^I \cdots \times^I L^+P_{i_m}/I = D_{\tilde{w}}. \end{aligned} \tag{6.4.1}$$

In particular, let $\tilde{w}[j]$ denote the subsequence of $\tilde{w}$ obtained by deleting s_{i_j}. Then

$$\sigma_{\tilde{w}[j],\tilde{w}} : D_{\tilde{w}[j]} \hookrightarrow D_{\tilde{w}}$$

is a divisor. This way, we obtain m divisors of $D_{\tilde{w}}$. If $\tilde{v}_1, \tilde{v}_2$ are two subsequences of $\tilde{w}$, then the scheme-theoretical intersection $D_{\tilde{v}_1} \cap D_{\tilde{v}_2}$ inside $D_{\tilde{w}}$ is $D_{\tilde{v}_1 \cap \tilde{v}_2}$.

For $w \in W_{\mathrm{aff}}$, let $m = \ell(w)$, let us fix a reduced expression of $w = s_{i_1} \cdots s_{i_m}$ and let $\tilde{w} = (s_{i_1}, s_{i_2}, \ldots, s_{i_m})$. Let $D_{\tilde{w}}$ be the corresponding BSDH variety so that $D_{\tilde{w}}$ is smooth and $\pi_{\tilde{w}} : D_{\tilde{w}} \to \mathcal{F}\ell_w$ is birational. By twisting by the I-torsor $\mathrm{Gr}_{\mathcal{G},\mu} \times_{\mathrm{Gr}_{\mathcal{G}}} \mathrm{Gr}_{\mathcal{G},\underline{0}}$, we have $\mathrm{Gr}_{\mathcal{G},\mu} \tilde{\times} D_{\tilde{w}} \to \mathrm{Gr}_{\mathcal{G},\mu} \tilde{\times} \mathcal{F}\ell_w$, still denoted by $\pi_{\tilde{w}}$.

By the standard argument, to prove Theorem 6.8, it is enough to prove that

PROPOSITION 6.19. *The variety $\mathrm{Gr}_{\mathcal{G},\mu} \tilde{\times} D_{\tilde{w}}$ is Frobenius split, compatibly with all $\mathrm{Gr}_{\mathcal{G},\mu} \tilde{\times} D_{\tilde{w}[j]}$ for all j, and with $(\mathrm{Gr}_{\mathcal{G},\mu})_{\tilde{0}} \tilde{\times} D_{\tilde{w}}$.*

Let $\omega_{D_{\tilde{w}}}$ be the canonical sheaf of $D_{\tilde{w}}$. It is known that there is an isomorphism (for example, see [Gör01, Prop. 3.19] for the SL_n case, [PR08, proof of Prop. 9.6], [Mat88, Ch. 8.18] for the general case)

$$\omega_{D_{\tilde{w}}}^{-1} \cong \mathcal{O}\Big(\sum_{j=1}^{m} D_{\tilde{w}[j]}\Big) \otimes \pi_{\tilde{w}}^{*}\mathcal{L}\Big(\sum_{i\in\mathbf{S}} \varepsilon_i\Big), \tag{6.4.2}$$

where $\mathcal{L}(\sum_{i\in\mathbf{S}} \varepsilon_i)$ is the line bundle on $\mathcal{F}\ell_{\mathrm{sc}}$ as defined in Section 2.2. (Recall that since we assume that $w \in W_{\mathrm{aff}}$, $\mathcal{F}\ell_w \subset \mathcal{F}\ell_{\mathrm{sc}} = (\mathcal{F}\ell)^0_{\mathrm{red}}$ by [PR08, §6].) If we endow $\mathcal{L}(\sum_{i\in\mathbf{S}} \varepsilon_i)$ with the I-equivariant structure such that I acts on its fiber over $* \in \mathcal{F}\ell_{\mathrm{sc}}$ trivially, then the isomorphism (6.4.2) is I-invariant. This observation allows us to formulate a relative version of this isomorphism.

Let us denote by $\mathcal{L}_{2c}$ the line bundle on $\mathcal{F}\ell$ that is the pullback of $\mathcal{L}_{\det}$ along $\mathcal{F}\ell \to \mathrm{Gr}_{\mathrm{GL}(\mathrm{Lie}\, I)}$ as in Section 4.1. (As before, by abuse of notation, $\mathrm{Lie}\, I$ is considered as an $\mathcal{O}$-module.) We endow it with the I-equivariant structure so that I acts its fiber over $* \in \mathcal{F}\ell$ trivially. By twisting by the I-torsor $\mathrm{Gr}_{\mathcal{G},\mu} \times_{\mathrm{Gr}_{\mathcal{G}}} \mathrm{Gr}_{\mathcal{G},\underline{0}}$, we obtain a line bundle on $\mathrm{Gr}_{\mathcal{G},\mu}\tilde{\times}\mathcal{F}\ell$, denoted by $\tilde{\mathcal{L}}_{2c}$. In addition, to simply the notation, let us denote the projection $\mathrm{Gr}_{\mathcal{G},\mu}\tilde{\times}D_{\tilde{w}} \to \mathrm{Gr}_{\mathcal{G},\mu}$ by $f : X \to V$. Then by the same argument as in the proof of (6.4.2) (i.e., by induction on the length of w), we have

$$\omega_{X/V}^{-2} \cong \mathcal{O}\Big(2\sum_{j=1}^{m} \mathrm{Gr}_{\mathcal{G},\mu}\tilde{\times}D_{\tilde{w}[j]}\Big) \otimes \pi_{\tilde{w}}^{*}\tilde{\mathcal{L}}_{2c}. \tag{6.4.3}$$

We will later prove the following lemma.

Lemma 6.20. *There is a section σ_0 of $\tilde{\mathcal{L}}_{2c}$ whose divisor $\mathrm{div}(\sigma_0) \subset \mathrm{Gr}_{\mathcal{G},\mu}\tilde{\times}\mathcal{F}\ell_w$ does not intersect $z(\mathrm{Gr}_{\mathcal{G},\mu}) = \mathrm{Gr}_{\mathcal{G},\mu}\tilde{\times}*$.*

Let us remark that the line bundle $\mathcal{L}(\sum_{i\in\mathbf{S}} \varepsilon_i)$ is very ample on $\mathcal{F}\ell_w$, and therefore there exists a section of $\mathcal{L}(\sum_{i\in\mathbf{S}} \varepsilon_i)$ that does not pass through $*$. However, $\tilde{\mathcal{L}}_{2c}$ is twisted by the I-torsor $\mathrm{Gr}_{\mathcal{G},\mu} \times_{\mathrm{Gr}_{\mathcal{G}}} \mathrm{Gr}_{\mathcal{G},\underline{0}}$, and it is not ample. Therefore, some detailed analysis of this line bundle is needed.

Let us first assume this lemma, and let σ be a section of $\omega_{X/V}^{-2}$ whose divisor is of the form

$$\mathrm{div}(\sigma) = 2\sum_{j=1}^{m} \mathrm{Gr}_{\mathcal{G},\mu}\tilde{\times}D_{\tilde{w}[j]} + \mathrm{div}(\pi_{\tilde{w}}^{*}\sigma_0). \tag{6.4.4}$$

We claim that

Lemma 6.21. *A nonzero scalar multiple of the section $\sigma^{\frac{p-1}{2}} \in \omega_{X/V}^{1-p}$ (recall that we assume that $p > 2$), when regarded as a morphism $(F_{X/V})_*\mathcal{O}_X \to \mathcal{O}_{X^{(p)}}$ via (9.3.3), will send 1 to 1.*

Proof. Let

$$h = \sigma^{\frac{p-1}{2}}(1) \in \Gamma(X^{(p)}, \mathcal{O}_{X^{(p)}})$$

be the function as in the lemma. By Corollary 6.18, we have $\Gamma(X^{(p)}, \mathcal{O}_{X^{(p)}}) = \Gamma(\tilde{C}, \mathcal{O}_{\tilde{C}})$ and so h is obtained by pullback from a function on $\tilde{C}$ that then has to be a constant. To see h is nowhere vanishing, let $x \in \mathrm{Gr}_{\mathcal{G},\mu}$ be a point. It is enough to show the restriction of h to $(D_{\tilde{w}})_x := \mathrm{Gr}_{\mathcal{G},\mu}\tilde{\times} D_{\tilde{w}}|_x \cong D_{\tilde{w}}$ is not zero. This is because the restriction of σ to $\mathrm{Gr}_{\mathcal{G},\mu}\tilde{\times} D_{\tilde{w}}|_x$ gives a divisor of the form $2\sum_{j=1}^{m} D_{\tilde{w}[j]} + D$ for some D that does not pass through $*$. Therefore, by (a slight variant of) [MR85, Prop. 8], $\sigma^{\frac{p-1}{2}}|_{(D_{\tilde{w}})_x}$, when regarded as a morphism from $F_*\mathcal{O}_{D_{\tilde{w}}}$ to $\mathcal{O}_{(D_{\tilde{w}})_x}$ via (9.3.1), will send 1 to a nonzero constant function on $(D_{\tilde{w}})_x$. Therefore, by (9.3.7),

$$h|_{(D_{\tilde{w}})_x} = \sigma^{p-1}|_{(D_{\tilde{w}})_x}(1)$$

is a nonzero constant. This finishes the proof of the lemma. □

Now let $\tau \in \omega_{\mathrm{Gr}_{\mathcal{G},\mu}}^{1-p}$ be a section that gives rise to a Frobenius splitting of $\mathrm{Gr}_{\mathcal{G},\mu}$, compatible with $(\mathrm{Gr}_{\mathcal{G},\mu})_{\tilde{0}}$ by Corollary 6.17. Consider $\sigma^{\frac{p-1}{2}} \otimes f^*\tau \in \omega_X^{1-p}$. By (9.3.6), it gives a splitting of $\mathrm{Gr}_{\mathcal{G},\mu}\tilde{\times} D_{\tilde{w}}$, compatible with $(\mathrm{Gr}_{\mathcal{G},\mu})_{\tilde{0}}\tilde{\times} D_{\tilde{w}}$. Again, by (a slight variant of) [MR85, Prop. 8], this splitting is also compatible with all $\mathrm{Gr}_{\mathcal{G},\mu}\tilde{\times} D_{\tilde{w}[j]}$. This finishes the proof of Theorem 6.8.

It remains to prove Lemma 6.20. Let us consider the surjective map

$$V_w = \Gamma(\mathcal{F}\ell_w, \mathcal{L}_{2c}) \to \Gamma(*, \mathcal{L}_{2c}) = V_1 \cong k.$$

By twisting with the I-torsor $\mathrm{Gr}_{\mathcal{G},\mu} \times_{\mathrm{Gr}_{\mathcal{G}}} \mathrm{Gr}_{\mathcal{G},\underline{0}}$, we obtain a surjective morphism of vector bundles $\mathcal{V}_w \to \mathcal{V}_1 \cong \mathcal{O}_{\mathrm{Gr}_{\mathcal{G},\mu}}$ over $\mathrm{Gr}_{\mathcal{G},\mu}$. Clearly, $\mathcal{V}_w$ is $\pi_*\tilde{\mathcal{L}}_{2c}$, where $\pi : \mathrm{Gr}_{\mathcal{G},\mu}\tilde{\times}\mathcal{F}\ell \to \mathrm{Gr}_{\mathcal{G},\mu}$ is the base change of $\pi : \mathrm{Gr}_{\mathcal{G}}^{\mathrm{conv}} \to \mathrm{Gr}_{\mathcal{G}}$. Then to prove Lemma 6.20 is equivalent to prove that there is a morphism $\mathcal{O}_{\mathrm{Gr}_{\mathcal{G},\mu}} \to \mathcal{V}_w$ (which determines the section σ_0 of $\mathcal{L}_{2c}$) such that the composition $\mathcal{O}_{\mathrm{Gr}_{\mathcal{G},\mu}} \to \mathcal{V}_w \to \mathcal{V}_1$ is an isomorphism.

To this goal, let us first observe that the I-torsor $\mathrm{Gr}_{\mathcal{G},\underline{0}} \times_C C^\circ \to \mathrm{Gr}_{\mathcal{G}} \times_C C^\circ$ has a canonical section. Namely, we associated an R-point $(y, \mathcal{E}, \beta)$ of $\mathrm{Gr}_{\mathcal{G}} \times_C C^\circ$ an R-point $(y, \mathcal{E}, \beta, \gamma)$ of $\mathrm{Gr}_{\mathcal{G},\underline{0}} \times_C C^\circ$ as follows. Since the graph Γ_y of $y : \mathrm{Spec}R \to C$ does not intersect with $\{0\} \times \mathrm{Spec}R \subset C \times \mathrm{Spec}R$, we can define

$$\gamma : \mathcal{E}|_{\widehat{\{0\}\times \mathrm{Spec}\, R}} \to \mathcal{E}^0|_{\widehat{\{0\}\times \mathrm{Spec}\, R}}$$

as the restriction of $\beta : \mathcal{E}|_{C_R - \Gamma_y} \cong \mathcal{E}^0|_{C_R - \Gamma_y}$. By base change, we get a canonical section (a canonical trivialization) ψ of the I-torsor $W \times_{\mathrm{Gr}_{\mathcal{G}}} \mathrm{Gr}_{\mathcal{G},\underline{0}} \to W$, where $W = \mathrm{Gr}_{\mathcal{G},\mu}|_{\tilde{C}^\circ} \cong \mathrm{Gr}_\mu \times \tilde{C}^\circ$. Therefore, $\mathrm{Gr}_{\mathcal{G},\mu}\tilde{\times}\mathcal{F}\ell_w|_W \cong W \times \mathcal{F}\ell_w$

canonically, and over W, we have

$$\begin{array}{ccc} V_w \otimes \mathcal{O}_W & \longrightarrow & V_1 \otimes \mathcal{O}_W \\ \cong\downarrow & & \cong\downarrow \\ \mathcal{V}_w|_W & \longrightarrow & \mathcal{V}_1|_W. \end{array}$$

To complete the proof of the lemma, it is enough to show

(1) the isomorphism $V_1 \otimes \mathcal{O}_W \to \mathcal{V}_1|_W$ extends to an isomorphism $V_1 \otimes \mathcal{O}_{\mathrm{Gr}_{\mathcal{G},\mu}} \to \mathcal{V}_1$;
(2) there is a splitting $V_1 \to V_w$ (equivalently, a section of $\mathcal{L}(2\sum_{i\in\mathbf{S}} \varepsilon_i)$ whose divisor does not pass through $* \in \mathcal{F}\ell_w$), such that the induced map
$$V_1 \otimes \mathcal{O}_W \to V_w \otimes \mathcal{O}_W \to \mathcal{V}_w|_W$$
extends to $V_1 \otimes \mathcal{O}_{\mathrm{Gr}_{\mathcal{G},\mu}} \to \mathcal{V}_w$.

Let us first prove (1). Let us consider the general situation: Let $E \to B$ be a torsor under some group K, and let M be a space with trivial K-action. Then there is a canonical isomorphism $t : E \times^K M \simeq E/K \times M = B \times M$. In addition, for any section $s : B \to E$, the induced isomorphism $E \times^K M \simeq (B \times K) \times^K M \simeq B \times M$ coincides with t. Back to our situation, as the I-module V_1 is trivial, we can apply this general remark to conclude that $\mathcal{V}_1$ is canonically trivialized, which restricts to its trivialization over W induced from the canonical trivialization of the I-torsor over W.

To prove (2), let us first complete the curve $\bar{C} = C \cup \{\infty\} \cong \mathbb{P}^1$ and $\tilde{\bar{C}} = \tilde{C} \cup \{\tilde{\infty}\}$. We extend $\mathcal{G}$ to a group scheme over $\bar{C}$ so that $\mathcal{G}_{\mathcal{O}_\infty}$ is the pro-unipotent radical of the Iwahori opposite to $\mathcal{G}_{\mathcal{O}_0}$. More precisely, the pinning of H (Sections 2.1 and 3.2) determines a unique Borel B^- such that $B_H \cap B^- = T_H$. Let $U^- = [B^-, B^-]$. Let $\tilde{\mathcal{G}}$ be the group scheme over $\tilde{\bar{C}}$ obtained by dilatation of $H \times \tilde{\bar{C}}$ along $B_H \times \{\tilde{0}\}$ and $U^- \times \{\tilde{\infty}\}$. Then $\mathcal{G}$ is the neutral connected component of $(\mathrm{Res}_{\tilde{\bar{C}}/\bar{C}}\tilde{\mathcal{G}})^\Gamma$. This group scheme is the same as the group scheme $\mathcal{G}(0,1)$ in [HNY13]. Let

$$I^{u,-} = \Gamma(\bar{C} - \{0\}, \mathcal{G}),$$

considered as an ind-group over k. Then the Birkhoff decomposition (cf. [HNY13, Prop. 1(4)]) implies that $\mathrm{Lie}\, G = \mathrm{Lie}\, I \oplus \mathrm{Lie}\, I^{u,-}$ as k-vector spaces. (This is the triangular decomposition in the Kac-Moody theory.) Recall that we identify $\mathcal{O} \simeq \mathcal{O}_0$ and $G \simeq \mathcal{G}_{F_0}$ (cf. Section 3.2). For an $\mathcal{O}$-lattice L in $\mathrm{Lie}\, G$, consider the complex of k-vector spaces

$$L \oplus \mathrm{Lie}\, I^{u,-} \to \mathrm{Lie}\, G.$$

As L varies, its determinant defines a section of $\mathcal{L}_{\det}$ over the neutral connected component of $\mathrm{Gr}_{\mathrm{GL}(\mathrm{Lie}\, I)}$ (see [Fal03, §2] for details), whose pullback defines

a section σ^0 of $\mathcal{L}_{2c}$ vanishing away from $* \in \mathcal{F}\ell$. This gives us a splitting $V_1 \to V_w$, which we claim is the desired splitting satisfying (2).

To prove this claim, we need two more ingredients. Let $\mathrm{Bun}_\mathcal{G}$ be the moduli stack of $\mathcal{G}$-bundles on $\bar{C}$. Let us express $\mathcal{F}\ell$ as the ind-scheme representing $(\mathcal{E}, \beta)$, where $\mathcal{E}$ is a $\mathcal{G}$-torsor on $\bar{C}$ and β is a trivialization of $\mathcal{E}$ away from $0 \in \bar{C}$. Let $\omega_{\mathrm{Bun}_\mathcal{G}}^{-1}$ be the anti-canonical bundle of $\mathrm{Bun}_\mathcal{G}$. Its fiber over a $\mathcal{G}$-torsor $\mathcal{E}$ is the inverse of the determinant of the cohomology $\det R\Gamma(\mathbb{P}^1, \mathrm{ad}\mathcal{E})^{-1}$. Therefore $\omega_{\mathrm{Bun}_\mathcal{G}}^{-1}$ is isomorphic to the pullback along $\mathrm{Bun}_\mathcal{G} \to \mathrm{Bun}_{\mathrm{GL}(\mathrm{Lie}\,\mathcal{G})}$ of the inverse of the determinant of cohomology line bundle. As is well known (e.g., [Fal03]), the pullback of the latter line bundle on $\mathrm{Bun}_{\mathrm{GL}(\mathcal{V})}$ to $\mathrm{Gr}_{\mathrm{GL}(\mathcal{V})}$ is the determinant line bundle $\mathcal{L}_{\det}$ we introduced in Section 4. Therefore, we have $\mathcal{L}_{2c} \cong h^*\omega_{\mathrm{Bun}_\mathcal{G}}^{-1}$.

The following lemma is the first ingredient we need.

LEMMA 6.22. *The section σ^0 of $\mathcal{L}_{2c}$ descends to a section $\Theta \in \omega_{\mathrm{Bun}_\mathcal{G}}^{-1}$.*

Proof. Clearly, the adjoint action of $I^{u,-}$ preserves the determinant of $L \oplus \mathrm{Lie}\, I^{u,-} \to \mathrm{Lie}\, G$ up to a scalar. As $I^{u,-}$ has no nontrivial characters, the left action of $I^{u,-}$ on $\mathcal{F}\ell$ preserves σ^0. As $\mathrm{Bun}_\mathcal{G}$ is the quotient of $\mathcal{F}\ell$ by $I^{u,-}$ (cf. [HNY13, Prop. 1]), σ^0 descends. □

By [HNY13, Cor. 1.2], we can translate Θ to obtain sections of $\omega_{\mathrm{Bun}_\mathcal{G}}^{-1}$ over other connected components of $\mathrm{Bun}_\mathcal{G}$; we will still denote these sections by Θ.

Next, consider the following morphisms

$$\mathrm{Bun}_\mathcal{G} \xleftarrow{h_1} \mathrm{Gr}_\mathcal{G} \xleftarrow{\pi} \mathrm{Gr}_\mathcal{G}^{\mathrm{Conv}} \xrightarrow{m} \mathrm{Gr}_\mathcal{G}^{\mathrm{BD}} \xrightarrow{h_2} \mathrm{Bun}_\mathcal{G}.$$

The second ingredient we need is as follows.

LEMMA 6.23. *Over $\mathrm{Gr}_\mathcal{G} \tilde{\times} \mathcal{F}\ell_w \subset \mathrm{Gr}_\mathcal{G}^{\mathrm{Conv}}$, there is an isomorphism*

$$\tilde{\mathcal{L}}_{2c} \cong m^* h_2^* \omega_{\mathrm{Bun}_\mathcal{G}}^{-1} \otimes \pi^* h_1^* \omega_{\mathrm{Bun}_\mathcal{G}}.$$

Proof. Since $\mathrm{Gr}_\mathcal{G} \tilde{\times} \mathcal{F}\ell_w$ is proper over $\mathrm{Gr}_\mathcal{G}$, by the see-saw principle, it is enough to show that (i) for each $x \in \mathrm{Gr}_\mathcal{G}$, the restrictions of $m^* h_2^* \omega_{\mathrm{Bun}_\mathcal{G}}^{-1} \otimes \pi^* h_1^* \omega_{\mathrm{Bun}_\mathcal{G}}$ and $\tilde{\mathcal{L}}_{2c}$ to $\mathcal{F}\ell_w \subset \pi^{-1}(x)$ are isomorphic; and (ii) when restricting both line bundles via the section $z : \mathrm{Gr}_\mathcal{G} \to \mathrm{Gr}_\mathcal{G}^{\mathrm{Conv}}$, they are isomorphic.

Indeed, recall that over C°, $\mathrm{Gr}_\mathcal{G}^{\mathrm{Conv}}|_{C^\circ} \cong \mathrm{Gr}_\mathcal{G}^{\mathrm{BD}}|_{C^\circ} \cong \mathrm{Gr}_\mathcal{G}|_{C^\circ} \times \mathcal{F}\ell$, and over $0 \in C(k)$, the morphisms $(\mathrm{Gr}_\mathcal{G})_0 \xleftarrow{\pi} (\mathrm{Gr}_\mathcal{G}^{\mathrm{Conv}})_0 \xrightarrow{m} (\mathrm{Gr}_\mathcal{G}^{\mathrm{BD}})_0$ identify with $\mathcal{F}\ell \xleftarrow{\pi} \mathcal{F}\ell \tilde{\times} \mathcal{F}\ell \xrightarrow{m} \mathcal{F}\ell$. Under these isomorphisms,

$$h_2^* \omega_{\mathrm{Bun}_\mathcal{G}}^{-1}|_{\mathrm{Gr}_\mathcal{G}^{\mathrm{BD}}|_{C^\circ}} \cong h_1^* \omega_{\mathrm{Bun}_\mathcal{G}}^{-1}|_{\mathrm{Gr}_\mathcal{G}|_{C^\circ}} \otimes h^* \omega_{\mathrm{Bun}_\mathcal{G}}^{-1}, \quad h_2^* \omega_{\mathrm{Bun}_\mathcal{G}}^{-1}|_{(\mathrm{Gr}_\mathcal{G}^{\mathrm{BD}})_0} \cong h^* \omega_{\mathrm{Bun}_\mathcal{G}}^{-1}.$$

Therefore, for all $x \in \mathrm{Gr}_\mathcal{G}$, the restriction of $m^* h_2^* \omega_{\mathrm{Bun}_\mathcal{G}}^{-1} \otimes \pi^* h_1^* \omega_{\mathrm{Bun}_\mathcal{G}}$ to $\mathcal{F}\ell_w \subset \pi^{-1}(x)$ is isomorphic to $\mathcal{L}_{2c}$, which is just the restriction of $\tilde{\mathcal{L}}_{2c}$ to $\pi^{-1}(x) \simeq \mathcal{F}\ell$. The first fact is established. For the second fact, one can easily see that when

restricting both line bundles via $z : \mathrm{Gr}_{\mathcal{G}} \to \mathrm{Gr}_{\mathcal{G}}^{\mathrm{Conv}}$, they are isomorphic to the trivial bundle. □

Finally, we prove that σ^0 gives the desired splitting satisfying (2). Indeed, since the I-torsor $\mathrm{Gr}_{\mathcal{G},\underline{0}} \times_C C^\circ \to \mathrm{Gr}_{\mathcal{G}} \times_C C^\circ$ has a canonical section, we can spread out σ^0 as a section of $\tilde{\mathcal{L}}_{2c}$ over $\mathrm{Gr}_{\mathcal{G},\mu}\tilde{\times}\mathcal{F}\ell_w|_W$, still denoted by σ^0. This induces a map $V_1 \otimes \mathcal{O}_W \to V_w \otimes \mathcal{O}_W$. Then to prove (2), it is equivalent to show that σ^0 indeed extends to a section of $\tilde{\mathcal{L}}_{2c}$ over the whole $\mathrm{Gr}_{\mathcal{G},\mu}\tilde{\times}\mathcal{F}\ell_w$. Otherwise, let $n > 0$ be the smallest integer such that $u^n\sigma^0$ extends. (Recall that we use u to denote the global coordinate on $\tilde{C}$ so that $u = 0$ defines the divisor $(\mathrm{Gr}_{\mathcal{G},\mu})_{\tilde{0}}\tilde{\times}\mathcal{F}\ell_w$ inside $\mathrm{Gr}_{\mathcal{G},\mu}\tilde{\times}\mathcal{F}\ell_w$.) Then $u^n\sigma^0|_{(\mathrm{Gr}_{\mathcal{G},\mu})_{\tilde{0}}\tilde{\times}\mathcal{F}\ell_w}$ would not be zero. Observe that by construction, over $\mathrm{Gr}_{\mathcal{G},\mu}\tilde{\times}\mathcal{F}\ell_w|_W$, we have

$$\pi^* h_1^*\Theta \otimes \sigma^0 = m^* h_2^*\Theta,$$

as sections in $m^*h_2^*\omega_{\mathrm{Bun}_{\mathcal{G}}}^{-1}|_{\mathrm{Gr}_{\mathcal{G},\mu}\tilde{\times}\mathcal{F}\ell_w|_W}$. Then as sections in $m^*h_2^*\omega_{\mathrm{Bun}_{\mathcal{G}}}^{-1}$ over the whole $\mathrm{Gr}_{\mathcal{G},\mu}\tilde{\times}\mathcal{F}\ell_w$, we would have

$$\pi^* h_1^*\Theta \otimes u^n\sigma^0 = u^n m^* h_2^*\Theta.$$

When restricting this equation to $(\mathrm{Gr}_{\mathcal{G},\mu})_{\tilde{0}}\tilde{\times}\mathcal{F}\ell_w$, the right-hand side is zero. However, the left-hand side is not since $\pi^*h_1^*\Theta|_{(\mathrm{Gr}_{\mathcal{G},\mu})_{\tilde{0}}\tilde{\times}\mathcal{F}\ell_w} \neq 0$. This is a contradiction. This finishes the proof of Lemma 6.20.

7. **Proofs II: the nearby cycles**

7.1. *The strategy.* In this section, we prove Theorem 3.8. As mentioned in the introduction, a direct proof would be to write down a moduli problem $\mathcal{M}_\mu$ over $\tilde{C}$, which is a closed subscheme of $\widetilde{\mathrm{Gr}}_{\mathcal{G}}$, such that (i) $\mathcal{M}_\mu|_{\tilde{C}^\circ} \cong \overline{\mathrm{Gr}}_{\mathcal{G},\mu}|_{\tilde{C}^\circ}$; and (ii) $(\mathcal{M}_\mu)_{\tilde{0}}(k) = \bigcup_{w\in \mathrm{Adm}^Y(\mu)} \mathcal{F}\ell_w^Y(k)$. Then by Lemma 3.7, Theorem 3.8 would follow. Unfortunately, so far, such a moduli functor is not available for general group G and general coweight μ. In certain cases, such a moduli problem is available. We refer to [PRS13] for a survey of the known results.

The proof presented here is indirect. Let (S, s, η) be a Henselian trait; i.e., S is the spectrum of a Henselian discrete valuation ring, s is the closed point of S and η is the generic point of S. Assume that the residue field $k(s)$ of s is algebraically closed, and let ℓ be a prime different from $\mathrm{char}\, k(s)$. Recall that if $p : V \to S$ is a morphism, where V is a scheme, separated and of finite type over S, then there is the so-called nearby cycle functor

$$\Psi_V : D_c^b(V_\eta, \overline{\mathbb{Q}}_\ell) \to D_c^b(V_s \times_s \eta, \overline{\mathbb{Q}}_\ell),$$

which restricts to an exact functor ([Ill94, §4]) between the categories of perverse sheaves

$$\Psi_V : P(V_\eta, \overline{\mathbb{Q}}_\ell) \to P(V_s \times_s \eta, \overline{\mathbb{Q}}_\ell).$$

If V is an equidimensional variety over a field whose characteristic is prime to ℓ, the intersection cohomology sheaf is the Goresky-MacPherson extension to V of the (shifted) constant sheaf $\mathbb{Q}_\ell[\dim V]$ on the smooth locus of V. We will use the following lemma.

LEMMA 7.1. *Let $f : V \to S$ be a proper flat morphism. Let* IC *be the intersection cohomology sheaf of $V_\eta := V \times_S \eta$, and let $\Psi_V(\mathrm{IC})$ be the nearby cycle of* IC. *Then the support of $\Psi_V(\mathrm{IC})$ is V_s.*

Proof. Let $x \in V$ be a point in the special fiber V_s and $\bar{x}$ be a geometric point over x. Then by definition $\Psi_V(\mathrm{IC})_{\bar{x}} \cong H^*((V_{(\bar{x})})_{\bar{\eta}}, \mathrm{IC}|_{(V_{(\bar{x})})_{\bar{\eta}}})$, where $V_{(\bar{x})}$ is the strict Henselization of V at $\bar{x}$ and $(V_{(\bar{x})})_{\bar{\eta}}$ is its fiber over $\bar{\eta}$, a geometric point over η. Let x be a generic point of V_s. Then $(V_{(\bar{x})})_{\bar{\eta}}$ is the union of finite many points and $\mathrm{IC}|_{(V_{(\bar{x})})_{\bar{\eta}}} \cong \mathbb{Q}_\ell[\dim V]^m$ for some $m > 0$. The lemma follows. □

Now, let ℓ be a prime different from p. Let IC_μ be the intersection cohomology sheaf of $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}|_{\tilde{C}^\circ}$. Then the nearby cycle $\Psi_{\overline{\mathrm{Gr}}_{\mathcal{G},\mu}}(\mathrm{IC}_\mu)$ is a perverse sheaf on $\mathcal{F}\ell^Y$ whose support is $(\overline{\mathrm{Gr}}_{\mathcal{G},\mu})_{\tilde{0}}$. Therefore, to prove the theorem, it is enough to determine the support of $\Psi_{\overline{\mathrm{Gr}}_{\mathcal{G},\mu}}(\mathrm{IC}_\mu)$. In fact, we will give a filtration of $\Psi_{\overline{\mathrm{Gr}}_{\mathcal{G},\mu}}(\mathrm{IC}_\mu)$ and describe the support of each associated graded piece.

When the group G is split, such a description can be deduced from [AB09, Th. 4] directly. In the nonsplit case, we will mostly follow their strategy but with the following difference. We will not make use of the results in [Bez04, appendix], and therefore we will not generalize the full version of [AB09, Th. 4] to the ramified case (but see Remark 7.2). In particular, we will not perform any categorical arguments as in loc. cit.

7.2. *Central sheaves.* Let us set $K^Y = L^+\mathcal{G}_{\sigma_Y}$ and let $P_{K^Y}(\mathcal{F}\ell^Y)$ denote the category of K^Y-equivariant perverse sheaves on $\mathcal{F}\ell^Y$. Recall that this category is defined as the direct limit of categories of K^Y-equivariant perverse sheaves supported on the K^Y-stable finite-dimensional subvarieties of $\mathcal{F}\ell^Y$. (See [Gai01, appendix] for details.)

LEMMA 7.2. *The sheaf $\Psi_{\overline{\mathrm{Gr}}_{\mathcal{G},\mu}}(\mathrm{IC}_\mu)$ has a natural K^Y-equivariant structure and thus gives an object of $P_{K^Y}(\mathcal{F}\ell^Y)$.*

Proof. Let $\mathcal{L}^+_n\mathcal{G}$ be the n-th jet group of $\mathcal{G}$, i.e., the group scheme over C, whose R-points classify (y, β) where $y \in C(R)$ and $\beta \in \mathcal{G}(\Gamma_{y,n})$, where $\Gamma_{y,n}$ is the n-th nilpotent thickening of Γ_y. It is clear that $\mathcal{L}^+_n\mathcal{G}$ is smooth over C and the action of $\widetilde{\mathcal{L}^+\mathcal{G}}$ on $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}$ factors through some $\mathcal{L}^+_n\mathcal{G} \times_C \tilde{C}$ for n sufficiently large.

Let $m : \mathcal{L}_n^+\mathcal{G} \times_{\tilde{C}} \overline{\mathrm{Gr}}_{\mathcal{G},\mu} \to \overline{\mathrm{Gr}}_{\mathcal{G},\mu}$ be the action map and p be the natural projection. Then there is a canonical isomorphism $m^*\mathrm{IC}_\mu \cong p^*\mathrm{IC}_\mu$ as sheaves on $\mathcal{L}_n^+\mathcal{G} \times_{\tilde{C}} \overline{\mathrm{Gr}}_{\mathcal{G},\mu}|_{\tilde{C}^\circ}$. By taking nearby cycles, we have a canonical isomorphism

$$\Psi_{\mathcal{L}_n^+\mathcal{G}\times_{\tilde{C}}\mathrm{Gr}_{\mathcal{G},\mu}}(m^*\mathrm{IC}_\mu) \cong \Psi_{\mathcal{L}_n^+\mathcal{G}\times_{\tilde{C}}\mathrm{Gr}_{\mathcal{G},\mu}}(p^*\mathrm{IC}_\mu).$$

Since both m and p are smooth morphisms and taking nearby cycle commutes with smooth base change, we have

$$m^*\Psi_{\overline{\mathrm{Gr}}_{\mathcal{G},\mu}}(\mathrm{IC}_\mu) \cong p^*\Psi_{\overline{\mathrm{Gr}}_{\mathcal{G},\mu}}(\mathrm{IC}_\mu). \tag{7.2.1}$$

The isomorphism $m^*\mathrm{IC}_\mu \cong p^*\mathrm{IC}_\mu$ satisfies the cocycle condition under the pullback to $\mathcal{L}_n^+\mathcal{G} \times_{\tilde{C}} \mathcal{L}_n^+\mathcal{G} \times_{\tilde{C}} \overline{\mathrm{Gr}}_{\mathcal{G},\mu}|_{\tilde{C}^\circ}$. This implies the cocycle condition for the isomorphism (7.2.1). The lemma follows. □

Let us set

$$\mathcal{Z}_\mu = \Psi_{\overline{\mathrm{Gr}}_{\mathcal{G},\mu}}(\mathrm{IC}_\mu) \tag{7.2.2}$$

as a K^Y-equivariant perverse sheaf in $P_{K^Y}(\mathcal{F}\ell^Y)$.

Let $D(\mathcal{F}\ell^Y)$ be the derived category of constructible sheaves on $\mathcal{F}\ell^Y$ and $D_{K^Y}(\mathcal{F}\ell^Y)$ be the K^Y-equivariant derived category on $\mathcal{F}\ell^Y$. Recall that $D_{K^Y}(\mathcal{F}\ell^Y)$ is a monoidal category and there is a monoidal action (the "convolution product") of $D_{K^Y}(\mathcal{F}\ell)$ on $D(\mathcal{F}\ell^Y)$ (cf. [MV07, §4]). Namely, we have the convolution diagram

$$\mathcal{F}\ell^Y \times \mathcal{F}\ell^Y \xleftarrow{q} LG \times \mathcal{F}\ell^Y \xrightarrow{p} LG \times^{K^Y} \mathcal{F}\ell^Y = \mathcal{F}\ell^Y \tilde{\times} \mathcal{F}\ell^Y \xrightarrow{m} \mathcal{F}\ell^Y.$$

Let $\mathcal{F}_1 \in D(\mathcal{F}\ell^Y)$, $\mathcal{F}_2 \in D_{K^Y}(\mathcal{F}\ell^Y)$, and let $\mathcal{F}_1 \tilde{\times} \mathcal{F}_2$ be the unique sheaf on $LG \times^{K^Y} \mathcal{F}\ell^Y$ such that

$$p^*(\mathcal{F}_1 \tilde{\times} \mathcal{F}_2) \cong q^*(\mathcal{F}_1 \boxtimes \mathcal{F}_2). \tag{7.2.3}$$

Then

$$\mathcal{F}_1 \star \mathcal{F}_2 = m_!(\mathcal{F}_1 \tilde{\times} \mathcal{F}_2), \tag{7.2.4}$$

where $m_!$ is the derived pushforward functor with compact support. In general, if $\mathcal{F}_1, \mathcal{F}_2$ are perverse sheaves, it is not necessarily the case that $\mathcal{F}_1 \star \mathcal{F}_2$ is perverse. However, we have

THEOREM 7.3. (i) *Let $\mathcal{F}$ be an arbitrary perverse sheaf on $\mathcal{F}\ell^Y$. Then $\mathcal{F} \star \mathcal{Z}_\mu$ is a perverse sheaf on $\mathcal{F}\ell^Y$.*

(ii) *If $\mathcal{F} \in D_{K^Y}(\mathcal{F}\ell^Y)$, then there is a canonical isomorphism $c_\mathcal{F} : \mathcal{F} \star \mathcal{Z}_\mu \cong \mathcal{Z}_\mu \star \mathcal{F}$.*

Remark 7.1. (i) The isomorphism $c_\mathcal{F}$ is the composition of the isomorphisms in Proposition 7.4 below.

(ii) In the case when $G = H$ is a split group, this theorem is proved by Gaitsgory (cf. [Gai01]).[7] The general case proved below follows his line of argument. Still, we take the opportunity to spell out all the details for the following reasons. First, the family $\widetilde{\mathrm{Gr}}_{\mathcal{G}}$ we use here is in fact different from Gaitsgory's family, which has no obvious generalization to ramified groups. On the other hand, this theorem for ramified groups is used in [Zhu14] to establish the geometric Satake correspondence for ramified groups. Second, the use of the nonconstant group schemes allows us to simplify Gaitsgory's argument. Namely, we can treat (i) and (ii) in Proposition 7.4 below equally. This argument is generalized to a mixed characteristic situation in [PZ13]. On the other hand, in [Gai01], the proof of part (i) of Proposition 7.4 is considerably harder than the proof of part (ii).

(iii) To simplify the notation, in the proof we only consider $Y = \mathbf{a}$ an alcove. In this case, we denote by $I = K^{\mathbf{a}} = L^+\mathcal{G}_{\sigma_Y}$ the corresponding Iwahori subgroup of LG, and we denote $\mathcal{F}\ell = \mathcal{F}\ell^{\mathbf{a}}$. However, the proof (with the only change replacing I by K^Y and $\mathcal{F}\ell$ by $\mathcal{F}\ell^Y$) is valid in *any* parahoric case.

Proof. Recall the Beilinson-Drinfeld Grassmannian $\mathrm{Gr}_{\mathcal{G}}^{\mathrm{BD}}$ as introduced in (6.2.1). We have

$$\mathrm{Gr}_{\mathcal{G}}^{\mathrm{BD}} \times_C \tilde{C}^\circ \cong \mathcal{F}\ell \times (\mathrm{Gr}_{\mathcal{G}} \times_C \tilde{C}^\circ).$$

For $\mathcal{F} \in D(\mathcal{F}\ell)$, let

$$\mathcal{F} \boxtimes \mathrm{IC}_\mu \subset D(\mathcal{F}\ell \times (\mathrm{Gr}_{\mathcal{G}} \times_C \tilde{C}^\circ)),$$

which can be therefore regarded as a complex on $\mathrm{Gr}_{\mathcal{G}}^{\mathrm{BD}} \times_C \tilde{C}^\circ$. Consider the nearby cycle functor $\Psi_{\mathrm{Gr}_{\mathcal{G}}^{\mathrm{BD}} \times_C \tilde{C}}$.

Proposition 7.4. (i) *If $\mathcal{F} \in D(\mathcal{F}\ell)$, then there is a canonical isomorphism*

$$\Psi_{\mathrm{Gr}_{\mathcal{G}}^{\mathrm{BD}} \times_C \tilde{C}}(\mathcal{F} \boxtimes \mathrm{IC}_\mu) \cong \mathcal{F} \star \mathcal{Z}_\mu.$$

(ii) *If $\mathcal{F} \in D_I(\mathcal{F}\ell)$, then there is a canonical isomorphism*

$$\Psi_{\mathrm{Gr}_{\mathcal{G}}^{\mathrm{BD}} \times_C \tilde{C}}(\mathcal{F} \boxtimes \mathrm{IC}_\mu) \cong \mathcal{Z}_\mu \star \mathcal{F}.$$

It is clear that this proposition will imply the theorem. The isomorphisms involved in the statement essentially come from the fact that nearby cycles commute with proper pushforward and smooth pullback. They will be constructed in the proof.

[7] In fact, part (ii) of the theorem was proved in [Gai01] under the assumption that $\mathcal{F}$ is perverse. We are not sure whether the argument applies to the case that $\mathcal{F}$ is an arbitrary object in $D_{K^Y}(\mathcal{F}\ell^Y)$.

We first prove (ii). Let $\mathrm{Gr}_{\mathcal{G}}^{\mathrm{Conv}}$ be the convolution Grassmannian as introduced in (6.2.2), which we recall is a fibration over $\mathrm{Gr}_{\mathcal{G}}$ with fibers isomorphic to $\mathcal{F}\ell$. Regard $\mathcal{F}\boxtimes \mathrm{IC}_\mu$ as a complex of sheaves on $\mathrm{Gr}_{\mathcal{G}}^{\mathrm{Conv}}\times_C \tilde{C}^\circ \cong \mathcal{F}\ell\times(\mathrm{Gr}_{\mathcal{G}}\times_C \tilde{C}^\circ)$. Since taking nearby cycles commutes with proper pushforward, it is enough to prove that as complex of sheaves on $\mathcal{F}\ell\tilde{\times}\mathcal{F}\ell$, there is a canonical isomorphism

$$\Psi_{\mathrm{Gr}_{\mathcal{G}}^{\mathrm{Conv}}\times_C\tilde{C}}(\mathcal{F}\boxtimes \mathrm{IC}_\mu)\cong \mathcal{Z}_\mu\tilde{\times}\mathcal{F},$$

where $\mathcal{Z}_\mu\tilde{\times}\mathcal{F}$ is the twisted product as defined in (7.2.3).

Recall the I-torsor $\mathrm{Gr}_{\mathcal{G},\underline{0}}$ over $\mathrm{Gr}_{\mathcal{G}}$ defined in (6.2.3) and $\mathrm{Gr}_{\mathcal{G}}^{\mathrm{Conv}}\cong \mathrm{Gr}_{\mathcal{G},\underline{0}}\times^I \mathcal{F}\ell$. Let $V\subset\mathcal{F}\ell$ be the support of $\mathcal{F}$, and let $I_n = L_n^+\mathcal{G}_{\mathcal{O}_0}$ (the n-th jet group as defined in the proof of Lemma 7.2) be the finite-dimensional quotient of I such that the action of I on V factors through I_n. Let $\mathrm{Gr}_{\mathcal{G},0,n}$ be the I_n-torsor over $\mathrm{Gr}_{\mathcal{G}}$ that classifies $(y,\mathcal{E},\beta,\gamma)$ where $(y,\mathcal{E},\beta)$ is as in the definition of $\mathrm{Gr}_{\mathcal{G}}$ and γ is a trivialization of $\mathcal{E}$ on the n-th infinitesimal neighborhood of $0\in C$. Then $\mathrm{IC}_\mu\tilde{\times}\mathcal{F}$ is supported on

$$(\tilde{C}\times_C \mathrm{Gr}_{\mathcal{G},\underline{0}})\times^I V\cong (\tilde{C}\times_C \mathrm{Gr}_{\mathcal{G},0,n})\times^{I_n} V\subset \mathrm{Gr}_{\mathcal{G}}^{\mathrm{Conv}}\times_C\tilde{C}.$$

Observe that over $\tilde{C}^\circ$, it makes sense to talk about $\mathrm{IC}_\mu\tilde{\times}\mathcal{F}$ (as defined via (7.2.3)), which is canonically isomorphic to $\mathcal{F}\boxtimes\mathrm{IC}_\mu$. We thus need to show that

$$\Psi_{\mathrm{Gr}_{\mathcal{G}}^{\mathrm{Conv}}\times_C\tilde{C}}(\mathrm{IC}_\mu\tilde{\times}\mathcal{F})\cong \mathcal{Z}_\mu\tilde{\times}\mathcal{F}. \tag{7.2.5}$$

Let us denote the pullback of IC_μ to $\mathrm{Gr}_{\mathcal{G},0,n}\times_C\tilde{C}^\circ$ by $\widetilde{\mathrm{IC}}_\mu$. Since $\mathrm{Gr}_{\mathcal{G},0,n}\to\mathrm{Gr}_{\mathcal{G}}$ is smooth, $\Psi_{\mathrm{Gr}_{\mathcal{G},0,n}\times_C\tilde{C}}(\widetilde{\mathrm{IC}}_\mu)$ is canonically isomorphic to the pullback of $\mathcal{Z}_\mu$, and

$$\Psi_{(\mathrm{Gr}_{\mathcal{G},0,n}\times_C\tilde{C})\times V}(\widetilde{\mathrm{IC}}_\mu\boxtimes\mathcal{F})\cong \Psi_{\mathrm{Gr}_{\mathcal{G},0,n}\times_C\tilde{C}}(\widetilde{\mathrm{IC}}_\mu)\boxtimes\mathcal{F}$$

is I_n-equivariant. We thus have (7.2.5).

Next we prove (i). There is another convolution affine Grassmannian $\mathrm{Gr}_{\mathcal{G}}^{\mathrm{Conv}'}$, which is an ind-scheme ind-proper over C and represents the functor that associates to every k-algebra R:

(7.2.6)

$$\mathrm{Gr}_{\mathcal{G}}^{\mathrm{Conv}'}(R)=\left\{(y,\mathcal{E},\mathcal{E}',\beta,\beta')\;\middle|\;\begin{array}{l} y\in C(R),\mathcal{E},\mathcal{E}' \text{ are two } \mathcal{G}\text{-torsors on } C_R,\\ \beta:\mathcal{E}|_{(C-\{0\})_R}\cong\mathcal{E}^0|_{(C-\{0\})_R} \text{ is a}\\ \text{trivialization, and } \beta':\mathcal{E}'|_{C_R-\Gamma_y}\cong\mathcal{E}|_{C_R-\Gamma_y}\end{array}\right\}.$$

Let us sketch the proof of the ind-representability of $\mathrm{Gr}_{\mathcal{G}}^{\mathrm{Conv}'}$. Let $\mathcal{L}_n^+\mathcal{G}$ be the n-th jet group of $\mathcal{G}$. As mentioned before, $\mathcal{L}_n^+\mathcal{G}$ is smooth over C. Then one can present $\mathrm{Gr}_{\mathcal{G}}$ as the inductive limit $\varinjlim Z_n$, where Z_n is a $\mathcal{L}^+\mathcal{G}$-stable closed subscheme and the action of $\mathcal{L}^+\mathcal{G}$ on Z_n factors through $\mathcal{L}_n^+\mathcal{G}$. Let us define the $\mathcal{L}_n^+\mathcal{G}$-torsor $\mathcal{P}_n$ over $\mathcal{F}\ell\times C$ as follows. Its R-points are quadruples

$(y, \mathcal{E}, \beta, \gamma)$, where $y \in C(R)$, $(\mathcal{E}, \beta)$ are as in the definition of $\mathcal{F}\ell$ (and therefore β is a trivialization of $\mathcal{E}$ on C_R°), and γ is a trivialization of $\mathcal{E}$ over $\Gamma_{y,n}$, the n-th nilpotent thickening of the graph Γ_y of y. Then it is not hard to see that $\mathrm{Gr}_{\mathcal{G}}^{\mathrm{Conv}'} = \varinjlim \mathcal{P}_n \times^{\mathcal{L}_n^+\mathcal{G}} Z_n$ is an ind-scheme ind-proper over C.

Clearly, we have $m' : \mathrm{Gr}_{\mathcal{G}}^{\mathrm{Conv}'} \to \mathrm{Gr}_{\mathcal{G}}^{\mathrm{BD}}$ by sending $(y, \mathcal{E}, \mathcal{E}', \beta, \beta')$ to $(y, \mathcal{E}', \beta' \circ \beta)$. This is a morphism over C, which is an isomorphism over $C - \{0\}$, and m'_0 again is the local convolution diagram

$$m : \mathcal{F}\ell \tilde{\times} \mathcal{F}\ell \to \mathcal{F}\ell.$$

Again, regarding $\mathcal{F} \boxtimes \mathrm{IC}_\mu$ as a sheaf on $\mathrm{Gr}_{\mathcal{G}}^{\mathrm{Conv}'}|_{\tilde{C}^\circ} \cong \mathcal{F}\ell \times (\mathrm{Gr}_{\mathcal{G}} \times_C \tilde{C}^\circ)$, it is enough to prove that as sheaves on $\mathcal{F}\ell \tilde{\times} \mathcal{F}\ell$,

$$\Psi_{\mathrm{Gr}_{\mathcal{G}}^{\mathrm{Conv}'} \times_C \tilde{C}}(\mathcal{F} \boxtimes \mathrm{IC}_\mu) \cong \mathcal{F} \tilde{\times} \mathcal{Z}_\mu.$$

Observe that the action of $\widetilde{\mathcal{L}^+\mathcal{G}}$ on $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}$ factors through some $\mathcal{L}_n^+\mathcal{G} \times_C \tilde{C}$ for n sufficiently large. Then we have the twisted product

$$(\mathcal{P}_n \times_C \tilde{C}) \times^{\mathcal{L}_n^+\mathcal{G} \times_C \tilde{C}} \overline{\mathrm{Gr}}_{\mathcal{G},\mu} \subset \mathrm{Gr}_{\mathcal{G}}^{\mathrm{Conv}'} \times_C \tilde{C}.$$

Over the restriction of this ind-scheme to $\tilde{C}^\circ$, we can form the twisted product $\mathcal{F}[1] \tilde{\times} \mathrm{IC}_\mu$ as in (7.2.3), which is canonically isomorphic to $\mathcal{F} \tilde{\boxtimes} \mathrm{IC}_\mu$. By the same argument as in the proof of (ii) (i.e., by pulling back everything to $\mathcal{P}_n \times_{\tilde{C}} \overline{\mathrm{Gr}}_{\mathcal{G},\mu}$), we have

$$\Psi_{(\mathcal{P}_n \times_C \tilde{C}) \times^{\mathcal{L}_n^+\mathcal{G} \times_C \tilde{C}} \overline{\mathrm{Gr}}_{\mathcal{G},\mu}}(\mathcal{F}[1] \tilde{\times} \mathrm{IC}_\mu) \cong \Psi_{\mathcal{F}\ell \times \tilde{C}}(\mathcal{F}[1]) \tilde{\times} \Psi_{\overline{\mathrm{Gr}}_{\mathcal{G},\mu}}(\mathrm{IC}_\mu) \cong \mathcal{F} \tilde{\times} \mathcal{Z}_\mu.$$

□

7.3. *Wakimoto filtrations.* Our goal is to prove that the support of $\mathcal{Z}_\mu$ is exactly the Schubert varieties in $\mathcal{F}\ell^Y$ labeled by the set $W^Y \setminus \mathrm{Adm}^Y(\mu)/W^Y$, which will imply Theorem 3.8 by Lemma 7.1. Clearly, it is enough to prove this in the case that $\mathcal{G}_{\mathcal{O}_0}$ is Iwahori.

Let us recall some standard objects in $P_I(\mathcal{F}\ell)$. Recall that I-orbits in $\mathcal{F}\ell$ are labeled by elements $w \in \widetilde{W}$. For any w, let $j_w : C(w) \to \mathcal{F}\ell_w$ be the open embedding of the Schubert cell to the Schubert variety. This is an affine embedding. Let us denote

$$j_{w*} = (j_w)_* \overline{\mathbb{Q}}_\ell[\ell(w)], \qquad j_{w!} = (j_w)_! \overline{\mathbb{Q}}_\ell[\ell(w)].$$

Then it is well known (e.g., [AB09, Lemma 8]) that there are canonical isomorphisms

$$j_{w*} \star j_{w'*} \cong j_{ww'*}, \quad j_{w!} \star j_{w'!} \cong j_{ww'!} \quad \text{if } \ell(ww') = \ell(w) + \ell(w'), \tag{7.3.1}$$

$$j_{w*} \star j_{w^{-1}!} \cong j_{w^{-1}!} \star j_{w*} \cong \delta_e.$$

In addition, if $\ell(ww'w'') = \ell(w) + \ell(w') + \ell(w'')$, then the two isomorphisms from $j_{w*} \star j_{w'*} \star j_{w''*}$ (resp. from $j_{w!} \star j_{w'!} \star j_{w''!}$) to $j_{ww'w''*}$ (resp. to $j_{ww'w''!}$) are the same.

Let us recall the following fundamental result due to I. Mirkovic (cf. [AB09, appendix]). The proof for ramified groups is exactly the same as for split groups. In fact, the proof works in the general affine Kac-Moody setting.

PROPOSITION 7.5. *Let $w, v \in \widetilde{W}$. Then both $j_{w*} \star j_{v!}$ and $j_{w!} \star j_{v*}$ are perverse sheaves. In addition, both sheaves are supported on the Schubert variety $\mathcal{F}\ell_{wv}$ and $j_{wv}^*(j_{w*} \star j_{v!}) \cong j_{wv}^*(j_{w!} \star j_{v*}) \cong \overline{\mathbb{Q}}_\ell[\ell(wv)]$.*

Fix $w \in W_0$ to be an element in the finite Weyl group of G. We are going to define the w-Wakimoto sheaves on $\mathcal{F}\ell$. Recall the definition of $\mathbb{X}_\bullet(T)_\Gamma^+$ in (2.1.5). For $\mu \in \mathbb{X}_\bullet(T)_\Gamma$, we write $\mu = \lambda - \nu$ with $\lambda, \nu \in w(\mathbb{X}_\bullet(T)_\Gamma^+)$. Define

$$J_\mu^w = j_{t_\lambda !} \star j_{t_\nu *}, \tag{7.3.2}$$

which is well defined up to a canonical isomorphism (by (7.3.1)). By Proposition 7.5, $J_\mu^w \in \mathrm{P}_I(\mathcal{F}\ell)$ and is supported on $\mathcal{F}\ell_\mu$ with $j_{t_\mu}^* J_\mu^w \cong \bar{\mathbb{Q}}_\ell[\ell(t_\mu)]$. Let us remark that for G being split and $w = w_0$ being the longest element in W_0, they are the Wakimoto sheaves considered in [AB09]. In addition, we have

$$J_\mu^w \star J_\lambda^w \cong J_{\mu+\lambda}^w. \tag{7.3.3}$$

In fact, by (7.3.1) and Lemma 9.1, this is true for μ, λ for $\mu, \lambda \in w(\mathbb{X}_\bullet(T)_\Gamma^+)$. The extension to all μ, λ is immediate.

One of the important applications of the Wakimoto sheaves is as follows. An object $\mathcal{F} \in P_I(\mathcal{F}\ell)$ is called convolution exact if $\mathcal{F}' \star \mathcal{F}$ is perverse for any $\mathcal{F}' \in P_I(\mathcal{F}\ell)$ and is called central if in addition $\mathcal{F} \star \mathcal{F}' \cong \mathcal{F}' \star \mathcal{F}$. For example, $\mathcal{Z}_\mu$ is central. The following proposition generalizes [AB09, Prop. 5], where the case $w = e$ is considered. The proof is basically the same.

PROPOSITION 7.6. *Fix $w \in W_0$. Any central object in $P_I(\mathcal{F}\ell)$ has a filtration whose associated graded pieces are $J_\lambda^w, \lambda \in \mathbb{X}_\bullet(T)_\Gamma$.*

Proof. We begin with some general notation and remarks following [AB09]. For a triangulated category D and a set of objects $S \subset \mathrm{Ob}(D)$, let $\langle S \rangle$ be the set of all objects obtained from elements of S by extensions; i.e., $\langle S \rangle$ is the smallest subset of $\mathrm{Ob}(D)$ containing $S \cup \{0\}$ and such that

(1) if $A \cong B$ and $A \in \langle S \rangle$, then $B \in \langle S \rangle$; and
(2) for all $A, B \in \langle S \rangle$ and an exact triangle $A \to C \to B \to A[1]$, we have $C \in \langle S \rangle$.

Let $\mathcal{F} \in D_I(\mathcal{F}\ell)$. The $*$-support of $\mathcal{F}$ is defined to be

$$W_{\mathcal{F}}^* := \{w \in \widetilde{W} \mid j_w^* \mathcal{F} \neq 0\},$$

and the !-support of $\mathcal{F}$ is the set

$$W_{\mathcal{F}}^! := \{w \in \widetilde{W} \mid j_w^! \mathcal{F} \neq 0\}.$$

By the induction on the dimension of the support of $\mathcal{F}$, it is easy to see that if $\mathcal{F} \in D_I(\mathcal{F}\ell)^{p,\leq 0}$ (p stands for the perverse t-structure), then $\mathcal{F}$ is contained in $\langle j_{v!}[n] \mid v \in W^*_{\mathcal{F}}, n \geq 0\rangle$. On the other hand, if $\mathcal{F} \in D_I(\mathcal{F}\ell)^{p,\geq 0}$, then $\mathcal{F} \in \langle j_{v*}[n] \mid v \in W^!_{\mathcal{F}}, n \leq 0\rangle$.

For any $\mathcal{F} \in D_I(\mathcal{F}\ell)$, there exists a finite subset $S_{\mathcal{F}} \subset \widetilde{W}$ such that

$$W^!_{j_{w*}\star\mathcal{F}}, W^*_{j_{w!}\star\mathcal{F}} \subset w \cdot S_{\mathcal{F}}; \qquad W^!_{\mathcal{F}\star j_{w*}}, W^*_{\mathcal{F}\star j_{w!}} \subset S_{\mathcal{F}} \cdot w.$$

Namely, let $\mathcal{F}\ell_v$ be a Schubert variety such that $\mathcal{F}$ is supported in $\mathcal{F}\ell_v$ (in both the $*$-sense and the !-sense). Then by the proper base change theorem, the above assertions will follow if we can show that there exists $S_v \subset \widetilde{W}$ such that

$$C(w)\tilde{\times}\mathcal{F}\ell_v \subset \bigcup_{v' \in wS_v} C(v'), \qquad \mathcal{F}\ell_v\tilde{\times}C(w) \subset \bigcup_{v' \in S_v w} C(v').$$

This can be proved easily by induction of the length of v.

Now we prove the proposition. Let $\mathcal{F} \in P_I(\mathcal{F}\ell)$ be a central object, and let $S_{\mathcal{F}} \subset \widetilde{W}$ be the finite set associated to $\mathcal{F}$ as above. Recall that we have the special vertex v_0 in the building of G that determines an isomorphism $\widetilde{W} = \mathbb{X}_\bullet(T)_\Gamma \rtimes W_0$ determined by v_0. Let $\mu \in w(\mathbb{X}_\bullet(T)^+_\Gamma)$ such that

$$t_\mu S_{\mathcal{F}} \subset w(\mathbb{X}_\bullet(T)^{++}_\Gamma)W_0, \qquad S_{\mathcal{F}} t_\mu \subset W_0 w(\mathbb{X}_\bullet(T)^{++}_\Gamma),$$

where $\mathbb{X}_\bullet(T)^{++}_\Gamma$ is the subset of regular elements in $\mathbb{X}_\bullet(T)^+_\Gamma$. This is always possible since $S_{\mathcal{F}}$ is a finite set. We have $J^w_\mu = j_{\mu!}$, and from $J^w_\mu \star \mathcal{F} \cong \mathcal{F} \star J^w_\mu$, we have

$$W^*_{J^w_\mu\star\mathcal{F}} \subset t_\mu S_{\mathcal{F}} \cap S_{\mathcal{F}} t_\mu \subset w(\mathbb{X}_\bullet(T)^{++}_\Gamma)W_0 \cap W_0 w(\mathbb{X}_\bullet(T)^{++}_\Gamma) = w(\mathbb{X}_\bullet(T)^{++}_\Gamma).$$

Therefore, $J^w_\mu \star \mathcal{F} \in \langle j_{t_\lambda!}[n] \mid \lambda \in w(\mathbb{X}_\bullet(T)^+_\Gamma), n \geq 0\rangle$. Observe that $J^w_\lambda = j_{t_\lambda!}$ for $\lambda \in w(\mathbb{X}_\bullet(T)^+_\Gamma)$. Then by (7.3.3), we have

$$\mathcal{F} \in \langle J^w_\lambda[n] \mid \lambda \in \mathbb{X}_\bullet(T)_\Gamma, n \geq 0\rangle.$$

By choosing $\mu \in w(-\mathbb{X}_\bullet(T)^+_\Gamma)$ large enough and using $J^w_\lambda = j_{t_\lambda *}$ for $\lambda \in w(-\mathbb{X}_\bullet(T)^+_\Gamma)$, we have

$$\mathcal{F}' := j_{t_\mu *} \star \mathcal{F} = J^w_\mu \star \mathcal{F} \in \langle j_{t_\lambda *}[n] \mid \lambda \in w(-\mathbb{X}_\bullet(T)^+_\Gamma), n \geq 0\rangle. \tag{7.3.4}$$

We claim that this already implies that $\mathcal{F}'$ has a filtration (in the category of perverse sheaves) with associated graded by $j_{t_\mu *}, \mu \in w(-\mathbb{X}_\bullet(T)^+_\Gamma)$. Therefore, this implies the proposition. Indeed, since $\mathcal{F}'$ is perverse, for any $\nu \in w(-\mathbb{X}_\bullet(T)^+_\Gamma)$, the !-stalk of $\mathcal{F}'$ at t_ν has homological degree $\geq -\ell(t_\nu)$. On the other hand, (7.3.4) implies that the !-stalk of $\mathcal{F}'$ at t_ν has homological degree $\leq -\ell(t_\nu)$. The claim follows. □

To proceed, we now study the category of perverse sheaves on $\mathcal{F}\ell$ that are generated by J^w_λ.

LEMMA 7.7. *For* $\lambda, \mu \in \mathbb{X}_\bullet(T)_\Gamma$, $R\mathrm{Hom}(J^w_\lambda, J^w_\mu) = 0$ *unless* $w^{-1}(\lambda) \preceq w^{-1}(\mu)$. *Furthermore,* $R\mathrm{Hom}(J^w_\mu, J^w_\mu) \cong \overline{\mathbb{Q}}_\ell$.

Proof. $R\mathrm{Hom}(J^w_\lambda, J^w_\mu) = R\mathrm{Hom}(J^w_{\lambda+\nu}, J^w_{\mu+\nu}) = R\mathrm{Hom}(j_{t_{\lambda+\nu}!}, j_{t_{\mu+\nu}!})$ for $\nu \in \mathbb{X}_\bullet(T)_\Gamma$ such that $\lambda+\nu, \mu+\nu \in w(\mathbb{X}_\bullet(T)^+_\Gamma)$. The above complex of ℓ-adic vector spaces is nonzero only if $\mathcal{F}\ell_{t_{\lambda+\nu}} \subset \mathcal{F}\ell_{t_{\mu_\nu}}$; i.e., $t_{\lambda+\nu} \leq t_{\mu+\nu}$ in the Bruhat order. This is equivalent to $t_{w^{-1}(\lambda+\nu)} \leq t_{w^{-1}(\mu+\nu)}$ by Lemma 9.6, which is in turn equivalent to $w^{-1}(\lambda+\nu) \preceq w^{-1}(\mu+\nu)$ by Lemma 9.4, which is equivalent to $w^{-1}(\lambda) \preceq w^{-1}(\mu)$. The second statement follows from $R\mathrm{Hom}(J^w_\mu, J^w_\mu) \cong R\mathrm{Hom}(J^w_0, J^w_0) \cong \overline{\mathbb{Q}}_\ell$. □

LEMMA 7.8. *Let* $\mathcal{F} \in D_I(\mathcal{F}\ell)$. *Then for any* $\mu \in \mathbb{X}_\bullet(T)_\Gamma$,

$$H^*(\mathcal{F}\ell, J^w_\mu \star \mathcal{F}) \cong H^{*-(w^{-1}(\mu),2\rho)}(\mathcal{F}\ell, \mathcal{F}).$$

In particular, $H^*(\mathcal{F}\ell, J^w_\mu) = H^{(w^{-1}(\mu),2\rho)}(\mathcal{F}\ell, J^w_\mu) \cong \bar{\mathbb{Q}}_\ell$.

Proof. For any $v \in \widetilde{W}$, let $C(v)$ be the Schubert cell in $\mathcal{F}\ell$ corresponding to v. Then we have $m : C(v)\tilde{\times}\mathcal{F}\ell \to \mathcal{F}\ell$, which is an affine bundle over $\mathcal{F}\ell$. Then the isomorphism $j_{v*} \star \mathcal{F} \cong m_*(\overline{\mathbb{Q}}_\ell[\ell(v)]\tilde{\times}\mathcal{F})$ induces $H^*(\mathcal{F}\ell, j_{v*} \star \mathcal{F}) \cong H^*(\mathcal{F}\ell, \mathcal{F})[\ell(v)]$. Therefore, for $\mu \in w(-\mathbb{X}_\bullet(T)^+_\Gamma)$, the lemma holds by the above fact and Lemma 9.1. If the lemma holds for λ, μ, then

$$H^*(\mathcal{F}\ell, \mathcal{F}) \cong H^*(\mathcal{F}\ell, J^w_\lambda \star J^w_{-\lambda} \star \mathcal{F}) \cong H^{*-(w^{-1}(\lambda),2\rho)}(\mathcal{F}\ell, J^w_{-\lambda} \star \mathcal{F}),$$

$$H^*(\mathcal{F}\ell, J^w_{\lambda+\mu} \star \mathcal{F}) \cong H^{*-(w^{-1}(\lambda),2\rho)}(\mathcal{F}\ell, J^w_\mu \star \mathcal{F}) \cong H^{*-(w^{-1}(\lambda+\mu),2\rho)}(\mathcal{F}\ell, \mathcal{F}).$$

Therefore, the lemma holds for $-\lambda$ and $\lambda+\mu$. Now any element in $\mathbb{X}_\bullet(T)_\Gamma$ can be written as $\lambda - \mu$ with $\lambda, \mu \in w(\mathbb{X}_\bullet(T)^+_\Gamma)$, and using this we complete the proof. □

Let $\mathrm{W}^w(\mathcal{F}\ell)$ be the full abelian subcategory of $P_I(\mathcal{F}\ell)$ generated by J^w_μ, $\mu \in \mathbb{X}_\bullet(T)_\Gamma$. Let $\mathrm{W}^w(\mathcal{F}\ell)_{\succeq\mu}$ be the category of $\mathrm{W}^w(\mathcal{F}\ell)$ that is generated by $J^w_\lambda, w^{-1}(\lambda) \succeq w^{-1}(\mu)$. For each object $\mathcal{F} \in \mathrm{W}^w(\mathcal{F}\ell)$, we define a filtration

$$\mathcal{F} = \bigcup_\mu \mathcal{F}^w_{\succeq\mu},$$

where $\mathcal{F}^w_{\succeq\mu} \in \mathrm{W}^w(\mathcal{F}\ell)_{\succeq\mu}$ is the maximal subobject of $\mathcal{F}$ that belongs to $\mathrm{W}^w(\mathcal{F}\ell)_{\succeq\mu}$. Then by Lemma 7.7,

$$\mathcal{F}^w_{\succeq\mu} / \bigcup_{w^{-1}(\mu') \succ w^{-1}(\mu)} \mathcal{F}^w_{\succeq\mu'} \cong J^w_\mu \otimes {}^wW^\mu_{\mathcal{F}},$$

where ${}^wW^\mu_{\mathcal{F}}$ is a finite-dimensional $\overline{\mathbb{Q}}_\ell$ vector space. A direct consequence of Lemma 7.8 is

COROLLARY 7.9. *Suppose that the notation is as above. Then for any* $\mathcal{F} \in \mathrm{W}^w(\mathcal{F}\ell)$, *we have*

$$H^*(\mathcal{F}\ell, \mathcal{F}) \cong \bigoplus_{\mu \in \mathbb{X}_\bullet(T)_\Gamma} H^*(\mathcal{F}\ell, J^w_\mu) \otimes {}^wW^\mu_{\mathcal{F}}.$$

7.4. *Proof of Theorem* 3.8. Finally, let us prove Theorem 3.8. Let $\mu \in \mathbb{X}_\bullet(T)^+_\Gamma$. Let $\mathrm{Supp}(\mu)$ denote the subset of $\widetilde{W}$ consisting of those w such that $\mathcal{F}\ell_w \subset (\overline{\mathrm{Gr}}_{\mathcal{G},\mu})_{\tilde{0}}$. We need to show that $\mathrm{Supp}(\mu) = \mathrm{Adm}(\mu)$. We already know that $\mathrm{Adm}(\mu) \subset \mathrm{Supp}(\mu)$ (Lemma 3.7). By Propositions 7.6 and 7.5, we also know that the maximal elements in $\mathrm{Supp}(\mu)$ (under the Bruhat order) belong to $\mathbb{X}_\bullet(T)_\Gamma \subset \widetilde{W}$. Let $t_{\mu'} \in \mathrm{Supp}(\mu)$ be a maximal element. Then there exists some $w \in W_0$ such that $\mu' \in w(\mathbb{X}_\bullet(T)^+_\Gamma)$. By Proposition 7.6, $\mathcal{Z}_\mu \in \mathrm{W}^w(\mathcal{F}\ell)$. Write $\mathcal{Z}_\mu = \cup_\lambda (\mathcal{Z}_\mu)^w_{\preceq\lambda}$ so that the associated graded pieces are $J^w_\lambda \otimes {}^wW^\lambda_\mu$ as above. (We write ${}^wW^\lambda_\mu$ instead of ${}^wW^\lambda_{\mathcal{Z}_\mu}$ for brevity.) By Lemma 7.1, ${}^wW^{\mu'}_\mu \neq 0$. In addition, being a maximal element in $\mathrm{Supp}(\mu)$, $t_{\mu'}$ must have length $(\mu, 2\rho)$. Therefore, $(w^{-1}(\mu'), 2\rho) = (\mu, 2\rho)$. On the other hand, $t_{w(\mu)} \in \mathrm{Adm}(\mu) \subset \mathrm{Supp}(\mu)$ is also a maximal element in $\mathrm{Supp}(\mu)$ since $\ell(t_{w(\mu)}) = (2\rho, \mu)$ by Lemma 9.1. Therefore, ${}^wW^{w(\mu)}_\mu \neq 0$. We claim that $\mu' = w(\mu)$. Otherwise, we would have

$$H^{(\mu,2\rho)}(\mathcal{F}\ell, \mathcal{Z}_\mu) \supset {}^wW^{\mu'}_\mu \oplus {}^wW^{w(\mu)}_\mu$$

whose dimension would be at least two.

On the other hand, the map $f : \overline{\mathrm{Gr}}_{\mathcal{G},\mu} \to \tilde{C}$ is proper, and therefore $H^*(\mathcal{F}\ell, \mathcal{Z}_\mu) \cong \Psi_{\tilde{C}}(f_*\mathrm{IC}_\mu)$. Since $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}|_{\tilde{C}^\circ} \cong \overline{\mathrm{Gr}}_\mu \times \tilde{C}^\circ$, we have $H^*(\mathcal{F}\ell, \mathcal{Z}_\mu) \cong IH^*(\overline{\mathrm{Gr}}_\mu)$, where IH^* denotes the intersection cohomology of $\overline{\mathrm{Gr}}_\mu$. It is well known (for example, see [MV07]) that $IH^{(\mu,2\rho)}(\overline{\mathrm{Gr}}_\mu) \cong \mathbb{Q}_\ell$, which contradicts the above unless $\mu' = w(\mu)$. In other words, all the maximal elements on $\mathrm{Supp}(\mu)$ are contained in $\mathrm{Adm}(\mu)$, which proves the theorem.

Remark 7.2. One should be able to generalize [AB09, Th. 4] to the ramified case, which will imply Theorem 3.8 directly. We sketch here a possible approach. First, $\mathrm{W}^w(\mathcal{F}\ell)$ is indeed a monoidal abelian subcategory of $P_I(\mathcal{F}\ell)$ because $J^w_\lambda \star J^w_\mu \cong J^w_{\lambda+\mu}$. Let $\mathrm{GrW}^w(\mathcal{F}\ell)$ be the submonoidal category whose objects are direct sums of J^w_λ. One can see that this category is equivalent to $\mathrm{Rep}(\hat{T}^\Gamma)$, where $\hat{T}$ is the dual torus of T defined over $\bar{\mathbb{Q}}_\ell$, and $\hat{T}^\Gamma$ is the Galois fixed subgroup. By taking the associated graded of the filtration of $\mathcal{F} \in \mathrm{W}^w(\mathcal{F}\ell)$ defined before, one obtains a well-defined functor $\mathrm{Gr} : \mathrm{W}^w(\mathcal{F}\ell) \to \mathrm{GrW}^w(\mathcal{F}\ell)$. As explained in [AB09, Lemma 16], this is a monoidal functor.

Since $\mathrm{Gr}_{\mathcal{G}}|_{\tilde{C}^\circ} \cong \mathrm{Gr}_H \times \tilde{C}^\circ$, the nearby cycle functor indeed gives a monoidal functor from $\mathcal{Z} : P_{L^+H}(\mathrm{Gr}_H) \to \mathrm{W}^w(\mathcal{F}\ell)$, where $P_{L^+H}(\mathrm{Gr}_H)$ is the category of L^+H-equivariant perverse sheaves on Gr_H, which is well known to be equivalent to the category of representations of the Langlands dual group $\hat{H}$. One can use a similar argument as in [Bez04, appendix by Gaitsgory] to show that this functor is in fact central (see Section 2 of loc. cit. for the definition). Then by the same argument as [AB09], one can show that $\mathrm{Gr} \circ \mathcal{Z} : P_{L^+H}(\mathrm{Gr}_H) \to$

$\mathrm{GrW}^w(\mathcal{F}\ell)$ is in fact a tensor functor that is indeed equivalent to the restriction functor from the representations of $\hat{H}$ to the representations of $\hat{T}^\Gamma$.

Remark 7.3. This remark is not used in the rest of the paper. As a sheaf of nearby cycles, $\mathcal{Z}_\mu$ carries the monodromy action of the Galois group of F_0. One can show that this action is purely unipotent. (See [Gai01] for the case when G is split and [PZ13, Th. 10.9] in general.)

8. **Appendix I: line bundles on the local models for ramified unitary groups**

Since Theorem 1 is not quite identical to the original coherence conjecture given by Pappas and Rapoport, we explain here how to apply it to the local models. First, if the group G is split of type A or C, we find that all $a_i^\vee = 1$ in this case, and the formulation of Theorem 1 coincides with the original conjecture of Pappas and Rapoport. Namely, the central charge of $\mathcal{L}(\sum_{i\in Y}\varepsilon_i)$ is $\sharp Y$. In fact, in these cases, it is proven in loc. cit. (using the results of [Gör01], [Gör03], [PR05]) that the coherence conjecture holds for μ a sum of minuscule coweights. In what follows, we mainly discuss the ramified unitary groups. As the main application of the coherence conjecture, general cases are treated in [PZ13].

Let us change the notation in the main body of the paper to the following. Let $\mathcal{O}_{F_0}$ be a completed discrete valuation ring with algebraically closed residue field k with $\mathrm{char}\, k \neq 2$ and fractional field F_0. Let π_0 be the uniformizer. For example, $\mathcal{O} = k[[t]]$ with $\pi_0 = t$ as in the main body of the paper, or $\mathcal{O} = \mathbb{Z}_p^{ur}$, the completion of the maximal unramified extension of $\mathbb{Z}_p$ and $\pi_0 = p$.

We will follow [PR09] (see also [PR08]). Let F/F_0 be a quadratic extension. Let (V,ϕ) be a split hermitian vector space over F of dimension ≥ 4. That is, V is a vector space over F and ϕ is a hermitian form such that there is a basis $e_1,\dots,e_n$ of V satisfying

$$\phi(e_i, e_{n+1-j}) = \delta_{ij}, \quad i,j = 1,\dots,n.$$

Let $G = \mathrm{GU}(V,\phi)$ be the group of unitary similitudes for (V,ϕ); i.e., for any F_0-algebra R,

$$G(R) = \{g \in \mathrm{GL}(V\otimes_{F_0} R) | \phi(gv,gw) = c(g)\phi(v,w) \text{ for some } c(g) \in R^\times\}.$$

Then $G\otimes_{F_0} F \cong \mathrm{GL}_n \times \mathbb{G}_m$. The derived group G_{der} is the ramified special unitary group $\mathrm{SU}(V,\phi)$ consisting of those $g \in G(R)$ such that $\det(g) = c(g) = 1$.

We fix a square root π of π_0. There are two associated F_0-bilinear forms,

$$(v,w) = \mathrm{Tr}_{F/F_0}(\phi(v,w)), \quad \langle v,w\rangle = \mathrm{Tr}_{F/F_0}(\pi^{-1}\phi(v,w)).$$

The form $(-,-)$ is symmetric while $\langle -,-\rangle$ is alternating. For $i=0,\dots,n-1$, set

$$\Lambda_i = \mathrm{span}_{\mathcal{O}_{\breve{F}}}\{\pi^{-1}e_1,\dots,\pi^{-1}e_i,e_{i+1},\dots,e_n\},$$

and complete this into a self-dual periodic lattice chain by setting $\Lambda_{i+kn}=\pi^{-k}\Lambda_i$. Then $\langle -,-\rangle:\Lambda_{-j}\times\Lambda_j\to\mathcal{O}_{F_0}$ is a perfect pairing. In particular, Λ_0 is self-dual for the alternating form $\langle -,-\rangle$.

Let us fix a minuscule coweight $\mu_{r,s}$ of G_F of signature (r,s) with $r\le s, r+s=n$. That is,

$$\mu_{r,s}(a)=(\mathrm{diag}\{a^{(s)},1^{(r)}\},a),$$

where $a^{(s)}$ denotes s-copies of a. Let $E=F$ if $r\neq s$ and $E=F_0$ if $r=s$. Let $m=[\frac{n}{2}]$. Let $I\subset\{0,\dots,m\}$ be a nonempty subset with the requirement that if n is even and $m-1\in I$, then $m\in I$ as well. (See [PR09, §1.b] or [PR08, Rem. 4.2.C] for the reason why we make this assumption.)

Let us define the following moduli scheme $\mathcal{M}_I^{\mathrm{naive}}$ over $\mathcal{O}_E$. A point of $\mathcal{M}_I^{\mathrm{naive}}$ with values in an $\mathcal{O}_E$-scheme S is given by an $\mathcal{O}_F\otimes_{\mathcal{O}_{F_0}}\mathcal{O}_S$-submodule $\mathcal{F}_j\subset\Lambda_j\otimes_{\mathcal{O}_{F_0}}\mathcal{O}_S$ for each $j\in\pm I+n\mathbb{Z}$ satisfying the following conditions:

(1) as an $\mathcal{O}_S$-module, $\mathcal{F}_j$ is locally on S a direct summand of rank n;
(2) for each $j<j', j,j'\in\pm I+n\mathbb{Z}$, the natural inclusion
$$\Lambda_j\otimes_{\mathcal{O}_{F_0}}\mathcal{O}_S\to\Lambda_{j'}\otimes_{\mathcal{O}_{F_0}}\mathcal{O}_S$$
induces a morphism $\mathcal{F}_j\to\mathcal{F}_{j'}$, and the isomorphism $\pi:\Lambda_j\to\Lambda_{j-n}$ induces an isomorphism of $\mathcal{F}_j$ with $\mathcal{F}_{j-n}$;
(3) under the perfect pairing induced by $\langle -,-\rangle:\Lambda_{-j}\times\Lambda_j\to\mathcal{O}_{F_0}$, $\mathcal{F}_{-j}=\mathcal{F}_j^{\perp}$, where $\mathcal{F}_j^{\perp}$ is the orthogonal complement of $\mathcal{F}_j$;
(4) the set $\{\mathcal{F}_j\}$ should satisfy the determinant condition as in [PR09, §1.5.1, d)].

As explained in loc. cit., for any I, $\mathcal{M}_I^{\mathrm{naive}}\otimes_{\mathcal{O}_E}E$ is isomorphic to the Grassmannian $\mathbb{G}(s,n)$ of s-planes in n-space. In addition, for $i\in I$, there is a natural projection $\mathcal{M}_I^{\mathrm{naive}}\to\mathcal{M}_{\{i\}}^{\mathrm{naive}}$. (If n is even and $i=m-1$, $\{i\}$ will mean $\{m-1,m\}$.) Now the local model $\mathcal{M}_I^{\mathrm{loc}}$ is defined as the flat closure of the generic fiber $\mathcal{M}_I^{\mathrm{naive}}\otimes E$ inside $\mathcal{M}_I^{\mathrm{naive}}$.

The special fiber $\mathcal{M}_I^{\mathrm{naive}}\otimes k$ (and therefore $\mathcal{M}_I^{\mathrm{loc}}\otimes k$) embeds into the (partial) affine flag variety of the unitary group over $k(\!(t)\!)$. Namely, let (V',ϕ') be a split hermitian space over $k(\!(u)\!)$ $(u^2=t)$ with a standard basis $e_1,\dots,e_n$ such that $\phi'(e_i,e_{n+1-j})=\delta_{ij}$. Let $\lambda_j, j\in\{0,1,\dots,n-1\}$, be the standard lattices in V' defined similarly to Λ_j (replacing π by u and $\mathcal{O}_F$ by $k[[u]]$ in the definition of Λ_j). For $I\subset\{0,\dots,m\}$ as before, write $I=i_0<i_1<\cdots<i_k$ and let P_I be the group scheme over $k[[t]]$ that is the stabilizer of the lattice chain

$$\lambda_{i_0}\subset\cdots\subset\lambda_{i_k}\subset u^{-1}\lambda_{i_0}$$

in $\mathrm{GU}(V',\phi')$. As explained in loc. cit., this is not always a connected group scheme over $k[[t]]$. But if it is, then it is a parahoric group scheme of $\mathrm{GU}(V',\phi')$. In any case, the neutral connected component P_I^0 of P_I is a parahoric group scheme.

Consider the ind-scheme $\mathcal{F}_I$ that to a k-algebra R associates the set of sequences of $R[[u]]$-lattice chains

$$L_{i_0} \subset \cdots \subset L_{i_k} \subset u^{-1}L_{i_0}$$

in $V'\otimes_{k((u))}R((u))$ together with an $R[[t]]$-lattice $L \subset R((t))$ satisfying conditions (a) and (b) as in [PR09, §3.b]. (Observe that we replace $\alpha \in R((t))^\times/R[[t]]^\times$ in loc. cit. by a lattice $L \subset R((t))$, which seems more natural.) Then

$$\mathcal{F}_I \cong L\mathrm{GU}(V',\phi')/L^+P_I,$$

and $L\mathrm{GU}(V',\phi')/L^+P_I^0$ is either isomorphic to $L\mathrm{GU}(V',\phi')/L^+P_I$ or to the disjoint union of two copies of $L\mathrm{GU}(V',\phi')/L^+P_I$. In addition, to such I, one can canonically associate a subset $Y \subset \mathbf{S}$ ($\mathbf{S}$ is the set of vertices in the local Dynkin diagram of $\mathrm{GU}(V',\phi')$) such that $\mathcal{F}\ell^Y = L\mathrm{GU}(V',\phi')/L^+P_I^0$. Indeed, by [PR08, Rem. 10.3] (see also [PR09, §1.2.3]), one can identify $\mathbf{S}$ with $\{0,1,\dots,m\}$, if $n = 2m+1$, resp. $\{0,1,\dots,m-2,m,m'\}$, if $n = 2m$, where m' is a formal symbol as defined in [PR08, §4], to which a lattice of V'

$$\lambda_{m'} = \mathrm{span}_{k[[u]]}\{u^{-1}e_1,\dots,u^{-1}e_{m-1},e_m,u^{-1}e_{m+1},e_{m+2},\dots,e_{2m}\}$$

is associated. Then $Y = I$ in all cases except when $n = 2m, \{m-1,m\} \subset I$, in which case $Y = (I \setminus \{m-1\}) \cup \{m'\}$.

Remark 8.1. (i) Observe that if $n = 2m+1$, under our identification of $\{0,1,\dots,m\}$ with $\mathbf{S}$ (the set of vertices of the local Dynkin diagram), i goes to the label $m-i$ in Kac's book ([Kac90, p. 55]), and if $n = 2m$, under the identification of $\{0,\dots,m-2,m,m'\}$ with $\mathbf{S}$, i goes to $m-i$ for $i \le m-2$ and $\{m,m'\}$ go to $\{0,1\}$.

(ii) As pointed out in [PR08], [PR09], if $n = 2m+1$, then $P_{\{0\}}$ and $P_{\{m\}}$ are the special parahoric group schemes, and if $n = 2m$, then $P_{\{m\}}, P_{\{m'\}}$ are the special parahoric group schemes. We further point out the following:

(1) let $n = 2m+1$. Then
   (a) $P_{\{0\}}$ is the special parahoric determined by a pinning of $\mathrm{GL}_{2n+1}\times\mathbb{G}_m$, i.e., the group scheme $\mathcal{G}_{v_0}$ as in (2.1.2), and its reductive quotient is GO_{2m+1}; and
   (b) the special parahoric $P_{\{m\}}$ has reductive quotient GSp_{2m}, but it is not of the form (2.1.2).

(2) Let $n = 2m$. Then both $P_{\{m\}}, P_{\{m'\}}$ are of the form (2.1.2), and their reductive quotients are both isomorphic to GSp_{2m}.

Fix the isomorphisms $\Lambda_j \otimes_{\mathcal{O}_0} k \cong \lambda_j \otimes_{k[[t]]} k$, compatible with the actions of π and u, by sending $e_i \to e_i$. Now we embed the special fiber $\mathcal{M}_I^{\mathrm{naive}} \otimes k$ into $\mathcal{F}_I$ as follows. For every k-algebra R,

$$\mathcal{F}_j \subset (\Lambda_j \otimes k) \otimes_k R \cong (\lambda_j \otimes k) \otimes_k R,$$

and let $L_j \subset \lambda_j \otimes R[[t]]$ be the inverse image of $\mathcal{F}_j$ under $\lambda_j \otimes R[[t]] \to \lambda_j \otimes \mathcal{O}_S$. In addition, let $L = t^{-1}R[[t]] \subset R(\!(t)\!)$. This gives the embedding

$$\iota_I : \mathcal{M}_I^{\mathrm{naive}} \otimes k \to \mathcal{F}_I.$$

It is proved in [PR09, Prop. 3.1] that $\mathcal{A}^I(\mu_{r,s})$ is contained in $\mathcal{M}_I^{\mathrm{loc}} \otimes_{\mathcal{O}_E} k$ under ι_I, where $\mathcal{A}^I(\mu_{r,s})$ is as defined in (2.2.2). Here we show the following result, which was shown in [PR09, Th. 0.1] to follow from a slightly different version of the coherence conjecture.

THEOREM 8.1. *One has the equality $\mathcal{A}^I(\mu_{r,s}) = \mathcal{M}_I^{\mathrm{loc}} \otimes_{\mathcal{O}_E} k$. Therefore, the special fiber of $\mathcal{M}_I^{\mathrm{loc}}$ is reduced and each irreducible component is normal, Cohen-Macaulay and Frobenius-split.*

To prove it, one needs to construct a natural line bundle on $\mathcal{M}_I^{\mathrm{naive}}$ and apply the coherence conjecture to compare the dimensions of the space of global sections of this line bundle over the generic and the special fibers. There are several choices of natural line bundles. One of them will be given in [PZ13], after we give a group theoretical description of $\mathcal{M}_I^{\mathrm{naive}}$. Here, we follow the original approach of [PR08], [PR09] to construct another line bundle $\mathcal{L}_I$, which is more explicit.

First, if $I = \{j\}$, we define the line bundle $\mathcal{L}_{\{j\}}$ over $\mathcal{M}_{\{j\}}^{\mathrm{naive}}$ whose value at the $\mathcal{O}_S$-point given by $\mathcal{F}_j \subset \Lambda_j \otimes_{\mathcal{O}_E} \mathcal{O}_S$ is $\det(\mathcal{F}_j)^{-1}$. If $n = 2m$, we also define $\mathcal{L}_{\{m-1,m\}}$ over $\mathcal{M}_{\{m-1,m\}}^{\mathrm{naive}}$ whose value at the $\mathcal{O}_S$-point given by $\mathcal{F}_{m-1} \subset \mathcal{F}_m$ is $\det(\mathcal{F}_{m-1})^{-1} \otimes \det(\mathcal{F}_m)^{-1}$. For general I, the line bundle $\mathcal{L}_I$ is defined as the tensor product of these $\mathcal{L}_{\{j\}}$ or $\mathcal{L}_{\{m-1,m\}}$ along all possible projections $\mathcal{M}_I^{\mathrm{naive}} \to \mathcal{M}_{\{j\}}^{\mathrm{naive}}$ or $\mathcal{M}_I^{\mathrm{naive}} \to \mathcal{M}_{\{m-1,m\}}^{\mathrm{naive}}$.

The restriction of $\mathcal{L}_{\{j\}}$ to the generic fiber $\mathcal{M}_{\{j\}}^{\mathrm{naive}} \otimes_E F \cong \mathrm{Gr}(s,n)$ is isomorphic to $\mathcal{L}_{\det}^{\otimes 2}$, where $\mathcal{L}_{\det}$ is the determinant line bundle on $\mathrm{Gr}(s,n)$, which is the positive generator of the Picard group of $\mathrm{Gr}(s,n)$. On the other hand, the restriction of $\mathcal{L}(\{m-1,m\})$ to the generic fiber of $\mathcal{M}_{\{m-1,m\}}^{\mathrm{naive}}$ is isomorphic to $\mathcal{L}_{\det}^{\otimes 4}$. Recall $\mathcal{A}^I(\mu_{r,s})^\circ$ as defined in (2.2.1), and recall that there is a canonical isomorphism $\mathcal{A}^I(\mu_{r,s})^\circ \cong \mathcal{A}^I(\mu_{r,s})$ as $G_{\mathrm{der}} = \mathrm{SU}_n$ is simply-connected.

PROPOSITION 8.2. *Under the canonical isomorphism $\mathcal{A}^I(\mu_{r,s})^\circ \cong \mathcal{A}^I(\mu_{r,s})$, the line bundle $\mathcal{L}_I$, when restricted to $\mathcal{A}^I(\mu_{r,s})$, is isomorphic to the restriction of $\mathcal{L}(\sum_{j\in Y} \kappa(j)\varepsilon_j)$ to $\mathcal{A}^I(\mu_{r,s})^\circ$, where*

(1) *if* $n = 2m+1$, *then* $\kappa(j) = 1$ *for* $j = 0, 1, \ldots, m-1$ *and* $\kappa(m) = 2$;
(2) *if* $n = 2m$, *then* $\kappa(j) = 1$ *for* $j = 0, \ldots, m-2$ *and* $\kappa(m) = \kappa(m') = 2$.

Proof. Let us first introduce a convention. In what follows, when we write λ_j, we consider it as a $k[[u]]$-lattice. If we just remember its $k[[t]]$-lattice structure, we denote it by $\lambda_j/k[[t]]$.

Clearly, we can assume that $I = \{j\}$, or when $n = 2m$, we shall also consider $I = \{m-1, m\}$. The latter case will be treated at the end of the proof. So we first assume that $j \neq m-1$.

Observe that we have a natural closed embedding of ind-schemes

$$L\mathrm{GU}(V', \phi')/L^+P_{\{j\}} \cong \mathcal{F}_{\{j\}} \to \mathrm{Gr}_{\mathrm{GL}(\lambda_j)} \times \mathrm{Gr}_{\mathbb{G}_m}$$

by just remembering the lattices $L_j \subset \lambda_j \otimes_{k(\!(u)\!)} R(\!(u)\!)$ and $L \subset R(\!(t)\!)$. By definition, the line bundle $\iota_{\{j\}}^*\mathcal{L}_{\{j\}}$ on $\mathcal{M}_{\{j\}}^{\mathrm{naive}} \otimes_{\mathcal{O}_E} k$ is the pullback of the determinant line bundle on $\mathrm{Gr}_{\mathrm{GL}(\lambda_j)}$ along the above map.

Let $\mathrm{SU}(V', \phi')$ be the special unitary group. As explained in [PR08, §4], $P'_I = P_I \cap \mathrm{SU}(V', \phi')$ is a parahoric group scheme of $\mathrm{SU}(V', \phi')$. By [PR08, §6], we have

$$\begin{array}{ccc} L\mathrm{SU}(V', \phi')/L^+P'_{\{j\}} & \longrightarrow & L\mathrm{GU}(V', \phi')/L^+P_{\{j\}} \\ \downarrow & & \downarrow \\ \mathrm{Gr}_{\mathrm{SL}(\lambda_j)} & \longrightarrow & \mathrm{Gr}_{\mathrm{GL}(\lambda_j)}, \end{array}$$

where the ind-schemes in the left column are identified with the reduced part of neutral connected components of the ind-schemes in the right column. Since the isomorphism $\mathcal{A}^I(\mu_{r,s})^\circ \cong \mathcal{A}^I(\mu_{r,s})$ is obtained from the translation by some $g \in \mathrm{GU}(V', \phi')(F)$, it is enough to prove

Lemma 8.3. *The pullback of* $\mathcal{L}_{\det}$ *by* $L\mathrm{SU}(V', \phi')/L^+P'_{\{j\}} \to \mathrm{Gr}_{\mathrm{SL}(\lambda_j)}$ *is* $\mathcal{L}(\kappa(j)\varepsilon_j)$.

Proof. Assume that $j \neq 0, m$, and in the case $n = 2m$, $j \neq m-1$. By (2.2.6), the pullback of $\mathcal{L}_{\det}$ is of the form $\mathcal{L}(m\varepsilon_j)$ for some m. Consider the rational line $\mathbb{P}^1_j \subset L\mathrm{SU}(V', \phi')/L^+P'_{\{j\}}$ given by the $\mathbb{A}^1 = \mathrm{Spec}k[s]$-family of lattices

$$\begin{aligned} L_{j,s} = {} & u^{-1}k[[u]]e_1 + \cdots + u^{-1}k[[u]]e_{j-1} \\ & + u^{-1}k[[u]](e_j + se_{j+1}) + k[[u]]e_{j+1} + \cdots + k[[u]]e_n. \end{aligned}$$

It is easy to see that the restriction of $\mathcal{L}_{\det}$ to this rational line is $\mathcal{O}(1)$. In fact, by the map

$$L_{j,s} \to L_{j,s}/\Big(\sum_{r \leq j-1} u^{-1}k[[u]]e_r + \sum_{r \geq j} k[[u]]e_r\Big),$$

this rational curve $\mathbb{P}^1_j$ is identified with the Grassmannian $\mathrm{Gr}(1,2)$ classifying lines in the two-dimensional k-vector space generated by $\{u^{-1}e_j, u^{-1}e_{j+1}\}$, and clearly the restriction of the determinant line bundle of $\mathrm{Gr}_{\mathrm{SL}(\lambda_j)}$ is the determinant line bundle on $\mathrm{Gr}(1,2)$. Therefore, $\kappa(j) = 1$ if $j \neq 0, m$ (and $j \neq m-1$ if $n = 2m$).

If $j = 0$, consider the rational line $\mathbb{P}^1_0 \subset L\mathrm{SU}(V', \phi')/L^+P'_{\{0\}}$ given by the $\mathbb{A}^1 = \mathrm{Spec}k[s]$-family of lattices

$$L_s = k[[u]]e_1 + \cdots + k[[u]]e_{n-1} + k[[u]](e_n + su^{-1}e_1). \tag{8.0.1}$$

By the same reasoning as above, the restriction of $\mathcal{L}_{\mathrm{det}}$ to this rational line is $\mathcal{O}(1)$. Therefore, $\kappa(0) = 1$.

Now, if $n = 2m+1$ and $j = m$ or $n = 2m$ and $j = m$ or m', we will prove that $2 \mid \kappa(j)$. Assuming this, to prove the lemma it is enough to find some rational line $\mathbb{P}^1_j \subset L\mathrm{SU}(V', \phi')/L^+P'_{\{j\}}$ such that the restriction of $\mathcal{L}_{\mathrm{det}}$ to it is $\mathcal{O}(2)$. If $n = 2m+1$, we can take the rational line $\mathbb{P}^1_m$ given by the $\mathbb{A}^1 = \mathrm{Spec}k[s]$-family of lattices

$$\begin{aligned} L_s = {} & u^{-1}k[[u]]e_1 + \cdots + u^{-1}k[[u]]e_{m-1} \\ & + u^{-1}k[[u]]\left(e_m + se_{m+1} - \frac{s^2}{2}e_{m+2}\right) + k[[u]]e_{m+1} + \cdots + k[[u]]e_n. \end{aligned}$$

To see that $\mathcal{L}_{\mathrm{det}}$ restricts to $\mathcal{O}(2)$, consider the map

$$L_s \to L_s/\left(\sum_{r \leq m-1} u^{-1}k[[u]]e_r + \sum_{r \geq m} k[[u]]e_r\right),$$

which gives rise to embeddings $\mathbb{P}^1_m \subset \mathrm{Gr}(1,3) \subset \mathrm{Gr}_{\mathrm{SL}(\lambda_m)}$. Here $\mathrm{Gr}(1,3)$ is the Grassmannian that classifies lines in the three-dimensional k-vector space generated by $\{u^{-1}e_m, u^{-1}e_{m+1}, u^{-1}e_{m+2}\}$. Since the pullback of $\mathcal{L}_{\mathrm{det}}$ along $\mathrm{Gr}(1,3) \to \mathrm{Gr}_{\mathrm{SL}(\lambda_m)}$ is the determinant line bundle and the embedding $\mathbb{P}^1_m \to \mathrm{Gr}(1,3)$ is quadratic, the claim follows.

If $n = 2m$ and $j = m$ (the case $j = m'$ is similar), we can take the rational line $\mathbb{P}^1_m$ given by the $\mathbb{A}^1 = \mathrm{Spec}k[s]$-family of lattices

$$\begin{aligned} L_s = {} & u^{-1}k[[u]]e_1 + \cdots + u^{-1}k[[u]]e_{m-2} + u^{-1}k[[u]](e_{m-1} + se_{m+1}) \\ & + u^{-1}k[[u]](e_m - se_{m+2}) + k[[u]]e_{m+1} + \cdots + k[[u]]e_n. \end{aligned}$$

To see that $\mathcal{L}_{\mathrm{det}}$ restricts to $\mathcal{O}(2)$, consider the map

$$L_s \to L_s/\left(\sum_{r \leq m-2} u^{-1}k[[u]]e_r + \sum_{r \geq m-1} k[[u]]e_r\right),$$

which gives rise to embeddings $\mathbb{P}^1_m \subset \mathrm{Gr}(2,4) \subset \mathrm{Gr}_{\mathrm{SL}(\lambda_m)}$. Here $\mathrm{Gr}(2,4)$ is the Grassmannian that classifies planes in the four-dimensional k vector space generated by $\{u^{-1}e_{m-1}, \ldots, u^{-1}e_{m+2}\}$. The restriction of $\mathcal{L}_{\mathrm{det}}$ to $\mathrm{Gr}(2,4)$ is the

determinant line bundle, and therefore it is enough to see that the restriction of the determinant line bundle on $\mathrm{Gr}(2,4)$ along $\mathbb{P}^1_m \to \mathrm{Gr}(2,4)$ is $\mathcal{O}(2)$. If we use the determinant line bundle on $\mathrm{Gr}(2,4)$ to embed $\mathrm{Gr}(2,4)$ into $\mathbb{P}(V)$, where V is generated by $\{\{u^{-1}e_i \wedge u^{-1}e_j \mid m-1 \leq i < j \leq m+2\}$, then the composition $\mathrm{Spec}k[s] \subset \mathbb{P}^1_m \to \mathrm{Gr}(2,4) \to \mathbb{P}(V) \setminus \{u^{-1}e_{m-1} \wedge u^{-1}e_m\}$ is given by

$$s \mapsto su^{-1}e_{m-1} \wedge u^{-1}e_{m+2} + su^{-1}e_m \wedge u^{-1}e_{m+1} - s^2u^{-1}e_{m+1} \wedge u^{-1}e_{m+2}.$$

The claim is clear from this description.

So it remains to prove $2 \mid \kappa(j)$ for $n = 2m+1, j = m$, or $n = 2m, j = m$ or m'. Recall that when regarding V' as a vector space over $k(\!(t)\!)$, it has a split symmetric bilinear form

$$(v,w) = \mathrm{Tr}_{k(\!(u)\!)/k(\!(t)\!)}(\phi'(v,w)).$$

Observe that when $n = 2m+1, j = m$, or $n = 2m, j = m$ or m', $\lambda_j/k[[t]]$ is maximal isotropic, i.e., $\lambda_j \subset \widehat{\lambda_j}^s$ and $\dim_k(\widehat{\lambda_j}^s/\lambda_j) = 0$ or 1, where

$$\widehat{\lambda_j}^s = \{v \in V' \mid (v, \lambda_j) \subset \mathcal{O}\}.$$

Let $\mathrm{Iso}(V') \subset \mathrm{Gr}_{\mathrm{SL}(\lambda_j/k[[t]])}$ denote the subspace of maximal isotropic lattices in V'. Then the morphism

$$L\mathrm{SU}(V',\phi')/L^+P'_{\{j\}} \to \mathrm{Gr}_{\mathrm{SL}(\lambda_j)} \to \mathrm{Gr}_{\mathrm{SL}(\lambda_j/k[[t]])}$$

factors through

$$L\mathrm{SU}(V',\phi')/L^+P'_{\{j\}} \to \mathrm{Lag}(V') \to \mathrm{Gr}_{\mathrm{SL}(\lambda_j/k[[t]])}.$$

By definition, the pullback of $\mathcal{L}_{\det}$ along $\mathrm{Gr}_{\mathrm{SL}(\lambda_j)} \to \mathrm{Gr}_{\mathrm{SL}(\lambda_j/k[[t]])}$ is $\mathcal{L}_{\det}$, and it is well known (for example, see [BD, §4]) that the pullback of $\mathcal{L}_{\det}$ along $\mathrm{Iso}(V') \to \mathrm{Gr}_{\mathrm{SL}(\lambda_j/k[[t]])}$ admits a square root (the Pffafian line bundle). The lemma follows. □

To deal with the case $n = 2m$ and $I = \{m-1, m\}$, observe there is a map

$$L\mathrm{GU}(V',\phi')/L^+P_I \to \mathrm{Gr}_{\mathrm{GL}(\lambda_m)} \times \mathrm{Gr}_{\mathrm{GL}(\lambda_{m'})}$$

by sending $L_{m-1} \subset L_m$ to L_m, gL_m, where g is the unitary transformation $e_m \mapsto e_{m+1}, e_{m+1} \mapsto e_m$ and $e_i \mapsto e_i$ for $i \neq m, m+1$. One observes that $\iota_I^*\mathcal{L}_I$ on $\mathcal{M}_I^{\mathrm{naive}} \otimes_{\mathcal{O}_E} k$ is the pullback along the above map of the tensor product of the determinant line bundles (on each factor). □

Finally, let us see why this proposition can be used to deduce Theorem 0.1 of [PR09]. First let $a_i^\vee$ be the Kac labeling as in [Kac90, §6.1]. Using Remark 8.1(i), by checking all the cases, we find that $a_i^\vee \kappa(i) = 2$. Let $\mathcal{L}_I$ be the line bundle on $\mathcal{M}_I^{\mathrm{loc}}$. Then for $a \gg 0$,

$$\dim \Gamma(\mathcal{M}_I^{\mathrm{loc}} \otimes_{\mathcal{O}_E} k, \mathcal{L}_I^a) = \dim \Gamma(\mathcal{M}_I^{\mathrm{loc}} \otimes_{\mathcal{O}_E} E, \mathcal{L}_I^a). \tag{8.0.2}$$

By the above proposition and [PR09, Prop. 3.1],

$$\dim \Gamma(\mathcal{M}_I^{\mathrm{loc}} \otimes_{\mathcal{O}_E} k, \mathcal{L}_I^a) \geq \dim \Gamma\bigg(\mathcal{A}^I(\mu_{r,s})^\circ, \mathcal{L}\bigg(a\sum_{i\in Y}\kappa(i)\varepsilon_i\bigg)\bigg),$$

and the central charge of $\mathcal{L}(a\sum_{i\in Y}\kappa(i)\varepsilon_i)$ is

$$\sum_{i\in Y} a a_i^\vee \kappa(i) = 2a\sharp I.$$

The line bundle on right-hand side of (8.0.2) is just the $2a\sharp I$-power of the ample generator of the Picard group of $\mathbb{G}(s,n)$. Then since Theorem 1 holds, Theorem 8.1 follows by the argument in [PR08].

9. **Appendix II: Some recollections and proofs**

We collect and strengthen various results, which exist in literature, in the form needed in the main body of the paper.

9.1. *Combinatorics of Iwahori-Weyl group.* We recall a few facts about the translation elements in the Iwahori-Weyl group that are used in the paper. We keep the notation as in Section 2.1. In particular, we identify the apartment $A(G,S)$ with $\mathbb{X}_\bullet(S)_\mathbb{R}$ via the special vertex v. We choose the alcove $\mathbf{a}$, whose closure $\overline{\mathbf{a}}$ contains v, and which is contained in the finite Weyl chamber of G determined by the chosen Borel subgroup. We write $\widetilde{W} = \mathbb{X}_\bullet(T)_\Gamma \rtimes W_0$ using the vertex v.

Let 2ρ be the sum of all positive roots (for H). Observe that given $\mu \in \mathbb{X}_\bullet(T)_\Gamma$, the integer $(\tilde{\mu}, 2\rho)$ is independent of its lifting $\tilde{\mu} \in \mathbb{X}_\bullet(T)$. By abuse of notation, we denote this number by $(\mu, 2\rho)$.

Lemma 9.1. *Let $\mu \in \mathbb{X}_\bullet(T)^+_\Gamma$, the set defined in* (2.1.5). *Let $\Lambda \subset \mathbb{X}_\bullet(T)_\Gamma$ be the W_0-orbit associated to μ as in Section* 2.1. *Then for all $\nu \in \Lambda$, $\ell(t_\nu) = (2\rho, \mu)$.*

Proof. Let $x \in \mathbf{a}$ be a point in the interior of the alcove $\mathbf{a}$. Then for any $w \in \widetilde{W}$,

$$\ell(w) = \{\alpha \text{ is an affine root } \mid \alpha(x) > 0, \alpha(w(x)) < 0\}. \tag{9.1.1}$$

If $w = t_\nu$ is a translation element, then this is the number of affine roots α such that $0 < \alpha(x) < (\dot{\alpha}, \nu)$, where $\dot{\alpha}$ is the vector part of α. (So $\dot{\alpha}$ is a finite root of G.) This number can be rewritten as

$$\sum_{a\in\Phi,(a,\nu)>0} \sharp\{\alpha \mid \dot{\alpha} = a, 0 < \alpha(x) < (a,\nu)\}.$$

Let $j : \Phi(H, T_H) \to \Phi(G,S)$ be the restriction of the root system of H (the absolute root system) to the root system of G (the relative root system) . Then

the lemma will follow from the equality

$$\sharp\{\alpha \mid \dot{\alpha} = a, 0 < \alpha(x) < (a, \nu)\} = \sum_{\tilde{a} \in j^{-1}(a)} (\tilde{a}, \tilde{\nu}).$$

This statement involves only one root of G. By checking the semi-simple subgroup of G of semi-simple F-rank one (which are Weil restrictions of either SL_2 or SU_3), we see that this equality holds. □

One can easily deduce the following lemma from (9.1.1).

LEMMA 9.2. *Let $w, w' \in \widetilde{W}$. Then $\ell(ww') = \ell(w) + \ell(w')$ if and only if the following two statements hold: for $a + m$ an affine root,*

(1) *if $\alpha(x) > 0$ and $\alpha(w'(x)) < 0$, then $\alpha(ww'(x)) < 0$;*
(2) *if $\alpha(x) > 0$ and $\alpha(w'^{-1}(x)) < 0$, then $\alpha(w(x)) > 0$.*

LEMMA 9.3. *Let $\mu \in \mathbb{X}_\bullet(T)^+_\Gamma$. Then $\ell(t_\mu w_f) = \ell(t_\mu) + \ell(w_f)$.*

Proof. Let $w = t_\mu$ with $\mu \in \mathbb{X}_\bullet(T)^+_\Gamma$ and $w_f \in W_0$. Assume that $\alpha(x) > 0$ and $\alpha(w_f(x)) < 0$. As $v = \overline{\mathbf{a}} \cap \overline{w_f(\mathbf{a})}$, $\alpha(v) = 0$. Let $a = \dot{\alpha}$, then $a(x - v) = \alpha(x) > 0$; i.e., a is a positive root of G. Therefore, $\alpha(t_\mu w_f(x)) = \alpha(w_f(x)) - (\mu, a) < 0$. On the other hand, assume that $\alpha(x) > 0$ and $\alpha(w_f^{-1}(x)) < 0$. Then $\alpha(v) = 0$, and $w_f(a)$ is negative. Therefore, $(\mu, a) \leq 0$. Then $\alpha(t_\mu(x)) = \alpha(x) - (\mu, a) > 0$. This proves that $\ell(t_\mu w_f) = \ell(t_\mu) + \ell(w_f)$. □

On the finitely generated abelian group $\mathbb{X}_\bullet(T)_\Gamma$, there are two partial orders. One is the restriction of the Bruhat order on $\widetilde{W}$, denoted by "$\leq$." The other, denoted by "$\preceq$," is defined as follows. Recall that the lattice $\mathbb{X}_\bullet(T_{\mathrm{sc}})$ is the coroot lattice of H. The Galois group Γ acts on $\mathbb{X}_\bullet(T_{\mathrm{sc}})$, under which positive coroots of H (determined by the pinning) are sent to positive coroots. Therefore, it makes sense to talk about positive elements in $\mathbb{X}_\bullet(T_{\mathrm{sc}})_\Gamma$. Namely, $\lambda \in \mathbb{X}_\bullet(T_{\mathrm{sc}})_\Gamma$ is positive if its preimage in $\mathbb{X}_\bullet(T_{\mathrm{sc}})$ is a sum of positive coroots (of (H, T_H)). Since $\mathbb{X}_\bullet(T_{\mathrm{sc}})_\Gamma \subset \mathbb{X}_\bullet(T)_\Gamma$, we can define $\lambda \preceq \mu$ if $\mu - \lambda$ is positive in $\mathbb{X}_\bullet(T_{\mathrm{sc}})_\Gamma$.

LEMMA 9.4. *Let $\lambda, \mu \in \mathbb{X}_\bullet(T)^+_\Gamma$. Then $\lambda \preceq \mu$ if and only if $t_\lambda \leq t_\mu$ in the Bruhat order.*

Proof. In the case that G is split, the proof is contained in [Rap05, Props. 3.2, 3.5]. The ramified case can be reduced to the same proof as shown in [Ric13, Cor. 1.8]. See [PRS13, Rem. 4.2.7]. □

Recall the following lemma.

LEMMA 9.5. *Let $x, y \in \widetilde{W}$ and $w \in W_{\mathrm{aff}}$. Assume that $\ell(xw) = \ell(x) + \ell(w)$ and $\ell(yw) = \ell(y) + \ell(w)$. Then $x \leq y$ if and only if $xw \leq yw$.*

Proof. By induction on the length of w, we can assume that w is a simple reflection. Then the lemma is clear. □

LEMMA 9.6. *Let $\lambda, \mu \in w(\mathbb{X}_\bullet(T)^+_\Gamma)$. Then $t_\lambda \leq t_\mu$ if and only if $t_{w^{-1}\lambda} \leq t_{w^{-1}\mu}$.*

Proof. Observe that $w^{-1}\lambda$ and $w^{-1}\mu$ are dominant. Combining Lemmas 9.1 and 9.3,

$$\ell(w^{-1}t_\lambda) = \ell(t_{w^{-1}\lambda}) + \ell(w^{-1}) = \ell(w^{-1}) + \ell(t_\lambda).$$

Therefore, by the above lemma, $t_{w^{-1}\lambda} \leq t_{w^{-1}\mu}$ if and only if $w^{-1}t_\lambda \leq w^{-1}t_\mu$ if and only if $t_\lambda \leq t_\mu$. □

9.2. *Deformation to the normal cone.* Let C be a smooth curve over an algebraically closed field k. Let $\mathcal{X}$ be a scheme faithfully flat and affine over C. Let $x \in C(k)$ be a point, and let $\mathcal{X}_x$ denote the fiber of $\mathcal{X}$ over x. Let $Z \subset \mathcal{X}_x$ be a closed subscheme. Consider the following functor $\mathcal{X}_Z$ on the category of flat C-schemes: for each $V \to C$,

$$\mathcal{X}_Z(V) = \{f \in \mathrm{Hom}_C(V, \mathcal{X}) \mid f_x : V_x \to \mathcal{X}_x \text{ factors through } V_x \to Z \subset \mathcal{X}_x\}.$$

It is well known that this functor is represented by a scheme affine and flat over C, usually called the deformation to the normal cone (or called the dilatation of $\mathcal{X}$ along Z; see [BLR90, §3.2]). Indeed, the construction is easy if $\mathcal{X}$ is affine over C. Namely, we can assume that C is affine and x is defined by a local parameter t. Assume that $\mathcal{A}$ is the $\mathcal{O}_C$-algebra defining $\mathcal{X}$ over C, and let $\mathcal{I} \subset \mathcal{A}$ be the ideal sheaf defining $Z \subset \mathcal{X}$. Then $t\mathcal{A} \subset \mathcal{I}$. Let $\mathcal{B} = \mathcal{A}[\frac{i}{t}, i \in \mathcal{I}] \subset \mathcal{A}[t^{-1}]$. It is easy to see that $\mathcal{B}$ is flat over $\mathcal{O}_C$ and Spec$\mathcal{B}$ represents $\mathcal{X}_Z$.

There is a natural morphism $\mathcal{X}_Z \to \mathcal{X}$ that induces an isomorphism over $C - \{x\}$, and over x it factors as $(\mathcal{X}_Z)_x \to Z \to \mathcal{X}_x$. If $\mathcal{X}$ is smooth over C, and Z is a smooth closed subscheme of $\mathcal{X}_x$, then $\mathcal{X}_Z$ is also smooth over C. Indeed, étale locally on $\mathcal{X}_x$, the map $(\mathcal{X}_Z)_x \to Z$ can be identified with the map from the normal bundle of Z inside $\mathcal{X}_x$ to Z, which justifies the name of the construction.

Now let $\mathcal{G}_1$ be a connected affine smooth group scheme over the curve C. Let $x \in C(k)$, and let $(\mathcal{G}_1)_x$ be the fiber of $\mathcal{G}_1$ at x. Let $P \subset (\mathcal{G}_1)_x$ be a smooth closed subgroup. Let $\mathcal{G}_2 = (\mathcal{G}_1)_P$. This is indeed a smooth connected affine group scheme over C. By restriction to x, we have $r : \mathrm{Bun}_{\mathcal{G}_2} \to \mathbb{B}(\mathcal{G}_2)_x$ and $r : \mathrm{Bun}_{\mathcal{G}_1} \to \mathbb{B}(\mathcal{G}_1)_x$. (Here we assume that C is a complete curve.)

PROPOSITION 9.7. *We have the following Cartesian diagram*:

$$\begin{array}{ccccc} \mathrm{Bun}_{\mathcal{G}_2} & \xrightarrow{r} & \mathbb{B}(\mathcal{G}_2)_x & \longrightarrow & \mathbb{B}P \\ \downarrow & & & & \downarrow \\ \mathrm{Bun}_{\mathcal{G}_1} & & \xrightarrow{\quad r \quad} & & \mathbb{B}(\mathcal{G}_1)_x. \end{array}$$

Proof. Let $V = \mathrm{Spec}R$ be a noetherian[8] affine scheme. Let $\mathcal{E}$ be a $\mathcal{G}_1$-torsor on C_R and $\mathcal{E}_P$ be a P-torsor on V together with an isomorphism $\mathcal{E}_P \times^P (\mathcal{G}_1)_x \cong \mathcal{E}|_{\{x\}\times \mathrm{Spec}\, R}$. We need to construct a $\mathcal{G}_2$-torsor $\mathcal{E}'$ satisfying the appropriate conditions. This construction will provide the inverse to the morphism $\mathrm{Bun}_{\mathcal{G}_2} \to \mathrm{Bun}_{\mathcal{G}} \times_{\mathbb{B}(\mathcal{G}_1)_x} \mathbb{B}P$.

As a scheme over C, $\mathcal{E}$ is faithfully flat. Its fiber over x is $\mathcal{E}|_{\{x\}\times \mathrm{Spec}\, R}$. Let Z be the closed subscheme of $\mathcal{E}_x$ given by the closed embedding

$$\mathcal{E}_P \subset \mathcal{E}_P \times^P (\mathcal{G}_1)_x \cong \mathcal{E}|_{\{x\}\times \mathrm{Spec}\, R}.$$

Then $\mathcal{E}_Z$ is a scheme affine and flat over C, together with a morphism $\mathcal{E}_Z \to \mathcal{E}$. Therefore, $\mathcal{E}_Z$ is a scheme over C_R. We claim that $\mathcal{E}_Z$ is a $\mathcal{G}_2$-torsor over C_R. First, $\mathcal{E}_Z$ is faithfully flat over C_R. Indeed, by the local criterion of flatness, it is enough to prove that $\mathcal{E}_Z|_{\{x\}\times \mathrm{Spec}\, R}$ is faithfully flat over $\mathrm{Spec}R$. But this is clear, since étale locally on $\mathcal{E}|_{\{x\}\times \mathrm{Spec}\, R}$, there is an isomorphism between $\mathcal{E}_Z|_{\{x\}\times \mathrm{Spec}\, R}$ and the normal bundle of $\mathcal{E}_P \subset \mathcal{E}_P \times^P (\mathcal{G}_1)_x$. Next, there is an action of $\mathcal{G}_2$ on $\mathcal{E}_Z$. Indeed, the map $\mathcal{E}_Z \times_{C_R} \mathcal{G}_2 \mapsto \mathcal{E} \times_{C_R} \mathcal{G}_1 \to \mathcal{E}$, when restricted to the fiber over x, factors through Z. Therefore, by the definition of $\mathcal{E}_Z$, it gives rise to a map

$$\mathcal{E}_Z \times_{C_R} \mathcal{G}_2 \mapsto \mathcal{E}_Z.$$

Finally, it is easy to see that

$$\mathcal{E}_Z \times_{C_R} \mathcal{E}_Z \cong \mathcal{E}_Z \times_{C_R} \mathcal{G}_2.$$

Indeed, the left-hand side represents the scheme $(\mathcal{E} \times_{C_R} \mathcal{E})_{Z\times_{\mathrm{Spec}\, R} Z}$ and the right-hand side represents the scheme $(\mathcal{E} \times_{C_R} \mathcal{G}_1)_{Z\times_{\mathrm{Spec}\, R} P}$. Then the desired isomorphism follows from

$$(\mathcal{E} \times_{C_R} \mathcal{E})_{Z\times_{\mathrm{Spec}\, R} Z} \cong (\mathcal{E} \times_{C_R} \mathcal{G}_1)_{Z\times_{\mathrm{Spec}\, R} P}. \qquad \square$$

9.3. *Frobenius morphisms.* Let us review some basic facts about the Frobenius morphisms of a variety X over an algebraically closed field of characteristic $p > 0$. The book [BK05, Chap. 1] provides a detailed account of the general theory.

[8]This suffices since all the stacks are locally of finite presentation.

First assume that X is smooth, and let ω_X be its canonical sheaf. Then there is the following isomorphism ([BK05, §§1.3.7–1.3.8]):

$$\mathfrak{D} : F_*\omega_X^{1-p} \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X), \tag{9.3.1}$$

where $F : X \to X$ is the absolute Frobenius map of X. The existence of this isomorphism follows from the Grothendieck duality theorem for finite morphisms (see [BK05, the discussion before §1.3.1]). Explicitly, the isomorphism is given as follows. Let $x \in X$ be a closed point, and let $x_1, \ldots, x_n$ be a sequence of regular parameters of the local ring $\mathcal{O}_{X,x}$. Then in an étale neighborhood of x in X, the above isomorphism is given by

$$\begin{aligned}\mathfrak{D}(x_1^{m_1}\cdots x_n^{m_n}(dx_1\cdots dx_n)^{1-p})(x_1^{\ell_1}\cdots x_n^{\ell_n}) \\ = \begin{cases} 0 & \text{if } p \nmid m_i + \ell_i + 1 \text{ for some } i, \\ x_1^{(m_1+\ell_1-p+1)/p}\cdots x_n^{(m_n+\ell_n-p+1)/p}. \end{cases}\end{aligned} \tag{9.3.2}$$

Next, assume that X is normal ([BK05, §1.3.12]). It is still make sense to talk about the canonical sheaf ω_X and its n-th power $\omega_X^{[n]}$ for all n. Namely, let $j : X^{\mathrm{sm}} \to X$ be the open immersion of the smooth locus into X. Then by definition $\omega_X^{[n]} := j_*\omega_{X^{\mathrm{sm}}}^n$. The isomorphism (9.3.1) still holds in this situation. Observe that there is a natural map $(\omega_X^{[\pm 1]})^{\otimes n} \to \omega_X^{[\pm n]} (n > 0)$ that is not necessarily an isomorphism. In what follows, we use ω_X^n to denote $\omega_X^{[n]}$ if no confusion will rise. Let us recall that if in addition X is Cohen-Macaulay, ω_X is the dualizing sheaf.

Next, we consider a flat family $f : X \to V$ of varieties that is fiberwise normal and Cohen-Macaulay. In addition, let us assume that V is smooth, so that the total space X is also normal and Cohen-Macaulay. In this case, the relative dualizing sheaf $\omega_{X/V}$ commutes with base change and is flat over V. We have $\omega_X \cong f^*\omega_V \otimes \omega_{X/V}$. Let $X^{(p)}$ be the Frobenius twist of X over V, i.e., the pullback of X along the absolute Frobenius endomorphism $F : V \to V$. Let $F_{X/V} : X \to X^{(p)}$ be the relative Frobenius morphism, and let $\varphi : X^{(p)} \to X$ be the map such that the composition $\varphi \circ F_{X/V}$ is the absolute Frobenius morphism F for X. Then

$$\mathfrak{RD} : (F_{X/V})_*\omega_{X/V}^{1-p} \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_{X^{(p)}}}((F_{X/V})_*\mathcal{O}_X, \mathcal{O}_{X^{(p)}}). \tag{9.3.3}$$

Here $\omega_{X/V}^n$, as in the absolute case, is the pushout of the n-th tensor power of the relative canonical sheaf on $X^{\mathrm{rel,sm}}$, the maximal open part of X such that $f|_{X^{\mathrm{rel,sm}}}$ is smooth. In addition, we have the following homorphisms:

$$f^*F_*\omega_V^{1-p} \xrightarrow{\sim} f^*\mathcal{H}om(F_*\mathcal{O}_V, \mathcal{O}_V) \cong \mathcal{H}om(\varphi_*\mathcal{O}_{X^{(p)}}, \mathcal{O}_X), \tag{9.3.4}$$

$$F_*\omega_{X/V}^{1-p} \otimes f^*F_*\omega_V^{1-p} \cong F_*\omega_{X/V}^{1-p} \otimes F_*f^*\omega_V^{1-p} \to F_*\omega_X^{1-p}. \tag{9.3.5}$$

The homorphisms (9.3.1), (9.3.3)–(9.3.5) fit into the following commutative diagram:

(9.3.6)

$$\begin{array}{ccccc}
\varphi_*\mathcal{H}om((F_{X/V})_*\mathcal{O}_X, \mathcal{O}_{X^{(p)}}) & \otimes_{\mathcal{O}_X} & \mathcal{H}om(\varphi_*\mathcal{O}_{X^{(p)}}, \mathcal{O}_X) & \longrightarrow & \mathcal{H}om(F_*\mathcal{O}_X, \mathcal{O}_X) \\
\uparrow & & \uparrow & & \uparrow \\
F_*\omega_{X/V}^{1-p} & \otimes_{\mathcal{O}_X} & f^*F_*\omega_V^{1-p} & \longrightarrow & F_*\omega_X^{1-p}.
\end{array}$$

Finally, let W be another smooth variety over k, and let $g : W \to V$ be a k-morphism (not necessarily flat). By abuse of notation, we still use g to denote the base change maps $X_W \to X$ and $(X_W)^{(p)} \cong X_W^{(p)} \to X^{(p)}$. Then the following diagram is commutative:

(9.3.7)

$$\begin{array}{ccc}
g^*(F_{X/V})_*\omega_{X/V}^{1-p} & \xrightarrow{\cong} & g^*\mathcal{H}om((F_{X/V})_*\mathcal{O}_X, \mathcal{O}_{X^{(p)}}) \\
\cong\downarrow & & \downarrow\cong \\
(F_{X_W/W})_*\omega_{X_W/W}^{1-p} & \xrightarrow{\cong} & \mathcal{H}om((F_{X_W/W})_*\mathcal{O}_{X_W}, \mathcal{O}_{X_W^{(p)}}).
\end{array}$$

To prove the isomorphism (9.3.3), and that (9.3.6) and (9.3.7) are commutative, one can first assume that X is smooth over V. In this case, the proof of (9.3.1) (as in [BK05, §1.3]) with obvious modifications applies to (9.3.3). In particular, étale locally on X, (9.3.3) can be described by the explicit formula as in (9.3.2), with $x_1, \ldots, x_n$ replaced by a system of local coordinates of X relative to V. Then (9.3.6) and (9.3.7) follow from the direct calculation. Then one can easily extend these to the case that X is flat over V with normal and Cohen-Macaulay fibers. Indeed, under our assumptions, all the sheaves appearing in (9.3.3), (9.3.6) and (9.3.7) have the following property: Let $\mathcal{F}$ be such a sheaf on X and $j : X^{\mathrm{rel,sm}} \to X$ be the open embedding as before. Then $\mathcal{F} \cong j_*(\mathcal{F}|_{X^{\mathrm{rel,sm}}})$.

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(Received: January 17, 2011)
(Revised: March 8, 2013)

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