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The c-function expansion of a basic hypergeometric function associated to root systems

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Abstract

We derive an explicit c-function expansion of a basic hypergeometric function associated to root systems. The basic hypergeometric function in question was constructed as an explicit series expansion in symmetric Macdonald polynomials by Cherednik in case the associated twisted affine root system is reduced. Its construction was extended to the nonreduced case by the author. It is a meromorphic Weyl group invariant solution of the spectral problem of the Macdonald q-difference operators. The c-function expansion is its explicit expansion in terms of the basis of the space of meromorphic solutions of the spectral problem consisting of q-analogs of the Harish-Chandra series. We express the expansion coefficients in terms of a q-analog of the Harish-Chandra c-function, which is explicitly given as product of q-Gamma functions. The c-function expansion shows that the basic hypergeometric function formally is a q-analog of the Heckman-Opdam hypergeometric function, which in turn specializes to elementary spherical functions on noncompact Riemannian symmetric spaces for special values of the parameters.

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1. Introduction

In this paper we establish the c-function expansion of a basic hypergeometric function $\mathcal{E}_+$ associated to root systems. Besides the base q, the basic hypergeometric function $\mathcal{E}_+$ depends on a choice of a multiplicity function k on

an affine root system naturally associated to the underlying based root system data. It will become apparent from the c-function expansion that $\mathcal{E}_+$ formally is a q-analog of the Heckman-Opdam [14], [15], [33] hypergeometric function, which in turn reduces to the elementary spherical functions on noncompact Riemannian symmetric spaces for special parameter values. We distinguish three important subclasses of the theory: the reduced case, the GL_m case and the nonreduced case.

In the reduced case $\mathcal{E}_+$ is Cherednik's global spherical function from [5], [7], [8], or a reductive extension thereof. It is a Weyl group invariant, meromorphic, self-dual common eigenfunction of the Macdonald q-difference operators, constructed as an explicit convergent series in symmetric Macdonald polynomials. In the rank one case $\mathcal{E}_+$ can be explicitly related to the basic hypergeometric series solutions of Heine's basic hypergeometric q-difference equation (see Section 5.3).

The GL_m case is a special case of the reduced case with the underlying root system of type A_{m-1}. It is of special interest since it relates to Ruijsenaars' [35] relativistic quantum trigonometric Calogero-Moser model. In fact, the associated Macdonald q-difference operators were first written down by Ruijsenaars [35] as the corresponding quantum Hamiltonians.

In the nonreduced case the associated affine root system is the nonreduced affine root system of type $C^\vee C_n$. The multiplicity function k now comprises five degrees of freedom (four if the rank n is equal to one). The associated basic hypergeometric function $\mathcal{E}_+$ was constructed in [42]. Duality of $\mathcal{E}_+$ now involves a nontrivial transformation of the multiplicity k to a dual multiplicity function k^d. (We use the convention that the dual multiplicity function k^d equals k in the reduced case.) The associated Macdonald q-difference operators include Koornwinder's [22] multivariable extension of the Askey-Wilson [1] second-order q-difference operator. It is the nonreduced case that is expected to be amenable to generalizations to the elliptic level; cf. [34].

The basic Harish-Chandra series $\widehat{\Phi}_\eta(\cdot,\gamma)$ with base point given by a torus element η is a meromorphic common eigenfunction of the Macdonald q-difference operators having a converging series expansion of the form

$$\widehat{\Phi}_\eta(t,\gamma)=\widehat{\mathcal{W}}_\eta(t,\gamma)\sum_{\mu\in Q_+}\Gamma_\mu(\gamma)t^{-\mu},\qquad \Gamma_0(\gamma)=1$$

deep in the appropriate asymptotic sector, where Q_+ consists of the elements in the root lattice that can be written as sum of positive roots. The prefactor $\widehat{\mathcal{W}}_\eta(t,\gamma)$ is an explicit quotient of theta functions satisfying the asymptotic Macdonald q-difference equations (see Section 3.3). It is normalized such that it reduces to the natural choice (3.7) of the prefactor when restricting t to the q-lattice containing $\eta\gamma_{0,d}$, where $\gamma_{0,d}^{-1}$ (respectively γ_0^{-1}) denotes the torus

element associated to the k^d-deformation (respectively k-deformation) of the half sum of positive roots; cf. (2.3). For the construction of the basic Harish-Chandra series $\widehat{\Phi}_\eta$ we closely follow [31], [30].

Let W_0 be the Weyl group of the underlying finite root system. The space of common meromorphic eigenfunctions of the Macdonald q-difference operators has, for generic γ, the W_0-translates $\widehat{\Phi}_\eta(\cdot, w\gamma)$ $(w \in W_0)$ as a linear basis over the field of quasiconstants. (This follows from combining and extending [31, Cor. 5.14], [30, Rem. 5.13] and [43, Thm. 5.16].) Hence, for generic γ, we have

$$\mathcal{E}_+(t,\gamma) = \widehat{c}_\eta(\gamma_0)^{-1} \sum_{w\in W_0} \widehat{c}_\eta(w\gamma)\widehat{\Phi}_\eta(t, w\gamma) \tag{1.1}$$

for a unique coefficient $\widehat{c}_\eta(\gamma)$, which turns out to be independent of t due to the particular choice $\widehat{\mathcal{W}}_\eta$ of the prefactor. We will call (1.1) the (monic form of) the c-function expansion of $\mathcal{E}_+$. We will prove the following explicit expression:

$$\widehat{c}_\eta(\gamma) = \frac{\vartheta((w_0\eta)^{-1}\xi\gamma)}{\vartheta(\xi\gamma)} c_{k^d,q}(\gamma) \tag{1.2}$$

for the expansion coefficient, where $w_0 \in W_0$ is the longest Weyl group element, $\vartheta(\cdot)$ is the theta-function (2.8) associated to the given root system data, ξ is an explicit torus element depending on the multiplicity function k (see Corollary 4.7 for the explicit expression of ξ; in the reduced case it is the unit element 1 of the complex torus) and $c_{k^d,q}(\gamma)$ (4.9) is "half" of the inverse of the dual weight function of the associated symmetric Macdonald-Koornwinder polynomials. The expression of $c_{k,q}(\cdot)$ as product of q-shifted factorials (equivalently, as product of q-Gamma functions) is given by (4.10) in the reduced case and by (4.11) in the nonreduced case. It is the q-analog of the Gindikin-Karpelevic [12] type product formula [14, Def. 6.4] of the Harish-Chandra c-function for the Heckman-Opdam hypergeometric function.

Note that for $\eta = 1$, the theta function factors in the expression for $\widehat{c}_\eta(\gamma)$ cancel out. The c-function expansion (1.1) thus simplifies to

$$\mathcal{E}_+(t,\gamma) = c_{k^d,q}(\gamma_0)^{-1} \sum_{w\in W_0} c_{k^d,q}(w\gamma)\widehat{\Phi}_1(t, w\gamma). \tag{1.3}$$

Comparing this formula for t on the q-lattice containing $\gamma_{0,d}$ to the c-function expansion [15, Part I, Def. 4.4.1] of the Heckman-Opdam hypergeometric function, it is apparent that $\mathcal{E}_+$ is formally a q-analog of the Heckman-Opdam hypergeometric function. The corresponding classical limit $q \to 1$ can be made rigorous if the underlying finite root system is of type A_1; see [21]. In this paper we will not touch upon making the limit rigorous in general; see [7, Thm. 4.5] for further results in this direction.

It is important to consider the c-function expansion for arbitrary η. In the rank one nonreduced case, a self-dual Fourier transform with Fourier kernel $\mathcal{E}_+$

and (Plancherel) density

$$\mu_\eta(\gamma) = \frac{1}{\widehat{c}_\eta(\gamma)\widehat{c}_\eta(\gamma^{-1})}$$

was defined and studied in [20], [19]. The extra theta function contributions in $\mu_\eta(\gamma)$ compared to the usual weight function $\mu_1(\gamma)$ of the Macdonald-Koornwinder polynomials (which, in the present nonreduced rank one setup, are the Askey-Wilson [1] polynomials) give rise to an infinite sequence of discrete mass points in the associated (Plancherel) measure. In the interpretation as the inverse of a spherical Fourier transform on the quantum $\mathrm{SU}(1,1)$ group these mass points account for the contributions of the strange series representations of the quantized universal enveloping algebra (see [20]).

The basic hypergeometric function $\mathcal{E}_+(t,\gamma)$ is self-dual,

$$\mathcal{E}_+(t,\gamma) = \mathcal{E}_{+,d}(\gamma^{-1},t^{-1}),$$

where $\mathcal{E}_{+,d}$ is the basic hypergeometric function with respect to the dual k^d of the multiplicity function k. This implies that $\mathcal{E}_+(t,\gamma)$ solves a bispectral problem, in which dual Macdonald q-difference equations acting on γ are added to the original Macdonald q-difference equations acting on t. We show that a suitable, explicit renormalization $\Phi(\cdot,\cdot) = \Phi(\cdot,\cdot;k,q)$ of the basic Harish-Chandra series $\widehat{\Phi}_\eta$ also becomes a self-dual solution of the bispectral problem. We will derive the c-function expansion (1.1) as a consequence of the more refined asymptotic expansion of $\mathcal{E}_+$,

$$\mathcal{E}_+(t,\gamma) = \sum_{w\in W_0} \mathfrak{c}(t,w\gamma)\Phi(t,w\gamma), \tag{1.4}$$

where $\mathfrak{c}(t,\gamma)$ now is an explicit meromorphic function, quasiconstant in both t and γ.

To prove the existence of an expansion of the form (1.4) we make essential use of Cherednik's [6] double affine Hecke algebra and of the bispectral quantum Knizhnik-Zamolodchikov (KZ) equations from [31], [30]. We show that $\mathcal{E}_+$ is the Hecke algebra symmetrization of a nonsymmetric analog $\mathcal{E}$ of the basic hypergeometric function, whose fundamental property is an explicit transformation rule relating the action of the double affine Hecke algebra on the first torus variable to the action of the double affine Hecke algebra on the second torus variable. (This goes back to [5] in the reduced case and [42] in the nonreduced case.) The Hecke algebra symmetrizer acting on such functions factorizes as $\phi\circ\psi$ with ψ mapping into the space $\mathcal{K}^{W_0\times W_0}$ of Weyl group invariant meromorphic solutions of the bispectral quantum KZ equations. The map ϕ is the difference Cherednik-Matsuo map from [3]. This implies that the basic hypergeometric function $\mathcal{E}_+$ is the image under ϕ of the Weyl group invariant meromorphic

solution $\psi(\mathcal{E}) \in \mathcal{K}^{W_0 \times W_0}$ of the bispectral quantum KZ equations. This observation is essential because it allows us to use the asymptotic analysis of the bispectral quantum KZ equations from [31], [30]. It implies that the space $\mathcal{K}$ of meromorphic solutions of the bispectral quantum KZ equations has a basis over the field of quasiconstants defined in terms of W_0-translates of a self-dual asymptotically free solution F. The image of F under the difference Cherednik-Matsuo map ϕ is the self-dual basic Harish-Chandra series Φ in (1.4).

In Theorem 4.6 we give an explicit expression of the quasiconstant coefficient $\mathfrak{c}(t,\gamma)$ in the expansion (1.4) as product of theta functions. The higher rank theta function $\vartheta(\cdot)$ (2.8) and Jacobi's one-variable theta function (2.7) are both involved. The coefficient $\mathfrak{c}(t,\gamma)$ splits in two factors; the first factor is an explicit product of higher rank theta functions, the second factor is $c_{k^d,q}(\gamma)\mathcal{S}_{k^d,q}(\gamma)/\mathcal{L}_q(\gamma)$ with $\mathcal{L}_q(\gamma)$ the leading term of the asymptotic series of $\Phi(\cdot,\gamma)$ and $\mathcal{S}_{k^d,q}(\gamma)$ the holomorphic function capturing the singularities of $\Phi(t,\gamma)$ in γ (see Theorem 3.6 and Definition 3.8). The appearance of the higher rank theta functions and of $c_{k^d,q}(\gamma)$ is due to the asymptotics of a suitable renormalization of the basic hypergeometric function $\mathcal{E}_+(\cdot,\gamma)$; see Corollary 4.3 and Proposition 4.5. (In the reduced case the asymptotics of the basic hypergeometric function was considered by Cherednik [7, §4.2].) Similarly to $c_{k^d,q}(\gamma)$, the factor $\mathcal{S}_{k^d,q}(\gamma)/\mathcal{L}_q(\gamma)$ can be explicitly expressed as a product of q-Gamma functions. By the Jacobi triple product identity their product $c_{k^d,q}(\gamma)\mathcal{S}_{k^d,q}(\gamma)/\mathcal{L}_q(\gamma)$ admits an expression as a product of Jacobi theta functions.

Recently [45] explicit connection coefficient formulas for the self-dual basic Harish-Chandra series Φ are derived. They do not lead to a new proof of the c-function expansion though; see [45, §1.5] for a detailed discussion.

As an application of the c-function expansion we establish pointwise asymptotics of the Macdonald-Koornwinder polynomials in Section 5.1. (The L^2-asymptotics was obtained by different methods in [36], [9], [10].) In addition, in Sections 5.2 and 5.3 we relate and compare our results to the classical theory of basic hypergeometric series [11] when the rank of the underlying root system is one.

2. **The basic hypergeometric function**

In this section we give the definition of the basic hypergeometric function associated to root systems. It was introduced by Cherednik in [5] for irreducible reduced twisted affine root systems. In [42] it was defined for the nonreduced case (sometimes called the Koornwinder case, or $C^\vee C$ case). We give a uniform treatment in which we allow extra freedom in the choice of the associated translation lattice. This enables us to include the GL_m-extension of the reduced type A case in our treatment.

2.1. *Affine root systems and extended affine Weyl groups.* In this subsection we recall well-known facts on affine root systems and affine Weyl groups. (For further details see, e.g., [6], [28].) Let V be an Euclidean space of dimension m with scalar product $\big(\cdot,\cdot\big)$ and corresponding norm $|\cdot|$. Let $R_0 \subset V$ be a finite set of nonzero vectors, and let V_0 be its real span. We suppose that $R_0 \subset V_0$ is a crystallographic, reduced irreducible root system. We write $R_{0,s}$ (respectively $R_{0,l}$) for the subset of R_0 of short (respectively long) roots. If all roots of R_0 have the same root length, then $R_{0,s} = R_0 = R_{0,l}$ by convention. Let $n = \dim(V_0) \leq m$ be the rank of R_0. Let $\Delta_0 = (\alpha_1, \ldots, \alpha_n)$ be an ordered basis of R_0 and $R_0 = R_0^+ \cup R_0^-$ the corresponding decomposition of R_0 in positive and negative roots. We order the basis elements in such a way that α_n is a short root. We write $\varphi \in R_0^+$ (respectively $\theta \in R_0^+$) for the corresponding highest root (respectively highest short root). They coincide if R_0 has only one root length. Let $Q = \bigoplus_{i=1}^n \mathbb{Z}\alpha_i$ be the root lattice, and set $Q_+ = \bigoplus_{i=1}^n \mathbb{Z}_{\geq 0}\alpha_i$.

View $\widehat{V} = V \oplus \mathbb{R}c$ as the space of real valued affine linear functions on V by

$$v + rc : v' \mapsto (v, v') + r, \qquad (v, v' \in V, r \in \mathbb{R}).$$

We extend the scalar product $\big(\cdot,\cdot\big)$ to a semi-positive definite form on $\widehat{V}$ such that the constant functions $\mathbb{R}c$ are in the radical. The canonical action of the affine linear group $\mathrm{GL}_{\mathbb{R}}(V) \ltimes V$ on V gives rise to a linear action on $\widehat{V}$ by transposition. We denote the resulting translation actions by τ. Thus $\tau(v)v' = v + v'$ and

$$\tau(v)(v' + rc) = v' + (r - (v, v'))c.$$

For $0 \neq \alpha \in V$ and $r \in \mathbb{R}$, let $s_{\alpha+rc}$ be the orthogonal reflection in the affine hyperplane $\{v \in V \mid (\alpha, v) = -r\}$. Then $s_{\alpha+rc} \in \mathrm{GL}_{\mathbb{R}}(V) \ltimes V$. In fact, $s_{\alpha+rc} = \tau(-r\alpha^\vee)s_\alpha$ with $\alpha^\vee := 2\alpha/|\alpha|^2$.

The twisted reduced affine root system $R^\bullet$ associated to R_0 is

$$R^\bullet := \left\{\alpha + r\frac{|\alpha|^2}{2}c \mid \alpha \in R_0,\ r \in \mathbb{Z}\right\} \subset \widehat{V}.$$

The affine Weyl group $W^\bullet$ of $R^\bullet$ is the subgroup of $\mathrm{GL}_{\mathbb{R}}(V) \ltimes V$ generated by s_a ($a \in R^\bullet$). It preserves $R^\bullet$. In addition, $W^\bullet \simeq W_0 \ltimes Q$ with W_0 the Weyl group of R_0. We extend the ordered basis Δ_0 of R_0 to an ordered basis

$$\Delta = (a_0, a_1, \ldots, a_n) := \left(\frac{|\theta|^2}{2}c - \theta, \alpha_1, \ldots, \alpha_n\right)$$

of $R^\bullet$. It results in the decomposition $R^\bullet = R^{\bullet,+} \cup R^{\bullet,-}$ of $R^\bullet$ in positive and negative roots.

The Weyl group W_0 and the affine Weyl group $W^\bullet$ are Coxeter groups, with Coxeter generators the simple reflections $s_i := s_{\alpha_i}$ ($1 \leq i \leq n$) respectively $s_0 := s_{a_0}, s_1, \ldots, s_n$.

Let $Q^\vee$ be the coroot lattice of R_0; i.e., it is the integral span of the coroots $\alpha^\vee = 2\alpha/|\alpha|^2$ ($\alpha \in R_0$). Fix a full lattice $\Lambda \subset V$ satisfying $Q \subseteq \Lambda$ and $\big(\Lambda, Q^\vee\big) \subseteq \mathbb{Z}$.

Remark 2.1. In this remark we relate the triples (R_0, Δ_0, Λ) to the notion of a based root datum (cf., e.g., [38, §1] for a survey on root data). Using the notations from [38, §1], suppose that $\Psi_0 = (X, \Phi, \Delta, X^\vee, \Phi^\vee, \Delta^\vee)$ is a nontoral based root datum with associated perfect pairing $\langle \cdot, \cdot \rangle : X \times X^\vee \to \mathbb{Z}$ and associated bijection $\alpha \mapsto \alpha^\vee$ of Φ onto $\Phi^\vee$. Assume that the root system Φ is reduced and irreducible. Choose a Weyl group invariant scalar product $\big(\cdot, \cdot\big)$ on $V := \mathbb{R} \otimes_{\mathbb{Z}} X$. Now we use the scalar product to embed $X^\vee$ as a full lattice in V. Thus $\xi \in X^\vee$, regarded as element of V, is characterized by the requirement that (ξ, x) equals $\langle x, \xi \rangle$ for all $x \in X$. The element $\alpha^\vee \in \Phi^\vee$ then corresponds to the coroot $2\alpha/|\alpha|^2$ in V. It follows that the triple $(R_0, \Delta_0, \Lambda) := (\Phi, \Delta, X)$ in V satisfies the desired properties.

Let

$$P := \{\lambda \in V_0 \mid (\lambda, \alpha^\vee) \in \mathbb{Z} \quad \forall\, \alpha \in R_0\}$$

be the weight lattice of R_0. Let $\widetilde{\varpi}_i \in P$ ($1 \leq i \leq n$) be the fundamental weights of P with respect to the ordered basis Δ_0 of R_0. In other words, $\widetilde{\varpi}_i \in V_0$ is characterized by $(\widetilde{\varpi}_i, \alpha_j^\vee) = \delta_{i,j}$ (Kronecker delta function) for $1 \leq i, j \leq n$. Since $Q \subseteq P$ with finite index and $Q \subseteq \Lambda$, there exists for each $i \in \{1, \ldots, n\}$ a smallest natural number m_i such that $\varpi_i := m_i \widetilde{\varpi}_i \in \Lambda$. Then $\{\varpi_i\}_{i=1}^n$ is a basis of a W_0-invariant, rank n sublattice of $\Lambda \cap V_0$ with the basis elements satisfying $(\varpi_i, \alpha_j^\vee) = 0$ if $j \neq i$ and $(\varpi_i, \alpha_i^\vee) = m_i \in \mathbb{Z}_{>0}$. Set

$$\Lambda_c := \Lambda \cap V_0^\perp.$$

Note that $L\Lambda \subseteq \Lambda_c \oplus \bigoplus_{i=1}^n \mathbb{Z}\varpi_i$ for $L := m_1 \cdots m_n$. In particular, Λ_c is a full sublattice of $V_0^\perp$.

We list here the three key examples of triples (R_0, Δ_0, Λ).

Example 2.2. (i) If $V_0 = V$, then a natural choice for Λ is the weight lattice P. Then $\varpi_i = \widetilde{\varpi}_i$.

(ii) Let $V = \mathbb{R}^m$ with orthonormal basis $\{\varepsilon_i\}_{i=1}^m$. Take as root system of type A_{m-1} the set $\{\varepsilon_i - \varepsilon_j\}_{1 \leq i \neq j \leq m}$ and take as ordered basis

$$\Delta_0 = (\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \ldots, \varepsilon_{m-1} - \varepsilon_m).$$

Then $V_0 \subset V$ is of codimension one and $n = m - 1$. In this case we can take $\Lambda = \bigoplus_{i=1}^m \mathbb{Z}\varepsilon_i$. The corresponding elements $\varpi_i \in \Lambda$ ($1 \leq i < m$) are given by $\varpi_i = m_i \widetilde{\varpi}_i$, with m_i the smallest natural number such that $im_i \in m\mathbb{Z}$ and with

$$\widetilde{\varpi}_i = \varepsilon_1 + \cdots + \varepsilon_i - \frac{i}{m}(\varepsilon_1 + \cdots + \varepsilon_m).$$

The lattice Λ_c is generated by $\varepsilon_1 + \cdots + \varepsilon_m$.

(iii) Let $R_0 \subset V_0 = V = \mathbb{R}^n$ be the root system of type A_1 if $n = 1$ and of type B_n if $n \geq 2$. In this case we can take $\Lambda = Q$. The corresponding elements ϖ_i $(1 \leq i \leq n)$ are given by $\varpi_i = \widetilde{\varpi}_i$ $(1 \leq i < n)$ and $\varpi_n = 2\widetilde{\varpi}_n$. They form a basis of Λ.

We call $W := W_0 \ltimes \Lambda$ the extended affine Weyl group associated to the triple (R_0, Δ_0, Λ). It preserves $R^\bullet$, and it contains the affine Weyl group $W^\bullet$ as normal subgroup.

The length function on W is defined by

$$l(w) = \#\big(R^{\bullet,+} \cap w^{-1}R^{\bullet,-}\big), \qquad w \in W.$$

Let $\Omega \subset W$ be the subgroup

$$\Omega = \{w \in W \mid l(w) = 0\}.$$

It normalizes $W^\bullet$, and $W \simeq \Omega \ltimes W^\bullet$. In particular, $\Omega \simeq W/W^\bullet \simeq \Lambda/Q$ as abelian groups.

The action of Ω on $R^\bullet$ restricts to an action on the unordered basis $\{a_0, \ldots, a_n\}$ of $R^\bullet$. We also view it as action on the indexing set $\{0, \ldots, n\}$ of the basis, so that $ws_iw^{-1} = s_{w(i)}$ for $w \in \Omega$ and $0 \leq i \leq n$. By the same formula, Ω acts on the affine braid group $\mathcal{B}$ associated to the Coxeter system $(W^\bullet, (s_0, \ldots, s_n))$ by group automorphisms.

The elements of the group Ω can alternatively be described as follows. For $\lambda \in \Lambda$, write $u(\lambda) \in W$ for the unique element in the coset $W_0\tau(\lambda) \subset W$ of minimal length. We have $u(\lambda) = \tau(\lambda)$ if $\lambda \in \Lambda^-$, where

$$\Lambda^\pm := \{\lambda \in \Lambda \mid \pm(\lambda, \alpha^\vee) \geq 0 \quad \forall\, \alpha \in R_0^+\}$$

is the set of dominant and antidominant weights in Λ respectively. Let

$$\Lambda^+_{\mathrm{min}} := \{\lambda \in \Lambda \mid 0 \leq (\lambda, \alpha^\vee) \leq 1 \quad \forall\, \alpha \in R_0^+\}$$

be the set of miniscule dominant weights in Λ. Then $\Omega = \{u(\lambda) \,|\, \lambda \in \Lambda^+_{\mathrm{min}}\}$.

Note that $(\Lambda, a^\vee) = \mathbb{Z}$ or $= 2\mathbb{Z}$ for $a \in R^\bullet$, where $a^\vee = 2a/(a, a)$. Define a subset $S = S(R_0, \Delta_0, \Lambda)$ of the index set $\{0, \ldots, n\}$ of the simple affine roots by

$$S := \{i \in \{0, \ldots, n\} \mid (\Lambda, a_i^\vee) = 2\mathbb{Z}\}.$$

Case-by-case verification shows that $S = \emptyset$ or $\#S = 2$. If $\#S = 2$ and $n = 1$ then R_0 is of type A_1. If $\#S = 2$ and $n \geq 2$, then R_0 is of type B_n and $S = \{0, n\}$ (recall that α_n is short). We call $S = \emptyset$ the *reduced case* and $\#S = 2$ the *nonreduced case*. Thus the GL_m case (corresponding to Example 2.2(ii)) will be regarded as a special case of the reduced case.

We define a Λ-dependent extension of the reduced irreducible affine root system $R^\bullet$ as follows.

Definition 2.3. The irreducible affine root system $R = R(R_0, \Delta_0, \Lambda) \subset \widehat{V}$ is defined by

$$R := R^\bullet \cup \bigcup_{j \in S} W^\bullet(2a_j).$$

In the reduced case we simply have $R = R^\bullet$. In this case the W-orbits of R are in one-to-one correspondence with the W_0-orbits of R_0. Concretely, the affine root $\alpha + r\frac{|\alpha|^2}{2}c \in R$ lies in the same W-orbit as $\beta + r'\frac{|\beta|^2}{2}c \in R$ if and only if $\alpha \in W_0\beta$.

In the nonreduced case, we have

$$R = R^\bullet \cup W(2a_0) \cup W(2a_n).$$

It is the nonreduced irreducible affine root system of type $C^\vee C_n$; cf. [26]. The basis Δ of $R^\bullet$ is also a basis of R, and W is still the associated affine Weyl group. Note that R now has five W-orbits:

$$W(a_0), W(2a_0), W(\varphi), W(\theta), W(2\theta).$$

In the nonreduced case, $\Lambda^+_{\min} = \Lambda_c$ and $\Lambda = Q \oplus \Lambda_c$; hence $W = \Lambda_c \times W^\bullet$. The reductive extension Λ_c of the root lattice will always play a trivial role in the nonreduced case. To simplify the presentation we will therefore assume in the remainder of the paper that $V_0 = V$, in particular $\Lambda = Q$, $\Lambda_c = \{0\}$ and $\Omega = \{1\}$, if we are dealing with the nonreduced case.

2.2. *The double affine Hecke algebra.* References for this subsection are [6], [28], [39]. We call a function $k : R \to \mathbb{C}^*$, denoted by $a \mapsto k_a$, a multiplicity function if $k_{wa} = k_a$ for $w \in W$ and $a \in R$. We will assume throughout the paper that

$$0 < k_a < 1 \quad \forall a \in R. \tag{2.1}$$

We write $k^\bullet$ for its restriction to $R^\bullet$ and $k_i := k^\bullet_{a_i}$ for $0 \le i \le n$. We set $k_{2a} := k_a$ if $a \in R$ and $2a \notin R$.

Definition 2.4. (i) The affine Hecke algebra $H^\bullet(k^\bullet) = H^\bullet(R_0, \Delta_0; k^\bullet)$ is the unique associative unital algebra over $\mathbb{C}$ with generators $T_0, \ldots, T_n$ satisfying the affine braid relations of $\mathcal{B}$ and satisfying the quadratic relations

$$(T_i - k_i)(T_i + k_i^{-1}) = 0, \qquad 0 \le i \le n.$$

(ii) The extended affine Hecke algebra $H(k^\bullet) = H(R_0, \Delta_0; k^\bullet)$ is the crossed product algebra $\Omega \ltimes H^\bullet(k^\bullet)$, where Ω acts by algebra automorphisms on $H^\bullet(k^\bullet)$ by $w(T_i) = T_{w(i)}$ for $w \in \Omega$ and $0 \le i \le n$.

Recall that the lattice Λ is W_0-stable; hence W_0 acts on the complex algebraic torus $T = \mathrm{Hom}(\Lambda, \mathbb{C}^*)$ by transposition. Writing t^λ for the value of $t \in T$ at $\lambda \in \Lambda$, we thus have $(w^{-1}t)^\lambda = t^{w\lambda}$ for $w \in W_0$.

Fix $0<q<1$. The W_0-action on T extends to a q-dependent left W-action $(w,t)\mapsto w_qt$ on T by

$$\tau(\lambda)_qt:=q^\lambda t,$$

where $q^\lambda\in T$ is defined by $\nu\mapsto q^{(\lambda,\nu)}$.

Let $\mathbb{C}[T]$ be the space of regular functions on T with $\mathbb{C}$-basis the monomials $t\mapsto t^\lambda$ ($\lambda\in\Lambda$). Let $\mathbb{C}(T)$ be the corresponding quotient field. Let $\mathcal{M}(T)$ be the field of meromorphic functions on T. By transposition the q-dependent W-action on T gives an action by field automorphisms on both $\mathbb{C}(T)$ and $\mathcal{M}(T)$. This action will also be denoted by $(w,p)\mapsto w_qp$. Let $\mathbb{C}(T)\rtimes_q W\subset\mathcal{M}(T)\rtimes_q W$ be the corresponding crossed product algebras. They canonically act on $\mathcal{M}(T)$ by q-difference reflection operators.

For $t\in T$ and $a=\alpha+r\frac{|\alpha|^2}{2}c\in R^\bullet$, write

$$t_q^a:=q_\alpha^r t^\alpha,$$

where $q_\alpha:=q^{\frac{|\alpha|^2}{2}}$. Define for $a\in R^\bullet$ the rational function $c_a=c_a(\cdot;k,q)\in\mathbb{C}(T)$ by

$$c_a(t):=\frac{(1-k_ak_{2a}t_q^a)(1+k_ak_{2a}^{-1}t_q^a)}{(1-t_q^a)(1+t_q^a)}. \tag{2.2}$$

It satisfies $c_a(w_q^{-1}t)=c_{wa}(t)$ for $a\in R^\bullet$ and $w\in W$. The following fundamental result is due to Cherednik in the reduced case (see [6, Thm. 3.2.1] and references therein) and due to Noumi [32] in the nonreduced case.

THEOREM 2.5. *There exists a unique faithful algebra homomorphism*

$$\pi=\pi_{k,q}:H(k^\bullet)\to\mathbb{C}(T)\rtimes_q W$$

satisfying

$$\begin{aligned}\pi_{k,q}(T_i)&=k_i+k_i^{-1}c_{a_i}(s_{i,q}-1), && 0\leq i\leq n,\\ \pi_{k,q}(w)&=w_q, && w\in\Omega.\end{aligned}$$

Remark 2.6. In the reduced case $\pi_{k,q}$ is a one-parameter family of algebra embeddings of $H(k^\bullet)$ (with q being the free parameter). In the nonreduced case, $\pi_{k,q}$ is a three-parameter family of algebra embeddings of $H(k^\bullet)$ (with $q,k_{2\theta},k_{2a_0}$ being the free parameters).

The double affine Hecke algebra $\mathbb{H}=\mathbb{H}(k,q)$ is the subalgebra of the crossed product algebra $\mathbb{C}(T)\rtimes_q W$ generated by $\mathbb{C}[T]$ and $\pi_{k,q}(H(k^\bullet))$. In the remainder of the paper we will often identify $H(k^\bullet)$ with its $\pi_{k,q}$-image in $\mathbb{H}(k,q)$. In addition, for $\lambda+rc$ ($\lambda\in\Lambda$ and $r\in\mathbb{R}$), we write

$$X_q^{\lambda+rc}=q^rX^\lambda$$

for the element in the double affine Hecke algebra corresponding to the regular function $t \mapsto q^r t^\lambda$ on T.

Under the canonical action of $\mathbb{C}(T) \rtimes_q W$ on $\mathbb{C}(T)$, the subspace $\mathbb{C}[T]$ is $\mathbb{H}$-stable. It is called the basic, or polynomial, representation of the double affine Hecke algebra.

2.3. *Nonsymmetric Macdonald-Koornwinder polynomials.* The results on nonsymmetric Macdonald and Koornwinder polynomials in this subsection are well known. In the reduced case they are due to Cherednik (the definition of the nonsymmetric Macdonald polynomial was independently given by Macdonald); see, e.g., [6, §3.3] and [28] and references therein. In the nonreduced case the results in this subsection are from [32], [37], [40]. The current uniform presentation of these results follows [39].

For $w \in W$ with reduced expression $w = u(\lambda)s_{i_1} \dots s_{i_l}$ ($\lambda \in \Lambda^+_{\min}$, $0 \leq i_j \leq n$ and $l = l(w)$), we write

$$T_w := u(\lambda)T_{i_1} \cdots T_{i_l} \in H(k^\bullet).$$

The expression is independent of the choice of reduced expression. By unpublished results of Bernstein and Zelevinsky (cf. [25]), there exists a unique injective algebra homomorphism $\mathbb{C}[T] \hookrightarrow H(k^\bullet)$, which we denote by $p \mapsto p(Y)$, such that $Y^\lambda = T_{\tau(\lambda)}$ for $\lambda \in \Lambda^+$. Its image in $H(k^\bullet)$ is denoted by $\mathbb{C}_Y[T]$. The center $Z(H(k^\bullet))$ of $H(k^\bullet)$ is $\mathbb{C}_Y[T]^{W_0}$.

For $x \in \mathbb{C}^*$ and $\alpha \in R_0$, define $x^{\alpha^\vee} \in T$ by $\lambda \mapsto x^{(\lambda,\alpha^\vee)}$ ($\lambda \in \Lambda$). Let $\gamma_0 = \gamma_0(k) \in T$ be the torus element

$$\gamma_0 := \prod_{\alpha \in R_0^+} \left(k_\alpha^{\frac{1}{2}} k_{\alpha+|\alpha|^2 c/2}^{\frac{1}{2}}\right)^{-\alpha^\vee} \in T. \tag{2.3}$$

More generally, define for $\lambda \in \Lambda$ the element $\gamma_\lambda := \gamma_\lambda(k, q) \in T$ by

$$\gamma_\lambda := u(\lambda)_q \gamma_0.$$

For $\lambda \in \Lambda^-$, we thus have $\gamma_\lambda = q^\lambda \gamma_0$.

THEOREM 2.7. *Let $\lambda \in \Lambda$. There exists a unique $P_\lambda = P_\lambda(\cdot; k, q) \in \mathbb{C}[T]$ such that*

$$\pi_{k,q}(p(Y))P_\lambda = p(\gamma_\lambda^{-1})P_\lambda \qquad \forall\, p \in \mathbb{C}[T]$$

and such that the coefficient of t^λ in the expansion of $P_\lambda(t)$ in monomials t^ν ($\nu \in \Lambda$) is one.

P_λ is the monic nonsymmetric Macdonald polynomial of degree λ in the reduced case and the nonsymmetric monic Koornwinder polynomial in the nonreduced case. We refer to P_λ in the remainder of the text as the monic nonsymmetric Macdonald-Koornwinder polynomial. (Similar terminology will be used later for the normalized and symmetrized versions of P_λ.)

Write k^{-1} for the multiplicity function $a \mapsto k_a^{-1}$. Similarly to Theorem 2.7, there exists, for $\lambda \in \Lambda$, a unique $P'_\lambda = P'_\lambda(\cdot;k,q) \in \mathbb{C}[T]$ such that

$$\pi_{k^{-1},q^{-1}}(p(Y))P'_\lambda = p(\gamma_\lambda)P'_\lambda \qquad \forall\, p \in \mathbb{C}[T]$$

and such that the coefficient of t^λ in the expansion of $P'_\lambda(t)$ in monomials t^ν ($\nu \in \Lambda$) is one.

Next we define the normalized versions of P_λ and P'_λ. For this we first need to recall the evaluation formulas for P_λ and P'_λ.

The multiplicity function k^d on R dual to k is defined as follows. In the reduced case, $k^d := k$. In the nonreduced case, $k_0^d := k_{2\theta}$, $k_{2\theta}^d := k_0$ and the values on the remaining W-orbits of R are unchanged. Set $\gamma_{\lambda,d} = \gamma_\lambda(k^d,q)$. The evaluation formulas for the nonsymmetric Macdonald-Koornwinder polynomials then read

$$\begin{aligned} P_\lambda(\gamma_{0,d}) &= \prod_{a \in R^{\bullet,+} \cap u(\lambda)^{-1}R^{\bullet,-}} (k_a^d)^{-1} c_a(\gamma_0;k^d,q), \\ P'_\lambda(\gamma_{0,d}^{-1}) &= \prod_{a \in R^{\bullet,+} \cap u(\lambda)^{-1}R^{\bullet,-}} k_a^d c_a(\gamma_0^{-1};(k^d)^{-1},q^{-1}). \end{aligned} \tag{2.4}$$

By the conditions on the parameters k_a and q, we have $P_\lambda(\gamma_{0,d}) \neq 0 \neq P'_\lambda(\gamma_{0,d}^{-1})$ for all $\lambda \in \Lambda$. Hence the following definition makes sense.

Definition 2.8. Let $\lambda \in \Lambda$. The normalized nonsymmetric Macdonald-Koornwinder polynomial $E(\gamma_\lambda;\cdot) = E(\gamma_\lambda;\cdot;k,q) \in \mathbb{C}[T]$ of degree λ is defined by

$$E(\gamma_\lambda;t) := \frac{P_\lambda(t)}{P_\lambda(\gamma_{0,d})}.$$

Similarly, we define $E'(\gamma_\lambda^{-1};\cdot) = E'(\gamma_\lambda^{-1};\cdot;k,q) \in \mathbb{C}[T]$ by

$$E'(\gamma_\lambda^{-1};t) := \frac{P'_\lambda(t)}{P'_\lambda(\gamma_{0,d}^{-1})}.$$

It is related to $E(\gamma_\lambda;t)$ by the formula

$$E'(\gamma_\lambda^{-1};t^{-1}) = k_{w_0}^{-1}\big(\pi_{k,q}(T_{w_0})E(\gamma_{-w_0\lambda};\cdot)\big)(t), \tag{2.5}$$

where $w_0 \in W_0$ is the longest Weyl group element and $k_w := \prod_{\alpha \in R_0^+ \cap w^{-1}R_0^-} k_\alpha$ for $w \in W_0$. (See [6, (3.3.26)] for a proof of (2.5) in the reduced case; its proof easily extends to the nonreduced case.)

An important property of the nonsymmetric Macdonald-Koornwinder polynomials is duality,

$$E(\gamma_\lambda;\gamma_{\nu,d};k,q) = E(\gamma_{\nu,d};\gamma_\lambda;k^d,q), \qquad \forall\, \lambda,\nu \in \Lambda.$$

A similar duality formula is valid for E'.

Define $N(\lambda) = N(\lambda; k, q)$ $(\lambda \in \Lambda)$ by

$$N(\lambda) := \prod_{a \in R^{\bullet,+} \cap u(\lambda)^{-1} R^{\bullet,-}} \frac{c_{-a}(\gamma_0; k^d, q)}{c_a(\gamma_0; k^d, q)}.$$

They appear as the quadratic norms of the nonsymmetric Macdonald-Koornwinder polynomials. Concretely, if the parameters satisfy the additional conditions $|k_a k_{2a}^{-1}| \leq 1$ for all $a \in R$ (this only gives additional constraints in the nonreduced case), then

$$\langle E(\gamma_\lambda; \cdot; k, q), E(\gamma_\nu^{-1}; k^{-1}, q^{-1}) \rangle_{k,q} = \langle 1, 1 \rangle_{k,q} N(\lambda; k, q) \delta_{\lambda,\nu}$$

for all $\lambda, \nu \in \Lambda$ with respect to the sesquilinear pairing

$$\langle p_1, p_2 \rangle_{k,q} := \int_{T_u} p_1(t) \overline{p_2(t)} \Big(\prod_{a \in R^{\bullet,+}} \frac{1}{c_a(t; k, q)} \Big) dt, \qquad p_1, p_2 \in \mathbb{C}[T].$$

Here $T_u := \mathrm{Hom}(\Lambda, S^1) \subset T$ with S^1 the unit circle in the complex plane, and dt is the normalized Haar measure on the compact torus T_u.

2.4. *Theta functions.* The results in this section are from [6, §3.2] in the reduced case and from [42] in the nonreduced case.

The q-shifted factorial is

$$\big(x; q\big)_r := \prod_{i=0}^{r-1} (1 - q^i x), \qquad r \in \mathbb{Z}_{\geq 0} \cup \{\infty\}.$$

(By convention, empty products are equal to one.) The q-Gamma function is

$$\Gamma_q(x) := (1-q)^{1-x} \frac{\big(q; q\big)_\infty}{\big(q^x; q\big)_\infty}; \tag{2.6}$$

see [11, §1.10]. Set

$$\theta(x; q) := \big(q; q\big)_\infty \big(x; q\big)_\infty \big(q/x; q\big)_\infty. \tag{2.7}$$

It is the Jacobi theta function $\sum_{r=-\infty}^{\infty} q^{\frac{r^2}{2}} (-q^{-\frac{1}{2}} x)^r$, written in multiplicative form via the Jacobi triple product identity. It satisfies the functional equations

$$\theta(q^r x; q) = (-x)^{-r} q^{-\frac{r(r-1)}{2}} \theta(x; q) \qquad \forall\, r \in \mathbb{Z}.$$

The theta function associated to the lattice Λ is the holomorphic W_0-invariant function $\vartheta(\cdot) = \vartheta_\Lambda(\cdot)$ on T defined by

$$\vartheta(t) := \sum_{\lambda \in \Lambda} q^{\frac{|\lambda|^2}{2}} t^\lambda. \tag{2.8}$$

Since the base for $\vartheta(\cdot)$ will always be q, we do not specify it in the notation. The theta function $\vartheta(\cdot)$ satisfies the functional equations $\vartheta(q^\lambda t) = q^{-\frac{|\lambda|^2}{2}} t^{-\lambda} \vartheta(t)$ for all $\lambda \in \Lambda$.

Remark 2.9. We will always specify the variable dependence, $\theta(\cdot;q)$ and $\vartheta(\cdot)$, to avoid confusion with the highest short root θ and the highest root ϑ of R_0.

Definition 2.10. Define $G=G_{k,q}\in\mathcal{M}(T)$ by

$$G(t):=\vartheta(t)^{-1}$$

in the reduced case and

$$G(t):=\bigl(q_\theta^2;q_\theta^2\bigr)_\infty^{-n}\prod_{\alpha\in R_{0,s}}\bigl(-q_\theta k_0k_{2a_0}^{-1}t^\alpha;q_\theta^2\bigr)_\infty^{-1}$$

in the nonreduced case.

Note that $G(\cdot)$ is W_0-invariant, and that $G(t)=G(t^{-1})$.

Remark 2.11. In the nonreduced case the set $R_0^{s,+}$ of positive short roots is an orthogonal basis of V and a $\mathbb{Z}$-basis of $\Lambda=Q$ (cf. Section 5.2). By the Jacobi triple product identity it then follows that $G(t)$ equals $\vartheta(t)^{-1}$ in the nonreduced case if $k_0=k_{2a_0}$.

We recall the most fundamental property of $G(\cdot)$ in the following proposition. It implies that $G(\cdot)$ serves as the analog of the Gaussian in the context of the double affine Hecke algebra. For proofs and more facts, we refer to [5], [42].

Proposition 2.12. (i) *Given a multiplicity function k on R, the assignment $k_{a_0}^\tau:=k_{2a_0}$, $k_{2a_0}^\tau=k_{a_0}$ and $k_{a_i}^\tau:=k_{a_i}$, $k_{2a_i}^\tau:=k_{2a_i}$ for $1\leq i\leq n$ determines a multiplicity function k^τ on R.*

(ii) *There exists a unique algebra isomorphism $\tau:\mathbb{H}(k,q)\xrightarrow{\sim}\mathbb{H}(k^\tau,q)$ satisfying*

$$\begin{aligned}\tau(T_0)&=X_q^{-a_0}T_0^{-1},&&\\ \tau(T_i)&=T_i,&&1\leq i\leq n,\\ \tau(X^\lambda)&=X^\lambda,&&\lambda\in\Lambda,\\ \tau(u(\lambda))&=q^{-\frac{|\lambda|^2}{2}}X^\lambda u(\lambda),&&\lambda\in\Lambda^+_{\min}.\end{aligned}$$

(iii) *For all $Z\in\mathbb{H}(k,q)$, we have*

$$G_{k,q}(\cdot)ZG_{k,q}(\cdot)^{-1}=\tau(Z)$$

in $\mathcal{M}(T)\rtimes_q W$, where we view both $\mathbb{H}(k,q)$ and $\mathbb{H}(k^d,q)$ as subalgebras of $\mathcal{M}(T)\rtimes_q W$.

2.5. *The nonsymmetric basic hypergeometric function.* The nonsymmetric Macdonald-Mehta weight $\Xi(\cdot) = \Xi(\cdot; k, q) : \Lambda \to \mathbb{C}$ is

$$\Xi(\lambda; k, q) := \frac{G_{k^{\tau d},q}(\gamma_{\lambda,\tau})}{G_{k^{\tau d},q}(\gamma_{0,\tau})N(\lambda; k^{d\tau}, q)},$$

where $\gamma_{\lambda,\tau} := \gamma_\lambda(k^\tau, q)$, $k^{\tau d} = (k^\tau)^d$ and $k^{d\tau} = (k^d)^\tau$. (See [5, §7] in the reduced case and [40, §6.1] in the nonreduced case.) We have $\Xi(0; k, q) = 1$ and

$$\Xi(\lambda; k, q) = \Xi(\lambda; k^d, q), \tag{2.9}$$
$$\Xi(-w_0\lambda; k, q) = \Xi(\lambda; k, q)$$

for all $\lambda \in \Lambda$. Due to the factor $G_{k^{\tau d},q}(\gamma_{\lambda,\tau})$ in the weight $\Xi(\lambda)$, the discrete Macdonald-Metha integral $M := M(k, q)$ defined by

$$M := G(\gamma_0; k^{d\tau}, q)G(\gamma_{0,d}; k^\tau, q) \sum_{\lambda\in\Lambda} \Xi(\lambda; k, q)$$

is convergent. It can be evaluated explicitly; see [5, Thm 1.1] in the reduced case and [42, Prop. 6.1] in the nonreduced case. It will play the role of normalization constant for the (nonsymmetric) basic hypergeometric function.

For $i \in \{1, \dots, n\}$, we denote the simple root $-w_0\alpha_i$ by α_{i^*}. Write $\mathbb{K}$ for the field of meromorphic functions on $T \times T$.

THEOREM 2.13. (i) *There exists a unique anti-isomorphism* $\xi : \mathbb{H}(k, q) \to \mathbb{H}((k^d)^{-1}, q^{-1})$ *satisfying*

$$\xi(T_i) = T_{i^*}^{-1}, \qquad 1 \le i \le n,$$
$$\xi(Y^\lambda) = X^{-w_0\lambda}, \qquad \lambda \in \Lambda,$$
$$\xi(X^\lambda) = T_{w_0}Y^\lambda T_{w_0}^{-1}, \qquad \lambda \in \Lambda.$$

(ii) *There exists a unique* $\mathcal{E}(\cdot, \cdot) = \mathcal{E}(\cdot, \cdot; k, q) \in \mathbb{K}$ *satisfying*

(1) $(t, \gamma) \mapsto G_{k^\tau,q}(t)^{-1}G_{k^{d\tau},q}(\gamma)^{-1}\mathcal{E}(t, \gamma; k, q)$ *is a holomorphic function on* $T \times T$,

(2) $\pi^t_{k,q}(Z)\mathcal{E} = \pi^\gamma_{(k^d)^{-1},q^{-1}}(\xi(Z))\mathcal{E}$ *for all* $Z \in \mathbb{H}(k, q)$, *where* $\pi^t(Z)$ *and* $\pi^\gamma(\xi(Z))$ *are the actions of* $\pi(Z)$ *and* $\pi(\xi(Z))$ *on the first and second torus variable respectively,*

(3) $\mathcal{E}(\gamma_{0,d}, \gamma_0) = 1$.

Explicitly,

$$\mathcal{E}(t, \gamma; k, q) = M_{k,q}^{-1}G_{k^\tau,q}(t)G_{k^{d\tau},q}(\gamma) \sum_{\lambda\in\Lambda} \Xi(\lambda; k, q)E(\gamma_{-w_0\lambda,\tau}; t; k^\tau, q)E'(\gamma_{\lambda,d\tau}^{-1}; \gamma; k^{d\tau}, q),$$

with $\gamma_{\lambda,d\tau} := \gamma_\lambda(k^{d\tau}, q)$. *The sum converges normally for* (t, γ) *in compacta of* $T \times T$.

Proof. (i) Set $(If)(t) := f(t^{-1})$ for $f \in \mathbb{C}(T)$. There exists a unique algebra isomorphism $\eta : \mathbb{H}(k,q) \xrightarrow{\sim} \mathbb{H}(k^{-1},q^{-1})$ such that $\pi_{k,q}(Z) \circ I = I \circ \pi_{k^{-1},q^{-1}}(\eta(Z))$ for all $Z \in \mathbb{H}(k,q)$ (both sides of the identity viewed as operators on $\mathbb{C}(T)$). Then

$$\xi(Z) = T_{w_0}(\delta(\eta(Z)))T_{w_0}^{-1},$$

with $\delta : \mathbb{H}(k^{-1},q^{-1}) \xrightarrow{\sim} \mathbb{H}((k^d)^{-1},q^{-1})$ the linear duality anti-isomorphism mapping T_i to T_i^{-1} $(1 \leq i \leq n)$, X^λ to $Y^{-\lambda}$ and Y^λ to $X^{-\lambda}$ $(\lambda \in \Lambda)$. See [6, §3.3.2] in the reduced case and [37] in the nonreduced case for further details on the duality anti-isomorphism δ, as well as [16], [13].

(ii) A nonsymmetric kernel function $\widetilde{\mathcal{E}} \in \mathcal{M}(T \times T)$ was defined and studied by Cherednik [5, §5] in the reduced case (denoted in [5] as $\mathcal{E}_{q^{-1}}$) and by the author [42, §5] in the nonreduced case (denoted in [42] as $\mathfrak{E}_\ddagger$). Its transformation property with respect to the actions of the double affine Hecke algebra is

$$\pi^t_{k^{-1},q^{-1}}(Z)\widetilde{\mathcal{E}} = \pi^\gamma_{(k^d)^{-1},q^{-1}}(\delta(Z))\widetilde{\mathcal{E}} \qquad \forall\, Z \in \mathbb{H}(k^{-1},q^{-1}).$$

Our kernel $\mathcal{E}$ can be expressed in terms of $\widetilde{\mathcal{E}}$ by

$$\mathcal{E}(t,\gamma) = \big(\pi^\gamma_{(k^d)^{-1},q^{-1}}(T_{w_0})\widetilde{\mathcal{E}}\big)(t^{-1},\gamma)$$

up to normalization; cf. the proof of (i). □

Definition 2.14. We call $\mathcal{E}(\cdot,\cdot) = \mathcal{E}(\cdot,\cdot;k,q)$ the nonsymmetric basic hypergeometric function associated to the triple (R_0,Δ_0,Λ).

Remark 2.15. As already noted in the proof of Theorem 2.13, the definition of the nonsymmetric basic hypergeometric function $\mathcal{E}$ differs slightly from the definitions of the kernel functions in [5], [42], [7]. With our definition of $\mathcal{E}$, the connection with meromorphic solutions of the bispectral quantum Knizhnik-Zamolodchikov equations will be more transparent (see Section 3.1). The difference between the definitions disappears upon symmetrization; cf. Section 2.6.

Note that $\mathcal{E}(\cdot,\gamma)$ for $\gamma \in T$ such that $G_{k^{d\tau},q}(\gamma) \neq 0$ is a meromorphic solution of the spectral problem

$$\pi_{k,q}(p(Y))f = (w_0 p)(\gamma^{-1})f \qquad \forall\, p \in \mathbb{C}[T].$$

In view of Theorem 2.13 we actually have $\mathcal{E} \in \mathcal{V} = \mathcal{V}_{k,q}$, where

$$\mathcal{V}_{k,q} := \{f \in \mathbb{K} \mid \pi^t_{k,q}(Z)f = \pi^\gamma_{(k^d)^{-1},q^{-1}}(\xi(Z))f \quad \forall\, Z \in \mathbb{H}\}. \tag{2.10}$$

$\mathcal{V}$ is a vectorspace over $\mathbb{F}^{W_0 \times W_0}$, with $\mathbb{F} \subset \mathbb{K}$ the subfield

$$\mathbb{F} := \{f \in \mathbb{K} \mid f(q^\lambda t, q^{\lambda'}\gamma) = f(t,\gamma) \quad \forall\, \lambda,\lambda' \in \Lambda\} \tag{2.11}$$

of quasiconstant meromorphic functions on $T \times T$.

PROPOSITION 2.16. (i) *The involution ι of $\mathbb{K}$, defined by $(\iota f)(t,\gamma) = f(\gamma^{-1},t^{-1})$, restricts to a complex linear isomorphism*

$$\iota|_{\mathcal{V}_{k,q}} : \mathcal{V}_{k,q} \xrightarrow{\sim} \mathcal{V}_{k^d,q}.$$

(ii) *The nonsymmetric basic hypergeometric function $\mathcal{E}$ is self-dual*:

$$\iota\big(\mathcal{E}(\cdot,\cdot;k,q)\big) = \mathcal{E}(\cdot,\cdot;k^d,q).$$

Proof. (i) Let $f \in \mathcal{V}_{k,q}$, and set $g := \iota f$. Recall the isomorphism η from the proof of Theorem 2.13. Denote η_d for the isomorphism η with respect to dual parameters (k^d,q). Then

$$\pi^t_{k^d,q}(Z)g = \pi_{k^{-1},q^{-1}}(\widetilde{\xi}(Z))g \qquad \forall\, Z \in \mathbb{H}(k^d,q),$$

with anti-isomorphism $\widetilde{\xi} = \eta^{-1} \circ \xi^{-1} \circ \eta_d : \mathbb{H}(k^d,q) \to \mathbb{H}(k^{-1},q^{-1})$. Since $\eta(T_i) = T_i^{-1}$, $\eta(X^\lambda) = X^{-\lambda}$ and $\eta(Y^\lambda) = T_{w_0} Y^{w_0\lambda} T_{w_0}^{-1}$ for $1 \le i \le n$ and $\lambda \in \Lambda$ (cf. [6, Prop. 3.2.2]), $\widetilde{\xi}$ is the anti-isomorphism ξ with respect to dual parameters (k^d,q). Hence $g \in \mathcal{V}_{k^d,q}$.

(ii) It follows from the explicit series expansion of $\mathcal{E}$, (2.5) and (2.9) that

$$\iota(\mathcal{E}(\cdot,\cdot;k,q)) = \pi^t_{k^d,q}(T_{w_0})\pi^\gamma_{k^{-1},q^{-1}}(T_{w_0})\mathcal{E}(\cdot,\cdot;k^d,q).$$

But this equals $\mathcal{E}(\cdot,\cdot;k^d,q)$ since $\xi(T_{w_0}) = T_{w_0}^{-1}$. □

2.6. *The basic hypergeometric function.* We first recall some well-known facts about symmetric Macdonald-Koornwinder polynomials from, e.g., [6], [28], [32], [37], [42]. For $p \in \mathbb{C}[T]^{W_0}$, we decompose the q-difference reflection operator $\pi_{k,q}(p(Y))$ associated to the central element $p(Y) \in Z(H(k^\bullet))$ as

$$\pi_{k,q}(p(Y)) = \sum_{w\in W_0} D^{k,q}_{p,w} w_q, \qquad D^{k,q}_{p,w} \in \mathbb{C}(T) \rtimes_q \tau(\Lambda).$$

The Macdonald operator $D_p = D^{k,q}_p$ associated to p is defined by

$$D^{k,q}_p := \sum_{w\in W_0} D^{k,q}_{p,w}.$$

The Macdonald operators D_p ($p \in \mathbb{C}[T]^{W_0}$) are pairwise commuting, W_0-equivariant, scalar q-difference operators (see, e.g., [24, Lemma 2.7]). Explicit expressions of D_p can be given for special choices of $p \in \mathbb{C}[T]^{W_0}$, in which case they reduce to the original definitions of the Macdonald, Koornwinder and Ruijsenaars q-difference operators from [27], [22] and [35] respectively (see [28, §4.4]).

The idempotent

$$C_+ := \frac{1}{\sum_{w\in W_0} k_w^2} \sum_{w\in W_0} k_w T_w \in H(k^\bullet)$$

satisfies $T_iC_+ = k_iC_+ = C_+T_i$ for $1 \leq i \leq n$. (We do not specify the $k^\bullet$-dependence of C_+, it will always be clear from the context.) It follows that $\pi_{k,q}(C_+) : \mathcal{M}(T) \to \mathcal{M}(T)$ is a projection operator with image $\mathcal{M}(T)^{W_0}$. Consequently, if $f \in \mathcal{M}(T)$ satisfies the q-difference reflection equations

$$\pi_{k,q}(p(Y))f = p(\gamma^{-1})f \qquad \forall\, p \in \mathbb{C}[T]^{W_0}$$

for some $\gamma \in T$, then $f_+ := \pi_{k,q}(C_+)f \in \mathcal{M}(T)^{W_0}$ satisfies

$$D_pf_+ = p(\gamma^{-1})f_+ \qquad \forall\, p \in \mathbb{C}[T]^{W_0}.$$

In particular, the *normalized symmetric Macdonald-Koornwinder polynomial*

$$E_+(\gamma_\lambda;\cdot) := \pi_{k,q}(C_+)E(\gamma_\lambda;\cdot) \in \mathbb{C}[T]^{W_0}, \qquad \lambda \in \Lambda^-$$

satisfies

$$D_p\big(E_+(\gamma_\lambda;\cdot)\big) = p(\gamma_\lambda^{-1})E_+(\gamma_\lambda;\cdot) \qquad \forall\, p \in \mathbb{C}[T]^{W_0}.$$

The monic symmetric Macdonald-Koornwinder polynomial

$$P_\lambda^+(\cdot) = P_\lambda^+(\cdot;k,q) \in \mathbb{C}[T]^{W_0}, \qquad \lambda \in \Lambda^-$$

is the renormalization of $E_+(\gamma_\lambda;\cdot)$ having an expression

$$P_\lambda^+(t) = \sum_{\mu \in Q_+} d_\mu t^{w_0\lambda-\mu}$$

in monomials with leading coefficient $d_0 = 1$. Then

$$E_+(\gamma_\lambda;\cdot) = P_\lambda^+(\gamma_{0,d})^{-1}P_\lambda^+(\cdot)$$

since $E_+(\gamma_\lambda;\gamma_{0,d}) = 1$. Self-duality and the evaluation formula for the symmetric Macdonald-Koornwinder follow from the corresponding results for the nonsymmetric Macdonald-Koornwinder polynomials by standard symmetrization arguments. Alternatively, they can be derived from the asymptotic analysis of the bispectral quantum Knizhnik-Zamolodchikov equations; see Remark 3.11.

Before symmetrizing the nonsymmetric basic hypergeometric function $\mathcal{E}$, we first introduce and analyze the natural space it will be contained in; cf. [31, Def. 6.13] for GL_m and [30, Def. 6.4] for the reduced case.

Definition 2.17. We set $\mathcal{U} := \mathcal{U}_{k,q}$ for the $\mathbb{F}$-vector space of meromorphic functions f on $T \times T$ satisfying

$$\begin{aligned} \big(D_p^tf\big)(t,\gamma) &= p(\gamma^{-1})f(t,\gamma), \\ \big(\widetilde{D}_p^\gamma f\big)(t,\gamma) &= p(t)f(t,\gamma) \end{aligned} \tag{2.12}$$

for all $p \in \mathbb{C}[T]^{W_0}$, where $\widetilde{D}_p = D_p^{(k^d)^{-1},q^{-1}}$. The superindices t and γ indicate that the q-difference operator is acting on the first and second torus component respectively.

Note that $\mathcal{U}$ is a $W_0 \times W_0$-invariant subspace of $\mathbb{K}$.

View $\pi^t_{k,q}(C_+)$ and $\pi^{\gamma}_{(k^d)^{-1},q^{-1}}(C_+)$ as projection operators on $\mathbb{K}$. Their images are $\mathbb{K}^{W_0\times\{1\}}$ and $\mathbb{K}^{\{1\}\times W_0}$ respectively.

LEMMA 2.18. (i) *The restrictions of the projection operators* $\pi^t_{k,q}(C_+)$ *and* $\pi^{\gamma}_{(k^d)^{-1},q^{-1}}(C_+)$ *to* $\mathcal{V}_{k,q}$ *coincide and map into* $\mathcal{U}^{W_0\times W_0}_{k,q}$.

(ii) *The involution* ι *of* $\mathbb{K}$ *restricts to a complex linear isomorphism*

$$\iota|_{\mathcal{U}_{k,q}} : \mathcal{U}_{k,q} \xrightarrow{\sim} \mathcal{U}_{k^d,q}.$$

(iii) *For all* $f \in \mathcal{V}$,

$$\pi^t_{k^d,q}(C_+)(\iota f) = \iota(\pi^t_{k,q}(C_+)f).$$

Proof. Since $\xi(C_+) = C_+$, the restrictions of $\pi^t_{k,q}(C_+)$ and $\pi^{\gamma}_{(k^d)^{-1},q^{-1}}(C_+)$ to $\mathcal{V}_{k,q}$ coincide. Let $f \in \mathcal{V}_{k,q}$, and set $f_+ := \pi^t_{k,q}(C_+)f$. Since $\xi(p(Y)) = p(X^{-1})$ for all $p \in \mathbb{C}[T]^{W_0}$ and since the projection operator $\pi_{k,q}(C_+)$ on $\mathbb{K}$ has range $\mathbb{K}^{W_0\times\{1\}}$, the meromorphic function f_+ satisfies the first set of equations from (2.12). For the second set of equations of (2.12), note that f_+ is W_0-invariant in the second torus component since $f_+ = \pi^{\gamma}_{(k^d)^{-1},q^{-1}}(C_+)f$. Then for $p \in \mathbb{C}[T]^{W_0}$,

$$\begin{aligned} D_p^{(k^d)^{-1},q^{-1}} f_+ &= \pi^{\gamma}_{(k^d)^{-1},q^{-1}}(C_+p(Y))f \\ &= \pi^{\gamma}_{(k^d)^{-1},q^{-1}}(C_+)\pi_{(k^d)^{-1},q^{-1}}(T_{w_0}p(Y)T_{w_0}^{-1})f \\ &= \pi^t_{k,q}(p(X))\pi^{\gamma}_{(k^d)^{-1},q^{-1}}(C_+)f \\ &= \pi^t_{k,q}(p(X))f_+ \end{aligned}$$

since $\xi(p(X)) = T_{w_0}p(Y)T_{w_0}^{-1}$. Hence $f_+ \in \mathcal{U}^{W_0\times W_0}_{k,q}$, proving (i). Part (iii) follows from (i) and the fact that

$$\iota \circ \pi^t_{k,q}(C_+) = \pi^{\gamma}_{k^{-1},q^{-1}}(C_+)$$

(which in turn follows from the fact that $\eta(C_+) = C_+$). It remains to prove (ii). It suffices to show that $I \circ D_p^{k,q} \circ I = D_p^{k^{-1},q^{-1}}$ for $p \in \mathbb{C}[T]^{W_0}$ as endomorphism of $\mathcal{M}(T)$, where $(Ig)(t) := g(t^{-1})$. This follows from $\eta(p(Y)) = p(Y)$ for $p \in \mathbb{C}[T]^{W_0}$; cf. the proof of Proposition 2.16 (see also [30, Lemma 6.2]). □

The nonsymmetric basic hypergeometric function $\mathcal{E}$ associated to the triple (R_0, Δ_0, Λ), being a distinguished element of $\mathcal{V}$, thus gives rise to a distinguished $W_0 \times W_0$-symmetric meromorphic solution of the bispectral problem (2.12).

Definition 2.19. We call $\mathcal{E}_+(\cdot,\cdot) = \mathcal{E}_+(\cdot,\cdot;k,q) := \pi^t_{k,q}(C_+)\mathcal{E}(\cdot,\cdot;k,q) \in \mathcal{U}^{W_0\times W_0}$ the basic hypergeometric function associated to the triple (R_0, Δ_0, Λ).

In the reduced case, $\mathcal{E}_+$ is Cherednik's [5], [7] global spherical function. In the nonreduced case, $\mathcal{E}_+$ was defined by the author in [42].

We list the key properties of the basic hypergeometric function in the following theorem.

THEOREM 2.20. (i) *Explicit series expansion*:

$$\begin{aligned}\mathcal{E}_+(t,\gamma;k,q) = M_{k,q}^{-1} G_{k^\tau,q}(t) G_{k^{d\tau},q}(\gamma) \\ \times \sum_{\lambda\in\Lambda^-} \Xi^+(\lambda;k,q) E_+(\gamma_{\lambda,\tau};t;k^\tau,q) E_+(\gamma_{\lambda,d\tau};\gamma^{-1};k^{d\tau},q)\end{aligned}$$

with $\Xi^+(\lambda;k,q) := \sum_{\nu\in W_0\lambda}\Xi(\nu;k,q)$. *The sum converges normally for* (t,γ) *in compacta of* $T\times T$.

(ii) *Inversion symmetry*:

$$\mathcal{E}_+(t,\gamma;k,q) = \mathcal{E}_+(t^{-1},\gamma^{-1};k,q).$$

(iii) *Duality*:

$$\iota\big(\mathcal{E}_+(\cdot,\cdot;k,q)\big) = \mathcal{E}_+(\cdot,\cdot;k^d,q).$$

(iv) *Reduction to symmetric Macdonald-Koornwinder polynomials*:

$$\mathcal{E}_+(t,\gamma_\lambda;k,q) = E_+(\gamma_{\lambda_-};t;k,q) \qquad \forall\,\lambda\in\Lambda,$$

with $\lambda_-\in\Lambda^-$ *the unique antidominant weight in the orbit* $W_0\lambda$.

Proof. We only sketch the proof. For detailed proofs, see [5] in the reduced case and [42] in the nonreduced case.

(i) This follows from rather standard symmetrization arguments, using the fact that $\pi(C_+)E(\gamma_\lambda;\cdot)$ only depends on the orbit $W_0\lambda$ of λ and that

$$E'_+(\gamma_\lambda^{-1};t) = E_+(\gamma_{-w_0\lambda};t^{-1}) \tag{2.13}$$

for $\lambda\in\Lambda^-$, where $E'_+(\gamma_\lambda^{-1};\cdot) := \pi_{k^{-1},q^{-1}}(C_+)E'(\gamma_\lambda^{-1};\cdot)$. Formula (2.13) is a direct consequence of (2.5).

(ii) This follows from (i) and the formula $E_+(\gamma_\lambda;t^{-1}) = E_+(\gamma_{-w_0\lambda};t)$ for $\lambda\in\Lambda^-$. The latter formula is a consequence of (2.13) and the fact that $E'_+(\gamma_\lambda^{-1};t) = E_+(\gamma_\lambda;t)$ for $\lambda\in\Lambda^-$ (see, e.g., [28, (5.3.2)]).

(iii) This follows from (i) and the self-duality $\Xi^+(\lambda;k,q) = \Xi^+(\lambda;k^d,q)$ of the weight Ξ^+. Alternatively, use Proposition 2.16 and Lemma 2.18(iii).

(iv) This is Cherednik's generalization of the Shintani-Casselman-Shalika formula in the reduced case (see [5, (7.13)],[7, (3.11)]). For the nonreduced case, see [42, Thm. 6.15(d)]. □

3. **Basic Harish-Chandra series**

In this section we generalize and analyze the basic Harish-Chandra series from [31] (GL_m case) and from [30] (reduced case). The basic Harish-Chandra series is a q-analog of the Harish-Chandra series solution of the Heckman-Opdam hypergeometric system associated to root systems (see [15, Part I, Chap. 4] and references therein).

Our approach differs from the classical treatment in the sense that we construct, following [31], [30], the basic Harish-Chandra series as matrix coefficient of a power series solution of a bispectral extension of Cherednik's [4], [3] quantum affine Knizhnik-Zamolodchikov (KZ) equations associated to the minimal principal series of $H(k^\bullet)$ (the extension being given by a compatible set of equations acting on the central character of the minimal principal series representation). This is essential for two reasons:

(1) Convergence issues: formal power series solutions of the (bispectral) quantum KZ equation are easily seen to converge deep in the asymptotic sector, in contrast to formal power series solutions of the spectral problem for the Macdonald q-difference operators.

(2) The formal power series solution of the bispectral quantum KZ equation gives rise to a self-dual, globally meromorphic q-analog of the classical Harish-Chandra series. The self-duality plays an important role in our proof of the c-function expansion of the basic hypergeometric function in Section 4.

Our approach also gives new proofs of the self-duality and the evaluation formula for the symmetric Macdonald-Koornwinder polynomials (see Remark 3.11).

3.1. *Bispectral quantum Knizhnik-Zamolodchikov equations.* In this subsection we show that the space $\mathcal{V}$ (see (2.10)) is isomorphic to the space of solutions of a bispectral extension of the quantum affine Knizhnik-Zamolodchikov (KZ) equations.

We will first introduce the bispectral quantum KZ equations, following and extending [31], [30]. Tensor products and endomorphism spaces will be over $\mathbb{C}$ unless stated explicitly otherwise. Let $\chi : R_0 \to \{0,1\}$ be the characteristic function of R_0^-. Set $M := \bigoplus_{w\in W_0} \mathbb{C}v_w$. Define elements

$$C^{k,q}_{(w,1)}, C^{k,q}_{(1,w)} \in \mathbb{K} \otimes \mathrm{End}(M)$$

for the generators $w = s_i$, $w = u(\lambda)$ ($0 \leq i \leq n$ and $\lambda \in \Lambda^+_{\mathrm{min}}$) of the extended affine Weyl group W by

$$C^{k,q}_{(s_0,1)}(t,\gamma)v_w := \frac{\gamma^{-w^{-1}\theta}v_{s_\theta w}}{k_0c_0(t;k^{-1},q)} + \left(\frac{c_0(t;k^{-1},q)-k_0^{-2\chi(w^{-1}\theta)}}{c_0(t;k^{-1},q)}\right)v_w,$$

$$C^{k,q}_{(s_i,1)}(t,\gamma)v_w := \frac{v_{s_iw}}{k_ic_i(t;k^{-1},q)} + \left(\frac{c_i(t;k^{-1},q)-k_i^{-2\chi(-w^{-1}\alpha_i)}}{c_i(t;k^{-1},q)}\right)v_w,$$

$$C^{k,q}_{(u(\lambda),1)}(t,\gamma)v_w := \gamma^{w^{-1}w_0\lambda}v_{v(\lambda)^{-1}w}$$

for $1 \leq i \leq n$, $\lambda \in \Lambda^+_{\min}$ and $w \in W_0$, where $v(\lambda) \in W_0$ is the element of minimal length such that $v(\lambda)\lambda \in \Lambda^-$, and

$$C^{k,q}_{(1,s_0)}(t,\gamma)v_w := \frac{t^{w\theta}v_{ws_\theta}}{k_0^dc_0(\gamma^{-1};(k^d)^{-1},q)} + \left(\frac{c_0(\gamma^{-1};(k^d)^{-1},q)-(k_0^d)^{-2\chi(w\theta)}}{c_0(\gamma^{-1};(k^d)^{-1},q)}\right)v_w,$$

$$C^{k,q}_{(1,s_i)}(t,\gamma)v_w := \frac{v_{ws_i}}{k_i^dc_i(\gamma^{-1};(k^d)^{-1},q)} + \left(\frac{c_i(\gamma^{-1};(k^d)^{-1},q)-(k_i^d)^{-2\chi(-w\alpha_i)}}{c_i(\gamma^{-1};(k^d)^{-1},q)}\right)v_w,$$

$$C^{k,q}_{(1,u(\lambda))}(t,\gamma)v_w := t^{-ww_0\lambda}v_{wv(\lambda)}.$$

The following theorem is [31, Cor. 3.4, Lemma 4.3] in the GL_m-case and [30, Cor. 3.8, Lemma 4.3] in the reduced case. The extension to the nonreduced case is straightforward.

THEOREM 3.1. *There exists a unique left $W \times W$-action $((w_1,w_2),g) \mapsto \nabla^{k,q}((w_1,w_2))g$ on $\mathbb{K} \otimes M$ satisfying*

$$\nabla^{k,q}(w,1)g = C^{k,q}_{(w,1)}w_q^tg,$$
$$\nabla^{k,q}(1,w)g = C^{k,q}_{(1,w)}w^\gamma_{q^{-1}}g$$

for $g \in \mathbb{K}$, $w = s_j$ $(0 \leq j \leq n)$ and $w = u(\lambda)$ $(\lambda \in \Lambda^+_{\min})$, where

$$(w_q^tg)(t,\gamma) = g(w_q^{-1}t,\gamma), \qquad (w^\gamma_{q^{-1}}g)(t,\gamma) = g(t,w^{-1}_{q^{-1}}\gamma).$$

We say that $g \in \mathbb{K}\otimes M$ satisfies the bispectral quantum Knizhnik-Zamolodchikov equations if g is a solution of the compatible system

$$\nabla(\tau(\lambda),\tau(\lambda'))g = g \qquad \forall\,(\lambda,\lambda') \in \Lambda \times \Lambda \tag{3.1}$$

of q-difference equations. Restricting the equations (3.1) to $\Lambda \times \{0\}$ and fixing the second torus variable $\gamma \in T$ gives, in the reduced case, Cherednik's [4], [3] quantum affine KZ equation associated to the minimal principal series representation of $H(k^\bullet)$ with central character γ.

Definition 3.2. We write $\mathcal{K} = \mathcal{K}_{k,q}$ for the $\mathbb{F}$-vector space consisting of $g \in \mathbb{K} \otimes M$ satisfying the bispectral quantum KZ equations (3.1).

Note that $\mathcal{K}$ is a $W_0 \times W_0$-module, with action the restriction of ∇ to $W_0 \times W_0$.

Let σ be the complex linear automorphism of $\mathbb{K} \otimes M$ defined by

$$\sigma(f \otimes v_w) := \iota f \otimes v_{w^{-1}}.$$

Then

$$\sigma \circ \nabla^{k,q}((w,w')) = \nabla^{k^d,q}((w',w)) \circ \sigma \qquad \forall\, w,w' \in W. \tag{3.2}$$

In particular, σ restricts to a complex linear isomorphism $\mathcal{K}_{k,q} \xrightarrow{\sim} \mathcal{K}_{k^d,q}$.

Define complex linear maps

$$\psi := \psi_{k,q} : \mathbb{K} \to \mathbb{K} \otimes M,$$
$$\phi := \phi_{k,q} : \mathbb{K} \otimes M \to \mathbb{K}$$

by

$$\psi(f) := \sum_{w \in W_0} (\pi^t_{k,q}(T_{ww_0})f) \otimes v_w,$$
$$\phi\Big(\sum_{w \in W_0} f_w \otimes v_w\Big) := \frac{k_{w_0}^{-1}}{\sum_{w \in W_0} k_w^{-2}} \sum_{w \in W_0} k_w^{-1} f_w.$$

Note that $\phi \circ \psi = \pi^t_{k,q}(C_+)$. Recall from Lemma 2.18 that $\pi^t_{k,q}(C_+)$ restricts to a complex linear map $\pi^t_{k,q}(C_+) : \mathcal{V} \to \mathcal{U}^{W_0 \times W_0}$. It factorizes through the solutions space $\mathcal{K}^{W_0 \times W_0}$ of the bispectral quantum KZ equations:

THEOREM 3.3. (i) *ψ restricts to a $\mathbb{F}^{W_0 \times W_0}$-linear isomorphism $\psi : \mathcal{V} \xrightarrow{\sim} \mathcal{K}^{W_0 \times W_0}$.*

(ii) *ϕ restricts to an injective $W_0 \times W_0$-equivariant $\mathbb{F}$-linear map $\phi : \mathcal{K} \hookrightarrow \mathcal{U}$.*

(iii) *$\psi_{k^d,q} \circ \iota|_{\mathcal{V}_{k,q}} = \sigma \circ \psi_{k,q}|_{\mathcal{V}_{k,q}}$ and $\phi_{k^d,q} \circ \sigma|_{\mathcal{K}_{k,q}^{W_0 \times W_0}} = \iota \circ \phi_{k,q}|_{\mathcal{K}_{k,q}^{W_0 \times W_0}}$.*

Proof. (i) The analogous statement in the reduced case for the usual quantum affine KZ equations was proved in [43, Thm. 4.9]. Its extension to the nonreduced case is straightforward. The bispectral extension follows by a repetition of the arguments for the dual part of the quantum KZ equations (i.e., the part acting on the second torus component).

(ii) This is the bispectral extension of the difference Cherednik-Matsuo correspondence [4, Thm. 3.4(a)]. See [31, Thm. 6.16, Cor. 6.21] for the GL_m-case and [30, Thm. 6.6] for the reduced case. (The injectivity follows from the asymptotic analysis of the bispectral quantum KZ equations, which we will also recall in Section 3.2.) The extension to the nonreduced case is straightforward. An alternative approach is to extend the techniques from [43, §5] to the present bispectral (and nonreduced) setting.

(iii) Using $\eta(T_{ww_0})=T^{-1}_{w_0w^{-1}}$ for $w\in W_0$, it follows that

$$\psi_{k^d,q}(\iota f)=\sigma\Big(\sum_{w\in W_0}\pi^{\gamma}_{(k^d)^{-1},q^{-1}}(T^{-1}_{w_0w})f\otimes v_w\Big)$$

for $f\in\mathbb{K}$. The first part then follows from the observation that

$$\psi_{k,q}(f)=\sum_{w\in W_0}\pi^{\gamma}_{(k^d)^{-1},q^{-1}}(T^{-1}_{w_0w})f\otimes v_w$$

if $f\in\mathcal{V}_{k,q}$, since $\xi(T_{ww_0})=T^{-1}_{w_0w}$ for $w\in W_0$. For the second equality, let $f\in\mathcal{K}^{W_0\times W_0}$ and set $g=\psi^{-1}(f)\in\mathcal{V}$. Then

$$\begin{aligned}\phi(\sigma(f))&=\pi^t(C_+)\psi^{-1}(\sigma(f))\\&=\pi^t(C_+)(\iota g)\\&=\iota(\pi^t(C_+)g)\\&=\iota(\phi(f)),\end{aligned}$$

where we use the first part of (ii) for the second equality and Lemma 2.18(iii) for the third equality. □

Corollary 3.4. $\mathcal{E}_+\in\phi(\mathcal{K}^{W_0\times W_0})$.

Proof. $\mathcal{E}_+=\pi^t(C_+)\mathcal{E}=\phi(\psi(\mathcal{E}))$ and $\psi(\mathcal{E})\in\mathcal{K}^{W_0\times W_0}$ since $\mathcal{E}\in\mathcal{V}$. □

3.2. *Asymptotically free solutions of the bispectral quantum KZ equations.* We recall the results on asymptotically free solutions of the bispectral quantum KZ equations from [31] (GL_m case) and [30] (reduced case). The extension to the nonreduced case presented here follows from straightforward adjustments of the arguments of [31], [30].

Define $\mathcal{W}(\cdot,\cdot)=\mathcal{W}(\cdot,\cdot;k,q)\in\mathbb{K}$ by

$$\mathcal{W}(t,\gamma):=\frac{\vartheta(t(w_0\gamma)^{-1})}{\vartheta(\gamma_0t)\vartheta(\gamma_{0,d}^{-1}\gamma)}.$$

There is some flexibility in the choice of $\mathcal{W}(\cdot,\cdot)$. The key properties we need it to satisfy are the functional equations

$$\mathcal{W}(q^{\lambda}t,\gamma)=\gamma_0^{\lambda}\gamma^{w_0\lambda}\mathcal{W}(t,\gamma),\qquad\lambda\in\Lambda \tag{3.3}$$

and the self-duality property

$$\iota\big(\mathcal{W}(\cdot,\cdot;k,q)\big)=\mathcal{W}(\cdot,\cdot;k^d,q).$$

For $\varepsilon>0$, set

$$B_{\varepsilon}:=\{t\in T\mid|t^{\alpha_i}|<\varepsilon\quad\forall\,i\in\{1,\dots,n\}\}$$

and $B_{\varepsilon}^{-1}:=\{t^{-1}\mid t\in B_{\varepsilon}\}$.

THEOREM 3.5. *There exists a unique* $F(\cdot,\cdot) = F(\cdot,\cdot;k,q) \in \mathcal{K}_{k,q}$ *such that* $F(t,\gamma) = \mathcal{W}(t,\gamma)H(t,\gamma)$ *with* $H(\cdot,\cdot) = H(\cdot,\cdot;k,q) \in \mathbb{K}\otimes M$ *satisfying, for* $\varepsilon > 0$ *sufficiently small,*

$$H(t,\gamma) = \sum_{\mu,\nu\in Q_+} H_{\mu,\nu}t^{-\mu}\gamma^{\nu} \quad (H_{\mu,\nu}\in M), \qquad H_{0,0} = v_{w_0}$$

for $(t,\gamma) \in B_\varepsilon^{-1}\times B_\varepsilon$, *with the series converging normally for* (t,γ) *in compacta of* $B_\varepsilon^{-1}\times B_\varepsilon$.

Proof. See [31, Thm. 5.3] (GL_m case) and [30, Thm. 5.4] (reduced case). The proofs are based on the asymptotic analysis of compatible systems of q-difference equations using classical methods that go back to Birkhoff [2] (see the appendix of [31]). These results extend immediately to the present setup if one restricts the bispectral quantum KZ equations (3.1) to λ,λ' in the sublattice $\bigoplus_{i=1}^n \mathbb{Z}\varpi_i$ of Λ. But the resulting function $F(\cdot,\cdot)$ then automatically satisfies (3.1) for all $\lambda,\lambda'\in\Lambda$ due to the compatibility of the bispectral quantum KZ equations (3.1) (cf. the proof of [45, Thm. 3.4]). □

For $a\in R^\bullet$, let $n_a(\cdot) = n_a(\cdot;k,q)$ be the rational function

$$n_a(t) = \begin{cases} 1-k_a^{-2}t_q^a & \text{if } 2a\notin R,\\ (1-k_a^{-1}k_{2a}^{-1}t_q^a)(1+k_a^{-1}k_{2a}t_q^a) & \text{if } 2a\in R.\end{cases}$$

Note that $c_a(t;k^{-1},q) = n_a(t;k,q)/n_a(t;\mathbf{1},q)$ for $a\in R^\bullet$, with $\mathbf{1}$ the multiplicity function identically equal to one. Let $\mathcal{L}(\cdot) = \mathcal{L}_q(\cdot)$ and $\mathcal{S}(\cdot) = \mathcal{S}_{k,q}(\cdot)$ be the holomorphic functions on T defined by

$$\mathcal{L}_q(t) := \prod_{\substack{\alpha\in R_0^+,\\ r\in\mathbb{Z}_{>0}}} n_{\alpha+r\frac{|\alpha|^2}{2}c}(t;\mathbf{1},q), \qquad \mathcal{S}_{k,q}(t) := \prod_{\substack{\alpha\in R_0^+,\\ r\in\mathbb{Z}_{>0}}} n_{\alpha+r\frac{|\alpha|^2}{2}c}(t;k,q).$$

We give the key properties of $F(\cdot,\cdot)$ in the following theorem. The proof follows from straightforward adjustments of the arguments in [31], [30] (which corresponds to the GL_m case and reduced case respectively).

THEOREM 3.6. (i) $F\in\mathcal{K}$ *is self-dual:* $\sigma(F(\cdot,\cdot;k,q)) = F(\cdot,\cdot;k^d,q)$.

(ii) $\{\nabla(1,w)F\}_{w\in W_0}$ *is a* $\mathbb{F}$*-basis of* $\mathcal{K}$.

(iii) $T\times T\ni(t,\gamma)\mapsto \mathcal{S}_{k,q}(t^{-1})\mathcal{S}_{k^d,q}(\gamma)H(t,\gamma;k,q)$ *is holomorphic.*

(iv) *For* $\varepsilon>0$ *sufficiently small, there exist unique holomorphic* M*-valued functions* $\Upsilon_\mu(\cdot)$ *on* T $(\mu\in Q_+)$ *such that*

$$\mathcal{S}_{k^d,q}(\gamma)H(t,\gamma;k,q) = \sum_{\mu\in Q_+}\Upsilon_\mu(\gamma)t^{-\mu}$$

for $(t,\gamma)\in B_\varepsilon^{-1}\times T$, *with the series converging normally for* (t,γ) *in compacta of* $B_\varepsilon^{-1}\times T$.

(v) $\Upsilon_0(\gamma) = \mathcal{L}_q(\gamma)v_{w_0}$.

From the third part of the theorem, we conclude

COROLLARY 3.7. *Let $Z_{k,q}\subseteq T$ be the zero locus of $\mathcal{S}_{k,q}(\cdot)$ and set $Z_{k,q}^{-1}:=\{t^{-1}\,|\,t\in Z_{k,q}\}$. Then $H(\cdot,\cdot;k,q)$ is holomorphic on $T\setminus Z_{k,q}^{-1}\times T\setminus Z_{k^d,q}$.*

In the reduced case,

$$Z_{k,q}=\{t\in T\mid t^{\alpha}\in k_{\alpha}^2q_{\alpha}^{-\mathbb{Z}_{>0}}\ \text{ for some }\ \alpha\in R_0^+\}.$$

In the nonreduced case,

$$Z_{k,q}=\big\{t\in T\mid t^{\alpha}\in\{aq_{\theta}^{-2\mathbb{Z}_{>0}},bq_{\theta}^{-2\mathbb{Z}_{>0}},cq_{\theta}^{-2\mathbb{Z}_{>0}},dq_{\theta}^{-2\mathbb{Z}_{>0}}\}\ \text{ for some }\ \alpha\in R_{0,s}^+\\ \text{or }\ t^{\beta}\in k_{\vartheta}^2q_{\vartheta}^{-\mathbb{Z}_{>0}}\ \text{ for some }\ \beta\in R_{0,l}^+\big\},$$

where

$$\{a,b,c,d\}:=\{k_{\theta}k_{2\theta},-k_{\theta}k_{2\theta}^{-1},q_{\theta}k_0k_{2a_0},-q_{\theta}k_0k_{2a_0}^{-1}\}. \tag{3.4}$$

3.3. *Basic Harish-Chandra series.* Following [31, §6.3] and [30, §7], we have the following fundamental definition.

Definition 3.8. The self-dual basic Harish-Chandra series

$$\Phi(\cdot,\cdot)=\Phi(\cdot,\cdot;k,q)\in\mathcal{U}_{k,q}$$

is defined by

$$\Phi:=\phi(F)=\mathcal{W}\phi(H). \tag{3.5}$$

The properties of F from Theorem 3.6 (singularities, self-duality, leading term) can immediately be transferred to the self-dual basic Harish-Chandra series Φ. In particular, by Theorem 3.3(iii) the self-duality of F gives the self-duality of Φ,

$$\iota\big(\Phi(\cdot,\cdot;k,q)\big)=\Phi(\cdot,\cdot;k^d,q).$$

In the derivation of the c-function expansion of the basic hypergeometric function, we initially make use of the self-dual basic Harish-Chandra series. To make the connection to the classical theory more transparent we will reformulate these results in terms of a renormalization of $\Phi(t,\gamma)$ that is closer to the standard normalization of the classical Harish-Chandra series (see the introduction). It is a γ-dependent renormalization of $\Phi(t,\gamma)$, which also depends on a base point $\eta\in T$ (indicating the choice of normalization of the prefactor). This renormalization of Φ breaks the duality symmetry.

To define the renormalized version of the basic Harish-Chandra series, consider first the renormalization $\widehat{H}(\cdot,\cdot)=\widehat{H}(\cdot,\cdot;k,q)\in\mathbb{K}\otimes M$ of $H(\cdot,\cdot)$ given by

$$\widehat{H}(t,\gamma):=\frac{\mathcal{S}_{k^d,q}(\gamma)\sum_{w\in W_0}k_w^2}{\mathcal{L}_q(\gamma)}H(t,\gamma).$$

Note that for $\varepsilon > 0$ sufficiently small,

$$\widehat{H}(t,\gamma) = \sum_{\mu\in Q_+} \widehat{\Upsilon}_\mu(\gamma)t^{-\mu}$$

for $(t,\gamma) \in B_\varepsilon^{-1} \times \{\gamma \in T \mid \mathcal{L}_q(\gamma) \neq 0\}$, and $\phi(\widehat{\Upsilon}_0) \equiv 1$.

The monic basic Harish-Chandra series $\widehat{\Phi}_\eta(\cdot,\cdot) = \widehat{\Phi}_\eta(\cdot,\cdot;k,q)$ with generic reference point $\eta \in T$ is now defined by

$$\widehat{\Phi}_\eta := \widehat{\mathcal{W}}_\eta \phi(\widehat{H}),$$

with prefactor $\widehat{\mathcal{W}}_\eta(\cdot,\cdot) = \widehat{\mathcal{W}}_\eta(\cdot,\cdot;k,q) \in \mathbb{K}$ defined as follows. Let

$$\rho_s^\vee := \frac{1}{2}\sum_{\beta\in R_{0,s}^+} \beta^\vee$$

with $R_{0,s}^+ \subset R_0^+$ the subset of positive short roots. For $x \in \mathbb{R}_{>0}$, let $x^{\rho_s^\vee} \in T$ be the torus element $\lambda \mapsto x^{(\rho_s^\vee,\lambda)}$ $(\lambda \in \Lambda)$. Then $\widehat{\mathcal{W}}_\eta$ is defined to be

$$\widehat{\mathcal{W}}_\eta(t,\gamma) = \frac{\widehat{\mathcal{W}}(t,\gamma)}{\widehat{\mathcal{W}}(\eta\gamma_{0,d},\gamma)}$$

with

$$\widehat{\mathcal{W}}(t,\gamma) = \frac{\vartheta\big(\gamma_0^{-1}(k_0^{-1}k_{2a_0})^{\rho_s^\vee} t(w_0\gamma)^{-1}\big)}{\vartheta((k_0^{-1}k_{2a_0})^{\rho_s^\vee} t\big)}.$$

(Note that $(k_0^{-1}k_{2a_0})^{\rho_s^\vee} = 1$ in the reduced case.) The prefactor $\widehat{\mathcal{W}}_\eta(t,\gamma)$ satisfies the same functional equations as function of $t \in T$ as the self-dual prefactor $\mathcal{W}(t,\gamma)$,

$$\widehat{\mathcal{W}}_\eta(q^\lambda t,\gamma) = \gamma_0^\lambda \gamma^{w_0\lambda}\widehat{\mathcal{W}}_\eta(t,\gamma) \qquad \forall\, \lambda \in \Lambda.$$

COROLLARY 3.9. *Let $\gamma \in T$ such that $\mathcal{L}_q(\gamma) \neq 0$. The monic basic Harish-Chandra series $\widehat{\Phi}_\eta(\cdot,\gamma)$ satisfies the Macdonald q-difference equations*

$$D_p\widehat{\Phi}_\eta(\cdot,\gamma) = p(\gamma^{-1})\widehat{\Phi}_\eta(\cdot,\gamma) \qquad \forall\, p \in \mathbb{C}[T]^{W_0} \tag{3.6}$$

and has, for $t \in B_\varepsilon^{-1}$ with $\varepsilon > 0$ sufficiently small, a convergent series expansion

$$\widehat{\Phi}_\eta(t,\gamma) = \widehat{\mathcal{W}}_\eta(t,\gamma)\sum_{\mu\in Q_+}\Gamma_\mu(\gamma)t^{-\mu},$$

where $\Gamma_\mu(\gamma) := \phi(\widehat{\Upsilon}_\mu(\gamma))$. (In particular, $\Gamma_0(\gamma) = 1$.) The series converges normally for t in compacta of B_ε^{-1}.

Since

$$\widehat{\mathcal{W}}_\eta(q^\lambda\eta\gamma_{0,d},\gamma) = \gamma_0^\lambda\gamma^{w_0\lambda} \qquad \forall\, \lambda \in \Lambda, \tag{3.7}$$

the monic basic Harish-Chandra series $\widehat{\Phi}_\eta(\cdot,\gamma)$ is the natural normalization of the basic Harish-Chandra series when restricting the Macdonald q-difference equations (3.6) to functions on the q-lattice $\eta\gamma_{0,d}q^\Lambda$.

PROPOSITION 3.10. *Fix $\lambda\in\Lambda^-$. For generic values of the multiplicity function k, we have*

$$\widehat{\Phi}_\eta(t,\gamma_\lambda)=(\eta\gamma_{0,d})^{-w_0\lambda}P_\lambda^+(t). \tag{3.8}$$

Proof. Fix $\lambda\in\Lambda^-$. Note that

$$\widehat{\mathcal{W}}(t,\gamma_\lambda)=q^{\frac{-|\lambda|^2}{2}}k_0^{(\lambda,\rho_s^\vee)}k_{2a_0}^{-(\lambda,\rho_s^\vee)}t^{w_0\lambda}.$$

Hence for $t\in B_\varepsilon^{-1}$ with $\varepsilon>0$ sufficiently small,

$$\widehat{\Phi}_\eta(t,\gamma_\lambda)=\sum_{\mu\in Q_+}d_\mu t^{w_0\lambda-\mu} \tag{3.9}$$

as normally convergent series for t in compacta of B_ε^{-1}, with leading coefficient

$$d_0=(\eta\gamma_{0,d})^{-w_0\lambda}. \tag{3.10}$$

(This requires $\mathcal{L}_q(\gamma_\lambda)\neq 0$, which we impose as one of the genericity conditions on the multiplicity function.) Since k is generic, this characterizes $\widehat{\Phi}(\cdot,\gamma_\lambda)$ within the class of formal power series $f\in\mathbb{C}[[X^{-\alpha_i}]]X^{w_0\lambda}$ satisfying the eigenvalue equations

$$\big(D_pf\big)(t)=p(\gamma_\lambda^{-1})f(t)\qquad\forall\,p\in\mathbb{C}[T]^{W_0}$$

(cf., e.g., [24, Thm. 4.6]). The result now follows since $f(t)=d_0P_\lambda^+(t)$ satisfies the same characterizing properties. □

Remark 3.11. The explicit evaluation formula [6, §3.3.2] for the symmetric Macdonald-Koornwinder polynomial $P_\lambda^+(\gamma_{0,d})=P_\lambda^+(w_0\gamma_{0,d})=P_\lambda^+(\gamma_{0,d}^{-1})$ ($\lambda\in\Lambda^-$) can be derived from Proposition 3.10 and the fundamental properties of the self-dual basic Harish-Chandra series

$$\Phi(t,\gamma)=\frac{\mathcal{W}(t,\gamma)}{\widehat{\mathcal{W}}_\eta(t,\gamma)}\frac{\mathcal{L}_q(\gamma)}{\mathcal{S}_{k^d,q}(\gamma)\sum_{w\in W_0}k_w^2}\widehat{\Phi}_\eta(t,\gamma) \tag{3.11}$$

as follows. By a direct computation using Proposition 3.10,

$$\Phi(\gamma_{\mu,d}^{-1},\gamma_\lambda;k,q)=\frac{\mathcal{W}(\gamma_{0,d}^{-1},\gamma_0;k,q)}{\sum_{w\in W_0}k_w^2}\gamma_{0,d}^{-\lambda}\frac{\mathcal{L}_q(\gamma_\lambda)}{\mathcal{S}_{k^d,q}(\gamma_\lambda)}P_\lambda^+(\gamma_{\mu,d}^{-1};k,q) \tag{3.12}$$

for $\lambda,\mu\in\Lambda^-$. By the self-duality of Φ and of $\mathcal{W}(\cdot,\cdot)$, it is also equal to

$$\Phi(\gamma_\lambda^{-1},\gamma_{\mu,d};k^d,q)=\frac{\mathcal{W}(\gamma_{0,d}^{-1},\gamma_0;k,q)}{\sum_{w\in W_0}k_w^2}\gamma_0^{-\mu}\frac{\mathcal{L}_q(\gamma_{\mu,d})}{\mathcal{S}_{k,q}(\gamma_{\mu,d})}P_\mu^+(\gamma_\lambda^{-1};k^d,q). \tag{3.13}$$

Setting $\lambda = \mu = 0$, we get

$$\frac{\mathcal{L}_q(\gamma_0)}{\mathcal{S}_{k^d,q}(\gamma_0)} = \frac{\mathcal{L}_q(\gamma_{0,d})}{\mathcal{S}_{k,q}(\gamma_{0,d})}.$$

Setting $\mu = 0$, we then get the evaluation formula

$$P_\lambda^+(\gamma_{0,d}^{-1}) = \gamma_{0,d}^\lambda \frac{\mathcal{L}_q(\gamma_0)}{\mathcal{S}_{k^d,q}(\gamma_0)} \frac{\mathcal{S}_{k^d,q}(\gamma_\lambda)}{\mathcal{L}_q(\gamma_\lambda)}.$$

Returning to (3.12) and (3.13) it yields the well-known self-duality

$$E_+(\gamma_\lambda; \gamma_{\mu,d}^{-1}; k, q) = E_+(\gamma_{\mu,d}, \gamma_\lambda^{-1}; k^d, q) \qquad \forall \lambda, \mu \in \Lambda^-$$

of the symmetric Macdonald-Koornwinder polynomials. Using $E_+(\gamma_\lambda; t) = E_+(\gamma_{-w_0\lambda}; t^{-1})$ and $\gamma_\lambda^{-1} = w_0\gamma_{-w_0\lambda}$ for $\lambda \in \Lambda^-$, the self-duality can be rewritten as

$$E_+(\gamma_\lambda; \gamma_{\mu,d}; k, q) = E_+(\gamma_{\mu,d}; \gamma_\lambda; k^d, q) \qquad \forall\, \lambda, \mu \in \Lambda^-.$$

4. **The c-function expansion**

The existence of an expansion of the basic hypergeometric function $\mathcal{E}_+$ in terms of basic Harish-Chandra series now follows readily.

PROPOSITION 4.1. *$\{\Phi(\cdot, w\cdot)\}_{w\in W_0}$ is a $\mathbb{F}$-basis of the subspace $\phi(\mathcal{K})$ of $\mathcal{U}$. Hence there exists a unique $\mathfrak{c}(\cdot,\cdot) = \mathfrak{c}(\cdot,\cdot; k, q) \in \mathbb{F}$ such that*

$$\mathcal{E}_+(t,\gamma) = \sum_{w\in W_0} \mathfrak{c}(t, w\gamma)\Phi(t, w\gamma). \tag{4.1}$$

Proof. Since $\phi : \mathcal{K} \to \mathcal{U}$ is $W_0 \times W_0$-equivariant, we have

$$\phi(\nabla(1, w)F) = \Phi(\cdot, w^{-1}\cdot), \qquad w \in W_0.$$

The first statement then follows from Theorem 3.6(ii). By Corollary 3.4, we have

$$\mathcal{E}_+(t,\gamma) = \sum_{w\in W_0} \mathfrak{c}_w(t,\gamma)\Phi(t, w\gamma)$$

in $\phi(\mathcal{K}) \subset \mathcal{U}$ for unique $\mathfrak{c}_w \in \mathbb{F}$ ($w \in W_0$). Since $\mathcal{E}_+$ is $W_0 \times W_0$-invariant, $\mathfrak{c}_w(t,\gamma) = \mathfrak{c}_1(t, w\gamma)$ for $w \in W_0$. □

We are now going to derive an explicit expression of the expansion coefficient $\mathfrak{c} \in \mathbb{F}$ in terms of theta functions. As a first step we will single out the t-dependence. The following preliminary lemma is closely related to [7, Thm. 4.1(i)] (reduced case).

Set

$$\widetilde{\rho} := \varpi_1 + \cdots + \varpi_n \in \Lambda^+.$$

LEMMA 4.2. *Fix generic $\gamma \in T$ with $|\gamma^{-\alpha_i}| \leq 1$ for $1 \leq i \leq n$. For $\lambda \in \Lambda^-$, define $h_\lambda \in \mathcal{M}(T)$ by*

$$h_\lambda(t) := \gamma_0^{-\lambda}\gamma^{-w_0\lambda}\left(\prod_{\alpha\in R_{0,s}^+}\frac{\left(-q_\theta k_0 k_{2a_0}^{-1}t^{-\alpha};q_\theta^2\right)_{-(\lambda,\alpha^\vee)/2}}{\left(-q_\theta k_{2a_0}k_0^{-1}t^{-\alpha};q_\theta^2\right)_{-(\lambda,\alpha^\vee)/2}}\right)\frac{\mathcal{E}_+(q^\lambda t,\gamma)}{G_{k^\tau,q}(t)G_{k^{d\tau},q}(\gamma)}.$$

(*In the reduced case we have $k_0 = k_{2a_0}$, hence in this case the product over $R_{0,s}^+$ is one; in the nonreduced case $(\lambda,\alpha^\vee)$ is even for all $\alpha \in R_{0,s}^+$.*) *Then h_λ is holomorphic on T and*

$$\lim_{r\to\infty} h_{-r\tilde{\rho}}(t)$$

converges to a holomorphic function $h_{-\infty}(t)$ in $t \in T$.

Proof. Observe that

$$\left(\prod_{\alpha\in R_{0,s}^+}\frac{\left(-q_\theta k_0 k_{2a_0}^{-1}t^{-\alpha};q_\theta^2\right)_{-(\lambda,\alpha^\vee)/2}}{\left(-q_\theta k_{2a_0}k_0^{-1}t^{-\alpha};q_\theta^2\right)_{-(\lambda,\alpha^\vee)/2}}\right)\frac{G_{k^\tau,q}(q^\lambda t)}{G_{k^\tau,q}(t)}$$

is a regular function in $t \in T$ and $G_{k^\tau,q}(q^\lambda t)^{-1}G_{k^{d\tau},q}(\gamma)^{-1}\mathcal{E}_+(q^\lambda t,\gamma)$ is holomorphic in $(t,\gamma) \in T\times T$. Hence $h_\lambda(t)$ is holomorphic. It remains to show that the $h_\lambda(t)$ $(\lambda \in \Lambda^-)$ are uniformly bounded for t in compacta of T. Without loss of generality it suffices to prove uniform boundedness for t in compacta of B_ε^{-1} for sufficiently small $\varepsilon > 0$.

Set

$$\mathcal{F}(t,\gamma) := \frac{\mathcal{E}(t,\gamma)}{G_{k^\tau,q}(t)G_{k^{d\tau},q}(\gamma)},$$

which is the holomorphic part of the nonsymmetric basic hypergeometric function $\mathcal{E}$. For $w \in W_0$, let v_w^* be the $\mathbb{K}$-linear functional on $\mathbb{K}\otimes M$ mapping $v_{w'}$ to $\delta_{w,w'}$. Recall from Theorem 3.3 that

$$\mathcal{E}_+ = \pi^t(C_+)\mathcal{E} = \phi(\psi\mathcal{E})$$

and $\psi\mathcal{E} \in \mathcal{K}$. Hence

$$(\psi\mathcal{E})(q^\lambda t,\gamma) = C_{(\tau(\lambda),1)}(q^\lambda t,\gamma)(\psi\mathcal{E})(t,\gamma),$$

so that

$$\begin{aligned}h_\lambda(t) = {}& \frac{k_{w_0}^{-1}}{\sum_{w\in W_0}k_w^{-2}}\prod_{\alpha\in R_{0,s}^+}\frac{\left(-q_\theta k_0 k_{2a_0}^{-1}t^{-\alpha};q_\theta^2\right)_{-(\lambda,\alpha^\vee)/2}}{\left(-q_\theta k_{2a_0}k_0^{-1}t^{-\alpha};q_\theta^2\right)_{-(\lambda,\alpha^\vee)/2}}\\ &\times\sum_{w,w'\in W_0}k_w^{-1}\big(\pi^t(T_{w'w_0})\mathcal{F}\big)(t,\gamma)v_w^*\big(\gamma_0^{-\lambda}\gamma^{-w_0\lambda}C_{(\tau(\lambda),1)}(q^\lambda t,\gamma)v_{w'}\big).\end{aligned}$$

It thus suffices to give bounds for $v_w^*(D_\lambda(t)v_{w'})$ $(\lambda \in \Lambda^-)$, uniform for t in compacta of B_ε^{-1}, where

$$D_\lambda(t) := \gamma_0^{-\lambda}\gamma^{-w_0\lambda}C_{(\tau(\lambda),1)}(q^\lambda t,\gamma).$$

Recall from Theorem 3.5 the asymptotically free solution $F(\cdot,\cdot) = \mathcal{W}(\cdot,\cdot)H(\cdot,\cdot)$ of the bispectral quantum KZ equations. Then

$$\gamma^{w_0\lambda - ww_0\lambda} D_\lambda(t) H_w(t) = H_w(q^\lambda t), \qquad w \in W_0,$$

with

$$H_w(t) := \big(\nabla(e,w)H\big)(t,\gamma) = C_{(1,w)}(t,\gamma)H(t,w^{-1}\gamma).$$

(Note that the $C_{(1,w)}(t,\gamma)$ ($w \in W_0$) do not depend on t.) Furthermore, writing

$$H_{w'}(t) = \sum_{w\in W_0} a_w^{w'}(t) v_w,$$

the matrix $A(t) := \big(a_w^{w'}(t)\big)_{w,w'\in W_0}$ is invertible (cf. proof of [31, Lemma 5.12]) and both $A(t)$ and $A(t)^{-1}$ are uniformly bounded for $t \in B_\varepsilon^{-1}$. Writing $N(t,\lambda)$ for the matrix $\big(v_w^*(D_\lambda(t)v_{w'})\big)_{w,w'\in W_0}$, we conclude that

$$N(t,\lambda) = A(q^\lambda t) M(\lambda) A(t)^{-1},$$

where $M(\lambda)$ is the diagonal matrix $\big(\delta_{w,w'}\gamma^{-w_0\lambda + ww_0\lambda}\big)_{w,w'\in W_0}$. The matrix coefficients of $M(\lambda)$ are bounded as function of $\lambda \in \Lambda^-$ since $|\gamma^{-\alpha_i}| \le 1$ for all i. This implies the required boundedness conditions for the matrix coefficients of $N(t,\lambda)$. □

Set

(4.2)

$$\mathfrak{c}^\vartheta(t,\gamma;k,q) := \frac{\vartheta(\gamma_0^{-1}(k_0^{-1}k_{2a_0})^{\rho_s^\vee} t(w_0\gamma)^{-1})\vartheta(\gamma_0 t)\vartheta(\gamma_{0,d}^{-1}\gamma)\vartheta((k_0^{-1}k_{2a_0})^{\rho_s^\vee}\gamma_{0,d})}{\vartheta((k_{2a_0}k_{2\theta}^{-1})^{\rho_s^\vee}\gamma)\vartheta(t(w_0\gamma)^{-1})\vartheta((k_0^{-1}k_{2a_0})^{\rho_s^\vee} t)}.$$

Observe that $\mathfrak{c}^\vartheta$ satisfies the functional equations

$$\mathfrak{c}^\vartheta(q^\lambda t,\gamma) = \mathfrak{c}^\vartheta(t,\gamma),$$
$$\mathfrak{c}^\vartheta(t,q^\lambda\gamma) = \gamma_{0,d}^{2\lambda}\mathfrak{c}^\vartheta(t,\gamma)$$

for $\lambda \in \Lambda$. Since $\gamma_0^{-1}\gamma_{0,d} = (k_0 k_{2\theta}^{-1})^{\rho_s^\vee}$, in addition we have

$$\mathfrak{c}^\vartheta(\gamma_{0,d},\gamma) = \frac{1}{\mathcal{W}(\gamma_{0,d},\gamma)}. \tag{4.3}$$

Corollary 4.3. *The expansion coefficient $\mathfrak{c} \in \mathbb{F}$ in (4.1) is of the form*

$$\mathfrak{c}(t,\gamma) = \mathfrak{c}^\vartheta(t,\gamma)\mathfrak{c}^\theta(\gamma) \tag{4.4}$$

for a unique $\mathfrak{c}^\theta(\cdot) = \mathfrak{c}^\theta(\cdot;k,q) \in \mathcal{M}(T)$ satisfying the functional equations

$$\mathfrak{c}^\theta(q^\lambda\gamma) = \gamma_{0,d}^{-2\lambda}\mathfrak{c}^\theta(\gamma) \qquad \forall\,\lambda\in\Lambda. \tag{4.5}$$

Proof. In view of the functional equations of $\mathfrak{c}^{\vartheta}(t,\gamma)$ in γ it suffices to prove the factorization for generic $t,\gamma\in T$ satisfying $|\gamma^{-\alpha_i}|<1$ for all $1\leq i\leq n$. Since

$$\frac{1}{G_{k^{\tau},q}(t)}\prod_{\alpha\in R_{0,s}^+}\frac{\bigl(-q_\theta k_0k_{2a_0}^{-1}t^{-\alpha};q_\theta^2\bigr)_\infty}{\bigl(-q_\theta k_{2a_0}k_0^{-1}t^{-\alpha};q_\theta^2\bigr)_\infty}=\vartheta\bigl((k_0^{-1}k_{2a_0})^{\rho_s^\vee}t\bigr) \tag{4.6}$$

(which is trivial in the reduced case and follows by a direct computation in the nonreduced case), we have

$$\begin{aligned} h_{-\infty}(t)&=\frac{\vartheta((k_0^{-1}k_{2a_0})^{\rho_s^\vee}t)}{G_{k^{d\tau},q}(\gamma)}\\ &\quad\times\sum_{w\in W_0}\mathfrak{c}(t,w\gamma)\mathcal{W}(t,w\gamma)\lim_{r\to\infty}\gamma^{r(w_0\widetilde{\rho}-w^{-1}w_0\widetilde{\rho})}(\phi H)(q^{-r\widetilde{\rho}}t,w\gamma)\\ &=\frac{\vartheta((k_0^{-1}k_{2a_0})^{\rho_s^\vee}t)\mathfrak{c}(t,\gamma)\mathcal{W}(t,\gamma)\mathcal{L}_q(\gamma)}{G_{k^{d\tau},q}(\gamma)\mathcal{S}_{k^d,q}(\gamma)\sum_{w\in W_0}k_w^2} \end{aligned} \tag{4.7}$$

by Theorem 3.6, Proposition 4.1, (3.3), (3.5) and the assumption that $|\gamma^{-\alpha_i}|<1$ for all $1\leq i\leq n$. It follows from this expression that the holomorphic function $h_{-\infty}$ satisfies

$$h_{-\infty}(q^\lambda t)=q^{-\frac{|\lambda|^2}{2}}\bigl(\gamma_0^{-1}(k_0^{-1}k_{2a_0})^{\rho_s^\vee}t(w_0\gamma)^{-1}\bigr)^{-\lambda}h_{-\infty}(t)\quad\forall\,\lambda\in\Lambda.$$

Consequently,

$$h_{-\infty}(t)=\mathrm{cst}\,\vartheta\bigl(\gamma_0^{-1}(k_0^{-1}k_{2a_0})^{\rho_s^\vee}t(w_0\gamma)^{-1}\bigr)$$

for some $\mathrm{cst}\in\mathbb{C}$ independent of $t\in T$. Combined with the second line of (4.7) one obtains the desired result. □

The factor $\mathfrak{c}^{\vartheta}(t,\gamma)$ is highly dependent on our specific choice of (self-dual, meromorphic) prefactor $\mathcal{W}$ in the self-dual basic Harish-Chandra series. We will see later that this factor simplifies when considering the expansion of the basic hypergeometric function in terms of the monic basic Harish-Chandra series. In particular it will no longer depend on the first torus variable $t\in T$.

The next step is to compute $\mathfrak{c}^{\theta}(\gamma)=\mathfrak{c}^{\theta}(\gamma;k,q)$ explicitly. We will obtain an expression in terms of the Jacobi theta function $\theta(\cdot;q)$. Recall that

$$\frac{\mathcal{S}_{k,q}(t)}{\mathcal{L}_q(t)}=\prod_{\substack{\alpha\in R_0^+,\\ r\in\mathbb{Z}_{>0}}}c_{\alpha+r\frac{|\alpha|^2}{2}c}(t;k^{-1},q). \tag{4.8}$$

We define a closely related meromorphic function $c(\cdot)=c_{k,q}(\cdot)\in\mathcal{M}(T)$ by

$$c(t):=\prod_{\substack{\alpha\in R_0^+,\\ r\in\mathbb{Z}_{\geq 0}}}c_{-\alpha+r\frac{|\alpha|^2}{2}c}(t;k,q). \tag{4.9}$$

An explicit computation yields expressions of both (4.8) and (4.9) in terms of of q-shifted factorials. For $c(t)$, it reads

$$c(t) = \prod_{\alpha\in R_0^+} \frac{\left(k_\alpha^2 t^{-\alpha}; q_\alpha\right)_\infty}{\left(t^{-\alpha}; q_\alpha\right)_\infty} \tag{4.10}$$

in the reduced case and

$$c(t) = \prod_{\alpha\in R_{0,l}^+} \frac{\left(k_\vartheta^2 t^{-\alpha}; q_\vartheta\right)_\infty}{\left(t^{-\alpha}; q_\vartheta\right)_\infty} \prod_{\beta\in R_{0,s}^+} \frac{\left(at^{-\beta}, bt^{-\beta}, ct^{-\beta}, dt^{-\beta}; q_\theta^2\right)_\infty}{\left(t^{-2\beta}; q_\theta^2\right)_\infty} \tag{4.11}$$

in the nonreduced case, where $R_{0,l}^+ \subset R_0^+$ is the subset of positive long roots and $\{a, b, c, d\}$ are given by (3.4). The product formula of $c_{k^d,q}(\gamma)$ in the reduced case is the q-analog of the Gindikin-Karpelevic [12] product formula of the Harish-Chandra c-function as well as of its extension to the Heckman-Opdam theory (see [15, Part I, Def. 3.4.2]).

Taking the product

$$\frac{\mathcal{S}_{k,q}(t)c_{k,q}(t)}{\mathcal{L}_q(t)}$$

of (4.8) and (4.9), the q-shifted factorials can be combined pairwise to yield the following explicit expression in terms of Jacobi's theta function $\theta(\cdot; q)$.

LEMMA 4.4. (i) *In the reduced case*,

$$\frac{\mathcal{S}_{k,q}(t)c_{k,q}(t)}{\mathcal{L}_q(t)} = \prod_{\alpha\in R_0^+} \frac{\theta(k_\alpha^2 t^{-\alpha}; q_\alpha)}{\theta(t^{-\alpha}; q_\alpha)}.$$

(ii) *In the nonreduced case*,

$$\begin{aligned}\frac{\mathcal{S}_{k,q}(t)c_{k,q}(t)}{\mathcal{L}_q(t)} &= \prod_{\alpha\in R_{0,l}^+} \frac{\theta(k_\vartheta^2 t^{-\alpha}; q_\vartheta)}{\theta(t^{-\alpha}; q_\vartheta)} \\ &\quad \times \prod_{\beta\in R_{0,s}^+} \frac{\theta(at^{-\beta}; q_\theta^2)\theta(bt^{-\beta}; q_\theta^2)\theta(ct^{-\beta}; q_\theta^2)\theta(dt^{-\beta}; q_\theta^2)}{\left(q_\theta^2; q_\theta^2\right)_\infty^3 \theta(t^{-2\beta}; q_\theta^2)}.\end{aligned}$$

In Section 3.2 we have seen that

$$\frac{\mathcal{L}_q(\gamma)}{\mathcal{S}_{k^d,q}(\gamma)\sum_{w\in W_0} k_w^2}$$

is the leading coefficient of the power series expansion of $(\phi H)(t, \gamma)$ in the variables $t^{-\alpha_i}$ $(1 \le i \le n)$. On the other hand, it is closely related to the evaluation formula for the symmetric Macdonald-Koornwinder polynomials, see Remark 3.11. In the next proposition we show that the meromorphic function $c(t)$ governs the asymptotics of the symmetric Macdonald-Koornwinder

polynomial. In the reduced case it is due to Cherednik [7, Lemma 4.3]. (For the rank one case see, e.g., [18].)

PROPOSITION 4.5. *For $\varepsilon > 0$ sufficiently small,*

$$\lim_{r\to\infty} \gamma_{0,d}^{r\widetilde{\rho}} t^{rw_0\widetilde{\rho}} E_+(\gamma_{-r\widetilde{\rho}}; t) = \frac{c(t)}{c(\gamma_{0,d})}, \tag{4.12}$$

normally converging for t in compacta of B_ε^{-1}.

Proof. The proof in the reduced case (see [7]) consists of analyzing the gauged Macdonald q-difference equations $t^{rw_0\widetilde{\rho}} \circ D_p \circ t^{-rw_0\widetilde{\rho}}$ ($p \in \mathbb{C}[T]^{W_0}$) in the limit $r \to \infty$ and observing that the left and right-hand side of (4.12) are the (up to normalization) unique solution of the resulting residual q-difference equations that have a series expansion in $t^{-\mu}$ ($\mu \in Q_+$), normally converging for t in compacta of B_ε^{-1}. This proof can be straightforwardly extended to the nonreduced case. □

THEOREM 4.6. *We have*

$$\mathcal{E}_+(t,\gamma;k,q) = \sum_{w\in W_0} \mathfrak{c}(t,w\gamma;k,q)\Phi(t,w\gamma;k,q),$$

with $\mathfrak{c}(\cdot,\cdot) = \mathfrak{c}(\cdot,\cdot;k,q) \in \mathbb{F}$ given by

$$\mathfrak{c}(t,\gamma;k,q) = \mathfrak{c}^\vartheta(t,\gamma;k,q)\mathfrak{c}^\theta(\gamma;k,q), \tag{4.13}$$

where $\mathfrak{c}^\vartheta(\cdot,\cdot) = \mathfrak{c}^\vartheta(\cdot,\cdot;k,q) \in \mathbb{K}$ is given by (4.2) and $\mathfrak{c}^\theta(\cdot) = \mathfrak{c}^\theta(\cdot;k,q) \in \mathcal{M}(T)$ is given by

$$\mathfrak{c}^\theta(\gamma;k,q) = \frac{\mathcal{S}_{k^d,q}(\gamma)c_{k^d,q}(\gamma)}{\mathcal{L}_q(\gamma)} \frac{\sum_{w\in W_0} k_w^2}{c_{k^d,q}(\gamma_0)}. \tag{4.14}$$

In view of Lemma 4.4, formula (4.14) provides an explicit expression of $\mathfrak{c}^\theta(\gamma)$ as product of Jacobi theta functions.

Proof. Using Lemma 4.4 it is easy to check that the right-hand side of (4.14) satisfies the functional equations (4.5). Hence it suffices to prove the explicit expression (4.14) of $\mathfrak{c}^\theta(\gamma)$ for generic $\gamma \in T$ such that $|\gamma^{-\alpha_i}|$ is sufficiently small for all $1 \leq i \leq n$. We fix such γ in the remainder of the proof. Recall the associated holomorphic function $h_\lambda(t)$ in $t \in T$ from Lemma 4.2. By Theorem 2.20, using (4.6) for the first equality and Proposition 4.5 for the second equality,

$$\begin{aligned} h_{-\infty}(\gamma_{0,d}) &= \frac{\vartheta\big((k_0^{-1}k_{2a_0})^{\rho_s^\vee}\gamma_{0,d}\big)}{G_{k^{d\tau},q}(\gamma)} \lim_{r\to\infty} \gamma_0^{r\widetilde{\rho}}\gamma^{rw_0\widetilde{\rho}} E_+(\gamma_{-r\widetilde{\rho},d};\gamma;k^d,q) \\ &= \frac{\vartheta\big((k_0^{-1}k_{2a_0})^{\rho_s^\vee}\gamma_{0,d}\big)c_{k^d,q}(\gamma)}{c_{k^d,q}(\gamma_0)G_{k^{d\tau},q}(\gamma)}. \end{aligned}$$

On the other hand, by (4.7), Corollary 4.3 and (4.3),

$$h_{-\infty}(\gamma_{0,d}) = \frac{\vartheta((k_0^{-1}k_{2a_0})^{\rho_s^\vee}\gamma_{0,d})\mathcal{L}_q(\gamma)}{G_{k^{d\tau},q}(\gamma)\mathcal{S}_{k^d,q}(\gamma)\sum_{w\in W_0}k_w^2}\mathfrak{c}^\theta(\gamma).$$

Combining these two formulas yields the desired expression for $\mathfrak{c}^\theta(\gamma)$. □

A direct computation using (3.11) now gives the following c-function expansion of the basic hypergeometric function $\mathcal{E}_+$ in terms of the monic basic Harish-Chandra series $\widehat{\Phi}_\eta$.

COROLLARY 4.7. *For generic $\eta \in T$, we have*

$$\mathcal{E}_+(t,\gamma;k,q) = \widehat{c}_\eta(\gamma_0;k,q)^{-1}\sum_{w\in W_0}\widehat{c}_\eta(w\gamma;k,q)\widehat{\Phi}_\eta(t,w\gamma;k,q), \tag{4.15}$$

with $\widehat{c}_\eta(\cdot) = \widehat{c}_\eta(\cdot;k,q) \in \mathcal{M}(T)$ explicitly given by

$$\widehat{c}_\eta(\gamma;k,q) = \frac{\vartheta\big((w_0\eta)^{-1}(k_{2a_0}k_{2\theta}^{-1})^{\rho_s^\vee}\gamma\big)}{\vartheta\big((k_{2a_0}k_{2\theta}^{-1})^{\rho_s^\vee}\gamma\big)}c_{k^d,q}(\gamma).$$

In the rank one case the c-function expansion of $\mathcal{E}_+$ was established by direct computations in [8] (GL_2 case) and in [19], [41] (nonreduced rank one case). We return to the rank one case and establish the connections to basic hypergeometric series in Section 5.

In the reduced case, by (4.10) the coefficient $\widehat{c}_\eta$ explicitly reads

$$\widehat{c}_\eta(\gamma) = \frac{\vartheta((w_0\eta)^{-1}\gamma)}{\vartheta(\gamma)}\prod_{\alpha\in R_0^+}\frac{\big(k_\alpha^2\gamma^{-\alpha};q_\alpha\big)_\infty}{\big(\gamma^{-\alpha};q_\alpha\big)_\infty}.$$

By (4.11) an explicit expression of the monic c-function $\widehat{c}_\eta$ can also be given in the nonreduced case; see (5.3) for the resulting expression.

Note that the formula for $\widehat{c}_\eta(\gamma)$ simplifies, for $\eta = 1$, to

$$\widehat{c}_1(\gamma;k,q) = c_{k^d,q}(\gamma).$$

As remarked in the introduction this shows that $\mathcal{E}_+$ formally is a q-analog of the Heckman-Opdam [14], [15], [33] hypergeometric function.

The η-dependence is expected to be of importance in the applications to harmonic analysis on noncompact quantum groups. In [20], [19] a self-dual spherical Fourier transform on the quantum $\mathrm{SU}(1,1)$ quantum group was defined and studied, whose Fourier kernel is given by the nonreduced rank one basic hypergeometric function $\mathcal{E}_+$, which is the Askey-Wilson function from [19] (see Section 5.2). The Fourier transform and the Plancherel measure were defined in terms of the Plancherel density function

$$\mu_\eta(\gamma) = \frac{1}{\widehat{c}_\eta(\gamma)\widehat{c}_\eta(\gamma^{-1})}. \tag{4.16}$$

The extra theta-factors compared to the familiar Macdonald density

$$\mu_1(\gamma)=\frac{1}{c_{k^d,q}(\gamma)c_{k^d,q}(\gamma^{-1})}$$

lead to an infinite set of discrete mass points in the associated (inverse of the) Fourier transform. In its interpretation as spherical Fourier transform, these mass points account for the contributions of the strange series representations to the Plancherel measure. (The strange series is a series of irreducible unitary representations of the quantized universal enveloping algebra that vanishes in the limit $q\to 1$; see [29].) Crucial ingredients for obtaining the Plancherel and inversion formulas are the explicit c-function expansion and the self-duality of the Askey-Wilson function $\mathcal{E}_+$. The generalization of these results to arbitrary root systems is not known.

5. **Special cases and applications**

5.1. *Asymptotics of symmetric Macdonald-Koornwinder polynomials.* As a consequence of the c-function expansion we can establish pointwise asymptotics of the symmetric Macdonald-Koornwinder polynomials when the degree tends to infinity. The L^2-asymptotics was established in [36] for GL_m, [9] for the reduced case and [10] for the nonreduced case. (For the rank one cases see, e.g., [18], [11, §7.4 & §7.5] and references therein.)

For $\lambda\in\Lambda^-$, set

$$m(\lambda):=\max((\lambda,\alpha_i)\mid 1\leq i\leq n)\in\mathbb{R}_{\leq 0}.$$

Corollary 5.1. *Fix $t\in T$ such that $\mathcal{S}_{k,q}(wt)\neq 0$ for all $w\in W_0$. Then*

$$E_+(\gamma_\lambda;t;k,q)=\sum_{w\in W_0}\frac{c_{k,q}(wt)}{c_{k,q}(\gamma_{0,d})}\gamma_{0,d}^{\lambda}t^{w^{-1}w_0\lambda}+\mathcal{O}(q^{-m(\lambda)}) \tag{5.1}$$

as $m(\lambda)\to-\infty$.

Proof. By the c-function expansion in self-dual form, Theorem 2.20(ii)–(iv), (3.11) and the expression $\widehat{\Phi}_\eta=\widehat{\mathcal{W}}_\eta\phi(\widehat{H})$, we have for $\lambda\in\Lambda^-$,

$$\begin{aligned}E_+(\gamma_\lambda;t;k,q)&=\mathcal{E}_+(\gamma_\lambda,t;k^d,q)\\&=\sum_{w\in W_0}\mathfrak{c}(\gamma_\lambda,wt;k^d,q)\mathcal{W}(\gamma_\lambda,wt;k^d,q)\frac{\mathcal{L}_q(wt)}{\mathcal{S}_{k,q}(wt)\sum_{v\in W_0}k_v^2}(\phi\widehat{H})(\gamma_\lambda,wt;k^d,q).\end{aligned}$$

The corollary now follows easily from the asymptotic series expansion in $\gamma_\lambda^{-\alpha}$ ($\alpha\in Q_+$) of $(\phi\widehat{H})(\gamma_\lambda,wt,k^d;q)$, together with (3.3), (4.3) and the explicit expression (4.13) of $\mathfrak{c}\in\mathbb{F}$. □

5.2. *The nonreduced case.* We realize the root system $R_0 \subset V_0 = V$ of type B_n as $R_0 = \{\pm\varepsilon_i\}_{i=1}^n \cup \{\pm(\varepsilon_i \pm \varepsilon_j)\}_{1\leq i<j\leq n}$, with $\{\varepsilon_i\}_{i=1}^n$ a fixed orthonormal basis of V. We take as ordered basis

$$\Delta_0 = (\varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{n-1} - \varepsilon_n, \varepsilon_n)$$

so that $R_{0,s}^+ = \{\varepsilon_i\}_{i=1}^n$, $R_{0,l}^+ = \{\varepsilon_i \pm \varepsilon_j\}_{1\leq i<j\leq n}$ and $\theta = \varepsilon_1$, $\vartheta = \varepsilon_1 + \varepsilon_2$. We include $n = 1$ as $R_0 = \{\pm\varepsilon_1\}$, the root system of type A_1. (It amounts to omitting the factors involving the long roots $\{\pm(\varepsilon_i \pm \varepsilon_j)\}_{i<j}$ in the formulas below.) We have $\Lambda = Q = \bigoplus_{i=1}^n \mathbb{Z}\varepsilon_i$. We identify $T \simeq \big(\mathbb{C}^*\big)^n$, taking $t_i := t^{\varepsilon_i}$ as the coordinates. Note that $q_\theta = q^{\frac{1}{2}}$ and $q_\vartheta = q$.

The q-difference operator D_p with respect to

$$p(t) := \sum_{i=1}^n (t_i + t_i^{-1})$$

was identified with Koornwinder's [22] multivariable extension of the Askey-Wilson second order q-difference operator by Noumi [32]. It can most conveniently be expressed in terms of the Askey-Wilson parameters

$$\{a, b, c, d\} = \Big\{k_\theta k_{2\theta}, -k_\theta k_{2\theta}^{-1}, q^{\frac{1}{2}} k_0 k_{2a_0}, -q^{\frac{1}{2}} k_0 k_{2a_0}^{-1}\Big\}$$

and the dual Askey-Wilson parameters

$$\Big\{\widetilde{a}, \widetilde{b}, \widetilde{c}, \widetilde{d}\Big\} = \{k_\theta k_0, -k_\theta k_0^{-1}, q^{\frac{1}{2}} k_{2\theta} k_{2a_0}, -q^{\frac{1}{2}} k_{2\theta} k_{2a_0}^{-1}\Big\}$$

(which are the Askey-Wilson parameters associated to the multiplicity function k^d dual to k) as

$$D_p = \widetilde{a}^{-1} k_\vartheta^{2(1-n)} \left(\mathcal{D} + \sum_{i=1}^n \Big(\widetilde{a}^2 k_\vartheta^{2(2n-i-1)} + k_\vartheta^{2(i-1)}\Big)\right)$$

with

$$\mathcal{D} = \sum_{i=1}^n \Big(A_i(t)(\tau(-\varepsilon_i)_q - 1) + A_i(t^{-1})(\tau(\varepsilon_i)_q - 1)\Big),$$

$$A_i(t) = \frac{(1 - at_i)(1 - bt_i)(1 - ct_i)(1 - dt_i)}{(1 - t_i^2)(1 - qt_i^2)} \prod_{j\neq i} \frac{(1 - k_\vartheta^2 t_i t_j)(1 - k_\vartheta^2 t_i t_j^{-1})}{(1 - t_i t_j)(1 - t_i t_j^{-1})}.$$

If follows that P_λ^+ ($\lambda \in \Lambda^-$) are the monic symmetric Koornwinder [22] polynomials and $E(\gamma_\lambda; \cdot)$ ($\lambda \in \Lambda^-$) are Sahi's [37] normalized symmetric Koornwinder polynomials.

We now make the (monic version of) the c-function expansion of the associated basic hypergeometric function $\mathcal{E}_+$ more explicit (see Corollary 4.7). First note that $\rho_s^\vee = \sum_{i=1}^n \varepsilon_i$ and that

$$\gamma_0^{-1} = \big(\widetilde{a} k_\vartheta^{2(n-1)}, \cdots, \widetilde{a} k_\vartheta^2, \widetilde{a}\big),$$

$$\gamma_{0,d}^{-1} = \big(a k_\vartheta^{2(n-1)}, \ldots, a k_\vartheta^2, a\big).$$

In the present nonreduced setup, the higher rank theta function $\vartheta(t)$ (2.8) can be written in terms of Jacobi's theta function,

$$\vartheta(t)=\prod_{i=1}^n\theta\big(-q^{\frac{1}{2}}t_i;q\big). \tag{5.2}$$

This allows us to rewrite the theta function factor of $\widehat{c}_\eta(\gamma)$ (see Corollary 4.7) as

$$\frac{\vartheta((w_0\eta)^{-1}(k_{2a_0}k_{2\theta}^{-1})^{\rho_s^\vee}\gamma)}{\vartheta((k_{2a_0}k_{2\theta}^{-1})^{\rho_s^\vee}\gamma)}=\prod_{i=1}^n\frac{\theta(q\eta_i\gamma_i/\widetilde{d})}{\theta(q\gamma_i/\widetilde{d})}.$$

The normalized c-function $\widehat{c}_\eta(\gamma)$ thus becomes

$$\begin{aligned}\widehat{c}_\eta(\gamma)=\prod_{i=1}^n&\frac{\big(\widetilde{a}\gamma_i^{-1},\widetilde{b}\gamma_i^{-1},\widetilde{c}\gamma_i^{-1},\widetilde{d}\gamma_i^{-1}/\eta_i,q\eta_i\gamma_i/\widetilde{d};q\big)_\infty}{\big(\gamma_i^{-2},q\gamma_i/\widetilde{d};q\big)_\infty}\\
&\times\prod_{1\leq i<j\leq n}\frac{\big(k_\vartheta^2\gamma_i^{-1}\gamma_j,k_\vartheta^2\gamma_i^{-1}\gamma_j^{-1};q\big)_\infty}{\big(\gamma_i^{-1}\gamma_j,\gamma_i^{-1}\gamma_j^{-1};q\big)_\infty},\end{aligned} \tag{5.3}$$

where we use the shorthand notation

$$\big(\alpha_1,\ldots,\alpha_j;q\big)_r:=\prod_{i=1}^j\big(\alpha_i;q\big)_r.$$

In the remainder of this subsection we set $n=1$. Then $\mathcal{D}$ is the Askey-Wilson [1] second-order q-difference operator. In this case the c-function expansion (Corollary 4.7) was proved in [19], [41] using the theory of one-variable basic hypergeometric series. Important ingredients are the explicit basic hypergeometric series expressions for $\mathcal{E}_+$ and $\widehat{\Phi}_\eta$, which we now recall.

The ${}_{r+1}\phi_r$ basic hypergeometric series [11] is the convergent series

$${}_{r+1}\phi_r\left(\begin{matrix}a_1,a_2,\ldots,a_{r+1}\\ b_1,b_2,\ldots,b_r\end{matrix};q,z\right):=\sum_{j=0}^\infty\frac{\big(a_1,a_2,\ldots,a_{r+1};q\big)_j}{\big(q,b_1,\ldots,b_r;q\big)_j}z^j,\qquad |z|<1. \tag{5.4}$$

The very-well-poised ${}_8\phi_7$ basic hypergeometric series is given by

$${}_8W_7(\alpha_0;\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5;q,z):=\sum_{r=0}^\infty\frac{1-\alpha_0q^{2r}}{1-\alpha_0}z^r\prod_{j=0}^5\frac{\big(\alpha_j;q\big)_r}{\big(q\alpha_0/\alpha_j;q\big)_r},\quad |z|<1.$$

Very-well-poised ${}_8\phi_7$ basic hypergeometric series solutions of the eigenvalue equation

$$\mathcal{D}f=(\widetilde{a}(\gamma+\gamma^{-1})-\widetilde{a}^2-1)f \tag{5.5}$$

were obtained in [17]. On the other hand, we already observed that $E_+(\gamma_{-r};\cdot)$ ($r\in\mathbb{Z}_{\geq 0}$) is the inversion invariant, Laurent polynomial solution of (5.5) with spectral point $\gamma=\gamma_{-r}$ and that both $\mathcal{E}_+(\cdot,\gamma)$ and $\widehat{\Phi}_\eta(\cdot,\gamma)$ satisfy (5.5). These solutions are related as follows.

PROPOSITION 5.2. *For the nonreduced case with $n=1$, we have*

$$E_+(\gamma_{-r};t) = {}_4\phi_3\left(\begin{matrix}q^{-r},q^{r-1}abcd,at,a/t\\ ab,ac,ad\end{matrix};q,q\right),\qquad r\in\mathbb{Z}_{\geq 0}, \tag{5.6}$$

$$\mathcal{E}_+(t,\gamma) = \frac{\left(\frac{qat\gamma}{\widetilde{d}},\frac{qa\gamma}{\widetilde{dt}},\frac{qa}{d},\frac{q}{ad};q\right)_\infty}{\left(\widetilde{a}\widetilde{b}\widetilde{c}\gamma,\frac{q\gamma}{\widetilde{d}},\frac{qt}{d},\frac{q}{dt};q\right)_\infty}\,{}_8W_7\Big(\frac{\widetilde{a}\widetilde{b}\widetilde{c}\gamma}{q};at,\frac{a}{t},\widetilde{a}\gamma,\widetilde{b}\gamma,\widetilde{c}\gamma;q,\frac{q}{\widetilde{d}\gamma}\Big) \tag{5.7}$$

for $|q/\widetilde{d}\gamma|<1$, and

$$\begin{aligned}\widehat{\Phi}_\eta(t,\gamma) = {}&\frac{\theta\big(\frac{ad}{\eta};q\big)\theta\big(\frac{\widetilde{qat}\gamma}{d};q\big)}{\theta\big(\frac{\widetilde{d}}{\eta\gamma};q\big)\theta\big(\frac{qt}{d};q\big)}\,\frac{\left(\frac{qa\gamma}{\widetilde{at}},\frac{qb\gamma}{\widetilde{at}},\frac{qc\gamma}{\widetilde{at}},\frac{\widetilde{qa}\gamma}{dt},\frac{d}{t};q\right)_\infty}{\left(\frac{q}{at},\frac{q}{bt},\frac{q}{ct},\frac{q}{dt},\frac{q^2\gamma^2}{dt};q\right)_\infty}\\ &\times {}_8W_7\Big(\frac{q\gamma^2}{dt};\frac{q\gamma}{\widetilde{a}},\frac{q\gamma}{\widetilde{d}},\widetilde{b}\gamma,\widetilde{c}\gamma,\frac{q}{dt};q,\frac{d}{t}\Big)\end{aligned} \tag{5.8}$$

for $|d/t|<1$.

Proof. Formula (5.6) follows from [41, Thm. 4.2], (5.7) from [41, Thm. 4.2] and (5.8) from [19, §4]. □

The proposition shows that $E_+(\gamma_{-r};\cdot)$ is the normalized symmetric Askey-Wilson [1] polynomials of degree $r\in\mathbb{Z}_{\geq 0}$ and that $\mathcal{E}_+$ coincides up to a constant multiple with the Askey-Wilson function from [19].

In the present nonreduced, rank one setting, the c-function expansion

$$\mathcal{E}_+(t,\gamma) = \widehat{c}_\eta(\gamma_0)^{-1}\big(\widehat{c}_\eta(\gamma)\widehat{\Phi}_\eta(t,\gamma)+\widehat{c}_\eta(\gamma^{-1})\widehat{\Phi}_\eta(t,\gamma^{-1})\big) \tag{5.9}$$

with

$$\widehat{c}_\eta(\gamma) = \frac{\left(\widetilde{a}\gamma^{-1},\widetilde{b}\gamma^{-1},\widetilde{c}\gamma^{-1},\widetilde{d}\gamma^{-1}/\eta,q\eta\gamma/\widetilde{d};q\right)_\infty}{\left(\gamma^{-2},q\gamma/\widetilde{d};q\right)_\infty}$$

is a special case of Bailey's three-term recurrence relation for very-well-poised ${}_8\phi_7$-series (see [11, (III.37)]). This follows by repeating the proof of [19, Prop. 1]. (Formula (5.9) is more general since it does not involve restriction to a q-interval.) See also [44] for a detailed discussion.

Remark 5.3. The explicit expressions of $\mathcal{E}_+$ and $\widehat{\Phi}_\eta$ as meromorphic functions on $\mathbb{C}^*\times\mathbb{C}^*$ can be obtained from the above explicit expressions by writing the ${}_8W_7$ series as the sum of two balanced ${}_4\phi_3$ series using Bailey's formula [11, (III.36)]; see, for instance, formula [19, (3.3)] for $\mathcal{E}_+$. (The basic ${}_{r+1}\phi_r$ series (5.4) is called balanced if $z=q$ and $qa_1a_2\cdots a_{r+1}=b_1b_2\cdots b_r$.)

5.3. *The* GL_m *case.* In this subsection we use the notations from Example 2.2(ii). We identify $T \simeq \bigl(\mathbb{C}^*\bigr)^m$, taking $t_i := t^{\varepsilon_i}$ as the coordinates $(1 \leq i \leq m)$. Note that the multiplicity function k is constant (its constant value will also be denoted by k). The q-difference operators D_{e_r} associated to the elementary symmetric functions

$$e_r(t) = \sum_{\substack{I \subseteq \{1,\ldots,m\} \\ \#I=r}} \prod_{j \in I} t_j, \qquad 1 \leq r \leq m$$

are Ruijsenaars' [35] quantum Hamiltonians of the relativistic quantum trigonometric Calogero-Moser-Sutherland model,

$$D_{e_r} = \sum_{\substack{I \subseteq \{1,\ldots,m\} \\ \#I=r}} \left(\prod_{i \in I, j \notin J} \frac{k^{-1}t_i - kt_j}{t_i - t_j} \right) \tau\bigl(\sum_{i \in I} \varepsilon_i\bigr)_q, \qquad 1 \leq r \leq m. \tag{5.10}$$

The monic version of the c-function (Corollary 4.7) becomes

$$\widehat{c}_\eta(\gamma) = \frac{\vartheta((w_0\eta)^{-1}\gamma)}{\vartheta(\gamma)} \prod_{1 \leq i < j \leq m} \frac{\bigl(k^2\gamma_j/\gamma_i;q\bigr)_\infty}{\bigl(\gamma_j/\gamma_i;q\bigr)_\infty}. \tag{5.11}$$

Also in the present GL_m case, the higher rank theta functions appearing in (5.11) can be expressed as product of Jacobi theta functions by (5.2).

Our results for GL_2 can be matched with the extensive literature on Heine's q-analog of the hypergeometric differential equation (see, e.g., [11, Chap. 1], [23, Chap. 3, §1.7] and [28, §6.3]). It leads to explicit expressions of $E_+(\gamma_\lambda;\cdot)$, Φ and $\widehat{\Phi}_\eta$ in terms of Heine's q-analog of the hypergeometric function. For completeness, we detail this link here.

Heine's basic hypergeometric q-difference equation is

$$\begin{aligned} z(c - abqz)(\partial_q^2 u)(z) + \left(\frac{1-c}{1-q} + \frac{(1-a)(1-b) - (1-abq)}{1-q} z \right) (\partial_q u)(z) \\ - \frac{(1-a)(1-b)}{(1-q)^2} u(z) = 0, \end{aligned} \tag{5.12}$$

with

$$(\partial_q u)(z) := \frac{u(z) - u(qz)}{(1-q)z}$$

the q-derivative. Note that (5.12) formally reduces to the hypergeometric differential equation

$$z(1-z)u''(z) + (c - (a+b+1)z)u'(z) - abu(z) = 0$$

by replacing in (5.12) the parameters a, b, c by q^a, q^b, q^c and taking the limit $q \to 1$. A distinguished solution of (5.12) is Heine's basic hypergeometric

function

$$\phi_H(z) := {}_2\phi_1\left(\begin{matrix}a,b\\c\end{matrix};q,z\right) \tag{5.13}$$

for $|z|<1$.

Note that for GL_2,

$$\widehat{\mathcal{W}}(t,\gamma)=\frac{\theta\big(-q^{\frac12}kt_1/\gamma_2;q\big)\theta\big(-q^{\frac12}t_2/k\gamma_1;q\big)}{\theta\big(-q^{\frac12}t_1;q\big)\theta\big(-q^{\frac12}t_2;q\big)}.$$

LEMMA 5.4. *Fix $\gamma\in T\simeq\big(\mathbb{C}^*\big)^2$. If $u\in\mathcal{M}(\mathbb{C}^*)$ satisfies Heine's basic hypergeometric q-difference equation* (5.12) *with the parameters a,b,c given by*

$$a=k^2,\quad b=k^2\gamma_1/\gamma_2,\quad c=q\gamma_1/\gamma_2, \tag{5.14}$$

then the meromorphic function $f(t):=\widehat{\mathcal{W}}_\eta(t,\gamma)u(qt_2/k^2t_1)$ satisfies

$$D_{e_r}f=e_r(\gamma^{-1})f,\qquad r=1,2, \tag{5.15}$$

where the D_{e_r} are the GL_2 Macdonald-Ruijsenaars q-difference operators. Conversely, if f is a meromorphic solution of (5.15) *of the form $f(t_1,t_2)=\widehat{\mathcal{W}}_\eta(t,\gamma)u(qt_2/k^2t_1)$ for some $u\in\mathcal{M}(\mathbb{C}^*)$, then u satisfies* (5.12) *with parameters a,b,c given by* (5.14).

Proof. Direct computation. □

COROLLARY 5.5 (GL_2 case). (i) *The normalized symmetric Macdonald polynomial $E_+(\gamma_\lambda;t)$ ($\lambda=\lambda_1\varepsilon_1+\lambda_2\varepsilon_2$ with $\lambda_i\in\mathbb{Z}$ and $\lambda_1\le\lambda_2$) is given by*

$$E_+(\gamma_\lambda;t)=k^{\lambda_1-\lambda_2}\frac{\big(q^{1+\lambda_1-\lambda_2}/k^2;q\big)_{\lambda_2-\lambda_1}}{\big(q^{1+\lambda_1-\lambda_2}/k^4;q\big)_{\lambda_2-\lambda_1}}t_1^{\lambda_2}t_2^{\lambda_1}\,{}_2\phi_1\left(\begin{matrix}k^2,q^{\lambda_1-\lambda_2}\\q^{1+\lambda_1-\lambda_2}/k^2\end{matrix};q,\frac{qt_2}{k^2t_1}\right). \tag{5.16}$$

(ii) *The monic basic Harish-Chandra series is explicitly given by*

$$\widehat{\Phi}_\eta(t,\gamma)=\widehat{\mathcal{W}}_\eta(t,\gamma)\,{}_2\phi_1\left(\begin{matrix}k^2,k^2\gamma_1/\gamma_2\\q\gamma_1/\gamma_2\end{matrix};q,\frac{qt_2}{k^2t_1}\right) \tag{5.17}$$

for $|qt_2/k^2t_1|<1$.

Proof. (i) Consider the solution

$$\phi_H(z)={}_2\phi_1\left(\begin{matrix}k^2,q^{\lambda_1-\lambda_2}\\q^{1+\lambda_1-\lambda_2}/k^2\end{matrix};q,z\right)$$

of the basic hypergeometric q-difference equation (5.12) with parameters a,b,c given by (5.14) and with $\gamma=\gamma_\lambda=(q^{\lambda_1}k^{-1},q^{\lambda_2}k)$. It is a polynomial in z of degree $\lambda_2-\lambda_1$. (It is essentially the continuous q-ultraspherical polynomial.) In addition, $\widehat{\mathcal{W}}(t,\gamma_\lambda)=q^{-\frac{|\lambda|^2}{2}}t_1^{\lambda_2}t_2^{\lambda_1}$. By the previous lemma we conclude

that $P^+_\lambda(t)=t_1^{\lambda_2}t_2^{\lambda_1}\phi_H(qt_2/k^2t_1)$; see also [28, §6.3] and [6, Chap. 2]. The normalization factor turning $P^+_\lambda(t)$ into the normalized symmetric Macdonald polynomial $E_+(\gamma_\lambda;t)$ can, for instance, be computed using the q-Vandermonde formula [11, (II.6)].

(ii) $\widehat{\mathcal{W}}_\eta(t,\gamma)\phi_H(qt_2/k^2t_1)$ with the parameters a,b,c in ϕ_H given by (5.14) and $\widehat{\Phi}_\eta(t,\gamma)$ both satisfy (5.15) and have the same asymptotic expansion for small $|t_2/t_1|$. This forces them to be equal (cf., e.g., [23, Chap. 3, §1.7] and [8, Thm. 2.3]). □

The monic GL_2 c-function expansion

$$\mathcal{E}_+(t,\gamma)=\widehat{c}_\eta(\gamma_0)^{-1}\big(\widehat{c}_\eta(\gamma)\widehat{\Phi}_\eta(t,\gamma)+\widehat{c}_\eta(w_0\gamma)\widehat{\Phi}_\eta(t,w_0\gamma)\big), \tag{5.18}$$

where $w_0\gamma=(\gamma_2,\gamma_1)$ and

$$\widehat{c}_\eta(\gamma)=\frac{\theta(-q^{\frac{1}{2}}\gamma_1/\eta_2;q)\theta(-q^{\frac{1}{2}}\gamma_2/\eta_1;q)}{\theta(-q^{\frac{1}{2}}\gamma_1;q)\theta(-q^{\frac{1}{2}}\gamma_2;q)}\frac{\big(k^2\gamma_2/\gamma_1;q\big)_\infty}{\big(\gamma_2/\gamma_1;q\big)_\infty},$$

thus yields an explicit expression of the GL_2 basic hypergeometric function $\mathcal{E}_+(t,\gamma)$ as the sum of two ${}_2\phi_1$ basic hypergeometric series.

Another solution of Heine's basic hypergeometric q-difference equation (5.12) is

$$\begin{aligned}v(x):=&\frac{\theta(ax;q)}{\theta(x;q)}{}_2\phi_1\left(\begin{matrix}a,qa/c\\qa/b\end{matrix};q,\frac{qc}{abx}\right)\\=&\frac{\theta(ax;q)\big(a,q^2/bx;q\big)_\infty}{\theta(x;q)\big(qa/b,qc/abx;q\big)_\infty}{}_2\phi_1\left(\begin{matrix}q/b,qc/abx\\q^2/bx\end{matrix};q,a\right);\end{aligned}$$

see, e.g., [23, Chap. 3, §1.7]. (The second formula follows from Heine's transformation formula [11, (III.1)].) By Lemma 5.4 it yields yet another solution,

$$f(t)=\widehat{\mathcal{W}}_\eta(t,\gamma)\frac{\theta(qt_2/t_1;q)}{\theta(qt_2/k^2t_1;q)}\frac{\big(qt_1\gamma_2/t_2\gamma_1;q\big)_\infty}{\big(qt_1/k^2t_2;q\big)_\infty}{}_2\phi_1\left(\begin{matrix}q\gamma_2/k^2\gamma_1,qt_1/k^2t_2\\qt_1\gamma_2/t_2\gamma_1\end{matrix};q,k^2\right),$$

of the system (5.15) of GL_2 Macdonald q-difference equations. The various ${}_2\phi_1$ basic hypergeometric series solutions of (5.15) are related by explicit connection coefficient formulas; see, e.g., [44].

We finish this subsection by relating the monic GL_2 basic Harish-Chandra series $\widehat{\Phi}_\eta$ (5.17) to the monic nonreduced rank one basic Harish-Chandra series (5.8), which we will denote here by $\widehat{\Phi}^{\mathrm{nr}}_\xi$. Recall that in the nonreduced rank one setting the associated multiplicity function is determined by the four values $k^{\mathrm{nr}}=(k^{\mathrm{nr}}_\theta,k^{\mathrm{nr}}_{2\theta},k^{\mathrm{nr}}_0,k^{\mathrm{nr}}_{2a_0})$. We write $\theta(x_1,\ldots,x_r;q)=\prod_{j=1}^r\theta(x_j;q)$ for products of Jacobi theta functions.

PROPOSITION 5.6. *Let* $\eta = (\eta_1, \eta_2) \in \big(\mathbb{C}^*\big)^2$ *and* $\xi \in \mathbb{C}^*$. *Let* k^{nr} *be the multiplicity function* $k^{\mathrm{nr}} = (k, k, 0, 0)$ *with* $0 < k < 1$. *For* $\gamma = (\gamma_1, \gamma_2) \in \big(\mathbb{C}^*\big)^2$, *set* $\gamma^{\pm 2} := (\gamma_1^{\pm 2}, \gamma_2^{\pm 2})$. *Then*

$$\widehat{\Phi}_\eta(t, \gamma^2; k, q) = C_{\eta,\xi}(t, \gamma)\widehat{\Phi}^{\mathrm{nr}}_\xi\Big(\frac{t_1}{t_2}, \frac{\gamma_1}{\gamma_2}; k^{\mathrm{nr}}, q\Big) \tag{5.19}$$

with $\widehat{\Phi}_\eta$ *the* GL_2 *monic basic Harish-Chandra series* (5.17) *and*

$$C_{\eta,\xi}(t, \gamma) = \frac{\theta(-q^{\frac{1}{2}}\eta_1/k, -q^{\frac{1}{2}}\eta_2 k, -q^{\frac{1}{2}}k\gamma_2/\xi\gamma_1, -q^{\frac{1}{2}}kt_1/\gamma_2^2, -q^{\frac{1}{2}}t_2/k\gamma_1^2, -q^{\frac{1}{2}}t_1/t_2; q)}{\theta(-q^{\frac{1}{2}}\eta_1/\gamma_2^2, -q^{\frac{1}{2}}\eta_2/\gamma_1^2, -q^{\frac{1}{2}}k^2/\xi, -q^{\frac{1}{2}}t_1, -q^{\frac{1}{2}}t_2, -q^{\frac{1}{2}}kt_1\gamma_1/t_2\gamma_2; q)}.$$

Proof. If the meromorphic function $f(x, z)$ in $(x, z) \in \mathbb{C}^* \times \mathbb{C}^*$ satisfies the Askey-Wilson second-order q-difference equation

$$\mathcal{D}f(\cdot, z) = (\widetilde{a}(z + z^{-1}) - \widetilde{a}^2 - 1)f(\cdot, z)$$

with respect to the multiplicity function $k^{\mathrm{nr}} = (k, k, 0, 0)$, then

$$g(t) := C_{\eta,\xi}(t, \gamma)f\Big(\frac{t_1}{t_2}, \frac{\gamma_1}{\gamma_2}\Big)$$

satisfies the GL_2 Macdonald-Ruijsenaars q-difference equations

$$\big(D_{e_r}g\big)(t) = e_r(\gamma^{-2})g(t), \qquad r = 1, 2; \tag{5.20}$$

cf. [44]. Here we use that the prefactor $C_{\eta,\xi}(t, \gamma)$ satisfies

$$C_{\eta,\xi}(\tau(-\varepsilon_r)_q t, \gamma) = \gamma_1^{-1}\gamma_2^{-1}C_{\eta,\xi}(t, \gamma)$$

for $r = 1, 2$. Hence both sides of (5.19) satisfy (5.20). In addition, both sides of (5.19) have an expansion of the form

$$\widehat{\mathcal{W}}_\eta(t, \gamma^2)\sum_{r=0}^{\infty}\Xi_r\Big(\frac{t_2}{t_1}\Big)^r, \qquad \Xi_0 = 1$$

for $|t_2/t_1|$ sufficiently small. This forces the identity (5.19); cf. the proof of (5.17). □

A similar statement is not true if the role of the basic Harish-Chandra series in Proposition 5.6 is replaced by the associated basic hypergeometric functions. This follows from a comparison of the associated c-function expansions.

Remark 5.7. By (5.17) and (5.8), formula (5.19) is an identity expressing a very-well-poised ${}_8\phi_7$ basic hypergeometric series as a ${}_2\phi_1$ basic hypergeometric series. After application of the transformation formula [11, (III.23)] to the very-well-poised ${}_8\phi_7$ series, this identity becomes a special case of [11, (3.4.7)].

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