Optimal asymptotic bounds for spherical designs

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Abstract

In this paper we prove the conjecture of Korevaar and Meyers: for each \( N \geq c_d t \), there exists a spherical \( t \)-design in the sphere \( S^d \) consisting of \( N \) points, where \( c_d \) is a constant depending only on \( d \).

1. Introduction

Let \( S^d \) be the unit sphere in \( \mathbb{R}^{d+1} \) with the Lebesgue measure \( \mu_d \) normalized by \( \mu_d(S^d) = 1 \).

A set of points \( x_1, \ldots, x_N \in S^d \) is called a spherical \( t \)-design if

\[
\int_{S^d} P(x) \, d\mu_d(x) = \frac{1}{N} \sum_{i=1}^{N} P(x_i)
\]

for all polynomials in \( d+1 \) variables, of total degree at most \( t \). The concept of a spherical design was introduced by Delsarte, Goethals, and Seidel [12]. For each \( t, d \in \mathbb{N} \), denote by \( N(d, t) \) the minimal number of points in a spherical \( t \)-design in \( S^d \). The following lower bound,

\[
N(d, t) \geq \begin{cases} 
\binom{d+k}{d} + \binom{d+k-1}{d} & \text{if } t = 2k, \\
2 \binom{d+k}{d} & \text{if } t = 2k + 1,
\end{cases}
\]

is proved in [12].

Spherical \( t \)-designs attaining this bound are called tight. The vertices of a regular \( t+1 \)-gon form a tight spherical \( t \)-design in the circle, so \( N(1, t) = t + 1 \). Exactly eight tight spherical designs are known for \( d \geq 2 \) and \( t \geq 4 \). All such configurations of points are highly symmetrical, and optimal from many © 2013 Department of Mathematics, Princeton University.
different points of view (see Cohn, Kumar [10] and Conway, Sloane [11]). Unfortunately, tight designs rarely exist. In particular, Bannai and Damerell [2], [3] have shown that tight spherical designs with \( d \geq 2 \) and \( t \geq 4 \) may exist only for \( t = 4, 5, 7, \) or \( 11 \). Moreover, the only tight 11-design is formed by minimal vectors of the Leech lattice in dimension 24. The bound (1) has been improved by Delsarte’s linear programming method for most pairs \((d, t)\); see [22].

On the other hand, Seymour and Zaslavsky [20] have proved that spherical \( t \)-designs exist for all \( d, t \in \mathbb{N} \). However, this proof is nonconstructive and gives no idea of how big \( N(d, t) \) is. So, a natural question is to ask how \( N(d, t) \) differs from bound (1). Generally, to find the exact value of \( N(d, t) \) even for small \( d \) and \( t \) is a surprisingly hard problem. For example, everybody believes that 24 minimal vectors of the \( D_4 \) root lattice form a 5-design with minimal number of points in \( S^3 \), although it is only proved that \( 22 \leq N(3, 5) \leq 24 \); see [6]. Further, Cohn, Conway, Elkies, and Kumar [9] conjectured that every spherical 5-design consisting of 24 points in \( S^3 \) is in a certain 3-parametric family. Recently, Musin [17] has solved a long standing problem related to this conjecture. Namely, he proved that the kissing number in dimension 4 is 24.

In this paper we focus on asymptotic upper bounds on \( N(d, t) \) for fixed \( d \geq 2 \) and \( t \to \infty \). Let us give a brief history of this question. First, Wagner [21] and Bajnok [1] proved that \( N(d, t) \leq C_d t^{Cd^3} \) and \( N(d, t) \leq C_d t^{Cd^3} \), respectively. Then, Korevaar and Meyers [14] have improved these inequalities by showing that \( N(d, t) \leq C_d t^{(d^2+d)/2} \). They have also conjectured that

\[
N(d, t) \leq C_d t^d.
\]

Note that (1) implies \( N(d, t) \geq c_d t^d \). Here and in what follows we denote by \( C_d \) and \( c_d \) sufficiently large and sufficiently small positive constants depending only on \( d \), respectively.

The conjecture of Korevaar and Meyers attracted the interest of many mathematicians. For instance, Kuijlaars and Saff [19] emphasized the importance of this conjecture for \( d = 2 \) and revealed its relation to minimal energy problems. Mhaskar, Narcowich, and Ward [16] have constructed positive quadrature formulas in \( S^d \) with \( C_d t^d \) points having almost equal weights. Very recently, Chen, Frommer, Lang, Sloan, and Womersley [7], [8] gave a computer-assisted proof that spherical \( t \)-designs with \((t+1)^2\) points exist in \( S^2 \) for \( t \leq 100 \).

For \( d = 2 \), there is an even stronger conjecture by Hardin and Sloane [13] saying that \( N(2, t) \leq \frac{1}{2} t^2 + o(t^2) \) as \( t \to \infty \). Numerical evidence supporting the conjecture was also given.

In [4], we have suggested a nonconstructive approach for obtaining asymptotic bounds for \( N(d, t) \) based on the application of the Brouwer fixed point theorem. This led to the following result:
For each \( N \geq C_d t^{2d/(d+2)} \), there exists a spherical \( t \)-design in \( S^d \) consisting of \( N \) points.

Instead of the Brouwer fixed point theorem, in this paper we use the following result from the Brouwer degree theory [18, Ths. 1.2.6 and 1.2.9].

**Theorem A.** Let \( f : \mathbb{R}^n \to \mathbb{R}^n \) be a continuous mapping and \( \Omega \) an open bounded subset, with boundary \( \partial \Omega \), such that \( 0 \in \Omega \subset \mathbb{R}^n \). If \( \langle x, f(x) \rangle > 0 \) for all \( x \in \partial \Omega \), then there exists \( x \in \Omega \) satisfying \( f(x) = 0 \).

We employ this theorem to prove the conjecture of Korevaar and Meyers.

**Theorem 1.** For each \( N \geq C_d t^{d} \), there exists a spherical \( t \)-design in \( S^d \) consisting of \( N \) points.

Note that Theorem 1 is slightly stronger than the original conjecture because it guarantees the existence of spherical \( t \)-designs for each \( N \) greater than \( C_d t^{d} \).

This paper is organized as follows. In Section 2 we explain the main idea of the proof. Then in Section 3 we present some auxiliary results. Finally, we prove Theorem 1 in Section 4.

### 2. Preliminaries and the main idea

Let \( \mathcal{P}_t \) be the Hilbert space of polynomials \( P \) on \( S^d \) of degree at most \( t \) such that
\[
\int_{S^d} P(x) d\mu_d(x) = 0,
\]
equipped with the usual inner product
\[
\langle P, Q \rangle = \int_{S^d} P(x) Q(x) d\mu_d(x).
\]
By the Riesz representation theorem, for each point \( x \in S^d \), there exists a unique polynomial \( G_x \in \mathcal{P}_t \) such that
\[
\langle G_x, Q \rangle = Q(x) \quad \text{for all} \quad Q \in \mathcal{P}_t.
\]
Then a set of points \( x_1, \ldots, x_N \in S^d \) forms a spherical \( t \)-design if and only if
\[
(2) \quad G_{x_1} + \cdots + G_{x_N} = 0.
\]

The gradient of a differentiable function \( f : \mathbb{R}^{d+1} \to \mathbb{R} \) is denoted by
\[
\frac{\partial f}{\partial x} := \left( \frac{\partial f}{\partial \xi_1}, \ldots, \frac{\partial f}{\partial \xi_{d+1}} \right), \quad x = (\xi_1, \ldots, \xi_{d+1}).
\]
For a polynomial \( Q \in \mathcal{P}_t \), we define the spherical gradient
\[
(3) \quad \nabla Q(x) := \frac{\partial}{\partial x} \left( Q \left( \frac{x}{|x|} \right) \right),
\]
where \( | \cdot | \) is the Euclidean norm in \( \mathbb{R}^{d+1} \).
We apply Theorem A to the open subset $\Omega$ of a vector space $\mathcal{P}_t$:

$$\Omega := \left\{ P \in \mathcal{P}_t \left| \int_{S^d} |\nabla P(x)| d\mu_d(x) < 1 \right. \right\}.$$  

Now we observe that the existence of a continuous mapping $F : \mathcal{P}_t \to (S^d)^N$, such that for all $P \in \partial \Omega$

$$\sum_{i=1}^{N} P(x_i(P)) > 0, \text{ where } F(P) = (x_1(P), \ldots, x_N(P)),$$

readily implies the existence of a spherical $t$-design in $S^d$ consisting of $N$ points. Indeed, consider a mapping $L : (S^d)^N \to \mathcal{P}_t$ defined by

$$(x_1, \ldots, x_N) \xrightarrow{L} G_{x_1} + \cdots + G_{x_N},$$

and the following composition mapping $f = L \circ F : \mathcal{P}_t \to \mathcal{P}_t$. Clearly

$$\langle P, f(P) \rangle = \sum_{i=1}^{N} P(x_i(P))$$

for each $P \in \mathcal{P}_t$. Thus, applying Theorem A to the mapping $f$, the vector space $\mathcal{P}_t$, and the subset $\Omega$ defined by (4), we obtain that $f(Q) = 0$ for some $Q \in \partial \Omega$. Hence, by (2), the components of $F(Q) = (x_1(Q), \ldots, x_N(Q))$ form a spherical $t$-design in $S^d$ consisting of $N$ points.

The most naive approach to construct such $F$ is to start with a certain well-distributed collection of points $x_i (i = 1, \ldots, N)$, put $F(0) := (x_1, \ldots, x_N)$, and then move each point along the spherical gradient vector field of $P$. Note that this is the most greedy way to increase each $P(x_i(P))$ and make $\sum_{i=1}^{N} P(x_i(P))$ positive for each $P \in \partial \Omega$. Following this approach we will give an explicit construction of $F$ in Section 4, which will immediately imply the proof of Theorem 1.

3. Auxiliary results

To construct the corresponding mapping $F$ for each $N \geq C_d t^d$, we extensively use the following notion of an area-regular partition.

Let $\mathcal{R} = \{R_1, \ldots, R_N\}$ be a finite collection of closed sets $R_i \subset S^d$ such that $\bigcup_{i=1}^{N} R_i = S^d$ and $\mu_d(R_i \cap R_j) = 0$ for all $1 \leq i < j \leq N$. The partition $\mathcal{R}$ is called area-regular if $\mu_d(R_i) = 1/N, i = 1, \ldots, N$. The partition norm for $\mathcal{R}$ is defined by

$$\|\mathcal{R}\| := \max_{R \in \mathcal{R}} \text{diam } R,$$

where diam $R$ stands for the maximum geodesic distance between two points in $R$. We need the following fact on area-regular partitions (see Bourgain, Lindenstrauss [5] and Kuijlaars, Saff [15]).
Theorem B. For each $N \in \mathbb{N}$, there exists an area-regular partition $\mathcal{R} = \{R_1, \ldots, R_N\}$ with $\|\mathcal{R}\| \leq B_d N^{-1/d}$ for some constant $B_d$ large enough.

We will also use a result that is an easy corollary of Theorem 3.1 in [16].

Theorem C. There exists a constant $r_d$ such that for each area-regular partition $\mathcal{R} = \{R_1, \ldots, R_N\}$ with $\|\mathcal{R}\| < \frac{r_d}{m}$, each collection of points $x_i \in R_i$ ($i = 1, \ldots, N$), and each polynomial $P$ of total degree $m$,

\[
\frac{1}{2} \int_{S^d} |P(x)| \, d\mu_d(x) \leq \frac{1}{N} \sum_{i=1}^N |P(x_i)| \leq \frac{3}{2} \int_{S^d} |P(x)| \, d\mu_d(x)
\]

holds.

Theorem 3.1 in [16] was stated for slightly different definition of an area-regular partition. Namely, it was additionally assumed that each $R_i$ is a spherical region. However the proof clearly works for our more general definition as well; see [16, §3.3].

Corollary 1. For each area-regular partition $\mathcal{R} = \{R_1, \ldots, R_N\}$ with $\|\mathcal{R}\| < \frac{r_d}{m+1}$, each collection of points $x_i \in R_i$ ($i = 1, \ldots, N$), and each polynomial $P$ of total degree $m$,

\[
\frac{1}{3\sqrt{d}} \int_{S^d} |\nabla P(x)| \, d\mu_d(x) \leq \frac{1}{N} \sum_{i=1}^N |\nabla P(x_i)| \leq 3\sqrt{d} \int_{S^d} |\nabla P(x)| \, d\mu_d(x).
\]

Proof. For a point $x = (\xi_1, \ldots, \xi_{d+1}) \in S^d$, we get by (3) that

\[
|\nabla P(x)| = \sqrt{P_1^2(x) + \cdots + P_{d+1}^2(x)}
\]

where

\[
P_j(x) := \frac{\partial P}{\partial \xi_j}(x) - \sum_{k=1}^{d+1} \xi_j \xi_k \frac{\partial P}{\partial \xi_k}(x)
\]

are polynomials of total degree at most $m + 1$. Thus, using a simple inequality

\[
\frac{1}{\sqrt{d+1}} \sum_{k=1}^{d+1} |P_k(x_i)| \leq \sqrt{\frac{1}{d+1} \sum_{k=1}^{d+1} P_k^2(x_i)} \leq \sum_{k=1}^{d+1} |P_k(x_i)|
\]

and then applying (6) to polynomials $P_k$, we obtain the statement of the corollary. \hfill \Box

4. Proof of Theorem 1

In this section we construct the map $F$ introduced in Section 2 and thereby finish the proof of Theorem 1.

For $d, t \in \mathbb{N}$, take $C_d > (54dB_d/r_d)^d$, where $B_d$ is as in Theorem B and $r_d$ is as in Theorem C, and fix $N \geq C_d t^d$. Now we are in a position to give an exact
construction of the mapping $F : \mathcal{P}_t \to (S^d)^N$, which satisfies condition (5).

Take an area-regular partition $\mathcal{R} = \{R_1, \ldots, R_N\}$ with

$$\|\mathcal{R}\| \leq B_d N^{-1/d} < \frac{r_d}{54d}$$

as provided by Theorem B, and choose an arbitrary $x_i \in R_i$ for each $i = 1, \ldots, N$. Put $\varepsilon = \frac{1}{6\sqrt{d}}$, and consider the function

$$h_\varepsilon(u) := \begin{cases} 
  u & \text{if } u > \varepsilon, \\
  \varepsilon & \text{otherwise}.
\end{cases}$$

Take a mapping $U : \mathcal{P}_t \times S^d \to \mathbb{R}^{d+1}$ such that

$$U(P, y) = \frac{\nabla P(y)}{h_\varepsilon(|\nabla P(y)|)}.$$ 

For each $i = 1, \ldots, N$, let $y_i : \mathcal{P}_t \times [0, \infty) \to S^d$ be the map satisfying the differential equation

$$\frac{d}{ds} y_i(P, s) = U(P, y_i(P, s))$$

with the initial condition

$$y_i(P, 0) = x_i$$

for each $P \in \mathcal{P}_t$. Note that each mapping $y_i$ has its values in $S^d$ by definition of spherical gradient (3). Since the mapping $U(P, y)$ is Lipschitz continuous in both $P$ and $y$, each $y_i$ is well defined and continuous in both $P$ and $s$, where the metric on $\mathcal{P}_t$ is given by the inner product. Finally, put

$$F(P) = (x_1(P), \ldots, x_N(P)) := \left( y_1(P, \frac{r_d}{3t}), \ldots, y_N(P, \frac{r_d}{3t}) \right).$$

By definition, the mapping $F$ is continuous on $\mathcal{P}_t$. So, as explained in Section 2, to finish the proof of Theorem 1 it suffices to prove

**Lemma 1.** Let $F : \mathcal{P}_t \to (S^d)^N$ be the mapping defined by (10). Then for each $P \in \partial \Omega$,

$$\frac{1}{N} \sum_{i=1}^{N} P(x_i(P)) > 0,$$

where $\Omega$ is given by (4).

**Proof.** Fix $P \in \partial \Omega$; that is,

$$\int_{S^d} |\nabla P(x)| d\mu_d(x) = 1.$$
For the sake of simplicity, we write $y_i(s)$ in place of $y_i(P, s)$. By the Newton-Leibniz formula, we have

\begin{equation}
\frac{1}{N} \sum_{i=1}^{N} P(x_i(P)) = \frac{1}{N} \sum_{i=1}^{N} P(y_i(r_d/3t))
= \frac{1}{N} \sum_{i=1}^{N} P(x_i) + \int_{0}^{r_d/3t} \frac{d}{ds} \left[ \frac{1}{N} \sum_{i=1}^{N} P(y_i(s)) \right] ds.
\end{equation}

Now to prove Lemma 1, we first estimate the value

\[ \left| \frac{1}{N} \sum_{i=1}^{N} P(x_i) \right| \]

from above and then estimate the value

\[ \frac{d}{ds} \left[ \frac{1}{N} \sum_{i=1}^{N} P(y_i(s)) \right] \]

from below for each $s \in [0, r_d/3t]$. We have

\[ \left| \frac{1}{N} \sum_{i=1}^{N} P(x_i) \right| = \sum_{i=1}^{N} \int_{R_i} P(x_i) - P(x) \, d\mu_d(x) \leq \sum_{i=1}^{N} \int_{R_i} |P(x_i) - P(x)| \, d\mu_d(x) \]

\[ \leq \|\mathcal{R}\| \sum_{i=1}^{N} \max_{z \in S^d, \text{dist}(z, x_i) \leq \|\mathcal{R}\|} |\nabla P(z)|, \]

where dist$(z, x_i)$ denotes the geodesic distance between $z$ and $x_i$. Hence, for $z_i \in S^d$ such that dist$(z_i, x_i) \leq \|\mathcal{R}\|$ and

\[ |\nabla P(z_i)| = \max_{z \in S^d, \text{dist}(z, x_i) \leq \|\mathcal{R}\|} |\nabla P(z)|, \]

we obtain

\[ \left| \frac{1}{N} \sum_{i=1}^{N} P(x_i) \right| \leq \frac{\|\mathcal{R}\|}{N} \sum_{i=1}^{N} |\nabla P(z_i)|. \]

Consider another area-regular partition $\mathcal{R}' = \{R'_1, \ldots, R'_N\}$ defined by $R'_i = R_i \cup \{z_i\}$. Clearly $\|\mathcal{R}'\| \leq 2\|\mathcal{R}\|$ and so, by (8), we get $\|\mathcal{R}'\| < r_d/(27d t)$. Applying inequality (7) to the partition $\mathcal{R}'$ and the collection of points $z_i$, we obtain that

\begin{equation}
\left| \frac{1}{N} \sum_{i=1}^{N} P(x_i) \right| \leq 3\sqrt{d} \|\mathcal{R}\| \int_{S^d} |\nabla P(x)| \, d\mu_d(x) < \frac{r_d}{18 \sqrt{d} t}.
\end{equation}
for any $P \in \partial \Omega$. On the other hand, the differential equation (9) implies

\[
\frac{d}{ds} \left[ \frac{1}{N} \sum_{i=1}^{N} P(y_i(s)) \right] = \frac{1}{N} \sum_{i=1}^{N} \frac{|\nabla P(y_i(s))|^2}{h_\varepsilon(|\nabla P(y_i(s))|)} \geq \frac{1}{N} \sum_{i: |\nabla P(y_i(s))| \geq \varepsilon} |\nabla P(y_i(s))| \geq \frac{1}{N} \sum_{i=1}^{N} |\nabla P(y_i(s))| - \varepsilon.
\]

Since

\[
\frac{\nabla P(y)}{h_\varepsilon(|\nabla P(y)|)} \leq 1
\]

for each $y \in S^d$, it follows again from (9) that $\left| \frac{dy_i(s)}{ds} \right| \leq 1$. Hence we arrive at

\[
\text{dist}(x_i, y_i(s)) \leq s.
\]

Now for each $s \in [0, r_d/3t]$, we consider the area-regular partition $\mathcal{R}'' = \{R''_1, \ldots, R''_N\}$ given by $R''_i = R_i \cup \{y_i(s)\}$. By (8), we have

\[
\|\mathcal{R}''\| < \frac{r_d}{54dt} + \frac{r_d}{3t},
\]

so we can apply (7) to the partition $\mathcal{R}''$ and the collection of points $y_i(s)$. This and inequality (13) yield

\[
\frac{d}{ds} \left[ \frac{1}{N} \sum_{i=1}^{N} P(y_i(s)) \right] \geq \frac{1}{N} \sum_{i=1}^{N} |\nabla P(y_i(s))| - \frac{1}{6\sqrt{d}} \geq \frac{1}{3\sqrt{d}} \int_{S^d} |\nabla P(x)| d\mu_d(x) - \frac{1}{6\sqrt{d}} = \frac{1}{6\sqrt{d}}
\]

for each $P \in \partial \Omega$ and $s \in [0, r_d/3t]$. Finally, equation (11) and inequalities (12) and (14) imply

\[
\frac{1}{N} \sum_{i=1}^{N} P(x_i(P)) > \frac{1}{6\sqrt{d}} \frac{r_d}{3t} - \frac{r_d}{18 \sqrt{d} t} = 0.
\]

Lemma 1 is proved. □

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