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The survival probability and r-point functions in high dimensions

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Abstract

In this paper we investigate the *survival probability*, θ_n, in high-dimensional statistical physical models, where θ_n denotes the probability that the model survives up to time n. We prove that if the r-point functions scale to those of the canonical measure of super-Brownian motion, and if certain self-repellence and total-population tail-bound conditions are satisfied, then $n\theta_n \to 2/(AV)$, where A is the asymptotic expected number of particles alive at time n, and V is the vertex factor of the model. Our results apply to spread-out lattice trees above 8 dimensions, spread-out oriented percolation above $4+1$ dimensions, and the spread-out contact process above $4+1$ dimensions. In the case of oriented percolation, this reproves a result by the first author, den Hollander, and Slade (which was proved using heavy lace expansion arguments), at the cost of losing explicit error estimates. We further derive several consequences of our result involving the scaling limit of the number of particles alive at time proportional to n. Our proofs are based on simple weak convergence arguments.

1. Introduction and results

A celebrated result by Kolmogorov [39] states that the probability θ_n that a Galton-Watson branching process with offspring distribution having mean 1 and variance γ, starting from a single initial particle, survives until time n satisfies $n\theta_n \to 2/\gamma$ as $n \to \infty$ (see also [48, Th. II.1.1]). A related classical result by Yaglom [52] states that the population size N_n at time n is such that, conditional on survival up to time n, the random variable $n^{-1}N_n$ converges weakly to a random variable Y having an exponential distribution with mean $\gamma/2$. Thus, the probability of survival up to time n decays like $1/n$, while on the event of survival, the number of particles alive grows proportional to n. In this paper, we study extensions of this result, and their ramifications, to general spatial statistical mechanical models in sufficiently high dimensions.

We next define the *scaling limit* of the particle numbers for critical Galton-Watson trees. The probability of the population surviving is rather small, and in the literature, two constructions have been investigated to resolve this problem. The first construction to deal with the vanishing survival probability is to start with a large number of particles; i.e., take $N_0 = \lceil nx \rceil$, where $x > 0$. In this case, at any time $t > 0$, the number of particles at time 0 whose lineage survives until time t has an approximate Poisson distribution with parameter $2x/\gamma$. Then, the process $(N_{tn}/n)_{t\geq 0}$ converges in distribution to *Feller's branching diffusion* [18], which is the unique solution to a stochastic differential equation describing a continuous-state branching process. (See also [42] for related results.) The second construction to deal with the vanishing survival probability is to multiply the measure by a factor of n, making sure that the measure of the event of survival to time proportional to n converges to a finite and positive limit. Then, the process $(N_{tn}/n)_{t\geq 0}$ converges in distribution, where the notion of convergence in distribution is defined in terms of convergence of integrals of bounded continuous functions having support on paths that survive up to time $\varepsilon > 0$. The resulting measure is a σ-finite measure rather than a probability measure, and it is called the *canonical measure* of the branching process in reference to canonical measures appearing in infinitely divisible processes (see, e.g., [38]). We can retrieve a probability measure by 'conditioning' the measure on surviving up to time 1.

While the two constructions are quite different, they are closely related. Indeed, in the first construction (conditionally upon survival to time 1) take any of the Poisson $2x/\gamma$ initial particles whose lineage survives until time 1. Then the distribution of its rescaled numbers of descendants is identical to that in the canonical measure conditioned to survive up to time 1.

The models we consider will be *spatial*. Embedding the branching process into $\mathbb{Z}^d$, with the initial particle located at the origin, $0 \in \mathbb{Z}^d$, and where the offspring of any given particle are independently located at neighbors of that particle in $\mathbb{Z}^d$, we obtain a *branching random walk*. Since multiple occupancy can occur, the state of this process at time n is best described by a (random) *measure*, where the measure of any subset of $\mathbb{R}^d$ is the number of particles of generation n located in that set. With appropriate rescaling of space, time, mass (associated to each particle), and of the underlying law, we obtain a sequence of finite (no longer probability) measures μ_n. It is well known that the measures μ_n converge weakly to a measure $\mathbb{N}_0$ on the space of measure valued paths $(X_t)_{t\geq 0}$ that survive for positive time, i.e., $S \equiv \inf\{t > 0\colon X_t(1) = 0\} > 0$ (where $X_t(f) \equiv \int f dX_t$). Although we have not found an explicit statement and proof of this result, it is implicit in Watanabe [51], and it is explicit in, e.g., [48] in the case of branching Brownian-motions. The measure $\mathbb{N}_0$ is called the *canonical measure of super-Brownian motion* and is σ-finite, with

$\mathbb{N}_0(S > \varepsilon) = 2/\varepsilon$ for every $\varepsilon > 0$. The notion of weak convergence is defined with respect to the finite measures $\mathbb{N}_0^{\varepsilon}(\cdot) \equiv \mathbb{N}_0(\cdot, S > \varepsilon)$ (see, e.g., [37]) and, in particular, $n\theta_{\lfloor nt \rfloor} \to \gamma^{-1}\mathbb{N}_0(S > t) = 2/(\gamma t)$. See [12], [48] for detailed surveys of super-processes and convergence towards them, and see [16], [17], [44] for introductions to super-processes and continuous-state branching processes.

In this paper, we study extensions of these results in the context of general spatial statistical mechanical models in sufficiently high dimensions that converge (or are conjectured to converge) to super-Brownian motion (SBM) in the sense of *convergence of* r*-point functions*. Convergence of r-point functions means that the (rescaled) joint moments of particle numbers and locations converge (to those of SBM). The use of r-point functions has a long history and tradition in statistical physics. The main result of this paper is that convergence of r-point functions, subject to two conditions that are valid in all our examples, implies that the classical results by Kolmogorov, Yaglom and (to some extent) Feller hold as well. As such, our result confirms that convergence of r-point functions is a relevant and important notion (see also [37]).

Let us introduce the general setting that we investigate. Let $\mathbb{P}$ denote the probability measure describing the law of our model. In contrast to the branching random walk setting, all our models are of single-occupancy type and have a notion of *intrinsic distance*, in which $x \xrightarrow{n} y$ means that the shortest path between x and y has length n. Let $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$ and $\mathbb{R}_+ = [0, \infty)$. Then for $\vec{x} \in \mathbb{Z}^{d(r-1)}$ and $\vec{n} \in \mathbb{Z}_+^{r-1}$ (or $\vec{n} \in \mathbb{R}_+^{r-1}$ for models where time is continuous), we let

$$t_{\vec{n}}^{(r)}(\vec{x}) = \mathbb{P}(0 \xrightarrow{n_i} x_i \; \forall i = 1, \dots, r-1) \tag{1.1}$$

denote the r-point function in the model. Further, for $\vec{k} = (k_1, \dots, k_{r-1}) \in ([-\pi, \pi]^d)^{r-1}$, we let

$$\hat{t}_{\vec{n}}^{(r)}(\vec{k}) = \sum_{\vec{x} \in \mathbb{Z}^{d(r-1)}} \mathrm{e}^{i\vec{k}\cdot\vec{x}} t_{\vec{n}}^{(r)}(\vec{x}) \tag{1.2}$$

denote its Fourier transform and

$$\theta_n = \mathbb{P}(\exists x \in \mathbb{Z}^d \colon 0 \xrightarrow{n} x) \tag{1.3}$$

the survival probability.

Let $A_n = \{x \colon 0 \xrightarrow{n} x\}$, $N_n = \#\{x \colon 0 \xrightarrow{n} x\} = \#A_n$, and $S_n = \{N_n > 0\} = \{A_n \neq \varnothing\}$, so that $\theta_n = \mathbb{P}(S_n)$. Here $\#A$ denotes the number of elements in A. When the underlying model is defined in discrete time, we define $n\vec{t}$ to be the vector $(\lfloor nt_1 \rfloor, \dots, \lfloor nt_r \rfloor)$.

In this paper, we investigate the asymptotics of the survival probability, assuming the asymptotic behavior of the r-point functions. These results apply to branching random walk in all dimensions, as well as to (a) lattice trees;

(b) oriented percolation; and (c) the contact process, all above their (model-dependent) upper critical dimension, where the general philosophy in statistical physics suggests that these models behave like branching random walk. In particular, when the allowed connections are sufficiently spread out, e.g., where all vertices within distance $L \gg 1$ of a vertex are considered to be neighbors of that vertex, the following condition holds as a theorem for each of these models, above their respective upper critical dimensions.

CONDITION 1.1 (Convergence of the r-point functions). (a) *There exist constants $A, V > 0$ both depending on L such that for each $r \geq 2$ and $\vec{t} \in \mathbb{R}_+^{(r-1)}$,*

$$\frac{1}{A(VA^2n)^{r-2}} \hat{t}^{(r)}_{n\vec{t}}(0) \to \widehat{M}^{(r-1)}_{\vec{t}}(0), \quad \text{as } n \to \infty, \tag{1.4}$$

where the quantities $\widehat{M}^{(r-1)}_{\vec{t}}(0)$ are the joint moments of the total mass at times $t_1, \ldots, t_{r-1}$ of the canonical measure of SBM. *In particular,* $\widehat{M}^{(r-1)}_{t\vec{1}_{r-1}}(\vec{0}) = t^{r-2}2^{-(r-2)}(r-1)!$.

(b) *There exist constants $A, V, v > 0$ all depending on L such that for each $r \geq 2$, $\vec{t} \in \mathbb{R}_+^{(r-1)}$, and $\vec{k} \in \mathbb{R}^{d(r-1)}$,*

$$\frac{1}{A(VA^2n)^{r-2}} \hat{t}^{(r)}_{n\vec{t}}\left(\frac{\vec{k}}{\sqrt{vn}}\right) \to \widehat{M}^{(r-1)}_{\vec{t}}(\vec{k}), \quad \text{as } n \to \infty, \tag{1.5}$$

where the quantities $\widehat{M}^{(r-1)}_{\vec{t}}(\vec{k})$ are the Fourier transforms of the moment measures of the canonical measure of SBM.

Condition 1.1(a) is the weaker of the above conditions, and it can be rephrased as

$$n\mathbb{E}\left[\prod_{i=1}^{r-1} \left(N_{t_in}/n\right)\right] \to A(VA^2)^{r-2}\widehat{M}^{(r-1)}_{\vec{t}}(0), \tag{1.6}$$

where $\widehat{M}^{(r-1)}_{\vec{t}}(0)$ are the limits of the joint moments of population sizes of critical branching processes with variance one offspring distributions. Note that the convergence in (1.6) makes no assumption on the *spatial* locations of the particles involved, however the evolution of N_n is affected by spatial interaction present in our models. Condition 1.1(b), which contains (a), can be rephrased as

$$\mathbb{E}_{\mu_n}\left[\prod_{j=1}^{r-1} X^{(n)}_{t_j}(\phi_{k_j})\right] \to \mathbb{E}_{\mathbb{N}_0}\left[\prod_{j=1}^{r-1} X_{t_j}(\phi_{k_j})\right], \tag{1.7}$$

where $\phi_{k_j}(x) = \mathrm{e}^{ik_j \cdot x}$ for $k_j \in \mathbb{R}^d$ and $x \in \mathbb{Z}^d$ and where

$$X_t^{(n)}(f) = \frac{1}{VA^2 n} \sum_{x \in A_{nt}} f(x/\sqrt{vn}) \quad \text{and} \quad \mu_n(\cdot) = nVA\mathbb{P}(\cdot). \tag{1.8}$$

Thus, Condition 1.1(b) states that certain moment measures of the rescaled processes under the measure μ_n converge to those of the canonical measure of SBM. Condition 1.1(b) is the condition that is typically proved in the literature.

Before stating our main result, let us formulate two further conditions. Recall the definitions of A_n and N_n following (1.3). Let $|N|$ denote the total population size, i.e.,

$$|N| = \int_0^\infty N_t dt, \tag{1.9}$$

which is equivalent to $|N| = \sum_{n \geq 0} N_n$ for discrete-time models. We make two central assumptions on our high-dimensional models.

CONDITION 1.2 (Cluster tail bound). *There exists a constant C_N such that*

$$\mathbb{P}(|N| \geq k) \leq C_N/\sqrt{k}. \tag{1.10}$$

CONDITION 1.3 (Self-repellent survival property). *Let $\mathcal{F}_m$ be the σ-field generated by the vertices at distance at most m from* 0, *i.e., by $\{(x,n) \colon 0 \xrightarrow{n} x, n \leq m\}$. Then there exists a constant C_θ such that, almost surely for every stopping time $M \leq n$,*

$$\mathbb{P}(A_M \longrightarrow n \mid \mathcal{F}_M) \leq C_\theta N_M \theta_{n-M}. \tag{1.11}$$

The cluster tail condition (1.10) follows from the literature for all models under consideration (as we will show in more detail below). In the case of branching random walk with critical geometric branching, (1.10) reduces to the return time tail for simple random walk (by Harris's identity for branching processes; see, for example, [25].)

The self-repellent survival property in (1.11) is elementary for branching random walk for which it holds by the strong Markov property, due to the independence of the offspring of particles alive at time M, which implies that

$$\mathbb{P}(A_M \longrightarrow n \mid \mathcal{F}_M) = 1 - (1 - \theta_{n-M})^{N_M} \leq N_M \theta_{n-M}. \tag{1.12}$$

It is not much harder to verify for our models (again, see below). The first of our main results is the following theorem.

THEOREM 1.4. *When Conditions 1.1(a), 1.2, and 1.3 hold, as $n \to \infty$,*

$$n\theta_n \to 2/(AV), \tag{1.13}$$

and consequently for each $t > 0$,

$$\mu_n(X_t^{(n)}(1) > 0) \to \mathbb{N}_0(X_t(1) > 0) = 2/t. \tag{1.14}$$

Moreover, as $n \to \infty$, *the rescaled total mass process* $(X_s^{(n)}(1))_{s \geq 0}$ *under* μ_n *converges (in the sense of finite-dimensional distributions) to the total mass* $(X_s(1))_{s\geq 0}$ *of* CSBM, *both unconditionally and conditional on survival up to time* t *(for any* $t > 0$*).*

Note that conditional on survival up to time t the total mass process $(X_s(1))_{s\geq t}$ under $\mathbb{N}_0$ is Feller's branching diffusion started from an exponential random variable with mean $t/2$, so this says that conditionally on $N_{tn} > 0$, $\{N_{sn}/n\}_{s\geq t}$ converges in the sense of finite-dimensional distributions to Feller's branching diffusion started from an exponential random variable with mean $A^2Vt/2$.

For oriented percolation, Theorem 1.4 contains the result from [29], [30] (but without the error estimates). See also [40], [41], [50] for related results on survival probabilities. Our setup is rather general, so that in the future, it might be applicable to other models, such as percolation, the voter model, and lattice animals above their respective upper critical dimensions as well.

Theorem 1.4 is particularly important, since the combination of the convergence of the r-point functions (as formulated in Condition 1.1(b)) and Theorem 1.4 imply that $\{\mu_n\}_{n\geq 1}$ converge in the sense of *finite-dimensional distributions* to $\mathbb{N}_0$ (see [37]). This is the second of our main results.

THEOREM 1.5. *When Conditions* 1.1(b), 1.2, *and* 1.3 *hold, the finite-dimensional distributions of the process* $(X_s^{(n)})_{s\geq 0}$ *under* μ_n *converge to those of* $(X_s)_{s\geq 0}$ *under the measure* $\mathbb{N}_0$. *The same is true under the measures conditioned on survival up to time* t *for each* $t > 0$.

We now present our three main examples, which all involve a function $D\colon \mathbb{Z}^d \to [0,1]$, with $\sum_{x\in\mathbb{Z}^d} D(x) = 1$. In order to apply our main result to these examples, we make the additional assumption, under which Condition 1.1 is verified in the literature (see below for precise references), that

$$D(x) = \frac{h(x/L)}{\sum_{x\in\mathbb{Z}^d} h(x/L)}, \tag{1.15}$$

where L is large and h is a nonnegative bounded function on $\mathbb{R}^d$ that is piecewise continuous, symmetric under the $\mathbb{Z}^d$-symmetries of reflection in coordinate hyperplanes and rotation by $\pi/2$, supported in $[-1,1]^d$, and normalised ($\int_{[-1,1]^d} h(x)d^dx = 1$). A basic example of where this holds is $D(x) = ((2L+1)^d - 1)^{-1}\mathbb{1}_{\{0<\|x\|_\infty \leq L\}}$; i.e., D is the uniform distribution on a box of radius L excluding the origin.

Spread-out lattice trees above 8 *dimensions.* A lattice tree is a finite connected set of lattice bonds (and their associated end vertices) containing no cycles. For fixed $z > 0$, every such tree $T \ni 0$ with bond set B is assigned a weight $W_z(T) = z^{|B|} \prod_{(x,y)\in B} D(y-x)$, and we define $\rho_z(x) = \sum_{T\ni 0,x} W_z(T)$. The radius of convergence z_c of $\sum_{x\in\mathbb{Z}^d} \rho_z(x)$ is finite. Let $W(\cdot) = W_{z_c}(\cdot)$ and $\rho = \rho_{z_c}(0)$. We define a probability measure on the (countable) set of lattice trees containing the origin by $\mathbb{P}(T) = W(T)/\rho$. Given a lattice tree $T \ni 0$, we define $A_n(T) = \{a_1, \dots, a_{N_n}\}$ to be the (ordered) set of vertices in T of tree distance $n \in \mathbb{Z}_+$ from the origin under some arbitrary but fixed ordering of $\mathbb{Z}^d$.

Condition 1.1 is the main result in [36]. Condition 1.2 follows from the detailed asymptotics for $\mathbb{P}(|T| = n) \sim cn^{-3/2}$ proved in [13], [14], where $|T|$ denotes the number of vertices in the lattice tree T. We next check Condition 1.3, for which it is enough to show that the result holds almost surely for every deterministic time $m \le n$. Letting T_m denote the tree up to tree distance m from the root, we have that $\mathbb{P}(A_m \longrightarrow n \mid T_m = \tau_m)$ is equal to

$$\frac{W(\tau_m)}{\sum_{T\colon T_m=\tau_m} W(T)} \sum_{R_1\ni a_1} \cdots \sum_{R_{N_m}\ni a_{N_m}} \times \prod_{i=1}^{N_m} W(R_i) \mathbb{1}_{\{R_i \text{ avoid each other and } \tau_m\}} \mathbb{1}_{\{\cup_j \{R_j \text{ survives at least until } n-m\}\}},$$

where $\sum_{R\ni a}$ is a sum over lattice trees R rooted at $a \in \mathbb{Z}^d$ (with survival measured in terms of tree distance from a), and we recall that $A_m = \{a_1, \dots, a_{N_m}\}$.

The final indicator function is bounded above by $\sum_j \mathbb{1}_{\{\mathcal{S}_{R_j} \ge n-m\}}$, where $\mathcal{S}_R$ is the survival time of R. By taking the sum over j outside and dropping the restriction that R_j avoids other R_i and τ_m, this is bounded above by

$$\begin{aligned} \sum_{j=1}^{N_m} \sum_{R_j\ni a_j} W(R_j) \mathbb{1}_{\{\mathcal{S}_{R_j}\ge n-m\}} \Bigg[\frac{W(\tau_m)}{\sum_{T\colon T_m=\tau_m} W(T)} \sum_{R_1\ni a_1} \cdots \sum_{R_{j-1}\ni a_{j-1}} \\ \times \sum_{R_{j+1}\ni a_{j+1}} \cdots \sum_{R_{N_m}\ni a_{N_m}} \prod_{i\ne j} W(R_i) \mathbb{1}_{\{R_i, i\ne j \text{ avoid each other and } \tau_m\}} \Bigg] \\ \le \sum_{j=1}^{N_m} \sum_{R_j\ni a_j} W(R_j) \mathbb{1}_{\{\mathcal{S}_{R_j}\ge n-m\}} = N_m \rho \theta_{n-m}, \end{aligned} \tag{1.16}$$

where we have used the fact that the interaction term makes the graph $\tau_m \cup_{i\ne j} R_i$ a lattice tree T with $T_m = \tau_m$, and weight $W(T) = W(\tau_m) \prod_{i\ne j} W(R_i)$, so the numerator in brackets is no more than the denominator. Since $\rho = \rho_{z_c}(0) < \infty$ [22], this verifies Condition 1.3.

Spread-out contact process above $4+1$ *dimensions.* We define the spread-out contact process as follows. Let $\mathcal{C}_n \subset \mathbb{Z}^d$ be the set of infected individuals at time $n \in \mathbb{R}_+$, and let $\mathcal{C}_0 = \{0\}$. An infected site x recovers in a small time interval $[n, n+\varepsilon]$ with probability $\varepsilon + o(\varepsilon)$ independently of n, where $o(\varepsilon)$ is a function that satisfies $\lim_{\varepsilon\to 0} o(\varepsilon)/\varepsilon = 0$. In other words, $x \in \mathcal{C}_n$ recovers at rate 1. A healthy site x gets infected, depending on the status of its neighbors, at rate $\lambda \sum_{y\in\mathcal{C}_n} D(x-y)$, where $\lambda \geq 0$ is the infection rate. We denote by $\mathbb{P}^\lambda$ the associated probability measure.

By [20], which extends the results in [3] to the spread-out contact process, there exists a unique critical value $\lambda_c \in (0,\infty)$ such that

$$\theta(\lambda) \equiv \lim_{n\to\infty} \mathbb{P}^\lambda(\mathcal{C}_n \neq \varnothing) \begin{cases} = 0 & \text{if } \lambda \leq \lambda_c, \\ > 0 & \text{if } \lambda > \lambda_c, \end{cases} \tag{1.17}$$

and we define

$$\theta_n = \theta_n(\lambda_c) = \mathbb{P}^{\lambda_c}(\mathcal{C}_n \neq \varnothing). \tag{1.18}$$

Condition 1.1 is proved in [32], [33]. Condition 1.2 holds by [2], [32], [33], [49], while Condition 1.3 morally follows from a union bound and the strong Markov property. To be precise, let the infected particles at time M be written as $A_M = \{x_1, \ldots, x_{N_M}\}$ (according to some fixed but arbitrary ordering). Relabel the infection at each $x \in A_M$ to be 1_x, so that there are now N_M different infections $\{1_{x_1}, \ldots, 1_{x_{N_M}}\}$ at time M. (These labels are fixed thereafter.) Let this modified process evolve as before, except that a site may carry more than one of the M distinct infections. When one particle infects another, it passes all of its infections onto that particle, and whenever a particle recovers (rate 1), it recovers from all its infections simultaneously. Then,

$$\begin{aligned} \mathbb{P}(A_M \longrightarrow n \mid \mathcal{F}_M) &= \mathbb{P}(\cup_{x\in A_M}\{\text{infection } 1_x \text{ survives until time } n\} \mid \mathcal{F}_M) \\ &\leq \sum_{x\in A_M} \mathbb{P}(\{\text{infection } 1_x \text{ survives until time } n\} \mid \mathcal{F}_M). \end{aligned}$$

However, any one of the infections $\{1_{x_1}, \ldots, 1_{x_{N_M}}\}$ spreads according to the ordinary contact process dynamics, so the latter probability is θ_{n-M}; i.e.,

$$\mathbb{P}(A_M \longrightarrow n \mid \mathcal{F}_M) \leq \sum_{x\in A_M} \theta_{n-M} = N_M \theta_{n-M}.$$

Convergence of the spread-out contact process to super-Brownian motion is proved in [15] in the setting where the range of the contact grows with the time until which the contact process is being considered.

Spread-out oriented percolation above $4+1$ *dimensions.* The spread-out oriented bond percolation model is defined as follows. Consider the graph with vertices $\mathbb{Z}^d \times \mathbb{Z}_+$ and with directed bonds $((x,n),(y,n+1))$ for $n \in \mathbb{Z}_+$ and $x, y \in \mathbb{Z}^d$. Let $p \in [0, \|D\|_\infty^{-1}]$, where $\|\cdot\|_\infty$ denotes the supremum norm, so that $pD(x) \leq 1$ for all $x \in \mathbb{Z}^d$. We associate to each directed bond $((x,n),(y,n+1))$

an independent random variable taking the value 1 with probability $pD(y-x)$ and the value 0 with probability $1-pD(y-x)$. We say that a bond is *occupied* when the corresponding random variable is 1 and *vacant* when it is 0. The joint probability distribution of the bond variables will be denoted by $\mathbb{P}_p$ and the corresponding expectation by $\mathbb{E}_p$.

We say that (x,n) is *connected* to (y,m), and we write $(x,n) \longrightarrow (y,m)$, if there is an oriented path from (x,n) to (y,m) consisting of occupied bonds. Note that this is only possible when $m \geq n$. By convention, (x,n) is connected to itself. We write $(x,n) \longrightarrow m$ if $m \geq n$ and there is a $y \in \mathbb{Z}^d$ such that $(x,n) \longrightarrow (y,m)$. The event $\{(0,0) \longrightarrow \infty\}$ is the event that $\{(0,0) \longrightarrow n\}$ occurs for all n. There is a critical threshold $p_c > 0$ such that the event $\{(0,0) \longrightarrow \infty\}$ has probability zero for $p < p_c$ and has positive probability for $p > p_c$. The survival probability at time n is defined by

$$\theta_n(p) = \mathbb{P}_p((0,0) \longrightarrow n), \tag{1.19}$$

and we let $\theta_n = \theta_n(p_c)$. General results of [3], [20] imply that $\lim_{n\to\infty} \theta_n = 0$.

Then, for $\mathbb{P} = \mathbb{P}_{p_c}$, Condition 1.1 is proved in [35]. Condition 1.2 holds by [1], [35], [46], [47], while Condition 1.3 follows from a union bound and the strong Markov property, via the same argument as for the contact process.

Our main results can be restated in terms of the above models as follows.

THEOREM 1.6. *Let $L \gg 1$, let $d > 4$ for oriented percolation and the contact process, and let $d > 8$ for lattice trees. Then, with $A, V, v > 0$ all depending on L such that for each $\vec{t} \in \mathbb{R}_+^{(r-1)}$ and $\vec{k} \in \mathbb{R}^{(r-1)}$,*

$$\frac{1}{A(VA^2 n)^{r-2}} \hat{t}^{(r)}_{n\vec{t}}\big(\vec{k}/\sqrt{vn}\big) \to \widehat{M}^{(r-1)}_{\vec{t}}(\vec{k}), \quad \text{as } n \to \infty, \tag{1.20}$$

the asymptotics

$$n\theta_n \to 2/(AV) \quad \text{and} \quad \mu_n(X_t^{(n)}(1) > 0) \to \mathbb{N}_0(X_t(1) > 0) = 2/t, \quad \text{as } n \to \infty \tag{1.21}$$

hold. As a consequence, the finite-dimensional distributions of the process $(X_s^{(n)})_{s>0}$ under μ_n converge to those of $(X_s)_{s>0}$ under the measure $\mathbb{N}_0$, and similarly for the measures conditioned on survival up to time t for any $t > 0$.

In work in progress [31] and jointly with Ed Perkins, we prove a tightness result for spread-out lattice trees in dimensions $d > 8$. Together with Theorem 1.5 and under the same conditions, this proves weak convergence of lattice trees to super-Brownian in the sense of measure-valued processes.

We close this section with some possible extensions to our results.

Long-range models. In all our models, we assume that D has finite range (in some cases this can be weakened to finite spatial variance), so that SBM can arise as the scaling limit. In the literature, *long-range* models have attracted considerable attention. See [7], [8], [9] for results on long-range oriented percolation, [26] for long-range self-avoiding walk, and [27] for percolation, self-avoiding walk and the Ising model. In long-range models, the random walk step distribution D has infinite variance. The simplest example arises when

$$D(x) = \frac{(1+|x|/L)^{-(d+\alpha)}}{\sum_{y\in\mathbb{Z}^d}(1+|y|/L)^{-(d+\alpha)}}, \qquad x \in \mathbb{Z}^d, \tag{1.22}$$

where $\alpha \in (0,2)$ and $|x|$ denotes the Euclidean norm of $x \in \mathbb{Z}^d$. The results in [7], [8], [9] suggest that the upper critical dimension of oriented percolation equals 2α, while [27] indicates that it is 3α for percolation, and 2α for self-avoiding walk and the Ising model.

We believe that Condition 1.1(a) holds for these models above their respective upper critical dimensions. Once this is proved, Theorem 1.4 then implies convergence of the survival probability in each case. However, random walk with step distribution D converges to α-stable motion rather than Brownian motion, a fact that is proved to hold for self-avoiding walk above 2α dimensions in [26]. Therefore, Condition 1.1(b) does *not* hold and should be replaced with convergence towards the canonical measure of super-stable motion.

By considering branching random walks, where the population size process is independent of the random walk step-distribution, it is easy to see that the law of the total mass process under the canonical measure of super-stable motion is the same as under $\mathbb{N}_0$. Thus by [37, Th. 2.6], in the long-range setting, convergence of the r-point functions and the survival probability still implies convergence in the sense of finite-dimensional distributions. Therefore to prove a version of Theorem 1.5 in the long-range setting, it is sufficient to prove the convergence of the r-point functions in Condition 1.1(b).

Voter model. In this model we start at time 0 with a single site (the origin) having opinion 1 and all other sites having opinion 0. Each site has a (standard) Poisson clock, and when the clock rings the site adopts the opinion of a random neighbour.

The voter model and its connections to super-Brownian motion have received substantial attention in the literature. The survival probability asymptotics are known in all dimensions, with (1.13) holding in dimensions $d \geq 3$ [6]. See [45] for a general introduction to particle systems including the voter model, and see [5], [10], [11] for convergence of rescaled voter models to super-Brownian motion.

Our proof might simplify the analysis of the survival probability for this model, although we have not found a statement of Condition 1.2 in the literature. As for the contact process, it is easy to verify Condition 1.3 for this model. Let $A_M = \{x_1, \ldots, x_{N_M}\}$ denote the particles with opinion 1 at time

M, and relabel the opinion at each $x \in A_M$ to be 1_x, so that there are now $N_M + 1$ different opinions $\{0, 1_{x_1}, \dots, 1_{x_{N_M}}\}$ at time M. Letting the process evolve as a voter model with $N_M + 1$ different opinions (each particle having exactly one opinion), we again have

$$\mathbb{P}(A_M \longrightarrow n \mid \mathcal{F}_M) \le \sum_{x \in A_M} \mathbb{P}(\{\text{opinion } 1_x \text{ survives until time } n\} \mid \mathcal{F}_M)$$
$$= \sum_{x \in A_M} \theta_{n-M} = N_M \theta_{n-M}$$

since the dynamics of a single type of opinion are those of the ordinary voter model. Thus, Condition 1.3 follows. While Condition 1.1 is unknown, closely related estimates have been obtained in [43] via the duality between the voter model and coalescing random walks.

Spread-out percolation above 6 *dimensions.* For a general introduction to percolation, we refer to [19]. We now introduce the model that we consider. Let $p \in [0, \|D\|_\infty^{-1}]$ be a parameter. We declare a bond $\{u, v\}$ to be *occupied* with probability $pD(v-u)$ and *vacant* with probability $1 - pD(v-u)$. The occupation status of all bonds are independent random variables. The law of the configuration of occupied bonds (at the critical percolation threshold) is denoted by $\mathbb{P}_{p_c}$ with corresponding expectation denoted by $\mathbb{E}_{p_c}$. Given a configuration, we say that x is connected to y, and we write $x \xrightarrow{n} y$ if there is a path of occupied bonds from x to y and the path with the minimal number of bonds connecting x and y has precisely n edges.

For percolation, Condition 1.1 is not known. Condition 1.2 follows from [21] together with [1]; see also [23], [24]. A form of Condition 1.3 can be established in a similar way as for lattice trees above. Evaluating the cluster up to generation m, we have observed a set of open edges T_m in our cluster of generation $\le m$, as well as a corresponding set of closed edges incident to those edges. Let $\overline{T}_m$ be the union of these edges. Then

$$\mathbb{P}(A_m \to n | \mathcal{F}_m) = \mathbb{P}\Big(\bigcup_{x \in A_m} \{x \text{ survives at least } n-m \text{ in } \overline{T}_m^c\} \mid \mathcal{F}_m\Big)$$

since $A_m \subset T_m$ contains all ancestors of generation m of all vertices of generation n. It is tempting to now conclude Condition 1.3 from a union bound and by dropping the restriction that the connections occur outside $\overline{T}_m$. Unfortunately, the function θ_n is not monotone in the graph on which we perform percolation. Indeed, for a set of edges B, it is not true that $\mathbb{P}(0 \text{ survives at least } n \text{ in } B^c) \le \theta_n$. This was cleverly resolved by Kozma and Nachmias [40] by instead studying $\overline{\theta}_n = \sup_B \mathbb{P}(0 \text{ survives at least } n \text{ in } B^c)$, for which the proof of Theorem 2.1 does apply. Since $\theta_n \le \overline{\theta}_n$, this does imply the fact that $n\theta_n$ is bounded. It is straightforward to check that this strategy can also be applied to the proof of Theorem 1.4 below. As a result, for percolation, our results hold as soon as Condition 1.1 is proved, even though this proof is substantially more subtle than the proof for lattice trees, the contact process, and oriented percolation.

The above discussion suggests the following research program to identify the right constants in arm-probabilities in high-dimensional percolation, both in the intrinsic as well as in the Euclidean or extrinsic distance: (1) prove the convergence of the r-point functions in Condition 1.1(b) (from which the right constant in the survival probability or intrinsic one-arm probability would follow, improving upon the results in [40]); (2) prove *tightness* for convergence towards SBM; (3) identify the right constant for the *extrinsic* one-arm probability, improving upon the result in [41]. For the last step, an important ingredient showing that it is unlikely that a short path exists to the boundary of a Euclidean ball is proved in [34, Th. 1.5].

The remainder of this paper is organized as follows. In Section 2, we prove an upper bound on θ_n that is of the correct order, but with the wrong constant. In Section 3, we use weak-convergence arguments to identify the correct constant and prove the consequences of convergence of the survival probability.

2. Weak upper bound on the survival probability

The following theorem gives a weak upper bound on the survival probability.

THEOREM 2.1. *When Conditions 1.2 and 1.3 hold, there exists a constant c_+ such that*

$$\theta_n \le c_+/n. \tag{2.1}$$

Proof. We follow [40], where a similar bound was proved for the intrinsic one-arm in percolation. We split θ_{4n} into two parts,

$$\begin{aligned}\theta_{4n} = &\ \mathbb{P}(N_m \ge \varepsilon n\ \forall m \in [n, 3n], 0 \longrightarrow 4n) \\ &+ \mathbb{P}(\exists m \in [n, 3n]\colon N_m < \varepsilon n, 0 \longrightarrow 4n).\end{aligned} \tag{2.2}$$

We can bound the first probability using (1.10) since $|N| \ge 2\varepsilon n^2$ if $N_m \ge \varepsilon n$ for all $m \in [n, 3n]$. Therefore,

$$\mathbb{P}(N_m \ge \varepsilon n\ \forall m \in [n, 3n], 0 \longrightarrow 4n) \le \mathbb{P}(|N| \ge 2\varepsilon n^2) \le \frac{C_N}{n\sqrt{2\varepsilon}}. \tag{2.3}$$

In the second probability in (2.2), we let $J \ge n$ be the first $m \in [n, 3n]$ such that $0 < N_m < \varepsilon n$, and we condition on $\mathcal{F}_J = \sigma((A_m)_{m\le J})$. Then, by (1.11),

$$\mathbb{P}(A_J \longrightarrow 4n \mid \mathcal{F}_J) \le N_J C_\theta \theta_n \le \varepsilon n C_\theta \theta_n. \tag{2.4}$$

As a result,

$$\mathbb{P}(\exists m \in [n, 3n]\colon N_m < \varepsilon n, 0 \longrightarrow 4n) \\ = \mathbb{E}[\mathbb{1}_{\{n \le J \le 3n\}} \mathbb{P}(A_J \longrightarrow 4n \mid \mathcal{F}_J)] \le \varepsilon C_\theta n \theta_n^2,$$

where we use the fact that $n \le J$ implies that $0 \longrightarrow n$. Thus, we end up with the inequality

$$\theta_{4n} \le \frac{C_N}{n\sqrt{2\varepsilon}} + \varepsilon C_\theta n \theta_n^2. \tag{2.5}$$

Take $\varepsilon = c_2^{-4/3}$, and take $c_2 > 1$ so large that

$$2^{-\frac{1}{2}} C_N c_2^{2/3} + C_\theta c_2^{2/3} \le c_2/4. \tag{2.6}$$

Then, it is easy to prove by induction that $\theta_{4^k} \le c_2 4^{-k}$ for every $k \ge 1$. By monotonicity of $n \mapsto \theta_n$, this immediately implies that $\theta_n \le (4c_2)/n$. This completes the proof of Theorem 2.1. □

3. Identifying the constant: Proof of Theorem 1.4

In this section, we make use of general weak convergence arguments to prove that $n\theta_n \to 2/(AV)$. We rely on a result that is essentially a special case of [37, Prop. 2.3], which requires the introduction of some more notation. Let $M_F(\mathbb{R}^d)$ (resp. $M_1(\mathbb{R}^d)$) denote the space of finite (resp. probability) measures on $\mathbb{R}^d$ equipped with the topology of weak convergence. Let $\mathcal{D}_G$ denote the set of discontinuities of a function G, and let $D(E)$ denote the space of càdlàg E-valued functions with the Skorohod topology. When we say that μ is a measure on (a topological space) E, this means that it is a measure with respect to the Borel σ-algebra on E.

LEMMA 3.1. *Suppose that Condition 1.1(a) holds. Then for every s, t, $\eta > 0$ and every bounded Borel measurable $H\colon \mathbb{R} \to \mathbb{R}$ such that $\mathbb{N}_0(X_t(1) \in \mathcal{D}_H) = 0$,*

$$\mathbb{E}_{\mu_n}\left[\mathbb{1}_{\{X_s^{(n)}(1) > \eta\}} H(X_t^{(n)}(1))\right] \to \mathbb{E}_{\mathbb{N}_0}\left[\mathbb{1}_{\{X_s(1) > \eta\}} H(X_t(1))\right], \quad \textit{as } n \to \infty. \tag{3.1}$$

Proof. We follow the proof of [37, Prop. 2.3]. For convenience, we drop the superscripts $^{(n)}$. By Condition 1.1(a), $\{\mu_n\}_{n \ge 1}$ is a sequence of finite measures on $D(M_F(\mathbb{R}^d))$ such that for every $r \ge 1$ and $\vec{t} \in [0, \infty)^r$, (1.7) holds when $\phi_{k_j} = 1$ for each j.

Fix $s, t, \eta > 0$. Let $Y_s = X_s(1)$, and define $P_n = P_{n,s,t} \in M_1(\mathbb{R}^2)$ and $P = P_{s,t} \in M_1(\mathbb{R}^2)$ by

$$P_n(A) = \frac{\mathbb{E}_{\mu_n}[Y_s \mathbb{1}_{\{(Y_s, Y_t) \in A\}}]}{\mathbb{E}_{\mu_n}[Y_s]} \qquad \text{and} \qquad P(A) = \frac{\mathbb{E}_{\mathbb{N}_0}[Y_s \mathbb{1}_{\{(Y_s, Y_t) \in A\}}]}{\mathbb{E}_{\mathbb{N}_0}[Y_s]},$$

where these measures are well defined since

$$\mathbb{E}_{\mu_n}[Y_s] \to \mathbb{E}_{\mathbb{N}_0}[Y_s] \in (0, \infty).$$

On each of these spaces, let (W, Z) be the canonical random vector; that is, $(W, Z)(\omega_1, \omega_2) = (\omega_1, \omega_2)$. Then, for every $m_1, m_2 \geq 0$,

(3.2)
$$\mathbb{E}_{P_n}\left[W^{m_1} Z^{m_2}\right] = \frac{\mathbb{E}_{\mu_n}\left[Y_s^{m_1+1} Y_t^{m_2}\right]}{\mathbb{E}_{\mu_n}[Y_s]} \to \frac{\mathbb{E}_{\mathbb{N}_0}\left[Y_s^{m_1+1} Y_t^{m_2}\right]}{\mathbb{E}_{\mathbb{N}_0}[Y_s]} = \mathbb{E}_P\left[W^{m_1} Z^{m_2}\right];$$

i.e., the moments of (W, Z) under P_n converge to those under P.

Furthermore (see, e.g., [37, Lemma 4.1]) there exists $\delta > 0$ such that

$$\mathbb{E}_P\left[\mathrm{e}^{\delta(W+Z)}\right] = \frac{\mathbb{E}_{\mathbb{N}_0}\left[Y_s\, \mathrm{e}^{\delta(Y_s+Y_t)}\right]}{\mathbb{E}_{\mathbb{N}_0}[Y_s]} < \infty; \tag{3.3}$$

i.e., the moment generating function of (W, Z) under P is finite in a neighborhood of $(0, 0)$. It then follows (see, e.g., [4, Ths. 30.1, 30.2 and Problems 30.5, 30.6]) that P_n converges weakly to P, and therefore for $G\colon \mathbb{R}^2 \to \mathbb{R}$ bounded and such that $P((W, Z) \in \mathcal{D}_G) = 0$,

$$\mathbb{E}_{P_n}[G(W, Z)] \to \mathbb{E}_P[G(W, Z)].$$

In other words, for each bounded $G\colon \mathbb{R}^2 \mapsto \mathbb{R}$ such that $\mathbb{N}_0((Y_s, Y_t) \in \mathcal{D}_G) = 0$,

$$\mathbb{E}_{\mu_n}\left[Y_s G(Y_s, Y_t)\right] \to \mathbb{E}_{\mathbb{N}_0}\left[Y_s G(Y_s, Y_t)\right].$$

Let H be as in the statement of the lemma, and define

$$G_H(x, y) = \begin{cases} \frac{H(y)}{x} & \text{if } x > \eta, \\ 0 & \text{otherwise.} \end{cases}$$

Then G_H is bounded, and $\mathcal{D}_{G_H} = \{(x, y)\colon y \in \mathcal{D}_H \text{ or } x = \eta\}$, whence

$$\mathbb{N}_0((X_s, X_t) \in \mathcal{D}_{G_H}) = 0.$$

The claim follows since $Y_s G_H(Y_s, Y_t) = \mathbb{1}_{\{Y_s > \eta\}} H(Y_t)$. □

Proof of Theorem 1.4. By Theorem 2.1, we have that $n\theta_n$ is bounded. In order to investigate the limit of $n\theta_n$, we split, for each fixed $\varepsilon > 0$,

$$n\theta_n = n\mathbb{P}(N_n > \varepsilon n) + n\mathbb{P}(0 < N_n \leq \varepsilon n). \tag{3.4}$$

The first term is equal to $(AV)^{-1}\mu_n(X_1^{(n)} > c\varepsilon)$, with $c = (VA^2)^{-1}$. From Lemma 3.1 with $s = 1$, $\eta = c\varepsilon$, and with the continuous function $H \equiv 1$ (and Condition 1.1(a)), we have that the first term on the right converges to $(AV)^{-1}\mathbb{N}_0(X_1(1) > c\varepsilon)$, and this converges to $(AV)^{-1}\mathbb{N}_0(X_1(1) > 0) = 2/(AV)$ as $\varepsilon \to 0$. Since $n\mathbb{P}(0 < N_n \leq \varepsilon n) \geq 0$, this immediately proves that

$$\liminf_{n\to\infty} n\theta_n \geq 2/(AV). \tag{3.5}$$

In order to identify the limit, we adapt [28, §5.3, proof of Th. 1.5]. Let $\delta \in (0,1)$, and let $\{n_k\} = \{n_k(\delta)\}$ be any subsequence of $\mathbb{N}$ such that $n_k\theta_{n_k} \to \bar{b}$, where $\bar{b} = \limsup_n n\theta_n$, and $(1-\delta)n_k\theta_{(1-\delta)n_k} \to b_\delta$ for some $b_\delta \geq 2/AV$. Similarly to (3.4), for $\delta, \varepsilon, \varepsilon' \in (0,1)$, we write

$$
\begin{aligned}
(3.6) \qquad n_k\theta_{n_k} &= n_k\mathbb{P}(N_{(1-\delta)n_k} > \varepsilon n_k, N_{n_k} > \varepsilon' n_k) \\
&\quad + n_k\mathbb{P}(N_{(1-\delta)n_k} > \varepsilon n_k, 0 < N_{n_k} \\
&\quad \leq \varepsilon' n_k) + n_k\mathbb{P}(0 < N_{(1-\delta)n_k} \leq \varepsilon n_k, N_{n_k} > 0) \\
&= A_{k,\delta,\varepsilon,\varepsilon'} + B_{k,\delta,\varepsilon,\varepsilon'} + D_{k,\delta,\varepsilon}.
\end{aligned}
$$

Since the above is true for each $\delta, \varepsilon, \varepsilon'$, it follows that also

$$
(3.7) \quad \bar{b} \leq \limsup_{\delta,\varepsilon,\varepsilon'\downarrow 0} \limsup_{k\to\infty} A_{k,\delta,\varepsilon,\varepsilon'} + \limsup_{\delta,\varepsilon,\varepsilon'\downarrow 0} \limsup_{k\to\infty} B_{k,\delta,\varepsilon,\varepsilon'} + \limsup_{\delta,\varepsilon\downarrow 0} \limsup_{k\to\infty} D_{k,\delta,\varepsilon},
$$

where the limits are taken in the order $k \to \infty$, $\varepsilon' \downarrow 0$, $\varepsilon \downarrow 0$, $\delta \downarrow 0$.

The term $A_{k,\delta,\varepsilon,\varepsilon'}$ can be rewritten as

$$
\begin{aligned}
&\frac{1}{AV}\mu_{n_k}(X^{(n_k)}_{1-\delta}(1) > c\varepsilon, X^{(n_k)}_1(1) > c\varepsilon') \\
&\qquad\qquad \to \frac{1}{AV}\mathbb{N}_0(X_{1-\delta}(1) > c\varepsilon, X_1(1) > c\varepsilon'), \qquad \text{as } k \to \infty,
\end{aligned}
$$

by Lemma 3.1. Letting $\varepsilon' \downarrow 0$ and then $\varepsilon \downarrow 0$, this converges to

$$
\frac{1}{AV}\mathbb{N}_0(X_{1-\delta}(1) > 0, X_1(1) > 0) = \frac{1}{AV}\mathbb{N}_0(X_1(1) > 0) = 2/AV,
$$

where we use that $\{X_1(1) > 0\} \subset \{X_{1-\delta}(1) > 0\}$ and which, in particular, does not depend on δ. Further, using Condition 1.3, the term $D_{k,\delta,\varepsilon}$ satisfies

$$
\begin{aligned}
D_{k,\delta,\varepsilon} &= n_k\mathbb{E}\Big[\mathbb{1}_{\{0<N_{(1-\delta)n_k}\leq\varepsilon n_k\}}\mathbb{P}(N_{n_k} > 0|\mathcal{F}_{(1-\delta)n_k})\Big] \\
&\leq C_\theta\varepsilon n_k\theta_{\delta n_k} n_k\theta_{(1-\delta)n_k} \leq \frac{C\varepsilon}{\delta(1-\delta)},
\end{aligned}
$$

uniformly in k, since $n\theta_n$ is bounded above uniformly in k. Letting $\varepsilon \downarrow 0$, this converges to 0.

We are left to investigate $B_{k,\delta,\varepsilon,\varepsilon'}$, for which we define, for each m, the measure $\mathbb{Q}_m = \mathbb{P}(\cdot \mid N_m > 0)$. Then, we can rewrite

$$
B_{k,\delta,\varepsilon,\varepsilon'} = n_k\theta_{(1-\delta)n_k}\mathbb{Q}_{(1-\delta)n_k}(N_{(1-\delta)n_k} > \varepsilon n_k, 0 < N_{n_k} \leq \varepsilon' n_k).
$$

Thus, since $n_k\theta_{(1-\delta)n_k}$ is bounded above by $\frac{C}{1-\delta} \leq 2C$ for $\delta < \frac{1}{2}$ (where C is independent of δ), proving that $\limsup_{\delta,\varepsilon,\varepsilon'\downarrow 0}\limsup_{k\to\infty} B_{k,\delta,\varepsilon,\varepsilon'} = 0$ is equivalent to proving that

$$
(3.8) \qquad \limsup_{\delta,\varepsilon,\varepsilon'\downarrow 0}\limsup_{k\to\infty} \mathbb{Q}_{(1-\delta)n_k}(N_{(1-\delta)n_k} > \varepsilon n_k, 0 < N_{n_k} \leq \varepsilon' n_k) = 0.
$$

To prove (3.8), we note that, for any integers $\ell_1, \ell_2 \geq 0$ such that $\ell_1+\ell_2 \geq 1$,

$$\begin{aligned}
(3.9) \qquad & \mathbb{E}_{\mathbb{Q}_{(1-\delta)n_k}}\Big[\Big(N_{(1-\delta)n_k}/n_k\Big)^{\ell_1}\Big(N_{n_k}/n_k\Big)^{\ell_2}\Big] \\
&= \frac{1}{\theta_{(1-\delta)n_k}}\mathbb{E}\Big[\Big(N_{(1-\delta)n_k}/n_k\Big)^{\ell_1}\Big(N_{n_k}/n_k\Big)^{\ell_2}\Big] \\
&= \frac{1}{n_k\theta_{(1-\delta)n_k}} n_k^{-(\ell_1+\ell_2-1)}\mathbb{E}[N^{\ell_1}_{(1-\delta)n_k}N^{\ell_2}_{n_k}] \\
&= \frac{1}{n_k\theta_{(1-\delta)n_k}} n_k^{-(\ell_1+\ell_2-1)}\hat{t}^{(\ell_1+\ell_2+1)}_{\vec{n}_k}(0),
\end{aligned}$$

where we use that $N_{(1-\delta)n_k} > 0$ when $N_{n_k} > 0$ and where $\vec{n}_k$ denotes a vector with precisely ℓ_1 coordinates equal to $(1-\delta)n_k$ and ℓ_2 coordinates equal to n_k. By Condition 1.1(a),

(3.10)

$$\begin{aligned}
n_k^{-(\ell_1+\ell_2-1)}\hat{t}^{(\ell_1+\ell_2+1)}_{\vec{n}_k}(0) &\to A(VA^2)^{\ell_1+\ell_2-1}\mathbb{E}_{\mathbb{N}_0}\left[X_{1-\delta}(1)^{\ell_1}X_1(1)^{\ell_2}\right] \\
&= \frac{2}{AV(1-\delta)}\mathbb{E}_{\mathbb{N}_0}\left[\Big(VA^2X_{1-\delta}(1)\Big)^{\ell_1}\Big(VA^2X_1(1)\Big)^{\ell_2}\Big| X_{1-\delta}(1)>0\right],
\end{aligned}$$

where the last equality follows from the fact that $\mathbb{N}_0(X_{1-\delta}(1) > 0) = 2/(1-\delta)$. Therefore, also using that $(1-\delta)n_k\theta_{(1-\delta)n_k} \to b_\delta$,

$$\begin{aligned}
(3.11) \qquad & \mathbb{E}_{\mathbb{Q}_{(1-\delta)n_k}}\Big[\Big(N_{(1-\delta)n_k}/n_k\Big)^{\ell_1}\Big(N_{n_k}/n_k\Big)^{\ell_2}\Big] \\
&\to \frac{2}{AVb_\delta}\mathbb{E}_{\mathbb{N}_0}\left[\Big(VA^2X_{1-\delta}(1)\Big)^{\ell_1}\Big(VA^2X_1(1)\Big)^{\ell_2}\Big| X_{1-\delta}(1)>0\right].
\end{aligned}$$

We recognize the above joint moments as the joint moments of (X,Y) with distribution $(1-\alpha_\delta)\delta_{(0,0)}+\alpha_\delta\nu_\delta$, where $\delta_{(0,0)}$ is the point measure on the vector $(0,0)$ and ν_δ is the law of $(A^2VX_{1-\delta}(1), A^2VX_1(1))$ under $\mathbb{N}_0(\cdot|X_{1-\delta}(1)>0)$, and with $\alpha_\delta = 2/(AVb_\delta) \in [0,1]$ (due to the lower bound (3.5)). For any $t > 1-\delta$,

$$(3.12) \qquad \mathbb{N}_0(X_t(1)=0|X_{1-\delta}(1)>0) = 1-(1-\delta)/t,$$

so that

$$(3.13) \qquad \nu_\delta(X_1(1)=0) = 1-(1-\delta) = \delta.$$

Let (X_n, Y_n) be a two-dimensional distribution. Again by [4, Ths. 30.1, 30.2 and Problems 30.5, 30.6], convergence of the joint moments of (X_n, Y_n) to those of (X, Y) implies convergence in distribution when the moment generating functions of both X and Y are finite in a neighborhood of 0. Under the conditional law $\mathbb{N}_0(\cdot|X_{1-\delta}(1) > 0)$, the distribution of $A^2VX_{1-\delta}(1)$ is exponential with mean $(1-\delta)A^2V/2$ (see, e.g., [28, Th. 1.4]), and by (3.13), $A^2VX_1(1)$ is 0 with probability δ and an exponential with mean $A^2V/2$ with probability $1-\delta$. As a result, the distribution of both limits X and Y are mixtures of point masses at 0 with probabilities $1-\alpha_\delta$ and $1-\alpha_\delta+\alpha_\delta\delta$ and exponentials with positive means λ_X and λ_Y. Therefore, their moment generating functions are finite in a neighborhood of zero, so that under $\mathbb{Q}_{(1-\delta)n_k}$, $\big(N_{(1-\delta)n_k}/n_k, N_{n_k}/n_k\big)$ converges in distribution to (X,Y) having distribution $(1-\alpha_\delta)\delta_{(0,0)}+\alpha_\delta\nu_\delta$.

Thus, as $k\to\infty$,

$$\begin{aligned}&\mathbb{Q}_{(1-\delta)n_k}(N_{(1-\delta)n_k}>\varepsilon n_k, N_{n_k}\le\varepsilon' n_k)\\&\qquad\qquad\to\alpha_\delta\nu_\delta(A^2VX_{1-\delta}(1)>\varepsilon, A^2VX_1(1)\le\varepsilon').\end{aligned}$$

When $\varepsilon'\downarrow 0$,

$$\begin{aligned}\nu_\delta(A^2VX_{1-\delta}(1)>\varepsilon, A^2VX_1(1)\le\varepsilon')&\to\nu_\delta(X_{1-\delta}(1)>\varepsilon c, X_1(1)=0)\\&\le\nu_\delta(X_1(1)=0)=\delta,\end{aligned}\tag{3.14}$$

where we use (3.13). Letting $\delta\downarrow 0$, we obtain (3.8). We conclude that $\limsup_{n\to\infty} n\theta_n=\bar b\le 2/(AV)$, which, together with (3.5) and as required, shows that $\lim_{n\to\infty} n\theta_n=2/(AV)$.

The fact that, conditionally on $X^{(n)}_t(1)>0$, the finite-dimensional distributions of $(X^{(n)}_s(1))_{s\ge0}$ converge to those of the total mass of CSBM conditionally on $X_t(1)>0$ can be obtained as follows. Again by [4, Ths. 30.1, 30.2 and Problems 30.5, 30.6], for the convergence under the conditional measures it is enough to show that for $\ell\ge0$, $\vec s\in[0,\infty)^\ell$ and $t>0$,

$$\mathbb{E}_{\mu_n}\left[\prod_{i=1}^{\ell}X_{s_i}(1)\Big|S>t\right]\to\mathbb{E}_{\mathbb{N}_0}\left[\prod_{i=1}^{\ell}X_{s_i}(1)\Big|S>t\right].\tag{3.15}$$

Since we have already shown convergence of the survival probabilities, it is sufficient to show that

$$\Big|\mathbb{E}_{\mu_n}[Y_{\vec s}\mathbb{1}_{\{S>t\}}]-\mathbb{E}_{\mathbb{N}_0}[Y_{\vec s}\mathbb{1}_{\{S>t\}}]\Big|\to 0,\tag{3.16}$$

where $Y_{\vec s}=\prod_{i=1}^{\ell}X_{s_i}(1)$ and $\ell>0$, and trivially this implies the unconditional version of the statement as well. Finally, (3.16) can be proved exactly as in [37, proof of Prop. 2.4] □

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