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Potential automorphy and change of weight

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Abstract

We prove an automorphy lifting theorem for l-adic representations where we impose a new condition at l, which we call "potentential diagonalizability." This result allows for "change of weight" and seems to be substantially more flexible than previous theorems along the same lines. We derive several applications. For instance we show that any irreducible, totally odd, essentially self-dual, regular, weakly compatible system of l-adic representations of the absolute Galois group of a totally real field is potentially automorphic and hence is pure and its L-function has meromorphic continuation to the whole complex plane and satisfies the expected functional equation.

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Introduction

Suppose that F and M are number fields, that S is a finite set of primes of F and that n is a positive integer. By a *weakly compatible system* of n-dimensional l-adic representations of G_F defined over M and unramified outside S

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we shall mean a family of continuous semi-simple representations

$$r_\lambda : G_F \longrightarrow \mathrm{GL}_n(\overline{M}_\lambda),$$

where λ runs over the finite places of M, with the following properties:

- If $v \notin S$ is a finite place of F, then for all λ not dividing the residue characteristic of v the representation r_λ is unramified at v and the characteristic polynomial of $r_\lambda(\mathrm{Frob}_v)$ lies in $M[X]$ and is independent of λ.
- Each representation r_λ is de Rham at all places above the residue characteristic of λ, and in fact crystalline at any place $v \notin S$ which divides the residue characteristic of λ.
- For each embedding $\tau : F \hookrightarrow \overline{M}$ the τ-Hodge–Tate numbers of r_λ are independent of λ.

In this paper we prove the following theorem. (See Theorem 5.4.1.)

THEOREM A. *Let $\{r_\lambda\}$ be a weakly compatible system of n-dimensional l-adic representations of G_F defined over M and unramified outside S, where for simplicity we assume that M contains the image of each embedding $F \hookrightarrow \overline{M}$. Suppose that $\{r_\lambda\}$ satisfies the following properties*:

(1) (Irreducibility). *Each r_λ is irreducible.*
(2) (Regularity). *For each embedding $\tau : F \hookrightarrow M$ the representation r_λ has n distinct τ-Hodge–Tate numbers.*
(3) (Odd essential self-duality). *F is totally real*; *and either each r_λ factors through a map to $\mathrm{GSp}_n(\overline{M}_\lambda)$ with a totally odd multiplier character*; *or each r_λ factors through a map to $\mathrm{GO}_n(\overline{M}_\lambda)$ with a totally even multiplier character. Moreover in either case the multiplier characters form a weakly compatible system.*

Then there is a finite, Galois, totally real extension of F over which all the r_λ's become automorphic. In particular for any embedding $\imath : M \hookrightarrow \mathbb{C}$ the partial L-function $L^S(\imath\{r_\lambda\}, s)$ converges in some right half plane and has meromorphic continuation to the whole complex plane.

This is not the first paper to prove potential automorphy results for compatible systems of l-adic representations of dimension greater than 2; see, for example, [HSBT10], [BLGHT11], [BLGG11]. However previous attempts only applied to very specific, though well known, examples (e.g., symmetric powers of the Tate modules of elliptic curves) and one had to exploit special properties of these examples. We believe this is the first general potential automorphy theorem in dimension greater than 2, and we are hopeful that it can be applied to many examples. We give an analogous theorem when F is an imaginary CM field. Other than this we do not see how to improve much on this theorem using current methods.

As one example application, suppose that $\mathcal{K}$ is a finite set of positive integers such that the $2^{\#\mathcal{K}}$ possible partial sums of elements of $\mathcal{K}$ are all distinct. For each $k \in \mathcal{K}$ let f_k be an elliptic modular newform of weight $k+1$ without complex multiplication. Then the $\#\mathcal{K}$-fold tensor product of the l-adic representations associated to the f_k is potentially automorphic and the $\#\mathcal{K}$-fold product L-function for the f_k has meromorphic continuation to the whole complex plane. (See Corollary 5.4.4.)

The proof of Theorem A follows familiar lines. One works with r_λ for one suitably chosen λ. One finds a motive X over some finite Galois totally real extension F'/F which realizes the reduction $\overline{r}_\lambda$ in its mod l cohomology and whose mod l' cohomology is induced from a character. One tries to argue that by automorphic induction the mod l' cohomology is automorphic over F', hence by an automorphy lifting theorem the l'-adic cohomology is automorphic over F', hence tautologically the mod l cohomology is automorphic over F' and hence, finally, by another automorphy lifting theorem r_λ is automorphic over F'. To find X one uses a lemma of Moret-Bailly [MB89], [GPR95] and for this one needs a family of motives with distinct Hodge numbers, which has large monodromy. Griffiths transversality tells us that this will only be possible if the Hodge numbers of the motives are consecutive (e.g., $0, 1, 2, \dots, n-1$). Thus the l-adic cohomology of X may be automorphic of a different weight (infinitesimal character) than r_λ and the second automorphy lifting theorem needs to incorporate a 'change of weight.' In addition it seems that we can in general only expect to find X over an extension F'/F which is highly ramified at l. Thus our second automorphy lifting theorem needs to work over a base which is highly ramified at l. These two related problems were the principal difficulties we faced. The original higher-dimensional automorphy lifting theorems (see [CHT08], [Tay08]) could handle neither of them. In the ordinary case one of us (D.G.) proved an automorphy lifting theorem that uses Hida theory and some new local calculations to handle both of these problems. (See [Ger10].) This has had important applications, but its applicability is still severely limited because we do not know how to prove that many compatible systems of l-adic representations are ordinary infinitely often.

The main innovation of this paper is a new automorphy lifting theorem that handles both these problems in significant generality. One of our key ideas is to introduce the notion of a potentially crystalline representation ρ of the absolute Galois group of a local field K being *potentially diagonalizable*: ρ is potentially diagonalizable if there is a finite extension K'/K such that $\rho|_{G_{K'}}$ lies on the same irreducible component of the universal crystalline lifting ring of $\overline{\rho}|_{G_{K'}}$ (with fixed Hodge–Tate numbers) as a sum of characters lifting $\overline{\rho}|_{G_{K'}}$. (We remark that this does not depend on the choice of integral model for ρ.)

Ordinary crystalline representations are potentially diagonalizable, as are crystalline representations in the Fontaine–Laffaille range (i.e., over an absolutely unramified base and with Hodge–Tate numbers in the range $[0, l-2]$). Potential diagonalizability is also preserved under restriction to the absolute Galois group of a finite extension. In this sense they behave better than 'crystalline representations in the Fontaine–Laffaille range' which require the ground field to be absolutely unramified. Finally "potentially diagonalizable" representations are perfectly suited to our method of proving automorphy lifting theorems that allow for a change of weight. It seems to us to be a very interesting question to clarify further the ubiquity of potential diagonalizability. Could every crystalline representation be potentially diagonalizable? (We have no reason to believe this, but we know of no counterexample.)

The following gives an indication of the sort of automorphy lifting theorems we are able to prove. (See Theorem 4.2.1 and also Section 2.1 for the definition of any notation or terminology which may be unfamiliar.)

THEOREM B. *Let F be an imaginary* CM *field with maximal totally real subfield F^+ and let c denote the nontrivial element of* $\mathrm{Gal}(F/F^+)$. *Let n denote a positive integer. Suppose that $l \geq 2(n+1)$ is a prime such that F does not contain a primitive l^{th} root of* 1. *Let*

$$r : G_F \longrightarrow \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$$

be a continuous irreducible representation and let $\overline{r}$ denote the semi-simplification of the reduction of r. Also let

$$\mu : G_{F^+} \longrightarrow \overline{\mathbb{Q}}_l^\times$$

be a continuous character. Suppose that r and μ enjoy the following properties:

(1) (Odd essential conjugate-self-duality). $r^c \cong r^\vee \mu$ *and* $\mu(c_v) = -1$ *for all* $v|\infty$.
(2) (Unramified almost everywhere). r *ramifies at finitely many primes.*
(3) (Potential diagonalizability and regularity). $r|_{G_{F_v}}$ *is potentially diagonalizable* (*and so in particular potentially crystalline*) *for all* $v|l$ *and for each embedding* $\tau : F \hookrightarrow \overline{\mathbb{Q}}_l$ *it has n distinct τ-Hodge–Tate numbers.*
(4) (Irreducibility). *The restriction* $\overline{r}|_{G_{F(\zeta_l)}}$ *is irreducible.*
(5) (Residual ordinary automorphy). *There is a regular algebraic, cuspidal, polarized automorphic representation* (π, χ) *of* $\mathrm{GL}_n(\mathbb{A}_F)$ *such that*

$$(\overline{r}, \overline{\mu}) \cong (\overline{r}_{l,\imath}(\pi), \overline{r}_{l,\imath}(\chi)\overline{\epsilon}_l^{1-n})$$

and π is $\imath$-ordinary.

Then (r, μ) is automorphic.

Theorem B implies the following potential automorphy theorem for a single l-adic representation, from which Theorem A can be deduced. (See Corollary 4.5.2 and Theorem 5.5.1.)

THEOREM C. *Suppose that F is a totally real field. Let n be a positive integer and let $l \geq 2(n+1)$ be a prime. Let*

$$r : G_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$$

be a continuous representation. We will write $\overline{r}$ for the semi-simplification of the reduction of r. Suppose that the following conditions are satisfied.

(1) (Unramified almost everywhere). *r is unramified at all but finitely many primes.*
(2) (Odd essential self-duality). *Either r maps to GSp_n with totally odd multiplier or it maps to GO_n with totally even multiplier.*
(3) (Potential diagonalizability and regularity). *r is potentially diagonalizable (and hence potentially crystalline) at each prime v of F above l and for each $\tau : F \hookrightarrow \overline{\mathbb{Q}}_l$ it has n distinct τ-Hodge–Tate numbers.*
(4) (Irreducibility). *$\overline{r}|_{G_{F(\zeta_l)}}$ is irreducible.*

Then we can find a finite Galois totally real extension F'/F such that $r|_{G_{F'}}$ is automorphic. Moreover r is part of a weakly compatible system of l-adic representations. (In fact, r is part of a strictly pure compatible system in the sense of Section 5.1.)

This theorem has other applications besides Theorem A. For instance we mention the following irreducibility result. (See Theorem 5.5.2.)

THEOREM D. *Suppose that F is a* CM *field and that π is a regular, algebraic, essentially conjugate self-dual, cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$. If π_∞ has sufficiently regular weight ("extremely regular" in the sense of Section* 2.1*), then for l in a set of rational primes of Dirichlet density* 1 *the n-dimensional l-adic representations associated to π are irreducible.*

To prove Theorem B we employ Harris' tensor product trick (see [Har09]), which was first employed in connection with change of weight in [BLGG11]. However the freedom that "potential diagonalizability" gives us to make highly ramified base changes in the nonordinary case means that this method becomes more powerful. More precisely, suppose that r is potentially diagonalizable, and that r_0 is a potentially diagonalizable, automorphic lift of $\overline{r}$ (with possibly different Hodge–Tate numbers to r). In fact making a finite soluble base change we can assume they are diagonalizable; i.e., we can take $K' = K$ in the definition of potential diagonalizability. We choose a cyclic extension M/F of degree n in which each prime above l splits completely, and two l-adic characters θ and θ_0 of G_M such that

- $\bar{\theta} = \overline{\theta}_0$,
- the restriction of $\operatorname{Ind}_{G_M}^{G_F}\theta$ to an inertia group at a prime $v|l$ realizes a diagonal point on the same component of the universal crystalline lifting ring of $\overline{r}|_{G_{F_v}}$ as $r|_{G_{F_v}}$,
- the restriction of $\operatorname{Ind}_{G_M}^{G_F}\theta_0$ to an inertia group at a prime $v|l$ realizes a diagonal point on the same component of the universal crystalline lifting ring of $\overline{r}|_{G_{F_v}}$ as $r_0|_{G_{F_v}}$.

Then $r_0 \otimes \operatorname{Ind}_{G_M}^{G_F}\theta$ is automorphic and has the same reduction as $r \otimes \operatorname{Ind}_{G_M}^{G_F}\theta_0$. Moreover the restrictions of these two representations to the decomposition group at a prime $v|l$ lie on the same component of the universal crystalline lifting ring of $(\overline{r} \otimes \operatorname{Ind}_{G_M}^{G_F}\overline{\theta}_0)|_{G_{F_v}}$. This is enough for the usual Taylor–Wiles–Kisin argument to prove that $r \otimes \operatorname{Ind}_{G_M}^{G_F}\theta_0$ is also automorphic, from which we can deduce (as in [BLGHT11]) the automorphy of r.

Things are a little more complicated than this because it seems to be hard to combine this with the "level changing" argument in [Tay08]. In addition a direct argument imposes minor, but unwanted, conditions on the Hodge–Tate numbers of r_0 and r. So instead of going directly from the automorphy of r_0 to that of r we create two ordinary lifts r_1 and r_2 of $\overline{r}$ (at least after a base change) where r_1 has the same local behavior away from l as r_0; r_2 has the same local behavior away from l as r; and where the Hodge–Tate numbers of r_1 and r_2 are chosen suitably. Our new arguments allow us to deduce the automorphy of r_1 from that of r_0. D.G.'s results in the ordinary case [Ger10] allow us to deduce the automorphy of r_2 from that of r_1. Finally applying our new argument again allows us to deduce the automorphy of r from the automorphy of r_2. To construct r_1 and r_2 we use the method of Khare and Wintenberger [KW09] based on potential automorphy (in the ordinary case, where it is already available; see, for example, [BLGHT11]).

Along the way we also prove a general theorem about the existence of l-adic lifts of a given mod l Galois representation with prescribed local behavior. (See Theorem 4.3.1.) We deduce a rather general theorem about change of weight and level (see Theorem 4.4.1) of which a very particular instance is the following.

THEOREM E. *Let n be a positive integer and let $l > 2(n+1)$ be a prime. Fix $\iota : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$. Let F be a* CM *field such that all primes of F above l are unramified over $\mathbb{Q}$ and split over the maximal totally real subfield of F. Let π be a regular, algebraic, polarizable, cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ satisfying the following conditions*:

- *π is unramified above l*;
- *π_∞ has weight $(a_{\tau,i})_{\tau:F\hookrightarrow\mathbb{C},\, i=1,\dots,n}$ with $l - n - 1 \geq a_{\tau,1} \geq a_{\tau,2} \geq \cdots \geq a_{\tau,n} \geq 0$ for all τ*;

- *the restriction to* $G_{F(\zeta_l)}$ *of the mod* l *Galois representation* $\overline{r}_{l,\imath}(\pi)$ *associated to* π *and* $\imath$ *is irreducible.*

Note that in this case $a_{\tau,i} + a_{\tau c,n+1-i} = w$ *is independent of* τ *and* i*. Suppose that we are given a second weight* $(a'_{\tau,i})_{\tau:F\hookrightarrow\mathbb{C},\ i=1,\dots,n}$ *with* $l - n - 1 \geq a'_{\tau,1} \geq a'_{\tau,2} \geq \cdots \geq a'_{\tau,n} \geq 0$ *for all* τ *such that*

- $a'_{\tau,i} + a'_{\tau c,n+1-i} = w$ *is also independent of* τ *and* i*;*
- *for all places* $v|l$ *of* F *the restriction* $\overline{r}_{l,\imath}(\pi)|_{G_{F_v}}$ *has a lift which is crystalline with* τ*-Hodge–Tate numbers* $\{a'_{\imath\tau,i} + n - i\}$*.*

Then there is a second regular, algebraic, polarizable, cuspidal automorphic representation π' *of* $\mathrm{GL}_n(\mathbb{A}_F)$ *giving rise (via* $\imath$*) to the same mod* l *Galois representation (i.e., "congruent to* π *mod* l*") such that* π' *is also unramified above* l *and* π_∞ *has weight* $(a'_{\tau,i})$*.*

We remark that combining the results of this paper with work of Caraiani, one can deduce full local global compatibility of the l-adic representations associated to regular algebraic, essentially conjugate self-dual, cuspidal automorphic representations of GL_n over a CM or totally real field. (See [BLGGT12] and [Car12b].)

We also remark that Stefan Patrikis and one of us (R.T.) recently combined the methods and results of this paper with one further idea, originating in [Pat12], and obtained variants of Theorems A and D which are perhaps more useful in practice. (See [PT12].) More specifically they proved a version of Theorem A where the irreducibility assumption is replaced by a purity assumption. This is useful because for many compatible systems arising from geometry purity is known by Deligne's theorem, but irreducibility can be hard to check. In particular one can deduce the meromorphic continuation and functional equation of the L-function of any regular, pure, self-dual motive over a totally real field. They also prove a version of Theorem D in which the hypothesis that π_∞ is "extremely regular" is weakened to "regular," but the conclusion is also weakened to give irreducibility only above a set of rational primes of positive Dirichlet density.

We now explain the structure of the paper. In Section 1 we collect some results about the deformation theory of Galois representations. These are mostly now fairly standard results but we recall them to fix notations and in some cases to make slight improvements. The main exception is the introduction of potential diagonalizability in Section 1.4, which is new and of key importance for us. In Section 2 we fix some notations and we recall the existing automorphy lifting theorems (or slight generalizations of them). Very little in this section is novel. Between the writing of the first and second versions of this paper, Jack Thorne [Tho12] has found improved versions of these theorems which allow one to remove the troublesome "bigness" conditions from [CHT08]

and the papers that followed it. Moreover Ana Caraiani [Car12a], [Car12b] has proved local-global compatibility in all $l \neq p$ cases, temperedness of all regular algebraic, polarizable, cuspidal automorphic representations, and the purity of all the Weil–Deligne representations associated to the l-adic representation associated to automorphic representations including the $l = p$ case. We have taken advantage of Caraiani's and Thorne's works to optimize our own results. In Section 3 we make use of the automorphy lifting theorems from Section 2 and the Dwork family to prove a potential automorphy theorem (in the ordinary case) and a theorem about lifting mod l Galois representations (again in the ordinary case). These arguments follow those of [BLGHT11] and will not surprise an expert.

In Section 4 we prove our main new theorems. Section 4.1 contains our main new argument. In Section 4.2 we combine this with the results of Sections 2 and 3 to obtain our optimal automorphy lifting theorem. In Section 4.3 we use the same ideas to deduce an improved result about the existence of l-adic lifts of mod l Galois representations with specified local behavior. Combining the results of Sections 4.2 and 4.3 we deduce in Section 4.4 a general theorem about change of weight and level for mod l automorphic forms on GL_n. Then in Section 4.5 we use the automorphy lifting theorem of Section 4.2 and our potential automorphy theorem from Section 3.3 to deduce our main new potential automorphy result for a single l-adic representation.

In Section 5 we turn to applications of our main results. In Section 5.1 we recall definitions connected to compatible systems of l-adic representations. In Sections 5.2 and 5.3 we prove some group theoretic lemmas about the images of compatible systems of l-adic representations. Then in Section 5.4 we deduce from the potential automorphy theorem of Section 4.5 our main theorem — a potential automorphy theorem for compatible systems of l-adic representations. Finally in Section 5.5 we give further applications of our main results — applications to fitting an l-adic representation into a compatible system and to the irreducibility of some l-adic representations associated to cusp forms on $\mathrm{GL}(n)$.

In the appendix we record some miscellaneous results which we use elsewhere in the paper. Some of these are results we suspect are "well known," but for which we could not find a reference. In these cases we give a proof. Others are results for which we know a reference. However we hope it may assist the reader to recall these here.

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Notation. We write all matrix transposes on the left; so ${}^tA$ is the transpose of A. Let $\mathfrak{gl}_n$ denote the space of $n \times n$ matrices with the adjoint action of GL_n and let $\mathfrak{sl}_n$ denote the subspace of trace zero matrices. We will write $O(n)$ (resp. $U(n)$) for the group of matrices $g \in \mathrm{GL}_n(\mathbb{R})$ (resp. $\mathrm{GL}_n(\mathbb{C})$) such that ${}^tg^c g = 1_n$.

If R is a local ring, we write $\mathfrak{m}_R$ for the maximal ideal of R.

If Δ is an abelian group, we will let Δ^{tor} denote its maximal torsion subgroup and Δ^{tf} its maximal torsion free quotient. If Γ is a profinite group, then Γ^{ab} will denote its maximal abelian quotient by a closed subgroup. If $\rho : \Gamma \to \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$ is a continuous homomorphism, then we will let $\overline{\rho} : \Gamma \to \mathrm{GL}_n(\overline{\mathbb{F}}_l)$ denote the *semi-simplification* of its reduction, which is well defined up to conjugacy.

If M is a field, we let $\overline{M}$ denote an algebraic closure of M and G_M the absolute Galois group $\mathrm{Gal}\,(\overline{M}/M)$. We will use ζ_n to denote a primitive n^{th}-root of 1. Let ϵ_l denote the l-adic cyclotomic character and $\overline{\epsilon}_l$ its reduction modulo l. We will also let $\omega_l : G_M \to \mu_{l-1} \subset \mathbb{Z}_l^\times$ denote the Teichmüller lift of $\overline{\epsilon}_l$. If N/M is a separable quadratic extension, we will let $\delta_{N/M}$ denote the nontrivial character of $\mathrm{Gal}\,(N/M)$.

If Γ is a profinite group and M is a topological abelian group with a continuous action of Γ, then by $H^i(\Gamma, M)$ we shall mean the continuous cohomology.

We will write $\mathbb{Q}_{l^r}$ for the unique unramified extension of $\mathbb{Q}_l$ of degree r and $\mathbb{Z}_{l^r}$ for its ring of integers. We will write $\mathbb{Q}_l^{\mathrm{nr}}$ for the maximal unramified extension of $\mathbb{Q}_l$ and $\mathbb{Z}_l^{\mathrm{nr}}$ for its ring of integers. We will also write $\widehat{\mathbb{Z}}_l^{\mathrm{nr}}$ for the l-adic completion of $\mathbb{Z}_l^{\mathrm{nr}}$ and $\widehat{\mathbb{Q}}_l^{\mathrm{nr}}$ for its field of fractions.

If K is a finite extension of $\mathbb{Q}_p$ for some p, we write K^{nr} for its maximal unramified extension; I_K for the inertia subgroup of G_K; $\mathrm{Frob}_K \in G_K/I_K$ for the geometric Frobenius; and W_K for the Weil group. If K'/K is a Galois extension we will write $I_{K'/K}$ for the inertia subgroup of $\mathrm{Gal}\,(K'/K)$. We will write $\mathrm{Art}_K : K^\times \xrightarrow{\sim} W_K^{\mathrm{ab}}$ for the Artin map normalized to send uniformizers to geometric Frobenius elements. We will let rec_K be the local Langlands correspondence of [HT01], so that if π is an irreducible complex admissible representation of $\mathrm{GL}_n(K)$, then $\mathrm{rec}_K(\pi)$ is a Frobenius semi-simple Weil–Deligne representation of the Weil group W_K. We will write rec for rec_K when the

choice of K is clear. If (r, N) is a Weil–Deligne representation of W_K, we will write $(r, N)^{F-\mathrm{ss}}$ for its Frobenius semi-simplification. If ρ is a continuous representation of G_K over $\overline{\mathbb{Q}}_l$ with $l \neq p$, then we will write $\mathrm{WD}(\rho)$ for the corresponding Weil–Deligne representation of W_K. (See, for instance, Section 1 of [TY07].) By a *Steinberg* representation of $\mathrm{GL}_n(K)$ we will mean a representation $\mathrm{Sp}_n(\psi)$ (in the notation of Section 1.3 of [HT01]), where ψ is an unramified character of $K^\times$. If π_i is an irreducible smooth representation of $\mathrm{GL}_{n_i}(K)$ for $i = 1, 2$, we will write $\pi_1 \boxplus \pi_2$ for the irreducible smooth representation of $\mathrm{GL}_{n_1+n_2}(K)$ with $\mathrm{rec}(\pi_1 \boxplus \pi_2) = \mathrm{rec}(\pi_1) \oplus \mathrm{rec}(\pi_2)$. If K'/K is a finite extension and if π is an irreducible smooth representation of $\mathrm{GL}_n(K)$ we will write $\mathrm{BC}_{K'/K}(\pi)$ for the base change of π to K' which is characterized by $\mathrm{rec}_{K'}(\mathrm{BC}_{K'/K}(\pi)) = \mathrm{rec}_K(\pi)|_{W_{K'}}$.

If ρ is a continuous de Rham representation of G_K over $\overline{\mathbb{Q}}_l$, then we will write $\mathrm{WD}(\rho)$ for the corresponding Weil–Deligne representation of W_K, and if $\tau : K \hookrightarrow \overline{\mathbb{Q}}_l$ is a continuous embedding of fields, then we will write $\mathrm{HT}_\tau(\rho)$ for the multiset of Hodge–Tate numbers of ρ with respect to τ. Thus $\mathrm{HT}_\tau(\rho)$ is a multiset of $\dim \rho$ integers. In fact if W is a de Rham representation of G_K over $\overline{\mathbb{Q}}_l$ and if $\tau : K \hookrightarrow \overline{\mathbb{Q}}_l$, then the multiset $\mathrm{HT}_\tau(W)$ contains i with multiplicity $\dim_{\overline{\mathbb{Q}}_l}(W \otimes_{\tau,K} \widehat{\overline{K}}(i))^{G_K}$. Thus for example $\mathrm{HT}_\tau(\epsilon_l) = \{-1\}$.

We will let c denote complex conjugation on $\mathbb{C}$. We will write $\mathrm{Art}_{\mathbb{R}}$ (resp. $\mathrm{Art}_{\mathbb{C}}$) for the unique continuous surjection

$$\mathbb{R}^\times \twoheadrightarrow \mathrm{Gal}(\mathbb{C}/\mathbb{R})$$

(resp. $\mathbb{C}^\times \twoheadrightarrow \mathrm{Gal}(\mathbb{C}/\mathbb{C})$). We will write $\mathrm{rec}_{\mathbb{C}}$ (resp. $\mathrm{rec}_{\mathbb{R}}$), or simply rec, for the local Langlands correspondence from irreducible admissible (Lie $\mathrm{GL}_n(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$, $O(n)$)-modules (resp. (Lie $\mathrm{GL}_n(\mathbb{C}) \otimes_{\mathbb{R}} \mathbb{C}, U(n)$)-modules) to continuous, semi-simple n-dimensional representations of the Weil group $W_{\mathbb{R}}$ (resp. $W_{\mathbb{C}}$). (See [Lan89].) If π_i is an irreducible admissible (Lie $\mathrm{GL}_{n_i}(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}, O(n_i)$)-module (resp. (Lie $\mathrm{GL}_{n_i}(\mathbb{C}) \otimes_{\mathbb{R}} \mathbb{C}, U(n_i)$)-module) for $i = 1, \dots, r$ and if $n = n_1 + \cdots + n_r$, then we define an irreducible admissible (Lie $\mathrm{GL}_n(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}, O(n)$)-module (resp. (Lie $\mathrm{GL}_n(\mathbb{C}) \otimes_{\mathbb{R}} \mathbb{C}, U(n)$)-module) $\pi_1 \boxplus \cdots \boxplus \pi_r$ by

$$\mathrm{rec}(\pi_1 \boxplus \cdots \boxplus \pi_r) = \mathrm{rec}(\pi_1) \oplus \cdots \oplus \mathrm{rec}(\pi_r).$$

If π is an irreducible admissible (Lie $\mathrm{GL}_n(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}, O(n)$)-module, then we define $\mathrm{BC}_{\mathbb{C}/\mathbb{R}}(\pi)$ to be the irreducible admissible (Lie $\mathrm{GL}_n(\mathbb{C}) \otimes_{\mathbb{R}} \mathbb{C}, U(n)$)-module defined by

$$\mathrm{rec}_{\mathbb{C}}(\mathrm{BC}_{\mathbb{C}/\mathbb{R}}(\pi)) = \mathrm{rec}_{\mathbb{R}}(\pi)|_{W_{\mathbb{C}}}.$$

We will write $||\ ||$ for the continuous homomorphism

$$||\ || = \prod_v |\ |_v : \mathbb{A}^\times/\mathbb{Q}^\times \longrightarrow \mathbb{R}^\times_{>0},$$

where each $|\ |_v$ has its usual normalization, i.e., $|p|_p = 1/p$.

Now suppose that $K/\mathbb{Q}$ is a finite extension. We will write $||\ ||_K$ (or simply $||\ ||$) for $||\ || \circ \mathbf{N}_{K/\mathbb{Q}}$. We will also write

$$\mathrm{Art}_K = \prod_v \mathrm{Art}_{K_v} : \mathbb{A}_K^\times / \overline{K^\times (K_\infty^\times)^0} \xrightarrow{\sim} G_K^{\mathrm{ab}}.$$

If v is a finite place of K, we will write $k(v)$ for its residue field, q_v for $\#k(v)$, and Frob_v for Frob_{K_v}. If v is a real place of K, then we will let $[c_v]$ denote the conjugacy class in G_K consisting of complex conjugations associated to v. If K'/K is a quadratic extension of number fields, we will denote by $\delta_{K'/K}$ the nontrivial character of $\mathbb{A}_K^\times / K^\times \mathbf{N}_{K'/K} \mathbb{A}_{K'}^\times$. (We hope that this will cause no confusion with the Galois character $\delta_{K'/K}$. One equals the composition of the other with the Artin map for K.) If K'/K is a soluble, finite Galois extension and if π is a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_K)$, we will write $\mathrm{BC}_{K'/K}(\pi)$ for its base change to K', an (isobaric) automorphic representation of $\mathrm{GL}_n(\mathbb{A}_{K'})$ satisfying

$$\mathrm{BC}_{K'/K}(\pi)_v = \mathrm{BC}_{K'_v/K_{v|_K}}(\pi_{v|_K})$$

for all places v of K'. If π_i is an automorphic representation of $\mathrm{GL}_{n_i}(\mathbb{A}_K)$ for $i = 1, 2$, we will write $\pi_1 \boxplus \pi_2$ for the automorphic representation of $\mathrm{GL}_{n_1+n_2}(\mathbb{A}_K)$ satisfying

$$(\pi_1 \boxplus \pi_2)_v = \pi_{1,v} \boxplus \pi_{2,v}$$

for all places v of K.

We will call a number field K a CM field if it has an automorphism c such that for all embeddings $i : K \hookrightarrow \mathbb{C}$ one has $c \circ i = i \circ c$. In this case either K is totally real, or a totally imaginary quadratic extension of a totally real field. In either case we will let K^+ denote the maximal totally real subfield of K.

Suppose that K is a number field and

$$\chi : \mathbb{A}_K^\times / K^\times \longrightarrow \mathbb{C}^\times$$

is a continuous character. If there exists $a \in \mathbb{Z}^{\mathrm{Hom}(K,\mathbb{C})}$ such that

$$\chi|_{(K_\infty^\times)^0} : x \longmapsto \prod_{\tau \in \mathrm{Hom}(K,\mathbb{C})} (\tau x)^{a_\tau},$$

we will call χ algebraic. In this case we can attach to χ and a rational prime l and an isomorphism $\imath : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$, a unique continuous character

$$r_{l,\imath}(\chi) : G_K \longrightarrow \overline{\mathbb{Q}}_l^\times$$

such that for all $v \not| l$ we have

$$\imath \circ r_{l,\imath}(\chi)|_{W_{K_v}} \circ \mathrm{Art}_{K_v} = \chi_v.$$

There is also an integer $\mathrm{wt}(\chi)$, the weight of χ, such that

$$|\chi| = ||\ ||_K^{-\mathrm{wt}(\chi)/2}.$$

(See the discussion at the start of Section A.2 for more details.)

If F is a totally real field, we call a continuous character

$$\chi : \mathbb{A}_K^\times / K^\times \longrightarrow \mathbb{C}^\times$$

totally odd if $\chi_v(-1) = -1$ for all $v|\infty$. Similarly we call a continuous character

$$\mu : G_K \longrightarrow \overline{\mathbb{Q}}_l^\times$$

totally odd if $\mu(c_v) = -1$ for all $v|\infty$.

1. **Deformations of Galois representations**

1.1. *The group $\mathcal{G}_n$.* We let $\mathcal{G}_n$ denote the semi-direct product of $\mathcal{G}_n^0 = \mathrm{GL}_n \times \mathrm{GL}_1$ by the group $\{1, \jmath\}$ where

$$\jmath(g, a)\jmath^{-1} = (a^t g^{-1}, a).$$

We let $\nu : \mathcal{G}_n \to \mathrm{GL}_1$ be the character which sends (g, a) to a and sends $\jmath$ to -1. We will also let $\mathrm{GSp}_{2n} \subset \mathrm{GL}_{2n}$ denote the symplectic similitude group defined by the anti-symmetric matrix

$$J_{2n} = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix},$$

and we will again let $\nu : \mathrm{GSp}_{2n} \to \mathrm{GL}_1$ denote the multiplier character. Finally let GO_n denote the orthogonal similitude group defined by the symmetric matrix 1_n.

There is a natural homomorphism

$$\mathcal{G}_n \times \mathcal{G}_m \longrightarrow \mathcal{G}_n/\mathcal{G}_n^0 \times \mathcal{G}_m/\mathcal{G}_m^0 \longrightarrow \{\pm 1\}$$

which sends both $(\jmath, 1)$ and $(1, \jmath)$ to -1. Let $(\mathcal{G}_n \times \mathcal{G}_m)^+$ denote the kernel of this map. There is a homomorphism

$$\begin{array}{rcl} \otimes : (\mathcal{G}_n \times \mathcal{G}_m)^+ & \longrightarrow & \mathcal{G}_{nm}, \\ (g, a) \times (g', a') & \longmapsto & (g \otimes g', aa'), \\ \jmath \times \jmath & \longmapsto & \jmath. \end{array}$$

There is also a homomorphism

$$\begin{array}{rcl} I : \mathcal{G}_n & \longrightarrow & \mathrm{GSp}_{2n}, \\ (g, a) & \longmapsto & \begin{pmatrix} g & 0 \\ 0 & a^t g^{-1} \end{pmatrix}, \\ \jmath & \longmapsto & \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix}. \end{array}$$

Suppose that Γ is a group with a normal subgroup Δ of index 2 and that $\gamma_0 \in \Gamma - \Delta$. Suppose also that A is a ring and that $r : \Gamma \to \mathcal{G}_n(A)$ is a homomorphism with $\Delta = r^{-1}\mathcal{G}_n^0(A)$. Write $\breve{r} : \Delta \to \mathrm{GL}_n(A)$ for the

composition of $r|_\Delta$ with projection to $\mathrm{GL}_n(A)$. Write $r(\gamma_0) = (a, -(\nu\circ r)(\gamma_0))\jmath$. Then

$$\check{r}(\gamma_0\delta\gamma_0^{-1})a^t\check{r}(\delta) = (\nu\circ r)(\delta)a$$

for all $\delta \in \Delta$, and

$$\check{r}(\gamma_0^2)^t a = -(\nu\circ r)(\gamma_0)a.$$

If $r : \Gamma \to \mathrm{GSp}_{2n}(A)$ is a homomorphism with multiplier μ, then it gives rise to a homomorphism $\hat{r}_\Delta : \Gamma \to \mathcal{G}_{2n}(A)$ which sends $\delta \in \Delta$ to $(r(\delta), \mu(\delta))$ and $\gamma \in \Gamma - \Delta$ to $(r(\gamma)J_{2n}^{-1}, -\mu(\gamma))\jmath$. Then $\Delta = \hat{r}_\Delta^{-1}\mathcal{G}_{2n}^0(A)$ and $\nu\circ\hat{r}_\Delta = \mu$. Similarly if $r : \Gamma \to \mathrm{GO}_n(A)$ is a homomorphism with multiplier μ, then it gives rise to a homomorphism $\hat{r}_\Delta : \Gamma \to \mathcal{G}_n(A)$ which sends $\delta \in \Delta$ to $(r(\delta), \mu(\delta))$ and $\gamma \in \Gamma - \Delta$ to $(r(\gamma), \mu(\gamma))\jmath$. Then $\Delta = \hat{r}_\Delta^{-1}\mathcal{G}_{2n}^0(A)$ and $\nu\circ\hat{r}_\Delta$ equals the product of μ with the nontrivial character of Γ/Δ.

If $r : \Gamma \to \mathcal{G}_n(A)$ (resp. $r' : \Gamma \to \mathcal{G}_m(A)$) is a homomorphism with $r^{-1}\mathcal{G}_n^0(A) = \Delta$ (resp. $(r')^{-1}\mathcal{G}_m^0(A) = \Delta$), then we define

$$I(r) = I\circ r : \Gamma \longrightarrow \mathrm{GSp}_{2n}(A)$$

and

$$r\otimes r' = \otimes\circ(r\times r') : \Gamma \longrightarrow \mathcal{G}_{nm}(A).$$

Note that the multiplier of $I(r)$ equals the multiplier of r and that the multiplier of $r\otimes r'$ differs from the product of the multipliers of r and r' by the nontrivial character of Γ/Δ.

If $\chi : \Delta \to A^\times$ and $\mu : \Gamma \to A^\times$ satisfy

- $\chi\chi^{\gamma_0} = \mu|_\Delta$,
- $\chi(\gamma_0^2) = -\mu(\gamma_0)$

(i.e., the composition of χ with the transfer map $\Gamma^{\mathrm{ab}} \to \Delta^{\mathrm{ab}}$ equals the product of μ and the nontrivial character of Γ/Δ), then there is a homomorphism

$$\begin{array}{rcl}(\chi,\mu) : \Gamma & \longrightarrow & \mathcal{G}_1(A),\\ \delta & \longmapsto & (\chi(\delta),\mu(\delta)),\\ \gamma & \longmapsto & (\chi(\gamma\gamma_0^{-1}), -\mu(\gamma))\jmath\end{array}$$

for all $\delta \in \Delta$ and $\gamma \in \Gamma - \Delta$. We have $\nu\circ(\chi,\mu) = \mu$.

(At the referee's suggestion we include a proof that (χ,μ) is indeed a homomorphism in an attempt to convince the reader that all the unsupported assertions of this section can be checked in an entirely elementary way. Suppose that $\delta_1, \delta_2 \in \Delta$ and $\gamma_1, \gamma_2 \in \Gamma - \Delta$. Then we have

$$(\chi,\mu)(\delta_1\delta_2) = (\chi,\mu)(\delta_1)(\chi,\mu)(\delta_2)$$

and

$$\begin{aligned}(\chi,\mu)(\delta_1\gamma_2) &= (\chi(\delta_1\gamma_2\gamma_0^{-1}), -\mu(\delta_1\gamma_2))\jmath\\ &= (\chi,\mu)(\delta_1)(\chi,\mu)(\gamma_2)\end{aligned}$$

and

$$\begin{aligned}(\chi,\mu)(\gamma_1\delta_2) &= (\chi(\gamma_1\delta_2\gamma_0^{-1}), -\mu(\gamma_1\delta_2))\jmath \\ &= (\chi(\gamma_1\gamma_0^{-1}(\gamma_0\delta_2\gamma_0^{-1})), -\mu(\gamma_1\delta_2))\jmath \\ &= (\chi,\mu)(\gamma_1)\jmath(\chi(\gamma_0\delta_2\gamma_0^{-1}), \mu(\delta_2))\jmath \\ &= (\chi,\mu)(\gamma_1)(\mu(\delta_2)\chi^{\gamma_0}(\delta_2)^{-1}, \mu(\delta_2)) \\ &= (\chi,\mu)(\gamma_1)(\chi,\mu)(\delta_2)\end{aligned}$$

and

$$\begin{aligned}(\chi,\mu)(\gamma_1\gamma_2) &= (\chi(\gamma_1\gamma_0^{-1})\chi(\gamma_0\gamma_2), \mu(\gamma_1)\mu(\gamma_2)) \\ &= (\chi,\mu)(\gamma_1)\jmath(\chi^{\gamma_0}(\gamma_2\gamma_0), -\mu(\gamma_2)) \\ &= (\chi,\mu)(\gamma_1)(-\mu(\gamma_2)\chi^{\gamma_0}(\gamma_2\gamma_0)^{-1}, -\mu(\gamma_2))\jmath \\ &= (\chi,\mu)(\gamma_1)(-\chi(\gamma_2\gamma_0)\mu(\gamma_0)^{-1}, -\mu(\gamma_2))\jmath \\ &= (\chi,\mu)(\gamma_1)(\chi,\mu)(\gamma_2)\end{aligned}$$

as desired.)

In the case that $\Gamma = G_{F^+}$ and $\Delta = G_F$ where F is an imaginary CM field with maximal totally real subfield F^+, we call $r : G_{F^+} \to \mathcal{G}_n(A)$ (resp. $\mathrm{GSp}_{2n}(A)$, resp. $\mathrm{GO}_n(A)$) *totally odd* if the multiplier character takes every complex conjugation to -1 (resp. -1, resp. 1). Note that if r is totally odd, so is $I(r)$ (resp. $\hat{r}_\Delta$, resp. $\hat{r}_\Delta$).

Suppose now that A is a field, that $r : \Delta \to \mathrm{GL}_n(A)$ is absolutely irreducible, and that $\mu : \Gamma \to A^\times$ is a character so that

$$r^{\gamma_0} \cong r^\vee \otimes \mu|_\Delta.$$

More precisely if $\gamma \in \Gamma - \Delta$, there is a $b_\gamma \in \mathrm{GL}_n(A)$, unique up to scalar multiples, such that

$$r(\gamma\delta\gamma^{-1}){b_\gamma}^t r(\delta) = \mu(\delta) b_\gamma$$

for all $\delta \in \Delta$. Computing $r(\gamma^2\delta\gamma^{-2})$ in two ways and using the absolute irreducibility of r, we deduce that $r(\gamma^2)$ is a scalar multiple of ${b_\gamma}^t b_\gamma^{-1}$. (Write

$$\begin{aligned}r(\gamma^2)r(\delta)r(\gamma^2)^{-1} &= r(\gamma^2\delta\gamma^{-2}) \\ &= \mu(\delta){b_\gamma}^t r(\gamma\delta\gamma^{-1})^{-1} b_\gamma^{-1} = ({b_\gamma}^t b_\gamma^{-1}) r(\delta) ({b_\gamma}^t b_\gamma^{-1})^{-1}\end{aligned}$$

and apply Schur's lemma.) Substituting $\delta = \gamma^2$ in the last displayed equation, we then deduce that

$$r(\gamma^2)^t b_\gamma = \pm\mu(\gamma) b_\gamma.$$

One can check that the sign in the above equation is independent of $\gamma \in \Gamma - \Delta$, and we will denote it $-\mathrm{sgn}\,(r,\mu)$. (To see this one uses the fact that one can take $b_{\delta\gamma} = r(\delta)b_\gamma$ for $\delta \in \Delta$ and $\gamma \in \Gamma - \Delta$.) Then we get a homomorphism

$$\tilde{r}_\mu : \Gamma \longrightarrow \mathcal{G}_n(A)$$

which sends $\delta \in \Delta$ to $(r(\delta), \mu(\delta))$ and sends γ_0 to $(b_{\gamma_0}, -\mathrm{sgn}\,(r,\mu)\mu(\gamma_0))_J$. In particular if $\mathrm{sgn}\,(r,\mu) = 1$, then $\nu \circ \tilde{r}_\mu = \mu$, while if $\mathrm{sgn}\,(r,\mu) = -1$, then $\mu^{-1}(\nu \circ \tilde{r}_\mu)$ is the nontrivial character of Γ/Δ. Moreover $\breve{\tilde{r}} = r$.

1.2. *Abstract deformation theory.* Fix a rational prime l and let $\mathcal{O}$ denote the ring of integers of a finite extension L of $\mathbb{Q}_l$ in $\overline{\mathbb{Q}}_l$. Let λ denote the maximal ideal of $\mathcal{O}$ and let $\mathbb{F} = \mathcal{O}/\lambda$. Let Γ denote a topologically finitely generated profinite group and let $\overline{\rho} : \Gamma \to \mathrm{GL}_n(\mathbb{F})$ be a continuous homomorphism.

We will denote by

$$\rho^{\square} = \rho^{\square}_{\mathcal{O}} : \Gamma \to \mathrm{GL}_n(R^{\square}_{\mathcal{O},\overline{\rho}})$$

the universal lifting (or "framed deformation") of $\overline{\rho}$ to a complete noetherian local $\mathcal{O}$-algebra with residue field $\mathbb{F}$. (We impose no equivalence condition on lifts other than equality.) We will write

$$R^{\square}_{\overline{\rho}} \otimes \overline{\mathbb{Q}}_l$$

for $R^{\square}_{\mathcal{O},\overline{\rho}} \otimes_{\mathcal{O}} \overline{\mathbb{Q}}_l$. The following lemma is presumably well known, but as we do not know a reference, we give a proof.

LEMMA 1.2.1. *Suppose that $\mathcal{O}'$ is the ring of integers of a finite extension L'/L. Then the map*

$$R^{\square}_{\mathcal{O}',\overline{\rho}} \longrightarrow R^{\square}_{\mathcal{O},\overline{\rho}} \otimes_{\mathcal{O}} \mathcal{O}'$$

coming from the universal property of $R^{\square}_{\mathcal{O}',\overline{\rho}}$ and $\rho^{\square}_{\mathcal{O}} \otimes \mathcal{O}'$ is an isomorphism. In particular, as the notation suggests, the ring $R^{\square}_{\overline{\rho}} \otimes \overline{\mathbb{Q}}_l$ does not depend on the choice of L.

Proof. Let $R^{\square}_{\mathcal{O}',\mathbb{F},\overline{\rho}}$ denote the closed subring of $R^{\square}_{\mathcal{O}',\overline{\rho}}$ consisting of elements which reduce to an element of $\mathbb{F}$ modulo the maximal ideal. Then $R^{\square}_{\mathcal{O}',\mathbb{F},\overline{\rho}}$ is a complete, noetherian local $\mathcal{O}$-algebra with residue field $\mathbb{F}$ and $\rho^{\square}_{\mathcal{O}'}$ is defined over $R^{\square}_{\mathcal{O}',\mathbb{F},\overline{\rho}}$. Thus the universal property of $R^{\square}_{\mathcal{O},\overline{\rho}}$ gives rise to a map $R^{\square}_{\mathcal{O},\overline{\rho}} \to R^{\square}_{\mathcal{O}',\mathbb{F},\overline{\rho}}$ under which $\rho^{\square}_{\mathcal{O}}$ pushes forward to $\rho^{\square}_{\mathcal{O}'}$. This map extends to a $\mathcal{O}'$-linear map $R^{\square}_{\mathcal{O},\overline{\rho}} \otimes_{\mathcal{O}} \mathcal{O}' \to R^{\square}_{\mathcal{O}',\overline{\rho}}$. We claim this is an inverse to our map $R^{\square}_{\mathcal{O}',\overline{\rho}} \longrightarrow R^{\square}_{\mathcal{O},\overline{\rho}} \otimes_{\mathcal{O}} \mathcal{O}'$.

Under the composite $R^{\square}_{\mathcal{O}',\overline{\rho}} \longrightarrow R^{\square}_{\mathcal{O}',\overline{\rho}}$ the representation $\rho^{\square}_{\mathcal{O}'}$ pushes forward to itself, and so this map must be the identity. Consider the composite

$$R^{\square}_{\mathcal{O},\overline{\rho}} \longrightarrow R^{\square}_{\mathcal{O}',\overline{\rho}} \longrightarrow R^{\square}_{\mathcal{O},\overline{\rho}} \otimes_{\mathcal{O}} \mathcal{O}'.$$

It factors through the subring $(R^{\square}_{\mathcal{O},\overline{\rho}} \otimes_{\mathcal{O}} \mathcal{O}')_{\mathbb{F}} \subset R^{\square}_{\mathcal{O},\overline{\rho}} \otimes_{\mathcal{O}} \mathcal{O}'$ consisting of elements which reduce modulo the maximal ideal to an element of $\mathbb{F}$. The representation $\rho^{\square}_{\mathcal{O}}$ pushes forward to itself, and so this composite must equal the canonical inclusion, and we have proved the lemma. □

The maximal ideals are dense in $\operatorname{Spec} R^{\square}_{\mathcal{O},\overline{\rho}}[1/l]$. (See [Gro66, 10.5.7].) A prime ideal $\wp$ of $R^{\square}_{\mathcal{O},\overline{\rho}}[1/l]$ is maximal if and only if the residue field $k(\wp) = R^{\square}_{\mathcal{O},\overline{\rho}}[1/l]_{\wp}/\wp$ is (topologically isomorphic to) a finite extension of L. (For the "if" part note that the image of $R^{\square}_{\mathcal{O},\overline{\rho}}$ in $k(\wp)$ is a compact $\mathcal{O}$-submodule of $k(\wp)$ with field of fractions $k(\wp)$. Thus $R^{\square}_{\mathcal{O},\overline{\rho}}[1/l] \twoheadrightarrow k(\wp)$. For the "only if" part see, for instance, Lemma 2.6 of [Tay08].) We get a continuous representation $\rho_{\wp} : \Gamma \to \mathrm{GL}_n(k(\wp))$. The formal completion $R^{\square}_{\mathcal{O},\overline{\rho}}[1/l]^{\wedge}_{\wp}$ is the universal lifting ring for $\rho_{\wp}$; i.e., if A is an Artinian local $k(\wp)$-algebra with residue field $k(\wp)$ and if $\rho : \Gamma \to \mathrm{GL}_n(A)$ is a continuous representation lifting $\rho_{\wp}$, then there is a unique continuous map of $k(\wp)$-algebras $R^{\square}_{\mathcal{O},\overline{\rho}}[1/l]^{\wedge}_{\wp} \to A$ so that $\rho^{\square}$ pushes forward to ρ. (Let R denote the image of $R^{\square}_{\mathcal{O},\overline{\rho}}$ in $A/\mathfrak{m}_A$. Let A^0 denote the R-subalgebra of A generated by the matrix entries of the image of ρ. Then A^0 is a complete noetherian local $\mathcal{O}$-algebra with residue field $\mathbb{F}$ and $\rho : \Gamma \to \mathrm{GL}_n(A^0)$. The assertion follows easily.) The map which associates a cocycle $c \in Z^1(\Gamma, \operatorname{ad}\rho_{\wp})$ to the lift $(1_n + c\epsilon)\rho_{\wp}$ of $\rho_{\wp}$ defines an isomorphism between $Z^1(\Gamma, \operatorname{ad}\rho_{\wp})$ and the tangent space $\operatorname{Hom}_{k(\wp)}(R^{\square}_{\mathcal{O},\overline{\rho}}[1/l]^{\wedge}_{\wp}, k(\wp)[\epsilon]/\epsilon^2)$. If $H^2(\Gamma, \operatorname{ad}\rho_{\wp}) = (0)$, then $R^{\square}_{\mathcal{O},\overline{\rho}}[1/l]$ is formally smooth at $\wp$ (by the argument of Proposition 2 of [Maz89], which can be easily adapted to our current situation) of dimension

$$\dim_{k(\wp)} Z^1(\Gamma, \operatorname{ad}\rho_{\wp}) = n^2 + \dim_{k(\wp)} H^1(\Gamma, \operatorname{ad}\rho_{\wp}) - \dim_{k(\wp)} H^0(\Gamma, \operatorname{ad}\rho_{\wp}),$$

where we use continuous cohomology. (We learned these observations from Mark Kisin.)

Let H denote the subgroup of $\mathrm{GL}_n(R^{\square}_{\mathcal{O},\overline{\rho}})$ consisting of elements which reduce modulo the maximal ideal to an element that centralizes the image of $\overline{\rho}$. If $h \in H$, then there is a unique continuous homomorphism

$$\phi_h : R^{\square}_{\mathcal{O},\overline{\rho}} \to R^{\square}_{\mathcal{O},\overline{\rho}}$$

such that $\rho^{\square}$ pushes forward to $h^{-1}\rho^{\square}h$. We have that

$$\phi_g \circ \phi_h = \phi_{g\phi_g(h)}.$$

(We remark that after Definition 2.2.1 of [CHT08] this action is defined for elements of $1_n + M_n(\mathfrak{m}_{R^{\square}_{\mathcal{O},\overline{\rho}}})$, but it is incorrectly stated that this defines an action of the group $1_n + M_n(\mathfrak{m}_{R^{\square}_{\mathcal{O},\overline{\rho}}})$ on $R^{\square}_{\mathcal{O},\overline{\rho}}$. This is not important in the rest of [CHT08].)

Lemma 1.2.2. *Keep the notation of the previous paragraph. If $h \in H$, then ϕ_h fixes each irreducible component of* $\operatorname{Spec} R^{\square}_{\mathcal{O},\overline{\rho}}[1/l]$.

Proof. Suppose that $h \in H$ and that $\mathcal{O}'$ is the ring of integers of a finite extension of L'/L in $\overline{\mathbb{Q}}_l$, with maximal ideal λ'. Suppose that $\rho : \Gamma \to \mathrm{GL}_n(\mathcal{O}')$ lifts $\overline{\rho}$. Then $h^{-1}\rho h$ also lifts $\overline{\rho}$. (We are using h both for an element of

$\mathrm{GL}_n(R^{\square}_{\mathcal{O},\overline{\rho}})$ and for its image in $\mathrm{GL}_n(\mathcal{O}')$ under the map $R^{\square}_{\mathcal{O},\overline{\rho}} \to \mathcal{O}'$ induced by ρ.) Recall (from Lemma A.1.2) that $A = \mathcal{O}'\langle s,t\rangle/(s\det(t1_n+(1-t)h)-1)$ with the λ'-adic topology is a complete topological domain. We have a continuous representation

$$\widetilde{\rho} = (t1_n + (1-t)h)^{-1}\rho(t1_n + (1-t)h) : \Gamma \longrightarrow \mathrm{GL}_n(A).$$

Let A^0 denote the closed subalgebra of A generated by the matrix entries of elements of the image of $\widetilde{\rho}$ and give it the subspace topology. Then

$$\widetilde{\rho} : \Gamma \longrightarrow \mathrm{GL}_n(A^0),$$

and $\widetilde{\rho} \bmod (\lambda' A \cap A^0) = \overline{\rho}$, and $\widetilde{\rho}$ pushes forward to ρ (resp. $h^{-1}\rho h$) under the continuous homomorphism $A^0 \to \mathcal{O}'$ induced by $t \mapsto 1$ (resp. $t \mapsto 0$). We will show that A^0 is a complete, noetherian local $\mathcal{O}$-algebra with residue field $\mathbb{F}$. It will follow that there is a natural map $R^{\square}_{\mathcal{O},\overline{\rho}} \to A^0$ through which the maps $R^{\square}_{\mathcal{O},\overline{\rho}} \to \mathcal{O}'$ corresponding to ρ and $h^{-1}\rho h$ both factor. As A^0 is a domain (being a sub-ring of A), we conclude that the points corresponding to ρ and $h^{-1}\rho h$ lie on the same irreducible component of $\operatorname{Spec} R^{\square}_{\mathcal{O},\overline{\rho}}[1/l]$. As any irreducible component contains an $\mathcal{O}'$-point which lies on no other irreducible component for some $\mathcal{O}'$ as above (because such points are Zariski dense in $\operatorname{Spec} R^{\square}_{\mathcal{O},\overline{\rho}}[1/l]$), we see that the lemma follows. (The referee remarks that it may be helpful to think of this argument as an instance of homotopy.)

It remains to show that A^0 is a complete, noetherian, local $\mathcal{O}$-algebra with residue field $\mathbb{F}$. Let $\gamma_1, \dots, \gamma_r$ denote topological generators of Γ. Write

$$\widetilde{\rho}(\gamma_i) = a_i + b_i,$$

where $a_i \in \mathrm{GL}_n(\mathcal{O})$ lifts $\overline{\rho}(\gamma_i)$ and where $b_i \in M_{n\times n}(\lambda' A)$. Then A^0 is the closure of the $\mathcal{O}$-subalgebra of A generated by the entries of the b_i. As these entries are topologically nilpotent in A, we get a continuous $\mathcal{O}$-algebra homomorphism

$$\mathcal{O}[[X_{ijk}]]_{i=1,\dots,r;\ j,k=1,\dots,n} \longrightarrow A$$

which sends X_{ijk} to the (j,k)-entry of b_i. Let J denote the kernel. As $\mathcal{O}[[X_{ijk}]]/J$ is compact (and A is Hausdorff), this map is a topological isomorphism of $\mathcal{O}[[X_{ijk}]]/J$ with its image in A and this image is closed. Thus the image is just A^0 and we have $\mathcal{O}[[X_{ijk}]]/J \xrightarrow{\sim} A^0$, so that A^0 is indeed a complete, noetherian, local $\mathcal{O}$-algebra with residue field $\mathbb{F}$. □

Now suppose that

$$\overline{r} : \Gamma \longrightarrow \mathcal{G}_n(\mathbb{F})$$

is a continuous homomorphism such that $\Gamma \twoheadrightarrow \mathcal{G}_n/\mathcal{G}_n^0$. Let Δ denote the kernel of $\Gamma \twoheadrightarrow \mathcal{G}_n/\mathcal{G}_n^0$ and suppose that $\breve{\overline{r}} : \Delta \longrightarrow \mathrm{GL}_n(\mathbb{F})$ is absolutely irreducible. Then there is a universal deformation

$$r^{\mathrm{univ}} : \Gamma \longrightarrow \mathcal{G}_n(R^{\mathrm{univ}}_{\mathcal{O},\overline{r}})$$

to a complete noetherian local $\mathcal{O}$-algebra with residue field $\mathbb{F}$, where now we consider two liftings as equivalent deformations if they are conjugate. (See Section 2.2 of [CHT08].)

LEMMA 1.2.3.

(1) *Suppose that $\Sigma \subset \Gamma$ has finite index, that Σ is not contained in Δ and that $\breve{\overline{r}}|_{\Delta\cap\Sigma}$ is absolutely irreducible. Then the natural map $R^{\mathrm{univ}}_{\mathcal{O},\overline{r}|_\Sigma} \to R^{\mathrm{univ}}_{\mathcal{O},\overline{r}}$ induced by $r^{\mathrm{univ}}|_\Sigma$ makes $R^{\mathrm{univ}}_{\mathcal{O},\overline{r}}$ a finitely generated $R^{\mathrm{univ}}_{\mathcal{O},\overline{r}|_\Sigma}$-module.*

(2) *Suppose that $s : \Gamma \to \mathcal{G}_m(\mathcal{O})$ is a continuous homomorphism such that $\Delta = s^{-1}\mathcal{G}^0_m(\mathcal{O})$ and $\breve{\overline{s}} \otimes \breve{\overline{r}}$ is absolutely irreducible. Then the natural map $R^{\mathrm{univ}}_{\mathcal{O},\overline{r}\otimes\overline{s}} \to R^{\mathrm{univ}}_{\mathcal{O},\overline{r}}$ induced by $r^{\mathrm{univ}} \otimes s$ makes $R^{\mathrm{univ}}_{\mathcal{O},\overline{r}}$ a finitely generated $R^{\mathrm{univ}}_{\mathcal{O},\overline{r}\otimes\overline{s}}$-module.*

(3) *Suppose $I(\overline{r})$ is absolutely irreducible and that Σ is another open subgroup of index two in Γ which does not contain $\ker I(\overline{r})$. Then the natural map $R^{\mathrm{univ}}_{\mathcal{O},\widehat{I(\overline{r})_\Sigma}} \to R^{\mathrm{univ}}_{\mathcal{O},\overline{r}}$ induced by $\widehat{I(r^{\mathrm{univ}})_\Sigma}$ makes $R^{\mathrm{univ}}_{\mathcal{O},\overline{r}}$ a finitely generated $R^{\mathrm{univ}}_{\mathcal{O},\widehat{I(\overline{r})_\Sigma}}$-module.*

Proof. This is essentially an abstraction of Lemma 3.2.1 of [BLGG12]. Write R for $R^{\mathrm{univ}}_{\mathcal{O},\overline{r}|_\Sigma}$ resp. $R^{\mathrm{univ}}_{\mathcal{O},\overline{r}\otimes\overline{s}}$ resp. $R^{\mathrm{univ}}_{\mathcal{O},\widehat{I(\overline{r})_\Sigma}}$ and write $\mathfrak{m}$ for the maximal ideal of R. We first verify that the image of Γ in $\mathcal{G}_n(R^{\mathrm{univ}}_{\mathcal{O},\overline{r}}/\mathfrak{m}R^{\mathrm{univ}}_{\mathcal{O},\overline{r}})$ is finite. In the first case we use the inclusions

$$\ker(r^{\mathrm{univ}} \bmod \mathfrak{m}) \supset \ker(r^{\mathrm{univ}}|_\Sigma \bmod \mathfrak{m}) = \ker(\overline{r}|_\Sigma);$$

in the second we use the inclusions

$$\ker(r^{\mathrm{univ}} \bmod \mathfrak{m}) \supset \ker((r^{\mathrm{univ}} \otimes s) \bmod \mathfrak{m}) \cap \ker(s \bmod \mathfrak{m}) = \ker(\overline{r} \otimes \overline{s}) \cap \ker \overline{s};$$

and in the third case we use the inclusions

$$\ker(r^{\mathrm{univ}} \bmod \mathfrak{m}) \supset \ker(\widehat{I(r^{\mathrm{univ}})_\Sigma} \bmod \mathfrak{m}) = \ker \widehat{I(\overline{r})_\Sigma}.$$

Let m denote the order of the image of Γ in $\mathcal{G}_n(R^{\mathrm{univ}}_{\mathcal{O},\overline{r}}/\mathfrak{m}R^{\mathrm{univ}}_{\mathcal{O},\overline{r}})$ and let $\gamma_1, \dots, \gamma_m$ be elements of Γ chosen so that their images in $\mathcal{G}_n(R^{\mathrm{univ}}_{\mathcal{O},\overline{r}}/\mathfrak{m}R^{\mathrm{univ}}_{\mathcal{O},\overline{r}})$ exhaust the image of Γ. Let

$$f(T) = \prod_{(\zeta_1,\dots,\zeta_n)\in\mu_m(\overline{\mathbb{F}})^n} (T - (\zeta_1 + \cdots + \zeta_n)) \in \mathbb{F}[T]$$

and let A denote the maximal quotient of $\mathbb{F}[X_{i,j}]_{i,j=1,\dots,n}$ over which the m^{th}-power of the matrix $(X_{i,j})$ is 1_n. If $\wp$ is a prime ideal of A, then all the roots of the characteristic polynomial of $(X_{i,j})$ over $A_\wp/\wp$ are m^{th} roots of unity and hence $f(\mathrm{tr}\,(X_{i,j})) = 0$ in $A/\wp \subset A_\wp/\wp$. Thus there is a positive integer a such

that $f(\mathrm{tr}\,(X_{i,j}))^a = 0$ in A. Then we get a map

$$\begin{array}{rcl} \mathbb{F}[T_1,\dots,T_m]/(f(T_1)^a,\dots,f(T_m)^a) & \longrightarrow & R^{\mathrm{univ}}_{\mathcal{O},\overline{r}}/\mathfrak{m}R^{\mathrm{univ}}_{\mathcal{O},\overline{r}} \\ T_i & \longmapsto & \mathrm{tr}\,\check{r}^{\mathrm{univ}}(\gamma_i). \end{array}$$

By Lemma 2.1.12 of [CHT08] (which shows that $R^{\mathrm{univ}}_{\mathcal{O},\overline{r}}$ is topologically generated as a $\mathcal{O}$-algebra by the $\mathrm{tr}\,r^{\mathrm{univ}}(\gamma_i)$) we see that this map has dense image. On the other hand the source has finite cardinality. We conclude that the map is surjective and that $R^{\mathrm{univ}}_{\mathcal{O},\overline{r}}/\mathfrak{m}R^{\mathrm{univ}}_{\mathcal{O},\overline{r}}$ is finite over $\mathbb{F}$. Hence by Nakayama's lemma we conclude that that $R^{\mathrm{univ}}_{\mathcal{O},\overline{r}}$ is finite over R, as desired. □

1.3. *Local theory*: $l \neq p$. Continue to fix a rational prime l and let $\mathcal{O}$ denote the ring of integers of a finite extension L of $\mathbb{Q}_l$ in $\overline{\mathbb{Q}}_l$. Let λ denote the maximal ideal of $\mathcal{O}$ and let $\mathbb{F} = \mathcal{O}/\lambda$. However in this section we specialize our discussion to the case $\Gamma = G_K$, where $K/\mathbb{Q}_p$ is a finite extension and $p \neq l$. Thus $\overline{\rho} : G_K \to \mathrm{GL}_n(\mathbb{F})$ is continuous. Write q for the order of the residue field of K.

In this case the tangent space to $R^{\square}_{\mathcal{O},\overline{\rho}}[1/l]$ at a maximal ideal $\wp$ has dimension

$$\begin{aligned} n^2 + \dim_{k(\wp)} H^1(G_K, \mathrm{ad}\,\rho_\wp) - \dim_{k(\wp)} H^0(G_K, \mathrm{ad}\,\rho_\wp) \\ = n^2 + \dim_{k(\wp)} H^2(G_K, \mathrm{ad}\,\rho_\wp) \\ = n^2 + \dim_{k(\wp)} H^0(G_K, (\mathrm{ad}\,\rho_\wp)(1)), \end{aligned}$$

by the local Euler characteristic formula for $\mathbb{Q}_l$-modules and local duality for $\mathbb{Q}_l$-modules. (The proof of Lemma 9.7 of [Kis03] shows that the usual Euler characteristic formula with finite coefficients implies the analogous statement in the case where the coefficients are finite $\mathbb{Q}_l$-modules. Theorem 1.4.1 of [Rub00] provides a reference for local duality for $\mathbb{Q}_l$-modules.) Moreover $R^{\square}_{\mathcal{O},\overline{\rho}}[1/l]$ is formally smooth at a maximal ideal $\wp$ if $H^0(G_K, (\mathrm{ad}\,\rho_\wp)(1)) = (0)$. We will call a continuous representation $\rho : G_K \to \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$ *robustly smooth* (resp. *smooth*) if $H^0(G_{K'}, (\mathrm{ad}\,\rho_\wp)(1)) = (0)$ for all finite extensions K'/K (resp. for $K' = K$). Our next aim is to show that the set of closed points of $\mathrm{Spec}\, R^{\square}_{\mathcal{O},\overline{\rho}}[1/l]$ which are robustly smooth is Zariski dense, which will imply that all irreducible components of $\mathrm{Spec}\, R^{\square}_{\mathcal{O},\overline{\rho}}[1/l]$ are generically formally smooth of dimension n^2. (The corresponding result for smooth points can be found in the proof of Theorem 2.1.6 of [Gee11] or in [Cho09], but our proof seems to be different even in this special case. This case is already sufficient to deduce that all irreducible components of $\mathrm{Spec}\, R^{\square}_{\mathcal{O},\overline{\rho}}[1/l]$ are generically formally smooth of dimension n^2.)

Define a partial order on $\overline{L}^\times$ by $a \geq b$ if a equals $\sigma(b)\zeta q^m$ where $\sigma \in \mathrm{Gal}\,(\overline{L}/L)$, where ζ is a root of unity and where $m \in \mathbb{Z}_{\geq 0}$. We will write $a \approx b$ (a "equivalent" to b) if $a \geq b$ and $b \geq a$; and we will write $a \sim b$ (a

"comparable" to b) if $a \geq b$ or $b \geq a$. These are both equivalence relations. Further we will write $a > b$ for $a \geq b$ but $a \not\approx b$. Choose $\phi \in W_K$ a lift of Frob_K. If V is a finite-dimensional L-vector space with an action of W_K with open kernel and if $a \in \overline{L}^\times$, then we define $V((a))$ (resp. $V(a)$) to be the L-subspace of V such that $V((a)) \otimes_L \overline{L}$ (resp. $V(a) \otimes_L \overline{L}$) is the sum of the b-generalized eigenspaces of ϕ in V as b runs over all elements of $\overline{L}^\times$ with $a \approx b$ (resp. $a \sim b$). This is independent of the choice of ϕ. (If ϕ' is another choice then the actions of ϕ^m and $(\phi')^m$ on V are equal for some $m \in \mathbb{Z}_{>0}$.) Thus $V(a)$ and $V((a))$ are W_K-invariant. We have decompositions

$$V = \bigoplus V((a)),$$

where a runs over $\overline{L}^\times / \approx$, and

$$V = \bigoplus V(a),$$

where a runs over $\overline{L}^\times / \sim$. We will say that V has type a if $V = V((a))$.

LEMMA 1.3.1. *Suppose that (r, N) is a Weil–Deligne representation of W_K on a finite-dimensional L-vector space V. Then we can write*

$$V = \bigoplus_{i=1}^{u} \bigoplus_{j=1}^{s_i} V_{i,j},$$

where

- *$V_{i,j}$ is invariant under I_K;*
- *$N : V_{i,j} \xrightarrow{\sim} V_{i,j+1}$ unless $j = s_i$ in which case $NV_{i,s_i} = (0)$;*
- *$W_K V_{i,j} \subset V_{i,j} \oplus \bigoplus_{i'=1}^{i-1} \bigoplus_{j'} V_{i',j'}$ and so we get an induced action of W_K on $V_{i,j}$;*
- *the action of W_K on $(V_{i,j} \oplus \bigoplus_{i'=1}^{i-1} \bigoplus_{j'} V_{i',j'}) / (\bigoplus_{i'=1}^{i-1} \bigoplus_{j'} V_{i',j'})$ is irreducible;*
- *$V_{i,j}$ has type $a_i q^{1-j}$ for some $a_i \in \overline{\mathbb{Q}}_l^\times$;*
- *if $i' < i$, then $a_i \not\approx a_{i'}$.*

Proof. We may suppose that $V = V(b)$ for some b (because N must take any $V(b)$ to itself). We will construct the $V_{i,j}$ by recursion on i. Suppose that we have constructed $V_{i,j}$ for $i < t$. Choose $a_t \in \overline{L}^\times$ such that

$$\left(V / \bigoplus_{i=1}^{t-1} \bigoplus_{j} V_{i,j} \right) ((a_t)) \neq (0)$$

and such that if $a > a_t$, then

$$\left(V / \bigoplus_{i=1}^{t-1} \bigoplus_{j} V_{i,j} \right) ((a)) = (0).$$

Also choose an irreducible W_K-submodule

$$\overline{V}_{t,1} \subset \left(V / \bigoplus_{i=1}^{t-1} \bigoplus_j V_{i,j} \right) ((a_t))$$

and choose s_t minimal such that $N^{s_t}\overline{V}_{t,1} = (0)$. Lift $\overline{V}_{t,1}$ to an I_K-submodule $V^0_{t,1} \subset V((a_t))$. Then $N^{s_t}V^0_{t,1} \subset \bigoplus_{i=1}^{t-1} \bigoplus_j V_{i,j}$. For each $i < t$ choose $j_i \in \mathbb{Z}_{>s_t}$ such that $a_t q^{-s_t} \approx a_i q^{1-j_i}$. (To see that $j_i > s_t$ we are using the fact that for $i \leq t$ we have $a_i \geq a_t$.) Then

$$N^{s_t}V^0_{t,1} \subset \bigoplus_{i<t} V_{i,j_i}.$$

Thus if $v \in V^0_{t,1}$, we can write

$$N^{s_t}v = \sum_{i<t} N^{s_t} v_i$$

for unique elements $v_i \in V_{i,j_i - s_t}$. Set $V_{t,1}$ to be the set of

$$v - \sum_{i<t} v_i.$$

We see that $V_{t,1} \subset V((a_t))$ is a $\overline{\mathbb{Q}}_l$-sub-vector space lifting $\overline{V}_{t,1}$, which is I_K-invariant and satisfies $N^{s_t}V_{t,1} = (0)$. Set $V_{t,j} = N^{j-1}V_{t,1}$. It is not hard to see that these $V_{i,j}$ have all the desired properties. □

We remark that if we define an increasing filtration on V by

$$\mathrm{Fil}_i V = \bigoplus_{i' \leq i} \bigoplus_j V_{i,j},$$

then

$$V^{F-\mathrm{ss}} \cong \bigoplus_{i=1}^{u} \mathrm{gr}_i V.$$

Lemma 1.3.2.

(1) *Suppose $\rho : G_K \to \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$ is a continuous representation, that $\imath : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$ and that π is an irreducible smooth representation of $\mathrm{GL}_n(K)$ over $\mathbb{C}$ with $\imath\mathrm{WD}(\rho)^{F-\mathrm{ss}} \cong \mathrm{rec}_K(\pi)$. If π is generic, then ρ is smooth.*

(2) *The closed points $\wp$ in $\mathrm{Spec}\, R^{\square}_{\mathcal{O},\overline{\rho}}[1/l]$ for which $\rho_\wp$ is robustly smooth are Zariski dense.*

Proof. For the first part write $\pi = \mathrm{Sp}_{s_1}(\pi_1) \boxplus \cdots \boxplus \mathrm{Sp}_{s_t}(\pi_t)$ for some supercuspidal representations π_i of $\mathrm{GL}_{n_i}(K)$ and positive integers s_i, n_i with $\sum s_i n_i = n$. (We are using the notation of [HT01].) Then ρ has a filtration with graded pieces ρ_i satisfying $\imath\mathrm{WD}(\rho_i)^{F-\mathrm{ss}} = \mathrm{rec}_K(\mathrm{Sp}_{s_i}(\pi_i))$, possibly after reordering the i's. Thus $(\mathrm{ad}\,\rho)(1)$ has a filtration with graded pieces $\mathrm{Hom}\,(\rho_i, \rho_j(1))$. If this had nonzero invariants, then $\pi_i \cong \pi_j \otimes |\det|^m$ for some

$\max\{1, 1+s_j-s_i\} \leq m \leq s_j$. Thus (π_i, s_i) and (π_j, s_j) are linked, contradicting the assumption that π is generic. (See page 36 of [HT01].)

For the second part suppose that $\wp$ is a closed point of $\operatorname{Spec} R^{\square}_{\mathcal{O},\overline{\rho}}[1/l]$. Set $\mathcal{O}'$ equal to the image of $R^{\square}_{\mathcal{O},\overline{\rho}}$ in $L' = R^{\square}_{\mathcal{O},\overline{\rho}}[1/l]/\wp$ and let λ' denote the maximal ideal of $\mathcal{O}'$. By Lemma 1.3.1 we can find a decomposition $(L')^n = \bigoplus_{i=1}^{u} V_i$ such that

- for each i the sub-space V_i is invariant under I_K;
- for each i the sub-space $\operatorname{Fil}_i V = \bigoplus_{i'=1}^{i} V_{i'}$ is invariant under G_K;
- for each i we have $\operatorname{WD}(\operatorname{gr}_i V) = \operatorname{Sp}_{s_i}(W_i)$, where W_i has some type a_i.

(By $\operatorname{Sp}_s(W)$ we mean the Weil–Deligne representation of W_K whose underlying representation of W_K is $W \oplus W(1) \oplus \cdots \oplus W(s-1)$ and where $N : W(i) \xrightarrow{\sim} W(i+1)$ for $i = 0, \ldots, s-2$.) Choose $M \in \mathbb{Z}_{>0}$ so that

$$\bigoplus_{i=1}^{u}((\mathcal{O}')^n \cap V_i) \supset (l^{M-1}\mathcal{O}')^n.$$

Let

$$A \in \ker(\mathrm{GL}_n(\mathcal{O}'[[X_1, \ldots, X_u]]) \longrightarrow \mathrm{GL}_n(\mathbb{F}))$$

be the unique element which preserves each $(V_i \cap (\mathcal{O}')^n) \otimes_{\mathcal{O}'} \mathcal{O}'[[X_1, \ldots, X_u]]$ and acts on it by multiplication by $(1 + l^M X_i)$. Note that A commutes with $\rho_\wp(I_K)$. Then there is a unique continuous representation

$$\rho : G_K \longrightarrow \mathrm{GL}_n(\mathcal{O}'[[X_1, \ldots, X_u]])$$

such that $\rho|_{I_K} = \rho_\wp|_{I_K} \otimes_{\mathcal{O}'} \mathcal{O}'[[X_1, \ldots, X_u]]$ and such that for any lift ϕ of Frob_K to W_K we have $\rho(\phi) = \rho_\wp(\phi)A$. Then ρ is a lift of $\overline{\rho}$. If $x \in (\lambda')^u$, write ρ_x for $\rho \bmod (X_1 - x_1, \ldots, X_u - x_u)$. Note that $\rho_0 = \rho_\wp$. We will show that for (Zariski) generic x that ρ_x is robustly smooth and the second part of the lemma will follow. (Note that if $0 \neq f \in \mathcal{O}'[[X_1, \ldots, X_u]]$, then f cannot vanish on all of $(\lambda')^u$ by, for instance, Lemma 3.1 of [BLGHT11].)

If $y \in (\mathcal{O}')^\times$, let $\nu_y : G_K/I_K \to (\mathcal{O}')^\times$ be the unramified character taking Frob_K to y. Then $(\operatorname{ad}\rho_x)(1)$ has a filtration with graded pieces

$$\operatorname{Hom}(V_i, V_j(\nu_{(1+l^M x_j)/q(1+l^M x_i)})).$$

Note that if $i = j$, then

$$\operatorname{Hom}_{G_{K'}}(V_i, V_i(\nu_{1/q})) = (0)$$

for any finite K'/K, because

$$\operatorname{Hom}_{W_{K'}}(W_i, W_i(\nu_{1/q^j})) = (0)$$

for $j = 1, \ldots, s_j + 1$ (because, in turn, W_i and $W_i(\nu_{1/q^j})$ will have different types). So it remains to show that for general x we will also have

$$\operatorname{Hom}_{G_{K'}}(V_i, V_j(\nu_{(1+l^M x_j)/q(1+l^M x_i)})) = (0)$$

for all $i \neq j$ and all finite K'/K. Let $\phi \in W_K$ denote a Frobenius lift and let L'' denote the compositum of all extensions of L' of degree less than or equal to n. Then L''/L' is finite. It will do to choose $(x_i) \in (\lambda')^u$ so that if $i \neq j$ and if α (resp. β) is an eigenvalue of ϕ on V_i (resp. V_j) and if ζ is a root of unity, then $q\alpha\zeta(1 + l^M x_i) \neq \beta(1 + l^M x_j)$. However if such an equality were to hold, then $\zeta \in L''$. As L'' contains only finitely many roots of unity, the x_i's need only satisfy finitely many inequalities, as desired. □

Suppose that $\mathcal{C}$ is a set of irreducible components of $\operatorname{Spec} R^{\square}_{\mathcal{O},\overline{\rho}}[1/l]$ and let $R^{\square}_{\mathcal{O},\overline{\rho},\mathcal{C}}$ denote the maximal quotient of $R^{\square}_{\mathcal{O},\overline{\rho}}$, which is reduced, l-torsion free and has $\operatorname{Spec} R^{\square}_{\mathcal{O},\overline{\rho},\mathcal{C}}[1/l]$ supported on the components in $\mathcal{C}$. Also let $\mathcal{D}_{\mathcal{C}}$ denote the set of liftings of $\overline{\rho}$ to complete local noetherian $\mathcal{O}$-algebras R with residue field $\mathbb{F}$ such that the induced map $R^{\square}_{\mathcal{O},\overline{\rho}} \to R$ factors through $R^{\square}_{\mathcal{O},\overline{\rho},\mathcal{C}}$. By Lemma 1.2.2 above and Lemma 3.2 of [BLGHT11] we see that $\mathcal{D}_{\mathcal{C}}$ is a deformation problem in the sense of Definition 2.2.2 of [CHT08].

If K'/K is finite and Galois, we will let $R^{\square}_{\mathcal{O},\overline{\rho},K'-\mathrm{nr}}$ denote the maximal quotient of $R^{\square}_{\mathcal{O},\overline{\rho}}$ over which $\rho^{\square}(I_{K'}) = \{1_n\}$.

LEMMA 1.3.3. *The ring $R^{\square}_{\mathcal{O},\overline{\rho},K'-\mathrm{nr}}[1/l]$ is either the zero ring or is formally smooth of dimension n^2.*

Proof. It suffices to show that for any maximal ideal $\wp$ of $R^{\square}_{\mathcal{O},\overline{\rho},K'-\mathrm{nr}}[1/l]$ we have

$$H^2(\operatorname{Gal}((K')^{\mathrm{nr}}/K), \operatorname{ad}\rho_{\wp}) = (0)$$

and

$$\dim_{k(\wp)} H^1(\operatorname{Gal}((K')^{\mathrm{nr}}/K), \operatorname{ad}\rho_{\wp}) = \dim_{k(\wp)} H^0(\operatorname{Gal}((K')^{\mathrm{nr}}/K), \operatorname{ad}\rho_{\wp}).$$

However

$$H^i(\operatorname{Gal}((K')^{\mathrm{nr}}/K), \operatorname{ad}\rho_{\wp}) = H^i(G_K/I_K, (\operatorname{ad}\rho_{\wp})^{I_{(K')^{\mathrm{nr}}/K}})$$
$$= \begin{cases} (\operatorname{ad}\rho_{\wp})^{G_K} & \text{if } i = 0, \\ ((\operatorname{ad}\rho_{\wp})^{I_K})_{G_K/I_K} & \text{if } i = 1, \\ (0) & \text{otherwise.} \end{cases}$$

The lemma follows. □

Thus $R^{\square}_{\mathcal{O},\overline{\rho},K'-\mathrm{nr}}[1/l] = R^{\square}_{\mathcal{O},\overline{\rho},\mathcal{C}_{K'-\mathrm{nr}}}[1/l]$ for some finite set of components $\mathcal{C}_{K'-\mathrm{nr}}$ of $\operatorname{Spec} R^{\square}_{\mathcal{O},\overline{\rho}}$. Let $\mathcal{C}_{\text{p-nr}}$ denote the union of $\mathcal{C}_{K'-\mathrm{nr}}$ over all finite Galois extensions K'/K and set $R^{\square}_{\mathcal{O},\overline{\rho},\text{p-nr}} = R^{\square}_{\mathcal{O},\overline{\rho},\mathcal{C}_{\text{p-nr}}}$. A $\overline{\mathbb{Q}}_l$-point of $R^{\square}_{\mathcal{O},\overline{\rho}}$ factors through $R^{\square}_{\mathcal{O},\overline{\rho},\text{p-nr}}$ if and only if it is potentially unramified. The ring $R^{\square}_{\mathcal{O},\overline{\rho},\text{p-nr}}[1/l]$ is either the zero ring or is formally smooth of dimension n^2.

For $i = 1, 2$, let

$$\rho_i : G_K \longrightarrow \mathrm{GL}_n(\mathcal{O}_{\overline{\mathbb{Q}}_l})$$

be a continuous representation. We say that ρ_1 *connects to* ρ_2, which we denote $\rho_1 \sim \rho_2$, if and only if

- the reduction $\overline{\rho}_1 = \rho_1 \bmod \mathfrak{m}_{\overline{\mathbb{Q}}_l}$ is equivalent to the reduction $\overline{\rho}_2 = \rho_2 \bmod \mathfrak{m}_{\overline{\mathbb{Q}}_l}$,
- ρ_1 and ρ_2 define points on a common irreducible component of $\operatorname{Spec}(R^{\square}_{\overline{\rho}_1} \otimes \overline{\mathbb{Q}}_l)$.

We say that ρ_1 *strongly connects to* ρ_2, which we write $\rho_1 \rightsquigarrow \rho_2$, if $\rho_1 \sim \rho_2$ and ρ_1 lies on a unique irreducible component of $\operatorname{Spec}(R^{\square}_{\overline{\rho}_1} \otimes \overline{\mathbb{Q}}_l)$.

We make the following remarks:

(1) By Lemma 1.2.2 the relations $\rho_1 \sim \rho_2$ and $\rho_1 \rightsquigarrow \rho_2$ do not depend on the equivalence chosen between the reductions $\overline{\rho}_1$ and $\overline{\rho}_2$, nor on the $\mathrm{GL}_n(\mathcal{O}_{\overline{\mathbb{Q}}_l})$-conjugacy class of ρ_1 or ρ_2.
(2) "Connects" is a symmetric relationship, but "strongly connects" may not be.
(3) "Strongly connects" is a transitive relationship, whereas "connects" may not be.
(4) If $\rho_1 \sim \rho_2$ and $\rho_2 \rightsquigarrow \rho_3$, then $\rho_1 \sim \rho_3$.
(5) If $\rho_1 \sim \rho_2$ and $H^0(G_K, (\operatorname{ad}\rho_1)(1)) = (0)$, then $\rho_1 \rightsquigarrow \rho_2$.
(6) Write $\mathrm{WD}(\rho_i) = (r_i, N_i)$. If $\rho_1 \sim \rho_2$, then $r_1|_{I_K} \cong r_2|_{I_K}$. If $\rho_1 \rightsquigarrow \rho_2$ and $\rho_2 \rightsquigarrow \rho_1$, then $(r_1|_{I_K}, N_1) \cong (r_2|_{I_K}, N_2)$.
(7) If ρ_1 and ρ_2 are unramified and have the same reduction, then $\rho_1 \sim \rho_2$.
(8) If K'/K is a finite extension and $\rho_1 \sim \rho_2$, then $\rho_1|_{G_{K'}} \sim \rho_2|_{G_{K'}}$.
(9) If $\rho_1 \sim \rho_2$ and $\rho_1' \sim \rho_2'$, then $\rho_1 \oplus \rho_1' \sim \rho_2 \oplus \rho_2'$ and $\rho_1 \otimes \rho_1' \sim \rho_2 \otimes \rho_2'$ and $\rho_1^\vee \sim \rho_2^\vee$.
(10) If $\mu : G_K \to \overline{\mathbb{Q}}_l^\times$ is a continuous character and if $\rho_1 \rightsquigarrow \rho_2$, then $\rho_1^\vee \rightsquigarrow \rho_2^\vee$ and $\rho_1 \otimes \mu \rightsquigarrow \rho_2 \otimes \mu$.
(11) If $\mu : G_K \to \overline{\mathbb{Q}}_l^\times$ is a continuous unramified character with $\overline{\mu} = 1$, then $\rho_1 \sim \rho_1 \otimes \mu$.
(12) Suppose that $\overline{\rho}_1$ is semi-simple and let Fil^i be an invariant decreasing filtration on ρ_1 by $\mathcal{O}_{\overline{\mathbb{Q}}_l}$-direct summands; then $\rho_1 \sim \bigoplus_i \mathrm{gr}^i \rho_1$.

Two of these assertions, (6) and (12), require proof, so we separate them out into lemmas. The first was proved in [Cho09], but as this is not yet easily available we give a proof here.

LEMMA 1.3.4 (Choi). *Suppose that* $\overline{\rho} : G_K \longrightarrow \mathrm{GL}_n(\mathbb{F})$.

(1) *If* $\wp_1$ *and* $\wp_2$ *are two maximal ideals on the same connected component of* $\operatorname{Spec} R^{\square}_{\overline{\rho}} \otimes \overline{\mathbb{Q}}_l$, *then*

$$(\rho^{\square} \bmod \wp_1)|^{\mathrm{ss}}_{I_K} \cong (\rho^{\square} \bmod \wp_2)|^{\mathrm{ss}}_{I_K}.$$

(2) *Suppose that $\wp_1$ and $\wp_2$ are two maximal ideals lying on the same irreducible component of* $\operatorname{Spec} R^{\square}_{\overline{\rho}} \otimes \overline{\mathbb{Q}}_l$ *and that neither $\wp_1$ nor $\wp_2$ lie on any other irreducible component of* $\operatorname{Spec} R^{\square}_{\overline{\rho}} \otimes \overline{\mathbb{Q}}_l$. *Then*

$$(\rho^{\square} \bmod \wp_1)|_{I_K} \cong (\rho^{\square} \bmod \wp_2)|_{I_K}.$$

Proof. Let $I_K^{(l)}$ denote the unique Sylow-pro-l-complement in I_K, which is a closed normal subgroup of G_K. Let $H_1 = \ker \overline{\rho}|_{I_K^{(l)}}$. Then H_1 is also a closed normal subgroup of G_K, and $\rho^{\square}$ factors through G_K/H_1. Then $(\ker \overline{\rho}|_{I_K})/H_1 \cong \mathbb{Z}_l$. Let $\varphi \in G_K$ denote a lift of Frob_K and let H_0 denote the unique closed subgroup of $\ker \overline{\rho}|_{I_K}$ which contains H_1 and satisfies

$$H_0/H_1 = (\varphi^{n!} - 1)((\ker \overline{\rho}|_{I_K})/H_1).$$

We see that H_0 is normal in G_K and that $H_0/H_1 \cong \mathbb{Z}_l$ and that I_K/H_0 is finite. We will let σ denote a generator of H_0/H_1.

We will first show that for any prime ideal $\wp$ of $R^{\square}_{\overline{\rho}} \otimes \overline{\mathbb{Q}}_l$ the restriction $(\rho^{\square} \bmod \wp)(\sigma)$ is unipotent. To see this first note that, because $\ker \overline{\rho}|_{I_K}$ is normal in G_K, the restriction $(\rho^{\square} \bmod \wp)^{\mathrm{ss}}|_{\ker \overline{\rho}|_{I_K}}$ is semi-simple and hence is the direct sum of n characters, $\chi_1 \oplus \cdots \oplus \chi_n$. Conjugation by φ must permute these characters, and hence each χ_i is invariant by $\varphi^{n!}$. Thus $\chi_i(\sigma) = 1$ for each i and so $(\rho^{\square} \bmod \wp)(\sigma)$ is unipotent.

Again using the fact that H_0 is normal in I_K we deduce that for any prime ideal $\wp$ of $R^{\square}_{\overline{\rho}} \otimes \overline{\mathbb{Q}}_l$ the representation $(\rho^{\square} \bmod \wp)|^{\mathrm{ss}}_{I_K}$ factors through the finite group I_K/H_0. For each of the finitely many isomorphism classes $[r]$ of n-dimensional representations of I_K/H_0 over $\overline{\mathbb{Q}}_l$ consider the locus in $\operatorname{Spec} R^{\square}_{\overline{\rho}} \otimes \overline{\mathbb{Q}}_l$ where $\operatorname{tr} \rho^{\square}|_{I_K} = \operatorname{tr} r$. This gives finitely many disjoint closed subsets of $\operatorname{Spec} R^{\square}_{\overline{\rho}} \otimes \overline{\mathbb{Q}}_l$, whose union contains every prime of $\operatorname{Spec} R^{\square}_{\overline{\rho}} \otimes \overline{\mathbb{Q}}_l$. Thus our closed sets are also all open, and hence a union of connected components of $\operatorname{Spec} R^{\square}_{\overline{\rho}} \otimes \overline{\mathbb{Q}}_l$. The first part of the lemma follows.

Now suppose that $\wp$ is a maximal prime ideal of $R^{\square}_{\overline{\rho}} \otimes \overline{\mathbb{Q}}_l$ and let S denote the formal power series ring in n^2 variables over $\overline{\mathbb{Q}}_l$. We will first exhibit a surjection

$$\phi : (R^{\square}_{\overline{\rho}} \otimes \overline{\mathbb{Q}}_l)^{\wedge}_{\wp} \twoheadrightarrow S$$

such that $(\rho^{\square})|_{I_K}$ pushes forward to a conjugate of $(\rho^{\square} \bmod \wp)|_{I_K}$. Choose a basis $e_1, \ldots, e_d$ of the centralizer of $(\rho^{\square} \bmod \wp)(I_K)$ in $M_{n\times n}(\overline{\mathbb{Q}}_l)$ and extend it to a basis $e_1, \ldots, e_d, f_1, \ldots, f_{n^2-d}$ of $M_{n\times n}(\overline{\mathbb{Q}}_l)$. Because

$$\bigcap_{\tau \in I_K} \ker(\operatorname{ad}(\rho^{\square} \bmod \wp)(\tau)^{-1} - 1) = \bigoplus_j e_j$$

we see that we can find $\tau_{ik} \in I_K$ and $\mu_{ik} \in \mathrm{Hom}\,(M_{n\times n}(\overline{\mathbb{Q}}_l), \overline{\mathbb{Q}}_l)$ such that for all $a \in M_{n\times n}(\overline{\mathbb{Q}}_l)$ and all $i = 1, \dots, n^2 - d$,

$$\sum_k \mu_{ik}(\mathrm{ad}\,(\rho^{\square} \bmod \wp)(\tau_{ik})^{-1} - 1)a$$

is the coefficient of f_i when a is written in terms of the basis $e_1, \dots, e_d, f_1, \dots, f_{n^2-d}$.

Consider the continuous representation

$$\begin{array}{rcl} \rho : G_K & \longrightarrow & \mathrm{GL}_n(\overline{\mathbb{Q}}_l[[X_1, \dots, X_d, Y_1, \dots, Y_{n^2-d}]]) = \mathrm{GL}_n(S) \\ \sigma & \longmapsto & (1 + \sum Y_i f_i)(\rho^{\square} \bmod \wp)(\sigma)(1 + \sum X_j e_j)^{v(\sigma)}(1 + \sum Y_i f_i)^{-1}, \end{array}$$

where $v : G_K/I_K \xrightarrow{\sim} \widehat{\mathbb{Z}}$ sends Frob_K to 1. It is a lifting of $\rho^{\square} \bmod \wp$, and so gives rise to a map

$$\phi : (R^{\square}_{\overline{\rho}} \otimes \overline{\mathbb{Q}}_l)^{\wedge}_{\wp} \longrightarrow S$$

under which $\rho^{\square}|_{I_K}$ pushes forward to a conjugate of $(\rho^{\square} \bmod \wp)|_{I_K}$. It just remains to show that ϕ is surjective. Modulo $\mathfrak{m}_S^2$ we have

$$\rho(\sigma) = (\rho^{\square} \bmod \wp)(\sigma)\left(1 + (\mathrm{ad}\,(\rho^{\square} \bmod \wp)(\sigma)^{-1} - 1)\sum_i Y_i f_i + v(\sigma)\sum_j X_j e_j\right).$$

Looking at

$$\sum_k \mu_{ik}((\rho^{\square} \bmod \wp)(\tau_{ik})^{-1}\rho(\tau_{ik}) - 1_n)$$

we see that for each i the element Y_i is in the image of $\wp \to \mathfrak{m}_S/\mathfrak{m}_S^2$. Next looking at

$$((\rho^{\square} \bmod \wp)(\varphi)^{-1}\rho(\varphi) - 1_n)$$

we see that for each j the element X_j is in the image of

$$\wp \longrightarrow \mathfrak{m}_S/(\mathfrak{m}_S^2, Y_1, \dots, Y_{n^2-d}).$$

We conclude that

$$\phi : \wp \twoheadrightarrow \mathfrak{m}_S/\mathfrak{m}_S^2$$

and hence that

$$\phi : (R^{\square}_{\overline{\rho}} \otimes \overline{\mathbb{Q}}_l)^{\wedge}_{\wp} \twoheadrightarrow S,$$

as desired.

Write R for $R^{\square}_{\overline{\rho}} \otimes \overline{\mathbb{Q}}_l$ and let I denote the kernel of $\phi|_R$. It is a prime ideal contained in $\wp$. Let T denote the integral closure of R/I in its field of fractions, which is finite over R/I and hence has the same Krull dimension as R/I. Then $\phi|_{R/I}$ extends to a map $\phi_T : T \hookrightarrow S$. Let $\mathfrak{n}$ denote the contraction to T of the maximal ideal of S. We have $\mathfrak{n} \cap R/I = \wp$. Moreover $T^{\wedge}_{\mathfrak{n}}$ is a domain. The map ϕ_T gives a map

$$T^{\wedge}_{\mathfrak{n}} \twoheadrightarrow S$$

extending ϕ. We have

$$n^2 = \dim R \geq \dim R/I = \dim T \geq \dim T^{\wedge}_{\mathfrak{n}} \geq \dim S = n^2,$$

where dim denotes the Krull dimension. Thus the inequalities are equalities and we deduce that I is a minimal prime ideal of R and that $T^{\wedge}_{\mathfrak{n}} \xrightarrow{\sim} S$. Thus $R/I \hookrightarrow S$ and so the residue field $k(I)$ of I injects into the field of fractions of S. As $(\rho^{\square})|_{I_K}$ and $(\rho^{\square} \bmod \wp)|_{I_K}$ are equivalent over S, they are also equivalent over $k(I)$.

We deduce that if $\wp_1$ and $\wp_2$ are two maximal primes of $R^{\square}_{\overline{\rho}_1} \otimes \overline{\mathbb{Q}}_l$ each containing a minimal prime I and each containing no other minimal prime, then

$$(\overline{\rho}_1^{\square} \bmod \wp_1)|_{I_K} \cong (\overline{\rho}_1^{\square} \bmod \wp_2)|_{I_K}$$

over $k(I)$ and hence over $\overline{\mathbb{Q}}_l$. □

LEMMA 1.3.5. *Suppose that $\overline{\rho}_1$ is semi-simple and let* $\mathrm{Fil}^{\,i}$ *be an invariant decreasing filtration on ρ_1 by $\mathcal{O}_{\overline{\mathbb{Q}}_l}$-direct summands; then $\rho_1 \sim \bigoplus_i \mathrm{gr}^{\,i} \rho_1$.*

Proof. We may suppose that L is chosen large enough that $\rho_1 : G_K \to \mathrm{GL}_n(\mathcal{O})$ and that $\mathrm{Fil}^{\,i}$ is defined over L. Then we may choose a basis $\{e_{i,j}\}$ of $\mathcal{O}^n$ such that

- for each i the set $\{e_{i',j} : \ i \leq i'\}$ is a basis of $\mathrm{Fil}^{\,i}\mathcal{O}^n$;
- for each i the set $\{\overline{e}_{i,j}\}$ (with only j varying) spans a G_K-submodule of $\mathbb{F}^n$.

(Use reverse induction on i.) Let $\mathcal{O}\langle t\rangle$ denote the algebra of power series over $\mathcal{O}$ with coefficients tending to 0, so that $\mathcal{O}\langle t\rangle$ is complete in the l-adic topology. Let h denote the element of $M_{n\times n}(\mathcal{O}\langle t\rangle)$ such that $he_{i,j} = t^i e_{i,j}$ for all i and j. Consider the continuous representation

$$\rho = h\rho_1 h^{-1} : G_K \longrightarrow \mathrm{GL}_n(\mathcal{O}\langle t\rangle).$$

Let A^0 denote the closed subalgebra of $\mathcal{O}\langle t\rangle$ generated by the matrix entries of the image of ρ. As in the proof of Lemma 1.2.2, we see that A^0 is a complete, noetherian local $\mathcal{O}$-algebra with residue field $\mathbb{F}$ and that there is a continuous homomorphism $R^{\square}_{\mathcal{O},\overline{\rho}_1} \to A^0$ under which the universal lifting of $\overline{\rho}_1$ pushes forward to ρ. Under the map $A^0 \to \mathcal{O}$ which sends t to 1, we see that ρ pushes forward to ρ_1. Under the map $A^0 \to \mathcal{O}$ which sends t to 0, we see that ρ pushes forward to $\bigoplus_i \mathrm{gr}^{\,i}\rho_1$. As A^0 is a domain, the claim follows. □

Important convention: Suppose that F is a global field and that $r : G_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$ is a continuous representation with irreducible reduction $\overline{r}$. In this case there is a model $r^{\circ} : G_F \to \mathrm{GL}_n(\mathcal{O}_{\overline{\mathbb{Q}}_l})$ of r, which is unique up to $\mathrm{GL}_n(\mathcal{O}_{\overline{\mathbb{Q}}_l})$-conjugation. If $v|p$ is a place of F, we write $r|_{G_{F_v}} \sim \rho_2$ (resp.

$r|_{G_{F_v}} \rightsquigarrow \rho_2$, resp. $\rho_1 \rightsquigarrow r|_{G_{F_v}}$) to mean $r^\circ|_{G_{F_v}} \sim \rho_2$ (resp. $r^\circ|_{G_{F_v}} \rightsquigarrow \rho_2$, resp. $\rho_1 \rightsquigarrow r^\circ|_{G_{F_v}}$).

We end this section by recalling some facts about Weil–Deligne representations. Recall (from the paragraph before Lemma 1.4 of [TY07]) that a Weil–Deligne representation (r, N) of the Weil group W_K of a p-adic field K on a complex vector space W is called *pure of weight w* if there is an exhaustive and separated ascending filtration Fil_i of W such that

- each $\mathrm{Fil}_i W$ is invariant under r;
- if $\sigma \in W_K$ maps to $\mathrm{Frob}_K^{v(\sigma)}$, then all eigenvalues of $r(\sigma)$ on $\mathrm{gr}_i W$ are Weil $q^{iv(\sigma)}$-numbers;
- for all j we have $N^j : \mathrm{gr}_{w+j} W \xrightarrow{\sim} \mathrm{gr}_{w-j} W$. (Note that necessarily we have $N\mathrm{Fil}_i W \subset \mathrm{Fil}_{i-2} W$.)

The following lemma is part of Lemma 1.4 of [TY07].

LEMMA 1.3.6.

(1) *A Weil–Deligne representation is pure of weight w if and only if its Frobenius semi-simplification is.*
(2) *Let K'/K be a finite extension. A Weil–Deligne representation of W_K is pure of weight w if and only if its restriction to $W_{K'}$ is.*
(3) *Two pure, Frobenius semi-simple, Weil–Deligne representations (r_1, N_1) and (r_2, N_2) are equivalent if and only if r_1 and r_2 are equivalent.*

1.4. *Local theory*: $l = p$. Continue to fix a rational prime l and let $\mathcal{O}$ denote the ring of integers of a finite extension L of $\mathbb{Q}_l$ in $\overline{\mathbb{Q}}_l$. Let λ denote the maximal ideal of $\mathcal{O}$ and let $\mathbb{F} = \mathcal{O}/\lambda$. However in this section we specialize our discussion to the case $\Gamma = G_K$, where $K/\mathbb{Q}_l$ is a finite extension. Thus $\overline{\rho} : G_K \to \mathrm{GL}_n(\mathbb{F})$ is continuous. We will assume that the image of each continuous embedding $K \hookrightarrow \overline{L}$ is contained in L. Let $\{H_\tau\}$ be a collection of n element multisets of integers parametrized by $\tau \in \mathrm{Hom}_{\mathbb{Q}_l}(K, \overline{\mathbb{Q}}_l)$.

We call a continuous representation $\rho : G_K \to \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$ *ordinary* if the following conditions are satisfied:

- there is a G_K-invariant decreasing filtration Fil^i on $\overline{\mathbb{Q}}_l^n$ such that for $i = 1, \dots, n$ the graded piece $\mathrm{gr}^i \overline{\mathbb{Q}}_l^n$ is one-dimensional and G_K acts on it by a character χ_i;
- and there are integers $b_{\tau,i} \in \mathbb{Z}$ for $\tau \in \mathrm{Hom}_{\mathbb{Q}_l}(K, \overline{\mathbb{Q}}_l)$ and $i = 1, \dots, n$ and an open subgroup $U \subset K^\times$ such that
  - $(\chi_i \circ \mathrm{Art}_K)|_U(\alpha) = \prod_{\tau : K \hookrightarrow \overline{\mathbb{Q}}_l} (\tau\alpha)^{b_{\tau,i}}$,
  - $b_{\tau,1} < b_{\tau,2} < \cdots < b_{\tau,n}$ for all τ.

We will call ρ *ss-ordinary* if we may take $U = \mathcal{O}_K^\times$ in the above definition. We will call ρ *cr-ordinary* if it is ordinary and crystalline (in which case it is

also ss-ordinary). If ρ is ordinary (resp. ss-ordinary), then it is de Rham (resp. semi-stable) and

$$\mathrm{HT}_\tau(\rho) = \{-b_{\tau,1}, \dots, -b_{\tau,n}\}.$$

If ρ is ss-ordinary and if for each i there exists a τ such that $b_{\tau,i}+1 < b_{\tau,i+1}$ then ρ is cr-ordinary. (See Propositions 1.24, 1.26 and 1.28 of [Nek93] or Lemma 3.1.4 of [GG12].)

Let K'/K denote a finite extension. The universal lifting ring $R^{\square}_{\mathcal{O},\overline{\rho}}$ has various important quotients $R^{\square}_{\mathcal{O},\overline{\rho},\{H_\tau\},*}$ which are uniquely characterized by requiring that they are reduced without l-torsion and that a $\overline{\mathbb{Q}}_l$-point of $R^{\square}_{\mathcal{O},\overline{\rho}}$ factors through $R^{\square}_{\mathcal{O},\overline{\rho},\{H_\tau\},*}$ if and only if it corresponds to a representation $\rho : G_K \to \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$ which is de Rham with Hodge–Tate numbers $\mathrm{HT}_\tau(\rho) = H_\tau$ for all $\tau : K \hookrightarrow \overline{\mathbb{Q}}_l$ and which has a further specified property $\mathcal{P}_*$. We will consider the following instances of this construction:

- $* =$ cris and $\mathcal{P}_*$ is "crystalline,"
- $* =$ ss and $\mathcal{P}_*$ is "semi-stable,"
- $* = K' -$ cris and $\mathcal{P}_*$ is "crystalline after restriction to $G_{K'}$,"
- $* = K' -$ ss and $\mathcal{P}_*$ is "semi-stable after restriction to $G_{K'}$,"
- $* =$ cr-ord and $\mathcal{P}_*$ is "cr-ordinary,"
- $* =$ ss-ord and $\mathcal{P}_*$ is "ss-ordinary."

(The existence of $R^{\square}_{\mathcal{O},\overline{\rho},\{H_\tau\},*}$ in the case $* =$ ss follows from Corollary 2.6.2 of [Kis08]. In the case $* =$ cris it follows from this using Theorem 2.5.5(2) of [Kis08], because $R^{\square}_{\mathcal{O},\overline{\rho},\{H_\tau\},\mathrm{cris}}$ is the maximal reduced, l-torsion free quotient of $R^{\square}_{\mathcal{O},\overline{\rho},\{H_\tau\},\mathrm{ss}}$ over which the N of Theorem 2.5.5(2) of [Kis08] becomes 0. Existence in the cases $* = K' -$ cris and $* = K' -$ ss also follow as in the first paragraph of the proof of Theorem 2.7.6 of [Kis08]. In the cases $* =$ cr-ord, ss-ord existence follows from Lemma 3.3.3(1) of [Ger10].)

We will write $R^{\square}_{\overline{\rho},\{H_\tau\},*} \otimes \overline{\mathbb{Q}}_l$ for $R^{\square}_{\mathcal{O},\overline{\rho},\{H_\tau\},*} \otimes_{\mathcal{O}} \overline{\mathbb{Q}}_l$. This definition is independent of the choice of $\mathcal{O}$ (using the same argument as in the proof of Lemma 1.2.1).

If H_τ has n distinct elements for each τ, then each ring $R^{\square}_{\mathcal{O},\overline{\rho},\{H_\tau\},*}$ is either zero or equi-dimensional of dimension

$$1 + n^2 + [K : \mathbb{Q}_l]n(n-1)/2.$$

(In the cases $* =$ cris, ss this is a special case of Theorem 3.3.4 of [Kis08], and the cases $K' -$ cris and $K' -$ ss are treated in the same way. The cases $* =$ cr-ord, ss-ord follow from this and from Lemma 3.3.3(2) of [Ger10].) We deduce that if $K'' \supset K'$, then $\operatorname{Spec} R^{\square}_{\mathcal{O},\overline{\rho},\{H_\tau\},K'-\mathrm{cris}}$ is a union of irreducible components of $\operatorname{Spec} R^{\square}_{\mathcal{O},\overline{\rho},\{H_\tau\},K''-\mathrm{cris}}$. Similarly, $\operatorname{Spec} R^{\square}_{\mathcal{O},\overline{\rho},\{H_\tau\},K'-\mathrm{ss}}$ is a union of irreducible components of $\operatorname{Spec} R^{\square}_{\mathcal{O},\overline{\rho},\{H_\tau\},K''-\mathrm{ss}}$. Each of the schemes $\operatorname{Spec} R^{\square}_{\mathcal{O},\overline{\rho},\{H_\tau\},\mathrm{cris}}[1/l]$, $\operatorname{Spec} R^{\square}_{\mathcal{O},\overline{\rho},\{H_\tau\},K'-\mathrm{cris}}[1/l]$ and $\operatorname{Spec} R^{\square}_{\mathcal{O},\overline{\rho},\{H_\tau\},\mathrm{cr\text{-}ord}}[1/l]$

are formally smooth. (The case $* = \text{cris}$ is a special case of Theorem 3.3.8 of [Kis08], the case $* = K' - \text{cris}$ is treated in the same way and the case $* = \text{cr-ord}$ follows from this and Lemma 3.3.3(2) of [Ger10].) Finally if $\overline{\rho}$ is trivial, then the scheme $\operatorname{Spec} R^{\square}_{\mathcal{O},\overline{\rho},\{H_\tau\},\text{cr-ord}}[1/l]$ is geometrically irreducible. (See Lemma 3.4.3 of [Ger10].)

Choose a finite set $\mathcal{C}$ of irreducible components of

$$\lim_{\to K'} \operatorname{Spec} R^{\square}_{\mathcal{O},\overline{\rho},\{H_\tau\},K'-\text{ss}}.$$

Let $R^{\square}_{\mathcal{O},\overline{\rho},\mathcal{C}}$ denote the maximal quotient of $R^{\square}_{\mathcal{O},\overline{\rho},\{H_\tau\},K'-\text{ss}}$ which is reduced, l-torsion free and has $\operatorname{Spec} R^{\square}_{\mathcal{O},\overline{\rho},\mathcal{C}}$ supported on the components in $\mathcal{C}$, for K' chosen sufficiently large. This is independent of the choice of K' (as long as K' is sufficiently large). Also let $\mathcal{D}_{\mathcal{C}}$ denote the set of liftings of $\overline{\rho}$ to complete local noetherian $\mathcal{O}$-algebras R with residue field $\mathbb{F}$ such that the induced map $R^{\square}_{\mathcal{O},\overline{\rho}} \to R$ factors through $R^{\square}_{\mathcal{O},\overline{\rho},\mathcal{C}}$. Again we see that $\mathcal{D}_{\mathcal{C}}$ is a deformation problem in the sense of Definition 2.2.2 of [CHT08].

If ρ_1 and ρ_2 are continuous representations $G_K \to \mathrm{GL}_n(\mathcal{O}_{\overline{\mathbb{Q}}_l})$, we say that ρ_1 *connects to* ρ_2, which we denote $\rho_1 \sim \rho_2$, if and only if

- the reduction $\overline{\rho}_1 = \rho_1 \bmod \mathfrak{m}_{\overline{\mathbb{Q}}_l}$ is equivalent to the reduction $\overline{\rho}_2 = \rho_2 \bmod \mathfrak{m}_{\overline{\mathbb{Q}}_l}$;
- ρ_1 and ρ_2 are both potentially crystalline;
- for each continuous $\tau : K \hookrightarrow \overline{\mathbb{Q}}_l$ we have $\mathrm{HT}_\tau(\rho_1) = \mathrm{HT}_\tau(\rho_2)$;
- ρ_1 and ρ_2 define points on the same irreducible component of the scheme $\operatorname{Spec}(R^{\square}_{\overline{\rho}_1,\{\mathrm{HT}_\tau(\rho_1)\},K'-\text{cris}} \otimes \overline{\mathbb{Q}}_l)$ for some (and hence all) sufficiently large K'.

We make the following remarks.

(1) By the proof of Lemma 1.2.2 we see that the relation $\rho_1 \sim \rho_2$ does not depend on the equivalence chosen between the reductions $\overline{\rho}_1$ and $\overline{\rho}_2$, nor on the $\mathrm{GL}_n(\mathcal{O}_{\overline{\mathbb{Q}}_l})$-conjugacy class of ρ_1 or ρ_2.

(2) "Connects" is an equivalence relation (because each $R^{\square}_{\mathcal{O},\overline{\rho}_i,\{H_\tau\},K'-\text{cris}}[1/l]$ is formally smooth).

(3) If $\rho_1 \sim \rho_2$, then $\mathrm{WD}(\rho_1)|_{I_K} \cong \mathrm{WD}(\rho_2)|_{I_K}$. (See the proof of Theorem 2.7.6 of [Kis08].)

(4) If K'/K is a finite extension and $\rho_1 \sim \rho_2$, then $\rho_1|_{G_{K'}} \sim \rho_2|_{G_{K'}}$.

(5) If $\rho_1 \sim \rho_2$ and $\rho_1' \sim \rho_2'$, then $\rho_1 \oplus \rho_1' \sim \rho_2 \oplus \rho_2'$ and $\rho_1 \otimes \rho_1' \sim \rho_2 \otimes \rho_2'$ and $\rho_1^\vee \sim \rho_2^\vee$.

(6) If $\mu : G_K \to \overline{\mathbb{Q}}_l^\times$ is a continuous unramified character with $\overline{\mu} = 1$ and ρ_1 is potentially crystalline, then $\rho_1 \sim \rho_1 \otimes \mu$.

(7) Suppose that ρ_1 is potentially crystalline and that $\overline{\rho}_1$ is semi-simple. Let Fil^i be an invariant filtration on ρ_1 by $\mathcal{O}_{\overline{\mathbb{Q}}_l}$ direct summands; then $\rho_1 \sim \bigoplus_i \mathrm{gr}^i \rho_1$. (This is proved in the same way as Lemma 1.3.5 of the previous section.)

We will call a representation $\rho : G_K \to \mathrm{GL}_n(\mathcal{O}_{\overline{\mathbb{Q}}_l})$ *diagonalizable* if it is crystalline and connects to some representation $\chi_1 \oplus \cdots \oplus \chi_n$ with $\chi_i : G_K \to \mathcal{O}^\times_{\overline{\mathbb{Q}}_l}$ crystalline characters. We will call a representation $\rho : G_K \to \mathrm{GL}_n(\mathcal{O}_{\overline{\mathbb{Q}}_l})$ *potentially diagonalizable* if there is a finite extension K'/K such that $\rho|_{G_{K'}}$ is diagonalizable. Note that if K''/K is a finite extension and ρ is diagonalizable (resp. potentially diagonalizable), then $\rho|_{G_{K''}}$ is diagonalizable (resp. potentially diagonalizable). It seems to us an interesting, and important, question to determine which potentially crystalline representations are potentially diagonalizable. We know no example of a potentially crystalline representation that is not potentially diagonalizable.

LEMMA 1.4.1. *If ρ_1 and ρ_2 are conjugate in $\mathrm{GL}_n(\overline{\mathbb{Q}}_l)$, then ρ_1 is potentially diagonalizable if and only if ρ_2 is potentially diagonalizable.*

Thus we can speak of a representation $\rho : G_K \to \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$ being potentially diagonalizable without needing to specify an invariant lattice.

Proof. If ρ_1 and ρ_2 are conjugate by an element of $\mathrm{GL}_n(\mathcal{O}_{\overline{\mathbb{Q}}_l})$, then after passing to a finite extension over which $\overline{\rho}_1 = \overline{\rho}_2 = 1$ we see that $\rho_1 \sim \rho_2$. Thus we may suppose that $\rho_1 = g\rho_2 g^{-1}$ where $g = \mathrm{diag}(d_1, \ldots, d_n)$ with $d_i \in \overline{\mathbb{Q}}_l^\times$ satisfying $d_n | d_{n-1} | \ldots | d_1$. Choosing $L \subset \overline{\mathbb{Q}}_l$ large enough we may assume that ρ_1 and ρ_2 are defined over $\mathcal{O}$ and that $d_1, \ldots, d_n \in \mathcal{O}$. (If d_n is not integral, multiply each d_i by a suitable element of $L^\times$.) Replacing K by a finite extension we may also assume that $\rho_2 \equiv 1 \bmod l d_1/d_n$, in which case we also have $\overline{\rho}_1 = 1$.

Consider the complete topological domain

$$A = \mathcal{O}\langle t_1, s_1, t_2, s_2, \ldots, t_{n-1}, s_{n-1}\rangle / (s_1 t_1 - (d_1/d_2), \ldots, s_{n-1}t_{n-1} - (d_{n-1}/d_n)).$$

(See Lemma A.1.4.) Let $\tilde{g} = \mathrm{diag}(t_1 \cdots t_{n-1}, t_2 \cdots t_{n-1}, \ldots, t_{n-1}, 1)$ and let

$$\widetilde{\rho} = \tilde{g}\rho_2\tilde{g}^{-1}.$$

If $j \geq i$, then the (i,j) entry of $\widetilde{\rho}(\sigma)$ is $t_i \ldots t_{j-1}$ times the (i,j) entry of $\rho_2(\sigma)$. If $i > j$, then the (i,j) entry of $\widetilde{\rho}(\sigma)$ is $s_j \ldots s_{i-1} d_i/d_j$ times the (i,j) entry of $\rho_2(\sigma)$, which is in A by our assumption that $\rho_2 \equiv 1 \bmod l d_1/d_n$. Thus we see that $\widetilde{\rho} : G_K \to \mathrm{GL}_n(A)$ is a continuous homomorphism. The specialization under $t_i \mapsto 1$ for all i is ρ_2. The specialization under $s_i \mapsto 1$ for all i is ρ_1. As in the proof of Lemma 1.2.2 we conclude that $\rho_1 \sim \rho_2$, and we are done. □

We will establish some cases of (potential) diagonalizability below, but first we must recall some results from the theory of Fontaine and Laffaille [FL82], normalized as in Section 2.4.1 of [CHT08]. Assume that $K/\mathbb{Q}_l$ is unramified and denote its ring of integers by $\mathcal{O}_K$. Let $\mathcal{MF}_\mathcal{O}$ denote the category of finite $\mathcal{O}_K \otimes_{\mathbb{Z}_l} \mathcal{O}$-modules M together with

- a decreasing filtration $\mathrm{Fil}^i M$ by $\mathcal{O}_K \otimes_{\mathbb{Z}_l} \mathcal{O}$-submodules which are $\mathcal{O}_K$-direct summands with $\mathrm{Fil}^0 M = M$ and $\mathrm{Fil}^{l-1} M = \{0\}$;
- $\mathrm{Frob}_p^{-1} \otimes 1$-linear maps $\Phi^i : \mathrm{Fil}^i M \to M$ with $\Phi^i|_{\mathrm{Fil}^{i+1} M} = l\Phi^{i+1}$ and $\sum_i \Phi^i \mathrm{Fil}^i M = M$.

Let $\mathrm{REP}_{\mathcal{O}}(G_K)$ denote the category of finite $\mathcal{O}$-modules with a continuous G_K-action. There is an exact, fully faithful, covariant functor of $\mathcal{O}$-linear categories $\mathbf{G}_K : \mathcal{MF}_{\mathcal{O}} \to \mathrm{REP}_{\mathcal{O}}(G_K)$. The essential image of $\mathbf{G}_K$ is closed under taking sub-objects and quotients. If M is an object of $\mathcal{MF}_{\mathcal{O}}$, then the length of M as an $\mathcal{O}$-module is $[K : \mathbb{Q}_l]$ times the length of $\mathbf{G}_K(M)$ as an $\mathcal{O}$-module.

Let $\mathbb{F}$ denote the residue field of $\mathcal{O}$, let $\mathcal{MF}_{\mathbb{F}}$ denote the full subcategory of $\mathcal{MF}_{\mathcal{O}}$ consisting of objects killed by the maximal ideal λ of $\mathcal{O}$ and let $\mathrm{REP}_{\mathbb{F}}(G_K)$ denote the category of finite $\mathbb{F}$-modules with a continuous G_K-action. Then $\mathbf{G}_K$ restricts to a functor $\mathcal{MF}_{\mathbb{F}} \to \mathrm{REP}_{\mathbb{F}}(G_K)$. If M is an object of $\mathcal{MF}_{\mathbb{F}}$ and τ is a continuous embedding $K \hookrightarrow \overline{\mathbb{Q}}_l$, we let $\mathrm{FL}_\tau(M)$ denote the multiset of integers i such that $\mathrm{gr}^i M \otimes_{\mathcal{O}_K \otimes_{\mathbb{Z}_l} \mathcal{O}, \tau \otimes 1} \mathcal{O} \neq \{0\}$ and i is counted with multiplicity equal to the $\mathbb{F}$-dimension of this space. If M is an l-torsion free object of $\mathcal{MF}_{\mathcal{O}}$, then $\mathbf{G}_K(M) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ is crystalline and for every continuous embedding $\tau : K \hookrightarrow \overline{\mathbb{Q}}_l$ we have

$$\mathrm{HT}_\tau(\mathbf{G}_K(M) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l) = \mathrm{FL}_\tau(M \otimes_{\mathcal{O}} \mathbb{F}).$$

Moreover, if Λ is a G_K-invariant lattice in a crystalline representation V of G_K with all its Hodge–Tate numbers in the range $[0, l-2]$, then Λ is in the image of $\mathbf{G}_K$. (See [FL82].)

LEMMA 1.4.2. *Let $K/\mathbb{Q}_l$ be unramified. Let $\overline{M}$ denote an object of $\mathcal{MF}_{\mathbb{F}}$ together with a filtration*

$$\overline{M} = \overline{M}_0 \supset \overline{M}_1 \supset \cdots \supset \overline{M}_{n-1} \supset \overline{M}_n = (0)$$

by $\mathcal{MF}_{\mathbb{F}}$-subobjects such that $\overline{M}_i/\overline{M}_{i+1}$ has $\mathbb{F}$-rank $[K : \mathbb{Q}_l]$ for $i = 0, \ldots, n-1$. Then we can find an object M of $\mathcal{MF}_{\mathcal{O}}$ which is l-torsion free together with a filtration by $\mathcal{MF}_{\mathcal{O}}$-subobjects

$$M = M_0 \supset M_1 \supset \cdots \supset M_{n-1} \supset M_n = (0)$$

and an isomorphism

$$M \otimes_{\mathcal{O}} \mathbb{F} \cong \overline{M}$$

under which $M_i \otimes_{\mathcal{O}} \mathbb{F}$ maps isomorphically to $\overline{M}_i$ for all i.

Proof. $\overline{M}$ has an $\mathbb{F}$ basis $\overline{e}_{i,\tau}$ for $i = 1, \ldots, n$ and $\tau \in \mathrm{Hom}(K, \overline{\mathbb{Q}}_l)$ such that

- the residue field k_K of K acts on $\overline{e}_{i,\tau}$ via τ;
- $\overline{M}_j$ is spanned over $\mathbb{F}$ by the $\overline{e}_{i,\tau}$ for $i > j$;

- for each j there is a subset $\Omega_j \subset \{1, \dots, n\} \times \mathrm{Hom}\,(K, \overline{\mathbb{Q}}_l)$ such that $\mathrm{Fil}^j \overline{M}$ is spanned over $\mathbb{F}$ by the $\overline{e}_{i,\tau}$ for $(i, \tau) \in \Omega_j$.

Then we define M to be the free $\mathcal{O}$-module with basis $e_{i,\tau}$ for $i = 1, \dots, n$ and $\tau \in \mathrm{Hom}\,(K, \overline{\mathbb{Q}}_l)$.

- We let $\mathcal{O}_K$ act on $e_{i,\tau}$ via τ;
- we define M_j to the sub $\mathcal{O}$-module generated by the $e_{i,\tau}$ with $i > j$;
- we define $\mathrm{Fil}^j M$ to be the $\mathcal{O}$-submodule spanned by the $e_{i,\tau}$ for $(i, \tau) \in \Omega_j$.

We define $\Phi^j : \mathrm{Fil}^j M \to M$ by reverse induction on j. If we have defined Φ^{j+1}, we define Φ^j as follows:

- If $(i, \tau) \in \Omega_{j+1}$, then $\Phi^j e_{i,\tau} = l \Phi^{j+1} e_{i,\tau}$.
- If $(i, \tau) \in \Omega_j - \Omega_{j+1}$, then $\Phi^j e_{i,\tau}$ is chosen to be an $\mathcal{O}$-linear combination of the $e_{i', \tau \circ \mathrm{Frob}_p}$ for $i' \geq i$ which lifts $\overline{\Phi}^j \overline{e}_{i,\tau}$.

It follows from Nakayama's lemma that M is an object of $\mathcal{MF}_{\mathcal{O}}$, and then it is easy to verify that it has the desired properties. □

We can now state and prove our potential diagonalizability criteria.

LEMMA 1.4.3. *Keep the above notation, including the assumption $l = p$. Suppose that $\rho : G_K \to \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$ is a potentially crystalline representation.*

(1) *If ρ has a G_K-invariant filtration with one-dimensional graded pieces (in particular if it is ordinary), then ρ is potentially diagonalizable.*
(2) *If $K/\mathbb{Q}_l$ is unramified, if ρ is crystalline and if for each $\tau : K \hookrightarrow \overline{L}$ the Hodge–Tate numbers $\mathrm{HT}_\tau(\rho) \subset [a_\tau, a_\tau + l - 2]$ for some integer a_τ, then ρ is potentially diagonalizable.*

Proof. After passing to a finite extension so that $\overline{\rho}$ becomes trivial and each $\mathrm{gr}^i \rho$ becomes crystalline, the first part follows from item (7) of the first numbered list of this section.

For the second we may assume (by twisting) that $a_\tau = 0$ for all τ. Note that every irreducible subquotient of $\overline{\rho}|_{I_K}$ is trivial on wild inertia and hence one-dimensional. Choose a finite unramified extension K'/K such that $\overline{\rho}(G_{K'}) = \overline{\rho}(I_K)$. Then $\overline{\rho}|_{G_{K'}}$ has a $G_{K'}$ invariant filtration with one-dimensional graded pieces. From Lemma 1.4.2 (and the discussion just proceeding it) we see that $\overline{\rho}|_{G_{K'}}$ has a crystalline lift ρ_2 with the same Hodge–Tate numbers as $\rho|_{G_{K'}}$ which also has a $G_{K'}$-invariant filtration with one-dimensional graded pieces. It follows from Lemma 2.4.1 of [CHT08] that $\rho|_{G_{K'}} \sim \rho_2$. (Note that in Section 2.4.1 of [CHT08] there is a running assumption that the Hodge–Tate numbers are distinct. However this assumption is not used in the proof of Lemma 2.4.1 of [CHT08].) From the first part of this lemma we see that ρ_2 is potentially diagonalizable. Hence ρ is also potentially diagonalizable. □

Important convention: Suppose that F is a global field and that $r : G_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$ is a continuous representation with irreducible reduction $\overline{r}$. In this case there is model $r^\circ : G_F \to \mathrm{GL}_n(\mathcal{O}_{\overline{\mathbb{Q}}_l})$ of r, which is unique up to $\mathrm{GL}_n(\mathcal{O}_{\overline{\mathbb{Q}}_l})$-conjugation. If $v|l$ is a place of F, we write $r|_{G_{F_v}} \sim \rho_2$ to mean $r^\circ|_{G_{F_v}} \sim \rho_2$. We will also say that $r|_{G_{F_v}}$ is (potentially) diagonalizable to mean that $r^\circ|_{G_{F_v}}$ is.

1.5. *Global theory.* Fix an odd rational prime l and let $\mathcal{O}$ denote the ring of integers of a finite extension L of $\mathbb{Q}_l$ in $\overline{\mathbb{Q}}_l$. Let λ denote the maximal ideal of $\mathcal{O}$ and let $\mathbb{F} = \mathcal{O}/\lambda$. Let F denote an imaginary CM field with maximal totally real subfield F^+. We suppose that each prime of F^+ above l splits in F and that L contains the image of each embedding $F \hookrightarrow \overline{L}$. Let S denote a finite set of places of F^+ which split in F and suppose that S contains all places of F^+ above l. For each $v \in S$ choose once and for all a prime $\tilde{v}$ of F above v and we let $\widetilde{S}$ denote the set of $\tilde{v}$ for $v \in S$.

Let

$$\overline{r} : G_{F^+} \longrightarrow \mathcal{G}_n(\mathbb{F})$$

be a continuous representation such that $G_F = \overline{r}^{-1}\mathcal{G}_n^0(\mathbb{F})$ and such that $\overline{r}$ is unramified outside S. Let

$$\mu : G_{F^+} \longrightarrow \mathcal{O}^\times$$

be a continuous character lifting $\nu \circ \overline{r}$. We suppose that μ is de Rham, so that there is an integer w such that $\mathrm{HT}_\tau(\mu) = \{w\}$ for all $\tau : F^+ \hookrightarrow L$. For $\tau : F \hookrightarrow L$ let H_τ denote a multiset of n integers such that

$$H_{\tau \circ c} = \{w - h : \ h \in H_\tau\}.$$

For $v \in S$ with $v \not| l$ choose a set $\mathcal{C}_v$ of irreducible components of the scheme $\mathrm{Spec}\, R^{\square}_{\mathcal{O},\breve{\overline{r}}|_{G_{F_{\tilde{v}}}}}[1/l]$, and let $\mathcal{D}_v$ denote the corresponding local deformation problem. For $v \in S$ with $v|l$ choose a finite set $\mathcal{C}_v$ of irreducible components of $\lim_{\to K'} \mathrm{Spec}\, R^{\square}_{\mathcal{O},\breve{\overline{r}}|_{G_{F_{\tilde{v}}}},\{H_\tau\},K'-\mathrm{ss}}$, and let $\mathcal{D}_v$ denote the corresponding local deformation problem.

Let

$$\mathcal{S} = (F/F^+, S, \widetilde{S}, \mathcal{O}, \overline{r}, \mu, \{\mathcal{D}_v\}_{v \in S}).$$

The following proposition is established in [CHT08]. (See Proposition 2.2.9 and Corollary 2.3.5 of that paper.)

PROPOSITION 1.5.1. *Keep the notation and assumptions established in this section. Suppose moreover that $\breve{\overline{r}}$ is absolutely irreducible.*

(1) *There is a universal deformation*

$$r^{\mathrm{univ}}_{\mathcal{S}} : G_{F^+} \longrightarrow \mathcal{G}_n(R^{\mathrm{univ}}_{\mathcal{S}})$$

of $\overline{r}$ of type $\mathcal{S}$ in the sense of Section 2.3 *of* [CHT08].

(2) *If $\mu(c_v) = -1$ for all $v|\infty$ and if each H_τ has n distinct elements, then $R^{\mathrm{univ}}_{\mathcal{S}}$ has Krull dimension at least* 1.

2. **Automorphy lifting theorems**

2.1. *Terminology.* Continue to fix a rational prime l and an isomorphism $\imath : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$. Suppose that F is a CM (or totally real) field and that F^+ is the maximal totally real subfield of F. By a *polarized* l-adic (resp. mod l) representation of G_F we will mean a pair (r, μ) (resp. $(\overline{r}, \overline{\mu})$), where

$$r : G_F \longrightarrow \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$$

(resp. $\overline{r} : G_F \longrightarrow \mathrm{GL}_n(\overline{\mathbb{F}}_l)$) and

$$\mu : G_{F^+} \longrightarrow \overline{\mathbb{Q}}_l^\times$$

(resp. $\overline{\mu} : G_{F^+} \longrightarrow \overline{\mathbb{F}}_l^\times$) are continuous homomorphisms, such that for some infinite place v of F^+ there is $\varepsilon_v \in \{\pm 1\}$ and a nondegenerate pairing $\langle\ ,\ \rangle_v$ on $\overline{\mathbb{Q}}_l^n$ (resp. $\overline{\mathbb{F}}_l^n$) such that

$$\langle x, y\rangle_v = \varepsilon_v \langle y, x\rangle_v$$

and

$$\langle r(\sigma)x, r(c_v \sigma c_v)y\rangle_v = \mu(\sigma)\langle x, y\rangle_v$$

(resp.

$$\langle \overline{r}(\sigma)x, \overline{r}(c_v \sigma c_v)y\rangle_v = \overline{\mu}(\sigma)\langle x, y\rangle_v)$$

for all $x, y \in \overline{\mathbb{Q}}_l^n$ (resp. $\overline{\mathbb{F}}_l^n$) and all $\sigma \in G_F$. In the case that F is imaginary we further require that $\varepsilon_v = -\mu(c_v)$. (This last condition can always be achieved by replacing μ by $\mu\delta_{F/F^+}$.)

Note the following:

- If the condition is true for one place $v|\infty$, it will be true for all places $v|\infty$: take $\varepsilon_{v'} = \mu(c_v c_{v'})\varepsilon_v$ and $\langle x, y\rangle_{v'} = \langle x, r(c_v c_{v'})y\rangle_v$ (resp. $\langle x, y\rangle_{v'} = \langle x, \overline{r}(c_v c_{v'})y\rangle_v$).
- If F is imaginary, then (r, μ) (resp. $(\overline{r}, \overline{\mu})$) is a polarized l-adic (resp. mod l) representation if and only if there is a continuous homomorphism $\widetilde{r} : G_{F^+} \to \mathcal{G}_n(\overline{\mathbb{Q}}_l)$ (resp. $\mathcal{G}_n(\overline{\mathbb{F}}_l)$) with $\breve{\widetilde{r}} = r$ (resp. $\overline{r}$) and with multiplier μ (resp. $\overline{\mu}$).
- If F is totally real, then (r, μ) is a polarized l-adic representation if and only if r factors through $\mathrm{GSp}_n(\overline{\mathbb{Q}}_l)$ (if $\mu(c_v) = -\varepsilon_v$) or $\mathrm{GO}_n(\overline{\mathbb{Q}}_l)$ (if $\mu(c_v) = \varepsilon_v$) with multiplier μ. (Define the pairing on $\overline{\mathbb{Q}}_l^n$ by $\langle x, y\rangle = \langle x, r(c_v)y\rangle_v$.) A similar assertion is true in the case of $\overline{\mathbb{F}}_l$ if $l > 2$.

We will call (r, μ) (resp. $(\overline{r}, \overline{\mu})$) *totally odd* if $\varepsilon_v = 1$ for all $v|\infty$ (equivalently, if $\mu(c_v)$ (resp. $\overline{\mu}(c_v)$) is independent of $v|\infty$ and $\varepsilon_v = 1$ for some $v|\infty$). We call an l-adic (resp. mod l) representation r (resp. $\overline{r}$) *polarizable* if there exists a μ (resp. $\overline{\mu}$) such that (r, μ) (resp. $(\overline{r}, \overline{\mu})$) is a polarized representation. We call an l-adic (resp. mod l) representation r (resp. $\overline{r}$) *totally odd polarizable* if there exists a μ (resp. $\overline{\mu}$) such that (r, μ) (resp. $(\overline{r}, \overline{\mu})$) is a totally odd polarized representation.

We will call (r, μ) (resp. r) *algebraic* if r is unramified at all but finitely many primes and if it is de Rham at all primes above l. We will call it *regular algebraic* if it is algebraic and if for all $\tau : F \hookrightarrow \overline{\mathbb{Q}}_l$ the multiset $\mathrm{HT}_\tau(r)$ has n distinct elements.

We now recall from [CHT08] and [BLGHT11] the notions of RAESDC and RAECSDC automorphic representations. In fact, it will be convenient for us to work with a slight variant of these definitions, where we keep track of the character which occurs in the essential (conjugate) self-duality. Moreover the referee suggested we use a less cumbersome name. We have followed this advice with some reservations. It is less cumbersome, but we have a slight worry it could be misleading. We hope not.

If F is a number field and π is an automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$, we will call π *regular algebraic* if π_∞ has the same infinitesimal character as an irreducible algebraic representation of the restriction of scalars from F to $\mathbb{Q}$ of GL_n.

Let F be a CM (or totally real) field with maximal totally real subfield F^+. By a *polarized automorphic representation* of $\mathrm{GL}_n(\mathbb{A}_F)$ we mean a pair (π, χ) where

- π is an automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$;
- $\chi : \mathbb{A}_{F^+}^\times/(F^+)^\times \to \mathbb{C}^\times$ is a continuous character such that $\chi_v(-1)$ is independent of $v|\infty$;
- $\pi^c \cong \pi^\vee \otimes (\chi \circ \mathbf{N}_{F/F^+} \circ \det)$.

(Here π^c denotes the composition of π with complex conjugation on $\mathrm{GL}_n(\mathbb{A}_F)$. If F is totally real, then $\pi^c = \pi$.) In the case that F is imaginary we further suppose that $\mu_v(-1) = (-1)^n$ for all $v|\infty$. (This last condition can always be achieved by replacing μ by $\mu\delta_{F/F^+}$.)

We will call an automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_F)$ *polarizable* if F is CM (or totally real) and there is a character μ such that (π, μ) is a polarized automorphic representation.

We will say that (π, χ) is *cuspidal* if π is. We will say that (π, χ) is *regular algebraic* if π is, in which case χ is also algebraic. We will say that (π, χ) has *level prime to l* (resp. *level potentially prime to l*) if for all $v|l$ the representation π_v is unramified (resp. becomes unramified after a finite base change).

If Ω is an algebraically closed field of characteristic 0, then we will write $(\mathbb{Z}^n)^{\mathrm{Hom}\,(F,\Omega),+}$ for the set of $a = (a_{\tau,i}) \in (\mathbb{Z}^n)^{\mathrm{Hom}\,(F,\Omega)}$ satisfying

$$a_{\tau,1} \geq \cdots \geq a_{\tau,n}.$$

Let $w \in \mathbb{Z}$. If F is totally real or CM (resp. if $\Omega = \mathbb{C}$), we will write $(\mathbb{Z}^n)_w^{\mathrm{Hom}\,(F,\Omega)}$ for the subset of elements $a \in (\mathbb{Z}^n)^{\mathrm{Hom}\,(F,\Omega)}$ with

$$a_{\tau,i} + a_{\tau\circ c,n+1-i} = w$$

(resp. $a_{\tau,i} + a_{c\circ\tau,n+1-i} = w$). (These definitions are consistent when F is totally real or CM and $\Omega = \mathbb{C}$.) If F'/F is a finite extension, we define

$a_{F'} \in (\mathbb{Z}^n)^{\mathrm{Hom}(F',\Omega),+}$ by

$$(a_{F'})_{\tau,i} = a_{\tau|_F,i}.$$

We will call $a \in (\mathbb{Z}^n)^{\mathrm{Hom}(F,\Omega),+}$ *extremely regular* if for some τ the $a_{\tau,i}$ have the following property: for any subsets H and H' of $\{a_{\tau,i}+n-i\}_{i=1}^n$ of the same cardinality, if $\sum_{h\in H} h = \sum_{h\in H'} h$, then $H = H'$.

If $a \in (\mathbb{Z}^n)^{\mathrm{Hom}(F,\mathbb{C}),+}$, let Ξ_a denote the irreducible algebraic representation of $\mathrm{GL}_n^{\mathrm{Hom}(F,\mathbb{C})}$ which is the tensor product over τ of the irreducible representations of GL_n with highest weights a_τ. We will say that a regular algebraic polarized (resp. regular algebraic) automorphic representation (π,χ) (resp. π) of $\mathrm{GL}_n(\mathbb{A}_F)$ has *weight* a if π_∞ has the same infinitesimal character as $\Xi_a^\vee$. Note that in this case a must lie in $(\mathbb{Z}^n)_w^{\mathrm{Hom}(F,\mathbb{C})}$ for some $w \in \mathbb{Z}$.

Suppose π is a regular algebraic automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ of weight $a \in (\mathbb{Z}^n)^{\mathrm{Hom}(F,\mathbb{C}),+}$. Let v be a place of F dividing l and ϖ_v a uniformizer in $\mathcal{O}_{F_v}$. For each integer $b > 0$ let $\mathrm{Iw}(v^{b,b})$ denote the subgroup of $\mathrm{GL}_n(\mathcal{O}_{F_v})$ consisting of elements that reduce to upper triangular unipotent matrices modulo v^b. The space $(\iota^{-1}\pi_v)^{\mathrm{Iw}(v^{b,b})}$ carries commuting actions of the Hecke operators

$$U_{\varpi_v}^{(j)} = \left[\mathrm{Iw}(v^{b,b}) \begin{pmatrix} \varpi 1_j & 0 \\ 0 & 1_{n-j} \end{pmatrix} \mathrm{Iw}(v^{b,b})\right].$$

(See, for instance, Lemma 2.3.3 of [Ger10].) We define rescaled Hecke operators

$$U_{\iota^* a,\varpi_v}^{(j)} := \left(\prod_{\tau:F_v\hookrightarrow\overline{\mathbb{Q}}_l} \prod_{i=1}^{j} \tau(\varpi_v)^{-a_{\iota\circ\tau|_F,n-i+1}}\right) U_{\varpi_v}^{(j)}$$

for $j = 1,\dots,n$. We define the ordinary part $(\iota^{-1}\pi_v)^{\mathrm{Iw}(v^{b,b}),\mathrm{ord}}$ of $(\iota^{-1}\pi_v)^{\mathrm{Iw}(v^{b,b})}$ to be the maximal subspace which is invariant under each $U_{\iota^* a,\varpi_v}^{(j)}$ and such that every eigenvalue of each $U_{\iota^* a,\varpi_v}^{(j)}$ is an l-adic unit. This definition does not depend on the choice of uniformizer ϖ_v. (If we choose another uniformizer ϖ_v', then $U_{\varpi_v'}^{(j)} = \langle u\rangle U_{\varpi_v}^{(j)}$ where $\langle u\rangle$ is an operator on $\pi_v^{\mathrm{Iw}(v^{b,b})}$ that commutes with $U_{\varpi_v}^{(j)}$ and whose b-th power is trivial.)

We say that π is *ι-ordinary* if for each $v|l$ there is an integer $b > 0$ such that the space $(\iota^{-1}\pi_v)^{\mathrm{Iw}(v^{b,b}),\mathrm{ord}}$ is nonzero. Note that if ψ is an algebraic character of $\mathbb{A}_F^\times/F^\times$, then π is ι-ordinary if and only if $\pi\otimes(\psi\circ\det)$ is ι-ordinary. Also recall (from Lemma 5.1.5 of [Ger10]) that if π has weight 0 and if π_v is Steinberg for all $v|l$, then π is ι-ordinary.

We recall that thanks to the work of many people (we mention in particular Bellaiche, Caraiani, Chenevier, Clozel, Harris, Kottwitz, Labesse, Shin and R.T., and the references [Kot92], [Clo90], [HT01], [BC11], [Shi11], [CHLN11],

[Clo13], [Car12a] and [Car12b]) we can associate l-adic representations to regular algebraic cuspidal polarized automorphic representations:

THEOREM 2.1.1. *Suppose that (π, χ) is a regular algebraic, cuspidal, polarized automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$. Then there is a continuous semi-simple representation*

$$r_{l,\imath}(\pi) : G_F \longrightarrow \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$$

and an integer w with the following properties:

(1) *$(r_{l,\imath}(\pi), \epsilon_l^{1-n} r_{l,\imath}(\chi))$ is a totally odd, polarized l-adic representation.*
(2) *If $v \nmid l$ is a place of F, then*

$$\imath\mathrm{WD}(r_{l,\imath}(\pi)|_{G_{F_v}})^{F-\mathrm{ss}} \cong \mathrm{rec}(\pi_v \otimes |\det|_v^{(1-n)/2}),$$

and these Weil–Deligne representations are pure of weight w.
(3) *$r_{l,\imath}(\pi)$ is de Rham and if $\tau : F \hookrightarrow \overline{\mathbb{Q}}_l$, then*

$$\mathrm{HT}_\tau(r_{l,\imath}(\pi)) = \{a_{\imath\tau,1} + n - 1, a_{\imath\tau,2} + n - 2, \ldots, a_{\imath\tau,n}\}.$$

Moreover

$$\mathrm{HT}_{\tau\circ c}(r_{l,\imath}(\pi)) = \{w - h : \ h \in \mathrm{HT}_\tau(r_{l,\imath}(\pi))\}.$$

(4) *If $v|l$ and π_v has an Iwahori fixed vector, then*

$$\imath\mathrm{WD}(r_{l,\imath}(\pi)|_{G_{F_v}})^{F-\mathrm{ss}} \cong \mathrm{rec}(\pi_v \otimes |\det|_v^{(1-n)/2}).$$

In particular $r_{l,\imath}(\pi)$ is semi-stable at v, and if π_v is unramified, then it is crystalline.

(For most of this theorem we refer to Theorems 1.1 and 1.2 of [BLGHT11], strengthened by incorporating the main theorems of [Car12a] and [Car12b]. We warn the reader that the main theorem of [Car12b] depends on the current paper, however there is no circularity, because it does not do so in the case that Π_y has an Iwahori fixed vector, which is the only case we are quoting here. The first part follows from Theorem 1.2 and Corollary 1.3 of [BC11]. Note that Theorem 1.2 of [BC11] can easily be extended to the case χ nontrivial by a twisting argument. Also note that irreducible factors r of $r_{l,\imath}(\pi)$ which do not satisfy $r^c \cong r^\vee \otimes \epsilon_l^{1-n} r_{l,\imath}(\chi)$ occur in pairs r and $(r^c)^\vee \otimes \epsilon_l^{1-n} r_{l,\imath}(\chi)$, and it is straightforward to put a pairing of the desired form on $r \oplus (r^c)^\vee \otimes \epsilon_l^{1-n} r_{l,\imath}(\chi)$. When F is CM, note that by definition $\epsilon_l^{1-n} r_{l,\imath}(\chi)$ takes every complex conjugation to -1.) We also have the following remarks.

(6) *If π is $\imath$-ordinary and $v|l$, then $r_{l,\imath}(\pi)|_{G_{F_v}}$ is ordinary.* (This follows from Lemma 5.2.1 of [Ger10] using the same twisting argument as in Section 1 of [BLGHT11].)
(7) *If π has level potentially prime to l and if $r_{l,\imath}(\pi)$ is ordinary, then π is $\imath$-ordinary.* (See Lemmas 5.1.6 and 5.2.1 of [Ger10].)

We remark that it is presumably both true and provable that π is $\imath$-ordinary if and only if $r_{l,\imath}(\pi)$ is ordinary, but to work out the details here would take us too far afield.

We will let $\overline{r}_{l,\imath}(\pi)$ denote the semi-simplification of the reduction of $r_{l,\imath}(\pi)$. If F is totally real and if $\chi_v(-1) = (-1)^{n-1}$ for all $v|\infty$, then $\overline{r}_{l,\imath}(\pi)$ factors through a map

$$\widetilde{\overline{r}}_{l,\imath}(\pi) : G_F \longrightarrow \mathrm{GO}_n(\overline{\mathbb{F}}_l)$$

with multiplier $\overline{\epsilon}_l^{1-n}\overline{r}_{l,\imath}(\chi)$. If F is totally real and if $\chi_v(-1) = (-1)^n$ for all $v|\infty$, then n is even and $\overline{r}_{l,\imath}(\pi)$ factors through a map

$$\widetilde{\overline{r}}_{l,\imath}(\pi) : G_F \longrightarrow \mathrm{GSp}_n(\overline{\mathbb{F}}_l)$$

with multiplier $\overline{\epsilon}_l^{1-n}\overline{r}_{l,\imath}(\chi)$. If F is imaginary CM, then it extends to a continuous homomorphism

$$\widetilde{\overline{r}}_{l,\imath}(\pi) : G_{F^+} \longrightarrow \mathcal{G}_n(\overline{\mathbb{F}}_l)$$

with multiplier $\overline{\epsilon}_l^{1-n}\overline{r}_{l,\imath}(\chi)$.

We will call (r, μ) (resp. $(\overline{r}, \overline{\mu})$, resp. r, resp. $\overline{r}$) *automorphic* if there is a regular algebraic, polarized cuspidal automorphic representation (π, χ) such that

$$(r, \mu) \cong (r_{l,\imath}(\pi), r_{l,\imath}(\chi)\epsilon_l^{1-n})$$

(resp.

$$(\overline{r}, \overline{\mu}) \cong (\overline{r}_{l,\imath}(\pi), \overline{r}_{l,\imath}(\chi)\overline{\epsilon}_l^{1-n}),$$

resp.

$$r \cong r_{l,\imath}(\pi),$$

resp.

$$\overline{r} \cong \overline{r}_{l,\imath}(\pi)).$$

We will say that (r, μ) or $(\overline{r}, \overline{\mu})$ or r or $\overline{r}$ is *automorphic of level prime to l* (resp. *automorphic of level potentially prime to l*, resp. *ordinarily automorphic*, resp. *potentially diagonalizably automorphic*) if (π, χ) has level prime to l (resp. has level potentially prime to l, resp. is $\imath$-ordinary, resp. has level potentially prime to l and $r_{l,\imath}(\pi)$ is potentially diagonalizable). By Theorem 3.13 of [Clo90] these definitions do not depend on the choice of $\imath$.

Finally recall the following definition from [Tho12]. We will call a subgroup $H \subset \mathrm{GL}_n(\overline{\mathbb{F}}_l)$ *adequate* if the following conditions are satisfied:

- $H^1(H, \overline{\mathbb{F}}_l) = (0)$ and $H^1(H, \mathfrak{sl}_n(\overline{\mathbb{F}}_l)) = (0)$;
- $H^0(H, \mathfrak{sl}_n(\overline{\mathbb{F}}_l)) = (0)$;
- the elements of H with order coprime to l span $M_{n\times n}(\overline{\mathbb{F}}_l)$ over $\overline{\mathbb{F}}_l$.

Note that this is not exactly the definition given in Definition 2.3 of [Tho12], however it is equivalent to it by Lemma 1 of [GHTT10]. Note also

that if H is adequate, then $\overline{\mathbb{F}}_l^n$ is an irreducible H-module (by the third condition) and $l \not| n$ (by the second condition, as when $l|n$ we have $\overline{\mathbb{F}}_l 1_n \subset \mathfrak{sl}_n(\overline{\mathbb{F}}_l)$). The following proposition is Theorem 9 of [GHTT10].

PROPOSITION 2.1.2. *Suppose that H is a finite subgroup of $\mathrm{GL}_n(\overline{\mathbb{F}}_l)$ such that the tautological representation of H is irreducible. Let H^0 denote the subgroup of H generated by all elements of l-power order and let d denote the maximal dimension of an irreducible H^0-submodule of $\overline{\mathbb{F}}_l^n$. If $l \geq 2(d+1)$, then H is adequate.*

(R.T. would like to take this opportunity to make two corrections to [CHT08]. Robert Guralnick points out that the assumption in Corollary 2.5.4 of [CHT08] should be $l > 2n+1$ and not $l > 2n-1$. The application of Lemma (2.7) c) of [CPS75] in the penultimate sentence of the proof of Corollary 2.5.4 of [CHT08] requires $l > 2n+1$, not $l > n+1$ as was stated there. The correct form of the Corollary was used in [HSBT10].

Florian Herzig points out that in the proof of Corollary 4.4.4 of [CHT08] we should have written $\mathrm{Sp}_n(\mathbb{F}_l)/\{\pm 1\}$ and not $\mathrm{PSp}_n(\mathbb{F}_l)$. R.T. would like to thank Guralnick and Herzig for these observations.)

2.2. *Lemmas on automorphy.* Our first lemma is elementary.

LEMMA 2.2.1. *Suppose that F is a* CM (*or totally real*) *field and that ψ is an algebraic character of G_F. If F is imaginary, let ϕ denote the composition of ψ with the transfer map $G_{F^+}^{\mathrm{ab}} \to G_F^{\mathrm{ab}}$. If F is totally real, let $\phi = \psi^2$. Then (r, μ) is automorphic if and only if $(r \otimes \psi, \mu\phi)$ is.*

The next lemma is proven in the same way as Lemma 4.2.2 of [CHT08]. (See also Lemmas 4.3.2 and 4.3.3 of [CHT08].)

LEMMA 2.2.2. *Suppose that M/F is a soluble Galois extension of fields and that M is* CM (*or totally real*). *Suppose that (r, μ) is a polarized l-adic representation of G_F with $r|_{G_M}$ irreducible. Then (r, μ) is automorphic if and only if $(r|_{G_M}, \mu|_{G_{M^+}})$ is automorphic.*

The next lemma is a generalization of Lemma 7.1 of [BLGHT11].

LEMMA 2.2.3. *Suppose that π is an irreducible, unitary, admissible module for $((\mathrm{Lie}\ \mathrm{GL}_n(\mathbb{R})) \otimes_{\mathbb{R}} \mathbb{C}, \mathrm{O}(n))$ or $((\mathrm{Lie}\ \mathrm{GL}_n(\mathbb{C})) \otimes_{\mathbb{R}} \mathbb{C}, \mathrm{U}(n))$ with half integral Harish-Chandra parameter* (*i.e., this parameter lies in $1/2$ the co-character group of a maximal torus in the complexified group*). *Then*

$$\pi^c \cong \pi^\vee.$$

Proof. In the second case this is Lemma 7.1 of [BLGHT11]. However, we will give here a uniform proof in both cases. From the classification of irreducible unitary admissible $((\mathrm{Lie}\ \mathrm{GL}_n(\mathbb{R})) \otimes_{\mathbb{R}} \mathbb{C}, \mathrm{O}(n))$ and $((\mathrm{Lie}\ \mathrm{GL}_n(\mathbb{C})) \otimes_{\mathbb{R}}$

$\mathbb{C}, \mathrm{U}(n))$-modules, we see that π is of the form

$$\pi_1 \boxplus \cdots \boxplus \pi_r,$$

where

$$\pi_i = (\sigma_i \otimes |\det|^{(1-m_i)/2}) \boxplus (\sigma_i \otimes |\det|^{(3-m_i)/2}) \boxplus \cdots \boxplus (\sigma_i \otimes |\det|^{(m_i-1)/2})$$

with $m_i \in \mathbb{Z}_{>0}$ and σ_i an irreducible, unitary, discrete series, admissible module for $((\mathrm{Lie}\ \mathrm{GL}_{n_i}(\mathbb{R})) \otimes_{\mathbb{R}} \mathbb{C}, \mathrm{O}(n_i))$ or $((\mathrm{Lie}\ \mathrm{GL}_{n_i}(\mathbb{C})) \otimes_{\mathbb{R}} \mathbb{C}, \mathrm{U}(n_i))$ with half-integral Harish-Chandra parameter. (We are using the form of the classification proposed by Tadic in 1985 and proved in [Tad09]. A similar classification was given earlier in [Vog86]. [Clo87] indicates how the formulation we are using can be deduced from the one given in [Vog86].)

The only possibilities for σ_i are

- the $((\mathrm{Lie}\ \mathrm{GL}_1(\mathbb{R})) \otimes_{\mathbb{R}} \mathbb{C}, \mathrm{O}(1))$-module associated to the trivial or sign representation of $\mathbb{R}^\times$;
- the unique discrete series $((\mathrm{Lie}\ \mathrm{GL}_2(\mathbb{R})) \otimes_{\mathbb{R}} \mathbb{C}, \mathrm{O}(2))$-module with the same infinitesimal character as the representation $\mathrm{Sym}^a \otimes |\det|^{-a/2}$ of $\mathrm{GL}_2(\mathbb{R})$ for some $a \in \mathbb{Z}_{\geq 0}$;
- the $((\mathrm{Lie}\ \mathrm{GL}_1(\mathbb{C})) \otimes_{\mathbb{R}} \mathbb{C}, \mathrm{U}(1))$-module associated to one of the representations $z \mapsto (z/|z|)^a$ of $\mathbb{C}^\times$ for some $a \in \mathbb{Z}$.

In each case we see that $\sigma_i^\vee \cong \sigma_i^c$. Thus $\pi^\vee \cong \pi^c$, as desired. (Note that, by known properties of the Langlands local reciprocity map rec, we have $(\pi_1 \boxplus \pi_2)^\vee \cong \pi_1^\vee \boxplus \pi_2^\vee$ and $(\pi_1 \boxplus \pi_2)^c \cong \pi_1^c \boxplus \pi_2^c$.) □

The next lemma formalizes and generalizes the argument of step 3 of the proof of Theorem 7.5 of [BLGHT11]. Partial generalizations of this argument have already been given in Proposition 5.2.1 of [BLGG11] and Proposition 5.1.1 of [BLGG12].

Lemma 2.2.4. *Let F be a* CM *(or totally real) field and M/F a soluble Galois,* CM *(or totally real) extension of degree m. Let $r : G_M \to \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$ be an irreducible continuous representation and $\mu : G_{F^+} \to \overline{\mathbb{Q}}_l^\times$ a continuous character such that $(\mathrm{Ind}_{G_M}^{G_F} r, \mu)$ is an automorphic polarized l-adic representation of G_F. Then $(r, \mu|_{G_{M^+}})$ is also automorphic and polarized.*

Proof. Inductively we may reduce to the case that m is prime. Let σ denote a generator of $\mathrm{Gal}\,(M/F)$ and κ a generator of $\mathrm{Gal}\,(M/F)^\vee$. Let (Π, χ) be a regular algebraic, cuspidal polarized automorphic representation of $\mathrm{GL}_{mn}(\mathbb{A}_F)$ such that $r_{l,\iota}(\Pi) \cong \mathrm{Ind}_{G_M}^{G_F} r$ and $\mu = \epsilon_l^{1-mn} r_{l,\iota}(\chi)$. Because

$$r_{l,\iota}(\Pi) \otimes \kappa \cong r_{l,\iota}(\Pi),$$

we deduce that

$$\Pi \otimes (\kappa \circ \mathrm{Art}_F \circ \det) \cong \Pi,$$

and so from Theorems 3.4.2 and 3.5.1 of [AC89] there is a cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_M)$ such that

$$\pi \boxplus \pi^\sigma \boxplus \cdots \boxplus \pi^{\sigma^{m-1}}$$

is a strong base change of Π in the sense of Definitions 1.6.1 and 3.1.2 and Section 1.7 of [AC89]. Lemma VII.2.6 of [HT01] then shows that

$$\mathrm{BC}_{M/F}(\Pi) \cong \pi \boxplus \pi^\sigma \boxplus \cdots \boxplus \pi^{\sigma^{m-1}}.$$

Note that, because Π is regular algebraic, $\pi \otimes ||\det||_M^{n(1-m)/2}$ is regular algebraic and that the representations π^{σ^i} for $i = 0, 1, \ldots, m-1$ are pairwise nonisomorphic.

Because $\Pi^c \cong \Pi^\vee \otimes (\chi \circ \mathbf{N}_{F/F^+} \circ \det)$, we see that

$$\pi^c \cong \pi^{\sigma^i,\vee} \otimes (\chi \circ \mathbf{N}_{M/F^+} \circ \det)$$

for some $i = 0, \ldots, m-1$. If ψ denotes the central character of π, which must also be algebraic, then we see that

$$2\mathrm{wt}(\psi) = n\mathrm{wt}(\chi).$$

(See the discussion at the start of Section A.2.) Thus the central character of $\pi \otimes ||\det||_F^{-\mathrm{wt}(\chi)/4}$ (i.e., $\psi||\ ||_F^{-\mathrm{wt}(\psi)/2}$) is unitary, and so $\pi \otimes ||\det||_F^{-\mathrm{wt}(\chi)/4}$ is also unitary.

Moreover $\mathrm{wt}(\chi)$ is even (as F^+ is totally real), and so for all $v|\infty$ the representation $\pi_v \otimes |\det|_v^{-\mathrm{wt}(\chi)/4}$ has half integral Harish-Chandra parameter. We conclude from the previous lemma that for $v|\infty$ we have

$$\pi_v^c \cong \pi_v^\vee \otimes |\det|_v^{\mathrm{wt}(\chi)/2}.$$

Thus

$$(\pi^{\sigma^i})_v \cong \pi_v \otimes ((\chi||\ ||_{F^+}^{-\mathrm{wt}(\chi)/2}) \circ \mathbf{N}_{M/F^+} \circ \det)_v.$$

By the regularity of $\mathrm{BC}_{M/F}(\Pi)$ we deduce that we must have $i = 0$, i.e.,

$$\pi^c \cong \pi^\vee \otimes (\chi \circ \mathbf{N}_{M/F^+} \circ \det).$$

In particular $(\pi \otimes ||\det||^{n(1-m)/2}, (\chi||\ ||_{F^+}^{n(1-m)}) \circ \mathbf{N}_{M^+/F^+})$ is a regular algebraic, cuspidal, polarized automorphic representation of $\mathrm{GL}_n(\mathbb{A}_M)$. Moreover

$$\begin{aligned} r_{l,\imath}(\Pi)|_{G_M} \cong\ & r_{l,\imath}(\pi \otimes ||\det||^{n(1-m)/2}) \\ & \oplus r_{l,\imath}(\pi \otimes ||\det||^{n(1-m)/2})^\sigma \oplus \cdots \oplus r_{l,\imath}(\pi \otimes ||\det||^{n(1-m)/2})^{\sigma^{m-1}}. \end{aligned}$$

On the other hand

$$r_{l,\imath}(\Pi)|_{G_M} \cong (\mathrm{Ind}_{G_M}^{G_F} r)|_{G_M} \cong r \oplus r^\sigma \oplus \cdots \oplus r^{\sigma^{m-1}}.$$

As r is irreducible we deduce that

$$r \cong r_{l,\imath}(\pi \otimes ||\det||^{n(1-m)/2})^{\sigma^j} \cong r_{l,\imath}(\pi^{\sigma^j} \otimes ||\det||^{n(1-m)/2})$$

for some j. Moreover

$$r_{l,\imath}((\chi||\ ||_{F^+}^{n(1-m)}) \circ \mathbf{N}_{M^+/F^+})\epsilon_l^{1-n} = r_{l,\imath}(\chi)|_{G_{M^+}}\epsilon_l^{1-nm} = \mu|_{G_{M^+}}.$$

The lemma follows. □

2.3. *Automorphy lifting: the "minimal" case.* In this section we present an automorphy lifting theorem which represents the natural output of the Taylor–Wiles–Kisin method. We incorporate improvements due to Thorne [Tho12] and Caraiani [Car12a]. This result is essentially Theorem 7.1 of [Tho12].

THEOREM 2.3.1. *Suppose that F is an imaginary* CM *field with maximal totally real subfield F^+, that l is an odd prime and that $n \in \mathbb{Z}_{\geq 1}$. Assume also that $\zeta_l \notin F$. Let (r, μ) be an n-dimensional, algebraic, polarized l-adic representation of G_F satisfying the following properties*:

(1) *The reduction $\overline{r}$ is irreducible and $\overline{r}(G_{F(\zeta_l)}) \subset \mathrm{GL}_n(\overline{\mathbb{F}}_l)$ is adequate.*
(2) *$(\overline{r}, \overline{\mu})$ is automorphic of level potentially prime to l, arising from a regular algebraic, cuspidal, polarized automorphic representation (π, χ) of level potentially prime to l, such that*

$$r_{l,\imath}(\pi)|_{G_{F_v}} \sim r|_{G_{F_v}}$$

for each finite place v of F. (In particular, $r|_{G_{F_v}}$ is potentially crystalline for $v|l$, and r has the same Hodge–Tate numbers as $r_{l,\imath}(\pi)$.)

Then (r, μ) is automorphic of level potentially prime to l.

If further π has level prime to l and if r is crystalline at all primes above l, then (r, μ) is automorphic of level prime to l.

Proof. The result follows from Theorem 7.1 of [Tho12] on noting that for $v \not| l$ we have $r_{l,\imath}(\pi)|_{G_{F_v}} \rightsquigarrow r|_{G_{F_v}}$ (by Lemma 1.3.2, because $\imath\mathrm{WD}(r_{l,\imath}(\pi)|_{G_{F_v}})^{F-\mathrm{ss}}$ $\cong \mathrm{rec}(\pi_v \otimes |\det|_v^{(1-n)/2})$ and π_v is generic). □

THEOREM 2.3.2. *Let F be an imaginary* CM *field with maximal totally real subfield F^+. Suppose that $n \in \mathbb{Z}_{\geq 1}$ and that l is an odd prime. Assume also that $\zeta_l \notin F$. Let S be a finite set of primes of F^+ including all primes above l. Suppose moreover that each prime in S splits in F and choose a prime $\tilde{v}$ of F above each $v \in S$. Write $\widetilde{S}$ for the set of $\tilde{v}$ with $v \in S$.*

Let (π, χ) be a regular algebraic, cuspidal, polarized automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ which is unramified outside S and has level potentially prime to l. Let $a \in (\mathbb{Z}^n)^{\mathrm{Hom}(F,\mathbb{C}),+}$ be the weight of π. Suppose that the image $\overline{r}_{l,\imath}(\pi)(G_{F(\zeta_l)})$ is adequate.

Suppose, for each $v|l$, that $\mathcal{C}_v$ is an irreducible component of

$$\varinjlim_{K'} \mathrm{Spec}\, R^{\square}_{\overline{r}_{l,\imath}(\pi)|_{G_{F_{\tilde{v}}}}, \{\{a_{\imath\tau,i}+n-i\}_i\}_\tau, K'-\mathrm{cris}} \otimes \overline{\mathbb{Q}}_l$$

containing $r_{l,\imath}(\pi)|_{G_{F_{\tilde{v}}}}$. *Suppose also that for each* $v \in S$ *with* $v \nmid l$, $\mathcal{C}_v$ *is the irreducible component of* $\operatorname{Spec} R^{\square}_{\overline{r}_{l,\imath}(\pi)|_{G_{F_{\tilde{v}}}}} \otimes \overline{\mathbb{Q}}_l$ *containing* $r_{l,\imath}(\pi)|_{G_{F_{\tilde{v}}}}$.

Let L *denote a finite extension of* $\mathbb{Q}_l$ *in* $\overline{\mathbb{Q}}_l$ *such that* L *contains the image of each embedding* $F \hookrightarrow \overline{\mathbb{Q}}_l$; *and* L *contains the image of* $r_{l,\imath}(\chi)$; *and* $r_{l,\imath}(\pi)$ *is defined over* L; *and each* $\mathcal{C}_v$ *is defined over* L. *For* $v \in S$ *let* $\mathcal{D}_v$ *be the deformation problem corresponding to* $\mathcal{C}_v$. *Also let*

$$\mathcal{S} = (F/F^+, S, \widetilde{S}, \mathcal{O}_L, \widetilde{\overline{r}}_{l,\imath}(\pi), r_{l,\imath}(\chi)\epsilon_l^{1-n}, \{\mathcal{D}_v\}_{v\in S}).$$

Then the ring $R^{\mathrm{univ}}_{\mathcal{S}}$ *is a finitely generated* $\mathcal{O}_L$*-module.*

Proof. Note that $r_{l,\imath}(\pi)|_{G_{F_{\tilde{v}}}}$ lies on a unique component of $\operatorname{Spec} R^{\square}_{\overline{r}_{l,\imath}(\pi)|_{G_{F_{\tilde{v}}}}} \otimes \overline{\mathbb{Q}}_l$ for each $v \in S$ with $v \nmid l$; use Lemma 1.3.2 and the fact that

$$\imath\mathrm{WD}(r_{l,\imath}(\pi)|_{G_{F_{\tilde{v}}}})^{F-\mathrm{ss}} \cong \mathrm{rec}(\pi_{\tilde{v}} \otimes |\det|_{\tilde{v}}^{(1-n)/2}).$$

Also by making a base change to a finite, soluble, Galois, CM extension F'/F which is linearly disjoint from $\overline{F}^{\ker \overline{r}_{l,\imath}(\pi)}(\zeta_l)$ over F, we may suppose that π is unramified above l and that $\mathcal{C}_v$ is a component of the spectrum $\operatorname{Spec} R^{\square}_{\overline{r}_{l,\imath}(\pi)|_{G_{F_{\tilde{v}}}},\{\{a_{\imath\tau,i}+n-i\}_i\}_\tau,\mathrm{cris}} \otimes \overline{\mathbb{Q}}_l$ for each $v|l$. (Use Lemma 1.2.3). In particular the character χ is unramified above l (as F/F^+ is unramified above l). The result now follows from Theorem 10.1 of [Tho12]. □

2.4. *Automorphy lifting*: *the ordinary case.* One can combine the Taylor–Wiles–Kisin method with the level changing method of [Tay08] and Hida theory to derive a stronger theorem in the ordinary case. This theorem allows for changes of level and weight. The first such theorem was obtained by D.G. (See Theorem 5.3.2 of [Ger10].) The "bigness" condition in Theorem 5.3.2 of [Ger10] was relaxed by Thorne. The theorem we present below, in the case that F is imaginary, is Theorem 9.1 of [Tho12]. The case that F is totally real follows immediately from the case that F is imaginary by base change.

THEOREM 2.4.1. *Suppose that* F *is a* CM (*or totally real*) *field, that* l *is an odd prime and that* $n \in \mathbb{Z}_{\geq 1}$. *Let* (r, μ) *be an* n*-dimensional, algebraic, polarized* l*-adic representation of* G_F *satisfying the following properties*:

(1) *the reduction* $\overline{r}$ *is irreducible and* $\overline{r}(G_{F(\zeta_l)}) \subset \mathrm{GL}_n(\overline{\mathbb{F}}_l)$ *is adequate*;
(2) $\zeta_l \notin F$;
(3) r *is ordinary at all primes above* l;
(4) $(\overline{r}, \overline{\mu})$ *is ordinarily automorphic.*

Then (r, μ) *is ordinarily automorphic. If* r *is also crystalline* (*resp. potentially crystalline*), *then* (r, μ) *is ordinarily automorphic of level prime to* l (*resp. potentially level prime to* l).

The next result is Theorem 10.2 of [Tho12], which generalizes Corollary 4.3.3 of [GG12]. It provides a finiteness theorem for universal deformation rings.

THEOREM 2.4.2. *Let F be an imaginary* CM *field with maximal totally real subfield F^+. Suppose that $n \in \mathbb{Z}_{\geq 1}$ and that l is an odd prime with $\zeta_l \notin F$. Let S be a finite set of primes of F^+ including all primes above l. Suppose moreover that each prime in S splits in F and choose a prime $\tilde{v}$ of F above each $v \in S$. Write $\widetilde{S}$ for the set of $\tilde{v}$ for $v \in S$.*

Let (π, χ) be an ι-ordinary, regular algebraic, cuspidal, polarized automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ which is unramified outside S. Suppose that the image $\overline{r}_{l,\iota}(\pi)(G_{F(\zeta_l)})$ is adequate.

Let

$$\mu : G_{F^+} \longrightarrow \overline{\mathbb{Q}}_l^\times$$

be an algebraic character satisfying $\overline{\mu} = \overline{r}_{l,\iota}(\chi)\bar{\epsilon}_l^{1-n}$. Note that $\mathrm{HT}_\tau(\mu) = \{w\}$ is independent of $\tau : F^+ \hookrightarrow \overline{\mathbb{Q}}_l$. For each $\tau : F \hookrightarrow \overline{\mathbb{Q}}_l$ choose a multiset of n distinct integers H_τ such that

$$H_{\tau \circ c} = \{w - h : \ h \in H_\tau\}.$$

Let L denote a finite extension of $\mathbb{Q}_l$ in $\overline{\mathbb{Q}}_l$ such that L contains the image of each embedding $F \hookrightarrow \overline{\mathbb{Q}}_l$; and L contains the image of μ; and $r_{l,\iota}(\pi)$ is defined over L. For $v \in S$ with $v \not| l$ let $\mathcal{D}_v$ consist of all lifts of $\overline{r}_{l,\iota}(\pi)|_{G_{F_{\tilde{v}}}}$. If $v|l$, let $\mathcal{D}_v$ consist of all lifts which factor through $R^{\square}_{\mathcal{O},\overline{r}_{l,\iota}(\pi)|_{G_{F_{\tilde{v}}}},\{H_\tau\},\text{ss-ord}}$. Also let

$$\mathcal{S} = (F/F^+, S, \widetilde{S}, \mathcal{O}_L, \widetilde{\overline{r}}_{l,\iota}(\pi), \mu, \{\mathcal{D}_v\}_{v \in S}).$$

Then the ring $R^{\mathrm{univ}}_{\mathcal{S}}$ is a finitely generated $\mathcal{O}_L$-module.

3. **Potential automorphy**

3.1. *The Dwork family.* In this section we show that a suitable symplectic, mod l representation is potentially automorphic. The theorem and its proof are slight generalizations of Section 6 of [BLGHT11] The arguments are also simpler because of the stronger automorphy lifting theorems that we now have available, particularly [Tho12]. (See in particular step 2 of the proof of Theorem 6.3 of [BLGHT11].) We start with another minor variant of a result of Moret-Bailly [MB89]. (See also [GPR95] and Proposition 6.2 of [BLGHT11].)

PROPOSITION 3.1.1. *Let $K^{(\mathrm{avoid})}/K/K_0$ be number fields with $K^{(\mathrm{avoid})}/K$ and K/K_0 Galois. Suppose also that S is a finite set of places of K_0 and let S^K denote the set of places of K above S. For $v \in S^K$ let L'_v/K_v be a finite Galois extension with $L'_{\sigma v} = \sigma L'_v$ for $\sigma \in G_{K_{0,v|_{K_0}}}$. Suppose also that T/K is a smooth, geometrically connected variety and that for each $v \in S^K$ we are given*

a nonempty, $\mathrm{Gal}\,(L'_v/K_v)$*-invariant, open subset* $\Omega_v \subset T(L'_v)$*. Then there is a finite Galois extension* L/K *and a point* $P \in T(L)$ *such that*

- L/K_0 *is Galois.*
- L/K *is linearly disjoint from* $K^{(\mathrm{avoid})}/K$*.*
- *If* $v \in S^K$ *and* w *is a prime of* L *above* v*, then* L_w/K_v *is isomorphic to* L'_v/K_v *and* $P \in \Omega_v \subset T(L'_v) \cong T(L_w)$*. (This makes sense as* Ω_v *is* $\mathrm{Gal}\,(L'_v/K_v)$*-invariant.)*

Proof. Let $K_1^{(\mathrm{avoid})}, \dots, K_r^{(\mathrm{avoid})}$ denote the intermediate fields between $K^{(\mathrm{avoid})}$ and K with $K_i^{(\mathrm{avoid})}/K$ Galois with simple Galois group. Combining Hensel's lemma with the Weil bounds we see that T has a K_v-rational point for all but finitely many primes v of K. Thus enlarging S we may assume that for each $i = 1, \dots, r$ there is $v \in S^K$ with $L'_v = K_v$ and v not split completely in $K_i^{(\mathrm{avoid})}$. Then we may suppress the second condition on L.

Let K'/K be a finite extension such that

- K'/K_0 is Galois;
- if $v \in S^K$ and $w|v$ is a place of K', then K'_w/K_v is isomorphic to L'_v/K_v.

(Apply Lemma A.2.1, with F (resp. S) of that lemma taken to be our K (resp. S^K). This produces a soluble extension K''/K. Then we take K' to be the normal closure of K'' over K_0.) Thus we may assume that $L'_v = K_v$ for all $v \in S^K$.

Then Theorem 1.3 of [MB89] tells us that we can find a finite Galois extension K'/K and a point $P \in T(K')$ such that every place v of S^K splits completely in K' and if w is a prime of K' above v, then $P \in \Omega_v \subset T(K'_w)$. Now take L to be the normal closure of K' over K_0. □

THEOREM 3.1.2. *Suppose that*

- F/F_0 *is a finite, Galois extension of totally real fields*;
- $\mathcal{I}$ *is a finite set*;
- *for each* $i \in \mathcal{I}$*,* n_i *is a positive even integer,* l_i *is an odd rational prime and* $\imath_i : \overline{\mathbb{Q}}_{l_i} \xrightarrow{\sim} \mathbb{C}$;
- $F^{(\mathrm{avoid})}/F$ *is a finite Galois extension*;
- $\bar{r}_i : G_F \to \mathrm{GSp}_{n_i}(\overline{\mathbb{F}}_{l_i})$ *is a mod* l_i *Galois representation with open kernel and multiplier* $\bar{\epsilon}_{l_i}^{1-n_i}$*.*

Then we can find a finite totally real extension F'/F *and for each* $i \in \mathcal{I}$ *a regular algebraic, cuspidal, polarized automorphic representation* (π_i, χ_i) *of* $\mathrm{GL}_{n_i}(\mathbb{A}_{F'})$ *such that*

(1) F'/F_0 *is Galois*;
(2) F' *is linearly disjoint from* $F^{(\mathrm{avoid})}$ *over* F;
(3) $(\overline{r}_{l_i,\imath_i}(\pi_i), \overline{r}_{l_i,\imath_i}(\chi_i)\bar{\epsilon}_{l_i}^{1-n_i}) \cong (\bar{r}_i|_{G_{F'}}, \bar{\epsilon}_{l_i}^{1-n_i})$ *for each* $i \in \mathcal{I}$;
(4) π_i *is* $\imath_i$*-ordinary of weight* 0 *for each* $i \in \mathcal{I}$*.*

Proof. The proof follows the proof of Theorem 6.3 of [BLGHT11], although the proof here is simpler. First note that $\overline{r}_i$ is actually valued in $\mathrm{GSp}_n(\mathbb{F}^{(i)})$ for some finite cardinality subfield $\mathbb{F}^{(i)} \subset \overline{\mathbb{F}}_{l_i}$. We choose a positive integer N such that

- N is coprime to $2\prod_i l_i$,
- $N > n_i + 1$ for all i,
- N is not divisible by any rational prime which ramifies in $F^{(\mathrm{avoid})}$,
- for each $i \in \mathcal{I}$ there is a prime λ_i of $\mathbb{Q}(\zeta_N)^+$ above l_i and an embedding $\mathbb{F}^{(i)} \hookrightarrow \mathbb{Z}[\zeta_N]^+/\lambda_i$.

(Use Lemma 6.1 of [BLGHT11].) Note that in particular $F^{(\mathrm{avoid})}$ is linearly disjoint from $\mathbb{Q}(\zeta_N)$ over $\mathbb{Q}$.

We next choose an imaginary CM field M_i for each $i \in \mathcal{I}$ such that $M_i/\mathbb{Q}$ is cyclic of degree n_i and unramified at all rational primes which ramify in $F^{(\mathrm{avoid})}$. For each i let τ_i denote a generator of $\mathrm{Gal}\,(M_i/\mathbb{Q})$. Choose a rational prime q such that

- q splits completely in $\prod_i M_i$,
- q is unramified in $F(\zeta_{4N})$.

Also choose primes $\mathfrak{q}_i$ of M_i above q for each i. Choose M' containing the compositum of the M_i's and, for each i, a character $\phi_i : \mathbb{A}_{M_i}^\times \to (M')^\times$ with open kernel such that

- if $\alpha \in M_i^\times$, then

$$\phi_i(\alpha) = \prod_{j=0}^{n_i/2-1} \tau_i^j(\alpha)^j \tau_i^{n_i/2+j}(\alpha)^{n_i-1-j};$$

- $\phi_i\phi_i^c = \prod_{v \not| \infty} |\ |_v^{1-n_i}$;
- ϕ_i is unramified above all rational primes which ramify in $F/\mathbb{Q}$;
- ϕ_i is unramified above N;
- ϕ_i is unramified at all primes above q except $\mathfrak{q}_i$ and $\mathfrak{q}_i^c$, but $q|\#\phi_i(\mathcal{O}^\times_{M_i,\mathfrak{q}_i})$.

(Apply Lemma A.2.4. We take S to be the set of primes of M_i above Nq or any rational prime that ramifies in F. We also take $\psi_S = \prod_{v\in S}\psi_v$, where ψ_v is the trivial character unless $v|\mathfrak{q}_i\mathfrak{q}_i^c$. Moreover we choose $\psi_{\mathfrak{q}_i}$ to be wildly ramified and take $\psi_{\mathfrak{q}_i^c} = (\psi^c_{\mathfrak{q}_i})^{-1}$.)

Next choose a rational prime l' such that

- l' splits completely in $M'(\zeta_N)$ (so that in particular $l' \equiv 1 \bmod N$);
- l' is unramified in F;
- $l' \not| 6qN\prod_i(l_i n_i)$.

In particular $\zeta_{l'} \notin F$. Also choose primes $\lambda'_{M'}$ of M' and λ' of $\mathbb{Q}(\zeta_N)^+$ above l'. For each i let

$$\theta_i : G_{M_i} \longrightarrow \mathcal{O}^\times_{M',\lambda'_{M'}} = \mathbb{Z}_{l'}^\times$$

be the algebraic character defined by

$$\theta_i(\mathrm{Art}_{M_i}\alpha) = \phi_i(\alpha)\prod_{j=0}^{n_i/2-1}\tau_i^j(\alpha_{l'})^{-j}\tau_i^{n_i/2+j}(\alpha_{l'})^{j+1-n_i}.$$

Note that $\theta_i\theta_i^c = \epsilon_{l'}^{1-n_i}$. Let $r_i' = \mathrm{Ind}_{G_{M_i}}^{G_{\mathbb{Q}}}\theta_i$ and note that

$$r_i' \cong (r_i')^\vee \otimes \epsilon_{l'}^{1-n_i}.$$

Choose a lifting $\widetilde{\tau}_i$ of τ_i to $\ker\epsilon_{l'} \subset G_{\mathbb{Q}}$. Then

$$\theta_i(\widetilde{\tau}_i^n) = \theta_i(c(c\widetilde{\tau}_i^{n_i/2})c(c\widetilde{\tau}_i^{n_i/2})) = (\theta\theta^c)(c\widetilde{\tau}_i^{n_i/2}) = \epsilon_{l'}^{1-n_i}(c\widetilde{\tau}_i^{n_i/2}) = -1.$$

The module underlying r_i' has a $\mathbb{Z}_{l'}$-basis $e_0, e_1, \dots, e_{n_i-1}$ such that

- $r_i'(\sigma)e_j = \theta_i^{\tau_i^j}(\sigma)e_j$ for all $\sigma \in G_M$;
- $r_i'(\widetilde{\tau}_i)e_i = e_{i-1}$ for $i = 1, \dots, n_i - 1$;
- $r_i'(\widetilde{\tau}_i)e_0 = \theta_i(\widetilde{\tau}_i^{n_i})e_{n_i-1} = -e_{n_i-1}$.

If we define a perfect pairing on r_i' by setting

$$\langle e_{j_1}, e_{j_2}\rangle = \begin{cases} 1 & \text{if } j_2 = j_1 + n_i/2, \\ -1 & \text{if } j_1 = j_2 + n_i/2, \\ 0 & \text{otherwise,} \end{cases}$$

we see that this pairing is preserved by r_i' up to scalar multiplication by $\epsilon_{l'}^{1-n_i}$. Thus r_i' factors through $\mathrm{GSp}_{n_i}(\mathbb{Z}_{l'})$ with multiplier $\epsilon_{l'}^{1-n_i}$.

Let $\overline{\theta}_i : G_{M_i} \to \mathbb{F}_{l'}^\times$ denote the reduction of θ_i and let $\overline{r}_i'$ denote the reduction of r_i'. We have the following observations:

- $\overline{\theta}_i^{\tau^j}|_{G_{M_i}(\zeta_{l'})} \neq \overline{\theta}_i^{\tau^{j'}}|_{G_{M_i}(\zeta_{l'})}$ for $j \neq j'$ in the range $0, 1, \dots, n_i - 1$. (Look at the ramification above q.)
- $\overline{r}_i'|_{\mathbb{Q}(\zeta_{l'})}$ is irreducible.
- $l' \not| \#\overline{r}_i'G_{\mathbb{Q}}$.
- $\overline{r}_i'(G_{\mathbb{Q}(\zeta_{l'})})$ is adequate (by Proposition 2.1.2).
- $\mathbb{Q}(\zeta_N)$ is linearly disjoint over $\mathbb{Q}$ from $F^{(\mathrm{avoid})}/\mathbb{Q}$ (because no rational prime ramifies in both fields).
- $\overline{r}_i'(G_{\mathbb{Q}(\zeta_{l'})}) = \overline{r}_i'(G_{F(\zeta_{Nl'})})$ (because $\overline{r}_i'$ is only ramified at primes which are unramified in $F(\zeta_N)$).

Let $T_0/\mathrm{Spec}\, F(\zeta_N)^+$ denote the scheme $\mathbb{P}^1 - (\{\infty\} \cup \mu_N)$. For each i there are

- lisse $\mathbb{Z}[\zeta_N]^+_{\lambda_i}$ (resp. $\mathbb{Z}[\zeta_N]^+_{\lambda'}$) sheaves $V_{n_i,\lambda_i}((N-1-n_i)/2)$ (resp. $V_{n_i,\lambda'}((N-1-n_i)/2)$) over T_0;
- locally free etale sheaves $V_{n_i}[\lambda_i]((N-n_i-1)/2)$ (resp. $V_{n_i}[\lambda']((N-n_i-1)/2)$) of $\mathbb{Z}[\zeta_N]^+/\lambda_i$ (resp. $\mathbb{Z}[\zeta_N]^+/\lambda'$) modules;
- a finite cover $T_{\overline{r}_i \times \overline{r}_i'}/(T_0 \times \mathrm{Spec}\, F(\zeta_N)^+)$

constructed as in Section 4 of [BLGHT11] using $N = N$ and $n = n_i$. (We have added the subscript n_i to the notation of Section 4 of [BLGHT11] as a reminder that we are taking $n = n_i$ in the constructions of that section. We would like to point out that in the definition of T_W on page 54 of [BLGHT11] we should have specified that it represents isomorphisms $W_S \xrightarrow{\sim} V[\mathfrak{n}]((N-1-n)/2)_S$ *compatible with the symplectic structures*. We would also like to point out that the fourth occurrence of S_2 in the statement of Proposition 6.2 of [BLGHT11] should be an M. We thank Kevin Buzzard for pointing out these corrections to [BLGHT11].)

Let $\widetilde{T}$ denote the product of the $T_{\overline{r}_i \times \overline{r}'_i}$ over $\operatorname{Spec} F(\zeta_N)^+$ and let t_i denote the i^{th} projection to T_0. By Proposition 4.2 of [BLGHT11] we see that $\widetilde{T}$ is geometrically irreducible. By Proposition 3.1.1 we can find a finite extension of $F'/F(\zeta_N)^+$ and a point $P \in \widetilde{T}(F')$ such that

- F'/F_0 is Galois,
- F' is totally real,
- F' is linearly disjoint from $F^{(\text{avoid})}\overline{F}^{\cap_i \ker \overline{r}'_i}(\zeta_{Nl'})$ over $F(\zeta_N)^+$,
- $v(t_i(P)) < 0$ for all places $v|l_i$ of F',
- $v(t_i(P)) > 0$ for all places $v|l'$ of F'.

(The only thing we need to check is that $T_{\overline{r}_i \times \overline{r}'_i}(F(\zeta_N)^+_v) \neq \emptyset$ for each i and each $v|\infty$. However, because $\mathrm{GSp}_{n_i}(\mathbb{Z}/l_i l'\mathbb{Z})$ has a unique conjugacy class of elements of order 2 and multiplier -1, we see that every $F(\zeta_N)^+_v$-point of T_0 lifts to a $F(\zeta_N)^+_v$-point of $T_{\overline{r}_i \times \overline{r}'_i}$.)

Then we have the following observations:

- F' is linearly disjoint from $F^{(\text{avoid})}$ over F (as $F(\zeta_N)^+$ is linearly disjoint from $F^{(\text{avoid})}$ over F).
- $V_{n_i}[\lambda_i]((N-1-n_i)/2)_{t_i(P)} \cong \overline{r}_i|_{G_{F'}}$.
- $V_{n_i}[\lambda']((N-1-n_i)/2)_{t_i(P)} \cong \overline{r}'_i|_{G_{F'}}$.
- $\overline{r}'_i(G_{F'(\zeta_{l'})})$ is adequate.
- $\zeta_{l'} \notin F'$ (as F' is totally real and $l' > 2$).
- $V_{n_i,\lambda'}((N-1-n_i)/2)_{t_i(P)}$ is ordinary at all primes above l'. (See Lemma 5.3(3) of [BLGHT11].)
- $\mathrm{HT}_\tau(V_{n_i,\lambda'}((N-1-n_i)/2)_{t_i(P)}) = \{0, 1, \ldots, n_i - 1\}$ for all $\tau : F' \hookrightarrow \overline{\mathbb{Q}}_{l'}$. (See Lemma 5.3(1) of [BLGHT11].)
- If v is a place of F' above l_i, then

$$\imath' \mathrm{WD}(V_{n_i,\lambda'}((N-1-n_i)/2)_{t_i(P)}|_{G_{F'_v}}) \cong \mathrm{rec}_{F'_v}(\mathrm{Sp}_{n_i}(\phi_i))$$

for some unramified character ϕ_i (and for any isomorphism $\imath' : \overline{\mathbb{Q}}_{l'} \xrightarrow{\sim} \mathbb{C}$). (See Lemma 5.1(2) of [BLGHT11].)

From Theorem 4.2 of [AC89] we see that $(r'_i|_{G_{F'}}, \epsilon_{l'}^{1-n_i})$ is automorphic of level potentially prime to l' and hence ordinarily automorphic. By Theorem 2.4.1 we conclude that $(V_{n_i,\lambda'}((N-1-n_i)/2)_{t_i(P)}, \epsilon_{l'}^{1-n_i})$ is automorphic over F' of weight 0, arising from a regular algebraic, cuspidal, polarized automorphic representation $(\pi_i, 1)$ with $\pi_{i,v}$ Steinberg for all $v|l_i$, and from an isomorphism $\imath'_i : \overline{\mathbb{Q}}_{l_i} \xrightarrow{\sim} \mathbb{C}$. Thus π_i is $\imath_i$-ordinary. The theorem follows. □

3.2. *Lifting Galois representations* I. We now use the method of Khare and Wintenberger [KW09] to show that certain mod l representations have ordinary lifts with prescribed local behavior. We will later improve upon this by weakening the ordinary hypothesis (see Theorem 4.3.1), but we will need to use this special case before we are in a position to prove the more general result.

Let n be a positive integer and l an odd prime. Suppose that F is an imaginary CM field not containing ζ_l and with maximal totally real subfield F^+. Let S be a finite set of finite places of F^+ which split in F and suppose that S includes all places above l. For each $v \in S$ choose a prime $\tilde{v}$ of F above v.

Let $\mu : G_{F^+} \to \overline{\mathbb{Q}}_l^\times$ be a continuous, crystalline character unramified outside S such that $\mu(c_v) = -1$ for all $v|\infty$. Then there is a $w \in \mathbb{Z}$ such that for each $\tau : F^+ \hookrightarrow \overline{\mathbb{Q}}_l$ we have $\mathrm{HT}_\tau(\mu) = \{w\}$. For each $\tau : F \hookrightarrow \overline{\mathbb{Q}}_l$ let H_τ be a set of n distinct integers such that $H_{\tau\circ c} = \{w - h : \ h \in H_\tau\}$.

Let

$$\overline{r} : G_{F^+} \longrightarrow \mathcal{G}_n(\overline{\mathbb{F}}_l)$$

be a continuous representation unramified outside S with $\nu \circ \overline{r} = \overline{\mu}$ and $\overline{r}^{-1}\mathcal{G}_n^0(\overline{\mathbb{F}}_l) = G_F$. For $v \in S$ with $v \not| l$ let $\rho_v : G_{F_{\tilde{v}}} \to \mathrm{GL}_n(\mathcal{O}_{\overline{\mathbb{Q}}_l})$ denote a lift of $\breve{\overline{r}}|_{G_{F_{\tilde{v}}}}$.

Proposition 3.2.1. *Keep the notation and assumptions already stated in this section. Also make the following additional assumptions*:

(a) *Suppose that* $\breve{\overline{r}}|_{G_{F(\zeta_l)}}$ *is irreducible. Also, writing* d *for the maximal dimension of an irreducible constituent of the restriction of* $\breve{\overline{r}}$ *to the closed subgroup of* G_{F^+} *generated by all Sylow pro-*l*-subgroups, suppose that* $l \geq 2(d+1)$.

(b) *Suppose that for* $u|l$ *a place of* F *the restriction* $\breve{\overline{r}}|_{G_{F_u}}$ *admits a lift* $\rho_u : G_{F_u} \to \mathrm{GL}_n(\mathcal{O}_{\overline{\mathbb{Q}}_l})$ *which is ordinary and crystalline with Hodge–Tate numbers* H_τ *for each* $\tau : F_u \hookrightarrow \overline{\mathbb{Q}}_l$.

Then there is a lift

$$r : G_{F^+} \longrightarrow \mathcal{G}_n(\mathcal{O}_{\overline{\mathbb{Q}}_l})$$

of $\overline{r}$ *such that*

(1) $\nu \circ r = \mu$;

(2) *if $u|l$ is a place of F, then $\check{r}|_{G_{F_u}}$ is ordinary and crystalline with Hodge–Tate numbers H_τ for each $\tau : F_u \hookrightarrow \overline{\mathbb{Q}}_l$;*
(3) *if $v \in S$ and $v \not| l$, then $\check{r}|_{G_{F_{\tilde{v}}}} \sim \rho_v$;*
(4) *r is unramified outside S.*

Proof. Choose a place v_q of F above an odd rational prime q such that v_q is split over F^+ and v_q does not divide any prime in S. Also choose integers b_τ for all $\tau : F \hookrightarrow \overline{\mathbb{Q}}_l$ such that

- $b_\tau + b_{\tau\circ c} = 2n - 1 - w$ for all τ;
- $|b_\tau - b_{\tau\circ c}| > |h - h'|$ for all τ and for all $h \in H_\tau, h' \in H_{\tau\circ c}$.

Now choose a character $\psi : G_F \to \overline{\mathbb{Q}}_l^\times$ such that

- $\psi|_{G_{F_{\tilde{v}}}}$ is unramified if $v \in S$ but $v \not| l$;
- ψ is crystalline at all primes above l with $\mathrm{HT}_\tau(\psi) = \{b_\tau\}$ for all $\tau : F \hookrightarrow \overline{\mathbb{Q}}_l$;
- $q|\#(\psi/\psi^c)(I_{F_{v_q}})$;
- $\psi\psi^c = \epsilon_l^{1-2n}\mu^{-1}|_{G_F}$.

(To do this, apply Lemma A.2.5 with the set S of that lemma equal to the primes of F above our S, plus v_q and v_q^c. For v in this set take ψ_v as follows:

- if $v \in S$ but $v \not| l$, then $\psi_{\tilde{v}} = 1$ and $\psi_{\tilde{v}^c} = \epsilon_l^{1-2n}\mu^{-1}|_{G_{F_{\tilde{v}^c}}}$;
- ψ_{v_q} is a wildly ramified character and $\psi_{v_q^c} = (\psi_{v_q}^c)^{-1}$;
- if $v|l$, then ψ_v is crystalline with $\mathrm{HT}_\tau(\psi_v) = \{b_\tau\}$ for all continuous $\tau : F_v \hookrightarrow \overline{\mathbb{Q}}_l$.)

In the notation of Section 1.1, we have a homomorphism

$$(\overline{\psi}, \bar{\epsilon}_l^{1-2n}\overline{\mu}^{-1}\delta_{F/F^+}) : G_{F^+} \to \mathcal{G}_1(\overline{\mathbb{F}}_l),$$

and we can consider the representations

$$\overline{r} \otimes (\overline{\psi}, \bar{\epsilon}_l^{1-2n}\overline{\mu}^{-1}\delta_{F/F^+}) : G_{F^+} \to \mathcal{G}_n(\overline{\mathbb{F}}_l)$$

and

$$I_{\overline{\psi}}(\bar{r}) := I(\overline{r} \otimes (\overline{\psi}, \bar{\epsilon}_l^{1-2n}\overline{\mu}^{-1}\delta_{F/F^+})) : G_{F^+} \to \mathrm{GSp}_{2n}(\overline{\mathbb{F}}_l).$$

Note that $I_{\overline{\psi}}(\bar{r})$ has multiplier $\bar{\epsilon}_l^{1-2n}$. By the third condition on ψ the representation $I_{\overline{\psi}}(\bar{r})|_{G_{F^+(\zeta_l)}}$ is irreducible. (As it is the induction of an irreducible representation from the index 2 subgroup $G_{F(\zeta_l)}$, it suffices to check that the restriction to $G_{F(\zeta_l)}$ is not the sum of two isomorphic representations, and this follows, as the two representations differ when restricted to $I_{F_{v_q}}$.) By Proposition 2.1.2, $I_{\overline{\psi}}(\bar{r})(G_{F^+(\zeta_l)})$ is adequate.

Let F_0/F^+ be a totally imaginary quadratic extension linearly disjoint from $\overline{F}^{\ker I_{\overline{\psi}}(\bar{r})}(\zeta_l)$ over F^+. By Theorem 3.1.2 there is a Galois totally real field extension F_1^+/F^+ and a regular algebraic, cuspidal, polarized automorphic representation $(\pi_1, 1)$ of $\mathrm{GL}_{2n}(\mathbb{A}_{F_1^+})$ such that

- F_1^+ is linearly disjoint from $\overline{F}^{\ker I_{\overline{\psi}}(\overline{r})}F_0(\zeta_l)$ over F^+;
- $\overline{r}_{l,\imath}(\pi_1) \cong I_{\overline{\psi}}(\overline{r})|_{G_{F_1^+}}$;
- π_1 is $\imath$-ordinary.

Set $F_1 = F_0F_1^+$. It is linearly disjoint from $\overline{F}^{\ker I_{\overline{\psi}}(\overline{r})}F_1^+(\zeta_l)$ over F_1^+. Set (again, in the notation of Section 1.1)

$$\overline{r}_1 = (I_{\overline{\psi}}(\overline{r})|_{G_{F_1^+}})^{\wedge}_{G_{F_1}} : G_{F_1^+} \longrightarrow \mathcal{G}_{2n}(\overline{\mathbb{F}}_l).$$

Then $\breve{\overline{r}}_1(G_{F_1(\zeta_l)})$ is adequate and $\zeta_l \notin F_1$.

Let $T' \supset S$ denote a finite set of primes of F^+ including all those above which $\overline{\psi}$, π_1 or F_1 is ramified. Let F_2^+/F^+ be a finite soluble Galois totally real extension, linearly disjoint from $\overline{F}_1^{\ker \breve{\overline{r}}_1}(\zeta_l)$ over F^+ such that all primes of $F_3^+ = F_1^+F_2^+$ above T' split in $F_3 = F_1F_2^+$. (We have introduced F_2^+ in order to be able to apply Theorem 2.4.2. The primes of F_1^+ above T' may not split in F_1.) Set

$$\overline{r}_3 = \overline{r}_1|_{G_{F_3^+}} : G_{F_3^+} \longrightarrow \mathcal{G}_{2n}(\overline{\mathbb{F}}_l)$$

so that $\overline{r}_3^{-1}\mathcal{G}_{2n}^0(\overline{\mathbb{F}}_l) = G_{F_3}$. Then $\breve{\overline{r}}_3(G_{F_3}(\zeta_l))$ is adequate and $\zeta_l \notin F_3$. Let T denote the set of places of F_3^+ lying over T' and for each $u \in T$ choose a prime $\tilde{u}$ of F_3 above u and let $\widetilde{T}$ denote the set of $\tilde{u}$ for $u \in T$.

For $v \in S$ with $v \not| l$ let $\mathcal{C}_v$ denote a component of $R^{\square}_{\breve{\overline{r}}|_{G_{F_{\tilde{v}}}}} \otimes \overline{\mathbb{Q}}_l$ containing ρ_v. Choose a finite extension L of $\mathbb{Q}_l$ in $\overline{\mathbb{Q}}_l$ with integers $\mathcal{O}$ and residue field $\mathbb{F}$ such that

- L contains the image of each embedding $F_3 \hookrightarrow \overline{\mathbb{Q}}_l$,
- for $v \in S$ the component $\mathcal{C}_v$ is defined over L,
- $\overline{r}$ and $\overline{\psi}$ are defined over $\mathbb{F}$,
- μ is defined over L.

For $v \in S$ with $v \not| l$ let $\mathcal{D}_v$ denote the deformation problem for $\breve{\overline{r}}|_{G_{F_{\tilde{v}}}}$ corresponding to $\mathcal{C}_v$. For $v \in S$ with $v|l$ let $\mathcal{D}_v$ consist of all lifts of $\breve{\overline{r}}|_{G_{F_{\tilde{v}}}}$ which factor through $R^{\square}_{\mathcal{O},\breve{\overline{r}}|_{G_{F_{\tilde{v}}}},\{H_\tau\},\text{cr-ord}}$. Set

$$\mathcal{S} = (F/F^+, S, \widetilde{S}, \mathcal{O}, \overline{r}, \mu, \{\mathcal{D}_v\}_{v\in S}).$$

For $u \in T$ with $u \not| l$ let $\mathcal{D}_{3,u}$ consist of all lifts of $\breve{\overline{r}}_3|_{G_{F_{3,\tilde{u}}}}$. For $u \in T$ with $u|l$ let $\mathcal{D}_{3,u}$ consist of all lifts of $\breve{\overline{r}}_3|_{G_{F_{3,\tilde{u}}}}$ which factor through $R^{\square}_{\mathcal{O},\breve{\overline{r}}_3|_{G_{F_{3,\tilde{u}}}},\{H_{3,\tau}\},\text{ss-ord}'}$ where

$$H_{3,\tau} = \{h + b_{\tau_1} : \ h \in H_{\tau_1}\} \cup \{h + b_{\tau_2} : \ h \in H_{\tau_2}\},$$

and τ_1 and τ_2 denote the two embeddings of $F \hookrightarrow \overline{\mathbb{Q}}_l$ lying above $\tau|_{F^+}$. Set

$$\mathcal{S}_3 = (F_3/F_3^+, T, \widetilde{T}, \mathcal{O}, \overline{r}_3, \epsilon_l^{1-2n}, \{\mathcal{D}_{3,u}\}_{u\in T}).$$

According to Theorem 2.4.2 the ring $R^{\mathrm{univ}}_{\mathcal{S}_3}$ is a finitely generated $\mathcal{O}$-module. Hence by (all three parts of) Lemma 1.2.3 the ring $R^{\mathrm{univ}}_{\mathcal{S}}$ is also a finitely generated $\mathcal{O}$-module. On the other hand by Proposition 1.5.1 $R^{\mathrm{univ}}_{\mathcal{S}}$ has Krull dimension at least 1 and so there is a continuous ring homomorphism $R^{\mathrm{univ}}_{\mathcal{S}} \to \overline{\mathbb{Q}}_l$. The push forward of the universal deformation of $\overline{r}$ by this homomorphism is our desired lift r. □

3.3. *Potential ordinary automorphy.* In this section we improve Theorem 3.1.2 to show that a suitable mod l representation is potentially ordinarily automorphic with prescribed "weight and level." The proof will combine Theorem 3.1.2 and Proposition 3.2.1. We will improve further on this result in Corollary 4.5.3.

PROPOSITION 3.3.1. *Suppose that we are in the following situation*:

(a) *Let F/F_0 be a finite, Galois extension of imaginary* CM *fields, and let F^+ and F_0^+ denote their maximal totally real subfields. Choose a complex conjugation $c \in G_{F^+}$.*
(b) *Let $\mathcal{I}$ be a finite set.*
(c) *For each $i \in \mathcal{I}$ let n_i be a positive integer and l_i be an odd rational prime with $\zeta_{l_i} \notin F$. Also choose $\iota_i : \overline{\mathbb{Q}}_{l_i} \xrightarrow{\sim} \mathbb{C}$ for each $i \in \mathcal{I}$.*
(d) *For each $i \in \mathcal{I}$ let $\mu_i : G_{F^+} \to \overline{\mathbb{Q}}_{l_i}^\times$ be a continuous, totally odd, de Rham character. Then there is a $w_i \in \mathbb{Z}$ such that for each $\tau : F^+ \hookrightarrow \overline{\mathbb{Q}}_{l_i}$ we have $\mathrm{HT}_\tau(\mu_i) = \{w_i\}$.*
(e) *For each $i \in \mathcal{I}$ let $\overline{r}_i : G_F \to \mathrm{GL}_{n_i}(\overline{\mathbb{F}}_{l_i})$ be an irreducible continuous representation such that $(\overline{r}_i, \overline{\mu}_i)$ is a totally odd polarized mod l representation. Let d_i denote the maximal dimension of an irreducible subrepresentation of the restriction of $\overline{r}_i$ to the subgroup of G_F generated by all Sylow pro-l_i-subgroups. Suppose that $\overline{r}_i|_{G_{F(\zeta_{l_i})}}$ is irreducible and that $l_i \geq 2(d_i + 1)$.*
(f) *For each $i \in \mathcal{I}$ and each $\tau : F \hookrightarrow \overline{\mathbb{Q}}_{l_i}$ let $H_{i,\tau}$ be a set of n_i distinct integers such that $H_{i,\tau\circ c} = \{w_i - h : \ h \in H_{i,\tau}\}$.*
(g) *Let S denote a finite $\mathrm{Gal}(F/F^+)$-invariant set of primes of F including all those dividing $\prod_i l_i$ and all those at which some $\overline{r}_i$ ramifies.*
(h) *For each $i \in \mathcal{I}$ and $v \in S$ with $v \nmid l_i$ let $\rho_{i,v} : G_{F_v} \to \mathrm{GL}_{n_i}(\mathcal{O}_{\overline{\mathbb{Q}}_{l_i}})$ denote a lift of $\overline{r}_i|_{G_{F_v}}$ such that $\rho^c_{i,cv} \cong \mu_i|_{G_{F_v}} \rho^\vee_{i,v}$.*
(i) *Let $F^{(\mathrm{avoid})}/F$ be a finite Galois extension.*

Then we can find a finite CM *extension F'/F and for each $i \in \mathcal{I}$ a regular algebraic, cuspidal, polarized automorphic representation (π_i, χ_i) of $\mathrm{GL}_{n_i}(\mathbb{A}_{F'})$ such that*

(1) *F'/F_0 is Galois*;
(2) *F' is linearly disjoint from $F^{(\mathrm{avoid})}$ over F*;
(3) *$(\overline{r}_{l_i,\iota_i}(\pi_i), \overline{r}_{l_i,\iota_i}(\chi_i)\overline{\epsilon}_{l_i}^{1-n_i}) \cong (\overline{r}_i|_{G_{F'}}, \overline{\mu}_i|_{G_{(F')^+}})$*;

(4) *π_i is unramified above l_i and outside S;*
(5) *π_i is ι_i-ordinary;*
(6) *if $\tau : F' \hookrightarrow \overline{\mathbb{Q}}_{l_i}$, then $\mathrm{HT}_\tau(r_{l_i,\iota_i}(\pi_i)) = H_{i,\tau|_F}$;*
(7) *if $u \not| l_i$ is a prime of F' lying above an element $v \in S$, then $r_{l_i,\iota_i}(\pi_i)|_{G_{F'_u}} \sim \rho_{i,v}|_{G_{F'_u}}$.*

Proof. Note that $(\overline{r}_i, \overline{\mu}_i|_{G_F})$ extends to a continuous homomorphism $\tilde{\overline{r}}_{i,\overline{\mu}_i} : G_{F^+} \to \mathcal{G}_{n_i}(\overline{\mathbb{F}}_{l_i})$ with $\nu \circ \tilde{\overline{r}}_{i,\overline{\mu}_i} = \overline{\mu}_i$. (See Section 1.1.)

Choose a finite totally real extension F_1^+/F^+ so that

- F_1^+/F_0^+ is Galois;
- F_1^+ is linearly disjoint from $\overline{F}^{\cap_i \ker \overline{r}_i}(\zeta_{\prod_i l_i})F^{(\mathrm{avoid})}$ over F^+;
- all places of $F_1 = FF_1^+$ above S are split over F_1^+;
- for all $i \in \mathcal{I}$ and all places $u|l_i$ of F_1 the restriction $\overline{r}_i|_{G_{F_{1,u}}}$ admits a lift $\rho_{i,u} : G_{F_{1,u}} \to \mathrm{GL}_{n_i}(\mathcal{O}_{\overline{\mathbb{Q}}_{l_i}})$ which is ordinary and crystalline with Hodge–Tate numbers $H_{i,\tau|_F}$ for each $\tau : F_{1,u} \hookrightarrow \overline{\mathbb{Q}}_l$.

(For all v a prime of F^+ below an element of S there is a finite Galois extension E_v/F_v^+ with the following property: The last two bullet points will be satisfied as long as, for all primes w of F_1^+ above a prime v of F^+ below an element of S, we have $(F_1^+)_w \supset E_v$. So we may replace the last two bullet points by this condition. Now the existence of F_1^+ follows from Lemma A.2.1.) Replacing F by F_1 (and $F^{(\mathrm{avoid})}$ by $F_1F^{(\mathrm{avoid})}$) we may reduce the theorem to the special case that all elements of S are split over F^+ and that for all $i \in \mathcal{I}$ and all places $u|l_i$ of F the restriction $\overline{r}_i|_{G_{F_u}}$ admits a lift $\rho_{i,u} : G_{F_u} \to \mathrm{GL}_{n_i}(\mathcal{O}_{\overline{\mathbb{Q}}_{l_i}})$ which is ordinary and crystalline with Hodge–Tate numbers $H_{i,\tau}$ for each $\tau : F_u \hookrightarrow \overline{\mathbb{Q}}_l$. (Note that if F'/F_1 is finite and linearly disjoint from $F^{(\mathrm{avoid})}F_1$ over F_1 and if F'/F_0 is Galois, then F'/F is linearly disjoint from $F^{(\mathrm{avoid})}$ over F; thus replacing F by F_1 does not affect the condition that F' is linearly disjoint from $F^{(\mathrm{avoid})}$ over F.)

In this case, using Proposition 3.2.1, we see that for each $i \in \mathcal{I}$ there is a lift

$$r_i : G_{F^+} \longrightarrow \mathcal{G}_{n_i}(\mathcal{O}_{\overline{\mathbb{Q}}_{l_i}})$$

of $\tilde{\overline{r}}_{i,\overline{\mu}_i}$ such that

- $\nu \circ r_i = \mu_i$;
- if $u|l_i$ is a place of F, then $\breve{r}_i|_{G_{F_u}}$ is ordinary and crystalline with Hodge–Tate numbers $H_{i,\tau}$ for each $\tau : F_u \hookrightarrow \overline{\mathbb{Q}}_{l_i}$;
- if $v \in S$ and $v \not| l_i$, then $\breve{r}_i|_{G_{F_v}} \sim \rho_{i,v}$;
- r_i is unramified outside S.

(Note that if we write $S = \widetilde{S} \coprod c\widetilde{S}$, then we only need check the penultimate assertion for $v \in \widetilde{S}$ and it will follow also for $v \in c\widetilde{S}$.)

Choose a place v_q of F which is split over F^+ and which lies above an odd rational prime q, which in turn does not lie under any prime in S. Also choose integers $b_{i,\tau}$ for all $i \in \mathcal{I}$ and all $\tau : F \hookrightarrow \overline{\mathbb{Q}}_l$ such that

- $b_{i,\tau} + b_{i,\tau\circ c} = 2n_i - 1 - w_i$ for all τ;
- $|b_{i,\tau} - b_{i,\tau\circ c}| > |h - h'|$ for all τ and for all $h \in H_{i,\tau}, h' \in H_{i,\tau\circ c}$.

Now choose a character $\psi_i : G_F \to \overline{\mathbb{Q}}_{l_i}^\times$ for $i \in \mathcal{I}$ such that

- ψ_i is unramified at places in S which do not divide l_i;
- ψ_i is crystalline at all places above l_i and $\mathrm{HT}_\tau(\psi_i) = \{b_{i,\tau}\}$ for all $\tau : F \hookrightarrow \overline{\mathbb{Q}}_{l_i}$;
- $q|\#(\psi_i/\psi_i^c)(I_{F_{v_q}})$;
- $\psi_i\psi_i^c = \epsilon_{l_i}^{1-2n_i}\mu_i^{-1}|_{G_F}$.

(To do this apply Lemma A.2.5 with the set S of that lemma equal to the union of our S and the set of primes of F above q. For v in this set take ψ_v as follows:

- if $\{v, v^c\} \subset S$ but $v \not| l_i$, then put $\psi_v = 1$ and $\psi_{v^c} = \epsilon_l^{1-2n}\mu^{-1}|_{G_{F_{v^c}}}$ (or the other way round),
- ψ_{v_q} is a wildly ramified character and $\psi_{v_q^c} = (\psi_{v_q}^c)^{-1}$,
- if $v|l_i$, then ψ_v is crystalline with $\mathrm{HT}_\tau(\psi_v) = \{b_{i,\tau}\}$ for all continuous $\tau : F_v \hookrightarrow \overline{\mathbb{Q}}_l$.)

Consider

$$I_{\overline{\psi}_i}(\tilde{\overline{r}}_{i,\overline{\mu}_i}) := I(\tilde{\overline{r}}_{i,\overline{\mu}_i} \otimes (\overline{\psi}_i, \overline{\epsilon}_{l_i}^{1-2n_i}\overline{\mu}_i^{-1}\delta_{F/F^+})) : G_{F^+} \to \mathrm{GSp}_{2n_i}(\overline{\mathbb{F}}_{l_i}),$$

which has multiplier $\overline{\epsilon}_{l_i}^{1-2n_i}$. As in the proof of Proposition 3.2.1 we see that $I_{\overline{\psi}_i}(\tilde{\overline{r}}_{i,\overline{\mu}_i})(G_{F^+(\zeta_{l_i})})$ is adequate. Theorem 3.1.2 tells us that there is a finite totally real field extension F_1^+/F^+ such that

- F_1^+/F_0^+ is Galois;
- F_1^+ is linearly disjoint from $\overline{F}^{\cap_i \ker I_{\overline{\psi}_i}(\tilde{\overline{r}}_{i,\overline{\mu}_i})}(\zeta_{\prod_i l_i})F^{(\mathrm{avoid})}$ over F^+;
- each $((\mathrm{Ind}_{G_F}^{G_{F^+}} \overline{r}_i \otimes \psi_i)|_{G_{F_1^+}}, \overline{\epsilon}_{l_i}^{1-2n_i})$ is ordinarily automorphic of weight 0.

By Theorem 2.4.1 we conclude that each $((\mathrm{Ind}_{G_F}^{G_{F^+}} \breve{r}_i \otimes \psi_i)|_{G_{F_1^+}}, \epsilon_{l_i}^{1-2n_i})$ is ordinarily automorphic of level prime to l_i.

Let $F' = FF_1^+$. By Lemma 2.2.4 we see that each $((\breve{r}_i \otimes \psi_i)|_{G_{F'}}, \epsilon_{l_i}^{1-n_i})$ is automorphic of level prime to l_i. Hence each $(\breve{r}_i|_{G_{F'}}, \mu_i|_{G_{(F')^+}})$ is automorphic of level prime to l_i. As these representations are also ordinary, they are ordinarily automorphic of level prime to l. The theorem follows (using local-global compatibility). □

4. **The main theorems**

4.1. *A preliminary automorphy lifting result.* The proof of the next proposition is our main innovation. The last two parts of assumption (3) are rather restrictive and mean that the proposition is not directly terribly useful. However in the next section we will see how we can combine this result with Theorem 2.4.1 to get a genuinely useful result. Our main tool will be Harris' tensor product trick. (See [Har09] and [BLGHT11].)

PROPOSITION 4.1.1. *Let F be an imaginary* CM *field with maximal totally real subfield F^+ and let c denote the nontrivial element of* $\mathrm{Gal}(F/F^+)$. *Suppose that l is an odd prime and let $n \in \mathbb{Z}_{\geq 1}$. Assume that $\zeta_l \notin F$. Let (r, μ) be a regular algebraic, irreducible, n-dimensional, polarized l-adic representation of G_F. Let $\overline{r}$ denote the semi-simplification of the reduction of r and let d denote the maximal dimension of an irreducible subrepresentation of the restriction of $\overline{r}$ to the closed subgroup of G_F generated by all Sylow pro-l-subgroups. Suppose that (r, μ) enjoys the following properties*:

(1) *$r|_{G_{F_v}}$ is potentially diagonalizable (and so in particular potentially crystalline) for all $v|l$.*
(2) *The restriction $\overline{r}|_{G_{F(\zeta_l)}}$ is irreducible and $l \geq 2(d+1)$.*
(3) *$(\overline{r}, \overline{\mu})$ is automorphic of level prime to l arising from a regular algebraic, cuspidal, polarized automorphic representation (π, χ) such that*
   - *$r_{l,\imath}(\pi)|_{G_{F_v}}$ is potentially diagonalizable for all $v|l$;*
   - *for all $\tau : F \hookrightarrow \overline{\mathbb{Q}}_l$ the set $\{h + h' : \ h \in \mathrm{HT}_\tau(r),\ h' \in \mathrm{HT}_\tau(r_{l,\imath}(\pi))\}$ has n^2 distinct elements;*
   - *if $v \nmid l$, then $r_{l,\imath}(\pi)|_{G_{F_v}} \sim r|_{G_{F_v}}$.*

Then (r, μ) is potentially diagonalizably automorphic (of level potentially prime to l).

Proof. Note that $\imath\mathrm{WD}(r_{l,\imath}(\pi)|_{G_{F_v}})^{F-\mathrm{ss}} = \mathrm{rec}(\pi_v |\,|_v^{(1-n)/2})$ for all $v \nmid l$. Moreover as π_v is generic we have $r_{l,\imath}(\pi)|_{G_{F_v}} \rightsquigarrow r|_{G_{F_v}}$ for all $v \nmid l$.

Also note that by Proposition 2.1.2, $\overline{r}(G_{F(\zeta_l)})$ is adequate and so (by the remark in the paragraph before the statement of Proposition 2.1.2) we see that $l \nmid n$.

Using Lemma 2.2.2 we see that it is enough to prove the theorem after replacing F by a soluble CM extension which is linearly disjoint from $\overline{F}^{\ker \overline{r}}(\zeta_l)$ over F. Thus (using Lemma A.2.1) we may suppose that

- F/F^+ is unramified at all finite primes;
- all primes dividing l and all primes at which π or r ramify are split over F^+;
- if u is a place of F above a rational prime which equals l or above which π ramifies, then $\overline{r}|_{G_{F_u}}$ is trivial;

- if u is a place of F above l, then $r|_{G_{F_u}}$ and $r_{l,\imath}(\pi)|_{G_{F_u}}$ are diagonalizable, and π_u is unramified.

For each prime v of F^+ which splits in F, choose once and for all a prime $\tilde{v}$ of F above v.

For u a prime of F above l suppose that

$$r|_{G_{F_u}} \sim \psi_1^{(u)} \oplus \cdots \oplus \psi_n^{(u)},$$

and

$$r_{l,\imath}(\pi)|_{G_{F_u}} \sim \phi_1^{(u)} \oplus \cdots \oplus \phi_n^{(u)},$$

for crystalline characters $\psi_i^{(u)}$ and $\phi_i^{(u)} : G_{F_u} \to \mathcal{O}_{\overline{\mathbb{Q}}_l}^\times$. We can, and shall, assume that the characters $\psi_i^{(u)}$ and $\phi_i^{(u)}$ satisfy $\psi_i^{(cu)}(\psi_i^{(u)})^c = \mu|_{G_{F_{cu}}}$ and $\phi_i^{(cu)}(\phi_i^{(u)})^c = (r_{l,\imath}(\chi)\epsilon_l^{1-n})|_{G_{F_{cu}}}$. For $\tau : F_u \hookrightarrow \overline{\mathbb{Q}}_l$ write $\mathrm{HT}_\tau(\psi_i^{(u)}) = \{h'_{\tau,i}\}$ and $\mathrm{HT}_\tau(\phi_i^{(u)}) = \{h_{\tau,i}\}$. There are integers w and w' such that for each $\tau : F^+ \hookrightarrow \overline{\mathbb{Q}}_l$ we have $\mathrm{HT}_\tau(\mu) = \{w'\}$ and $\mathrm{HT}_\tau(r_{l,\imath}(\chi)) = \{w + 1 - n\}$. Then

$$h'_{\tau,i} + h'_{\tau c,i} = w'$$

and

$$h_{\tau,i} + h_{\tau c,i} = w$$

for all τ and i.

Using Corollary A.2.3 we may choose a CM extension M/F such that

- M/F is cyclic of degree n,
- M is linearly disjoint from $\overline{F}^{\ker \overline{r}}(\zeta_l)$ over F,
- all primes of F above l or at which π ramifies split completely in M.

Choose a prime u_q of F above a rational prime q such that

- $q \neq l$ and q splits completely in M;
- r, μ, π and χ are unramified above q.

If $v|ql$ is a prime of F, we label the primes of M above v as $v_{M,1}, \ldots, v_{M,n}$ so that $(cv)_{M,i} = c(v_{M,i})$. Choose continuous characters

$$\theta, \theta' : G_M \longrightarrow \overline{\mathbb{Q}}_l^\times$$

such that

- the reductions $\overline{\theta}$ and $\overline{\theta}'$ are equal;
- $\theta\theta^c = r_{l,\imath}(\chi)\epsilon_l^{1-n}$ and $\theta'(\theta')^c = \mu$;
- θ and θ' are de Rham;
- if $\tau : M \hookrightarrow \overline{\mathbb{Q}}_l$ lies above a place $v_{M,i}|l$ of M, then $\mathrm{HT}_\tau(\theta) = \{h_{\tau|_F,i}\}$ and $\mathrm{HT}_\tau(\theta') = \{h'_{\tau|_F,i}\}$;
- θ and θ' are unramified at $u_{q,M,i}$ for $i > 1$, but q divides $\#\theta(I_{M_{u_{q,M,1}}})$ and $\#\theta'(I_{M_{u_{q,M,1}}})$.

(First use the first part of Lemma A.2.5 to choose (say) θ and then use the second part of Lemma A.2.5 to choose θ'.) Note that if $u|l$ is a place of F and if K/F_u is a finite extension over which θ and θ' become crystalline and $\overline{\theta} = \overline{\theta}'$ become trivial, then

$$(\mathrm{Ind}\,_{G_M}^{G_F}\theta)|_{G_K} \sim \phi_1^{(u)}|_{G_K} \oplus \cdots \oplus \phi_n^{(u)}|_{G_K}$$

and

$$(\mathrm{Ind}\,_{G_M}^{G_F}\theta')|_{G_K} \sim \psi_1^{(u)}|_{G_K} \oplus \cdots \oplus \psi_n^{(u)}|_{G_K}.$$

(To see this, note that both sides are residually trivial by the choice of K, and both sides are sums of crystalline characters with the same Hodge–Tate numbers. The result then follows from points (5) and (6) of Section 1.4.)

Now let F_1/F be a solvable CM extension such that

- $\theta|_{G_{F_1M}}$ and $\theta'|_{G_{F_1M}}$ are unramified away from l and crystalline at all primes above l,
- $\overline{\theta}|_{G_{F_1M}}$ is trivial at all primes above l,
- F_1 is linearly disjoint from $\overline{F}^{\ker(\overline{r}\otimes\mathrm{Ind}\,_{G_M}^{G_F}\overline{\theta})}M(\zeta_l)$ over F,
- MF_1/F_1 is unramified at all finite places.

(Use Lemma A.2.1.) Note that MF_1/F_1 is split completely above all places of F at which π is ramified.

Put

$$R := (r \otimes (\mathrm{Ind}\,_{G_M}^{G_F}\theta))|_{G_{F_1}},$$
$$R' := (r_{l,\imath}(\pi) \otimes (\mathrm{Ind}\,_{G_M}^{G_F}\theta'))|_{G_{F_1}}.$$

Note that we have the following facts:

- $\overline{R} \cong \overline{R}'$.
- $R^c \cong (r^\vee \otimes \mathrm{Ind}\,_{G_M}^{G_F}\theta^c \otimes \mu)|_{G_{F_1}} \cong R^\vee \otimes (\mu r_{l,\imath}(\chi)\epsilon_l^{1-n}\delta_{F_1/F_1^+})|_{G_{F_1}}$.
- $(R')^c \cong (R')^\vee \otimes (\mu r_{l,\imath}(\chi)\epsilon_l^{1-n}\delta_{F_1/F_1^+})|_{G_{F_1}}$.
- $\overline{R}$ is irreducible and $\overline{R}(G_{F_1(\zeta_l)})$ is adequate.

  (As $\overline{r}|_{G_{M(\zeta_l)}}$ is irreducible, we see that the restriction to $G_{M(\zeta_l)}$ of any constituent of $(\overline{r} \otimes \mathrm{Ind}\,_{G_M}^{G_F}\overline{\theta})|_{G_{F(\zeta_l)}}$ is a sum of $\overline{r}|_{G_{M(\zeta_l)}}\overline{\theta}^\tau|_{G_{M(\zeta_l)}}$ as τ runs over some subset of $\mathrm{Gal}\,(M/F)$. Looking at ramification above u_q we see that the $\overline{r}|_{G_{M(\zeta_l)}}\overline{\theta}^\tau|_{G_{M(\zeta_l)}}$ are permuted transitively by $\mathrm{Gal}\,(M/F)$ and hence $(\overline{r} \otimes \mathrm{Ind}\,_{G_M}^{G_F}\overline{\theta})|_{G_{F(\zeta_l)}}$ is irreducible. Since F_1 is linearly disjoint from $\overline{F}^{\ker(\overline{r}\otimes\mathrm{Ind}\,_{G_M}^{G_F}\overline{\theta})}M(\zeta_l)$ over F, we see that $\overline{R}|_{G_{F_1(\zeta_l)}}$ is irreducible. As $l \not| n$, every Sylow pro-l subgroup of $G_{F(\zeta_l)}$ is a subgroup of $G_{M(\zeta_l)}$. By Proposition 2.1.2, we see that $\overline{R}(G_{F_1(\zeta_l)})$ is adequate.)

- $(R', \mu r_{l,\imath}(\chi)\epsilon_l^{1-n}\delta_{F_1/F_1^+})$ is automorphic of level prime to l, say

$$(R', \mu r_{l,\imath}(\chi)\epsilon_l^{1-n}\delta_{F_1/F_1^+}) \cong (r_{l,\imath}(\pi_1), r_{l,\imath}(\chi_1)\epsilon_l^{1-n^2}).$$

  Moreover π_1 only ramifies at places of F_1 where $\mathrm{BC}_{F_1/F}(\pi)$ is ramified. (π_1 is constructed as the automorphic induction of

$$\mathrm{BC}_{F_1M/F}(\pi) \otimes (\phi'|\ |^{n(n-1)/2} \circ \det)$$

  to F_1, where $r_{l,\imath}(\phi') = \theta'|_{G_{F_1M}}$. Note that if $\sigma \in \mathrm{Gal}(F_1M/F_1)$, then $\overline{r}_{l,\imath}(\pi)|_{G_{F_1M}}\overline{\theta}'|^{\sigma}_{G_{F_1M}} \not\cong \overline{r}_{l,\imath}(\pi)|_{G_{F_1M}}\overline{\theta}'|_{G_{F_1M}}$, so that

$$(\mathrm{BC}_{F_1M/F}(\pi) \otimes (\phi' \circ \det))^{\sigma} \not\cong \mathrm{BC}_{F_1M/F}(\pi) \otimes (\phi' \circ \det)$$

  and π_1 is cuspidal.)
- For all places $u|l$ of F_1,

$$\begin{aligned} R|_{G_{F_{1,u}}} &\sim (\psi_1^{(u|_F)} \oplus \cdots \oplus \psi_n^{(u|_F)})|_{G_{F_{1,u}}} \otimes (\phi_1^{(u|_F)} \oplus \cdots \oplus \phi_n^{(u|_F)})|_{G_{F_{1,u}}} \\ &\sim R'|_{G_{F_{1,u}}}. \end{aligned}$$

- For all places $u \nmid l$ of F_1 we have $R'|_{G_{F_{1,u}}} \sim R_{G_{F_{1,u}}}$ (because we know that $r_{l,\imath}(\mathrm{BC}_{F_1/F}(\pi))|_{G_{F_{1,u}}} \sim r|_{G_{F_{1,u}}}$ and $(\mathrm{Ind}_{G_M}^{G_F}\theta')|_{G_{F_{1,u}}} \sim (\mathrm{Ind}_{G_M}^{G_F}\theta)|_{G_{F_{1,u}}}$, the latter because they are both unramified).

We now apply Theorem 2.3.1, with

- F the present F_1;
- l as in the present setting;
- n the present n^2;
- r the present R;
- μ the present $\mu r_{l,\imath}(\chi)\epsilon_l^{1-n}\delta_{F_1/F_1^+}$;
- (π, χ) the present (π_1, χ_1).

We conclude that $(R, (\mu r_{l,\imath}(\chi)\epsilon_l^{1-n}\delta_{F_1/F_1^+})|_{G_{(F_1M)^+}})$ is automorphic of level prime to l. By Lemma 2.2.4, $(r|_{G_{F_1M}} \otimes \theta|_{G_{F_1M}}, (\mu r_{l,\imath}(\chi)\epsilon_l^{1-n}\delta_{F_1/F_1^+})|_{G_{(F_1M)^+}})$ is automorphic of level prime to l. Using Lemma 2.2.1 we deduce that the polarized l-adic representation $(r|_{G_{F_1M}}, \mu|_{G_{(F_1M)^+}})$ is automorphic of level prime to l. Finally by Lemma 2.2.2, r is automorphic of level potentially prime to l, and hence potentially diagonalizably automorphic. □

4.2. *Automorphy lifting*: *the potentially diagonalizable case.* In this section we will prove our main automorphy lifting theorem. It generalizes Theorem 2.4.1 from the ordinary case to the potentially diagonalizable case. It is proved by combining Theorem 2.4.1 and Propositions 3.2.1 and 4.1.1.

THEOREM 4.2.1. *Let F be an imaginary* CM *field with maximal totally real subfield F^+ and let c denote the nontrivial element of* $\mathrm{Gal}(F/F^+)$. *Suppose that that l is an odd prime, and that (r, μ) is a regular algebraic, irreducible, n-dimensional, polarized representation of G_F. Let $\overline{r}$ denote the semi-simplification of the reduction of r and let d denote the maximal dimension of an irreducible subrepresentation of the restriction of $\overline{r}$ to the closed subgroup of G_F generated by all Sylow pro-l-subgroups. Suppose that (r, μ) enjoys the following properties*:

(1) $r|_{G_{F_v}}$ *is potentially diagonalizable* (*and so in particular potentially crystalline*) *for all $v|l$.*
(2) *The restriction $\overline{r}|_{G_{F(\zeta_l)}}$ is irreducible, $l \geq 2(d+1)$, and $\zeta_l \notin F$.*
(3) *$(\overline{r}, \overline{\mu})$ is either ordinarily automorphic or potentially diagonalizably automorphic.*

Then (r, μ) is potentially diagonalizably automorphic (*of level potentially prime to l*).

We remark that condition (1) of the theorem will be satisfied if, in particular, l is unramified in F and r is crystalline at all primes above l, and $\mathrm{HT}_\tau(r)$ is contained in an interval of the form $[a_\tau, a_\tau + l - 2]$ for all τ (the "Fontaine–Laffaille" case). We also remark that the reason we cannot immediately apply Proposition 4.1.1 to deduce this theorem is the last two parts of assumption 3 in Proposition 4.1.1 (i.e., roughly speaking r and $r_{l,\imath}(\pi)$ may have different level or $r \otimes r_{l,\imath}(\pi)$ may have repeated Hodge–Tate weights). To get round this problem we use Proposition 3.2.1 to create two ordinary intermediate lifts of $\overline{r}$, one r_1 with similar behavior ('level') to r, and one r_2 with similar behavior to $r_{l,\imath}(\pi)$. We also ensure that $r_1 \otimes r$ and $r_2 \otimes r_{l,\imath}(\pi)$ are Hodge–Tate regular. Theorem 2.4.1 tells us that if r_2 is automorphic, so is r_1. On the other hand Proposition 4.1.1 allows us to show that r_2 is automorphic and that if r_1 is automorphic, then so is r.

Proof. Using Lemma 2.2.2 it is enough to prove the theorem after replacing F by a soluble CM extension which is linearly disjoint from $\overline{F}^{\ker \overline{r}}(\zeta_l)$ over F. Thus we may suppose that

- F/F^+ is unramified at all finite primes;
- all primes dividing l and all primes at which π or r ramify are split over F^+;
- if u is a place of F above l, then F_u contains a primitive l^{th} root of unity, and $\overline{r}|_{G_{F_u}}$ and $\overline{r}_{l,\imath}(\pi)|_{G_{F_u}}$ are trivial.

Let S denote the set of primes of F^+ which divide l or above which r or π ramifies. For each prime $v \in S$ choose once and for all a prime $\tilde{v}$ of F above v.

Note that $\mu(c) = -1$ for all complex conjugations c and that we may extend $\overline{r} \cong \overline{r}_{l,\imath}(\pi)$ to a homomorphism

$$\tilde{\overline{r}}_{\overline{\mu}} : G_{F^+} \longrightarrow \mathcal{G}_n(\overline{\mathbb{F}}_l)$$

with multiplier $\overline{\mu}$.

Choose an integer m strictly greater than $|h - h'|$ for all h and h' Hodge–Tate numbers for r or $r_{l,\imath}(\pi)$. If $\tau : F \hookrightarrow \mathbb{C}$, set

$$H_\tau = \{0, m, \ldots, (n-1)m\}.$$

Note that if $u|l$, then both $\overline{r}|_{G_{F_u}}$ and $\overline{r}_{l,\imath}(\pi)|_{G_{F_u}}$ have ordinary and crystalline lifts $1 \oplus \epsilon_l^{-m} \oplus \cdots \oplus \epsilon_l^{(1-n)m}$ with τ-Hodge–Tate numbers $H_{\tau|_F}$ for each $\tau : F_u \hookrightarrow \overline{\mathbb{Q}}_l$. (It is here that we use the assumption that F_u contains a primitive l^{th} root of unity, and $\overline{r}|_{G_{F_u}}$ and $\overline{r}_{l,\imath}(\pi)|_{G_{F_u}}$ are trivial.) Applying Proposition 3.2.1 we see that there is a continuous homomorphism

$$r_1 : G_{F^+} \longrightarrow \mathcal{G}_n(\overline{\mathbb{Q}}_l)$$

lifting $\tilde{\overline{r}}_{\overline{\mu}}$ and such that

- $\nu \circ r_1 = \epsilon_l^{(1-n)m} \omega_l^{(n-1)m} \widetilde{\mu}$, where $\widetilde{\mu}$ denotes the Teichmüller lift of $\overline{\mu}$;
- if $u|l$, then $\breve{r}_1|_{G_{F_u}}$ is ordinary and crystalline with Hodge–Tate numbers $H_{\tau|_F}$ for each $\tau : F_u \hookrightarrow \overline{\mathbb{Q}}_l$;
- r_1 is unramified outside S;
- if $v \in S$ and $v \not| l$, then $r|_{G_{F_{\tilde{v}}}} \sim \breve{r}_1|_{G_{F_{\tilde{v}}}}$.

First we treat the case that $(\overline{r}, \overline{\mu})$ is ordinarily automorphic. In this case Theorem 2.4.1 tells us that $(\breve{r}_1, \epsilon_l^{(1-n)m} \omega_l^{(n-1)m} \widetilde{\mu})$ is automorphic of level prime to l. Then Proposition 4.1.1 tells us that (r, μ) is potentially diagonalizably automorphic, and we have completed the proof of the theorem in this case.

Secondly we treat the case that $(\overline{r}, \overline{\mu})$ is potentially diagonalizably automorphic, say $(\overline{r}, \overline{\mu}) = (r_{l,\imath}(\pi), r_{l,\imath}(\chi))$. In doing so we are free to make use of the first case, which we have already proved. Again applying Proposition 3.2.1 we find a continuous homomorphism

$$r_2 : G_{F^+} \longrightarrow \mathcal{G}_n(\overline{\mathbb{Q}}_l)$$

lifting $\tilde{\overline{r}}_{\overline{\mu}}$ and such that

- $\nu \circ r_2 = \epsilon_l^{(1-n)m} \omega_l^{(n-1)(m-1)} \widetilde{\chi}$, where $\widetilde{\chi}$ denotes the Teichmüller lift of $\overline{r}_{l,\imath}(\chi)$;
- if $u|l$, then $\breve{r}_2|_{G_{F_u}}$ is ordinary and crystalline with Hodge–Tate numbers $H_{\tau|_F}$ for each $\tau : F_u \hookrightarrow \overline{\mathbb{Q}}_l$;
- r_2 is unramified outside S;
- if $v \in S$ and $v \not| l$, then $r_{l,\imath}(\pi)|_{G_{F_{\tilde{v}}}} \sim \breve{r}_2|_{G_{F_{\tilde{v}}}}$.

By Proposition 4.1.1, $(r_2, \epsilon_l^{(1-n)m} \omega_l^{(n-1)(m-1)} \widetilde{\chi})$ is automorphic of level potentially prime to l, say $r_2 = r_{l,\imath}(\pi_2)$. As r_2 is ordinary and π_2 has level potentially

prime to l, we can conclude that π_2 is $\imath$-ordinary. The second case of our theorem now follows from the first case. □

4.3. *Lifting Galois representations* II. We now use the same idea that we used to prove Theorem 4.2.1 to prove a strengthening of Proposition 3.2.1. Let n be a positive integer and l an odd prime. Suppose that F is an imaginary CM field not containing ζ_l and with maximal totally real subfield F^+. Let S be a finite set of finite places of F^+ which split in F and suppose that S includes all places above l. For each $v \in S$ choose a prime $\tilde{v}$ of F above v.

Let $\mu : G_{F^+} \to \overline{\mathbb{Q}}_l^\times$ be an algebraic character unramified outside S such that $\mu(c_v) = -1$ for all $v|\infty$.

Also let

$$\overline{r} : G_{F^+} \longrightarrow \mathcal{G}_n(\overline{\mathbb{F}}_l)$$

be a continuous representation unramified outside S with $\nu \circ \overline{r} = \overline{\mu}$ and $\overline{r}^{-1}\mathcal{G}_n^0(\overline{\mathbb{F}}_l) = G_F$. For $v \in S$ let $\rho_v : G_{F_{\tilde{v}}} \to \mathrm{GL}_n(\mathcal{O}_{\overline{\mathbb{Q}}_l})$ denote a lift of $\breve{\overline{r}}|_{G_{F_{\tilde{v}}}}$.

THEOREM 4.3.1. *Keep the notation and assumptions already stated in this section. Also make the following assumptions*:

- *Suppose that* $\breve{\overline{r}}|_{G_{F(\zeta_l)}}$ *is irreducible. Also, writing* d *for the maximal dimension of an irreducible subrepresentation of the restriction of* $\overline{r}$ *to the closed subgroup of* G_F *generated by all Sylow pro-*l*-subgroups, suppose that* $l \geq 2(d+1)$.
- *If* $v|l$, *we suppose that* ρ_v *is potentially diagonalizable and that, for all* $\tau : F_{\tilde{v}} \hookrightarrow \overline{\mathbb{Q}}_l$, *the multiset* $\mathrm{HT}_\tau(\rho_v)$ *consists of* n *distinct integers.*

Then there is a lift

$$r : G_{F^+} \longrightarrow \mathcal{G}_n(\mathcal{O}_{\overline{\mathbb{Q}}_l})$$

of $\overline{r}$ *such that*

(1) $\nu \circ r = \mu$;

(2) *if* $v \in S$, *then* $\breve{r}|_{G_{F_{\tilde{v}}}} \sim \rho_v$;

(3) r *is unramified outside* S.

Proof. We may suppose that for $v \in S$ with $v \not| l$ the representation ρ_v is robustly smooth (see Lemma 1.3.2) and hence lies on a unique component $\mathcal{C}_v$ of $R^{\square}_{\breve{\overline{r}}|_{G_{F_{\tilde{v}}}}} \otimes \overline{\mathbb{Q}}_l$. If $v|l$ is a place of F^+, then choose a finite extension $K_v/F_{\tilde{v}}$ over which ρ_v becomes crystalline, and let $\mathcal{C}_v$ denote the unique component of $R^{\square}_{\breve{\overline{r}}|_{G_{F_{\tilde{v}}}},\{\mathrm{HT}_\tau(\rho_v)\},K_v-\mathrm{cris}} \otimes \overline{\mathbb{Q}}_l$ on which ρ_v lies. Let $\widetilde{\mu}$ denote the Teichmüller lift of $\overline{\mu}$. Choose a positive integer m which is greater than one plus the difference of any two Hodge–Tate numbers of ρ_v for every $v|l$.

Choose (using Lemma A.2.1) a finite, soluble, Galois, CM extension F_1/F which is linearly disjoint from $\overline{F}^{\ker \overline{r}}(\zeta_l)$ over F such that

- for all u lying above S we have $\overline{r}(G_{F_{1,u}}) = \{1\}$;
- for all $u|l$ we have $\zeta_l \in F_{1,u}$;
- $\mu|_{G_{F_1^+}}$ is crystalline above l;
- if $u|\tilde{v}|l$ with $v \in S$, then $\rho_v|_{G_{F_{1,u}}}$ is crystalline and $\rho_v|_{G_{F_{1,u}}} \sim \psi_1^{(u)} \oplus \cdots \oplus \psi_n^{(u)}$ with each $\psi_i^{(u)}$ a crystalline character.

If $u|\tilde{v}|l$ with $v \in S$, then for $i = 1, \ldots, n$, we define $\psi_i^{(cu)} : G_{F_{1,cu}} \to \overline{\mathbb{Q}}_l^\times$ by $(\psi_i^{(cu)})^c \psi_i^{(u)} = \mu|_{G_{F_{1,u}}}$.

By Proposition 3.3.1 there is a finite, Galois, CM extension F_2/F_1 linearly disjoint from $F_1\overline{F}^{\ker \overline{r}}(\zeta_l)$ over F_1 and a regular algebraic, cuspidal, polarized automorphic representation (π_2, χ_2) of $\mathrm{GL}_n(\mathbb{A}_{F_2})$ such that

- $\overline{r}_{l,\imath}(\pi_2) \cong \breve{\overline{r}}|_{G_{F_2}}$;
- $r_{l,\imath}(\chi_2) = \widetilde{\mu}|_{G_{F_2^+}} \omega_l^{(n-1)m} \epsilon_l^{(1-n)(m-1)}$;
- π_2 is $\imath$-ordinary and unramified above l;
- if $\tau : F_2 \hookrightarrow \overline{\mathbb{Q}}_l$, then $\mathrm{HT}_\tau(r_{l,\imath}(\pi_2)) = \{0, m, 2m, \ldots, (n-1)m\}$;
- π_2 is unramified outside S;
- if $v \not| l$ is in S and if u is a prime of F_2 above $\tilde{v}$, then $r_{l,\imath}(\pi_2)|_{G_{F_{2,u}}} \sim \rho_v|_{G_{F_{2,u}}}$.

In particular if $u|l$ is a place of F_2, then

$$r_{l,\imath}(\pi_2)|_{G_{F_{2,u}}} \sim 1 \oplus \epsilon_l^{-m} \oplus \cdots \oplus \epsilon_l^{(1-n)m}.$$

Choose (using Corollary A.2.3) a CM extension M/F_2 such that

- M/F_2 is cyclic of degree n,
- M is linearly disjoint from $\overline{F}^{\ker \overline{r}}(\zeta_l)$ over F,
- all primes of F_2 above l split completely in M.

Choose a prime u_q of F_2 above a rational prime q such that

- $q \neq l$ and q splits completely in M,
- $\overline{r}$ is unramified above q.

If $v|ql$ is a prime of F_2, we label the primes of M above v as $v_{M,1}, \ldots, v_{M,n}$ so that $(cv)_{M,i} = c(v_{M,i})$. Choose continuous characters

$$\theta, \theta' : G_M \longrightarrow \overline{\mathbb{Q}}_l^\times$$

such that

- the reductions $\overline{\theta}$ and $\overline{\theta}'$ are equal;
- $\theta\theta^c = r_{l,\imath}(\chi_2)\epsilon_l^{1-n}$ and $\theta'(\theta')^c = \mu$;
- θ and θ' are de Rham;
- if $\tau : M \hookrightarrow \overline{\mathbb{Q}}_l$ lies above a place $v_{M,i}|l$ of M, then $\mathrm{HT}_\tau(\theta) = \{(i-1)m\}$ and $\mathrm{HT}_\tau(\theta') = \mathrm{HT}_{\tau|_{F_1}}(\psi_i^{(v_{M,i}|_{F_1})})$;

- θ and θ' are unramified at $u_{q,M,i}$ for $i > 1$, but q divides $\#\theta(I_{M_{u_{q,M,1}}})$ and $\#\theta'(I_{M_{u_{q,M,1}}})$.

(First choose (say) θ using the first part of Lemma A.2.5, then choose θ' using the second part of Lemma A.2.5.)

Note the following:

- If $u|l$ is a place of F_2 and if $K/F_{2,u}$ is a finite extension over which θ and θ' become crystalline and $\overline{\theta} = \overline{\theta}'$ become trivial, then
$$(\mathrm{Ind}\,_{G_M}^{G_{F_2}}\theta)|_{G_K} \sim 1 \oplus \epsilon_l^{-m} \oplus \cdots \oplus \epsilon_l^{(1-n)m}$$
and
$$(\mathrm{Ind}\,_{G_M}^{G_{F_2}}\theta')|_{G_K} \sim \psi_1^{(u|_{F_1})}|_{G_K} \oplus \cdots \oplus \psi_n^{(u|_{F_1})}|_{G_K}.$$
(To see this, note that both sides are residually trivial by the choice of K, and both sides are sums of crystalline characters with the same Hodge–Tate numbers. The result then follows from points (5) and (6) of Section 1.4.)
- $(\mathrm{Ind}\,_{G_M}^{G_{F_2}}\theta)^c \cong (\mathrm{Ind}\,_{G_M}^{G_{F_2}}\theta)^\vee \otimes r_{l,\imath}(\chi_2)\epsilon_l^{1-n}$ and $(\mathrm{Ind}\,_{G_M}^{G_{F_2}}\theta')^c \cong (\mathrm{Ind}\,_{G_M}^{G_{F_2}}\theta')^\vee \otimes \mu$.
- The representation
$$\breve{\overline{r}}|_{G_{F_2(\zeta_l)}} \otimes (\mathrm{Ind}\,_{G_M}^{G_{F_2}}\overline{\theta})|_{G_{F_2(\zeta_l)}} \cong \overline{r}_{l,\imath}(\pi_2)|_{G_{F_2(\zeta_l)}} \otimes (\mathrm{Ind}\,_{G_M}^{G_{F_2}}\overline{\theta}')|_{G_{F_2(\zeta_l)}}$$
is irreducible, and hence by Proposition 2.1.2
$$(\breve{\overline{r}}|_{G_{F_2}} \otimes (\mathrm{Ind}\,_{G_M}^{G_{F_2}}\overline{\theta}))(G_{F_2(\zeta_l)})$$
is adequate.

  (As $\breve{\overline{r}}|_{G_{M(\zeta_l)}}$ is irreducible, we see that the restriction to $G_{M(\zeta_l)}$ of any constituent of $(\breve{\overline{r}}|_{G_{F_2}} \otimes \mathrm{Ind}\,_{G_M}^{G_{F_2}}\overline{\theta})|_{G_{F_2(\zeta_l)}}$ is a sum of $\breve{\overline{r}}|_{G_{M(\zeta_l)}}\overline{\theta}^\tau|_{G_{M(\zeta_l)}}$ as τ runs over some subset of $\mathrm{Gal}\,(M/F_2)$. Looking at ramification above u_q we see that the $\breve{\overline{r}}|_{G_{M(\zeta_l)}}\overline{\theta}^\tau|_{G_{M(\zeta_l)}}$ are permuted transitively by $\mathrm{Gal}\,(M/F_2)$ and hence $(\breve{\overline{r}}|_{G_{F_2}} \otimes \mathrm{Ind}\,_{G_M}^{G_{F_2}}\overline{\theta})|_{G_{F_2(\zeta_l)}}$ is irreducible.)

Let F_3/F_2 be a finite, soluble, Galois, CM extension linearly disjoint from $\overline{F}_2^{\ker \mathrm{Ind}\,_{G_M}^{G_{F_2}}\overline{\theta}}\overline{F}^{\ker \overline{r}}(\zeta_l)$ over F_2 such that

- $\theta|_{G_{F_3M}}$ and $\theta'|_{G_{F_3M}}$ are crystalline above l and unramified away from l;
- MF_3/F_3 is unramified everywhere.

(Use Lemma A.2.1.)

Then there is a regular algebraic, cuspidal, polarized automorphic representation (π_3, χ_3) of $\mathrm{GL}_{n^2}(\mathbb{A}_{F_3})$ such that

- $r_{l,\imath}(\pi_3) \cong (r_{l,\imath}(\pi_2) \otimes \mathrm{Ind}\,_{G_M}^{G_{F_2}}\theta')|_{G_{F_3}}$;
- $r_{l,\imath}(\chi_3) = \mu r_{l,\imath}(\chi_2)\epsilon_l^{n(n-1)}\delta_{F_3/F_3^+}$;
- π_3 is unramified above l and outside S.

(The representation π_3 is the automorphic induction of $(\pi_2)_{MF_3} \otimes (\phi'| \ |^{n(n-1)/2} \circ \det)$ to F_3, where $r_{l,\imath}(\phi') = \theta'|_{G_{F_3M}}$. The first two properties are clear. The third property follows by the choice of F_3 and the fact that π_2 is unramified above l and outside S.)

Let $\widetilde{S}$ denote the set of $\tilde{v}$ as v runs over S and let S_3 denote the primes of F_3^+ above S and $\widetilde{S}_3$ the primes of F_3 above $\widetilde{S}$. If $v \in S_3$, let $\tilde{v}$ denote the element of $\widetilde{S}_3$ lying above it. For $v \in S_3$ with $v \not| l$ (resp. $v|l$) let $\mathcal{C}_{3,v}$ denote the unique component of $R^{\square}_{\overline{r}_{l,\imath}(\pi_3)|_{G_{F_{3,\tilde{v}}}}} \otimes \overline{\mathbb{Q}}_l$ (resp. $R^{\square}_{\overline{r}_{l,\imath}(\pi_3)|_{G_{F_{3,\tilde{v}}}},\{\mathrm{HT}_\tau(r_{l,\imath}(\pi_3)|_{G_{F_{3,\tilde{v}}}})\},\mathrm{cris}} \otimes \overline{\mathbb{Q}}_l$) containing $r_{l,\imath}(\pi_3)|_{G_{F_{3,\tilde{v}}}}$. Choose a finite extension $L/\mathbb{Q}_l$ in $\overline{\mathbb{Q}}_l$ such that

- L contains the image of each embedding $F_3 \hookrightarrow \overline{\mathbb{Q}}_l$;
- L contains the image of μ and of θ;
- $r_{l,\imath}(\pi_3)$ is defined over L;
- each of the components $\mathcal{C}_v$ for $v \in S$ and $\mathcal{C}_{3,v}$ for $v \in S_3$ is defined over L.

Set

$$s = \mathrm{Ind}^{G_{F_2^+}, G_{F_2}, r_{l,\imath}(\chi_2)\epsilon_l^{1-n}}_{G_{M^+}, G_M} (\theta, r_{l,\imath}(\chi_2)\epsilon_l^{1-n}) : G_{F_2^+} \longrightarrow \mathcal{G}_n(\mathcal{O}_L)$$

in the notation of Section 1.1 of this paper and Section 2.1 of [CHT08]. Thus $\nu \circ s = r_{l,\imath}(\chi_2)\epsilon_l^{1-n}$. For $v \in S$ (resp. $v \in S_3$) let $\mathcal{D}_v$ (resp. $\mathcal{D}_{3,v}$) denote the deformation problem for $\overline{r}|_{G_{F_{\tilde{v}}}}$ (resp. $\overline{r}_{l,\imath}(\pi_3)|_{G_{F_{3,\tilde{v}}}}$) over $\mathcal{O}_L$ corresponding to $\mathcal{C}_v$ (resp. $\mathcal{C}_{3,v}$). Also let

$$\mathcal{S} = (F/F^+, S, \widetilde{S}, \mathcal{O}_L, \overline{r}, \mu, \{\mathcal{D}_v\})$$

and

$$\mathcal{S}_3 = (F_3/F_3^+, S_3, \widetilde{S}_3, \mathcal{O}_L, \overline{r}_{l,\imath}(\pi_3), \mu r_{l,\imath}(\chi_2)\epsilon_l^{1-n}\delta_{F_3^+/F_3}, \{\mathcal{D}_{3,v}\}).$$

We next check that if $u \in S_3$, then $\check{r}^{\mathrm{univ}}_{\mathcal{S}}|_{G_{F_{3,\tilde{u}}}} \otimes (\mathrm{Ind}^{G_{F_2}}_{G_M}\theta)|_{G_{F_{3,\tilde{u}}}} \in \mathcal{D}_{3,u}$. To this end, let $v = u|_{F^+}$ and let $\rho^{\square}_{v,\mathcal{C}_v}$ denote the universal lift of $\overline{r}|_{G_{F_{\tilde{v}}}}$ to $R^{\square}_{\mathcal{O}_L,\overline{r}|_{G_{F_{\tilde{v}}}},\mathcal{C}_v}$. It suffices to show that $\rho^{\square}_{v,\mathcal{C}_v}|_{G_{F_{3,\tilde{u}}}} \otimes (\mathrm{Ind}^{G_{F_2}}_{G_M}\theta)|_{G_{F_{3,\tilde{u}}}} \in \mathcal{D}_{3,u}$. For this, it suffices to show if $\rho : G_{F_{\tilde{v}}} \to \mathrm{GL}_n(\mathcal{O}_{\overline{\mathbb{Q}}_l})$ is a lift of $\overline{r}|_{G_{F_{\tilde{v}}}}$ lying on $\mathcal{C}_v$, then $\rho|_{G_{F_{3,\tilde{u}}}} \otimes (\mathrm{Ind}^{G_{F_2}}_{G_M}\theta)|_{G_{F_{3,\tilde{u}}}}$ lies on $\mathcal{C}_{3,u}$. If $u|l$, then we have $\rho|_{G_{F_{3,\tilde{u}}}} \sim (\mathrm{Ind}^{G_{F_2}}_{G_M}\theta')|_{G_{F_{3,\tilde{u}}}}$ and $(\mathrm{Ind}^{G_{F_2}}_{G_M}\theta)|_{G_{F_{3,\tilde{u}}}} \sim r_{l,\imath}(\pi_2)|_{G_{F_{3,\tilde{u}}}}$ and hence

$$\rho|_{G_{F_{3,\tilde{u}}}} \otimes (\mathrm{Ind}^{G_{F_2}}_{G_M}\theta)|_{G_{F_{3,\tilde{u}}}} \sim (r_{l,\imath}(\pi_2) \otimes \mathrm{Ind}^{G_{F_2}}_{G_M}\theta')|_{G_{F_{3,\tilde{u}}}} \cong r_{l,\imath}(\pi_3)|_{G_{F_{3,\tilde{u}}}}.$$

If $u \not| l$, note that $\rho|_{G_{F_{3,\tilde{u}}}} \sim r_{l,\imath}(\pi_2)|_{G_{F_{3,\tilde{u}}}}$. (Since ρ_v is robustly smooth, we have $\rho_v|_{G_{F_{3,\tilde{u}}}} \rightsquigarrow \rho|_{G_{F_{3,\tilde{u}}}}$ and $\rho_v|_{G_{F_{3,\tilde{u}}}} \rightsquigarrow r_{l,\imath}(\pi_2)|_{G_{F_{3,\tilde{u}}}}$.) By the choice of F_3 we have $(\mathrm{Ind}^{G_{F_2}}_{G_M}\theta)|_{G_{F_{3,\tilde{u}}}} \sim (\mathrm{Ind}^{G_{F_2}}_{G_M}\theta')|_{G_{F_{3,\tilde{u}}}}$. Hence

$$\rho|_{G_{F_{3,\tilde{u}}}} \otimes (\mathrm{Ind}^{G_{F_2}}_{G_M}\theta)|_{G_{F_{3,\tilde{u}}}} \sim (r_{l,\imath}(\pi_2) \otimes \mathrm{Ind}^{G_{F_2}}_{G_M}\theta')|_{G_{F_{3,\tilde{u}}}} \cong r_{l,\imath}(\pi_3)|_{G_{F_{3,\tilde{u}}}}$$

and we are done.

We deduce that there is a natural map

$$R^{\mathrm{univ}}_{\mathcal{S}_3} \longrightarrow R^{\mathrm{univ}}_{\mathcal{S}}$$

induced by $r^{\mathrm{univ}}_{\mathcal{S}}|_{G_{F_3^+}} \otimes s|_{G_{F_3^+}}$. It follows from Lemma 1.2.3 that this map makes $R^{\mathrm{univ}}_{\mathcal{S}}$ a finite $R^{\mathrm{univ}}_{\mathcal{S}_3}$-module. By Theorem 2.3.2, $R^{\mathrm{univ}}_{\mathcal{S}_3}$ is a finite $\mathcal{O}_L$-module, and hence $R^{\mathrm{univ}}_{\mathcal{S}}$ is a finite $\mathcal{O}_L$-module. On the other hand by Proposition 1.5.1, $R^{\mathrm{univ}}_{\mathcal{S}}$ has Krull dimension at least 1. Hence $\operatorname{Spec} R^{\mathrm{univ}}_{\mathcal{S}}$ has a $\overline{\mathbb{Q}}_l$-point. This point gives rise to the desired lifting of $\overline{r}$. □

4.4. *Change of weight and level.* In this section we combine Theorems 4.2.1 and 4.3.1 to obtain a general "change of weight and level" result in the potentially diagonalizable case.

THEOREM 4.4.1. *Let F be an imaginary* CM *field with maximal totally real subfield F^+. Let $n \in \mathbb{Z}_{\geq 1}$ be an integer, and let $l > 2(n+1)$ be an odd prime, such that $\zeta_l \notin F$ and all primes of F^+ above l split in F. Let S be a finite set of finite places of F^+, including all places above l, such that each place in S splits in F. For each place $v \in S$ choose a place $\tilde{v}$ of F lying over v.*

Let μ be an algebraic character of G_{F^+} and let $\overline{r} : G_F \to \mathrm{GL}_n(\overline{\mathbb{F}}_l)$ be a continuous representation such that

- *$(\overline{r}, \overline{\mu})$ is a polarized mod l representation unramified outside S, which either we suppose is ordinarily automorphic or we suppose is potentially diagonalizably automorphic;*
- *$\overline{r}|_{G_{F(\zeta_l)}}$ is irreducible.*

For $v \in S$ let $\rho_v : G_{F_{\tilde{v}}} \to \mathrm{GL}_n(\mathcal{O}_{\overline{\mathbb{Q}}_l})$ be a lift of $\bar{r}_{l,\imath}(\pi)|_{G_{F_{\tilde{v}}}}$. If $v|l$, assume further that ρ_v is potentially diagonalizable, and that for all $\tau : F_{\tilde{v}} \hookrightarrow \overline{\mathbb{Q}}_l$, $\mathrm{HT}_\tau(\rho_v)$ consists of n distinct integers.

Then there is a regular algebraic, cuspidal, polarized automorphic representation (π, χ) of $\mathrm{GL}_n(\mathbb{A}_F)$ such that

(1) $\overline{r}_{l,\imath}(\pi) \cong \overline{r}$;
(2) $r_{l,\imath}(\chi)\epsilon_l^{1-n} = \mu$;
(3) *π has level potentially prime to l;*
(4) *π is unramified outside S;*
(5) *for $v \in S$ we have $\rho_v \sim r_{l,\imath}(\pi)|_{G_{F_{\tilde{v}}}}$.*

Proof. By Theorem 4.3.1 there is a continuous homomorphism

$$r : G_{F^+} \to \mathcal{G}_n(\mathcal{O}_{\overline{\mathbb{Q}}_l})$$

such that

- $\breve{\overline{r}} \cong \overline{r}$;
- r is unramified outside S;
- $\nu \circ r = \mu$;

- if $v|l$, then $\check{r}|_{G_{F_{\tilde{v}}}}$ is potentially diagonalizable;
- if $v \in S$, then $\check{r}|_{G_{F_{\tilde{v}}}} \sim \rho_v$.

By Theorem 4.2.1 $(\check{r}, \mu)$ is automorphic of level potentially prime to l and our present theorem follows (using local-global compatibility to establish point (4)). □

4.5. *Potential automorphy* II. We can now turn to our main potential automorphy theorem for (finite collections of) single l-adic representations. We will treat the case of compatible systems in the next section.

THEOREM 4.5.1. *Suppose that we are in the following situation*:

(a) *Let F/F_0 be a finite, Galois extension of imaginary* CM *fields and let F^+ and F_0^+ denote their maximal totally real subfields.*
(b) *Let $\mathcal{I}$ be a finite set.*
(c) *For each $i \in \mathcal{I}$ let n_i and d_i be positive integers and l_i be an odd rational prime such that $l_i \geq 2(d_i + 1)$ and $\zeta_{l_i} \notin F$. Also choose $\imath_i : \overline{\mathbb{Q}}_{l_i} \xrightarrow{\sim} \mathbb{C}$ for each $i \in \mathcal{I}$.*
(d) *For each $i \in \mathcal{I}$ let (r_i, μ_i) be a* **totally odd, regular algebraic,** *n_i-dimensional,* **polarized** *l_i-adic representation of G_F such that d_i is the maximum dimension of an irreducible constituent of the restriction of $\overline{r}_i$ to the closed subgroup of G_F generated by all Sylow pro-l_i-subgroups.*
(e) *Let $F^{(\mathrm{avoid})}/F$ be a finite Galois extension.*

Suppose moreover that the following conditions are satisfied for every $i \in \mathcal{I}$:

(1) (Potential diagonalizability at primes above l_i). *r_i is potentially diagonalizable (and hence potentially crystalline) at each prime v of F^+ above l_i.*
(2) (Irreducibility). *$\bar{r}_i|_{G_{F(\zeta_{l_i})}}$ is irreducible.*

Then we can find a finite CM *extension F'/F and for each $i \in \mathcal{I}$ a regular algebraic, cuspidal, polarized automorphic representation (π_i, χ_i) of $\mathrm{GL}_{n_i}(\mathbb{A}_{F'})$ such that*

(i) *F'/F_0 is Galois*;
(ii) *F' is linearly disjoint from $F^{(\mathrm{avoid})}$ over F*;
(iii) *π_i is unramified above l_i*;
(iv) $(r_{l_i,\imath_i}(\pi_i), r_{l_i,\imath_i}(\chi_i)\epsilon_{l_i}^{1-n_i}) \cong (r_i|_{G_{F'}}, \mu_i|_{G_{(F')^+}})$.

We remark that by Lemma 1.4.2 the hypothesis of potential diagonalizability will hold if l_i is unramified in F^+, and r_i is crystalline at all primes $v|l_i$, and $\mathrm{HT}_\tau(r_i)$ is contained in an interval of the form $[a_\tau, a_\tau + l - 2]$ for all τ (the "Fontaine–Laffaille" case).

Proof. By Proposition 3.3.1 there is a finite CM extension F'/F and regular algebraic, cuspidal, polarized automorphic representations (π_i', χ_i') of $\mathrm{GL}_{n_i}(\mathbb{A}_{F'})$ such that

- F'/F_0 is Galois;
- F' is linearly disjoint from $\overline{F}^{\bigcap_i \ker \overline{r}_i} F^{(\mathrm{avoid})}(\zeta_{\prod_i l_i})$ over F;
- $\overline{r}_{l_i,\imath_i}(\pi_i') \cong \overline{r}_i|_{G_{F'}}$;
- $r_{l_i,\imath_i}(\chi_i')\epsilon_{l_i}^{1-n_i} = \mu_i|_{G_{(F')^+}}$;
- π_i' is unramified above l;
- π_i' is $\imath_i$-ordinary.

Then the current theorem follows from Theorem 4.2.1. □

We can immediately deduce a version over totally real fields. For instance we have the following.

COROLLARY 4.5.2. *Suppose F^+ is a totally real field and $n \in \mathbb{Z}_{\geq 1}$. Suppose that $l \geq 2(n+1)$ is a rational prime. Suppose also that (r, μ) is a* **totally odd, regular algebraic**, *n-dimensional*, **polarized** *l-adic representation of G_{F^+}. Let $\overline{r}$ denote the semi-simplification of the reduction of r and suppose that the following conditions hold*:

(1) (Potential diagonalizability and regularity at primes above l). *r is potentially diagonalizable* (*and hence potentially crystalline*) *at each prime v of F^+ above l.*
(2) (Irreducibility). *$\overline{r}|_{G_{F^+(\zeta_l)}}$ is irreducible.*

Then there is a Galois totally real extension $F^{+,\prime}/F^+$ such that $(r|_{G_{F^{+,\prime}}}, \mu|_{G_{F^{+,\prime}}})$ is automorphic of level prime to l.

Proof. Choose F/F^+ a totally imaginary quadratic extension in which all the places lying over l split completely and which is linearly disjoint from $(\overline{F}^+)^{\ker \mathrm{ad}\, \overline{r}}(\zeta_l)$ over F^+. The representation $r|_{G_F}$ satisfies the hypotheses of Theorem 4.5.1, so that there is a finite Galois CM extension F'/F such that $(r|_{G_{F'}}, \mu|_{G_{F',+}})$ is automorphic of level prime to l. By Lemma 2.2.2, $(r|_{G_{F',+}}, \mu|_{G_{F',+}})$ is also automorphic of level prime to l, as required. □

Combining Theorem 4.5.1 with Theorem 4.3.1 we get a potential automorphy theorem for mod l Galois representations which strengthens Proposition 3.3.1.

COROLLARY 4.5.3. *Suppose that we are in the following situation*:

(a) *Let F/F_0 be a finite, Galois extension of imaginary* CM *fields, and let F^+ and F_0^+ denote their maximal totally real subfields. Choose a complex conjugation $c \in G_{F^+}$.*
(b) *Let $\mathcal{I}$ be a finite set.*
(c) *For each $i \in \mathcal{I}$ let n_i and d_i be positive integers and $l_i \geq 2(d_i+1)$ be an odd rational prime such that $\zeta_{l_i} \notin F$. Also choose $\imath_i : \overline{\mathbb{Q}}_{l_i} \xrightarrow{\sim} \mathbb{C}$ for each $i \in \mathcal{I}$.*

(d) *For each* $i \in \mathcal{I}$ *let* $\mu_i : G_{F^+} \to \overline{\mathbb{Q}}_{l_i}^\times$ *be a continuous, totally odd, de Rham character.*

(e) *For each* $i \in \mathcal{I}$ *let* $\overline{r}_i : G_F \to \mathrm{GL}_{n_i}(\overline{\mathbb{F}}_{l_i})$ *be an irreducible continuous representation so that* d_i *is the maximal dimension of an irreducible constituent of the restriction of* $\overline{r}_i$ *to the closed subgroup of* G_F *generated by all Sylow pro-*l_i*-subgroups. Suppose also that* $(\overline{r}_i, \overline{\mu}_i)$ *is a polarized mod* l_i *representation and that* $\overline{r}_i|_{G_{F(\zeta_{l_i})}}$ *is irreducible.*

(f) *Let* S *denote a finite* $\mathrm{Gal}(F/F^+)$*-invariant set of primes of* F *including all those dividing* $\prod_i l_i$ *and all those at which some* $\overline{r}_i$ *ramifies.*

(g) *For each* $i \in \mathcal{I}$ *and* $v \in S$ *let* $\rho_{i,v} : G_{F_v} \to \mathrm{GL}_{n_i}(\mathcal{O}_{\overline{\mathbb{Q}}_{l_i}})$ *denote a lift of* $\overline{r}_i|_{G_{F_v}}$ *such that* $\rho_{i,cv}^c \cong \mu_i|_{G_{F_v}} \rho_{i,v}^\vee$. *If* $v|l_i$, *further assume that* $\rho_{i,v}$ *is potentially diagonalizable and that for each* $\tau : F_v \hookrightarrow \overline{\mathbb{Q}}_{l_i}$ *the set* $\mathrm{HT}_\tau(\rho_{v,i})$ *has* n *distinct elements.*

(h) *Let* $F^{(\mathrm{avoid})}/F$ *be a finite Galois extension.*

Then we can find a finite CM *extension* F'/F *and for each* $i \in \mathcal{I}$ *a regular algebraic, cuspidal, polarized automorphic representation* (π_i, χ_i) *of* $\mathrm{GL}_{n_i}(\mathbb{A}_{F'})$ *such that*

(1) F'/F_0 *is Galois*;

(2) F' *is linearly disjoint from* $F^{(\mathrm{avoid})}$ *over* F;

(3) $\overline{r}_{l_i,\imath_i}(\pi_i) \cong \overline{r}_i|_{G_{F'}}$;

(4) $r_{l_i,\imath_i}(\chi_i)\epsilon_{l_i}^{1-n_i} = \mu_i|_{G_{(F')^+}}$;

(5) π_i *has level potentially prime to* l_i;

(6) *if* u *is a prime of* F' *not lying above a prime in* S, *then* $\pi_{i,u}$ *is unramified*;

(7) *if* u *is a prime of* F' *lying above an element* $v \in S$, *then* $r_{l_i,\imath_i}(\pi_i)|_{G_{F'_u}} \sim \rho_v|_{G_{F'_u}}$.

Proof. Note that $(\overline{r}_i, \overline{\mu}_i)$ corresponds to a continuous homomorphism $\tilde{\overline{r}}_{i,\overline{\mu}_i} : G_{F^+} \to \mathcal{G}_{n_i}(\overline{\mathbb{F}}_{l_i})$ with $\nu \circ \tilde{\overline{r}}_i = \overline{\mu}_i$. (See Section 1.1.) As in the proof of Proposition 3.3.1, we may reduce to the case where all elements of S are split over F^+. Then by Theorem 4.3.1, we see that for each $i \in \mathcal{I}$ there exists a lift

$$r_i : G_F \to \mathrm{GL}_{n_i}(\mathcal{O}_{\overline{\mathbb{Q}}_l})$$

of $\overline{r}_i$ such that

- $r_i^c \cong r_i^\vee \mu_i|_{G_F}$;
- if $v \in S$, then $r_i|_{G_{F_v}} \sim \rho_{i,v}$;
- r_i is unramified outside S.

The result now follows from Theorem 4.5.1. □

5. **Compatible systems**

5.1. *Compatible systems*: *definitions.* Let F denote a number field. By a *rank n weakly compatible system of l-adic representations $\mathcal{R}$ of G_F defined over M* we shall mean a 5-tuple

$$(M, S, \{Q_v(X)\}, \{r_\lambda\}, \{H_\tau\})$$

where

(1) M is a number field;
(2) S is a finite set of primes of F;
(3) for each prime $v \notin S$ of F, $Q_v(X)$ is a monic degree n polynomial in $M[X]$;
(4) for each prime λ of M (with residue characteristic l say)

$$r_\lambda : G_F \longrightarrow \mathrm{GL}_n(\overline{M}_\lambda)$$

is a continuous, semi-simple, representation such that
- if $v \notin S$ and $v \nmid l$ is a prime of F, then r_λ is unramified at v and $r_\lambda(\mathrm{Frob}_v)$ has characteristic polynomial $Q_v(X)$,
- while if $v|l$ then $r_\lambda|_{G_{F_v}}$ is de Rham and in the case $v \notin S$ crystalline;

(5) for $\tau : F \hookrightarrow \overline{M}$, H_τ is a multiset of n integers such that for any $\overline{M} \hookrightarrow \overline{M}_\lambda$ over M we have $\mathrm{HT}_\tau(r_\lambda) = H_\tau$.

We will refer to a rank 1 weakly compatible system of representations as a weakly compatible system of characters.

We make the following subsidiary definitions. We define the usual linear algebra and group theoretic operations on weakly compatible systems by applying the corresponding operation to each r_λ. For instance

$$\mathcal{R}^\vee = (M, S, \{X^n Q_v(0)^{-1} Q_v(X^{-1})\}, \{r_\lambda^\vee\}, \{-H_\tau\}),$$

where $-H_\tau = \{-h : h \in H_\tau\}$.

We will call $\mathcal{R}$ *regular* if for each $\tau : F \hookrightarrow \overline{M}$ every element of H_τ has multiplicity 1.

We will call $\mathcal{R}$ *extremely regular* if it is regular and for some $\tau : F \hookrightarrow \overline{M}$ the multiset H_τ has the following property: if H and H' are subsets of H_τ of the same cardinality and if $\sum_{h \in H} h = \sum_{h \in H'} h$, then $H = H'$.

If F is totally real and if $n = 1$, then we will call $\mathcal{R}$ *totally odd* (resp. *totally even*) if for some place λ of M we have $r_\lambda(c_v) = -1$ (resp. 1) for all infinite places v of F. In this case this will also be true for all places λ of M.

If F is CM and if $\mathcal{M} = (M, S_{F^+}, \{X - \alpha_v\}, \{\mu_\lambda\}, \{w\})$ is a weakly compatible system of characters of G_{F^+}, then we will call $(\mathcal{R}, \mathcal{M})$ a *polarized* (resp. *totally odd*, *polarized*) weakly compatible system if for all primes λ of M the pair (r_λ, μ_λ) is a polarized (resp. totally odd polarized) l-adic representation. (Here S_{F^+} denotes the set of places of F^+ lying below an element of S.) We will call $\mathcal{R}$ *polarizable* (resp. *totally odd*, *polarizable*) if there exists a $\mathcal{M}$ such that $(\mathcal{R}, \mathcal{M})$ is a polarized (resp. totally odd polarized) weakly compatible system.

We will call $\mathcal{R}$ *irreducible* if there is a set $\mathcal{L}$ of rational primes of Dirichlet density 1 such that for $\lambda|l \in \mathcal{L}$ the representation r_λ is irreducible.

We will call $\mathcal{R}$ *strictly compatible* if for each finite place v of F there is a Weil–Deligne representation $\mathrm{WD}_v(\mathcal{R})$ of W_{F_v} over $\overline{M}$ with the following property: for each place λ of M not dividing the residue characteristic of v and every M-linear embedding $\varsigma : \overline{M} \hookrightarrow \overline{M}_\lambda$, the push forward $\zeta WD_v(\mathcal{R})$ is isomorphic to $WD(r_\lambda|G_{F_v})^{F-\mathrm{ss}}$.

We will call $\mathcal{R}$ *pure* of weight w if

- for each $v \notin S$, each root α of $Q_v(X)$ in $\overline{M}$ and each $\imath : \overline{M} \hookrightarrow \mathbb{C}$ we have
$$|\imath\alpha|^2 = q_v^w;$$
- for each $\tau : F \hookrightarrow \overline{M}$ and each complex conjugation c in $\mathrm{Gal}\,(\overline{M}/\mathbb{Q})$ we have
$$H_{c\tau} = \{w - h : \ h \in H_\tau\}.$$

We will call $\mathcal{R}$ *strictly pure* of weight w if

- $\mathcal{R}$ is strictly compatible and for each prime v of F the Weil–Deligne representation $\mathrm{WD}_v(\mathcal{R})$ is pure of weight w;
- for each $\tau : F \hookrightarrow \overline{M}$ and each complex conjugation c in $\mathrm{Gal}\,(\overline{M}/\mathbb{Q})$ we have
$$H_{c\tau} = \{w - h : \ h \in H_\tau\}.$$

If $\imath : M \hookrightarrow \mathbb{C}$, we define the partial L-function
$$L^S(\imath\mathcal{R}, s) = \prod_{v \notin S} (q_v^{ns}/\imath Q_v(q_v^s)).$$

This may or may not converge. If $\mathcal{R}$ is pure of weight w then it will converge to an analytic function in $\mathrm{Re}\, s > 1 + w/2$. If $\lambda|l$ and every place of F above l lies in S, then $L^S(\imath\mathcal{R}, s)$ depends only on r_λ so, if $\widetilde{\imath} : \overline{M}_\lambda \xrightarrow{\sim} \mathbb{C}$ extends $\imath$, we will sometimes write $L^S(\widetilde{\imath} r_\lambda, s)$ instead of $L^S(\imath\mathcal{R}, s)$. This makes sense even for r_λ not part of a weakly compatible system, provided that S contains all primes above l and all primes at which r_λ ramifies.

If $\mathcal{R}$ is strictly compatible, then we can define the L-function
$$L(\imath\mathcal{R}, s) = \prod_{v \nmid \infty} L(\imath \mathrm{WD}_v(\mathcal{R}), s),$$
which differs from $L^S(\imath\mathcal{R}, s)$ only by the addition of finitely many Euler factors.

Suppose that $\mathcal{R}$ is strictly compatible, pure of weight w and regular. Also let $\psi = \prod_v \psi_v : \mathbb{A}_F/F \to \mathbb{C}^\times$ be the nontrivial additive character such that if v is real, then $\psi_v(x) = e^{2\pi i x}$; if v is complex, then $\psi_v(x) = e^{2\pi i(x + c_v x)}$; while if v is p-adic, then $\psi_v(x) = \psi_p(\mathrm{tr}_{F_v/\mathbb{Q}_p}(x))$ where $\psi_p|_{\mathbb{Z}_p} = 1$ and $\psi_p(1/p) = e^{-2\pi i/p}$. We wish to define the completed L-function and ϵ-factors of $\mathcal{R}$. (See [Tat79].)

- Write $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$ and $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s) = \Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s+1)$.
- Suppose that v is a complex infinite place of F and that $\tau, \tau' \in \mathrm{Hom}\,(F, \overline{M})$ are the two distinct elements such that $\imath \circ \tau$ and $\imath \circ \tau'$ extend to continuous isomorphisms $F_v \xrightarrow{\sim} \mathbb{C}$. Then we set

$$\begin{aligned} L_v(\imath\mathcal{R}, s) \quad = \quad & \Gamma_{\mathbb{C}}(s - w/2)^{\dim \mathcal{R}} \textstyle\prod_{h \in H_\tau,\ h<w/2} (\Gamma_{\mathbb{C}}(s-h)/\Gamma_{\mathbb{C}}(s-w/2)) \\ & \textstyle\prod_{h \in H_{\tau'},\ h<w/2} (\Gamma_{\mathbb{C}}(s-h)/\Gamma_{\mathbb{C}}(s-w/2)) \end{aligned}$$

  and

$$\epsilon_v(\imath\mathcal{R}, \psi_v, s) = i^{\sum_{h \in H_\tau} |h - w/2| + \sum_{h \in H_{\tau'}} |h - w/2|}.$$

- Suppose that v is a real infinite place of F and that $\tau : F \hookrightarrow \overline{M}$ is such that $\imath \circ \tau$ extends to a continuous isomorphism $F_v \xrightarrow{\sim} \mathbb{R} \subset \mathbb{C}$. As $\{\det r_\lambda\}$ is a weakly compatible system of characters, $\det r_\lambda(c_v) = \pm 1$ is independent of λ. We set $\det \mathcal{R}(c_v) = \det r_\lambda(c_v) \in \{\pm 1\}$.

  If $\dim \mathcal{R}$ is even, set $d_\pm = \dim \mathcal{R}/2$, while, if $\dim \mathcal{R}$ is odd (in which case $w/2 \in H_\tau$ so that w is even), set

$$d_\pm = (\dim \mathcal{R} \pm (-1)^{w/2} (\det \mathcal{R})(c_v))/2.$$

  (The reader might like to think of $d_\pm$ as $\dim r_\lambda^{c_v = \pm(-1)^{w/2}}$. Because we did not make a compatibility assumption between the r_λ at the infinite place v this does not make sense directly. However we make use of our regularity assumption to give this alternate definition, which does make sense and suffices for our purposes.)

  Now define

$$L_v(\imath\mathcal{R}, s) = \Gamma_{\mathbb{R}}(s - w/2)^{d_+} \Gamma_{\mathbb{R}}(s + 1 - w/2)^{d_-} \prod_{h \in H_\tau,\ h<w/2} (\Gamma_{\mathbb{C}}(s-h)/\Gamma_{\mathbb{C}}(s-w/2))$$

  and

$$\epsilon_v(\imath\mathcal{R}, \psi_v, s) = i^{d_- + \sum_{h \in H_\tau} |h - w/2|}.$$

- Finally we define the completed L-function

$$\Lambda(\imath\mathcal{R}, s) = L(\imath\mathcal{R}, s) \prod_{v|\infty} L_v(\imath\mathcal{R}, s)$$

  and the epsilon factor

$$\epsilon(\imath\mathcal{R}, s) = \left(\prod_{v \not| \infty} \epsilon(\imath \mathrm{WD}_v(\mathcal{R}), \psi_v, s) \right) \left(\prod_{v|\infty} \epsilon_v(\imath\mathcal{R}, \psi_v, s) \right).$$

  The latter does not depend on the choice of ψ.

We will call $\mathcal{R}$ *automorphic* if there is a regular algebraic, cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_F)$ and an embedding $\imath : M \hookrightarrow \mathbb{C}$, such that if $v \notin S$, then π_v is unramified and $\mathrm{rec}(\pi_v |\det|_v^{(1-n)/2})(\mathrm{Frob}_v)$ has characteristic polynomial $\imath(Q_v(X))$. Note that if $\mathcal{R}$ is polarizable, then so is π. It

follows from Theorem 3.13 of [Clo90] that, when $\mathcal{R}$ is automorphic, for any embedding $\imath' : M \hookrightarrow \mathbb{C}$ there is a regular, algebraic, cuspidal automorphic representation π' of $\mathrm{GL}_n(\mathbb{A}_F)$ such that if $v \notin S$ then π'_v is unramified and $\mathrm{rec}(\pi'_v |\det|_v^{(1-n)/2})(\mathrm{Frob}_v)$ has characteristic polynomial $\imath'(Q_v(X))$. (It would be more natural not to include the assumption that π is regular. However for the purposes of this paper our definition will be more convenient.)

We will call an n-dimensional polarized weakly compatible system $(\mathcal{R}, \mathcal{M})$ *automorphic* if there is a regular algebraic, cuspidal, polarized automorphic representation (π, χ) of $\mathrm{GL}_n(\mathbb{A}_F)$ and an embedding $\imath : M \hookrightarrow \mathbb{C}$ such that

- if $v \notin S$, then π_v is unramified and $\mathrm{rec}(\pi_v |\det|_v^{(1-n)/2})(\mathrm{Frob}_v)$ has characteristic polynomial equal to the image under $\imath$ of the polynomial $Q_v(X)$ for $\mathcal{R}$;
- if $v \notin S_{F^+}$, then χ_v is unramified and $\mathrm{rec}(\chi_v |\ |_v^{1-n})(\mathrm{Frob}_v)$ has characteristic polynomial equal to the image under $\imath$ of the polynomial $Q_v(X)$ for $\mathcal{M}$.

If F'/F is a finite extension, we define $\mathcal{R}|_{G_{F'}}$ to be the weakly compatible system of representations of $G_{F'}$:

$$(M, S^{(F')}, \{Q_v^{(F')}(X)\}, \{r_\lambda|_{G_{F'}}\}, \{H_\tau^{(F')}\}),$$

where

- $S^{(F')}$ is the set of primes of F' above S;
- $H_\tau^{(F')} = H_{\tau|_F}$;
- $Q_v^{(F')}(X)$ is the monic polynomial in $M[X]$ of degree n with roots the $\alpha^{[k(v):k(v|_F)]}$ as α runs over roots of $Q_{v|_F}(X)$.

We remark that if (π, χ) is a regular algebraic, cuspidal, polarized automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$, then $\{r_{l,\imath}(\chi)\}$ is a strictly pure compatible system of some necessarily even weight (because F^+ is totally real), which we will write $2(w+1-n)$. Moreover $\{r_{l,\imath}(\pi)\}$ is a strictly pure compatible system of weight w. (See Theorem 2.1.1.)

5.2. *Rational compatible systems.* In this section we present a formulation of the results of Larsen from [Lar95]. We will consider a weakly compatible system $\mathcal{R} = (\mathbb{Q}, S, \{Q_v(X)\}, \{r_l\}, \{H_\tau\})$, where for each l we have

$$r_l : G_F \longrightarrow \mathrm{GL}_n(\mathbb{Q}_l).$$

We are going to make a number of definitions which depend on $\mathcal{R}$, but for simplicity we will not put $\mathcal{R}$ in the notation. We hope this will cause no confusion.

We will write V_l for the $\mathbb{Q}_l$-vector space underlying r_l. We will also write G_l for the Zariski closure of the image of r_l in $\mathrm{GL}_n/\mathbb{Q}_l$, and G_l^0 for the connected component of the identity in G_l. Thus G_l^0 is a reductive group over $\mathbb{Q}_l$. Let Γ_l

denote the image of G_F in $G_l(\mathbb{Q}_l)$ and set $\Gamma_l^0 = \Gamma_l \cap G_l^0(\mathbb{Q}_l)$. There is a finite Galois extension F^0/F such that $\mathrm{Gal}\,(F^0/F) \xrightarrow{\sim} \Gamma_l/\Gamma_l^0$ for all l (Proposition 6.14 of [LP92]). Since r_l is Hodge-Tate, the Lie algebra of Γ_l is algebraic (as defined in Section II.7 of [Bor91]) and the group Γ_l is open in $G_l(\mathbb{Q}_l)$ ([Bog80]).

Let Z_l (resp. G_l^{der}) denote the centre (resp. derived subgroup) of G_l^0. Let $G_l^{\mathrm{ad}} = G_l^0/Z_l$ and let $C_l = G_l^0/G_l^{\mathrm{der}}$. Also let G_l^{sc} denote the simply connected cover of G_l^{ad}. We have surjective maps with finite kernels

$$G_l^{\mathrm{sc}} \longrightarrow G_l^{\mathrm{der}} \longrightarrow G_l^{\mathrm{ad}}.$$

Because the dimension of G_l^{ad} is bounded only depending on n, we see that there is a positive constant $A(n)$ depending only on n (and not on $\mathcal{R}$ or l) such that

$$\#\ker(G_l^{\mathrm{sc}} \to G_l^{\mathrm{ad}})|A(n).$$

The natural map $Z_l \to C_l$ is surjective with finite kernel of order dividing $A(n)$. We will write Γ_l^Z for $\Gamma_l^0 \cap Z_l(\mathbb{Q}_l)$ and Γ_l^C for the image of Γ_l^0 in $C_l(\mathbb{Q}_l)$. As the cokernel of the map $G_l^0(\mathbb{Q}_l) \to C_l(\mathbb{Q}_l)$ is finite (because it is a quotient of $C_l(\mathbb{Q}_l)/Z_l(\mathbb{Q}_l) \subset H^1(G_{\mathbb{Q}_l}, \ker(Z_l \to C_l))$), we see that Γ_l^C is open in $C_l(\mathbb{Q}_l)$.

Set $H_l = G_l^{\mathrm{sc}} \times Z_l$. Then there is a natural surjection of algebraic groups $H_l \twoheadrightarrow G_l^0$, with a finite, central kernel with order dividing $A(n)$ (as it equals the kernel of $G_l^{\mathrm{sc}} \to G_l^{\mathrm{ad}}$). Using Galois cohomology we see that the cokernel of the map $H_l(\mathbb{Q}_l) \to G_l^0(\mathbb{Q}_l)$ is a finite abelian group of order dividing $A(n)^3$ and exponent dividing $A(n)$. (The cokernel embeds in $H^1(G_{\mathbb{Q}_l}, \ker(H_l \to G_l^0))$, which by the local Euler characteristic formula has order dividing $(\#\ker(H_l \to G_l^0))^3$.) We will write $\Gamma_l^{00} = \Gamma_l^0 \cap \mathrm{Im}\,(H_l(\mathbb{Q}_l) \to G_l^0(\mathbb{Q}_l))$ and Γ_l^H for the preimage of Γ_l^{00} in $H_l(\mathbb{Q}_l)$. Thus the kernels of

$$\Gamma_l^H \longrightarrow \Gamma_l^0$$

and

$$\Gamma_l^Z \longrightarrow \Gamma_l^C$$

are both finite abelian groups of order dividing $A(n)$, while the cokernels are both finite abelian groups of order dividing $A(n)^3$ and exponent dividing $A(n)$. It will often be convenient to work with Γ_l^H in place of Γ_l^0, because it is easier to control.

As G_l acts by conjugation on G_l^0, it also acts on Z_l, G_l^{der}, G_l^{ad}, C_l, G_l^{sc} and H_l. Thus Γ_l does as well. Moreover, Γ_l^{00} is a normal subgroup of Γ_l and the conjugation action of Γ_l on Γ_l^{00} lifts to an action on Γ_l^H.

Let T_l denote a maximal torus in G_l^0 which we assume to be chosen unramified whenever G_l^0 is unramified. Let $T_l^{\mathrm{ad}} = T_l/Z_l$, let $T_l^{\mathrm{der}} = \ker(T_l \to C_l)^0$, let T_l^{sc} equal the connected preimage of T_l^{ad} in G_l^{sc} and let $T_l^H = T_l^{\mathrm{sc}} \times Z_l$. We

have natural embeddings

$$X^*(T_l^{\mathrm{ad}}) \subset X^*(T_l^{\mathrm{der}}) \subset X^*(T_l^{\mathrm{sc}}) \subset (1/A(n))X^*(T_l^{\mathrm{ad}}),$$
$$A(n)X^*(T_l^H) \subset X^*(T_l) \subset X^*(T_l^H)$$

and

$$A(n)X^*(Z_l) \subset X^*(C_l) \subset X^*(Z_l).$$

Let $\Delta \subset X^*(T_l^{\mathrm{ad}}) \subset X^*(T_l)$ denote a basis for the root system of G_l^0.

The dimensions of V_l and G_l^{sc} are bounded only in terms of n. Hence, if $(1/A(n))\sum_{\delta\in\Delta} m_\delta\delta$ is a weight of G_l^{sc} on V_l, then the $|m_\delta|$ can be bounded by a constant $B(n)$ depending only on n (and not on $\mathcal{R}$ or l).

If $\mu \in X^*(Z_l)$ is a Z_l-weight of V_l, then we can find $m_{\mu,\delta} \in \mathbb{Z}$ for $\delta \in \Delta$ such that $(1/A(n))\sum_{\delta\in\Delta} m_{\mu,\delta}\delta$ is a weight of T_l^{der} on V_l^μ. Thus

$$\left((1/A(n))\sum_{\delta\in\Delta} m_{\mu,\delta}\delta,\ \mu\right) \in X^*(T_l^H)$$

is a weight of V_l and $|m_{\mu,\delta}| \le B(n)$ for all $\delta \in \Delta$.

Let $\widetilde{S}_{F^0,l}$ denote the restriction of scalars from $\mathcal{O}_{F^0,l}$ to $\mathbb{Z}_l$ of $\mathbb{G}_m$ and let $S_{F^0,l} = \widetilde{S}_{F^0,l} \times \mathbb{Q}_l$. Note that $\mathrm{Hom}\,(F^0, \overline{\mathbb{Q}}_l)$ gives a natural basis of $X^*(S_{F^0,l})$. There is a homomorphism

$$\theta_l : S_{F^0,l} \longrightarrow C_l$$

such that θ_l agrees with $(r_l \bmod G_l^{\mathrm{der}}(\mathbb{Q}_l)) \circ \mathrm{Art}_{F^0}$ on an open subgroup of $\mathcal{O}_{F^0,l}^\times$. (See Sections III.1.2 and III.2.1 of [Ser68]. To aid comparison with [Ser68] we remark that i of Section II.1.1 of [Ser68] is the inverse of our Art_{F^0}. Thus θ_l is constructed from $r_l \bmod G_l^{\mathrm{der}}(\mathbb{Q}_l)$ in just the same way as r is constructed from ρ in Section II.1.1 of [Ser68].)

LEMMA 5.2.1. *Keep the notation and assumptions established earlier in this section.*

(1) $\theta_l : S_{F^0,l} \to C_l$ *is surjective.*
(2) *If* $l \notin S$, *then* $\theta_l = (r_l \bmod G_l^{\mathrm{der}}(\mathbb{Q}_l)) \circ \mathrm{Art}_{F^0}$ *on all of* $\mathcal{O}_{F^0,l}^\times$.
(3) *There is a constant* $C(\mathcal{R})$ *depending only on* $\mathcal{R}$ (*and not on* l) *such that if* $\mu \in X^*(Z_l)$ *is a weight of* Z_l *on* V_l, *then*

$$(A(n)\mu) \circ \theta_l = \sum_{\sigma\in\mathrm{Hom}\,(F^0,\overline{\mathbb{Q}}_l)} m_{\mu,\sigma}\sigma$$

with $|m_{\mu,\sigma}| < C(\mathcal{R})$.
(4) *There is a constant* $D(\mathcal{R})$ *depending only on* $\mathcal{R}$ (*and not on* l) *such that*

$$\#(X^*(S_{F^0,l})/\theta_l^* X^*(C_l))^{\mathrm{tor}} \le D(\mathcal{R}).$$

Proof. For the first part let U denote the open subgroup of $\mathcal{O}_{F^0,l}^\times$ where θ_l and $(r_l \bmod G_l^{\mathrm{der}}(\mathbb{Q}_l)) \circ \mathrm{Art}_{F^0}$ agree. As $C_l(\mathbb{Q}_l)$ has an open subgroup which is pro-l, we see that

$$((r_l \bmod G_l^{\mathrm{der}}(\mathbb{Q}_l)) \circ \mathrm{Art}_{F^0})(U) = \theta_l(U)$$

is open, and hence of finite index, in

$$((r_l \bmod G_l^{\mathrm{der}}(\mathbb{Q}_l)) \circ \mathrm{Art}_{F^0})(\mathbb{A}_{F^0}^\times) = (r_l \bmod G_l^{\mathrm{der}})(G_{F^0}).$$

As the image of G_{F^0} in $C_l(\mathbb{Q}_l)$ is Zariski dense and C_l is connected, we deduce that $\theta_l(U)$ is Zariski dense in $C_l/\mathbb{Q}_l$ and the result follows.

For the second part, if $l \notin S$, then $r_l : G_F \to G_l(\mathbb{Q}_l)$ is crystalline at each prime above l, and hence so is $r_l \bmod G_l^{\mathrm{der}}(\mathbb{Q}_l) : G_F \to C_l(\mathbb{Q}_l)$. Lemma 3.7.7 of [CCO13] then implies that for each $\mu \in X^*(C_l)$ the characters $\mu \circ \theta_l$ and $\mu \circ (r_l \bmod G_l^{\mathrm{der}}(\mathbb{Q}_l)) \circ \mathrm{Art}_{F^0}$ agree on all of $\mathcal{O}_{F_l^0}^\times$. Since this holds for all $\mu \in X^*(C_l)$, it follows immediately that θ_l and $(r_l \bmod G_l^{\mathrm{der}}(\mathbb{Q}_l)) \circ \mathrm{Art}_{F^0}$ agree on all of $\mathcal{O}_{F_l^0}^\times$.

For the third part note that $-m_{\mu,\sigma}$ is the σ-Hodge–Tate number of $(A(n)\mu) \circ (r_l \bmod G_l^0(\mathbb{Q}_l))$. If $v|l$ is a prime of F^0, then there is an element $\nu_{\mathrm{HT},v} \in X_*(T_l)$ such that for any algebraic representation ρ of G_l^0 defined over $\mathbb{Q}_l$ the Hodge–Tate numbers (with respect to any continuous $\sigma : F_v^0 \hookrightarrow \overline{\mathbb{Q}}_l$) of $\rho \circ r_l|_{G_{F_v^0}}$ are the $\langle \mu, \nu_{\mathrm{HT},v} \rangle$, where μ runs over the weights of ρ in $X^*(T_l)$. (See Section 1.2 of [Win86].) Thus what we have to show is that if $\mu \in X^*(Z_l)$ is a weight of Z_l on V_l, then

$$|\langle A(n)\mu, \nu_{\mathrm{HT},v} \rangle|$$

is bounded independently of l, μ and v. (Here we think of $A(n)\mu \in A(n)X^*(Z_l) \subset X^*(C_l) \subset X^*(T_l)$.) However

$$\Big((1/A(n)) \sum_{\delta \in \Delta} m_{\mu,\delta}\delta, \mu\Big) \in X^*(T_l) \subset X^*(T_l^H)$$

is a weight of T_l on V_l and so

$$\Big|\Big\langle \Big(\sum_{\delta \in \Delta} m_{\mu,\delta}\delta, A(n)\mu \Big), \nu_{\mathrm{HT},v} \Big\rangle\Big|$$

is bounded independently of l, μ and v (because the Hodge–Tate numbers of r_l are independent of l). As the $m_{\mu,\delta}$ are also bounded independently of l and μ, we see that it suffices to show that each

$$|\langle \delta, \nu_{\mathrm{HT},v} \rangle|$$

is bounded independently of l, $\delta \in \Delta$ and v. But $\mathrm{Lie}\, G_l^0 \subset \mathrm{Hom}_{\mathbb{Q}_l}(V_l, V_l)$ so that each $\delta \in \Delta$ is the difference of two weights of T_l on V_l. It follows that

$$|\langle \delta, \nu_{\mathrm{HT},v} \rangle|$$

is bounded independently of l, $\delta \in \Delta$ and v, as desired (again because the Hodge–Tate numbers of r_l are independent of l).

The fourth part follows from the third. □

PROPOSITION 5.2.2. *Keep the notation and assumptions established earlier in this section. There is a Dirichlet density* 1 *set* $\mathcal{L}$ *of rational primes with the following properties*:

(1) *If* $l \in \mathcal{L}$, *then* G_l^0, *and hence also* Z_l, C_l, G_l^{sc} *and* H_l, *are unramified. Write* $\widetilde{Z}_l$ (*resp.* $\widetilde{C}_l$) *for the torus over* $\mathbb{Z}_l$ *with generic fibre* Z_l (*resp.* C_l).
(2) *If* $l \in \mathcal{L}$, *then there is a semi-simple group scheme* $\widetilde{G}_l^{\mathrm{sc}}/\mathbb{Z}_l$ *with generic fibre* G_l^{sc} *such that* $\Gamma_l^H = \widetilde{G}_l^{\mathrm{sc}}(\mathbb{Z}_l) \times \Gamma_l^Z$. *Write* $\widetilde{H}_l = \widetilde{G}_l^{\mathrm{sc}} \times \widetilde{Z}_l$.
(3) *The index* $[\widetilde{Z}_l(\mathbb{Z}_l) : \Gamma_l^Z]$ *is bounded independently of* $l \in \mathcal{L}$.
(4) *If* $l \in \mathcal{L}$, *then the conjugation action of* Γ_l *on* H_l *extends* (*uniquely*) *to an action on* $\widetilde{H}_l$. *This makes* V_l *into an* $\widetilde{H}_l \rtimes \Gamma_l$*-module.*
(5) *If* $l \in \mathcal{L}$, *then* V_l *contains an* $\widetilde{H}_l \rtimes \Gamma_l$ *invariant* $\mathbb{Z}_l$*-lattice.*
(6) *Suppose that* $l \in \mathcal{L}$. *There is a finite unramified extension* $M_\lambda/\mathbb{Q}_l$ (*of degree bounded independently of* $l \in \mathcal{L}$) *such that all* G_l^{sc}*-irreducible subquotients of* $V_l \otimes \overline{\mathbb{Q}}_l$ *can be defined over* M_λ. *Choose such a field* M_λ *and let*

$$V_l \otimes M_\lambda = \bigoplus_i V_{\lambda,i}$$

be the decomposition of $V_l \otimes M_\lambda$ *into maximal* G_l^{sc}*-isotypical submodules. Suppose also that* $\Lambda \subset V_l \otimes M_\lambda$ *is a* $\widetilde{H}_l$*-invariant* $\mathcal{O}_{M_\lambda}$*-lattice. Then*

$$\Lambda = \bigoplus_i (\Lambda \cap V_{\lambda,i}).$$

Moreover all irreducible $\widetilde{G}_l^{\mathrm{sc}}(\mathbb{Z}_l)$*-subquotients of* $\Lambda \cap V_{\lambda,i}$ *are absolutely irreducible and isomorphic, say to* $\overline{\rho}_i$. *Moreover* $\dim_{k(\lambda)} \overline{\rho}_i$ *equals the dimension* (*over* M_λ) *of an irreducible constituent of* $V_{\lambda,i}$. *If* $\overline{\rho}_i \cong \overline{\rho}_j$, *then* $i = j$.

Proof. Proposition 8.9 of [LP92] tells us that we can find a Dirichlet density 1 set $\mathcal{L}$ of rational primes such that for $l \in \mathcal{L}$ the group G_l^0 is unramified. As the dimension of T_l is bounded independently of l, we see that the order of any finite order element of $\mathrm{Aut}\,(X^*(T_l))$ is independent of l. Hence for $l \in \mathcal{L}$ the group splits over an unramified extension $M_\lambda/\mathbb{Q}_l$ of degree bounded independently of l.

Theorem 3.17 of [Lar95] tells us that we can replace $\mathcal{L}$ by a possibly smaller set of Dirichlet density 1, which we will also denote $\mathcal{L}$, such that for $l \in \mathcal{L}$ there is a semi-simple group scheme $\widetilde{G}_l^{\mathrm{sc}}/\mathbb{Z}_l$ with generic fibre G_l^{sc} such that

$$\Gamma_l^H \cap G_l^{\mathrm{sc}}(\mathbb{Q}_l) = \widetilde{G}_l^{\mathrm{sc}}(\mathbb{Z}_l).$$

As $\widetilde{G}_l^{\mathrm{sc}}(\mathbb{Z}_l)$ is a maximal compact subgroup of $G_l^{\mathrm{sc}}(\mathbb{Q}_l)$, we see that $\widetilde{G}_l^{\mathrm{sc}}(\mathbb{Z}_l)$ is also the projection of Γ_l^H to $G_l^{\mathrm{sc}}(\mathbb{Q}_l)$ and so $\Gamma_l^H = \widetilde{G}_l^{\mathrm{sc}}(\mathbb{Z}_l) \times \Gamma_l^Z$ for some compact subgroup $\Gamma_l^Z \subset Z_l(\mathbb{Q}_l)$. As $\widetilde{Z}_l(\mathbb{Z}_l)$ is the unique maximal compact subgroup of Z_l, we see that $\Gamma_l^Z \subset \widetilde{Z}_l(\mathbb{Z}_l)$. As the image of Γ_l^Z in Γ_l^C has finite index we see that Γ_l^Z is open in $\widetilde{Z}_l(\mathbb{Z}_l)$, and we have verified the first two parts of the proposition.

Removing finitely many primes from $\mathcal{L}$ we may suppose that for all $l \in \mathcal{L}$ the representation r_l is crystalline at l and l is unramified in F^0. Then for $l \in \mathcal{L}$ we have

$$\Gamma_l^C \supset \theta_l(\widetilde{S}_{F^0,l}(\mathbb{Z}_l)).$$

The final part of Lemma 5.2.1, together with Lemma A.1.6, shows that the index of Γ_l^C in $\widetilde{C}_l(\mathbb{Z}_l)$ is bounded by $D(\mathcal{R})$ (independently of $l \in \mathcal{L}$). However the index of Γ_l^Z in $\widetilde{Z}_l(\mathbb{Z}_l)$ is bounded by the product

$$\#\ker(Z_l \to C_l) \times [\Gamma_l^C : \Gamma_l^Z] \times [\widetilde{C}_l(\mathbb{Z}_l) : \Gamma_l^C] \leq A(n)^4 D(\mathcal{R}),$$

and so the third part of the proposition follows.

Suppose that $\gamma \in \Gamma_l$; then we have seen that γ acts on G_l^{sc}. We will consider the reductive group scheme

$${}^{\gamma}\widetilde{G}_l^{\mathrm{sc}} = \mathrm{Spec}\left((\gamma^*)^{-1}\mathcal{O}_{\widetilde{G}_l^{\mathrm{sc}}}(\widetilde{G}_l^{\mathrm{sc}})\right),$$

where $(\gamma^*)^{-1}\mathcal{O}_{\widetilde{G}_l^{\mathrm{sc}}}(\widetilde{G}_l^{\mathrm{sc}}) \subset \mathcal{O}_{G_l^{\mathrm{sc}}}(G_l^{\mathrm{sc}})$ is the pre-image of $\mathcal{O}_{\widetilde{G}_l^{\mathrm{sc}}}(\widetilde{G}_l^{\mathrm{sc}}) \subset \mathcal{O}_{G_l^{\mathrm{sc}}}(G_l^{\mathrm{sc}})$ under γ^*. By Proposition 5.1.40 of [BT84] we see that $\widetilde{G}_l^{\mathrm{sc}}$ and ${}^{\gamma}\widetilde{G}_l^{\mathrm{sc}}$ must be the group schemes (with connected fibres) attached to special points x and x' in the building of G_l^{sc} in [BT84]. (The group schemes $\mathfrak{G}_x^{0\natural}$ and $\mathfrak{G}_{x'}^{0\natural}$ in the notation of [BT84].) The sets $\{x\}$ and $\{x'\}$ are facets for the building of G_l^{sc} (as G_l^{sc} is semi-simple). As

$$({}^{\gamma}\widetilde{G}_l)(\mathbb{Z}_l) = {}^{\gamma}(\widetilde{G}_l(\mathbb{Z}_l)) = \widetilde{G}_l(\mathbb{Z}_l),$$

we deduce from the proof of Proposition 5.2.8 of [BT84] that $\{x\} = \{x'\}$, and hence ${}^{\gamma}\widetilde{G}_l^{\mathrm{sc}} = \widetilde{G}_l^{\mathrm{sc}}$. Thus the action of Γ_l on G_l^{sc} extends to one on $\widetilde{G}_l^{\mathrm{sc}}$. (We thank Jiu-Kang Yu for valuable help with this argument.) As $\widetilde{Z}_l$ is the unique torus with generic fibre Z_l, we also see that the action of Γ_l on Z_l extends to an action on $\widetilde{Z}_l$. Thus the action of Γ_l on H_l extends to one on $\widetilde{H}_l$, and the fourth part of the proposition follows.

As in the fourth paragraph of Section 1.12 of [Lar95] we can find a $\mathbb{Z}_l$-lattice $\Lambda \subset V_l$ such that $\Lambda \otimes \mathbb{Z}_l^{\mathrm{nr}}$ is $\widetilde{H}_l(\mathbb{Z}_l^{\mathrm{nr}})$-invariant. Replacing Λ by the sum of its Γ_l-translates, which is again a lattice because Γ_l is compact, we see that we may suppose Λ to also be Γ_l-invariant. Again as in the fourth paragraph of Section 1.12 of [Lar95] we see that the natural map $H_l \to \mathrm{GL}(V_l)$ extends to a map $\widetilde{H}_l \to \mathrm{GL}(\Lambda)$. This establishes the fifth part of the proposition.

Let $\{\mu_i\}$ be the set of highest weights (with respect to Δ) of irreducible G_l^{sc}-submodules of $V_l \otimes \overline{\mathbb{Q}}_l$. Let ρ_{μ_i} be the corresponding irreducible representations of G_l^{sc}. As in the third paragraph of Section 1.12 of [Lar95] we see that each ρ_{μ_i} can be defined over M_λ. The first assertion of part six follows. Moreover ρ_{μ_i} extends to a representation $\widetilde{\rho}_{\mu_i}$ of $\widetilde{G}_l^{\mathrm{sc}}$ over $\mathcal{O}_{M_\lambda}$ (by the argument of paragraph four of Section 1.12 of [Lar95]). Let $\overline{\rho}_{\mu_i}$ denote the reduction of $\widetilde{\rho}_{\mu_i}$ modulo λ and let $\overline{\rho}^+_{\mu_i}$ denote the unique absolutely irreducible subquotient of $\overline{\rho}_{\mu_i}$ which contains μ_i. (See paragraph five of Section 1.12 of [Lar95].)

If $\delta \in \Delta$, let $\nu_\delta \in X^*(T_l^{\mathrm{sc}})$ denote the fundamental weight corresponding to δ, so that $\{\nu_\delta\}$ is the basis of $X^*(T_l^{\mathrm{sc}})$ dual to the basis of $X_*(T_l^{\mathrm{sc}})$ consisting of the coroots corresponding to $\delta \in \Delta$. If $\mu_i = \sum_{\delta \in \Delta} m_{i,\delta}\nu_\delta$, then the $m_{i,\delta}$ are bounded independently of l. (As $\dim G_l^{\mathrm{sc}}$ is bounded independently of l, there are only finitely many possibilities for the change of basis matrix between Δ and $\{\nu_\delta : \delta \in \Delta\}$.) Thus after removing a finite number of elements from $\mathcal{L}$ we may assume that for all $l \in \mathcal{L}$ and all i we have $0 \le m_{i,\delta} < l$. As in the first paragraph of Section 1.13 of [Lar95] we see that $\overline{\rho}^+_{\mu_i}$ is an absolutely irreducible representation of $\widetilde{G}_l^{\mathrm{sc}}(\mathbb{F}_l)$ and that $\overline{\rho}^+_{\mu_i} \cong \overline{\rho}^+_{\mu_j}$ implies $i = j$. Again removing a finite number of primes from $\mathcal{L}$ we can further assure, as in the second paragraph of Section 1.13 of [Lar95], that $\overline{\rho}^+_{\mu_i} = \overline{\rho}_{\mu_i}$ (for all $l \in \mathcal{L}$ and all i).

Now suppose that Λ is as in part 6 of the proposition. (We no longer use Λ to denote the lattice constructed in the proof of the fifth part.) We see that $\overline{\rho}^+_{\mu_i}$ is the only irreducible subquotient of $\Lambda \cap V_{\lambda,i}$. Suppose that

$$\Lambda \neq \bigoplus_i (\Lambda \cap V_{\lambda,i}).$$

Choose a $\widetilde{G}_l^{\mathrm{sc}}$-invariant lattice Λ' with

$$\Lambda \supset \Lambda' \supset \bigoplus_i (\Lambda \cap V_{\lambda,i}),$$

with $\Lambda' / (\bigoplus_i (\Lambda \cap V_{\lambda,i}))$ simple and nontrivial. Then

$$\Lambda' \Big/ \left(\bigoplus_i (\Lambda \cap V_{\lambda,i}) \right)$$

must be equivalent to $\overline{\rho}^+_{\mu_j}$ for some j. Suppose $i \neq j$. Then we have a commutative diagram

$$\begin{array}{ccc} \Lambda' & \twoheadrightarrow & \Lambda' / (\bigoplus_i (\Lambda \cap V_{\lambda,i})) \\ \downarrow & & \downarrow \\ V_{\lambda,i} & \twoheadrightarrow & V_{\lambda,i}/(\Lambda \cap V_{\lambda,i}), \end{array}$$

and we see that the right-hand vertical arrow must be zero. Thus the image of Λ' under projection to $V_{\lambda,i}$ is $\Lambda \cap V_{\lambda,i}$. We conclude that

$$\Lambda' = \left(\bigoplus_{i \neq j} \Lambda \cap V_{\lambda,i} \right) + \bigcap_{i \neq j} \ker(\Lambda' \to V_{\Lambda,i}).$$

However

$$\bigcap_{i\neq j}\ker(\Lambda'\to V_{\Lambda,i})=\Lambda\cap V_{\lambda,j}$$

and so

$$\Lambda'=\bigoplus_i(\Lambda\cap V_{\lambda,i}),$$

a contradiction. This completes the proof of the proposition. □

We remark that we will not need part (3) of the proposition, but we thought it was worth recording it anyway. We also remark that using part (3) of the proposition one can prove a version of part (6) in which H_l replaces G_l^{sc} and Γ_l^H replaces $\widetilde{G}_l^{\mathrm{sc}}(\mathbb{Z}_l)$. However, as we will not need it, we chose not to present the details.

5.3. *Compatible systems*: *lemmas.* We now return to more general compatible systems.

LEMMA 5.3.1. *Suppose that $\mathcal{R}$ is a weakly compatible system of l-adic representations of G_F of dimension n. Let G_λ denote the Zariski closure of $r_\lambda(G_F)$ in $\mathrm{GL}_n/\overline{M}_\lambda$ and let G_λ^0 denote its connected component. Then*

(1) *there is a finite Galois extension F^1/F such that for all λ the map $G_F\to G_\lambda(\overline{M}_\lambda)$ induces an isomorphism $\mathrm{Gal}\,(F^1/F)\xrightarrow{\sim}G_\lambda/G_\lambda^0$;*
(2) *if $\mathcal{R}$ is regular and H is an open subgroup of G_F, then any irreducible H-subrepresentation of r_λ has multiplicity one;*
(3) *if $\mathcal{R}$ is regular, then after replacing M by a finite extension we may suppose that for any open subgroup $H\subset G_F$ and any λ and any H-subrepresentation s of r_λ, the representation s is defined over $\mathcal{O}_{M,\lambda}$.*

Proof. The proof of the first part is the same as the proof of Proposition 6.14 of [LP92]. (One must replace $\mathcal{G}$ by G_F, and ρ_l by r_λ, and G_l^0 by G_λ^0; and $\mathbb{Q}_l$ by $\overline{M}_\lambda$, and $\mathbb{Q}$ by M in Section 6 of [LP92]. These arguments are due to Serre.)

As r_λ is semi-simple, we see that G_λ^0 is reductive. As $\mathrm{tr}\,r_\lambda$ is continuous on G_F and M_λ is closed in $\overline{M}_\lambda$, we see that $\mathrm{tr}\,r_\lambda$ is valued in M_λ.

Now assume that $\mathcal{R}$ is regular. By Theorem 1 of [Sen73] there is an element of $(\mathrm{Lie}\,G_\lambda^0)\otimes\widehat{\overline{M}_\lambda}$ with n distinct eigenvalues. Thus if T_λ is a maximal torus in G_λ^0, it has n distinct weights under r_λ. In particular all irreducible G_λ^0-subrepresentations of r_λ have multiplicity one. If H is any open subgroup of G_F then its Zariski closure in G_λ contains G_λ^0. Thus any irreducible H-subrepresentation of r_λ has multiplicity one, and the second part of the lemma is proved. (We note that the second part follows immediately from a consideration of the Hodge-Tate weights but we shall need the facts deduced in this paragraph below.)

Moreover the set of elements of G_λ^0 with n distinct eigenvalues under r_λ is a nonempty Zariski open subset. As the images of Frobenius elements at primes which split completely in F^1/F are Zariski dense in G_λ^0, we conclude that infinitely many $Q_v(X)$, for v splitting completely in F^1/F, have n distinct roots. Replace M by the splitting field over M of the product $Q_v(X)Q_{v'}(X)$ for two such v, v' with distinct residue characteristic. Then for all λ the image $r_\lambda(G_{F^1})$ contains an element with n distinct M_λ-rational eigenvalues.

Let H be any open subgroup of G_F. We must show that any H-subrepresentation of r_λ is defined over $\mathcal{O}_{M,\lambda}$. As H is compact, it suffices to show that it is defined over M_λ. As the Zariski closure of H in G_λ contains G_λ^0, it equals the Zariski closure of HG_{F^1}. Thus if s is an H-subrepresentation of r_λ, it is also a HG_{F^1}-subrepresentation. So we are reduced to the case $H \supset G_{F^1}$. In this case the result follows from Lemma A.1.5. □

PROPOSITION 5.3.2. *Suppose that $\mathcal{R}$ is a regular, weakly compatible system of l-adic representations of G_F defined over M. If s is a subrepresentation of r_λ, then we will write $\overline{s}$ for the semi-simplification of the reduction of s. Also write l for the rational prime below λ. Then there is a set of rational primes $\mathcal{L}$ of Dirichlet density 1 (depending only on $\mathcal{R}$) such that if s is any irreducible subrepresentation of r_λ for any λ dividing any element of $\mathcal{L}$, then $\overline{s}|_{G_{F(\zeta_l)}}$ is irreducible.*

Proof. If need be replace M by a finite extension so that

- $M/\mathbb{Q}$ is Galois;
- for every λ and every open subgroup $H \subset G_F$ any H-subrepresentation s of r_λ is defined over $\mathcal{O}_{M,\lambda}$ (Lemma 5.3.1).

For a rational prime l define

$$r_l = \epsilon_l \oplus \bigoplus_{\lambda|l} r_\lambda : G_F \longrightarrow \mathrm{GL}_{1+n[M:\mathbb{Q}]}(\mathbb{Q}_l).$$

Let

$$H_\mathbb{Q} = \{-1\} \amalg \coprod_{\tau: F \hookrightarrow \overline{M}} \coprod_{i=1}^{[M:F]} H_\tau$$

and, for v a prime of F not in S, let

$$Q_{\mathbb{Q},v}(X) = (X - q_v^{-1}) \prod_{\sigma \in \mathrm{Gal}\,(M/\mathbb{Q})} {}^\sigma Q_v(X).$$

We are going to apply the results of Section 5.2 to

$$\mathcal{R}_\mathbb{Q} = (\mathbb{Q}, S, \{Q_{\mathbb{Q},v}\}, \{r_l\}, \{H_\mathbb{Q}\}),$$

and we will use the notation established there without comment. If need be, we replace M by a finite extension so that M contains the image of every

embedding $F^0 \hookrightarrow \overline{M}$. Also choose a set $\mathcal{L}$ of rational primes of Dirichlet density one as in Proposition 5.2.2. Removing a finite number of primes from $\mathcal{L}$ we may further assume that

(1) if $l \in \mathcal{L}$, then l is unramified in $F^0/\mathbb{Q}$;
(2) if $l \in \mathcal{L}$, then $l > 4A(1+[M:\mathbb{Q}]n)C(\mathcal{R}_{\mathbb{Q}})+2$;
(3) $l \notin S_{\mathbb{Q}}$, where $S_{\mathbb{Q}}$ denotes the set of rational primes which lie below an element of S.

Let Λ_l be a $\widetilde{H}_l \rtimes \Gamma_l$-invariant lattice in V_l (Proposition 5.2.2).

The character ϵ_l of Γ_l extends to an algebraic character

$$\epsilon_l : G_l \twoheadrightarrow \mathbb{G}_m.$$

(It is surjective because $\epsilon_l(G_F)$ is infinite.) We will write Z_l^1, C_l^1 and H_l^1 for the kernel of ϵ_l on Z_l, C_l and H_l. The character ϵ_l extends to a character of $\widetilde{C}_l$. We will write $\widetilde{C}_l^1$ (resp. $\widetilde{Z}_l^1$) for the unique torus over $\mathbb{Z}_l$ with generic fibre C_l^1 (resp. Z_l^1) and set $\widetilde{H}_l^1 = \widetilde{G}_l^{\mathrm{sc}} \times \widetilde{Z}_l^1$. Set $\Gamma_l^{Z,1} = \Gamma_l^Z \cap Z_l^1(\mathbb{Q}_l)$ and $\Gamma_l^{C,1} = \Gamma_l^C \cap C_l^1(\mathbb{Q}_l)$ so that $\Gamma_l^{Z,1}$ is the pre-image of $\Gamma_l^{C,1}$ under $\Gamma_l^Z \to \Gamma_l^C$. Also set $\Gamma_l^{H,1} = \widetilde{G}_l^{\mathrm{sc}}(\mathbb{Z}_l) \times \Gamma_l^{Z,1}$. The conjugation action of Γ_l on $\widetilde{Z}_l$, $\widetilde{C}_l$ and $\widetilde{H}_l$ preserves $\widetilde{Z}_l^1$, $\widetilde{C}_l^1$ and $\widetilde{H}_l^1$.

We will next prove the following claim:

Suppose that $l \in \mathcal{L}$, that $\lambda | l$ is a prime of M, and that W_1 and W_2 are two irreducible $M_\lambda[\Gamma_l^0]$-sub-modules of $V_l \otimes M_\lambda$. Then (by Zariski density) W_i is G_l^0-invariant. Write s_i for the representation of $H_l \rtimes \Gamma_l^0$ on W_i, and further assume that the semi-simplified reductions $\overline{s}_i$ of s_i are isomorphic as $\Gamma_l^{H,1}$-modules. Then $s_1 \cong s_2 \otimes \epsilon_l^a$ (as representations of Γ_l^0) for some $a \in \mathbb{Z}$.

Proof. Let $\tilde{s}_i$ denote the action of $\widetilde{H}_l$ on the intersection of $\Lambda_l \otimes \mathcal{O}_{M_\lambda}$ with the space underlying s_i and let $\overline{s}_i$ denote its reduction modulo λ. From Proposition 5.2.2 we see that $\overline{s}_i|_{\widetilde{G}_l^{\mathrm{sc}}(\mathbb{Z}_l)}$ is absolutely irreducible and that $s_1|_{G_l^{\mathrm{sc}}} \cong s_2|_{G_l^{\mathrm{sc}}}$.

Let $\mu_i : Z_l \to \mathbb{G}_m$ denote the action of Z_l on s_i and let $\widetilde{\mu}_i$ denote the extension of μ_i to $\widetilde{Z}_l$ and $\overline{\mu}_i$ the reduction of $\widetilde{\mu}_i$ modulo λ. Then

$$(A(1+[M:\mathbb{Q}]n)\mu_i) \circ \theta_l = \sum m_{i,\sigma}\sigma$$

with $|m_{i,\sigma}| < C(\mathcal{R}_{\mathbb{Q}})$. We have

$$\overline{\mu}_1|_{\Gamma_l^{Z,1}} = \overline{\mu}_2|_{\Gamma_l^{Z,1}}.$$

Hence

$$(A(1+[M:\mathbb{Q}]n)^2\overline{\mu}_1)|_{\Gamma_l^{C,1}} = (A(1+[M:\mathbb{Q}]n)^2\overline{\mu}_2)|_{\Gamma_l^{C,1}},$$

and for some $(1-l)/2 < b < (l-1)/2$ we have

$$(A(1+[M:\mathbb{Q}]n)^2\overline{\mu}_1)|_{\Gamma_l^{C}} = (A(1+[M:\mathbb{Q}]n)^2\overline{\mu}_2 + b\epsilon_l)|_{\Gamma_l^{C}}.$$

By assumption (3) on $\mathcal{L}$ we know that $\mu_i^{A(1+[M:\mathbb{Q}]n)}$ is a crystalline character of G_{F^0}. Thus

$$\prod_{\sigma\in\mathrm{Hom}\,(F^0,\mathbb{Q}_l^{\mathrm{nr}})}\overline{\sigma}^{A(1+n[M:\mathbb{Q}])(m_{1,\sigma}-m_{2,\sigma})}=\prod_{\sigma\in\mathrm{Hom}\,(F^0,\mathbb{Q}_l^{\mathrm{nr}})}\overline{\sigma}^{b}$$

on $(\mathcal{O}_{F^0}/l)^\times$. If $v|l$ is a prime of F^0, choose an embedding $\sigma_v : F^0 \hookrightarrow \mathbb{Q}_l^{\mathrm{nr}}$ above v. Then the embeddings of F_v^0 into $\mathbb{Q}_l^{\mathrm{nr}}$ are $\mathrm{Frob}_l^i\circ\sigma_v$ for $i = 0,\dots,f_v-1$ with $f_v = [k(v):\mathbb{F}_l]$. Then

$$\sum_{i=0}^{f_v-1}A(1+[M:\mathbb{Q}]n)(m_{1,\mathrm{Frob}_l^i\circ\sigma_v}-m_{2,\mathrm{Frob}_l^i\circ\sigma_v})l^i\equiv b(l^{f_v}-1)/(l-1)\bmod(l^{f_v}-1).$$

As $(1-l)/2 < A(1+[M:\mathbb{Q}]n)(m_{1,\sigma}-m_{2,\sigma}) < (l-1)/2$ for all σ (by assumption (2) on $\mathcal{L}$), both sides lie in the range $((1-l^{f_v})/2,(l^{f_v}-1)/2)$, and so

$$\sum_{i=0}^{f_v-1}A(1+[M:\mathbb{Q}]n)(m_{1,\mathrm{Frob}_l^i\circ\sigma_v}-m_{2,\mathrm{Frob}_l^i\circ\sigma_v})l^i=b(l^{f_v}-1)/(l-1).$$

Then we see that $A(1+[M:\mathbb{Q}]n)(m_{1,\mathrm{Frob}_l^0\circ\sigma_v}-m_{2,\mathrm{Frob}_l^0\circ\sigma_v})\equiv b \bmod l$ and again using the bounds on both sides we conclude that

$$A(1+[M:\mathbb{Q}]n)(m_{1,\mathrm{Frob}_l^0\circ\sigma_v}-m_{2,\mathrm{Frob}_l^0\circ\sigma_v})=b.$$

Subtracting these terms, dividing by l and arguing recursively we see that

$$A(1+[M:\mathbb{Q}]n)(m_{1,\mathrm{Frob}_l^i\circ\sigma_v}-m_{2,\mathrm{Frob}_l^i\circ\sigma_v})=b$$

for all i. Thus $A(1+[M:\mathbb{Q}]n)$ divides b and

$$\mu_1^{A(1+[M:\mathbb{Q}]n)}\circ\theta_l=(\mu_2^{A(1+[M:\mathbb{Q}]n)}\otimes\epsilon_l^{b/A(1+[M:\mathbb{Q}]n)})\circ\theta_l$$

as characters of $S_{F^0,l}$. Thus $\mu_1=\mu_2$ as characters of Z_l^1 and so, for some $a\in\mathbb{Z}$, we have $\mu_1=\mu_2\epsilon_l^a$ as characters of Z_l. We deduce that $s_1 = s_2\epsilon_l^a$ as representations of H_l and hence of G_l^0, and hence again of Γ_l^0. The claim follows. □

We now return to the proof of the proposition. Let $\lambda|l\in\mathcal{L}$ be a prime of M. Let s be an irreducible subrepresentation of r_λ. Then s occurs on a $M_\lambda[\Gamma_l]$-submodule $W\subset V_l\otimes M_\lambda$. Let W_0 denote an irreducible Γ_l^0-submodule of W and let s_0 denote the representation of Γ_l^0 on W_0. Note that W_0 is also H_l-invariant and G_l^{sc}-irreducible. Let Γ_l' denote the set of γ in Γ_l with $s_0^\gamma\cong s_0$, or what comes to the same thing (by regularity) $\gamma W_0 = W_0\subset W$. (See Lemma 5.3.1.) Then s_0 extends to a representation of Γ_l' and

$$s\cong\mathrm{Ind}_{\Gamma_l'}^{\Gamma_l}s_0.$$

Write $\overline{s}$ (resp. $\overline{s}_0$) for the semi-simplified reduction of s (resp. s_0).

Let W_1 denote the W_0-isotypical component of $V_l \otimes M_\lambda$ for the action of G_l^{sc}. By Proposition 5.2.2 we see that

$$(W_1 \cap (\Lambda_l \otimes \mathcal{O}_{M_\lambda}))/\lambda(W_1 \cap (\Lambda_l \otimes \mathcal{O}_{M_\lambda}))$$

is isotypical for the action of $\widetilde{G}_l^{\mathrm{sc}}(\mathbb{Z}_l)$ corresponding to an absolutely irreducible representation of dimension equal to $\dim_{M_\lambda} W_0$. Thus

$$(W_0 \cap (\Lambda_l \otimes \mathcal{O}_{M_\lambda}))/\lambda(W_0 \cap (\Lambda_l \otimes \mathcal{O}_{M_\lambda}))$$

must be absolutely irreducible as a representation of $\widetilde{G}_l^{\mathrm{sc}}(\mathbb{Z}_l)$.

As $F^0/\mathbb{Q}$ is unramified above l (assumption (1) on $\mathcal{L}$), we see that $F(\zeta_l)$ is linearly disjoint from F^0 over F. Thus

$$\overline{s}|_{\ker \bar{\epsilon}_l|_{\Gamma_l}} \cong (\mathrm{Ind}_{\ker \bar{\epsilon}_l|_{\Gamma'_l}}^{\ker \bar{\epsilon}_l|_{\Gamma_l}} \overline{s}_0)^{\mathrm{ss}}.$$

Hence it suffices to show that for $\gamma \in \Gamma_l - \Gamma'_l$ we have

$$\overline{s}_0^{\gamma}|_{\Gamma_l^{H,1}} \ncong \overline{s}_0|_{\Gamma_l^{H,1}}.$$

If they were equivalent then by the claim we would have

$$s_0^{\gamma} \cong s_0 \otimes \epsilon_l^a$$

as Γ_l^0-modules (for some $a \in \mathbb{Z}$). As γ has finite order in Γ_l/Γ_l^0, but ϵ_l has infinite order, we see that $a = 0$, and so $\gamma \in \Gamma'_l$, a contradiction. The proposition follows. □

5.4. *Potential automorphy for weakly compatible systems.* In this section we prove a potential automorphy theorem for weakly compatible systems of l-adic representations of the absolute Galois group of a CM field.

THEOREM 5.4.1. *Suppose that F/F_0 is a finite Galois extension of* CM (*resp. totally real*) *fields and that $F^{(\mathrm{avoid})}/F$ is a finite Galois extension. Suppose also for $i = 1, \dots, r$ that $(\mathcal{R}_i, \mathcal{M}_i)$ are totally odd, polarized weakly compatible systems of l-adic representations of G_F, with each $\mathcal{R}_i$ regular and irreducible. Then there is a finite,* CM (*resp. totally real*), *extension F'/F linearly disjoint from $F^{(\mathrm{avoid})}$ over F and with F'/F_0 Galois, such that each $(\mathcal{R}i|_{G_{F'}}, \mathcal{M}_i|_{G_{(F')^+}})$ is automorphic.*

Proof. The totally real case follows easily from the imaginary CM case by Lemma 2.2.2. (The only thing to check is that we can find an imaginary CM extension F'/F such that each $\mathcal{R}_i$ remains irreducible upon restriction to $G_{F'}$. It will be enough to choose F' linearly disjoint from the fields F_i^1/F obtained by applying part (1) of Lemma 5.3.1 to each $\mathcal{R}_i$. Such a choice is possibly by Lemma A.2.1.) Thus we treat only the imaginary CM case.

Replacing each of the fields M_i associated to $\mathcal{R}_i$ by their compositum, we may assume that $M_i = M$ is independent of i. Let $\mathcal{L}_{i,1}$ be the Dirichlet

density 1 set of rational primes provided by applying Proposition 5.3.2 to $\mathcal{R}_i$ and let $\mathcal{L}_{i,2}$ be the Dirichlet density 1 set of rational primes above which $\mathcal{R}$ is irreducible. Let $\mathcal{L} = \bigcap_i \mathcal{L}_{i,1} \cap \mathcal{L}_{i,2}$. Then $\mathcal{L}$ also has Dirchlet density 1 (Lemma A.1.7). For $\lambda|l \in \mathcal{L}$ we see that each $\overline{r}_{i,\lambda}|_{G_{F(\zeta_l)}}$ is irreducible.

Removing finitely many primes from $\mathcal{L}$ we may further suppose that

- $l \in \mathcal{L}$ implies $l \geq 2(\dim \mathcal{R}_i + 1)$ for each i;
- $l \in \mathcal{L}$ implies l is unramified in F and l lies below none of the elements of the sets S_i of bad primes for $\mathcal{R}_i$;
- if $\lambda|l \in \mathcal{L}$ is a place of M, then all the Hodge–Tate numbers of $r_{i,\lambda}$ lie in a range of the form $[a, a + l - 2]$.

We deduce, by Lemma 1.4.3, that each $r_{i,\lambda}$ is potentially diagonalizable. Our theorem now follows by applying Theorem 4.5.1 to the $r_{i,\lambda}$ for any $\lambda|l \in \mathcal{L}$. □

We state a simple special case separately.

COROLLARY 5.4.2. *Suppose that F is a* CM (*resp. totally real*) *field and that $(\mathcal{R}, \mathcal{M})$ is a* **totally odd, polarized weakly compatible system** *of l-adic representations of G_F, with $\mathcal{R}$* **regular and irreducible**. *Then there is a finite,* CM (*resp. totally real*), *Galois extension F'/F such that $(\mathcal{R}|_{G_{F'}}, \mathcal{M}|_{G_{(F')^+}})$ is automorphic.*

COROLLARY 5.4.3. *Keep the assumptions of the last corollary.*

(1) *If $\imath : M \hookrightarrow \mathbb{C}$, then $L^S(\imath\mathcal{R}, s)$ converges (uniformly absolutely on compact subsets) on some right half plane and has meromorphic continuation to the whole complex plane.*
(2) *The compatible system $\mathcal{R}$ is strictly pure. Moreover*
$$\Lambda(\imath\mathcal{R}, s) = \epsilon(\imath\mathcal{R}, s)\Lambda(\imath\mathcal{R}^\vee, 1 - s).$$
(3) *If F is totally real, n is odd, and $v|\infty$, then $\operatorname{tr} r_\lambda(c_v) = \pm 1$ and is independent of λ.*

Proof. The strict purity follows from Theorem 5.4.1, Theorem 2.1.1 and the usual Brauer's theorem argument as in the last paragraph of the proof of Theorem 5.5.1 below. The convergence and meromorphic continuation and functional equation of the L-function follow from the theorem and a Brauer's theorem argument as in Theorem 4.2 of [HSBT10]. The last part generalizes an observation of F. Calegari [Cal11]. The theorem reduces the question to the automorphic case where it is the main result of [Tay12]. □

As one example of the above theorem we state the following result.

COROLLARY 5.4.4. *Suppose that $\mathcal{K}$ is a finite set of positive integers with the property that the $2^{\#\mathcal{K}}$ partial sums of elements of $\mathcal{K}$ are all distinct. For each $k \in \mathcal{K}$ let f_k be an elliptic modular newform of weight $k+1$ without complex*

multiplication and let π_k be the corresponding automorphic representation of $\mathrm{GL}_2(\mathbb{A})$. *Then there is a totally real Galois extension $F/\mathbb{Q}$ and a regular algebraic, polarizable, cuspidal automorphic representation Π of $\mathrm{GL}_{2^{\#\mathcal{K}}}(\mathbb{A}_F)$ such that for all but finitely many primes v of F we have*

$$\mathrm{rec}(\Pi_v |\det|_v^{(1-2^{\#\mathcal{K}})/2}) = \left(\bigotimes_{k\in\mathcal{K}} \mathrm{rec}(\pi_{k,v|_{\mathbb{Q}}} |\det|_{v|_{\mathbb{Q}}}^{-1/2})\right)\Big|_{W_{F_v}}.$$

In particular the "multiple product" L-function $L(\times_{k\in\mathcal{K}}\pi_k, s)$ has meromorphic continuation to the whole complex plane.

Proof. Let M denote the compositum of the fields of coefficients of the f_k's. Let λ be any prime of M and let $r_{k,\lambda} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{M}_\lambda)$ be the λ-adic representation associated to f_k. Because f_k is not CM, we know that $r_{k,\lambda}$ has Zariski dense image. We will apply Theorem 5.4.1 to the weakly compatible system

$$\bigotimes_{\mathcal{K}} r_{k,\lambda}.$$

The only assumption that is perhaps not clear is that this system is irreducible. So it only remains to check this property. The argument is a variant of Goursat's lemma.

Now let H denote the Zariski closure of $(\prod_{\mathcal{K}} r_{k,\lambda})(G_{\mathbb{Q}})$ in $\mathrm{GL}_2(\overline{M}_\lambda)^{\mathcal{K}}$ and let $\overline{H}$ denote its image in $\mathrm{PGL}_2(\overline{M}_\lambda)^{\mathcal{K}}$. Note that the projection of $\overline{H}$ to each factor is surjective. As PGL_2 is a simple algebraic group and all its automorphisms are inner, $\overline{H}$ must be of the form $\mathrm{PGL}_2(\overline{M}_\lambda)^{\mathcal{I}}$ for some set $\mathcal{I}$. Moreover we can decompose $\mathcal{K} = \coprod_{i\in\mathcal{I}} \mathcal{K}_i$, and the mapping $\overline{H} \to \mathrm{PGL}_2(\overline{M}_\lambda)^{\mathcal{K}}$ is conjugate to the mapping which sends the i^{th} factor of $\mathrm{PGL}_2(\overline{M}_\lambda)^{\mathcal{I}}$ diagonally into $\prod_{i\in\mathcal{K}_i} \mathrm{PGL}_2(\overline{M}_\lambda)$. If for some i one had $\#\mathcal{K}_i > 1$, then we would have $r_{k,\lambda} = r_{k',\lambda} \otimes \chi$ for some $k \neq k'$ in $\mathcal{K}_i$ and some character χ. We can conclude that χ is de Rham, and then looking at Hodge–Tate numbers gives a contradiction. Thus we must have $\overline{H} = \mathrm{PGL}_2(\overline{M}_\lambda)^{\mathcal{K}}$ and $H \supset \mathrm{SL}_2(\overline{M}_\lambda)^{\mathcal{K}}$. As the tensor product representation of $\mathrm{SL}_2(\overline{M}_\lambda)^{\mathcal{K}}$ on $\overline{M}_\lambda^{2^{\#\mathcal{K}}}$ is irreducible, we conclude that $\bigotimes_{\mathcal{K}} r_{k,\lambda}$ is also irreducible, as desired. □

We now turn to a proposition which will be useful in the next section. Its proof is essentially the same as the proof of Theorem 5.4.1, once we have established the following lemma.

LEMMA 5.4.5. *Suppose that F is an imaginary* CM *field, that $(\mathcal{R}, \mathcal{M})$ is a polarized weakly compatible system of l-adic representations of G_F defined over M and that $\mathcal{R}$ is pure and extremely regular. If F'/F is a finite extension and if s is a subrepresentation of $r_\lambda|_{G_{F'}}$ for some prime λ of M, then there is a* CM *field F'' with $F \subset F'' \subset F'$ such that the space of s is invariant by $G_{F''}$.*

Moreover (s, μ_λ) is a polarized l-adic representation of $G_{F''}$. It is totally odd if $(\mathcal{R}, \mathcal{M})$ is.

Proof. Let F_1 denote the normal closure of F'/F^+. Let $\tau : F \hookrightarrow \overline{M}$ be an embedding with the property that if H and H' are different subsets of H_τ of the same cardinality, then $\sum_{h\in H} h \neq \sum_{h\in H'} h$. Choose an embedding $\tau_1 : F_1 \hookrightarrow \overline{M}$ extending τ. Note that $\mathcal{R}$ is pure of some weight w.

If s_1 and s_2 are two G_{F_1}-submodules of some r_λ of the same dimension, we see that $s_1 = s_2$ if and only if $\mathrm{HT}_{\tau_1}(s_1) = \mathrm{HT}_{\tau_1}(s_2)$ if and only if $\mathrm{HT}_{\tau_1}(\det s_1) = \mathrm{HT}_{\tau_1}(\det s_2)$. (The first equivalence is by regularity and the second by extreme regularity. Note that, in particular, any irreducible submodule of $r_\lambda|_{G_{F_1}}$ has multiplicity 1.)

For $\sigma \in \mathrm{Gal}\,(F_1/F^+)$ write $\mathrm{HT}_{\tau_1}((\det s)^\sigma) = \mathrm{HT}_{\tau_1\circ\sigma^{-1}}(\det s) = \{h_\sigma\}$. As $\det s$ is de Rham and pure of weight $w \dim s$, we deduce that $h_\sigma + h_{\sigma c} = w \dim s$ for all $\sigma \in \mathrm{Gal}\,(F_1/F^+)$ and all complex conjugations $c \in \mathrm{Gal}\,(F_1/F^+)$. Thus if $c, c' \in \mathrm{Gal}\,(F_1/F^+)$ are complex conjugations, then $h_{\sigma cc'} = w \dim s - h_{\sigma c} = h_\sigma$ and so $s^{\sigma cc'} = s^\sigma$.

Let $H \subset \mathrm{Gal}\,(F_1/F^+)$ be the normal subgroup generated by all elements cc' with $c, c' \in \mathrm{Gal}\,(F_1/F^+)$ complex conjugations. The maximal CM subfield of F_1 is the maximal subfield on which all complex conjugations agree, i.e., F_1^H. Hence $F'' = F_1^H \cap F' = (F_1)^{H\mathrm{Gal}\,(F_1/F')}$ is the maximal CM sub-field of F'. Moreover if $\sigma \in H$, then $s^\sigma = s$ and so s extends to a representation of $G_{F''}$.

If $c \in \mathrm{Gal}\,(F_1/F^+)$ is a complex conjugation, then

$$\mathrm{HT}_{\tau_1}(\det(\mu_\lambda(s^\vee)^c)) = \mathrm{HT}_{\tau_1 c}(\det(\mu_\lambda s^\vee)) = \{w \dim s - h_c\} = \{h_1\}.$$

As $\mu_\lambda(s^\vee)^c$ is also a constituent of r_λ, we see that $s^c \cong \mu_\lambda s^\vee$ as representations of $G_{F''}$. Let v be an infinite place of F and $\langle\ ,\ \rangle_v$ a pairing on $\overline{M}_\lambda^n$ as in the definition of polarization for (r_λ, μ_λ). Then $\langle\ ,\ \rangle_v$ restricts to a perfect pairing on the space of s, as otherwise there would be a second irreducible constituent $s' \neq s$ of $r|_{G_{F''}}$ with $s' \cong \mu_\lambda(s^\vee)^c \cong s$, a contradiction. The lemma follows. □

(We remark that we have not made use of the whole weakly compatible system, only of a single l-adic representation with the desired properties.)

Proposition 5.4.6. *Suppose that F is a* CM *field and that $(\mathcal{R}, \mathcal{M})$ is a totally odd, polarized weakly compatible system of l-adic representations. Suppose moreover that $\mathcal{R}$ is pure and extremely regular. Write $r_\lambda = r_{\lambda,1} \oplus \cdots \oplus r_{\lambda,j_\lambda}$ with each $r_{\lambda,\alpha}$ irreducible. Then there is a set of rational primes $\mathcal{L}$ of Dirichlet density* 1 *such that if λ is a prime of M lying above $l \in \mathcal{L}$, then there is a finite,* CM, *Galois extension F'/F such that each $(r_{\lambda,\alpha}|_{G_{F'}}, \mu_\lambda|_{G_{(F')^+}})$ is irreducible and automorphic.*

Proof. By Lemma 5.4.5 we see that for all λ and α the pair $(r_{\lambda,\alpha}, \mu_\lambda)$ is a totally odd polarized l-adic representation.

Let $\mathcal{L}$ be the Dirichlet density 1 set of rational primes obtained by applying Proposition 5.3.2 to $\mathcal{R}$. Then for $\lambda|l \in \mathcal{L}$ we see that $\overline{r}_{\lambda,\alpha}|_{G_{F(\zeta_l)}}$ is irreducible. Removing finitely many primes from $\mathcal{L}$ we may further suppose that

- $l \in \mathcal{L}$ implies $l \geq 2(\dim \mathcal{R} + 1)$;
- $l \in \mathcal{L}$ implies l is unramified in F and l lies below no element of the set S of bad primes for $\mathcal{R}$;
- if $\lambda|l \in \mathcal{L}$, then all the Hodge–Tate numbers of r_λ lie in a range of the form $[a, a + l - 2]$.

We deduce, by Lemma 1.4.3, that $r_{\lambda,\alpha}$ is potentially diagonalizable. Our theorem now follows by applying Theorem 4.5.1 to $\{r_{\lambda,\alpha}\}$ with $F^{(\mathrm{avoid})}$ equal to the compositum of the $\overline{F}^{\ker \overline{r}_{\lambda,\alpha}}$ for $\alpha = 1, \ldots, j_\lambda$. Then $\overline{r}_{\lambda,\alpha}|_{G_{F'}}$ will be irreducible for $\alpha = 1, \ldots, j_\lambda$, and so $r_{\lambda,\alpha}|_{G_{F'}}$ will also be irreducible. □

5.5. *Irreducibilty results.* We will first recall some basic group theory. If F is a number field and l is a rational prime, we will let $\mathrm{GG}_{F,l}$ denote the category of semi-simple, continuous representations of G_F on finite-dimensional $\overline{\mathbb{Q}}_l$-vector spaces which ramify at only finitely many primes. If U, V and W are objects of $\mathrm{GG}_{F,l}$ with $U \oplus W \cong V \oplus W$, then $U \cong V$ (because they have the same traces). We will let $\mathrm{Rep}_{F,l}$ denote the Grothendieck group of $\mathrm{GG}_{F,l}$. If V is an object of $\mathrm{GG}_{F,l}$, we will denote by $[V]$ its class in $\mathrm{Rep}_{F,l}$. We have the following functorialities:

(1) The rule $[U][V] = [U \otimes V]$ makes $\mathrm{Rep}_{F,l}$ a commutative ring with 1.
(2) If $\sigma \in G_F$, then there is a ring homomorphism $\mathrm{tr}_\sigma : \mathrm{Rep}_{F,l} \to \overline{\mathbb{Q}}_l$ defined by $\mathrm{tr}_\sigma[V] = \mathrm{tr}\,\sigma|_V$. If $A \in \mathrm{Rep}_{F,l}$, then the function $\sigma \mapsto \mathrm{tr}_\sigma A$ is a continuous class function $G_F \to \overline{\mathbb{Q}}_l$. If $A, B \in \mathrm{Rep}_{F,l}$ and $\mathrm{tr}_\sigma A = \mathrm{tr}_\sigma B$ for all $\sigma \in G_F$ (or even for a dense set of σ), then $A = B$.
(3) We will write dim for tr_1. Then in fact $\dim : \mathrm{Rep}_{F,l} \to \mathbb{Z}$ and $\dim[V] = \dim_{\overline{\mathbb{Q}}_l} V$.
(4) There is a perfect symmetric $\mathbb{Z}$-valued pairing $(\ ,\ )_{F,l}$ on $\mathrm{Rep}_{F,l}$ defined by

$$([U], [V])_{F,l} = \dim_{\overline{\mathbb{Q}}_l} \mathrm{Hom}_{G_F}(U, V).$$

If $A = \sum_i n_i[V_i]$ with the V_i irreducible and distinct, then $(A, A)_{F,l} = \sum_i n_i^2$. In particular if $A \in \mathrm{Rep}_{F,l}$ and $\dim A \geq 0$ and $(A, A)_{F,l} = 1$, then $A = [V]$ for some irreducible object V of $\mathrm{GG}_{F,l}$.
(5) Suppose that G_1 and G_2 are algebraic groups over $\overline{\mathbb{Q}}_l$ and that $\theta : G_F \to G_1(\overline{\mathbb{Q}}_l) \times G_2(\overline{\mathbb{Q}}_l)$ is a continuous homomorphism with Zariski dense image. Suppose also that ρ_i and ρ_i' are semi-simple algebraic representations of G_i

over $\overline{\mathbb{Q}}_l$. Then

$$\begin{aligned}([(\rho_1 \otimes \rho_2) \circ \theta],&[(\rho_1' \otimes \rho_2') \circ \theta])_{F,l} \\ &= \dim \mathrm{Hom}_{G_1 \times G_2}(\rho_1 \otimes \rho_2, \rho_1' \otimes \rho_2') \\ &= (\dim \mathrm{Hom}_{G_1}(\rho_1, \rho_1'))(\dim \mathrm{Hom}_{G_2}(\rho_2, \rho_2')) \\ &= ([\rho_1 \circ \theta], [\rho_1' \circ \theta])_{F,l}([\rho_2 \circ \theta], [\rho_2' \circ \theta])_{F,l}.\end{aligned}$$

(6) If $\sigma \in G_{\mathbb{Q}}$, then there is a ring isomorphism conj_σ from $\mathrm{Rep}_{F,l}$ to $\mathrm{Rep}_{\sigma^{-1}F,l}$ such that $\mathrm{conj}_\sigma[V]$ equals the class of the representation of $G_{\sigma^{-1}F}$ on V, under which τ acts by $\sigma\tau\sigma^{-1}$. It preserves dimension and takes $(\ ,\ )_{F,l}$ to $(\ ,\ )_{\sigma^{-1}F,l}$. We have $\mathrm{tr}_{\ \tau}\mathrm{conj}_\sigma A = \mathrm{tr}_{\ \sigma\tau\sigma^{-1}} A$. Also if $\sigma \in G_F$, then conj_σ is the identity on $\mathrm{Rep}_{F,l}$.

(7) If F'/F is a finite extension, then the formula $\mathrm{res}_{F'/F}[V] = [V|_{G_{F'}}]$ defines a ring homomorphism $\mathrm{res}_{F'/F} : \mathrm{Rep}_{F,l} \to \mathrm{Rep}_{F',l}$. Note that if $\sigma \in G_{F'}$, then $\mathrm{tr}_{\ \sigma}\mathrm{res}_{F'/F}A = \mathrm{tr}_{\ \sigma}A$ (so in particular $\dim \mathrm{res}_{F'/F}A = \dim A$). If $\sigma \in G_{\mathbb{Q}}$, then $\mathrm{conj}_\sigma \circ \mathrm{res}_{F'/F} = \mathrm{res}_{\sigma^{-1}F'/\sigma^{-1}F} \circ \mathrm{conj}_\sigma$.

(8) If F'/F is a finite extension, there is a $\mathbb{Z}$-linear map $\mathrm{ind}_{F'/F} : \mathrm{Rep}_{F',l} \to \mathrm{Rep}_{F,l}$ defined by $\mathrm{ind}_{F'/F}[V] = [\mathrm{Ind}_{G_{F'}}^{G_F} V]$. Note the following:

(a) $\mathrm{tr}_{\ \sigma}\mathrm{ind}_{F'/F}A = \sum_{\tau \in G_F/G_{F'}:\ \tau\sigma\tau^{-1} \in G_{F'}} \mathrm{tr}_{\ \tau\sigma\tau^{-1}} A$.

(b) $\dim \mathrm{ind}_{F'/F}A = [F' : F] \dim A$.

(c) $\mathrm{ind}_{F'/F}(A(\mathrm{res}_{F'/F}B)) = (\mathrm{ind}_{F'/F}A)B$.

(d) $(\mathrm{ind}_{F'/F}A, B)_{F,l} = (A, \mathrm{res}_{F'/F}B)_{F',l}$ (by Frobenius reciprocity).

(e) If F''/F is another finite extension, then

$$\mathrm{res}_{F''/F} \circ \mathrm{ind}_{F'/F} = \sum_{[\sigma] \in G_{F'} \backslash G_F / G_{F''}} \mathrm{ind}_{(\sigma^{-1}F').F''/F''} \circ \mathrm{conj}_\sigma \circ \mathrm{res}_{F'.(\sigma F'')/F'}$$

(by Mackey's formula).

(f) If F'/F is a finite Galois extension, then there is a finite collection of intermediate fields $F'/F_i'/F$ with F'/F_i' soluble together with characters $\psi_i : \mathrm{Gal}\,(F'/F_i') \to \mathbb{C}^\times$ and integers n_i such that

$$1 = \sum_i n_i \mathrm{ind}_{F_i'/F}[\psi_i]$$

in the Grothendieck group of finite dimensional representation of the finite group $\mathrm{Gal}\,(F'/F)$ over $\mathbb{C}$. (This is just Brauer's theorem for $\mathrm{Gal}\,(F'/F)$.) If $\imath : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$, then applying $\imath^{-1}$ and multiplying by any $A \in \mathrm{Rep}_{F,l}$ we conclude that

$$A = \sum_i n_i \mathrm{ind}_{F_i'/F}([\imath^{-1}\psi_i]\mathrm{res}_{F_i'/F}A).$$

Writing for any i, j

$$G_F = \coprod_k G_{F_i'} \sigma_{ijk} G_{F_j'}$$

we see further that if

$$A = \sum_i n_i \mathrm{ind}_{F_i'/F}([\imath^{-1}\psi_i]B_i),$$

then

$$(A,A)_{F,l} = \sum_{i,j,k} n_i n_j \big((\mathrm{conj}_{\sigma_{ijk}} \circ \mathrm{res}_{F_i'.(\sigma_{ijk}F_j')/F_i'})([\imath^{-1}\psi_i]B_i), \\ \mathrm{res}_{(\sigma_{ijk}^{-1}F_i').F_j'/F_j'}([\imath^{-1}\psi_j]B_j)\big)_{(\sigma_{ijk}^{-1}F_i').F_j',l}.$$

(9) If S is a finite set of primes of F including all those above l, we will say that $A \in \mathrm{Rep}_{F,l}$ is unramified outside S if we can write $A = \sum_i n_i[V_i]$ with each V_i unramified outside S. In this case we can define, for each $\imath : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$,

$$L^S(\imath A, s) = \prod_i L^S(\imath V_i, s)^{n_i}$$

at least as a formal Euler product, which will converge in some right half plane if, for each i, the Weil–Deligne representation $\mathrm{WD}(V_i|_{G_{F,v}})$ is pure of weight w_i for all but finitely many primes $v \notin S$ of F. This definition is independent of the choices and we have

$$L^S(\imath(A+B), s) = L^S(\imath A, s)L^S(\imath B, s)$$

and

$$L^S(\imath \mathrm{ind}_{F'/F} A, s) = L^{S'}(\imath A, s),$$

where S' denotes the set of primes of F' above S.

Our first result is not really an irreducibility result, but it uses similar methods so we include it here. It is a generalization of results of Dieulefait [Die04] in dimension 2. The key ingredient is Theorem 4.5.1.

THEOREM 5.5.1. *Suppose that F is a* CM *field, that n is a positive integer and that $l \geq 2(n+1)$ is a rational prime such that $\zeta_l \notin F$. Suppose that (r, μ) is an n-dimensional,* **totally odd, regular algebraic, polarized l-adic representation** *of G_F. Suppose moreover that the following conditions are satisfied*:

(1) (Potential diagonalizability). *r is potentially diagonalizable (and hence potentially crystalline) at each prime v of F above l.*
(2) (Irreducibility). *$\bar{r}|_{G_{F(\zeta_l)}}$ is irreducible.*

Then r is part of a strictly pure compatible system of l-adic representations of G_F.

Proof. Let G denote the Zariski closure of the image of r, let G^0 denote the connected component of G and let $F^0 = \overline{F}^{r^{-1}G^0(\overline{\mathbb{Q}}_l)}$. By Theorem 4.5.1 (or Corollary 4.5.2) we can find a finite Galois CM extension F'/F, which is

totally real if F is totally real and which is linearly disjoint from $F^0\overline{F}^{\ker \overline{r}}$ over F, an isomorphism $\imath : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$, and a cuspidal, regular algebraic, polarized automorphic representation (π, χ) of $\mathrm{GL}_n(\mathbb{A}_{F'})$ such that

$$(r_{l,\imath}(\pi), r_{l,\imath}(\chi)\epsilon_l^{1-n}) \cong (r|_{G_{F'}}, \mu|_{G_{(F')^+}}).$$

For the rest of this proof we will fix F'. Note that F' is automatically Galois over any intermediate field between F' and F. Suppose that $F' \supset F'' \supset F$ with F'/F'' soluble. Then by Lemma 2.2.2 there is a cuspidal, regular algebraic, polarized automorphic representation $(\pi^{(F'')}, \chi^{(F'')})$ of $\mathrm{GL}_n(\mathbb{A}_{F''})$ such that

$$(r_{l,\imath}(\pi^{(F'')}), r_{l,\imath}(\chi)\epsilon_l^{1-n}) \cong (r|_{G_{F''}}, \mu|_{G_{(F'')^+}}).$$

Let l' be a rational prime and let $\imath' : \overline{\mathbb{Q}}_{l'} \xrightarrow{\sim} \mathbb{C}$. Note that if $\sigma \in G_F$ and if $F' \supset F'' \supset F''' \supset F$ with F'/F''' soluble (in which case F'/F'' is also soluble), then $r_{l',\imath'}(\pi^{(F''')})|_{G_{F''}} \cong r_{l',\imath'}(\pi^{(F'')})$ and $r_{l',\imath'}(\pi^{(F'')})^\sigma \cong r_{l',\imath'}(\pi^{(\sigma^{-1}F'')})$. Let G' (resp. $G^{(F'')} \supset G'$) denote the Zariski closure of $r_{l',\imath'}(\pi)$ (resp. $r_{l',\imath'}(\pi^{(F'')})$) and let $(G')^0$ (resp. $(G^{(F'')})^0$) denote the connected component of G' (resp. $G^{(F'')}$). It follows from Lemma 5.3.1 that $G^{(F'')}/(G^{(F'')})^0$ and the map $G_{F''} \twoheadrightarrow G^{(F'')}/(G^{(F'')})^0$ is independent of l'. In the case $(l', \imath') = (l, \imath)$ we see, by the choice of F', that $\mathrm{Gal}\,(F^0F''/F'') \xrightarrow{\sim} G^{(F'')}/(G^{(F'')})^0$. Thus this is true for all $(l', \imath')$. We deduce that (for any $(l', \imath')$) we have $G^{(F'')} = G'$ and that the natural map

$$G_{F''} \longrightarrow G'(\overline{\mathbb{Q}}_{l'}) \times \mathrm{Gal}\,(F'/F'')$$

has Zariski dense image (where we consider $\mathrm{Gal}\,(F'/F'')$ as a finite algebraic group). If we decompose $r_{l',\imath'}(\pi)$ into irreducibles as

$$r_{l',\imath'}(\pi) = r_{l',\imath'}(\pi)_1 \oplus \cdots \oplus r_{l',\imath'}(\pi)_t,$$

then this induces a unique decomposition

$$r_{l',\imath'}(\pi^{(F'')}) = r_{l',\imath'}(\pi^{(F'')})_1 \oplus \cdots \oplus r_{l',\imath'}(\pi^{(F'')})_t$$

with $r_{l',\imath'}(\pi^{(F'')})_\alpha|_{G_{F'}} \cong r_{l',\imath'}(\pi)_\alpha$. (Note that by regularity the $r_{l',\imath'}(\pi)_\alpha$ are pairwise nonisomorphic, as they will have different Hodge–Tate numbers.) We deduce that if $\sigma \in G_F$ and if $F' \supset F'' \supset F''' \supset F$ with F'/F''' soluble (so that F'/F'' is also soluble), then $r_{l',\imath'}(\pi^{(F''')})_\alpha|_{G_{F''}} \cong r_{l',\imath'}(\pi^{(F'')})_\alpha$ and $r_{l',\imath'}(\pi^{(F'')})_\alpha^\sigma \cong r_{l',\imath'}(\pi^{(\sigma^{-1}F'')})_\alpha$. Moreover if ρ_1 and ρ_2 are representations of $\mathrm{Gal}\,(F'/F'')$ over $\overline{\mathbb{Q}}_{l'}$, then

$$([r_{l'\imath'}(\pi^{(F'')})_\alpha][\rho_1], [r_{l'\imath'}(\pi^{(F'')})_\alpha][\rho_2])_{F'',l'} = ([\rho_1], [\rho_2])_{F'',l'}.$$

Choose intermediate fields F_i', characters ψ_i, integers n_i, and elements $\sigma_{ijk} \in G_F$, as in item (8f) above. Write F_{ijk} for $(\sigma_{ijk}^{-1}F_i').F_j'$. Then

$$[r] = \sum_i n_i \mathrm{ind}_{F_i'/F}[r_{l,\imath}(\pi^{(F_i')} \otimes (\psi_i \circ \mathrm{Art}_{F_i'} \circ \det))]$$

in $\mathrm{Rep}_{F,l}$. This motivates us to set

$$A_{l',\imath',\alpha} = \sum_i n_i \mathrm{ind}_{F_i'/F}([r_{l',\imath'}(\pi^{(F_i')})_\alpha][(\imath')^{-1}\circ\psi_i]) \in \mathrm{Rep}_{F,l'}.$$

Note that

$$\dim A_{l',\imath',\alpha} = \sum_i n_i [F_i' : F] \dim r_{l',\imath'}(\pi)_\alpha = \dim r_{l',\imath'}(\pi)_\alpha.$$

Also note that

$$\begin{aligned}(A_{l',\imath',\alpha}, A_{l',\imath',\alpha})_{F,l'} &= \sum_{i,j,k} n_i n_j \Big([r_{l',\imath'}(\pi^{(F_{ijk})})_\alpha][(\imath')^{-1}\psi_i^{\sigma_{ijk}}|_{G_{F_{ijk}}}], \\ &\qquad\qquad [r_{l',\imath'}(\pi^{(F_{ijk})})_\alpha][(\imath')^{-1}\psi_j|_{G_{F_{ijk}}}]\Big)_{F_{ijk},l'} \\ &= \sum_{i,j,k} n_i n_j ([(\imath')^{-1}\psi_i^{\sigma_{ijk}}|_{G_{F_{ijk}}}], [(\imath')^{-1}\psi_j|_{G_{F_{ijk}}}])_{F_{ijk},l'} \\ &= (1,1)_{F,l'} \\ &= 1.\end{aligned}$$

(The first and third equality follow from item (8f) above, while the second equality follows from items (4) and (5) above.) Thus $A_{l',\imath',\alpha} = [r_{l',\imath',\alpha}]$ for some irreducible continuous representation $r_{l',\imath',\alpha}$ of G_F on a $\overline{\mathbb{Q}}_{l'}$-vector space of dimension $\dim r_{l',\imath'}(\pi)_\alpha$. Set

$$r_{l',\imath'} = r_{l',\imath',1} \oplus \cdots \oplus r_{l',\imath',t}.$$

We see that $r_{l,\imath} \cong r$ and that

$$\mathrm{tr}\, r_{l',\imath'}(\sigma) = \sum_i n_i \sum_{\tau \in G_F/G_{F_i'},\ \tau\sigma\tau^{-1}\in G_{F_i'}} ((\imath')^{-1}\psi_i(\tau\sigma\tau^{-1}))\mathrm{tr}\, r_{l',\imath'}(\pi^{(F_i')})(\tau\sigma\tau^{-1}).$$

Let v' denote a prime of F' and set $v = v'|_F$. By Theorem 2.1.1, if $v \not| l'$, then

$$\imath'\mathrm{WD}(r_{l',\imath'}|_{G_{F'_{v'}}})^{F-\mathrm{ss}} \cong \mathrm{rec}(\pi_{v'} \otimes |\det|_{v'}^{(1-n)/2})$$

is pure. Hence $\imath'\mathrm{WD}(r_{l',\imath'}|_{G_{F_v}})$ is also pure. (See Lemma 1.3.6.) Moreover if $\sigma \in W_{F_v}$, then

$$\begin{aligned}&\mathrm{tr}\, \imath'\mathrm{WD}(r_{l',\imath'}|_{G_{F_v}})(\sigma) = \imath'\mathrm{tr}\, r_{l',\imath'}(\sigma) \\ &= \sum_i n_i \sum_{\tau \in G_F/G_{F_i'},\ \tau\sigma\tau^{-1}\in G_{F_i'}} \psi_i(\tau\sigma\tau^{-1})\imath'\mathrm{tr}\, r_{l',\imath'}(\pi^{(F_i')})(\tau\sigma\tau^{-1}) \\ &= \sum_i n_i \sum_{\tau \in G_F/G_{F_i'},\ \tau\sigma\tau^{-1}\in G_{F_i'}} \psi_i(\tau\sigma\tau^{-1})\mathrm{tr}\, \mathrm{rec}(\pi^{(F_i')}_{(\tau v')|_{F_i'}} \otimes |\det|^{(1-n)/2}_{(\tau v')|_{F_i'}})(\tau\sigma\tau^{-1})\end{aligned}$$

(again by Theorem 2.1.1). If l'' is another prime with $v \not| l''$ and if $\imath'' : \overline{\mathbb{Q}}_{l''} \xrightarrow{\sim} \mathbb{C}$, then we conclude that

$$\imath' \mathrm{WD}(r_{l',\imath'}|_{G_{F_v}})^{\mathrm{ss}} \cong \imath'' \mathrm{WD}(r_{l'',\imath''}|_{G_{F_v}})^{\mathrm{ss}}.$$

As both are pure, we conclude from Lemma 1.3.6 that

$$\imath' \mathrm{WD}(r_{l',\imath'}|_{G_{F_v}})^{F-\mathrm{ss}} \cong \imath'' \mathrm{WD}(r_{l'',\imath''}|_{G_{F_v}})^{F-\mathrm{ss}}.$$

Thus the $r_{l',\imath'}$ form a strictly pure compatible system. □

Recall that if (π, χ) is a cuspidal, regular algebraic, polarized automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$, then $(\{r_{l,\imath}(\pi)\}, \{r_{l,\imath}(\chi)\epsilon_l^{1-n}\})$ is a strictly pure polarized compatible system of weight w. (See Theorem 2.1.1 and the discussion at the end of Section 5.1.) Then $|\chi| = ||\,||_{F^+}^{n-1-w}$. If π has central character χ_π, then we see that $|\chi_\pi| = ||\,||_F^{n(n-1-w)/2}$, and so $\pi \otimes ||\det||_F^{(w+1-n)/2}$ has a unitary central character and so is unitary. If (π', χ') is a cuspidal, regular algebraic, polarized automorphic representation of $\mathrm{GL}_{n'}(\mathbb{A}_F)$ and if $\{r_{l,\imath}(\pi')\}$ has weight w' and if S is a finite set of finite places of F, then

$$\begin{aligned} &L^S(\pi \times (\pi')^\vee, s + (w - w' + n' - n)/2) \\ &\qquad = L^S\left((\pi||\det||_F^{(w+1-n)/2}) \times (\pi'||\det||_F^{(w'+1-n')/2})^\vee, s\right) \end{aligned}$$

is meromorphic and is holomorphic and nonzero at $s = 1$ unless

$$\pi \cong \pi'||\det||_F^{(w'-w+n-n')/2},$$

in which case it has a simple pole at $s = 1$. (See [Sha81] and [JS81].)

THEOREM 5.5.2. *Suppose that F is a* CM *field and that π is a regular algebraic, polarizable, cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$. If π has extremely regular weight, then there is a set of rational primes $\mathcal{L}$ of Dirichlet density 1 such that if $l \in \mathcal{L}$ and $\imath : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$, then $r_{l,\imath}(\pi)$ is irreducible.*

Proof. Let $\mathcal{L}$ be the set of rational primes of Dirichlet density 1 provided by Proposition 5.4.6 applied to the compatible system $\mathcal{R} := \{r_{l,\imath}(\pi)\}$. Suppose $l \in \mathcal{L}$ and $\imath : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$. Let

$$r_{l,\imath}(\pi) = r_{l,\imath}(\pi)_1 \oplus \cdots \oplus r_{l,\imath}(\pi)_j$$

be a decomposition into irreducibles. Let F'/F and π_α for $\alpha = 1, \ldots, j$ be as in Proposition 5.4.6 for $\{r_{l',\imath'}(\pi)\}$ and $(l, \imath)$. Let S denote the finite set of primes of F which divide l or above which π ramifies or above which F' ramifies. Then

$$\mathrm{ord}_{s=1} L^S(\imath(\mathcal{R} \otimes \mathcal{R}^\vee), s) = \mathrm{ord}_{s=1} L^S(\pi \times \pi^\vee, s) = -1.$$

We will show that $\mathrm{ord}_{s=1} L^S(\imath(\mathcal{R} \otimes \mathcal{R}^\vee), s)$ also equals $-j$, and the theorem will follow.

Suppose that we are given an intermediate field $F' \supset F'' \supset F$ with F'/F'' soluble. By Lemma 2.2.2 there is a regular algebraic, polarizable, cuspidal automorphic representation $\pi_\alpha^{(F'')}$ of $\mathrm{GL}_{n_\alpha}(\mathbb{A}_{F''})$ such that $r_{l,\imath}(\pi_\alpha^{(F'')}) \cong r_{l,\imath}(\pi)_\alpha|_{G_{F''}}$. Moreover $r_{l,\imath}(\pi)_\alpha|_{G_{F''}}$ is irreducible. Let $n_\alpha = \dim r_{l,\imath}(\pi)_\alpha$. If $\psi : \mathrm{Gal}\,(F'/F'') \to \overline{\mathbb{Q}}_l^\times$ is a character, then

$$\begin{aligned}
&\mathrm{ord}_{s=1}\, L^S(\imath r_{l,\imath}(\pi)_\alpha|_{G_{F''}} \otimes r_{l,\imath}(\pi)_\beta^\vee|_{G_{F''}} \otimes \psi, s) \\
&\quad = \mathrm{ord}_{s=1}\, L^S(\pi_\alpha^{(F'')} \times (\pi_\beta^{(F'')})^\vee \times (\imath \circ \psi \circ \mathrm{Art}_{F''}), s + (n_\beta - n_\alpha)/2) \\
&\quad = -\delta_{\alpha,\beta}\delta_{\psi,1} \\
&\quad = -([r_{l,\imath}(\pi)_\alpha|_{G_{F''}}][\psi], [r_{l,\imath}(\pi)_\beta|_{G_{F''}}])_{F'',l},
\end{aligned}$$

where $\delta_{\alpha,\beta} = 1$ if $\alpha = \beta$ and equals 0 otherwise, and where $\delta_{\psi,1} = 1$ if $\psi = 1$ and equals 0 otherwise. (Note that the $\pi_\gamma^{(F'')}||\det||_{F''}^{(1-n_\gamma)/2}$ have different weights so that if $\pi_\gamma^{(F'')}||\det||_{F''}^{(1-n_\gamma)/2} \cong \pi_{\gamma'}^{(F'')}||\det||^{(1-n_{\gamma'})/2}(\imath \circ \psi \circ \mathrm{Art}_{F''} \circ \det)$, then $\gamma = \gamma'$. Moreover, we would also have $r_{l,\imath}(\pi)_\gamma|_{G_{F''}} \cong r_{l,\imath}(\pi)_\gamma|_{G_{F''}} \otimes \psi$, and so, as $r_{l,\imath}(\pi)_\gamma|_{G_{F'}}$ is irreducible, $\psi = 1$. Similarly the $r_{l,\imath}(\pi)_\gamma|_{G_{F''}}$ have different Hodge–Tate numbers, so if $r_{l,\imath}(\pi)_\gamma|_{G_{F''}} \cong r_{l,\imath}(\pi)_{\gamma'}|_{G_{F''}} \otimes \psi$, then $\gamma = \gamma'$. Moreover, as $r_{l,\imath}(\pi)_\gamma|_{G_{F'}}$ is irreducible, we see that we also have $\psi = 1$.) Thus

$$\begin{aligned}
&\mathrm{ord}_{s=1}\, L^S(\imath(r_{l,\imath}(\pi)|_{G_{F''}} \otimes r_{l,\imath}(\pi)|_{G_{F''}}^\vee \otimes \psi, s) \\
&\qquad\qquad = -(\mathrm{res}_{F''/F}[r_{l,\imath}(\pi)][\psi], \mathrm{res}_{F''/F}[r_{l,\imath}(\pi)])_{F'',l}.
\end{aligned}$$

Now let F_i', n_i and ψ_i be as in item (8f) of the list at the start of this section. Then

$$L^S(\imath(\mathcal{R} \otimes \mathcal{R}^\vee), s) = \prod_i L^S(\imath(r_{l,\imath}(\pi)|_{G_{F_i'}} \otimes r_{l,\imath}(\pi)|_{G_{F_i'}}^\vee \otimes \psi_i, s)^{n_i}$$

and

$$\begin{aligned}
\mathrm{ord}_{s=1}\, L^S(\imath(\mathcal{R} \otimes \mathcal{R}^\vee), s &= -\sum_i n_i([\psi_i]\mathrm{res}_{F_i'/F}[r_{l,\imath}(\pi)], \mathrm{res}_{F_i'/F}[r_{l,\imath}(\pi)])_{F_i',l} \\
&= -\sum_i n_i(\mathrm{ind}_{F_i'/F}([\psi_i]\mathrm{res}_{F_i'/F}[r_{l,\imath}(\pi)]), [r_{l,\imath}(\pi)])_{F,l} \\
&= -([r_{l,\imath}(\pi)], [r_{l,\imath}(\pi)])_{F,l} \\
&= -j,
\end{aligned}$$

as desired. □

THEOREM 5.5.3. *Suppose that F is a* CM *field and that $\mathcal{R}$ is a pure, extremely regular, totally odd, polarizable weakly compatible system of l-adic representations of G_F. Then we can write $\mathcal{R} = \mathcal{R}_1 \oplus \cdots \oplus \mathcal{R}_s$ where each $\mathcal{R}_i$ is an irreducible, strictly pure, totally odd, polarizable compatible system of l-adic representations of G_F.*

Proof. Choose a set $\mathcal{L}$ of rational primes of Dirichlet density 1 which simultaneously works for Propositions 5.3.2 and 5.4.6. (See Lemma A.1.7.) Choose $\lambda|l \in \mathcal{L}$ such that l is unramified in F and $l \geq 2(n+1)$ and r_λ is crystalline with Hodge–Tate numbers all in an interval of the form $[a, a+l-2]$. Decompose r_λ into irreducible subrepresentations

$$r_\lambda = r_{\lambda,1} \oplus \cdots \oplus r_{\lambda,j_\lambda}.$$

By Theorem 5.5.1 each $r_{\lambda,\alpha}$ is part of a strictly pure compatible system $\mathcal{R}_\alpha$. Let F'/F and π_α for $\alpha = 1, \ldots, j_\alpha$ be as in Proposition 5.4.6 for $\mathcal{R}$ and λ. Then $\mathcal{R}_\alpha|_{G_{F'}}$ is the compatible system associated to π_α. Moreover π_α is extremely regular. By Theorem 5.5.2 there is a set $\mathcal{L}_\alpha$ of rational primes of Dirichlet density 1 such that if $\lambda'|l' \in \mathcal{L}_\alpha$, then $r_{\alpha,\lambda'}|_{G_{F'}}$ is irreducible. Thus $\mathcal{R}_\alpha$ is irreducible. □

Appendix A. Algebraic lemmas

The results recorded in this appendix make no reference to results proved elsewhere in this paper.

A.1. *Some algebra.* The results of this section are probably well known to experts. However as we could not find references we include proofs. We start with some commutative algebra.

Suppose that L is a finite extension of $\mathbb{Q}_l$ and let $|\ |_L$ denote the l-adic norm on L normalized by $|l|_L = l^{-[L:\mathbb{Q}_l]}$. We will denote the l-adic completion of the polynomial ring $\mathcal{O}_L[s_1, \ldots, s_r]$ by $\mathcal{O}_L\langle s_1, \ldots, s_r\rangle$ and we will let $L\langle s_1, \ldots, s_r\rangle = \mathcal{O}_L\langle s_1, \ldots, s_r\rangle[1/l]$ denote the Tate algebra. The Gauss norm $|f|_L$ of an element $f \in L\langle s_1, \ldots, s_r\rangle$ is defined to be the maximum of the $|\ |_L$ norm of any coefficient of a monomial in the s_i's. The Gauss norm is multiplicative ([BGR84, 5.1.2]) and $L\langle s_1, \ldots, s_r\rangle$ is a UFD ([BGR84, 5.2.6]). From these two facts one can deduce without difficulty that $\mathcal{O}_L\langle s_1, \ldots, s_r\rangle$ is also a UFD. (The units are the units in $L\langle s_1, \ldots, s_r\rangle$ with Gauss norm 1; and the irreducibles are the irreducibles in $L\langle s_1, \ldots, s_r\rangle$ with Gauss norm 1 together with a uniformizer in $\mathcal{O}_L$ times any unit in $\mathcal{O}_L\langle s_1, \ldots, s_r\rangle$.) The Tate algebra $L\langle s_1, \ldots, s_r\rangle$ is noetherian ([BGR84, 5.2.6]); all its ideals are closed ([BGR84, 5.2.7]); and a prime ideal $\wp$ of $L\langle s_1, \ldots, s_r\rangle$ is maximal if and only if $L\langle s_1, \ldots, s_r\rangle/\wp$ is a finite extension of L ([BGR84, 6.1.2]).

By an affinoid algebra we shall mean a quotient of a Tate algebra by some ideal. Maximal ideals are dense in the spectrum of an affinoid algebra ([BGR84, 6.1.1]). The completed tensor product of two affinoid algebras is again an affinoid algebra ([BGR84, 6.1.1]). We will call an affinoid algebra A *geometrically connected* if $\operatorname{Spec} A \otimes_L L'$ is connected for all finite extensions L'/L. It suffices to check this for a cofinal collection of finite extensions L'/L

(because, for any field L', any L'-algebra B and any finite extension L''/L, if $\operatorname{Spec} B \otimes_{L'} L''$ is connected, then so is $\operatorname{Spec} B$).

LEMMA A.1.1. *Suppose that A_1 and A_2 are two geometrically connected affinoid algebras over L. Then $A_1 \widehat{\otimes}_L A_2$ is also geometrically connected.*

Proof. Suppose that

$$\operatorname{Spec} A_1 \widehat{\otimes}_L A_2 \otimes_L L' = U^{(1)} \coprod U^{(2)}$$

is a decomposition into two nonempty open subsets for some finite extension L'/L. We will derive a contradiction. Each $U^{(j)}$ contains a maximal ideal and after replacing L' by a finite extension we may suppose that each $U^{(j)}$ contains a maximal ideal $\mathfrak{m}^{(j)}$ with residue field L'. Let $\mathfrak{m}_j$ denote the contraction of $\mathfrak{m}^{(j)}$ into $A_j \otimes_L L'$. Then $\mathfrak{m}_j$ is a prime ideal and $(A_j \otimes_L L')/\mathfrak{m}_j = L'$. Thus $\mathfrak{m}_j$ is maximal. We have an isomorphism

$$(A_1 \widehat{\otimes}_L A_2 \otimes_L L')/\mathfrak{m}_j \cong A_{j'} \otimes_L L',$$

where $j' \neq j$. Thus the fibre $\operatorname{Spec} (A_1 \widehat{\otimes}_L A_2 \otimes_L L')/\mathfrak{m}_j$ is connected, and so

$$\operatorname{Spec} (A_1 \widehat{\otimes}_L A_2 \otimes_L L')/\mathfrak{m}_j \subset U^{(j)}.$$

However

$$(A_1 \widehat{\otimes}_L A_2 \otimes_L L')/(\mathfrak{m}_1, \mathfrak{m}_2) \cong (A_1 \otimes L')/\mathfrak{m}_1 \otimes_{L'} (A_2 \otimes L')/\mathfrak{m}_2 \cong L',$$

and so we see $(\mathfrak{m}_1, \mathfrak{m}_2)$ is a maximal ideal of $A_1 \widehat{\otimes}_L A_2 \otimes_L L'$ lying in $U^{(1)} \cap U^{(2)}$, a contradiction. □

LEMMA A.1.2. *Suppose that $f(t) \in \mathcal{O}_L[t]$ and $f(0) \in \mathcal{O}_L^\times$. Then*

$$A = \mathcal{O}_L\langle s, t\rangle/(sf(t) - 1)$$

is an integral domain complete in the l-adic topology.

Proof. As $\mathcal{O}_L\langle s, t\rangle$ is a UFD, it suffices to prove that $sf(t)-1$ is irreducible. Suppose $sf(t) - 1 = gh$ in $\mathcal{O}_L\langle s, t\rangle$. Reducing modulo λ and using the fact that $sf(t) - 1$ is irreducible in $(\mathcal{O}_L/\lambda)[s, t]$, we see that one of g and h reduces to a constant and hence is itself a unit. The lemma follows. □

COROLLARY A.1.3. *If $\alpha \in \mathcal{O}_L - \{0\}$, then $\mathcal{O}_L\langle s, t\rangle/(st - \alpha)$ is an integral domain.*

Proof. Consider the map

$$\begin{array}{rcl} \theta : \mathcal{O}_L\langle s, t\rangle/(st - \alpha) & \longrightarrow & \mathcal{O}_L\langle s', t\rangle/(s't - 1), \\ h(s, t) & \longmapsto & h(\alpha s', t). \end{array}$$

By Lemma A.1.2 it suffices to show that θ is injective.

If $\beta \in \mathcal{O}_L - \{0\}$, then any element of $\mathcal{O}_L\langle s,t\rangle/(st-\beta)$ has a representative of the form

$$\sum_{i=0}^{\infty} a_i t^i + \sum_{i=1}^{\infty} b_i s^i.$$

Moreover we claim this representative is unique. Indeed if

$$\sum_{i=0}^{\infty} a_i t^i + \sum_{i=1}^{\infty} b_i s^i = (st-\beta) \sum_{i,j=0}^{\infty} c_{i,j} s^i t^j,$$

then

$$c_{i+1,j+1} = \beta^{-1} c_{i,j}$$

for all $i,j \in \mathbb{Z}_{\geq 0}$. If $\sum_{i,j=0}^{\infty} c_{i,j} s^i t^j \in \mathcal{O}_L\langle s,t\rangle$, this forces $c_{i,j} = 0$ for all $i,j \in \mathbb{Z}_{\geq 0}$, which proves the claim.

Applying these observations for $\beta = \alpha$ and for $\beta = 1$, the injectivity of θ follows. □

LEMMA A.1.4. *Suppose that* $\alpha_1, \dots, \alpha_r \in \mathcal{O} - \{0\}$. *Then*

$$A = \mathcal{O}_L\langle s_1, t_1, s_2, \dots, t_r\rangle/(s_1t_1 - \alpha_1, \dots, s_rt_r - \alpha_r)$$

is an integral domain complete in the l*-adic topology.*

Proof. By the exactness of completion (and noetherianness of the polynomial ring $\mathcal{O}_L[s_1, t_1, s_2, \dots, t_r]$) we see that A is the l-adic completion of

$$A^0 = \mathcal{O}_L[s_1, t_1, s_2, \dots, t_r]/(s_1t_1 - \alpha_1, \dots, s_rt_r - \alpha_r).$$

The ring A^0 is noetherian and flat over $\mathcal{O}_L$ (because it is free over $\mathcal{O}_L$ with basis $\{\prod s_i^{a_i} t_i^{b_i} : \ a_i b_i = 0\}$), and so A is also flat over $\mathcal{O}_L$. Thus it suffices to show that

$$A[1/l] = L\langle s_1, t_1, s_2, \dots, t_r\rangle/(s_1t_1 - \alpha_1, \dots, s_rt_r - \alpha_r)$$

is a domain.

The ring A^0 is also excellent (being a finitely generated $\mathcal{O}_L$-algebra), Cohen–Macaulay (being a complete intersection in a polynomial ring over $\mathcal{O}_L$) and normal (being Cohen–Macaulay and regular in codimension 1). Hence A is normal ([Mat80, 33.I]), and so $A[1/l]$ is also normal. Thus it suffices to show that $A[1/l]$ is geometrically connected (or even just connected).

In the case $r = 1$ we see that, for any finite extension L'/L, the ring

$$L'\langle s_1, t_1\rangle/(s_1t_1 - \alpha) = (\mathcal{O}_{L'}\langle s_1, t_1\rangle/(s_1t_1 - \alpha))[1/l]$$

is a domain by Corollary A.1.3. Hence $L\langle s_1, t_1\rangle/(s_1t_1 - \alpha_1)$ is geometrically connected.

In general we have

$$A[1/l] \cong L\langle s_1, t_1\rangle/(s_1t_1 - \alpha_1) \widehat{\otimes} \cdots \widehat{\otimes} L\langle s_r, t_r\rangle/(s_rt_r - \alpha_r),$$

and so the case $r = 1$ and Lemma A.1.1 imply that $A[1/l]$ is geometrically connected. □

Next we turn to representation theory.

LEMMA A.1.5. *Let Γ be a group and M a field of characteristic 0. Also let*

$$r : \Gamma \longrightarrow \mathrm{GL}_n(\overline{M})$$

be a semi-simple representation. Suppose that for all $\gamma \in \Gamma$ we have $\operatorname{tr} r(\gamma) \in M$ and that for some $\gamma \in \Gamma$ the characteristic polynomial of $r(\gamma)$ has distinct, M-rational roots. Then r is conjugate to a representation into $\mathrm{GL}_n(M)$.

Proof. Let B denote the M-span of the image of r in $M_{n\times n}(\overline{M})$. Note that the $\overline{M}$-span $B_{\overline{M}}$ of B is a finite-dimensional, semi-simple $\overline{M}$-algebra. Let $e_1, \dots, e_r$ be an $\overline{M}$-basis of $B_{\overline{M}}$ consisting of elements of B and let $e_1', \dots, e_r'$ be the dual basis for the trace pairing. Then $B \supset \sum_i M e_i$. If $b \in B$, we have $\operatorname{tr}(be_i) \in M$ for all i, and so $B \subset \sum_i M e_i'$. Thus $B = \sum M e_i = \sum M e_i'$ and

$$B \otimes_M \overline{M} \xrightarrow{\sim} B_{\overline{M}} \subset M_{n\times n}(\overline{M}).$$

In particular B is a finite-dimensional semi-simple M-algebra and $\overline{M}^n$ is a faithful $B \otimes_M \overline{M}$-module.

We have

$$B \cong \bigoplus_j M_{m_j}(D_j),$$

where each D_j is a division algebra with centre a finite extension $Z(D_j)$ of M. Let $r_j^2 = \dim_{Z(D_j)} D_j$. As a representation of $B \otimes_m \overline{M} \cong \bigoplus_{j,\tau} M_{m_j r_j}(\overline{M})$, where τ runs over M-embeddings $Z(D_j) \hookrightarrow \overline{M}$ for each j, we have

$$\overline{M}^n \cong \bigoplus_{j,\tau} \overline{M}^{m_j r_j n_{j,\tau}}$$

for some nonnegative integers $n_{j,\tau}$. Since $\operatorname{tr}(B) \subset M$, we see that $n_{j,\tau} = n_j$ is independent of τ. The existence of an element of B with n-distinct M-rational eigenvalues implies that $r_j = n_j = [Z(D_j) : M] = 1$ for all j. Thus $D_j = M$ for all j, and the lemma follows. □

Next we have a result about unramified tori.

LEMMA A.1.6. *Suppose that $\widetilde{T}_1$ and $\widetilde{T}_2$ are unramified tori over $\mathbb{Z}_l$ and that $\phi : \widetilde{T}_1 \to \widetilde{T}_2$ is a surjection. Then the cokernel of $\widetilde{T}_1(\mathbb{Z}_l) \to \widetilde{T}_2(\mathbb{Z}_l)$ is finite of order dividing the order of the torsion subgroup of $X^*(\widetilde{T}_1)/\phi^* X^*(\widetilde{T}_2)$.*

Proof. We have an exact (in the centre) sequence

$$\widetilde{T}_1(\mathbb{Z}_l) \xrightarrow{\phi} \widetilde{T}_2(\mathbb{Z}_l) \longrightarrow \operatorname{Hom}\left(X^*(\widetilde{T}_1)/\phi^* X^*(\widetilde{T}_2), (\widehat{\mathbb{Z}}_l^{\mathrm{nr}})^\times\right)/(\mathrm{Frob}_l - 1).$$

(Recall that $(\widehat{\mathbb{Z}}_l^{\mathrm{nr}})^{\mathrm{Frob}_l^r=1} = \mathbb{Z}_{l^r}$.) There is a positive integer m so that Frob_l^m acts trivially on $X^*(\widetilde{T}_1)$. There is also a surjection

$$\begin{aligned}\mathrm{Hom}\,(X^*(\widetilde{T}_1)/\phi^*X^*(\widetilde{T}_2), (\widehat{\mathbb{Z}}_l^{\mathrm{nr}})^\times)/(\mathrm{Frob}_l^m - 1)\\ \twoheadrightarrow \mathrm{Hom}\,(X^*(\widetilde{T}_1)/\phi^*X^*(\widetilde{T}_2), (\widehat{\mathbb{Z}}_l^{\mathrm{nr}})^\times)/(\mathrm{Frob}_l - 1).\end{aligned}$$

Thus it suffices to show that

$$\mathrm{Hom}\,(X^*(\widetilde{T}_1)/\phi^*X^*(\widetilde{T}_2), (\widehat{\mathbb{Z}}_l^{\mathrm{nr}})^\times)/(\mathrm{Frob}_l^m - 1)$$

is a finite group and has order dividing the order of the torsion subgroup of $X^*(\widetilde{T}_1)/\phi^*X^*(\widetilde{T}_2)$.

There is also an exact (in the centre) sequence

$$\begin{aligned}\mathrm{Hom}\,((X^*(\widetilde{T}_1)/\phi^*X^*(\widetilde{T}_2))^{\mathrm{tf}}, (\widehat{\mathbb{Z}}_l^{\mathrm{nr}})^\times) &\longrightarrow \mathrm{Hom}\,(X^*(\widetilde{T}_1)/\phi^*X^*(\widetilde{T}_2), (\widehat{\mathbb{Z}}_l^{\mathrm{nr}})^\times)\\ &\longrightarrow \mathrm{Hom}\,((X^*(\widetilde{T}_1)/\phi^*X^*(\widetilde{T}_2))^{\mathrm{tor}}, (\widehat{\mathbb{Z}}_l^{\mathrm{nr}})^\times)\end{aligned}$$

and hence an exact (in the centre) sequence

$$\begin{aligned}&\mathrm{Hom}\,((X^*(\widetilde{T}_1)/\phi^*X^*(\widetilde{T}_2))^{\mathrm{tf}}, (\widehat{\mathbb{Z}}_l^{\mathrm{nr}})^\times)\\ &\quad\longrightarrow \mathrm{Hom}\,(X^*(\widetilde{T}_1)/\phi^*X^*(\widetilde{T}_2), (\widehat{\mathbb{Z}}_l^{\mathrm{nr}})^\times)/(\mathrm{Frob}_l^m - 1)\\ &\quad\longrightarrow \mathrm{Hom}\,((X^*(\widetilde{T}_1)/\phi^*X^*(\widetilde{T}_2))^{\mathrm{tor}}, (\widehat{\mathbb{Z}}_l^{\mathrm{nr}})^\times)/B,\end{aligned}$$

where B denotes the image of $(\mathrm{Frob}_l^m - 1)\mathrm{Hom}\,(X^*(\widetilde{T}_1)/\phi^*X^*(\widetilde{T}_2), (\widehat{\mathbb{Z}}_l^{\mathrm{nr}})^\times)$ in the group $\mathrm{Hom}\,((X^*(\widetilde{T}_1)/\phi^*X^*(\widetilde{T}_2))^{\mathrm{tor}}, (\widehat{\mathbb{Z}}_l^{\mathrm{nr}})^\times)$. As

$$\mathrm{Hom}\,((X^*(\widetilde{T}_1)/\phi^*X^*(\widetilde{T}_2))^{\mathrm{tor}}, (\widehat{\mathbb{Z}}_l^{\mathrm{nr}})^\times)$$

is finite with order the prime-to-l part of $\#(X^*(\widetilde{T}_1)/\phi^*X^*(\widetilde{T}_2))^{\mathrm{tor}}$, it suffices to show that

$$\mathrm{Hom}\,((X^*(\widetilde{T}_1)/\phi^*X^*(\widetilde{T}_2))^{\mathrm{tf}}, (\widehat{\mathbb{Z}}_l^{\mathrm{nr}})^\times)/(\mathrm{Frob}_l^m - 1) = (0).$$

However this follows from the fact that $\mathrm{Frob}_l^m - 1$ is surjective on $(\widehat{\mathbb{Z}}_l^{\mathrm{nr}})^\times$ (which is proved recursively modulo higher and higher powers of l). □

Finally we recall the following observation, which is not really algebraic.

LEMMA A.1.7. *The intersection of a finite number of sets of rational primes of Dirichlet density* 1 *has Dirichlet density* 1.

Proof. A set of rational primes has Dirichlet density 1 if and only if its complement has Dirichlet density 0. Thus the lemma follows from the fact that the union of a finite number of sets of rational primes of Dirichlet density 0 has Dirichlet density 0. □

A.2. *Building fields and characters.* For the convenience of the reader we recall some results about the construction of fields and characters, but we will start by recalling some facts about algebraic characters.

Suppose that F is a number field, l is a rational prime and $\imath : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$. Let F_0 denote the maximal CM subfield of F. Suppose also that χ is an algebraic character of $\mathbb{A}_F^\times / F^\times$ with

$$\chi|_{(F_\infty^\times)^0} : x \longmapsto \prod_{\tau \in \mathrm{Hom}\,(F,\mathbb{C})} (\tau x)^{-a_\tau}.$$

Then

$$\imath \left((r_{l,\imath}(\chi) \circ \mathrm{Art}_F)(x) \prod_{\tau \in \mathrm{Hom}\,(F,\mathbb{C})} (\imath^{-1}\tau)(x_l)^{a_\tau} \right) = \chi(x) \prod_{\tau \in \mathrm{Hom}\,(F,\mathbb{C})} (\tau x)^{a_\tau}$$

for $x \in \mathbb{A}_F^\times$. Moreover $r_{l,\imath}(\chi)$ is de Rham at all primes above l and, if $\tau : F \hookrightarrow \overline{\mathbb{Q}}_l$, then $\mathrm{HT}_\tau(r_{l,\imath}(\chi)) = \{a_{\imath\circ\tau}\}$. (See [Ser68].) Moreover we have that

(1) $a_\tau + a_{\tau'} = \mathrm{wt}(\chi)$ for all $\tau, \tau' \in \mathrm{Hom}\,(F, \mathbb{C})$ with $\tau|_{F_0} = \tau'|_{F_0} \circ c$;
(2) a_τ only depends on $\tau|_{F_0}$;
(3) $\mathrm{wt}(\chi_1\chi_2) = \mathrm{wt}(\chi_1) + \mathrm{wt}(\chi_2)$;
(4) if σ is an automorphism of F, then $\mathrm{wt}(\chi^\sigma) = \mathrm{wt}(\chi)$ (as $\sigma|_{F_0}$ commutes with c);
(5) $\mathrm{wt}(\chi \circ \mathbf{N}_{K'/K}) = \mathrm{wt}(\chi)$;
(6) if F_0 is totally real, then $\mathrm{wt}(\chi)$ is even;
(7) $\mathrm{wt}(|\ |_F) = -2$;
(8) for all places v of F the Weil–Deligne representation $\mathrm{WD}(r_{l,\imath}(\chi))|_{G_{F_v}}$ is pure of weight $\mathrm{wt}(\chi)$.

Any algebraic l-adic character of G_F arises in this way.

Next we recall Lemma 4.1.2 of [CHT08].

LEMMA A.2.1. *Suppose that F is a number field, that $F^{(\mathrm{avoid})}/F$ is a finite Galois extension and that S is a finite set of places of F. For $v \in S$ let E_v/F_v be a finite Galois extension. Then we can find a finite, soluble Galois extension E/F linearly disjoint from $F^{(\mathrm{avoid})}$ such that for each $v \in S$ and each prime w of E above v, the extension E_w/F_v is isomorphic to E_v/F_v.*

In a somewhat similar vein we have the following result.

LEMMA A.2.2. *Suppose that F is a number field, that S is a finite set of places of F, that $F^{(\mathrm{avoid})}/F$ is a finite Galois extension and that N is a positive integer. Then we can find a finite cyclic extension E/F of degree N in which all the elements of S split completely and which is linearly disjoint from $F^{(\mathrm{avoid})}$ over F.*

Proof. Let $\{F_i^{(\mathrm{avoid})}\}$ denote the intermediate fields between $F^{(\mathrm{avoid})}$ and F for which $\mathrm{Gal}\,(F_i^{(\mathrm{avoid})}/F)$ is simple. Augment S to include, for each i, a prime which does not split in $F_i^{(\mathrm{avoid})}$. Then the linear disjointness of E from $F^{(\mathrm{avoid})}$ will be automatic. Choose a prime v_0 of F with $v_0 \notin S$. Now use Lemma 4.1.1 of [CHT08] to choose a finite order character

$$\chi : \mathbb{A}_F^\times / F^\times \longrightarrow \overline{\mathbb{Q}}^\times$$

such that

- $\chi|_{\prod_{v\in S} F_v^\times} = 1$,
- $\chi|_{F_{v_0}^\times}$ has order N.

Then $\overline{F}^{\ker \chi \circ \mathrm{Art}_F}/F$ is a cyclic extension of degree divisible by N in which all the primes in S split completely. The unique sub-extension E/F of degree N satisfies the requirements of the lemma. □

Corollary A.2.3. *Suppose that F is an imaginary* CM *field, that S is a finite set of places of F, that $F^{(\mathrm{avoid})}/F$ is a finite Galois extension and that N is a positive integer. Then we can find a cyclic* CM *extension E/F of degree N in which all the elements of S split completely and which is linearly disjoint from $F^{(\mathrm{avoid})}$ over F.*

Proof. Let F^+ denote the maximal totally real subfield of F. By the lemma we can find a totally real cyclic extension E^+/F^+ of degree N in which all the primes of F^+ below S split completely and which is linearly disjoint from the normal closure of $F^{(\mathrm{avoid})}/F^+$ over F^+. Then we can take $E = E^+F$. □

We now turn to building characters and record three results, all of which have a similar feel but which are slightly different. We start by restating (a special case of) Lemma 2.2 of [HSBT10].

Lemma A.2.4. *Suppose that F is an imaginary* CM *field with maximal totally real subfield F^+ and that S is a finite set of primes of F. Let*

$$\chi : (\mathbb{A}_{F^+}^\infty)^\times \longrightarrow \overline{\mathbb{Q}}^\times$$

and

$$\psi_S : \mathcal{O}_{F,S}^\times \longrightarrow \overline{\mathbb{Q}}^\times$$

be continuous characters such that

$$\psi_S|_{(\mathbb{A}_{F^+}^\infty)^\times \cap \mathcal{O}_{F,S}^\times} = \chi|_{(\mathbb{A}_{F^+}^\infty)^\times \cap \mathcal{O}_{F,S}^\times}.$$

Also let

$$\phi_0 : F^\times \longrightarrow \overline{\mathbb{Q}}^\times$$

be a character such that

$$\phi_0|_{(F^+)^\times} = \chi|_{(F^+)^\times}.$$

Then there is a continuous character

$$\phi : \mathbb{A}_F^\times \longrightarrow \overline{\mathbb{Q}}^\times$$

such that

- $\phi|_{F^\times} = \phi_0$,
- $\phi|_{(\mathbb{A}_{F^+}^\infty)^\times} = \chi$,
- $\phi|_{\mathcal{O}_{F,S}^\times} = \psi_S$.

When we invoke this lemma we may not specify S or the character ψ_S, but instead we may list local conditions to be satisfied by ϕ at a finite number of primes. We hope that the reader will have no difficulty in finding a set S and a character ψ_S, so that if we apply the lemma with these choices, it produces a character ϕ with the desired local properties.

We next record two similar results about algebraic characters of G_F.

LEMMA A.2.5. *Suppose that l is a rational prime, that F is an imaginary* CM *field with maximal totally real subfield F^+ and that S is a finite set of primes of F containing all primes above l and satisfying $S^c = S$. Let*

$$\chi : G_{F^+} \longrightarrow \overline{\mathbb{Q}}_l^\times$$

be a continuous character and, for $v \in S$, let

$$\psi_v : G_{F_v} \longrightarrow \overline{\mathbb{Q}}_l^\times$$

be a continuous character such that

$$(\psi_v \psi_{v^c}^c)|_{I_{F_v}} = \chi|_{I_{F_v}}.$$

For $v|l$ suppose that the character ψ_v is de Rham.

(1) *Suppose further that every element of S is unramified over F^+ and that $\chi(c_v)$ is independent of $v|\infty$. Then there is a continuous character*

$$\theta : G_F \longrightarrow \overline{\mathbb{Q}}_l^\times$$

such that

$$\theta\theta^c = \chi|_{G_F}$$

and, for all $v \in S$,

$$\theta|_{I_{F_v}} = \psi_v|_{I_{F_v}}.$$

(2) *Alternatively, suppose further that $l > 2$ and that*

$$\overline{\theta} : G_F \longrightarrow \overline{\mathbb{F}}_l^\times$$

is a continuous character such that

- *$\overline{\theta\theta^c}$ equal to the reduction of $\chi|_{G_F}$,*
- *for $v \in S$ the restriction $\overline{\theta}|_{G_{F_v}}$ equals the reduction of ψ_v.*

Then there is a continuous character

$$\theta : G_F \longrightarrow \overline{\mathbb{Q}}_l^\times$$

lifting $\overline{\theta}$ *and such that*

$$\theta\theta^c = \chi|_{G_F}$$

and, for all $v \in S$,

$$\theta|_{I_{F_v}} = \psi_v|_{I_{F_v}}.$$

Proof. We deduce the first part from Lemma A.2.4. Replacing χ by $\chi\delta_{F/F^+}$ if need be, we may suppose that $\chi(c_v) = 1$ for all $v|\infty$. Note that χ is algebraic and so, because F^+ is totally real, all its Hodge–Tate numbers are equal to some integer w. By class field theory we may think of $\chi : \mathbb{A}_{F^+}^\times \to \overline{\mathbb{Q}}_l^\times$ and $\phi_v : F_v^\times \to \overline{\mathbb{Q}}_l^\times$ and look for $\theta : \mathbb{A}_F^\times / F^\times \to \overline{\mathbb{Q}}_l^\times$. If $v|l$, then

$$\psi_v = \prod_{\tau : F_v \hookrightarrow \overline{\mathbb{Q}}_l} \tau^{-m_\tau}$$

on a some nonempty open subgroup of $F_v^\times$. Moreover $m_\tau + m_{\tau\circ c} = w$ for all τ.

Let $\chi' : \mathbb{A}_{F^+}^\times \to \overline{\mathbb{Q}}_l^\times$ be defined by

$$\chi'(\alpha) = \chi(\alpha)(\mathbf{N}_{F^+/\mathbb{Q}}\alpha_l)^w.$$

Then χ' has open kernel containing $(F_\infty^+)^\times$. Let

$$\phi_0 = \prod_{\tau : F \hookrightarrow \overline{\mathbb{Q}}_l} \tau^{m_\tau} : F^\times \to \overline{\mathbb{Q}}_l^\times.$$

We see that

$$\phi_0|_{(F^+)^\times} = \chi'|_{(F^+)^\times}.$$

Thus, in particular, χ' is actually valued in the algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ in $\overline{\mathbb{Q}}_l$. Also define

$$\psi_v' : F_v^\times \longrightarrow \overline{\mathbb{Q}}_l^\times$$

to be ψ_v if $v \not| l$ and

$$\psi_v \prod_{\tau : F_v \hookrightarrow \overline{\mathbb{Q}}_l} \tau^{m_\tau}$$

if $v|l$. In either case ψ_v' has open kernel and is valued in $\overline{\mathbb{Q}}$.

Now apply Lemma A.2.4 to χ', ϕ_0 and $\prod_{v\in S}\psi_v'$. (Note that the norm map $\mathbf{N}_{F/F^+}$ from $\prod_{v\in S}\mathcal{O}_{F,v}$ to the intersection of this group with $\mathbb{A}_{F^+}^\times$ is surjective.) We get a character

$$\phi : \mathbb{A}_F^\times \longrightarrow \overline{\mathbb{Q}}_l^\times$$

with open kernel such that

- $\phi|_{F^\times} = \phi_0$,
- $\phi|_{(\mathbb{A}_{F^+}^\infty)^\times} = \chi'$,
- $\phi|_{\mathcal{O}_{F,v}^\times} = \psi_v'$ for all $v \in S$.

The character

$$\theta : \mathbb{A}_F^\times \longrightarrow \overline{\mathbb{Q}}_l^\times$$

defined by

$$\theta(\alpha) = \phi(\alpha) \prod_{\tau : F \hookrightarrow \overline{\mathbb{Q}}_l} (\tau\alpha_l)^{-m_\tau}$$

satisfies the requirements of the first part of the present lemma.

The second part follows easily from Lemma 4.1.6 of [CHT08]. If $\tau : F \hookrightarrow \overline{\mathbb{Q}}_l$ lies above $v \in S$, then we will let m_τ denote the τ Hodge–Tate number of ψ_v. We need only verify that $m_\tau + m_{\tau c}$ is independent of τ. However the character χ must be algebraic and for each $\tau : F \hookrightarrow \overline{\mathbb{Q}}_l$ the τ Hodge–Tate number of χ is $m_\tau + m_{\tau c}$. The result follows from the facts recalled at the start of this section. □

Probably the assumption $l > 2$ in the second part of this lemma could be replaced by the assumption that $\chi(c_v)$ is independent of $v|\infty$ as in the first part of the lemma.

Again we will often invoke this lemma without specifying S or the characters ψ_v, but instead we may list local conditions to be satisfied by θ at a finite number of primes. We hope that the reader will have no difficulty in finding a set S and characters ψ_v, so that if we apply the lemma with these choices, it produces a character θ with the desired local properties.

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