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A class of superrigid group von Neumann algebras

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Abstract

We prove that for any group G in a fairly large class of generalized wreath product groups, the associated von Neumann algebra $\mathrm{L}G$ completely "remembers" the group G. More precisely, if $\mathrm{L}G$ is isomorphic to the von Neumann algebra $\mathrm{L}\Lambda$ of an arbitrary countable group Λ, then Λ must be isomorphic to G. This represents the first superrigidity result pertaining to group von Neumann algebras.

1. Introduction and statement of main results

A countable discrete group G gives rise to a variety of rings and algebras, studied in several areas of mathematics, such as algebra, finite group theory, geometric group theory, representation theory, noncommutative geometry, C*- and von Neumann operator algebras. A common underlying theme is the investigation of how the isomorphism class of the ring/algebra depends on the group G.

Thus, by letting the (complex) group algebra $\mathbb{C}G$ act on the Hilbert space $\ell^2 G$ by (left) convolution and then taking its closure in the operator norm, one obtains the *reduced group* C*-*algebra* $\mathrm{C}^*_{\mathrm{r}}G$, an important object of study in noncommutative geometry (e.g., related to the Novikov conjecture; see [Con94]). In turn, by taking the closure of $\mathbb{C}G$ in the weak operator topology one obtains the *group von Neumann algebra* $\mathrm{L}G$, introduced and studied by Murray and von Neumann in [MvN36], [MvN43].

When passing from $\mathbb{C}G$ to $\mathrm{L}G$, the memory of G tends to fade away. This is best seen in the torsion free abelian case, where $\mathbb{C}G$ remembers G completely

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(see, e.g., [Hig40]), while all $\mathrm{L}G$ are isomorphic (because $\mathrm{L}G = \mathrm{L}^\infty(\hat{G}) \cong \mathrm{L}^\infty([0,1])$). Conjecturally, if G is an arbitrary torsion free group, then the only unitary elements in $\mathbb{C}G$ are the multiples of the canonical unitaries $(u_g)_{g\in G}$. (See [Hig40], [Kap70], where in fact the conjecture was checked for all orderable groups.) On the other hand, the weak closure $\overline{\mathbb{C}G}^w = \mathrm{L}G$ entirely wipes out this structure. The intermediate case $\mathrm{C}^*_\mathrm{r}G$ appears to be closer to $\mathbb{C}G$ than to $\mathrm{L}G$. Indeed, if G is abelian torsion free, then the group of connected components of $\mathcal{U}(\mathrm{C}^*_\mathrm{r}G)$ coincides with G so that $\mathrm{C}^*_\mathrm{r}G$ completely remembers G. (This is obvious when $G = \mathbb{Z}^n$ and passes to inductive limits $\mathbb{Z}^{n_1} \hookrightarrow \mathbb{Z}^{n_2} \hookrightarrow \cdots$.) The noncommutative case is very poorly understood. It seems not even known whether $\mathrm{C}^*_\mathrm{r}G$ always remembers a torsion free group G. This question is particularly interesting for free groups, $G = \mathbb{F}_n$, where a result in [PV82] already shows that $\mathrm{C}^*_\mathrm{r}\mathbb{F}_n$ are nonisomorphic for different n's. In fact, when combined with results in [DHR97], [Rie87], if follows that the group of connected components of $\mathcal{U}(\mathrm{C}^*_\mathrm{r}\mathbb{F}_n)$ is isomorphic to $\mathbb{Z}^n$.

For von Neumann algebras, the really interesting case is when $\mathrm{L}G$ has trivial center, i.e., when $\mathrm{L}G$ is a II_1 factor, corresponding to G having infinite conjugacy classes (icc); see [MvN43]. Here again, like in the abelian case, a celebrated result of Connes [Con76] shows that all II_1 factors coming from icc amenable groups are isomorphic to the *hyperfinite* II_1 factor of Murray and von Neumann. While nonamenable groups G were known since [MvN43], [Sch63] to produce nonhyperfinite factors $\mathrm{L}G$ and an uncountable family of icc groups with the associated II_1 factors nonisomorphic was constructed in [McD69], very little is known of how $\mathrm{L}G$ depends on the group G, especially when G is a "classical" group like $\mathrm{SL}(n,\mathbb{Z})$, or a free group $\mathbb{F}_n$. For instance, it is a famous open problem whether the factors $\mathrm{L}\mathbb{F}_n$, $n \geqslant 2$, are nonisomorphic. In the same vein, a well-known conjecture of Connes [Con82] asks whether $\mathrm{L}G \cong \mathrm{L}\Lambda$ for icc property (T) groups G, Λ implies $G \cong \Lambda$. This conjecture remains wide open, notably for $G = \mathrm{SL}(n,\mathbb{Z})$, $n \geqslant 3$. Note however that by [CH89], if G, Λ are lattices in $\mathrm{Sp}(n,1)$, respectively $\mathrm{Sp}(m,1)$, then $\mathrm{L}G \cong \mathrm{L}\Lambda$ implies $n = m$. Along these lines, several recent results in deformation rigidity theory provide classes of groups $\mathcal{G}$ for which any isomorphism $\mathrm{L}G \simeq \mathrm{L}\Lambda$, with $G, \Lambda \in \mathcal{G}$, entails isomorphism of the groups $G \simeq \Lambda$ (see, e.g., [Pop06a], [Pop06d], [IPP08], [PV08], etc.). This is, for instance, the case for the class $\mathcal{G}$ of all wreath product groups $\mathbb{Z}/2\mathbb{Z} \wr \Gamma$ with Γ having property (T) [Pop06d]. At the opposite end, using [Con76] and free probability it has been shown that $\mathrm{L}(\Gamma_1 * \Gamma_2 * \cdots * \Gamma_n) \simeq \mathrm{L}\mathbb{F}_n$ for any infinite amenable groups Γ_i and $n \geqslant 2$; see [Dyk93]. Other unexpected isomorphisms between group factors can be found in Section 9.

In fact, more than just distinguishing between property (T) group factors, a positive answer to Connes' rigidity conjecture implies that the II_1 factor $\mathrm{L}G$

of an icc property (T) group G uniquely determines the group G. Indeed, by [CJ85], if $\mathrm{L}G \simeq \mathrm{L}\Lambda$ and G has property (T), then Λ automatically has this property, showing that in Connes' conjecture it is sufficient to assume property (T) only on the group G. This gives its statement a W*-superrigidity flavor, in the same spirit as the recent superrigidity results for group measure space II_1 factors ([PV10], [Ioa11]), showing that certain classes of free ergodic probability measure preserving group actions $G \curvearrowright (X, \mu)$ can be completely recovered from their associated II_1 factors $\mathrm{L}^\infty(X) \rtimes G$.

However, the superrigidity question for group factors is much harder, and all this progress in group measure space factors could not be exploited to obtain even one single example of a W*-*superrigid* icc *group* G, i.e., for which $\mathrm{L}G$ completely remembers G, in the sense that any isomorphism of $\mathrm{L}G$ and an arbitrary group factor $\mathrm{L}\Lambda$ forces the groups G, Λ to be isomorphic.

In this paper we provide a large class of generalized wreath product groups G that are W*-superrigid. For instance, we show that given ANY nonamenable group Γ, its canonical "augmentation" $G = (Z/2\mathbb{Z})^{(I)} \rtimes (\Gamma \wr \mathbb{Z})$ is superrigid, where the set I is the quotient $(\Gamma \wr \mathbb{Z})/\mathbb{Z}$ on which the group $\Gamma \wr \mathbb{Z} = \Gamma^{(\mathbb{Z})} \rtimes \mathbb{Z}$ acts by left multiplication. In fact, we show that any isomorphism between $\mathrm{L}G$ and an arbitrary group factor $\mathrm{L}\Lambda$ is implemented by an isomorphism of the groups. More precisely, we prove the following general result.

THEOREM 1.1. *Let Γ_0 be any nonamenable group, and let S be any infinite amenable group. Define the wreath product group $\Gamma = \Gamma_0^{(S)} \rtimes S$, and consider the action of Γ on $I = \Gamma/S$ by left multiplication. Let n be a square-free integer, and define the generalized wreath product group*

$$G = \left(\tfrac{\mathbb{Z}}{n\mathbb{Z}}\right)^{(I)} \rtimes \Gamma.$$

If Λ is any countable group and $\pi : \mathrm{L}\Lambda \to \mathrm{L}(G)^t$ is a surjective $*$-isomorphism for some $t > 0$, then $t = 1$ and $\Lambda \cong G$.

In the special case where $n = 2, 3$, the $*$-isomorphism π is necessarily group-like: there exists an isomorphism of groups $\delta : \Lambda \to G$, a character $\omega : \Lambda \to \mathbb{T}$ and a unitary $w \in \mathrm{L}G$ such that

$$\pi(v_s) = \omega(s)\, w\, u_{\delta(s)}\, w^* \quad \forall s \in \Lambda.$$

Here $(v_s)_{s\in\Lambda}$ and $(u_g)_{g\in G}$ denote the canonical generating unitaries of $\mathrm{L}\Lambda$, respectively $\mathrm{L}G$.

Theorem 8.3 below provides a much wider class of generalized wreath product groups $G = \frac{\mathbb{Z}}{n\mathbb{Z}} \wr_I \Gamma$ such that the group factor $\mathrm{L}G$ remembers the group G.

The conclusions of Theorem 1.1 do not hold, however, for plain wreath products $G = \frac{\mathbb{Z}}{n\mathbb{Z}} \wr \Gamma$. Nevertheless, we will see in Theorem 8.2 that it is still possible to describe more or less explicitly all groups Λ with $\mathrm{L}G \cong \mathrm{L}\Lambda$. But this description does not allow us to classify these groups Λ up to isomorphism. The groups Λ with $\mathrm{L}G \cong \mathrm{L}\Lambda$ can be quite different from G, as illustrated by the following result that we prove in Section 9.

THEOREM 1.2. *Let Γ be a nontrivial torsion free group, and let H_0 be a nontrivial finite abelian group. Then, there exists a torsion free group Λ with $\mathrm{L}(\Lambda) \cong \mathrm{L}(H_0 \wr \Gamma)$. In particular, $\Lambda \not\cong H_0 \wr \Gamma$.*

Let $n \geqslant 2$, and let H_0 be a nontrivial finite abelian group. There are infinitely many nonisomorphic groups Λ for which $\mathrm{L}\Lambda \cong \mathrm{L}(H_0 \wr \mathrm{PSL}(n, \mathbb{Z}))$.

We mention that in the final Section 10 we show that some of our methods allow us to extend [Ioa11, Th. A] and prove W*-superrigidity for Bernoulli actions $\Gamma \curvearrowright (X_0, \mu_0)^\Gamma$ of groups Γ that admit an infinite normal subgroup with nonamenable centralizer. We refer to Theorem 10.1 for a precise statement.

Structure of the article and comments on the proofs. The fact that large classes of generalized wreath product groups turn out to be W*-superrigid should come as no surprise, since such groups have been recognized for some time to be "exceptionally rigid" in the von Neumann algebra context (cf. [Pop06b], [Pop06c], [Pop06d], [Pop08], [PV08], [PV10], [Ioa07], [CI10], [Ioa11]). This is due to the Bernoulli type crossed-product decomposition that a wreath product group has, $G = H \wr_I \Gamma = H^{(I)} \rtimes \Gamma$, a feature that makes its associated von Neumann algebra $M = \mathrm{L}G$ "distinctly soft" on the side of $\mathrm{L}H^{(I)} \subset M$, once H is assumed amenable. Such softness is a consequence of the *malleable deformations* that II_1 factors arising from Bernoulli actions were shown to have ([Pop06b], [Pop06c]). This property allows the recovery of all "rigid parts" of $\mathrm{L}\Gamma$, such as subalgebras generated by subgroups $\Gamma_0 \subset \Gamma$ having either relative property (T), or nonamenable centralizer. Playing rigidity against deformability properties of an algebra in this manner became a paradigm of deformation/rigidity theory (see [Pop06b], [Pop06c], [Pop06d], [Pop08], [Pop07]). Then in [PV08] it was realized that if $\Gamma \curvearrowright I$ is of the form $\Gamma \curvearrowright \Gamma/\Gamma_0$, with Γ_0 a "malnormal" subgroup of Γ, the overall rigidity of M can be considerably enhanced, while the discovery in [Ioa07] of a new malleable deformation for generalized Bernoulli actions and wreath product groups unraveled more of their rigidity properties. The recent work in [PV10], [Ioa11] pushed the deformation/rigidity analysis of such group actions even deeper, notably through the systematic usage of "comultiplication"-type embeddings $\Delta : M \hookrightarrow M \overline{\otimes} M$ in [Ioa11] (cf. also [PV10]).

In order to prove Theorem 1.1 we use the entire arsenal of ideas and techniques developed in these previous papers. Yet recovering the discrete structure $G = H \wr_I \Gamma$ (rather than the action $\Gamma \curvearrowright \mathrm{L}H^{(I)}$, as in [Pop06d], [PV10], [Ioa11]) inside the algebra $\mathrm{L}G$ requires more intricate deformation/rigidity arguments and a lot of technical effort. This work, which takes the entire Sections 4 through 8, leads us to a crucial correlation between G and any other group implementing the same von Neumann algebra. More precisely, we show that the comultiplication on $\mathrm{L}G$ induced by an arbitrary group $\Lambda \subset \mathrm{L}G$ satisfying $\mathrm{L}\Lambda = \mathrm{L}G$ is unitarily conjugate to the initial comultiplication induced by G, with the corresponding unitary satisfying a "dual" 2-cocycle relation.[1] One of the big novelties in this paper is how we derive an isomorphism of the groups G, Λ out of this 2-cocycle. We do this in Theorem 3.3, which is essentially a vanishing of 2-cohomology result.

One should mention that a particular case of this result, which we emphasize separately as Theorem 3.1, provides a surprising characterization for the unitary conjugacy of arbitrary icc groups Λ, G giving the same II_1 factor, $\mathrm{L}\Lambda = \mathrm{L}G$. To state it, we use the (asymmetric) Hausdorff distance between subgroups $\mathcal{U}$ and $\mathcal{V}$ of the unitary group $\mathcal{U}(M)$ of a II_1 factor, defined by

$$\mathrm{dist}_{\|\cdot\|_2}(\mathcal{U}, \mathcal{V}) := \sup_{u \in \mathcal{U}} \Big(\inf_{v \in \mathcal{V}} \|u - v\|_2 \Big).$$

Denote by $\mathbb{T}\mathcal{U}$ the group of unitaries λu, $\lambda \in \mathbb{T}$, $u \in \mathcal{U}$ and notice that $\mathrm{dist}_{\|\cdot\|_2}(\mathbb{T}\mathcal{U}, \mathbb{T}\mathcal{V}) \leqslant \sqrt{2}$ for any subgroups $\mathcal{U}, \mathcal{V} \subset \mathcal{U}(M)$. We prove in Theorem 3.1 that if $M = \mathrm{L}G = \mathrm{L}\Lambda$ are two group von Neumann algebra decompositions of the same II_1 factor M then $\mathrm{dist}_{\|\cdot\|_2}(\mathbb{T}G, \mathbb{T}\Lambda) < \sqrt{2}$ if and only if $\mathbb{T}\Lambda$ and $\mathbb{T}G$ are conjugate by a unitary in M.

To describe the content of Sections 4–8 in more detail, let $G = H_0 \wr_I \Gamma$ be a generalized wreath product group as in Theorem 1.1 (or the more general Theorem 8.3). Write $M := \mathrm{L}G$, and assume that $M = \mathrm{L}\Lambda$ is another group von Neumann algebra decomposition. Denote by $\Delta : M \to M \,\overline{\otimes}\, M : \Delta(v_s) = v_s \otimes v_s, s \in \Lambda$, the comultiplication corresponding to the decomposition $M = \mathrm{L}\Lambda$. Observe that $M = \mathrm{L}G$ can be viewed as the group measure space construction $M = \mathrm{L}^\infty(X_0^I) \rtimes \Gamma$, where $X_0 = \widehat{H_0}$ is the Pontryagin dual of H_0 equipped with the Haar probability measure and where $\Gamma \curvearrowright X_0^I$ is the generalized Bernoulli action.

In [Ioa11], a classification result for embeddings $\Delta : M \to M \,\overline{\otimes}\, M$ was obtained in the case where $M = \mathrm{L}^\infty(X_0^\Gamma) \rtimes \Gamma$ is the group measure space II_1 factor given by the *plain* Bernoulli action of an icc property (T) group Γ, or more generally an icc group Γ that admits an infinite normal subgroup with the relative property (T). We extend these results to generalized Bernoulli actions. This generalization is technically painful, but unavoidable in light of Theorem 1.2.

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[1] To be more precise, we only find a unitary Ω satisfying the formulas (1.1) on page 236, but this suffices to deduce that Ω is a dual 2-cocycle.

We analyze the embedding $\Delta : M \to M \mathbin{\overline{\otimes}} M$ in three different steps, corresponding to Sections 4, 5 and 6. In this analysis we use much of the ideas and techniques developed in deformation/rigidity theory over the last years. Nevertheless, apart from the preliminary Section 2, where we also recall the notion of intertwining bimodules [Pop06c], our article is essentially self-contained and Sections 4, 5 and 6 contain independent results, each having an interest of its own.

We write $A = \mathrm{L}^\infty(X_0^I)$ and denote by $(u_g)_{g\in\Gamma}$ the canonical unitaries in the crossed product $M = A \rtimes \Gamma$.

In Section 4 we elaborate results from [Pop06c], [Ioa07] implying that under the right assumptions, rigid subalgebras of generalized Bernoulli crossed products $M = \mathrm{L}^\infty(X_0^I) \rtimes \Gamma$ have an intertwining bimodule into $\mathrm{L}\Gamma$; see Corollary 4.3. Following [Ioa07] we consider the "tensor length deformation" $\theta_\rho : M \to M$, which is roughly defined as $\theta_\rho(Fu_g) = \rho^n F u_g$ when $g \in \Gamma$ and $F \in \mathrm{L}^\infty(X_0^I)$ only depends on n variables in I. In Theorem 4.2 we describe which subalgebras $Q \subset M$ have the property that θ_ρ converges uniformly to the identity on the unit ball of Q. This result readily applies when $Q \subset M$ has the relative property (T), but also when Q has a nonamenable relative commutant, by the spectral gap argument from [Pop08].

Applied to the above comultiplication $\Delta : M \to M \mathbin{\overline{\otimes}} M$, we will be able to assume that after a unitary conjugacy, $\Delta(\mathrm{L}\Gamma) \subset \mathrm{L}(\Gamma \times \Gamma)$.

In Section 5 we prove the following. Assume that $M = \mathrm{L}^\infty(X_0^I) \rtimes \Gamma$ is a generalized Bernoulli crossed product, and write $A = \mathrm{L}^\infty(X_0^I)$. If $D \subset M \mathbin{\overline{\otimes}} M$ is an abelian von Neumann subalgebra that is normalized by many unitaries in $\mathrm{L}(\Gamma \times \Gamma)$ and if a number of conditions are satisfied, then the relative commutant $D' \cap M \mathbin{\overline{\otimes}} M$ can be essentially unitarily conjugated into $A \mathbin{\overline{\otimes}} A$. This result and its proof are very similar to [Ioa11, Th. 6.1] and very much inspired by the clustering sequences techniques from [Pop06d, §§1–4].

Applied to the above comultiplication $\Delta : M \to M \mathbin{\overline{\otimes}} M$ we may essentially assume that after a unitary conjugacy, $\Delta(A)' \cap M \mathbin{\overline{\otimes}} M = A \mathbin{\overline{\otimes}} A$.

In Section 6 we provide a very general conjugacy criterion for actions. Let $N = B \rtimes \Lambda$ and $M = A \rtimes \Gamma$ be group measure space II_1 factors. Assume that $N \subset M$ in such a way that there exist intertwining bimodules from B into A and from $\mathrm{L}\Lambda$ into $\mathrm{L}\Gamma$. Under a few extra conditions, we conclude that there exists a unitary $\Omega \in M$ such that $\operatorname{Ad}\Omega$ maps B into A and $\mathbb{T}\Lambda$ into $\mathbb{T}\Gamma$. We refer to Theorem 6.1 for a precise statement.

Applied to the above comultiplication $\Delta : M \to M \mathbin{\overline{\otimes}} M$, it ultimately follows that there exists a unitary $\Omega \in M \mathbin{\overline{\otimes}} M$ such that

$$(1.1) \qquad \Omega^*\Delta(u_g)\Omega = \omega(g)\, u_{\delta_1(g)} \otimes u_{\delta_2(g)} \;\; \forall g \in \Gamma \quad \text{and} \quad \Omega^*\Delta(A)\Omega \subset A \mathbin{\overline{\otimes}} A$$

for some group homomorphisms $\delta_i : \Gamma \to \Gamma$, $\omega : \Gamma \to \mathbb{T}$.

Once (1.1) above is established, we conclude that $\Omega \in \mathrm{L}\Lambda \mathbin{\overline{\otimes}} \mathrm{L}\Lambda$ satisfies a 2-cocycle and a symmetry relation. By the above mentioned vanishing of 2-cohomology Theorem 3.3, the main Theorems 1.1 and 8.3 will follow.

We finally refer to the lecture notes [Vae11] for an introduction to the results and techniques of this paper and [Ioa11].

2. **Preliminaries**

2.1. *Intertwining-by-bimodules.* We recall from [Pop06c, Th. 2.1, Cor. 2.3] the theory of *intertwining-by-bimodules*, summarized in the following definition.

Definition 2.1. Let (M, τ) be a tracial von Neumann algebra with separable predual and $P, Q \subset M$ possibly nonunital von Neumann subalgebras. We write $P \prec_M Q$ (or $P \prec Q$ if there is no risk of confusion) when one of the following equivalent conditions is satisfied:

- There exist projections $p \in P$, $q \in Q$, a $*$-homomorphism $\varphi : pPp \to qQq$ and a nonzero partial isometry $v \in pMq$ such that $xv = v\varphi(x)$ for all $x \in pPp$.
- There exist a projection $q \in \mathrm{M}_n(\mathbb{C}) \otimes Q$, a $*$-homomorphism $\varphi : P \to q(\mathrm{M}_n(\mathbb{C}) \otimes Q)q$ and a nonzero partial isometry $v \in (\mathrm{M}_{1,n}(\mathbb{C}) \otimes 1_P M)q$ such that $xv = v\varphi(x)$ for all $x \in P$.
- It is impossible to find a sequence $u_n \in \mathcal{U}(P)$ satisfying $\|E_Q(xu_ny^*)\|_2 \to 0$ for all $x, y \in 1_Q M 1_P$.
- There exists a subgroup $\mathcal{U} \subset \mathcal{U}(P)$ generating P as a von Neumann algebra for which it is impossible to find a sequence $u_n \in \mathcal{U}$ satisfying $\|E_Q(xu_ny^*)\|_2 \to 0$ for all $x, y \in 1_Q M 1_P$.

Remark 2.2. We freely use the following facts about the embedding property $\prec$. If $Q_k \subset M$ is a sequence of von Neumann subalgebras and $P \not\prec Q_k$ for all k, considering the diagonal inclusion of P into matrices over M together with the subalgebra $Q_1 \oplus \cdots \oplus Q_l$, we find a sequence of unitaries $u_n \in \mathcal{U}(P)$ such that for all k and all $x, y \in 1_{Q_k} M 1_P$, we have $\|E_{Q_k}(xu_ny^*)\|_2 \to 0$ (see, e.g., [Vae08, Rem. 3.3] for details).

If $p \in P$ is a nonzero projection and $pPp \prec Q$, then $P \prec Q$ (see, e.g., [Vae08, Lemma 3.4]). Also, if $P \prec Q$ and $B \subset Q$ has finite index, then $P \prec B$ (see, e.g., [Vae08, Lemma 3.9]). Finally, although $\prec$ is not transitive, the following holds for von Neumann subalgebras P and $B \subset Q$. If $P \prec Q$ and $P \not\prec B$, the $*$-homomorphism φ in Definition 2.1 can be chosen in such a way that the subalgebra $\varphi(P) \subset Q$ satisfies $\varphi(P) \not\prec_Q B$ (see, e.g., [Vae08, Rem. 3.8]).

2.2. *Bimodules and weak containment.* Let M, N be tracial von Neumann algebras. An *M-N-bimodule* ${}_M\mathcal{H}_N$ is a Hilbert space $\mathcal{H}$ equipped with a normal unital $*$-homomorphism $\pi : M \to \mathrm{B}(\mathcal{H})$ and a normal unital $*$-anti-homomorphism $\pi' : N \to \mathrm{B}(\mathcal{H})$ such that $\pi(M)$ and $\pi'(N)$ commute. We write $x\xi y$ instead of $\pi(x)\pi'(y)\xi$. The bimodule ${}_M\mathrm{L}^2(M)_M$ is called the *trivial bimodule*, and ${}_{(M\otimes 1)}\mathrm{L}^2(M \mathbin{\overline{\otimes}} M)_{(1\otimes M)}$ is called the *coarse bimodule.* Given the bimodules ${}_M\mathcal{H}_N$ and ${}_N\mathcal{K}_P$, one can define the M-P-bimodule $\mathcal{H} \otimes_N \mathcal{K}$, which is called the *Connes tensor product*; see [Con94, V.App. B].

Every M-N-bimodule ${}_M\mathcal{H}_N$ gives rise to a $*$-homomorphism $\pi_\mathcal{H} : M \otimes_{\mathrm{alg}} N^{\mathrm{op}} \to \mathrm{B}(\mathcal{H})$ given by $\pi_\mathcal{H}(x \otimes y)\xi = x\xi y$. We say that ${}_M\mathcal{H}_N$ is *weakly contained* in ${}_M\mathcal{K}_N$, and write $\mathcal{H} \subset_{\mathrm{weak}} \mathcal{K}$, if $\|\pi_\mathcal{H}(T)\| \leqslant \|\pi_\mathcal{K}(T)\|$ for all $T \in M \otimes_{\mathrm{alg}} N^{\mathrm{op}}$. Recall that $\mathcal{H} \subset_{\mathrm{weak}} \mathcal{K}$ if and only if ${}_M\mathcal{H}_N$ lies in the closure (for the Fell topology) of all finite direct sums of copies of ${}_M\mathcal{K}_N$.

For later use we record the following easy lemma and give a proof for the convenience of the reader.

LEMMA 2.3. *Let (M, τ) be a tracial von Neumann algebra, and let $P \subset pMp$ be a von Neumann subalgebra. Let ${}_P\mathcal{H}_M$ be a P-M-bimodule, and let $\kappa > 0$. Assume that $\xi_n \in \mathcal{H}$ satisfies*

$$\|a\xi_n - \xi_n a\| \to 0 \ \ \forall a \in P, \ \ \|\xi_n x\| \leqslant \kappa \|x\|_2 \ \ \forall n \in \mathbb{N}, \ x \in M, \ \ \limsup_n \|\xi_n p\| > 0.$$

Then there is a nonzero projection $p_1 \in P' \cap pMp$ such that ${}_P\mathrm{L}^2(p_1M)_M$ is weakly contained in ${}_P\mathcal{H}_M$.

Proof. Replacing ξ_n by $\xi_n p$, we may assume that $\xi_n p = \xi_n$ for all n. Since $\|\xi_n x\| \leqslant \kappa\|x\|_2$ for all $x \in M$, define $T_n \in pM^+p$ satisfying $\|T_n\| \leqslant \kappa$ and $\langle \xi_n, \xi_n x\rangle = \tau(T_n x)$ for all $x \in M$. We have $\|[a, T_n]\|_1 \to 0$ for all $a \in P$. Since $\tau(T_n) = \|\xi_n\|^2$ and $\|T_n\|$ is bounded, we can pass to a subsequence and assume that $T_n \to T$ weakly with $T \in pM^+p$, $\tau(T) > 0$. Note that $T \in P' \cap pM^+p$. Take $S \in P' \cap pM^+p$ such that $T^{1/2}S$ is a nonzero projection p_1. Define $\eta_n = \xi_n S$. It follows that $\langle \eta_n, a\eta_n x\rangle \to \tau(p_1 a x)$ for all $a \in P$, $x \in M$. Hence, ${}_P\mathrm{L}^2(p_1M)_M$ is weakly contained in ${}_P\mathcal{H}_M$. $\square$

2.3. *Relative property* (T). Let (M, τ) be a tracial von Neumann algebra, and let $P \subset M$ be a von Neumann subalgebra. Following [Pop06a, Prop. 4.1], we say that $P \subset M$ has the *relative property* (T) if every sequence $\varphi_n : M \to M$ of normal completely positive maps that are sub-unital ($\varphi_n(1) \leqslant 1$), subtracial ($\tau \circ \varphi_n \leqslant \tau$) and satisfy $\|\varphi_n(x) - x\|_2 \to 0$ for all $x \in M$, converges to the identity uniformly on the unit ball of P; i.e.,

$$\sup_{x \in P, \|x\| \leqslant 1} \|\varphi_n(x) - x\|_2 \to 0.$$

If $\Gamma_0 < \Gamma_1$ are countable groups, by [Pop06a, Proposition 5.1] the inclusion $\mathrm{L}\Gamma_0 \subset \mathrm{L}\Gamma_1$ has the relative property (T) if and only if $\Gamma_0 < \Gamma_1$ has the relative property (T) in the usual group theoretic sense.

2.4. *Relative amenability.* Recall that a tracial von Neumann algebra (M,τ) is called *amenable* if the trivial M-M-bimodule is weakly contained in the coarse M-M-bimodule.

Fix a tracial von Neumann algebra (M, τ) and a von Neumann subalgebra $Q \subset M$. *Jones' basic construction* $\langle M, e_Q\rangle$ is defined as the von Neumann subalgebra of $\mathrm{B}(\mathrm{L}^2(M))$ generated by M (acting on the left) and the orthogonal projection e_Q of $\mathrm{L}^2(M)$ onto $\mathrm{L}^2(Q)$. Note that $\langle M, e_Q\rangle$ equals the commutant of the right Q-action on $\mathrm{L}^2(M)$. The basic construction $\langle M, e_Q\rangle$ comes with a semi-finite faithful trace Tr satisfying $\mathrm{Tr}(ae_Qb) = \tau(ab)$ for all $a, b \in M$. We denote, for $p = 1, 2$, by $\mathrm{L}^p(\langle M, e_Q\rangle)$ the corresponding L^p-spaces.

Following [OP10, Def. 2.2], a von Neumann subalgebra $P \subset pMp$ is said to be *amenable relative to* Q if ${}_P\mathrm{L}^2(p\langle M, e_Q\rangle)_M$ weakly contains ${}_P\mathrm{L}^2(pM)_M$. By [OP10, Th. 2.1], P is amenable relative to Q if and only if there exists a sequence $T_n \in p\mathrm{L}^1(\langle M, e_Q\rangle)^+p$ satisfying

$$\|aT_n - T_na\|_1 \to 0 \ \ \forall a \in P \ \text{ and } \ \mathrm{Tr}(T_nx) \to \tau(x) \ \ \forall x \in pMp.$$

We say that a von Neumann subalgebra $P \subset pMp$ is *strongly nonamenable relative to* Q if for all nonzero projections $p_1 \in P' \cap pMp$, the von Neumann algebra Pp_1 is nonamenable relative to Q. Equivalently, none of the bimodules ${}_P\mathrm{L}^2(p_1M)_M$ with p_1 a nonzero projection in $P' \cap pMp$ is weakly contained in ${}_P\mathrm{L}^2(p\langle M, e_Q\rangle)_M$.

If $P \subset pMp$ is amenable relative to Q and if $A \subset eMe$ is a von Neumann subalgebra satisfying $A \prec_M P$, then there exists a nonzero projection $f \in A' \cap eMe$ such that Af is amenable relative to Q.

Note that ${}_M\mathrm{L}^2(\langle M, e_Q\rangle)_M \cong {}_M\big(\mathrm{L}^2(M) \otimes_Q \mathrm{L}^2(M)\big)_M$. In particular, a von Neumann subalgebra $P \subset p(N \mathbin{\overline{\otimes}} M)p$ is amenable relative to $N \otimes 1$ if and only if ${}_P\mathrm{L}^2(p(N \mathbin{\overline{\otimes}} M))_{(N \mathbin{\overline{\otimes}} M)}$ is weakly contained in

$${}_{(P \otimes 1)}\mathrm{L}^2(p(N \mathbin{\overline{\otimes}} M) \mathbin{\overline{\otimes}} M)_{(N \mathbin{\overline{\otimes}} 1 \mathbin{\overline{\otimes}} M)}.$$

3. Symmetric dual 2-cocycles and isomorphism of group von Neumann algebras

The main aim of this section is to prove the following result. Whenever Λ is a countable group and $(v_s)_{s\in\Lambda}$ are the canonical unitaries generating $\mathrm{L}\Lambda$, we denote by $\mathbb{T}\Lambda$ the group of unitaries in $\mathrm{L}\Lambda$ of the form λv_s for $\lambda \in \mathbb{T}$ and $s \in \Lambda$.

THEOREM 3.1. *Let* Γ *and* Λ *be* icc *groups, and let* $\mathrm{L}\Gamma = \mathrm{L}\Lambda$. *Denote by* $(u_g)_{g\in\Gamma}$ *and* $(v_s)_{s\in\Lambda}$ *the respective canonical unitaries. Denote by*

$$\operatorname{dist}_{\|\cdot\|_2}(\mathbb{T}\Gamma, \mathbb{T}\Lambda) = \sup_{u\in\mathbb{T}\Gamma}\Big(\inf_{v\in\mathbb{T}\Lambda} \|u-v\|_2\Big)$$

the (*asymmetric*) *upper Hausdorff distance. Then the following two statements are equivalent*:

- $\operatorname{dist}_{\|\cdot\|_2}(\mathbb{T}\Gamma, \mathbb{T}\Lambda) < \sqrt{2}$.
- *There exist a unitary* $w \in \mathrm{L}\Lambda$, *a character* $\gamma : \Gamma \to \mathbb{T}$ *and an isomorphism of groups* $\delta : \Gamma \to \Lambda$ *such that*

$$w u_g w^* = \gamma(g) v_{\delta(g)} \quad \forall g \in \Gamma.$$

Defining the *height* of an element $x \in \mathrm{L}\Lambda$ as

$$h_\Lambda(x) := \max\{|\tau(x v_s^*)| \mid s \in \Lambda\}, \tag{3.1}$$

it is an easy exercise to check that

$$\operatorname{dist}_{\|\cdot\|_2}(x, \mathbb{T}\Lambda) = \sqrt{1 + \|x\|_2^2 - 2h_\Lambda(x)}.$$

In particular, the assumption $\operatorname{dist}_{\|\cdot\|_2}(\mathbb{T}\Gamma, \mathbb{T}\Lambda) < \sqrt{2}$ in Theorem 3.1 is equivalent to the existence of a $\delta > 0$ such that $h_\Lambda(u_g) \geqslant \delta$ for all $g \in \Gamma$.

Remark 3.2. Assume that Γ and Λ are countable groups and that $\mathrm{L}\Gamma$ is a von Neumann subalgebra of $\mathrm{L}\Lambda$. Assume that $\operatorname{dist}_{\|\cdot\|_2}(\mathbb{T}\Gamma, \mathbb{T}\Lambda) < \sqrt{2}$. We do not know whether it is still true that there exist a unitary $w \in \mathrm{L}\Lambda$, a character $\gamma : \Gamma \to \mathbb{T}$ and an injective group homomorphism $\delta : \Gamma \to \Lambda$ such that $w u_g w^* = \gamma(g) v_{\delta(g)}$ for all $g \in \Gamma$.

We will not be able to prove our main Theorem 8.3 by a direct application of Theorem 3.1. Rather, we need the following *vanishing of cohomology theorem*, which at the same time will lead to a proof of Theorem 3.1.

Recall that every group von Neumann algebra $\mathrm{L}\Lambda$ is equipped with a natural normal unital $*$-homomorphism, called *comultiplication*, $\Delta : \mathrm{L}\Lambda \to \mathrm{L}\Lambda \mathbin{\overline{\otimes}} \mathrm{L}\Lambda$ given by $\Delta(v_s) = v_s \otimes v_s$ for all $s \in \Lambda$. Observe that $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$ and that $\sigma \circ \Delta = \Delta$, where $\sigma(x \otimes y) = y \otimes x$ is the flip automorphism. We also use the *tensor leg numbering notation* for operators in tensor products. In this manner, $X_{21} = \sigma(X)$, $X_{23} = 1 \otimes X$, $X_{13} = (\sigma \otimes \mathrm{id})(1 \otimes X)$, etc.

THEOREM 3.3. *Let* Λ *be a countable group, and let* $\Delta : \mathrm{L}\Lambda \to \mathrm{L}\Lambda \mathbin{\overline{\otimes}} \mathrm{L}\Lambda$ *be the comultiplication. Suppose that* $\Omega \in \mathrm{L}\Lambda \mathbin{\overline{\otimes}} \mathrm{L}\Lambda$ *is a unitary satisfying*

$$\Omega_{21} = \mu\Omega \quad \textit{and} \quad (\Delta \otimes \mathrm{id})(\Omega)(\Omega \otimes 1) = \eta(\mathrm{id} \otimes \Delta)(\Omega)(1 \otimes \Omega)$$

for some $\mu, \eta \in \mathbb{T}$*. Then,* $\mu = \eta = 1$ *and there exists a unitary* $w \in \mathrm{L}\Lambda$ *such that*

$$\Omega = \Delta(w^*)(w \otimes w).$$

Proof. Put $M = \mathrm{L}\Lambda$ and $H = \ell^2(\Lambda)$. Define the unitary operators λ_h, ρ_h, $h \in \Lambda$ by the formulae $\lambda_h \delta_k = \delta_{hk}$ and $\rho_h \delta_k = \delta_{kh^{-1}}$. Realize $M := \{\rho_h \mid h \in \Lambda\}''$.

We view $\ell^\infty(\Lambda)$ acting on H by multiplication operators. We define the unitary

$$W \in \ell^\infty(\Lambda) \mathbin{\overline{\otimes}} M \quad \text{given by} \quad W(\delta_g \otimes \delta_h) = \delta_g \otimes \rho_g \delta_h = \delta_g \otimes \delta_{hg^{-1}}.$$

Define the unitary

$$X \in \mathrm{B}(H) \mathbin{\overline{\otimes}} M : X = W\Omega.$$

It is easy to check that $\Delta(x) = W^*(x \otimes 1)W$ for all $x \in M$. Also,

$$(\mathrm{id} \otimes \Delta)(X)(1 \otimes \Omega) = \overline{\eta} X_{13} X_{12}. \tag{3.2}$$

Whenever $\mathcal{V} \subset \mathrm{B}(H)$, we denote by $[\mathcal{V}]$ the norm closed linear span of $\mathcal{V}$ inside $\mathrm{B}(H)$. Define

$$A := [(\mathrm{id} \otimes \omega)(X) \mid \omega \in M_*].$$

Step 1. The norm closed linear subspace $A \subset \mathrm{B}(H)$ is actually a C*-algebra acting nondegenerately on H (i.e., [A H] = H). Moreover, $\lambda_g A \lambda_g^* = A$ for all $g \in \Lambda$. Applying $\mathrm{id} \otimes \omega_1 \otimes \omega_2$ to (3.2), we get

$$\begin{aligned}
[AA] &= [(\mathrm{id} \otimes \omega_1 \otimes \omega_2)\big((\mathrm{id} \otimes \Delta)(X)(1 \otimes \Omega)\big) \mid \omega_1, \omega_2 \in M_*] \\
&= [(\mathrm{id} \otimes \Omega\omega)(\mathrm{id} \otimes \Delta)(X) \mid \omega \in (M \mathbin{\overline{\otimes}} M)_*] \\
&= [(\mathrm{id} \otimes \omega\Delta)(X) \mid \omega \in (M \mathbin{\overline{\otimes}} M)_*] = A.
\end{aligned}$$

Since $\Delta(x) = W^*(x \otimes 1)W$, we can rewrite (3.2) in the form

$$\overline{\eta} X_{12} X_{23}^* = X_{13}^* W_{23}^* X_{12}. \tag{3.3}$$

Applying $\mathrm{id} \otimes \omega_1 \otimes \omega_2$, $\omega_1, \omega_2 \in \mathrm{B}(H)_*$, we get

$$A = [(\mathrm{id} \otimes \omega_1 \otimes \omega_2)(X_{13}^* W_{23}^* X_{12}) \mid \omega_1, \omega_2 \in \mathrm{B}(H)_*].$$

Denote by $P_g \in \ell^\infty(\Lambda)$ the natural minimal projections. Then $\mathrm{B}(H)_* = [\omega P_g \mid \omega \in \mathrm{B}(H)_*, g \in \Lambda]$. Hence,

$$\begin{aligned}
A &= [(\mathrm{id} \otimes \omega_1 P_g \otimes \omega_2)(X_{13}^* W_{23}^* X_{12}) \mid \omega_1, \omega_2 \in \mathrm{B}(H)_*, g \in \Lambda] \\
&= [(\mathrm{id} \otimes \omega_1 \otimes \omega_2)(X_{13}^*(1 \otimes P_g \otimes 1) W_{23}^* X_{12}) \mid \omega_1, \omega_2 \in \mathrm{B}(H)_*, g \in \Lambda].
\end{aligned}$$

Since $(P_g \otimes 1)W^* = P_g \otimes \rho_g^*$, we get

$$\begin{aligned}
A &= [(\mathrm{id} \otimes \omega_1 P_g \otimes \rho_g^* \omega_2)(X_{13}^* X_{12}) \mid \omega_1, \omega_2 \in \mathrm{B}(H)_*, g \in \Lambda] \\
&= [(\mathrm{id} \otimes \omega_1 \otimes \omega_2)(X_{13}^* X_{12}) \mid \omega_1, \omega_2 \in \mathrm{B}(H)_*] = [A^* A].
\end{aligned}$$

Since $A = [AA]$ and $A = [A^*A]$, it follows that A is a C*-algebra. Also,

$$[AH] = [(\mathrm{id} \otimes \omega)(X)H \mid \omega \in M_*] = [(1 \otimes \xi_1^*)X(H \otimes \xi_2) \mid \xi_1, \xi_2 \in H] = H$$

since X is a unitary operator. So, the C*-algebra A acts nondegenerately on H.

Since $X(\lambda_g \otimes 1)X^* = \lambda_g \otimes \rho_g$, also

$$\lambda_g^*(\mathrm{id} \otimes \omega)(X)\lambda_g = (\mathrm{id} \otimes \omega\rho_g)(X).$$

Hence, λ_g normalizes A.

Step 2. We have $\mu = 1$, and A is an abelian C*-algebra. Applying $\mathrm{id} \otimes \sigma$ to (3.2) and using the fact that $\Delta = \sigma \circ \Delta$, one gets $X_{12}X_{13} = \mu X_{13}X_{12}$. Applying $\mathrm{id} \otimes \omega_1 \otimes \omega_2$ to this formula, we get that $ab = \mu ba$ for all $a, b \in A$. So, this formula also holds when a and b belong to A'', which contains 1. But then, $\mu = 1$ and A follows abelian.

Step 3. The closed linear span $B := [A\lambda_g \mid g \in \Lambda]$ is a C*-algebra that is ultraweakly dense in $\mathrm{B}(H)$. Since the unitaries λ_g normalize A, it follows that B is a C*-algebra. Also, $A \subset B$ and hence, B acts nondegenerately on H. It suffices to prove that $B' = \mathbb{C}1$. Since the commutant of $\{\lambda_g \mid g \in \Lambda\}$ equals M, we have to prove that $M \cap A' = \mathbb{C}1$. Take $x \in M \cap A'$. Denote $\mathcal{A} = A''$, and note that $X \in \mathcal{A} \mathbin{\overline{\otimes}} M$. Since $\mathcal{A}$ is abelian, we have $X_{12}X_{13} = X_{13}X_{12}$. Combining with (3.3), we have

$$W_{23}^* X_{12} X_{23} = \overline{\eta} X_{12} X_{13}.$$

Hence,

$$W_{23}^* X_{12} X_{23} (1 \otimes x \otimes 1) X_{23}^* X_{12}^* W_{23} = X_{12} X_{13} (1 \otimes x \otimes 1) X_{13}^* X_{12}^*.$$

Since $x \in \mathcal{A}'$, the left-hand side equals $(\mathrm{id} \otimes \Delta)(X(1 \otimes x)X^*)$, while the right-hand side equals $X(1 \otimes x)X^* \otimes 1$. Denote by τ the natural trace on $M = \mathrm{L}\Lambda$. Then, $(\mathrm{id} \otimes \tau)\Delta(y) = \tau(y)1$ for all $y \in M$. Applying $\mathrm{id} \otimes \mathrm{id} \otimes \tau$ to the equality $(\mathrm{id} \otimes \Delta)(X(1 \otimes x)X^*) = X(1 \otimes x)X^* \otimes 1$, we find $y \in \mathcal{A}$ such that $X(1 \otimes x)X^* = y \otimes 1$. But then,

$$1 \otimes x = X^*(y \otimes 1)X = y \otimes 1.$$

We finally conclude that x is a scalar multiple of 1.

Step 4. The formula $E(x) = (\mathrm{id} \otimes \tau)(X(x \otimes 1)X^*)$ provides a normal conditional expectation of $\mathrm{B}(H)$ onto $\mathcal{A}$. Since $\mathcal{A}$ is abelian, we have $E(x) = x$ for all $x \in \mathcal{A}$. So, it remains to prove that $E(x) \in \mathcal{A}$ for all $x \in \mathrm{B}(H)$. By Step 3 it suffices to check this for $x = a\lambda_g$, $a \in A$, $g \in \Lambda$. Since $a \otimes 1$ and X commute, we have

$$E(a\lambda_g) = a(\mathrm{id} \otimes \tau)(X(\lambda_g \otimes 1)X^*) = a(\mathrm{id} \otimes \tau)(\lambda_g \otimes \rho_g) = \begin{cases} a & \text{if } g = e, \\ 0 & \text{if } g \neq e. \end{cases}$$

End of the proof. Step 4 implies that $\mathcal{A}$ is a discrete von Neumann algebra. Let $p \in \mathcal{A}$ be a nonzero minimal projection. Since $\mathcal{A}$ is abelian, define the unitary $w \in M$ such that $X(p \otimes 1) = \eta p \otimes w$. Multiplying (3.2) with $p \otimes 1 \otimes 1$, we get that $\Delta(w)\Omega = w \otimes w$. So, $\Omega = \Delta(w^*)(w \otimes w)$. Then also $(\Delta \otimes \mathrm{id})(\Omega)(\Omega \otimes 1) = (\mathrm{id} \otimes \Delta)(\Omega)(1 \otimes \Omega)$, implying that $\eta = 1$. □

Before proving Theorem 3.1, we state and prove the following lemma which has some interest of its own. Recall that a unitary representation of a countable group is called *weakly mixing* if $\{0\}$ is the only finite dimensional invariant subspace.

LEMMA 3.4. *Let Γ, Λ be countable groups, and assume that $\mathrm{L}\Gamma \subset \mathrm{L}\Lambda$. Denote by $(u_g)_{g \in \Gamma}$ the canonical unitaries in $\mathrm{L}\Gamma$. Denote $M = \mathrm{L}\Lambda$, and let $(v_s)_{s \in \Lambda}$ be the canonical unitaries in Λ. Let $\Delta : \mathrm{L}\Lambda \to \mathrm{L}\Lambda \mathbin{\overline{\otimes}} \mathrm{L}\Lambda$ be the comultiplication. Assume that the unitary representation $\mathrm{Ad}\, u_g$ of Γ on $\mathrm{L}^2(M) \ominus \mathbb{C}1$ is weakly mixing.*

If $\Omega \in M \mathbin{\overline{\otimes}} M$ is a unitary satisfying $\Omega(u_g \otimes u_g)\Omega^* \in \Delta(M)$ for all $g \in \Gamma$, there exist unitaries $w, v \in M$, a character $\gamma : \Gamma \to \mathbb{T}$ and an injective group homomorphism $\rho : \Gamma \to \Lambda$ such that

$$w u_g w^* = \gamma(g) v_{\rho(g)} \;\; \forall g \in \Gamma \quad \textit{and} \quad \Omega = \Delta(v^*)(w \otimes w).$$

Proof. Define $\pi : \Gamma \to \mathcal{U}(M)$ such that $\Delta(\pi(g))\Omega = \Omega(u_g \otimes u_g)$ for all $g \in \Gamma$. Write $X = (\Delta \otimes \mathrm{id})(\Omega^*)(\mathrm{id} \otimes \Delta)(\Omega)$. Then, $X \in M \mathbin{\overline{\otimes}} M \mathbin{\overline{\otimes}} M$ is unitary and satisfies

$$(\Delta(u_g) \otimes u_g)X = X(u_g \otimes \Delta(u_g)) \tag{3.4}$$

for all $g \in \Gamma$. Define $Y = (X \otimes 1)(1 \otimes X)$, which is a unitary in $M \mathbin{\overline{\otimes}} M \mathbin{\overline{\otimes}} M \mathbin{\overline{\otimes}} M$ satisfying

$$(\Delta(u_g) \otimes u_g \otimes u_g)Y = Y(u_g \otimes u_g \otimes \Delta(u_g))$$

for all $g \in \Gamma$. It follows that the unitary representation $\xi \mapsto (u_g \otimes u_g)\xi\Delta(u_g)^*$ of Γ on $\mathrm{L}^2(M \mathbin{\overline{\otimes}} M)$ is not weakly mixing. This yields a finite dimensional unitary representation $\eta : \Gamma \to \mathcal{U}(\mathbb{C}^n)$ and a nonzero vector $\xi \in \mathbb{C}^n \otimes \mathrm{L}^2(M \mathbin{\overline{\otimes}} M)$ satisfying

$$(\eta(g) \otimes u_g \otimes u_g)\xi = \xi\Delta(u_g)$$

for all $g \in \Gamma$. We may assume that η is irreducible. Since $\mathrm{Ad}(u_g \otimes u_g)$ is weakly mixing on $\mathrm{L}^2(M \mathbin{\overline{\otimes}} M) \ominus \mathbb{C}1$ and since η is irreducible, it follows that $\xi\xi^*$ is a multiple of 1. Hence, $n = 1$ and we have found a unitary $Z \in M \mathbin{\overline{\otimes}} M$ and a character $\gamma : \Gamma \to \mathbb{T}$ satisfying $\gamma(g)\Delta(u_g)Z = Z(u_g \otimes u_g)$ for all $g \in \Gamma$.

Since $\sigma \circ \Delta = \Delta$, it follows that $Z_{21}^* Z$ commutes with $u_g \otimes u_g, g \in \Gamma$ and hence, is a scalar multiple of 1. Since $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$, it also follows that $(1 \otimes Z)^*(\mathrm{id} \otimes \Delta)(Z)^*(\Delta \otimes \mathrm{id})(Z)(Z \otimes 1)$ commutes with $u_g \otimes u_g \otimes u_g$ for all

$g \in \Gamma$ and hence, is a scalar multiple of 1. By Theorem 3.3, we find a unitary $w \in M$ such that $Z = \Delta(w^*)(w \otimes w)$.

It follows that $\gamma(g)\Delta(wu_g w^*) = wu_g w^* \otimes wu_g w^*$ for all $g \in \Gamma$. This means that $wu_g w^* = \gamma(g) v_{\rho(g)}$ for an injective group homomorphism $\rho : \Gamma \to \Lambda$. (See Lemma 7.1 below for this well-known fact.) Put $\Lambda_0 = \rho(\Gamma)$. Since $\operatorname{Ad} u_g$ is a weakly mixing representation of Γ on $\mathrm{L}^2(M)$, also $(\operatorname{Ad} v_s)_{s \in \Lambda_0}$ is weakly mixing, meaning that $\Lambda_0 \subset \Lambda$ has the relative icc property: $\{sts^{-1} \mid s \in \Lambda_0\}$ is infinite for all $t \in \Lambda - \{e\}$.

Since $\Omega(u_g \otimes u_g)\Omega^* \in \Delta(M)$ for all $g \in \Gamma$, it follows that

$$\Omega(w^* \otimes w^*)\,(v_s \otimes v_s)\,(w \otimes w)\Omega^* \in \Delta(M) \tag{3.5}$$

for all $s \in \Lambda_0$. Since $\Lambda_0 \subset \Lambda$ has the relative icc property, we can take a sequence $s_n \in \Lambda_0$ such that $s_n t s_n^{-1} \to \infty$ for all $t \in \Lambda - \{e\}$. It follows that

$$\|E_{\Delta(M)}(a(v_{s_n} \otimes v_{s_n})b) - E_{\Delta(M)}(a)\,\Delta(v_{s_n})\,E_{\Delta(M)}(b)\|_2 \to 0$$

for all $a, b \in M \mathbin{\overline{\otimes}} M$. Indeed, it suffices to check this for a and b of the form $v_r \otimes v_t$, $r, t \in \Lambda$. Together with (3.5), it follows that $\|E_{\Delta(M)}(\Omega(w^* \otimes w^*))\|_2 = 1$, meaning that $\Omega(w^* \otimes w^*) \in \Delta(M)$. We have found the required unitary $v \in M$ such that $\Omega = \Delta(v^*)(w \otimes w)$. □

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. Denote by $\Delta : \mathrm{L}\Lambda \to \mathrm{L}\Lambda \mathbin{\overline{\otimes}} \mathrm{L}\Lambda$ the canonical comultiplication. Put $\mathrm{L}\Gamma = M = \mathrm{L}\Lambda$, and denote by τ the trace on M. Whenever $x \in M$, we denote by x^s, $s \in \Lambda$ the Fourier coefficient $x^s := \tau(x v_s^*)$. As above, we define for all $x \in M$ the height $h_\Lambda(x) = \max\{|(x)^s| \mid s \in \Lambda\}$.

First assume that $\operatorname{dist}_{\|\cdot\|_2}(\mathbb{T}\Gamma, \mathbb{T}\Lambda) < \sqrt{2}$. By the discussion after the formulation of Theorem 3.1, we find a $\delta > 0$ such that $h_\Lambda(u_g) \geqslant \delta$ for all $g \in \Gamma$. A straightforward computation then gives

$$(\tau \otimes \tau)((\Delta(u_g) \otimes u_g)(u_g \otimes \Delta(u_g))^*) = \sum_{s \in \Lambda} |(u_g)^s|^4 \geqslant \delta^4$$

for all $g \in \Gamma$. So, there exists a nonzero $X \in M \mathbin{\overline{\otimes}} M \mathbin{\overline{\otimes}} M$ satisfying

$$(\Delta(u_g) \otimes u_g)X = X(u_g \otimes \Delta(u_g))$$

for all $g \in \Gamma$.

We also have $M = \mathrm{L}\Gamma$. So, Γ is an icc group and $(\operatorname{Ad} u_g)_{g \in \Gamma}$ is a weakly mixing representation of Γ on $\mathrm{L}^2(M) \ominus \mathbb{C}1$. Since XX^* commutes with all $\Delta(u_g) \otimes u_g$, $g \in \Gamma$, it follows that $XX^* \in (\Delta(M)' \cap M \mathbin{\overline{\otimes}} M) \otimes 1$. Since Λ is an icc group, $\Delta(M)$ has trivial relative commutant in $M \mathbin{\overline{\otimes}} M$. Hence, XX^* is a nonzero multiple of 1 and we may assume that X is a unitary element of $M \mathbin{\overline{\otimes}} M \mathbin{\overline{\otimes}} M$.

We can now start reading the proof of Lemma 3.4 at formula (3.4) and find a unitary $w \in M$, a character $\gamma : \Gamma \to \mathbb{T}$ and an injective group morphism $\delta : \Gamma \to \Lambda$ such that

$$wu_g w^* = \gamma(g) u_{\delta(g)} \quad \forall g \in \Gamma.$$

But then, δ follows onto as well.

Conversely assume that $\mathrm{dist}_{\|\cdot\|_2}(\mathbb{T}\Gamma, \mathbb{T}\Lambda) = \sqrt{2}$. So we can take a sequence $g_n \in \Gamma$ such that $h_\Lambda(u_{g_n}) \to 0$. We claim that $h_\Lambda(a u_{g_n} b) \to 0$ for all $a, b \in M$. The claim is trivial if a and b are finite linear combinations of $v_s, s \in \Lambda$ and follows in general by approximating in $\|\cdot\|_2$ arbitrary $a, b \in M$ by such finite linear combinations a_0, b_0 satisfying $\|a_0\| \leqslant \|a\|$ and $\|b_0\| \leqslant \|b\|$. If $w \in M$ would be a unitary satisfying $wu_g w^* \in \mathbb{T}\Lambda$ for all $g \in \Gamma$, we would arrive at the contradiction that $1 = h_\Lambda(w u_{g_n} w^*) \to 0$. □

Out of Connes' rigidity paper [Con80] grew a series of rigidity results "up to countable classes" (see, e.g., [Pop06b, Th 5.3(2)],[Pop06a, Th. 4.4], [Oza04, Th. 2], etc.). In particular, it was pointed out in [Pop07, §4] that Connes' rigidity conjecture ([Con82]) does hold true up to countable classes. More precisely, given an icc property (T) group Γ, there are at most countably many nonisomorphic groups Λ_i satisfying $\mathrm{L}\Gamma \cong \mathrm{L}\Lambda_i$. Besides "separability arguments," the proof in [Pop07] makes crucial use of a result in [Sha00, theorem, p. 5], which shows that every property (T) group is the quotient of a finitely presented property (T) group and thus us allows to assume (when arguing by contradiction) that all Λ_i are a quotient of one and the same property (T) group. As a corollary of Theorem 3.3 we can give an alternative proof, not relying on Shalom's theorem.

PROPOSITION 3.5. *Let* Γ *be an* icc *property* (*T*) *group. There are at most countably many nonisomorphic groups* Λ_i *satisfying* $\mathrm{L}\Gamma \cong \mathrm{L}\Lambda_i$.

Proof. Put $M = \mathrm{L}\Gamma$ with corresponding canonical unitaries $(u_g)_{g\in\Gamma}$. Assume that $(\Lambda_i)_{i\in I}$ is an uncountable family of groups such that $M = \mathrm{L}\Lambda_i$. Denote by $(u^i_g)_{g\in\Lambda_i}$ the corresponding canonical unitaries. We need to find $i \neq j$ such that $\Lambda_i \cong \Lambda_j$. Note that all Λ_i are icc groups. Denote by $\Delta_i : M \to M \overline{\otimes} M$ the comultiplication that corresponds to the group von Neumann algebra decomposition $M = \mathrm{L}\Lambda_i$.

Since Γ has property (T), take a finite subset $K \subset \Gamma$ and $\varepsilon > 0$ such that every unitary representation of Γ that admits a (K, ε)-invariant unit vector actually admits a nonzero invariant vector. Here, given a unitary representation $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$, a unit vector ξ is called (K, ε)-invariant if $\|\pi(g)\xi - \xi\| \leqslant \varepsilon$ for all $g \in K$.

Since the Hilbert space $\mathrm{L}^2(M \overline{\otimes} M) = \ell^2(\Gamma \times \Gamma)$ is separable, we can take $i \neq j$ such that $\|\Delta_i(u_g) - \Delta_j(u_g)\|_2 \leqslant \varepsilon$ for all $g \in K$. Define the unitary representation $\pi : \Gamma \to \mathcal{U}(\ell^2(\Gamma \times \Gamma))$ given by $\pi(g)x = \Delta_i(u_g)x\Delta_j(u_g)^*$ for all $x \in M \mathbin{\overline{\otimes}} M$. By construction, the vector $1 \otimes 1$ is (K, ε) invariant. Hence, π admits a nonzero invariant vector $\Omega \in \mathrm{L}^2(M \mathbin{\overline{\otimes}} M)$. So, $\Delta_i(a)\Omega = \Omega\Delta_j(a)$ for all $a \in M$. Since Λ_i is an icc group, the relative commutant of $\Delta_i(\mathrm{L}\Lambda_i)$ inside $\mathrm{L}(\Lambda_i \times \Lambda_i)$ equals $\mathbb{C}1$. It follows that Ω is a nonzero multiple of a unitary element in $M \mathbin{\overline{\otimes}} M$. Hence, we may assume that $\Omega \in \mathcal{U}(M \mathbin{\overline{\otimes}} M)$.

Since $\Delta_j = \mathrm{Ad}\,\Omega^* \circ \Delta_i$, one deduces, as in the proof of Lemma 3.4, that $\Omega \in \mathrm{L}(\Lambda_i) \overline{\otimes} \mathrm{L}(\Lambda_i)$ satisfies the 2-cocycle and symmetry relation of Theorem 3.3. So, by Theorem 3.3, we find a unitary $w \in M$ such that $\Omega = \Delta_i(w^*)(w \otimes w)$. Hence, for all $g \in \Lambda_j$,

$$wu^j_g w^* \otimes wu^j_g w^* = (w \otimes w)\Delta_j(u^j_g)(w^* \otimes w^*) = \Delta_i(wu^j_g w^*).$$

So, by Lemma 7.1 below, we find for every $g \in \Lambda_j$ an element $\delta(g) \in \Lambda_i$ such that $wu^j_g w^* = u^i_{\delta(g)}$. It follows that δ is an isomorphism of groups and hence $\Lambda_i \cong \Lambda_j$. □

4. **Support length deformation and intertwining of rigid subalgebras**

Let $\Gamma \curvearrowright I$ be an action of a countable group Γ on a countable set I, and let (A_0, τ) be a tracial von Neumann algebra. We denote $(A_0^I, \tau) := \bigotimes_{i \in I}(A_0, \tau)$. Put $(A, \tau) = (A_0^I, \tau)$ and $M = A \rtimes \Gamma$.

The following *tensor length deformation* of $M = A_0^I \rtimes \Gamma$ was introduced in [Ioa07]. For $0 < \rho < 1$, we define

$$\theta_\rho : M \to M : \theta_\rho(au_g) = \rho^n a u_g$$

whenever

$$g \in \Gamma, \ \ a \in (A_0 \ominus \mathbb{C}1)^J \ \text{ and } \ J \subset I, \ \ |J| = n.$$

By [Ioa07, §2] there is an embedding $M \hookrightarrow \widetilde{M}$ and a 1-parameter group of automorphisms $(\alpha_t)_{t \in \mathbb{R}}$ of $\widetilde{M}$ such that

$$E_M(\alpha_t(x)) = \theta_{\rho_t}(x) \ \ \forall x \in M. \tag{4.1}$$

We will recall this construction in the proof of Theorem 4.2. It follows, in particular, that θ_ρ is a well defined normal completely positive map on M. Also note that $\rho_t \to 1$ when $t \to 0$.

The length deformation θ_ρ is a variant of the malleable deformation that was discovered in [Pop06c]. Both the length deformation and the malleable deformation allow us to prove, under certain conditions, that rigid subalgebras of M can be conjugated into $\mathrm{L}\Gamma \subset M$. Theorem 4.2 below is an adaptation of [Pop06c, Th. 4.1] and [Ioa11, Th. 2.1]. We first need a technical lemma and some terminology.

Recall that if $Q \subset M$ is a von Neumann subalgebra, we define $\mathrm{QN}_M(Q) \subset M$ consisting of the elements $x \in M$ for which there exist $x_1, \dots, x_n, y_1, \dots, y_m$ satisfying

$$xQ \subset \sum_{i=1}^{n} Qx_i \quad \text{and} \quad Qx \subset \sum_{j=1}^{m} y_j Q.$$

Then, $\mathrm{QN}_M(Q)$ is a $*$-subalgebra of M containing Q. Its weak closure is called the *quasi-normalizer of Q inside M*. By construction, both Q and $Q' \cap M$ are subalgebras of $\mathrm{QN}_M(Q)$.

If $\Gamma \curvearrowright I$ and $\mathcal{F} \subset I$, we denote by $\operatorname{Stab} \mathcal{F}$ the subgroup of Γ given by $\operatorname{Stab} \mathcal{F} := \{g \in \Gamma \mid g \cdot i = i \ \ \forall i \in \mathcal{F}\}$. We also write $\operatorname{Norm} \mathcal{F} := \{g \in \Gamma \mid g \cdot \mathcal{F} = \mathcal{F}\}$. If $\mathcal{F}$ is finite, $\operatorname{Stab} \mathcal{F}$ is a finite index subgroup of $\operatorname{Norm} \mathcal{F}$.

LEMMA 4.1. *Let $\Gamma \curvearrowright I$ be an action. Let $A_0 \subset B_0$ and N be tracial von Neumann algebras. Consider $\mathcal{M} := N \mathbin{\overline{\otimes}} (A_0^I \rtimes \Gamma)$ and $\widetilde{\mathcal{M}} = N \mathbin{\overline{\otimes}} (B_0^I \rtimes \Gamma)$. Note that $\mathcal{M} \subset \widetilde{\mathcal{M}}$.*

(1) *If $P \subset p\mathcal{M}p$ is a von Neumann subalgebra such that $P \not\prec_{\mathcal{M}} N \mathbin{\overline{\otimes}} (A_0^I \rtimes \operatorname{Stab} i)$ for all $i \in I$, then the quasi-normalizer of P inside $p\widetilde{\mathcal{M}}p$ is contained in $p\mathcal{M}p$.*

(2) *If $\mathcal{F} \subset I$ is a finite subset and $Q \subset q(N \mathbin{\overline{\otimes}} A_0^{\mathcal{F}})q$ is a von Neumann subalgebra such that for all proper subsets $\mathcal{G} \subset \mathcal{F}$ we have $Q \not\prec_{N \overline{\otimes} A_0^{\mathcal{F}}} N \mathbin{\overline{\otimes}} A_0^{\mathcal{G}}$, then the quasi-normalizer of Q inside $q\mathcal{M}q$ is contained in $q(N \mathbin{\overline{\otimes}} (A \rtimes \operatorname{Norm} \mathcal{F}))q$.*

(3) *If $\mathcal{G} \subset I$ is a finite subset and $Q \subset q(N \mathbin{\overline{\otimes}} (A \rtimes \operatorname{Stab} \mathcal{G}))q$ is a von Neumann subalgebra such that for all strictly larger subsets $\mathcal{G} \subset \mathcal{G}'$ we have $Q \not\prec_{N \overline{\otimes} (A \rtimes \operatorname{Stab} \mathcal{G})} N \mathbin{\overline{\otimes}} (A \rtimes \operatorname{Stab} \mathcal{G}')$, then the quasi-normalizer of Q inside $q\mathcal{M}q$ is contained in $q(N \mathbin{\overline{\otimes}} (A \rtimes \operatorname{Norm} \mathcal{G}))q$.*

Proof. Analogous to the proof of [Vae08, Lemma 4.2]. $\square$

THEOREM 4.2. *Let $\Gamma \curvearrowright I$ be an action, and let (A_0, τ) be a tracial von Neumann algebra. Assume that $\kappa \in \mathbb{N}$ such that $\operatorname{Stab} J$ is finite whenever $J \subset I$ and $|J| \geqslant \kappa$. Put $M = A_0^I \rtimes \Gamma$ as above.*

Let (N, τ) be a tracial von Neumann algebra, and let $Q \subset p(N \mathbin{\overline{\otimes}} M)p$ be a von Neumann subalgebra. Denote by $P \subset p(N \mathbin{\overline{\otimes}} M)p$ the quasi-normalizer of Q. If for some $0 < \rho < 1$ and $\delta > 0$ we have

$$\tau(b^*(\mathrm{id} \otimes \theta_\rho)(b)) \geqslant \delta \ \ \forall b \in \mathcal{U}(Q), \tag{4.2}$$

then at least one of the following statements is true:

- *$Q \prec N \otimes 1$.*
- *$P \prec N \mathbin{\overline{\otimes}} (A \rtimes \operatorname{Stab} i)$ for some $i \in I$.*

- *There exists a nonzero partial isometry $v \in p(N \overline{\otimes} M)$ with $vv^* \in P$ and $v^*Pv \subset N \overline{\otimes} \mathrm{L}\Gamma$. If Γ is* icc *and N is a factor, we may assume that $vv^* \in \mathcal{Z}(P)$.*

Proof. We recall from [Ioa07, §2] the following construction. Put $B_0 = A_0 * \mathrm{L}\mathbb{Z}$, with respect to the natural traces. Denote by $v \in \mathrm{L}\mathbb{Z}$ the canonical unitary generator, and choose a self-adjoint element $h \in \mathrm{L}\mathbb{Z}$ with spectrum $[-\pi, \pi]$ such that $v = \exp(ih)$. Denote by $\alpha_t^0 \in \mathrm{Aut}(B_0)$ the inner automorphism given by $\alpha_t^0 = \mathrm{Ad}\exp(ith)$. Put $B = B_0^I$ and $\alpha_t = \bigotimes_{i \in I} \alpha_t^0$. Since α_t commutes with the generalized Bernoulli action, we extend α_t to an automorphism of $\widetilde{M} := B \rtimes \Gamma$ satisfying $\alpha_t(u_g) = u_g$ for all $g \in \Gamma$. Then (4.1) above holds with $\rho_t = \left|\frac{\sin(\pi t)}{\pi t}\right|^2$.

Denote by $\beta_0 \in \mathrm{Aut}(B_0)$ the automorphism given by $\beta_0(a) = a$ for all $a \in A_0$ and $\beta_0(v) = v^*$. Define $\beta = \bigotimes_{i \in I} \beta_0$, and extend β_0 to $\widetilde{M}$ by acting trivially on $\mathrm{L}\Gamma$. By construction, $\beta^2 = \mathrm{id}$ and $\beta \circ \alpha_t \circ \beta = \alpha_{-t}$. We continue writing α_t, β instead of $\mathrm{id} \otimes \beta$ and $\mathrm{id} \otimes \alpha_t$ on $N \overline{\otimes} \widetilde{M}$. Write $\mathcal{M} := N \overline{\otimes} M$ and $\widetilde{\mathcal{M}} := N \overline{\otimes} \widetilde{M}$. By (4.1), for all $x \in \mathcal{M}$, we have $E_{\mathcal{M}}(\alpha_t(x)) = (\mathrm{id} \otimes \theta_{\rho_t})(x)$.

Assume now that Q and P are as in the formulation of the theorem and that (4.2) holds. Assume that for all $i \in I$, we have $P \not\prec N \overline{\otimes}(A \rtimes \mathrm{Stab}\, i)$. Given von Neumann subalgebras $Q_1, Q_2 \subset \widetilde{\mathcal{M}}$, we say that $x \in \widetilde{\mathcal{M}}$ is Q_1-Q_2-finite if there exist $x_1, \ldots, x_n, y_1, \ldots, y_m \in \widetilde{\mathcal{M}}$ such that

$$xQ_2 \subset \sum_{i=1}^n Q_1 x_i \quad \text{and} \quad Q_1 x \subset \sum_{j=1}^m y_j Q_2.$$

Note that by definition $\mathrm{QN}_{p\mathcal{M}p}(Q)$ equals the set of Q-Q-finite elements in $p\mathcal{M}p$.

We follow the lines of [Vae08, proof of Lemma 5.2] to prove the following claim: there exists a nonzero Q-$\alpha_1(Q)$-finite element in $p\widetilde{\mathcal{M}}\alpha_1(p)$. Combining (4.2) and (4.1), we find an $n \in \mathbb{N}$ such that writing $t = 2^{-n}$, we have $\tau(b^*\alpha_t(b)) \geqslant \delta$ for all unitaries $b \in Q$. Define $v \in \widetilde{\mathcal{M}}$ as the element of minimal 2-norm in the $\|\cdot\|_2$-closed convex hull of $\{b^*\alpha_t(b) \mid b \in \mathcal{U}(Q)\}$. Then, $\tau(v) \geqslant \delta$ and hence, $v \neq 0$. By construction, $v \in p\widetilde{\mathcal{M}}\alpha_t(p)$ and $bv = v\alpha_t(b)$ for all $b \in Q$. Hence, v is Q-$\alpha_t(Q)$-finite.

To conclude the proof of the claim, it suffices to show the following statement: if there exists a nonzero Q-$\alpha_t(Q)$-finite element $v \in p\widetilde{\mathcal{M}}\alpha_t(p)$, then the same is true for $2t$ instead of t. For all $d \in \mathrm{QN}_{p\mathcal{M}p}(Q)$, we have that $\alpha_t(\beta(v^*)dv)$ is a Q-$\alpha_{2t}(Q)$-finite element in $p\widetilde{\mathcal{M}}\alpha_{2t}(p)$. So, we have to prove that there exists a $d \in \mathrm{QN}_{p\mathcal{M}p}(Q)$ such that $\beta(v^*)dv \neq 0$. If this is not the case and if we denote by $q \in p\widetilde{\mathcal{M}}p$ the projection onto the closed linear span of all $\{\mathrm{Im}(dv) \mid d \in \mathrm{QN}_{p\mathcal{M}p}(Q)\}$, it follows that q and $\beta(q)$ are orthogonal. By construction, q commutes with P. By Lemma 4.1(1), $q \in p\mathcal{M}p$. Hence,

$q = \beta(q)$ and it follows that $q = 0$. But then, $v = 0$, a contradiction. Hence, the claim is proven.

Since there is a nonzero Q-$\alpha_1(Q)$-finite element in $\widetilde{\mathcal{M}}$ we have, in particular, that $\alpha_1(Q) \prec_{\widetilde{\mathcal{M}}} \mathcal{M}$.

For every finite subset $\mathcal{F} \subset I$, define $\mathcal{M}(\mathcal{F}) := N \mathbin{\overline{\otimes}} (A_0^{\mathcal{F}} \rtimes \operatorname{Stab} \mathcal{F})$. By convention, $\mathcal{M}(\emptyset) = N \mathbin{\overline{\otimes}} \mathrm{L}\Gamma$. We now prove that there exists a finite, possibly empty, subset $\mathcal{F} \subset I$ such that $Q \prec_{\mathcal{M}} \mathcal{M}(\mathcal{F})$. Assume the contrary, and take a sequence of unitaries $v_n \in Q$ such that

$$\|E_{\mathcal{M}(\mathcal{F})}(a v_n b^*)\|_2 \to 0 \ \ \forall a, b \in \mathcal{M} \quad \text{and all finite subsets} \quad \mathcal{F} \subset I. \tag{4.3}$$

We will deduce from this that

$$\|E_{\mathcal{M}}(x \alpha_1(v_n) y^*)\|_2 \to 0 \ \ \forall x, y \in \widetilde{\mathcal{M}}. \tag{4.4}$$

Formula (4.4) implies that $\alpha_1(Q) \not\prec_{\widetilde{\mathcal{M}}} \mathcal{M}$, contradicting the statement $\alpha_1(Q) \prec_{\widetilde{\mathcal{M}}} \mathcal{M}$ proven above.

We now deduce (4.4) from (4.3). Let $\mathcal{F} \subset I$ be a finite subset, and let $x_i \in B_0 \ominus A_0 \alpha_1(A_0)$ for all $i \in \mathcal{F}$. Put $x_i = 1$ when $i \in I - \mathcal{F}$, and define $x = 1_N \otimes \bigotimes_{i \in I} x_i$. The linear span of all $\mathcal{M} x (1_N \otimes \alpha_1(A))$ forms a dense $*$-subalgebra of $\widetilde{\mathcal{M}}$. So it suffices to prove (4.4) for x, y having such a special form: x as above and $y = 1 \otimes \bigotimes_{j \in I} y_j$, where $y_j \in B_0 \ominus A_0 \alpha_1(A_0)$ when j belongs to a finite subset $\mathcal{G} \subset I$ and $y_j = 1$ when $j \notin \mathcal{G}$.

Denote $v_n = \sum_{g \in \Gamma} (v_n)^g (1 \otimes u_g)$, where $(v_n)^g \in N \mathbin{\overline{\otimes}} A$, and observe that

$$E_{\mathcal{M}}(x \alpha_1(v_n) y^*) = \sum_{g \in \Gamma} E_{N \overline{\otimes} A}\big(x \alpha_1((v_n)^g) \sigma_g(y^*)\big)(1 \otimes u_g).$$

If $g \cdot \mathcal{G} \neq \mathcal{F}$, we have $E_{N \overline{\otimes} A}\big(x \alpha_1((v_n)^g) \sigma_g(y^*)\big) = 0$. If $g \cdot \mathcal{G} = \mathcal{F}$, we have

$$E_{N \overline{\otimes} A}\big(x \alpha_1((v_n)^g) \sigma_g(y^*)\big) = E_{N \overline{\otimes} A}\Big(x \ \alpha_1\big(E_{N \overline{\otimes} A_0^{\mathcal{F}}}((v_n)^g)\big) \ \sigma_g(y^*)\Big).$$

Take finitely many $g_1, \ldots, g_k \in \Gamma$ such that $g_i \cdot \mathcal{G} = \mathcal{F}$ for all $i \in \{1, \ldots, k\}$ and such that $\{g \in \Gamma \mid g \cdot \mathcal{G} = \mathcal{F}\}$ is the disjoint union of $(\operatorname{Stab} \mathcal{F}) g_1, \ldots, (\operatorname{Stab} \mathcal{F}) g_k$. Put

$$z_n = \sum_{i=1}^{k} E_{N \overline{\otimes} (A_0^{\mathcal{F}} \rtimes \operatorname{Stab} \mathcal{F})}\big(v_n (1 \otimes u_{g_i}^*)\big)(1 \otimes u_{g_i}).$$

We have shown that

$$E_{\mathcal{M}}(x \alpha_1(v_n) y^*) = E_{\mathcal{M}}(x \alpha_1(z_n) y^*).$$

Since by (4.3), $\|z_n\|_2 \to 0$, we get (4.4).

So, take a finite subset $\mathcal{F} \subset I$ such that $Q \prec N \mathbin{\overline{\otimes}} (A_0^{\mathcal{F}} \rtimes \operatorname{Stab} \mathcal{F})$. We already assumed that for all $i \in I$ we have $P \not\prec N \mathbin{\overline{\otimes}} (A \rtimes \operatorname{Stab} i)$. We now also assume that $Q \not\prec N \otimes 1$, and we prove that the third statement of the theorem holds.

Take a larger finite subset $\mathcal{G} \supset \mathcal{F}$ such that $Q \prec N \mathbin{\overline{\otimes}} (A_0^{\mathcal{F}} \rtimes \operatorname{Stab} \mathcal{G})$ where $\mathcal{G}$ satisfies one of two alternatives: $\operatorname{Stab} \mathcal{G}$ is finite or $Q \not\prec N \mathbin{\overline{\otimes}} (A_0^{\mathcal{F}} \rtimes \operatorname{Stab} \mathcal{G}')$

whenever $\mathcal{G}'$ is strictly larger than $\mathcal{G}$. We claim that the first alternative does not occur. If it would, we would get that $Q \prec N \mathbin{\overline{\otimes}} A_0^{\mathcal{F}}$. Since $Q \not\prec N \otimes 1$, we have $\mathcal{F} \neq \emptyset$ and we can make $\mathcal{F}$ smaller, but still nonempty, until for all proper subsets $\mathcal{F}' \subset \mathcal{F}$ we have $Q \not\prec N \mathbin{\overline{\otimes}} A_0^{\mathcal{F}'}$. So we can take projections $p_0 \in Q$, $q \in N \mathbin{\overline{\otimes}} A_0^{\mathcal{F}}$, a $*$-homomorphism $\varphi : p_0 Q p_0 \to q(N \mathbin{\overline{\otimes}} A_0^{\mathcal{F}})q$ and a nonzero partial isometry $v \in p_0 \mathcal{M} q$ such that $bv = v\varphi(b)$ for all $b \in p_0 Q p_0$ and such that

$$\varphi(p_0 Q p_0) \not\prec_{N \mathbin{\overline{\otimes}} A_0^{\mathcal{F}}} N \mathbin{\overline{\otimes}} A_0^{\mathcal{F}'}$$

whenever $\mathcal{F}' \subset \mathcal{F}$ is a proper subset. Lemma 4.1(2) implies that $v^* P v \subset N \mathbin{\overline{\otimes}} (A \rtimes \operatorname{Norm} \mathcal{F})$ and hence $P \prec N \mathbin{\overline{\otimes}} (A \rtimes \operatorname{Norm} \mathcal{F})$. Since $\mathcal{F}$ is finite and nonempty, $\operatorname{Stab} \mathcal{F}$ has finite index in $\operatorname{Norm} \mathcal{F}$ and we reach the contradiction that $P \prec N \mathbin{\overline{\otimes}} (A \rtimes \operatorname{Stab} i)$ for some $i \in I$. This contradiction proves the claim above. We conclude that $Q \prec N \mathbin{\overline{\otimes}} (A_0^{\mathcal{F}} \rtimes \operatorname{Stab} \mathcal{G})$ and that $Q \not\prec N \mathbin{\overline{\otimes}} (A_0^{\mathcal{F}} \rtimes \operatorname{Stab} \mathcal{G}')$ whenever $\mathcal{G}'$ is strictly larger than $\mathcal{G}$.

Take projections $p_0 \in Q$, $q \in N \mathbin{\overline{\otimes}} (A_0^{\mathcal{F}} \rtimes \operatorname{Stab} \mathcal{G})$, a $*$-homomorphism $\varphi : p_0 Q p_0 \to q(N \mathbin{\overline{\otimes}} (A_0^{\mathcal{F}} \rtimes \operatorname{Stab} \mathcal{G}))q$ and a nonzero partial isometry $v \in p_0 \mathcal{M} q$ such that $bv = v\varphi(b)$ for all $b \in p_0 Q p_0$ and such that

$$\varphi(Q) \not\prec_{N \mathbin{\overline{\otimes}} (A_0^{\mathcal{F}} \rtimes \operatorname{Stab} \mathcal{G})} N \mathbin{\overline{\otimes}} (A_0^{\mathcal{F}} \rtimes \operatorname{Stab} \mathcal{G}') \tag{4.5}$$

whenever $\mathcal{G}'$ is strictly larger than $\mathcal{G}$.

We claim that $\mathcal{G} = \emptyset$ (and hence also $\mathcal{F} = \emptyset$). Assume the contrary. Then (4.5) implies, in particular, that

$$\varphi(Q) \not\prec_{N \mathbin{\overline{\otimes}} (A \rtimes \operatorname{Stab} \mathcal{G})} N \mathbin{\overline{\otimes}} (A \rtimes \operatorname{Stab} \mathcal{G}')$$

whenever $\mathcal{G}'$ is strictly larger than $\mathcal{G}$. Then Lemma 4.1(3) implies that $v^* P v \subset N \mathbin{\overline{\otimes}} (A \rtimes \operatorname{Norm} \mathcal{G})$ and hence $P \prec N \mathbin{\overline{\otimes}} (A \rtimes \operatorname{Norm} \mathcal{G})$, which leads as above to the contradiction that $P \prec N \mathbin{\overline{\otimes}} (A \rtimes \operatorname{Stab} i)$ for some $i \in I$. This proves the claim.

By the claim above, $\mathcal{F} = \mathcal{G} = \emptyset$. Note that vv^* commutes with $p_0 Q p_0$ and hence belongs to P. Also, by (4.5) and Lemma 4.1(1) we get that $v^* P v \subset N \mathbin{\overline{\otimes}} L\Gamma$. Finally, assume that N is a II_1 factor and that Γ is icc. Take partial isometries $v_1, \ldots, v_n \in P$ with $v_i^* v_i \leqslant p$ and such that $\sum_{i=1}^n v_i v_i^*$ is a central projection in P. Since $N \mathbin{\overline{\otimes}} L\Gamma$ is a II_1 factor, take partial isometries $w_1, \ldots, w_n \in N \mathbin{\overline{\otimes}} L\Gamma$ such that $w_i w_i^* = v^* v_i^* v_i v$ and such that the projections $w_i^* w_i$ are orthogonal. Define $x = \sum_{i=1}^n v_i v w_i$. Then, x is a partial isometry satisfying $xx^* \in \mathcal{Z}(P)$ and $x^* P x \subset N \mathbin{\overline{\otimes}} L\Gamma$. $\square$

Recall from Sections 2.3 and 2.4 the concepts of relative property (T) and relative amenability of von Neumann subalgebras.

Corollary 4.3. *Let Γ be an* icc *group, and let $\Gamma \curvearrowright I$ be an action. Assume that $\kappa \in \mathbb{N}$ such that* $\operatorname{Stab} J$ *is finite whenever $J \subset I$ and $|J| \geqslant \kappa$. Assume that* $\operatorname{Stab} i$ *is amenable for all $i \in I$. Put $A = A_0^I$ and $M = A \rtimes \Gamma$ as* *above. Let (N,τ) be a II_1 factor, and let $Q \subset p(N \mathbin{\overline{\otimes}} M)p$ be a von Neumann subalgebra satisfying at least one of the following rigidity properties:*

- *$Q \subset p(N \mathbin{\overline{\otimes}} M)p$ has the relative property (T).*
- *$Q' \cap p(N \mathbin{\overline{\otimes}} M)p$ is strongly nonamenable relative to $N \otimes 1$.*

Denote by $P \subset p(N \mathbin{\overline{\otimes}} M)p$ the quasi-normalizer of Q inside $p(N \mathbin{\overline{\otimes}} M)$. Then, at least one of the following statements is true:

- *$Q \prec N \otimes 1$.*
- *$P \prec N \mathbin{\overline{\otimes}} (A \rtimes \operatorname{Stab} i)$ for some $i \in I$.*
- *There exists $v \in N \mathbin{\overline{\otimes}} M$ with $vv^* = p$ and $v^*Pv \subset N \mathbin{\overline{\otimes}} \mathrm{L}\Gamma$.*

Proof. Assume that $Q \not\prec N \otimes 1$ and that for all $i \in I$, we have $P \not\prec N \mathbin{\overline{\otimes}} (A \rtimes \operatorname{Stab} i)$. It is sufficient to prove the following statement: for every nonzero central projection $p_0 \in \mathcal{Z}(P)$, there exists a $0 < \rho < 1$ and a $\delta > 0$ such that

$$\tau(b^*(\mathrm{id} \otimes \theta_\rho)(b)) \geqslant \delta \quad \forall b \in \mathcal{U}(Qp_0). \tag{4.6}$$

Indeed, in these circumstances Theorem 4.2 provides a nonzero partial isometry v such that $vv^* \in \mathcal{Z}(P)p_0$ and $v^*Pv \subset N \mathbin{\overline{\otimes}} \mathrm{L}\Gamma$. Moreover, since $N \mathbin{\overline{\otimes}} \mathrm{L}\Gamma$ is a II_1 factor, we can make sure that v^*v is any projection with the same trace as vv^*. As a result, a maximality argument allows us to put together several v's and find a partial isometry $v \in N \mathbin{\overline{\otimes}} M$ such that $vv^* = p$ and $v^*Pv \subset N \mathbin{\overline{\otimes}} \mathrm{L}\Gamma$.

Choose a nonzero central projection $p_0 \in \mathcal{Z}(P)$.

If $Q \subset p(N \mathbin{\overline{\otimes}} M)p$ has the relative property (T), the same is true for $Qp_0 \subset p_0(N \mathbin{\overline{\otimes}} M)p_0$. When $\rho \to 1$, the completely positive maps θ_ρ tend pointwise to the identity. The relative property (T) yields the existence of $0 < \rho < 1$ and $\delta > 0$ such that (4.6) holds for all $b \in \mathcal{U}(Qp_0)$.

If $Q' \cap p(N \mathbin{\overline{\otimes}} M)p$ is strongly nonamenable relative to $N \otimes 1$, the same is true for $(Qp_0)' \cap p_0(N \mathbin{\overline{\otimes}} M)p_0$.

Consider the von Neumann algebra $\widetilde{M}$ as in the proof of Theorem 4.2. Recall that $M \subset \widetilde{M} = B \rtimes \Gamma$ where $B = B_0^I$ and $B_0 = A_0 * \mathrm{L}\mathbb{Z}$. Also, $\theta_{\rho_t}(x) = E_M(\alpha_t(x))$ for all $x \in M$.

As explained in Section 2.2, we denote by $\subset_{\text{weak}}$ the weak containment of bimodules. We claim that

$${}_M\mathrm{L}^2(\widetilde{M} \ominus M)_M \subset_{\text{weak}} {}_{(M \otimes 1)}\mathrm{L}^2(M \mathbin{\overline{\otimes}} M)_{(1 \otimes M)}. \tag{4.7}$$

In the case of plain Bernoulli actions, this claim has been proven in [CI10, Lemma 5]. For the convenience of the reader we include a proof in our generalized Bernoulli case, using the amenability of all $\operatorname{Stab} i$, $i \in I$.

Denote by u the canonical unitary generator of $\mathrm{L}\mathbb{Z} \subset B_0$. Choose a subset $\mathcal{A}_0 \subset A_0 \ominus \mathbb{C}1$ such that $\mathcal{A}_0$ forms an orthonormal basis of $\mathrm{L}^2(A_0) \ominus \mathbb{C}1$. Define the subset $\mathcal{B}_0 \subset B_0$ given by

$$\mathcal{B}_0 := \{u^{n_1} a_1 u^{n_2} \cdots a_{k-1} u^{n_k} \mid k \geqslant 1,\ n_1, \dots, n_k \in \mathbb{Z} - \{0\},\ a_1, \dots, a_{k-1} \in \mathcal{A}_0\}.$$

By construction, we have a decomposition

$$\mathrm{L}^2(B_0) = \mathrm{L}^2(A_0) \oplus \bigoplus_{b \in \mathcal{B}_0} \overline{A_0 b A_0}$$

of $\mathrm{L}^2(B_0)$ into orthogonal A_0-A_0-subbimodules.

Whenever $\mathcal{F} \subset I$ is a nonempty finite subset and $(b_i)_{i \in \mathcal{F}}$ are elements in $\mathcal{B}_0$, we define the element $b \in B$ as

$$b = \Big(\bigotimes_{i \in \mathcal{F}} b_i\Big) \otimes \Big(\bigotimes_{i \in I - \mathcal{F}} 1\Big). \tag{4.8}$$

Define the subgroup $S < \Gamma$ given by

$$S := \{g \in \Gamma \mid g \cdot \mathcal{F} = \mathcal{F} \ \text{ and } \ b_{g \cdot i} = b_i \ \ \forall i \in \mathcal{F}\}.$$

Define $M_0 = A_0^{I - \mathcal{F}} \rtimes S$. One checks that the map $x \otimes y \to xby$ defines an M-M-bimodular unitary operator

$$\mathrm{L}^2(M) \otimes_{M_0} \mathrm{L}^2(M) \to \overline{MbM}.$$

Since $\mathcal{F}$ is finite, $S \cap \operatorname{Stab} i < S$ has finite index for all $i \in \mathcal{F}$ and so, S is amenable. It follows that M_0 is amenable and hence,

$${}_M\big(\mathrm{L}^2(M) \otimes_{M_0} \mathrm{L}^2(M)\big)_M \subset_{\text{weak}} {}_{(M \otimes 1)}\mathrm{L}^2(M \mathbin{\overline{\otimes}} M)_{(1 \otimes M)}.$$

Since the $\overline{MbM}$, b as above, form an orthogonal decomposition of $\mathrm{L}^2(\widetilde{M} \ominus M)$ into M-M-subbimodules, the claim (4.7) follows.

As in the proof of Theorem 4.2, denote $\mathcal{M} := N \mathbin{\overline{\otimes}} M$ and $\widetilde{\mathcal{M}} := N \mathbin{\overline{\otimes}} \widetilde{M}$. By claim (4.7), we have

$${}_{\mathcal{M}}\mathrm{L}^2(\widetilde{\mathcal{M}} \ominus \mathcal{M})_{\mathcal{M}} \subset_{\text{weak}} {}_{\mathcal{M}_{12}}\mathrm{L}^2(N \mathbin{\overline{\otimes}} M \mathbin{\overline{\otimes}} M)_{\mathcal{M}_{13}}.$$

Write $T := (Qp_0)' \cap p_0 \mathcal{M} p_0$. Since T is strongly nonamenable relative to $N \otimes 1$, it follows that for all nonzero projections $p_1 \in T' \cap p_0 \mathcal{M} p_0$, the bimodule ${}_T\mathrm{L}^2(p_1 \mathcal{M})_{\mathcal{M}}$ is not weakly contained in ${}_T\mathrm{L}^2(p_0(\widetilde{\mathcal{M}} \ominus \mathcal{M}))_{\mathcal{M}}$. By Lemma 2.3, we get a finite number of elements $a_1, \dots, a_n \in T$ and $\varepsilon > 0$ such that

$$\begin{aligned} &\text{if } \ x \in p_0 \widetilde{\mathcal{M}} p_0, \ \ \|x\| \leqslant 1 \ \text{ and } \ \|a_i x - x a_i\|_2 \leqslant \varepsilon \ \ \forall i = 1, \dots, n, \\ &\text{then } \ \|x - E_{\mathcal{M}}(x)\|_2 \leqslant \frac{1}{4}\|p_0\|_2. \end{aligned} \tag{4.9}$$

Taking t close enough to 0, we can make $\|a_i - \alpha_t(a_i)\|_2$ and $\|p_0 - \alpha_t(p_0)\|_2$ so small that, using the commutation of Qp_0 with $a_1, \dots, a_n$, we get

$$\begin{aligned} &\|a_i \, p_0 \alpha_t(b) p_0 - p_0 \alpha_t(b) p_0 \, a_i\|_2 \leqslant \varepsilon \\ &\quad \text{and} \quad \|\alpha_t(b) - p_0 \alpha_t(b) p_0\|_2 \leqslant \frac{1}{4}\|p_0\|_2 \ \ \forall b \in \mathcal{U}(Qp_0). \end{aligned}$$

Applying (4.9) to $x = p_0\alpha_t(b)p_0$, we conclude that $\|x - E_{\mathcal{M}}(x)\|_2 \leqslant \frac{1}{4}\|p_0\|_2$ and hence,

$$\|\alpha_t(b) - E_{\mathcal{M}}(\alpha_t(b))\|_2 \leqslant \frac{3}{4}\|p_0\|_2 \ \ \forall b \in \mathcal{U}(Qp_0).$$

Put $\rho = \rho_t^2$. For all $b \in \mathcal{U}(Qp_0)$, we get

$$\begin{aligned}\tau(p_0) - \tau(b^*(\mathrm{id}\otimes\theta_\rho)(b)) &= \tau(p_0) - \|(\mathrm{id}\otimes\theta_{\rho_t})(b)\|_2^2 \\ &= \|\alpha_t(b) - E_{\mathcal{M}}(\alpha_t(b))\|_2^2 \leqslant \frac{9}{16}\tau(p_0).\end{aligned}$$

Hence, (4.6) holds with $\delta = \frac{7}{16}\tau(p_0)$. □

Remark 4.4. We make the following two observations about Corollary 4.3, but we do not use them in the rest of the paper. In the situation where $Q \subset p(N \mathbin{\overline{\otimes}} M)p$ has the relative property (T), Corollary 4.3 can be strengthened in two ways. First of all, the same conclusion holds without the assumption that $\operatorname{Stab} i$ is amenable for all $i \in I$. In the relative property (T) part of the proof, we did not use the amenability of $\operatorname{Stab} i$. Secondly, if we assume that $\operatorname{Stab} i$ is amenable and that $Q \subset P$ has the relative property (T), then it is easy to see that the option $P \prec N \mathbin{\overline{\otimes}} (A \rtimes \operatorname{Stab} i)$ actually implies that $Q \prec N \otimes 1$.

5. Clustering sequences techniques and intertwining of abelian subalgebras

Throughout this section assume that $\Gamma \curvearrowright I$ is an action of the countable group Γ on the countable set I. Assume that $\kappa > 0$ such that the stabilizer $\operatorname{Stab} \mathcal{F}$ is finite whenever $\mathcal{F} \subset I$ is a subset with $|\mathcal{F}| \geqslant \kappa$. Let (X_0, μ_0) be a nontrivial standard probability space and put $A := \mathrm{L}^\infty(X_0^I)$, together with the action $\Gamma \curvearrowright A$ given by the generalized Bernoulli shift. Define $M = A \rtimes \Gamma$.

We prove a strong structural result for abelian von Neumann subalgebras $D \subset (M \mathbin{\overline{\otimes}} M)^t$ that are normalized by many unitaries in $(\mathrm{L}\Gamma \mathbin{\overline{\otimes}} \mathrm{L}\Gamma)^t$. Later we shall apply this structural result to $D = \Delta(A)$ whenever $\Delta : M \to (M \mathbin{\overline{\otimes}} M)^t$ is (the amplification of) the comultiplication given by another group von Neumann algebra or group measure space decomposition of M. This structural result and its proof are very similar to [Ioa11, Th. 6.1]. However, we give all the details because the generalization from plain Bernoulli to generalized Bernoulli actions is not totally innocent. Both here and in [Ioa11] the technique is very much inspired by the *clustering sequences techniques* from [Pop06d, §§1–4]. For a more gentle introduction to these matters, we refer to the lecture notes [Vae11].

THEOREM 5.1. *As above let* $\Gamma \curvearrowright I$ *be such that* $\operatorname{Stab} \mathcal{F}$ *is finite whenever* $\mathcal{F} \subset I$ *and* $|\mathcal{F}| \geqslant \kappa$. *Put* $A := \mathrm{L}^\infty(X_0^I)$ *and* $M = A \rtimes \Gamma$.

Assume that $t > 0$ *and that* $D \subset (M \mathbin{\overline{\otimes}} M)^t$ *is an abelian von Neumann subalgebra that is normalized by a group of unitaries* $(\gamma(s))_{s\in\Lambda}$ *that belong* *to* $(\mathrm{L}\Gamma \mathbin{\overline{\otimes}} \mathrm{L}\Gamma)^t$*. Denote by* $P \subset (M \mathbin{\overline{\otimes}} M)^t$ *the quasi-normalizer of* D *inside* $(M \mathbin{\overline{\otimes}} M)^t$*. Make the following assumptions*:

(1) $D \not\prec M \otimes 1$ *and* $D \not\prec 1 \otimes M$.
(2) *For all* $i \in I$*, we have* $P \not\prec M \mathbin{\overline{\otimes}} (A \rtimes \operatorname{Stab} i)$ *and* $P \not\prec (A \rtimes \operatorname{Stab} i) \mathbin{\overline{\otimes}} M$.
(3) $P \not\prec M \mathbin{\overline{\otimes}} \mathrm{L}\Gamma$ *and* $P \not\prec \mathrm{L}\Gamma \mathbin{\overline{\otimes}} M$.
(4) *For all* $i \in I$*, we have* $\gamma(\Lambda)'' \not\prec \mathrm{L}(\Gamma) \mathbin{\overline{\otimes}} \mathrm{L}(\operatorname{Stab} i)$ *and* $\gamma(\Lambda)'' \not\prec \mathrm{L}(\operatorname{Stab} i) \mathbin{\overline{\otimes}} \mathrm{L}(\Gamma)$.

Denote $C := D' \cap (M \mathbin{\overline{\otimes}} M)^t$*. Then for every nonzero projection* $q \in \mathcal{Z}(C)$*, we have that* $Cq \prec A \mathbin{\overline{\otimes}} A$.

Remark 5.2. To avoid unnecessary notational complexity we did not formulate the obvious more general result for subalgebras of $(M_1 \mathbin{\overline{\otimes}} M_2)^t$ where $M_i = \mathrm{L}^\infty(X_i^{I_i}) \rtimes \Gamma_i$ and where both $\Gamma_i \curvearrowright I_i$ satisfy the finiteness assumption on the stabilizer groups. Also there is an obvious version of the theorem for subalgebras $D \subset M^t$ that are normalized by unitaries $\gamma(s) \in \mathrm{L}(\Gamma)^t$.

Proof. Note that because D is abelian, we have $\mathcal{Z}(C) = C' \cap (M \mathbin{\overline{\otimes}} M)^t$. The main part of the proof consists in showing that for every nonzero projection $q \in \mathcal{Z}(C)$, we have that $Cq \prec M \mathbin{\overline{\otimes}} A$. At the end we then deduce that actually $Cq \prec A \mathbin{\overline{\otimes}} A$ for every nonzero projection $q \in \mathcal{Z}(C)$. Consider

$$\mathcal{P} := \{q_1 \in \mathcal{Z}(C) \mid q_1 \text{ is a projection and for every nonzero projection } q \in \mathcal{Z}(C)q_1 \text{ we have that } Cq \prec M \mathbin{\overline{\otimes}} A\}.$$

One easily checks that $\mathcal{P}$ admits a maximum q_2 and that this maximum commutes with the normalizer of C; in particular, with the unitaries $(\gamma(s))_{s\in\Lambda}$. (See [Vae13, Prop. 2.5] for details.) We have to prove that $q_2 = 1$. If not, we can replace D by $D(1-q_2)$ and $\gamma(s)$ by $\gamma(s)(1-q_2)$. So in the end, we only need to prove that $\mathcal{P}$ is nonempty. This means that we have to prove that $C \prec M \mathbin{\overline{\otimes}} A$.

We split the proof of the statement $C \prec M \mathbin{\overline{\otimes}} A$ into several steps. We use the following notation. We use the letter Q to denote all kind of orthogonal projections related to the infinite tensor product $A = A_0^I$ and the letter P to denote all kind of orthogonal projections related to the group Γ. All these projections Q and P project onto subspaces of the form $\mathrm{L}^2(M) \otimes \mathcal{K}$ and they all commute.

- For every subset $\mathcal{F} \subset I$, we denote by $Q_{\mathcal{F}}$ the orthogonal projection onto the closed linear span of $\{M \otimes A_0^{\mathcal{F}} u_g \mid g \in \Gamma\}$.
- For every $\ell \in \mathbb{N}$, we denote by $Q^{\geqslant \ell}$ the orthogonal projection onto the closed linear span of $\{M \otimes (A_0 \ominus \mathbb{C}1)^{\mathcal{F}} u_g \mid \mathcal{F} \subset I, \ell \leqslant |\mathcal{F}| < \infty, g \in \Gamma\}$.
- For every subset $S \subset \Gamma$, we denote by P_S the orthogonal projection onto the closed linear span of $\{M \otimes A u_g \mid g \in S\}$.

We denote by $Q_{\mathcal{F}}^{\geqslant \ell}$ the product of $Q^{\geqslant \ell}$ and $Q_{\mathcal{F}}$.

In general, the projection $Q_{\mathcal{F}}$ does not behave well with respect to the operator norm $\| \cdot \|$. Because of the formula

$$P_S(Q_{\mathcal{F}}(x)) = \sum_{g \in S} E_{M \overline{\otimes} A_0^{\mathcal{F}}}(x(1 \otimes u_g)^*)\,(1 \otimes u_g),$$

we do get $\|P_S(Q_{\mathcal{F}}(x))\| \leqslant |S|\,\|x\|$ and $\|P_S(x)\| \leqslant |S|\,\|x\|$ for all $x \in M \overline{\otimes} M$ and all subsets $\mathcal{F} \subset I$.

In a few cases, we use the same notation $Q_{\mathcal{F}}, Q^{\geqslant \ell}, P_S$ to denote projections of $\mathrm{L}^2(M)$ onto the corresponding obvious subspaces.

To avoid a too heavy notation, we assume that $t \leqslant 1$. So we have a projection $p \in \mathrm{L}(\Gamma \times \Gamma)$ such that $D \subset p(M \overline{\otimes} M)p$ and $\gamma(s) \in p\mathrm{L}(\Gamma \times \Gamma)p$. This simplification does not hide any essential part of the argument.

Step 1. For every $\varepsilon > 0$ and every $\ell \in \mathbb{N}$, there exists a unitary $a \in D$ such that

$$\|a - (Q^{\geqslant \ell} \otimes Q^{\geqslant \ell})(a)\|_2 < \varepsilon.$$

Proof. Denote by $\sigma : M \overline{\otimes} M \to M \overline{\otimes} M$ the flip automorphism $\sigma(a \otimes b) = b \otimes a$. Consider the projection

$$\widetilde{p} := \begin{pmatrix} p & 0 \\ 0 & \sigma(p) \end{pmatrix} \in \mathrm{M}_2(\mathbb{C}) \overline{\otimes} M \overline{\otimes} M.$$

Define the von Neumann subalgebra $\widetilde{D} \subset \widetilde{p}(\mathrm{M}_2(\mathbb{C}) \overline{\otimes} M \overline{\otimes} M)\widetilde{p}$ given by

$$\widetilde{D} := \left\{ \begin{pmatrix} a & 0 \\ 0 & \sigma(a) \end{pmatrix} \;\middle|\; a \in D \right\}.$$

Denote by $\widetilde{P}$ the quasi-normalizer of $\widetilde{D}$ inside $\widetilde{p}(\mathrm{M}_2(\mathbb{C}) \overline{\otimes} M \overline{\otimes} M)\widetilde{p}$. By assumption (1), we have $\widetilde{D} \not\prec M \otimes 1$. By assumption (2), we have for all $i \in I$ that $\widetilde{P} \not\prec M \overline{\otimes} (A \rtimes \operatorname{Stab} i)$. By assumption (3) we have $\widetilde{P} \not\prec M \overline{\otimes} \mathrm{L}\Gamma$. We now apply Theorem 4.2. We conclude that (4.2) in Theorem 4.2 cannot hold. So, given $\varepsilon > 0$ and $\ell \in \mathbb{N}$, we find a unitary $b \in \widetilde{D}$ such that $\|d - (1 \otimes Q^{\geqslant \ell})(d)\|_2 < \varepsilon/2$. Writing

$$d = \begin{pmatrix} a & 0 \\ 0 & \sigma(a) \end{pmatrix},$$

we have found a unitary $a \in D$ such that $\|a - (1 \otimes Q^{\geqslant \ell})(a)\|_2 < \varepsilon/2$ and $\|a - (Q^{\geqslant \ell} \otimes 1)(a)\|_2 < \varepsilon/2$. Hence also $\|a - (Q^{\geqslant \ell} \otimes Q^{\geqslant \ell})(a)\|_2 < \varepsilon$. ☐

Step 2. There is a sequence of group elements $g_n \in \Lambda$ such that for all $i \in I$ and $g, h \in \Gamma \times \Gamma$, we have

(5.1)
$$\|E_{\mathrm{L}(\Gamma \times \operatorname{Stab} i)}(u_g \gamma(g_n) u_h)\|_2 \to 0 \quad \text{and} \quad \|E_{\mathrm{L}(\operatorname{Stab} i \times \Gamma)}(u_g \gamma(g_n) u_h)\|_2 \to 0.$$

Proof. This follows immediately from assumption (4), Definition 2.1 and Remark 2.2. □

From now on, we fix a sequence (g_n) in Λ satisfying (5.1). We put $v_n := \gamma(g_n)$.

Step 3. For all $x \in M \mathbin{\overline{\otimes}} M$ and all finite subsets $\mathcal{F} \subset I$, we have

$$\|v_n x v_n^* - Q_{I-\mathcal{F}}(v_n x v_n^*)\|_2 \to 0. \tag{5.2}$$

Proof. It suffices to check (5.2) when x is of the form $x = x_0 \otimes a u_g$ with $x_0 \in M$, $g \in \Gamma$ and $a \in A_0^{\mathcal{G}}$ for some finite subset $\mathcal{G} \subset I$. Fix a finite subset $\mathcal{F} \subset I$. Define

$$K := \{k \in \Gamma \mid k\mathcal{G} \cap \mathcal{F} = \emptyset\}.$$

Define $w_n = P_K(v_n)$. Then, $w_n \in \mathrm{L}^2(M \mathbin{\overline{\otimes}} M)$ and by (5.1), $\|v_n - w_n\|_2 \to 0$. Hence, $\|v_n x v_n^* - w_n x v_n^*\|_2 \to 0$. Since by construction $w_n x v_n^*$ lies in the image of $Q_{I-\mathcal{F}}$, the formula (5.2) follows. □

Step 4. For all $a \in D$ and all $\varepsilon > 0$, there exists a finite subset $S \subset \Gamma$ such that

$$\|v_n a v_n^* - (P_S \otimes P_S)(v_n a v_n^*)\|_2 \leqslant \varepsilon \ \ \forall n.$$

Proof. Choose a unitary $a \in \mathcal{U}(D)$, and put $a_n := v_n a v_n^*$. Since the projections $P_S \otimes 1$ and $1 \otimes P_S$ commute, by symmetry it suffices to prove that for all $\varepsilon > 0$, there exists a finite subset $S \subset \Gamma$ such that $\|(1 \otimes P_S)(a_n)\|_2 \geqslant \|p\|_2 - 4\varepsilon$ for all n large enough.

Write $\delta = \varepsilon \|p\|_2$. By Step 1 take a unitary $b \in \mathcal{U}(D)$ such that $\|b - Q^{\geqslant\kappa}(b)\|_2 \leqslant \delta$. By the Kaplansky density theorem, take a finite subset $\mathcal{G} \subset I$ and an element

$$b_0 \in \operatorname{span}\{x_0 \otimes x_1 u_g \mid x_0 \in M, x_1 \in A_0^{\mathcal{G}}, g \in \Gamma\}$$

such that $\|b_0\| \leqslant 1$, $\|b - b_0\|_2 \leqslant \delta$ and $\|b_0\|_2 \leqslant \|b\|_2 = \|p\|_2$. Put $\eta = Q^{\geqslant\kappa}(b_0)$, and observe that $\|\eta\|_2 \leqslant \|b_0\|_2 \leqslant \|p\|_2$, that $\|b - \eta\|_2 \leqslant 2\delta$ and that

$$\eta \in \operatorname{span}\{y_0 \otimes y_1 u_h \mid y_0 \in M, y_1 \in (A_0 \ominus \mathbb{C}1)^J, J \subset \mathcal{G}, |J| \geqslant \kappa, h \in \Gamma\}.$$

Since a_n and b are commuting unitaries in $p(M \mathbin{\overline{\otimes}} M)p$, we have $\langle a_n\, b, b\, a_n\rangle = \tau(p)$ and hence,

$$\Big|\tau(p) - \langle a_n\, b_0, \eta\, a_n\rangle\Big| \leqslant 3\delta \tag{5.3}$$

for all n. Put $S := \{g \in \Gamma \mid \ |g \cdot \mathcal{G} \cap \mathcal{G}| \geqslant \kappa\}$. By our assumption on the action $\Gamma \curvearrowright I$, the set S is finite.

Claim. We have that $\langle P_{\Gamma - S}(a_n)\, b_0, \eta\, a_n\rangle \to 0$. Given the special form of b_0 and η, it suffices to prove the claim for $b_0 = x_0 \otimes x_1 u_g$ and $\eta = y_0 \otimes y_1 u_h$ where $x_0, y_0 \in M$, $x_1 \in A_0^{\mathcal{G}}$, $y_1 \in (A_0 \ominus \mathbb{C}1)^J$, $J \subset \mathcal{G}$, $|J| \geqslant \kappa$ and $g, h \in \Gamma$.

Put $d_n := Q_{I-(\mathcal{G}\cup h^{-1}\mathcal{G})}(a_n)$. By construction, ηd_n lies in the closed linear span of

$$M \otimes (A_0 \ominus \mathbb{C}1)^J A_0^{I-\mathcal{G}} u_k, k \in \Gamma.$$

On the other hand, $P_{\Gamma-S}(d_n) b_0$ lies in the closed linear span of

$$M \otimes A_0^{r\mathcal{G}\cup(I-\mathcal{G})} u_k, r \in \Gamma - S, k \in \Gamma.$$

Since $|r\mathcal{G} \cap J| < \kappa$ for all $r \in \Gamma - S$, the two subspaces are orthogonal. Hence, $\langle P_{\Gamma-S}(d_n)\, b_0, \eta d_n\rangle = 0$ for all n. By Step 3, $\|a_n - d_n\|_2 \to 0$. Hence, the claim follows.

Combining the claim with (5.3), we can take n_0 such that

$$\Big|\tau(p) - \langle P_S(a_n)\, b_0, \eta\, a_n\rangle\Big| \leqslant 4\delta$$

for all $n \geqslant n_0$. It follows that

$$\tau(p) - 4\delta \leqslant |\langle P_S(a_n)\, b_0, \eta\, a_n\rangle| \leqslant \|P_S(a_n)\|_2\, \|b_0\|\, \|\eta\|_2\, \|a_n\| \leqslant \|P_S(a_n)\|_2\, \|p\|_2.$$

Since $\tau(p) - 4\delta = \|p\|_2(\|p\|_2 - 4\varepsilon)$, we have shown that $\|P_S(a_n)\|_2 \geqslant \|p\|_2 - 4\varepsilon$ for all $n \geqslant n_0$. $\square$

Recall from (3.1) the notion of the height of an element in a group von Neumann algebra. We now use this notion in the group von Neumann algebra $\mathrm{L}(\Gamma \times \Gamma)$. So, for all $v \in \mathrm{L}(\Gamma \times \Gamma)$, we consider

$$h(v) = \max\{|\tau(v u_g^*)| \mid g \in \Gamma \times \Gamma\}.$$

Step 5. There exists a $\delta > 0$ such that $h(v_n) \geqslant \delta$ for all n.

Proof. If the assertion does not hold, we can pass to a subsequence and assume that $h(v_n) \to 0$.

Claim. Take $J_1, J_2 \subset I$ with $|J_i| \geqslant \kappa$. For all $a \in (A_0 \ominus \mathbb{C}1)^{J_1} \otimes (A_0 \ominus \mathbb{C}1)^{J_2}$ and for all sequences w_n in the unit ball of $\mathrm{L}(\Gamma \times \Gamma)$, we have

$$\|E_{A \overline{\otimes} A}(v_n a w_n^*)\|_2 \to 0.$$

To prove the claim, denote by $(v)_g$, $g \in \Gamma \times \Gamma$, the Fourier coefficients of an element $v \in \mathrm{L}(\Gamma \times \Gamma)$. So, by definition and with $\|\cdot\|_2$-convergence, we have

$$v = \sum_{g \in \Gamma \times \Gamma} (v)_g u_g.$$

Take finite sets $\mathcal{F}_i \subset \Gamma$ such that for all $g \in \Gamma - \mathcal{F}_i$, we have $|g \cdot J_i \cap J_i| < \kappa$. Put $\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2$. So, whenever $g \in (\Gamma \times \Gamma) - \mathcal{F}$, we have $a \perp \sigma_g(a)$. As a result, we get

$$
\begin{aligned}
&\|E_{A\overline{\otimes}A}(v_n a w_n^*)\|_2^2 \\
&\quad = \sum_{k\in\mathcal{F}} \sum_{g\in\Gamma\times\Gamma} (v_n)_g \, \overline{(v_n)_{gk}} \, \overline{(w_n)_g} \, (w_n)_{gk} \, \tau(a\sigma_k(a^*)) \\
&\quad \leqslant \|a\|_2^2 \, h(v_n)^2 \sum_{k\in\mathcal{F}} \sum_{g\in\Gamma\times\Gamma} |(w_n)_g| \, |(w_n)_{gk}| \\
&\quad \leqslant \|a\|_2^2 \, h(v_n)^2 \sum_{k\in\mathcal{F}} \left(\Big(\sum_{g\in\Gamma\times\Gamma} |(w_n)_g|^2 \Big)^{1/2} \Big(\sum_{g\in\Gamma\times\Gamma} |(w_n)_{gk}|^2 \Big)^{1/2} \right) \\
&\quad \leqslant \|a\|_2^2 \, |\mathcal{F}| \, h(v_n)^2 \to 0.
\end{aligned}
$$

This proves the claim. Applying the claim to w_n of the form $w_n = u_g v_n u_h^*$, we get the following: for all $\eta \in \mathrm{L}^2(M \mathbin{\overline{\otimes}} M)$ satisfying $\eta = (Q^{\geqslant\kappa} \otimes Q^{\geqslant\kappa})(\eta)$ and for all finite subsets $S \subset \Gamma$, we have

$$
\|(P_S \otimes P_S)(v_n \eta v_n^*)\|_2 \to 0. \tag{5.4}
$$

By Step 1 take a unitary $a \in \mathcal{U}(D)$ such that $\|a - (Q^{\geqslant\kappa} \otimes Q^{\geqslant\kappa})(a)\|_2 \leqslant \|p\|_2/2$. Formula (5.4) implies that for all $S \subset \Gamma$ finite, we have

$$
\limsup_n \|(P_S \otimes P_S)(v_n a v_n^*)\|_2 \leqslant \|p\|_2/2.
$$

This is a contradiction with Step 4. □

Step 6. Take $\delta > 0$ such that $h(v_n) \geqslant \sqrt{6\delta}$ for all n.

For every $\varepsilon > 0$, there exists a unitary $a \in \mathcal{U}(D)$, finite subsets $S \subset \Gamma$, $\mathcal{F} \subset I$ and a sequence $h_n \in \Gamma$ such that, writing $x_n = v_n a v_n^*$, we have for all n,

- $\|x_n - P_S(x_n)\|_2 \leqslant \varepsilon$,
- $\|x_n - Q^{\geqslant\kappa}(x_n)\|_2 \leqslant \varepsilon$,
- $\|x_n - Q_{h_n\cdot\mathcal{F}}(x_n)\|_2 \leqslant \|p\|_2 - 2\delta$,
- $\|x_n - Q_{I-\mathcal{G}}(x_n)\|_2 \to 0$ for every finite subset $\mathcal{G} \subset I$.

Proof. Choose $\varepsilon > 0$. By Step 1 take $a \in \mathcal{U}(D)$ such that $\|a - Q^{\geqslant\kappa}(a)\|_2 \leqslant \varepsilon$. Put $x_n = v_n a v_n^*$. Since the image of $Q^{\geqslant\kappa}$ is an $(M\overline{\otimes}\mathrm{L}(\Gamma))$-$(M\overline{\otimes}\mathrm{L}(\Gamma))$-bimodule, we have

$$
\|x_n - Q^{\geqslant\kappa}(x_n)\|_2 = \|a - Q^{\geqslant\kappa}(a)\|_2 \leqslant \varepsilon
$$

for all n. By Step 4, take a finite subset $S \subset \Gamma$ such that $\|x_n - P_S(x_n)\|_2 \leqslant \varepsilon$ for all n. By Step 3, we have $\|x_n - Q_{I-\mathcal{G}}(x_n)\|_2 \to 0$ for every finite subset $\mathcal{G} \subset I$.

Take a finite subset $\mathcal{F} \subset I$ such that $\|a - Q_{\mathcal{F}}(a)\|_2 \leqslant \delta$. Choose elements $k_n \in \Gamma \times \Gamma$ such that $|\tau(v_n u_{k_n}^*)| \geqslant \sqrt{6\delta}$ for all n. Denote by $h_n \in \Gamma$ the second component of k_n.

Denote $w_n = \tau(v_n u_{k_n}^*) u_{k_n}$ and $y_n = w_n a v_n^*$. It follows that

$$\begin{aligned}\|x_n - Q_{h_n \cdot \mathcal{F}}(x_n)\|_2 &= \|(1 - Q_{h_n \cdot \mathcal{F}})(x_n - y_n) + (1 - Q_{h_n \cdot \mathcal{F}})(y_n)\|_2 \\ &\leqslant \|x_n - y_n\|_2 + \|y_n - Q_{h_n \cdot \mathcal{F}}(y_n)\|_2.\end{aligned}$$

We consecutively get

$$\|x_n - y_n\|_2 \leqslant \|v_n - w_n\|_2 \leqslant \sqrt{\|p\|_2^2 - 6\delta} \leqslant \|p\|_2 - 3\delta$$

and

$$\|y_n - Q_{h_n \cdot \mathcal{F}}(y_n)\|_2 = \|w_n(a - Q_{\mathcal{F}}(a))v_n^*\|_2 \leqslant \|a - Q_{\mathcal{F}}(a)\|_2 \leqslant \delta.$$

Altogether we have $\|x_n - Q_{h_n \cdot \mathcal{F}}(x_n)\|_2 \leqslant \|p\|_2 - 2\delta$. □

We finally gathered enough results to prove that $C \prec M \mathbin{\overline{\otimes}} A$.

Step 7. We have that $C \prec M \mathbin{\overline{\otimes}} A$.

Proof. Assume that $C \not\prec M \mathbin{\overline{\otimes}} A$. Note that $M \mathbin{\overline{\otimes}} M = (M \mathbin{\overline{\otimes}} A) \rtimes \Gamma$, where Γ acts trivially on M. By [Ioa11, Th. 1.3.2], for every $\varepsilon > 0$ and every $k \in \mathbb{N}$, there exists a unitary $d \in \mathcal{U}(C)$ such that $\|P_{\mathcal{G}}(d)\|_2 < \varepsilon$ for all subsets $\mathcal{G} \subset \Gamma$ with $|\mathcal{G}| \leqslant k$.

Take $a \in \mathcal{U}(D)$, finite subsets $S \subset \Gamma$, $\mathcal{F} \subset I$ and a sequence $h_n \in \Gamma$ satisfying the conclusion of Step 6 with $\varepsilon \leqslant \delta/8$. Whenever $Z \subset \Gamma$ is finite, we define the orthogonal projection

$$R_Z = \bigvee_{g \in Z} Q_{g \cdot \mathcal{F}}.$$

Claim. Whenever Z_n is a sequence of finite subsets of Γ such that $\sup_n |Z_n| < \infty$, there exists a sequence of larger finite subsets $Z_n' \supset Z_n$ such that $\sup_n |Z_n'| < \infty$ and

$$\liminf_n \|R_{Z_n'}(x_n) - R_{Z_n}(x_n)\|_2 \geqslant \delta. \tag{5.5}$$

Once the claim is proven, we inductively construct $Z_n^1 \subset Z_n^2 \subset \cdots$. Since the vectors $R_{Z_n^{k+1}}(x_n) - R_{Z_n^k}(x_n)$ are orthogonal for different k, we arrive at the contradiction

$$\|p\|_2 = \liminf_n \|x_n\|_2^2 \geqslant k\delta^2 \ \ \forall k \in \mathbb{N}.$$

We now prove the claim. Let the sequence Z_n be given. For every n, denote

$$L_n := \{g \in \Gamma \mid \exists k \in Z_n \text{ such that } |gh_n\mathcal{F} \cap k\mathcal{F}| \geqslant \kappa\}.$$

Since $\operatorname{Stab} J$ is finite whenever $|J| \geqslant \kappa$, it follows that $\sup_n |L_n| < \infty$. So, we can take a unitary $d \in \mathcal{U}(C)$ such that $\|P_{L_n}(d)\|_2 \leqslant \varepsilon/(2|S|)$ for every n. Take a finite set $S' \subset \Gamma$ such that $\|d - P_{S'}(d)\|_2 \leqslant \varepsilon/(2|S|)$. Put $K_n = S' - L_n$. We retain that $\|d - P_{K_n}(d)\|_2 \leqslant \varepsilon/|S|$ for all n and that $|gh_n\mathcal{F} \cap k\mathcal{F}| < \kappa$ for all $g \in K_n$ and all $k \in Z_n$. Put $Z'_n = K_n h_n \cup Z_n$. We prove that Z'_n satisfies (5.5).

Using the Kaplansky density theorem, take a finite subset $\mathcal{G} \subset I$ and $d_0 \in M \mathbin{\overline{\otimes}} M$ such that $d_0 = Q_{\mathcal{G}}(d_0)$, $\|d_0\| \leqslant 1$ and $\|d - d_0\|_2 \leqslant \varepsilon/|S|$. Write $d_n := P_{K_n}(d_0)$. Hence, $\|d - d_n\|_2 \leqslant 2\varepsilon/|S|$. Also write $x'_n := Q_{h_n \cdot \mathcal{F}}(P_S(x_n)) = P_S(Q_{h_n \cdot \mathcal{F}}(x_n))$. Note that $\|x'_n\| \leqslant |S|$ and $\|x_n - x'_n\|_2 \leqslant \|p\|_2 - 2\delta + \varepsilon$ for all n. As a result,

$$\|dx_n - d_n x'_n\|_2 \leqslant \|d\|\,\|x_n - x'_n\|_2 + \|x'_n\|\,\|d - d_n\|_2 \leqslant \|p\|_2 - 2\delta + 3\varepsilon.$$

Define the orthogonal projection

$$R_n := \bigvee_{g \in K_n} Q_{\mathcal{G} \cup gh_n\mathcal{F}}.$$

Since $d_n x'_n$ lies in the image of R_n, it follows that $\|(1 - R_n)(dx_n)\|_2 \leqslant \|p\|_2 - 2\delta + 3\varepsilon$. But $dx_n = x_n d$. Hence, $\|R_n(x_n d)\|_2 \geqslant 2\delta - 3\varepsilon$.

Observe that

$$\|x_n d - P_S(x_n) d_0\|_2 \leqslant \|x_n - P_S(x_n)\|_2\,\|d\| + \|P_S(x_n)\|\,\|d - d_0\|_2 \leqslant 2\varepsilon.$$

So, $\|R_n(P_S(x_n)d_0)\|_2 \geqslant 2\delta - 5\varepsilon$. Write

$$R'_n := \bigvee_{g \in K_n} Q_{\mathcal{G} \cup S\mathcal{G} \cup gh_n\mathcal{F}}.$$

Since $R_n \leqslant R'_n$, we have $\|R'_n(P_S(x_n)d_0)\|_2 \geqslant 2\delta - 5\varepsilon$. But, $R'_n(P_S(x_n)d_0) = R'_n(P_S(x_n))d_0$ and $\|d_0\| \leqslant 1$. It follows that

$$\begin{aligned}\|R'_n(x_n)\|_2 \geqslant \|P_S(R'_n(x_n))\|_2 &= \|R'_n(P_S(x_n))\|_2 \geqslant \|R'_n(P_S(x_n))d_0\|_2 \\ &= \|R'_n(P_S(x_n)d_0)\|_2 \geqslant 2\delta - 5\varepsilon.\end{aligned}$$

Since $\|x_n - Q^{\geqslant\kappa}(x_n)\|_2 \leqslant \varepsilon$ and since $\|x_n - Q_{I - (\mathcal{G} \cup S\mathcal{G})}(x_n)\|_2 \to 0$, we can take n_0 such that

$$\|R''_n(x_n)\|_2 \geqslant 2\delta - 7\varepsilon \ \ \forall n \geqslant n_0, \quad \text{where } \ R''_n := \bigvee_{g \in K_n} Q^{\geqslant\kappa}_{gh_n\mathcal{F}}.$$

Whenever $g \in K_n$ and $k \in Z_n$, we have $|gh_n\mathcal{F}_n \cap k\mathcal{F}| < \kappa$. So, the projections $Q^{\geqslant\kappa}_{gh_n\mathcal{F}}$ and $Q_{k\mathcal{F}}$ have orthogonal ranges. Hence, R''_n and R_{Z_n} are orthogonal as well. By construction, $R''_n \leqslant R'_n$. It follows that

$$\begin{aligned}\|R_{Z'_n}(x_n) - R_{Z_n}(x_n)\|_2 &\geqslant \|R''_n(R_{Z'_n}(x_n) - R_{Z_n}(x_n))\|_2 \\ &= \|R''_n(x_n)\|_2 \geqslant 2\delta - 7\varepsilon \geqslant \delta\end{aligned}$$

for all $n \geqslant n_0$. So, we have proven (5.5). □

Step 8. End of the proof of Theorem 5.1.

We have shown that $Cq \prec M \overline{\otimes} A$ for every nonzero projection $q \in \mathcal{Z}(C) = C' \cap p(M \overline{\otimes} M)p$. This means that the following holds (see [Vae13, Lemma 2.4 and Proposition 2.5] for details): for every $\varepsilon > 0$, there exists a finite set $S \subset \Gamma$ such that $\|d - (1 \otimes P_S)(d)\|_2 \leqslant \varepsilon/2$ for every unitary $d \in \mathcal{U}(C)$. By symmetry, we also find a finite set $S' \subset \Gamma$ such that $\|d - (P_{S'} \otimes 1)(d)\|_2 \leqslant \varepsilon/2$ for all $d \in \mathcal{U}(C)$. Taking the union of S and S', we have found a finite set $S \subset \Gamma$ such that $\|d - (P_S \otimes P_S)(d)\|_2 \leqslant \varepsilon$ for all $d \in \mathcal{U}(C)$. This means that $Cq \prec A \overline{\otimes} A$ for all nonzero projections $q \in \mathcal{Z}(C)$. □

6. A conjugacy criterion for group actions

Suppose that we are given an embedding of group measure space factors $B \rtimes \Lambda \hookrightarrow A \rtimes \Gamma$ such that $B = A$ and such that $v\mathrm{L}\Lambda v^* \subset \mathrm{L}\Gamma$ for some unitary $v \in A \rtimes \Gamma$. Under the right conditions, one can deduce from this information the existence of a unitary $w \in A \rtimes \Gamma$ such that $wBw^* = A$ and $wv_sw^* = \omega(s)u_{\delta(s)}$ for all $s \in \Lambda$, where $\delta : \Lambda \to \Gamma$ is a group morphism and $\omega : \Lambda \to \mathbb{T}$ is a character. Such a result was first proven in [Pop06d, Th. 5.2] and generalized in [Ioa11, Th. 7.1]. We now prove a further generalization, involving arbitrary amplifications and weaker assumptions. We give a more elementary proof in the spirit of [Vae07, Prop. 9.3].

THEOREM 6.1. *Let $\Gamma \curvearrowright (X, \mu)$ be a free ergodic probability measure-preserving action. Put $A = \mathrm{L}^\infty(X)$ and $M = A \rtimes \Gamma$. Let $p \in \mathrm{M}_n(\mathbb{C}) \otimes \mathrm{L}\Gamma$ be a projection. Assume that $C \subset p(\mathrm{M}_n(\mathbb{C}) \otimes M)p$ is a von Neumann subalgebra and $\gamma : \Lambda \to \mathcal{U}(p(\mathrm{M}_n(\mathbb{C}) \otimes \mathrm{L}\Gamma)p)$ is a group morphism such that the following conditions hold*:

(1) *$C \prec A$ and $C' \cap p(\mathrm{M}_n(\mathbb{C}) \otimes M)p = \mathcal{Z}(C)$.*

(2) *The unitaries $\gamma(s)$ normalize C, and the action $(\mathrm{Ad}\,\gamma(s))_{s\in\Lambda}$ on $\mathcal{Z}(C)$ is weakly mixing.*

Then there exist

- *a subgroup $\Gamma_1 < \Gamma$, a finite normal subgroup $K \lhd \Gamma_1$ and a finite-dimensional unitary representation $\rho : K \to \mathcal{U}(\mathrm{M}_d(\mathbb{C}))$ with corresponding projection $p_K := |K|^{-1} \sum_{k\in K} \rho(k) \otimes u_k$;*
- *a group homomorphism $\delta : \Lambda \to \mathcal{G}/L$ where*

  $$\mathcal{G} := \{u \otimes u_g \mid u \in \mathcal{U}(\mathrm{M}_d(\mathbb{C})),\ g \in \Gamma_1,\ \rho(gkg^{-1}) = u\rho(k)u^*\ \forall k \in K\}$$

  and where the normal subgroup $K \cong L \lhd \mathcal{G}$ is given by $L := \{\rho(k) \otimes u_k \mid k \in K\}$;
- *a Γ_1-invariant projection $q \in A$;*
- *a partial isometry $v \in \mathrm{M}_{n,d}(\mathbb{C}) \otimes \mathrm{L}\Gamma$ with $vv^* = p$ and v^*v commuting with $\delta(\Lambda)L \subset \mathcal{G}$*

such that the composition of δ and the quotient homomorphism $\mathcal{G}/L \to \Gamma_1/K$ is surjective and such that $w := \tau(q)^{-1/2} v(1 \otimes q)$ is a partial isometry with left support p and right support $p_K(1 \otimes q)$ satisfying

$$w^* C w = (\mathrm{M}_d(\mathbb{C}) \otimes Aq)^{\mathrm{Ad}\, L}\, p_K \quad \textit{and} \quad w^* \gamma(s) w = \delta(s) p_K (1 \otimes q) \;\; \forall s \in \Lambda.$$

Proof. Define the automorphism $\beta_s \in \mathrm{Aut}(C)$ as $\beta_s = \mathrm{Ad}\,\gamma(s)$. Since $C \prec A$, the von Neumann algebra C has a direct summand that is finite of type I. Since $(\beta_s)_{s \in \Lambda}$ is ergodic on $\mathcal{Z}(C)$, we find an integer d such that $C \cong \mathrm{M}_d(\mathbb{C}) \otimes \mathcal{Z}(C)$. So, we can take matrix units $(e_{ij})_{i,j=1,\dots,d}$ in C with $e := e_{11}$ satisfying $eCe = \mathcal{Z}(C)e$. By construction, $\mathcal{Z}(C)e$ is a maximal abelian subalgebra of $e(\mathrm{M}_n(\mathbb{C}) \otimes M)e$ that is semi-regular; the normalizer of $\mathcal{Z}(C)e$ acts ergodically on $\mathcal{Z}(C)e$. Also, $\mathcal{Z}(C)e \prec A$.

Denote by $\mathrm{D}_m(\mathbb{C}) \subset \mathrm{M}_m(\mathbb{C})$ the subalgebra of diagonal matrices. Take an integer m and a projection $q_1 \in \mathrm{D}_m(\mathbb{C}) \otimes A$ such that $(\mathrm{Tr} \otimes \tau)(q_1) = (\mathrm{Tr} \otimes \tau)(e)$. Write $B := \mathrm{D}_m(\mathbb{C}) \otimes A$. By [Pop06a, Th. A.1], we find $V_1 \in \mathrm{M}_{n,m}(\mathbb{C}) \otimes M$ such that $V_1 V_1^* = e$, $V_1^* V_1 = q_1$ and $V_1^* \mathcal{Z}(C) e V_1 = B q_1$. Put the elements $V_i = e_{i1} V_1$, $i = 1, \dots, d$ next to each other, yielding

$$V \in \mathrm{M}_{n,dm}(\mathbb{C}) \otimes M$$

such that

$$VV^* = p, \;\; V^*V = 1 \otimes q_1 \;\text{ and }\; V^* C V = \mathrm{M}_d(\mathbb{C}) \otimes B q_1.$$

For every $s \in \Lambda$, the unitary $V^* \gamma(s) V \in \mathrm{M}_d(\mathbb{C}) \otimes q_1(\mathrm{M}_m(\mathbb{C}) \otimes M) q_1$ normalizes $\mathrm{M}_d(\mathbb{C}) \otimes B q_1$. One can describe all unitaries $w \in \mathrm{M}_d(\mathbb{C}) \otimes q_1(\mathrm{M}_m(\mathbb{C}) \otimes M) q_1$ normalizing $\mathrm{M}_d(\mathbb{C}) \otimes B q_1$ as follows. Then, w also normalizes $1 \otimes B q_1$ and we define the automorphism β_w of Bq_1 given by $1 \otimes \beta_w(b) = w(1 \otimes b) w^*$. Denote by $e_1, \dots, e_m$ the standard minimal projections in $\mathrm{D}_m(\mathbb{C})$. Write $q_1 = \sum_{k=1}^m e_k \otimes q_k$. For all $k, l \in \{1, \dots, m\}$ and $g \in \Gamma$, we find a projection $q_l^{k,g} \in A q_l$ such that

$$\sum_{k=1}^m \sum_{g \in \Gamma} q_l^{k,g} = q_l \quad \text{and} \quad \beta_w(e_l \otimes a q_l^{k,g}) = e_k \otimes \sigma_g(a q_l^{k,g}) \;\; \forall a \in A.$$

It follows that

$$w_1 := \sum_{k,l=1}^m \sum_{g \in \Gamma} e_{kl} \otimes u_g q_l^{k,g}$$

is a unitary element in $q_1(\mathrm{M}_m(\mathbb{C}) \otimes M) q_1$ satisfying $\beta_w(b) = w_1 b w_1^*$ for all $b \in Bq_1$. It follows that $w_0 := w(1 \otimes w_1^*)$ commutes with $1 \otimes B q_1$ and hence belongs to $\mathcal{U}(\mathrm{M}_d(\mathbb{C}) \otimes B q_1)$. By construction, $w = w_0(1 \otimes w_1)$.

Define $X_m = X \sqcup \dots \sqcup X$ as the disjoint union of m copies of X. Identify $\mathrm{L}^\infty(X) = B$. Let $Y \subset X_m$ be the support of the projection q_1. Define the closed subgroup $\mathcal{G}_1 \subset \mathcal{U}(\mathrm{M}_d(\mathbb{C}) \otimes \mathrm{L}(\Gamma))$ given by

$$\mathcal{G}_1 := \{ u \otimes u_g \mid u \in \mathcal{U}(\mathrm{M}_d(\mathbb{C})) \;\text{ and }\; g \in \Gamma \}.$$

We can view w_0 as a measurable function from Y to $\mathcal{U}(\mathrm{M}_d(\mathbb{C}))$. We then denote by $\Omega_w : Y \to \mathcal{G}_1$ the measurable function given by

$\Omega_w(y) = w_0(y) \otimes u_h$ whenever y belongs to the support of
$e_k \otimes \sigma_h(q_l^{k,h}) = \beta_w(e_l \otimes q_l^{k,h})$.

To make computations easier, we provide an alternative description of Ω_w. Define the Hilbert space $\mathcal{K} := \mathrm{M}_{n,dm}(\mathbb{C}) \otimes \mathrm{L}^2(M)$ that we view as an $(\mathrm{M}_n(\mathbb{C}) \otimes M)$-$(\mathrm{M}_{dm}(\mathbb{C}) \otimes M)$-bimodule. Define the Hilbert space $\mathcal{H} := \mathrm{M}_{n,d}(\mathbb{C}) \mathbin{\overline{\otimes}} \ell^2(\Gamma) \mathbin{\overline{\otimes}} \mathrm{L}^2(B)$ that we view as an $(\mathrm{M}_n(\mathbb{C}) \mathbin{\overline{\otimes}} \mathrm{L}\Gamma \mathbin{\overline{\otimes}} B)$-$(\mathrm{M}_d(\mathbb{C}) \mathbin{\overline{\otimes}} \mathrm{L}\Gamma \mathbin{\overline{\otimes}} B)$-bimodule. Define the unitary operator

$$\eta : \mathcal{K} \to \mathcal{H} : \eta(e_{i,jk} \otimes u_g a) = e_{ij} \otimes \delta_g \otimes (e_k \otimes a) \quad \text{for all indices } i, j, k,\ g \in \Gamma, a \in A.$$

Viewing $\mathrm{M}_n(\mathbb{C}) \otimes \mathrm{L}\Gamma \subset \mathrm{M}_n(\mathbb{C}) \otimes M$ and $\mathrm{M}_d(\mathbb{C}) \otimes B \subset \mathrm{M}_{dm}(\mathbb{C}) \otimes M$, we have for all $\xi \in \mathcal{K}$ the obvious formulae

$$\eta(a\xi) = (a \otimes 1)\xi \ \text{ when } \ a \in \mathrm{M}_n(\mathbb{C}) \otimes \mathrm{L}\Gamma$$

and

$$\eta(\xi b) = \eta(\xi) b_{13} \ \text{ when } \ b \in \mathrm{M}_d(\mathbb{C}) \otimes B.$$

An elementary computation yields

$$\eta(\xi)\Omega_w = (\mathrm{id} \otimes \mathrm{id} \otimes \beta_w)\eta(\xi w) \ \ \forall \xi \in \mathcal{K}(1 \otimes q_1). \tag{6.1}$$

The following 1-cocycle relation is then an immediate consequence.

$$\Omega_{wv} = \Omega_w\,(\mathrm{id} \otimes \mathrm{id} \otimes \beta_w)(\Omega_v) \quad \text{when both } \ w, v \ \text{ normalize } \ \mathrm{M}_d(\mathbb{C}) \otimes B q_1. \tag{6.2}$$

For $s \in \Lambda$, define $w_s := V^* \gamma(s) V$. Since $\mathrm{M}_d(\mathbb{C}) \otimes B q_1 = V^* C V$, the unitaries w_s normalize $\mathrm{M}_d(\mathbb{C}) \otimes B q_1$. So we can define the action $(\beta_s)_{s \in \Lambda}$ on $B q_1$ given by $\beta_s = \beta_{w_s}$. We denote by $s * y$, $s \in \Lambda$, $y \in Y$, the corresponding action of Λ on Y. By assumption, $\Lambda \curvearrowright Y$ is weakly mixing.

Thanks to the construction above, we can define the measurable function $\omega_1 : \Lambda \times Y \to \mathcal{G}_1$ given by $\omega_1(s, y) = \Omega_{w_s}(s * y)$. The 1-cocycle relation (6.2) now becomes

$$\omega_1(st, y) = \omega_1(s, t * y)\, \omega_1(t, y) \ \ \forall s, t \in \Lambda \ \text{ and a.e. } \ y \in Y.$$

Hence, ω_1 is a 1-cocycle for the action $\Lambda \curvearrowright Y$ with values in $\mathcal{G}_1$. Define the vector

$$\varphi \in \mathrm{M}_{n,d}(\mathbb{C}) \mathbin{\overline{\otimes}} \ell^2(\Gamma) \mathbin{\overline{\otimes}} \mathrm{L}^2(B q_1) \quad \text{given by} \quad \varphi = \eta(V).$$

View φ as a measurable function from Y to $\mathrm{M}_{n,d}(\mathbb{C}) \otimes \ell^2(\Gamma)$ and view the latter as an $(\mathrm{M}_n(\mathbb{C}) \otimes \mathrm{L}\Gamma)$-$(\mathrm{M}_d(\mathbb{C}) \otimes \mathrm{L}\Gamma)$-bimodule. By definition $pV = V$ and $\gamma(s)V = V w_s$ for all $s \in \Lambda$. The properties of η imply that $p\varphi(y) = \varphi(y)$ almost everywhere and that $\eta(\gamma(s)V)$ equals almost everywhere the function given by $y \mapsto \gamma(s)\varphi(y)$. By (6.1) we have that $\eta(V w_s)$ equals almost everywhere the function given by $y \mapsto \varphi(s * y)\omega_1(s, y)$. So, we conclude that

$$\varphi : Y \to p(\mathrm{M}_{n,d}(\mathbb{C}) \otimes \ell^2(\Gamma)) \quad \text{and} \quad \gamma(s)\varphi(y) = \varphi(s * y)\omega_1(s, y) \quad \text{a.e.}$$

From now on, identify

$$p(\mathrm{M}_{n,d}(\mathbb{C}) \otimes \ell^2(\Gamma)) = p\mathrm{L}^2(\mathrm{M}_{n,d}(\mathbb{C}) \otimes \mathrm{L}\Gamma).$$

So, we can define $P(y) := \varphi(y)\varphi(y)^*$ as an element in $p\mathrm{L}^1(\mathrm{M}_n(\mathbb{C}) \otimes \mathrm{L}\Gamma)p$. We have $P(s * y) = \gamma(s)P(y)\gamma(s)^*$. Since $\Lambda \curvearrowright Y$ is weakly mixing, [PV11, Lemma 5.4] implies that P is essentially constant. So, we have found an element $P \in p\mathrm{L}^1(\mathrm{M}_n(\mathbb{C}) \otimes \mathrm{L}\Gamma)p$ such that $P(y) = P$ almost everywhere. We claim that $P = (\mathrm{Tr} \otimes \tau)(q_1)^{-1}p$. Indeed, for an arbitrary projection $f \in p(\mathrm{M}_n(\mathbb{C}) \otimes \mathrm{L}\Gamma)p$, we get

$$\begin{aligned}(\mathrm{Tr} \otimes \tau)(f) = \langle fV, V\rangle = \langle \eta(fV), \eta(V)\rangle &= \int_Y \langle f\varphi(y), \varphi(y)\rangle \, d\rho(y)\\ &= \int_Y (\mathrm{Tr} \otimes \tau)(fP) \, d\rho(y) = (\mathrm{Tr} \otimes \tau)(fP) \; (\mathrm{Tr} \otimes \tau)(q_1).\end{aligned}$$

Since this holds for all projections f, the claim follows.

Define $\psi_1(y) := (\mathrm{Tr} \otimes \tau)(q_1)^{1/2}\varphi(y)$. Denote by $\mathcal{I}$ the set of all partial isometries in $\mathrm{M}_{n,d}(\mathbb{C}) \otimes \mathrm{L}\Gamma$ with left projection equal to p. So, $\psi_1 : Y \to \mathcal{I}$ and ψ_1 satisfies

$$\gamma(s)\psi_1(y) = \psi_1(s * y)\omega_1(s, y) \quad \text{almost everywhere.}$$

The $\|\cdot\|_2$-distance turns $\mathcal{I}$ into a Polish space on which $\mathcal{U}(p(\mathrm{M}_n(\mathbb{C}) \otimes \mathrm{L}\Gamma)p)$ acts by left multiplication and $\mathcal{G}_1$ by right multiplication. Both actions are isometric. The action of $\mathcal{G}_1$ on $\mathcal{I}$ by right multiplication is proper, so that the set $\mathcal{I}/\mathcal{G}_1$ of $\mathcal{G}_1$-orbits equipped with the distance between orbits is still a Polish space on which $\mathcal{U}(p(\mathrm{M}_n(\mathbb{C}) \otimes \mathrm{L}\Gamma)p)$ acts isometrically. Since $\psi_1(s * y)\mathcal{G}_1 = \gamma(s)\psi_1(y)\mathcal{G}_1$ almost everywhere and since $\Lambda \curvearrowright Y$ is weakly mixing, [PV11, Lemma 5.4] implies that $y \mapsto \psi_1(y)\mathcal{G}_1$ is essentially constant. Take $v \in \mathcal{I}$ such that $\psi_1(y) \in v\mathcal{G}_1$ almost everywhere and denote $p_1 := v^*v$.

Define the compact subgroup $L \subset \mathcal{G}_1$ consisting of the unitaries $u \otimes u_g$ that satisfy $p_1(u \otimes u_g) = p_1$. Define the measurable map $\psi_2 : Y \to L\backslash\mathcal{G}_1$ such that $\psi_1(y) = v\psi_2(y)$ almost everywhere. Composing ψ_2 with a measurable cross-section $L\backslash\mathcal{G}_1 \to \mathcal{G}_1$, we find a measurable map $\psi : Y \to \mathcal{G}_1$ satisfying $\psi_1(y) = v\psi(y)$ almost everywhere. Define the 1-cocycle $\omega : \Lambda \times Y \to \mathcal{G}_1$ given by $\omega(s, y) = \psi(s * y)\omega_1(s, y)\psi(y)^{-1}$. Define the group morphism $\pi : \Lambda \to \mathcal{U}(p_1(\mathrm{M}_d(\mathbb{C}) \otimes \mathrm{L}\Gamma)p_1)$ given by $\pi(s) = v^*\gamma(s)v$. By construction,

$$\pi(s) = p_1\omega(s, y) \quad \text{almost everywhere.}$$

Define the closed subgroup $\mathcal{G}_2 \subset \mathcal{G}_1$ consisting of the unitaries $u \otimes u_g$ that commute with p_1. It follows that ω takes values almost everywhere in $\mathcal{G}_2$ and hence $\pi(s) \in \mathcal{G}_2 p_1$ for all $s \in \Lambda$. Note that L is a normal subgroup of $\mathcal{G}_2$. We get a well-defined group morphism $\delta : \Lambda \to \mathcal{G}_2/L$ such that $\pi(s) = \delta(s)p_1$. So, $\delta(\Lambda)L$ commutes with $p_1 = v^*v$ and $v^*\gamma(s)v = \delta(s)p_1$ for all $s \in \Lambda$.

Write $\psi(y)=\zeta(y)\otimes u_{\theta(y)}$. View ζ as a unitary element in $\mathrm{M}_d(\mathbb{C})\otimes Bq_1$. Replacing V by $V\zeta^*$, we may assume that $\psi(y) = 1\otimes u_{\theta(y)}$. Define the projection $q_g \in Bq_1$ with support $\{y \in Y \mid \theta(y) = g\}$. Write

$$q_g = \sum_{k=1}^{m} e_k \otimes q_g^k \quad \text{and} \quad v = \sum_{ij} e_{ij} \otimes v_{ij}.$$

Since $\eta(V)$ equals almost everywhere the function $y \mapsto (\mathrm{Tr}\otimes\tau)(q_1)^{-1/2}v(1 \otimes u_{\theta(y)})$, it follows that

$$V = (\mathrm{Tr}\otimes\tau)(q_1)^{-1/2} \sum_{g\in\Gamma} \sum_{i,j,k} e_{i,jk} \otimes v_{ij} u_g q_g^k. \tag{6.3}$$

Define the projections $\tilde{q}_g^k := u_g q_g^k u_g^*$. Since $V^*V = 1 \otimes q_1$ we have

$$(1 \otimes q_h)V^*V(1 \otimes q_g) = \delta_{g,h} 1 \otimes q_g$$

so that by (6.3), it follows that

$$(1 \otimes \tilde{q}_h^j)p_1(1 \otimes \tilde{q}_g^i) = (\mathrm{Tr}\otimes\tau)(q_1)\delta_{i,j}\delta_{g,h} 1 \otimes \tilde{q}_g^i.$$

Applying $\mathrm{Tr}\otimes E_A$, it follows that the projections $\tilde{q}_g^i$ are orthogonal. In particular, the sum of their traces is at most 1, so that $(\mathrm{Tr}\otimes\tau)(q_1) \leqslant 1$. Hence, we may assume from the beginning that $m = 1$ and that $q_1 \in A$. We do not write the upper indices i, j, k any more. Since the projections $\tilde{q}_g$ are orthogonal, $u := \sum_{g\in\Gamma} u_g q_g$ is a partial isometry in M with right support q_1, with left support in A and such that $u\,Aq_1\,u^* = A\,uu^*$. Replacing V by $V(1 \otimes u^*)$ and q_1 by uu^*, we may further assume that $V = \tau(q_1)^{-1/2}v(1 \otimes q_1)$. By construction, the 1-cocycle ω that corresponds to the group of unitaries $(V^*\gamma(s)V)_{s\in\Lambda}$ normalizing $\mathrm{M}_d(\mathbb{C}) \otimes Aq_1$ satisfies

$$p_1\omega(s, y) = \pi(s) = p_1\delta(s) \quad \text{and hence,} \quad \omega(s, y)L = \delta(s)L. \tag{6.4}$$

Let $p_1 = \sum_{g\in\Gamma} P_g \otimes u_g$, with $P_g \in \mathrm{M}_d(\mathbb{C})$, be the Fourier decomposition of p_1. Since $V = \tau(q_1)^{-1/2}v(1 \otimes q_1)$, we have

$$(1 \otimes q_1)p_1(1 \otimes q_1) = \tau(q_1)\, 1 \otimes q_1. \tag{6.5}$$

Applying $\mathrm{id}\otimes E_A$, we get that $P_e = \tau(q_1)\,1$. So, when $u \otimes u_k \in L$, the formula $p_1(u^* \otimes u_k^*) = p_1$ implies that $P_k = \tau(q_1)u$. In particular, the homomorphism $L \to \Gamma : u \otimes u_k \mapsto k$ is injective. We denote the image by K and define the unitary representation $\rho : K \to \mathcal{U}(\mathrm{M}_d(\mathbb{C}))$ such that $L = \{\rho(k) \otimes u_k \mid k \in K\}$. Define Γ_1 as the image of $\delta(\Lambda)L$ in Γ. By construction, K is a finite normal subgroup of Γ_1. Define $\mathcal{G}$ as in the formulation of the theorem, i.e., as the unitaries $u \otimes u_g$, $g \in \Gamma_1$, that normalize L. So, $\delta(\Lambda)L \subset \mathcal{G}$.

Let $k \in K - \{e\}$. Multiplying (6.5) on the right by $\rho(k)^* \otimes u_k^*$ and applying $\mathrm{id}\otimes E_A$, it follows that $q_1\,\sigma_k(q_1) = 0$. Define the projection $q = \sum_{k\in K} \sigma_k(q_1)$. We claim that q is Γ_1-invariant. Recall that $s * y$ denotes the action of $s \in \Lambda$

on $y \in Y$ implemented by $\operatorname{Ad} V^*\gamma(s)V$. Denote by $\mu : \Lambda \times Y \to \Gamma$ and $\delta_1 : \Lambda \to \Gamma_1/K$ the compositions of the 1-cocycle ω and the group morphism δ with the natural morphism $\mathcal{G} \to \Gamma$. By (6.4), we have $\mu(s,y)K = \delta_1(s)K$, so that μ takes values in Γ_1 and

$$\delta_1(s)K \cdot y = \mu(s,y)K \cdot y = K\mu(s,y) \cdot y = K \cdot (s * y).$$

Hence $\delta_1(\Lambda)K \cdot Y = K \cdot Y$, proving the claim.

Define the projection $p_K = |K|^{-1} \sum_{k \in K} \rho(k) \otimes u_k$. Put $w := \tau(q)^{-1/2} v(1 \otimes q)$. We make several computations to check that all the conclusions of the theorem hold. We freely use that $\tau(q) = |K|\,\tau(q_1)$, that $V = \tau(q_1)^{-1/2} v(1 \otimes q_1)$ and that $vu = vp_1 u = v$ for all $u \in L$. First we get that

$$\begin{aligned} ww^* &= \tau(q)^{-1}\, v(1 \otimes q)v^* = \sum_{u \in L} \tau(q)^{-1}\, vu(1 \otimes q_1)u^*v^* \\ &= |L|\,\tau(q)^{-1}\, v(1 \otimes q_1)v^* = VV^* = p. \end{aligned}$$

On the other hand,

$$\begin{aligned} w^*w &= \tau(q)^{-1}\,(1 \otimes q)\, v^*v\,(1 \otimes q) = \tau(q)^{-1} \sum_{u_1,u_2 \in L} u_1(1 \otimes q_1)u_1^*\, p_1\, u_2(1 \otimes q_1)u_2^* \\ &= \tau(q)^{-1} \sum_{u_1,u_2 \in L} u_1(1 \otimes q_1)p_1(1 \otimes q_1)u_2^* = |L|^{-1} \sum_{u_1,u_2 \in L} u_1 V^*V u_2^* \\ &= |L|^{-1} \sum_{u_1,u_2 \in L} u_1(1 \otimes q_1)u_2^* = p_K(1 \otimes q). \end{aligned}$$

Since $\delta(\Lambda)L$ commutes with $1 \otimes q$ and $v^*\gamma(s)v = \delta(s)p_1$, it follows that $w^*\gamma(s)w = \delta(s)p_K(1 \otimes q)$ for all $s \in \Lambda$. Finally,

$$\begin{aligned} w^*Cw &= (1 \otimes q)v^*Cv(1 \otimes q) = \sum_{u_1,u_2 \in L} u_1(1 \otimes q_1)v^*Cv(1 \otimes q_1)u_2^* \\ &= \sum_{u_1,u_2 \in L} u_1 V^*CV u_2^* = p_K(\mathrm{M}_d(\mathbb{C}) \otimes Aq_1)p_K = (\mathrm{M}_d(\mathbb{C}) \otimes Aq)^{\operatorname{Ad} L} p_K. \end{aligned}$$

This ends the proof of the theorem. $\square$

COROLLARY 6.2. *The conclusions of Theorem* 6.1 *can be strengthened if we impose extra conditions. Denote by N the von Neumann algebra generated by C and $\gamma(\Lambda)$.*

(1) *If we impose the extra condition that $N \not\prec A \rtimes \operatorname{Centr} g$ whenever $g \neq e$, it follows that $K = \{e\}$, $q = 1$, $w = v$ and $v^*\gamma(s)v = \pi(s) \otimes u_{\delta_1(s)}$ for all $s \in \Lambda$, where $\pi : \Lambda \to \mathcal{U}(\mathrm{M}_d(\mathbb{C}))$ and $\delta_1 : \Lambda \to \Gamma$ are group morphisms. If, moreover, the weak mixing assumption is strengthened by imposing that $\mathbb{C}1$ is the only nonzero, finite dimensional, globally $(\operatorname{Ad}\gamma(s))_{s \in \Lambda}$-invariant vector subspace of C, then it follows that $d = 1$ and that $\pi : \Lambda \to \mathbb{T}$ is a character.*

(2) *If we impose the extra condition that* $N \not\prec A \rtimes \Gamma_1$ *whenever* $\Gamma_1 \curvearrowright (X,\mu)$ *is nonergodic, it follows that* $q = 1$ *and* $v^*v = p_K$.

Proof. (1) Choose a projection $q_1 \in Aq$ such that $q = \sum_{k\in K} \sigma_k(q_1)$. It follows that $w(\mathrm{M}_d(\mathbb{C}) \otimes q_1)w^*$ is a globally $(\mathrm{Ad}\,\gamma(s))_{s\in\Lambda}$-invariant vector subspace of C. So, $d = 1$ and the rest follows immediately.

(2) Denote by $\delta_1 : \Lambda \to \Gamma_1/K$ the composition of δ and the natural homomorphism $\mathcal{G}/L \to \Gamma_1/K$. Replacing Γ_1 by $\delta_1(\Lambda)K$, we may assume that δ_1 is surjective. The conclusions of Theorem 6.1 say in particular that $w^*Nw \subset \mathrm{M}_d(\mathbb{C}) \otimes (A \rtimes \Gamma_1)$. The extra condition $N \not\prec A \rtimes \mathrm{Centr}\, g$ whenever $g \neq e$ then implies that $\{hgh^{-1} \mid h \in \Gamma_1\}$ is infinite for all $g \neq e$. So, we can take a sequence $h_n \in \Gamma_1$ such that $h_n g h_n^{-1} \to \infty$ for all $g \neq e$. Take $u_n \in \mathcal{U}(\mathrm{M}_d(\mathbb{C}))$ such that $u_n \otimes u_{h_n} \in \delta(\Lambda)L$. Since v^*v commutes with $\delta(\Lambda)L$, it follows that $v^*v = p_0 \otimes 1$ for some projection $p_0 \in \mathrm{M}_d(\mathbb{C})$. But then $w^*w = \tau(q)^{-1}p_0 \otimes q$. Since w^*w actually equals $p_K(1 \otimes q)$, it follows that $q = 1$ and $K = \{e\}$.

(3) As in the proof of (2), we get that $w^*Nw \subset \mathrm{M}_d(\mathbb{C}) \otimes (A \rtimes \Gamma_1)$. Since q is Γ_1-invariant, the extra condition (3) implies that $q = 1$. Hence $w = v$ and $v^*v = p_K$. □

7. **Some properties of the comultiplication**

Throughout this section, we fix a countable group Λ and put $M = \mathrm{L}\Lambda$. We denote by $(u_g)_{g\in\Lambda}$ the canonical unitaries generating $\mathrm{L}\Lambda$. We consider the comultiplication $\Delta : M \to M \mathbin{\overline{\otimes}} M$ given by $\Delta(u_g) = u_g \otimes u_g$ for all $g \in \Lambda$.

We start with the following elementary and well-known lemma.

LEMMA 7.1. *A nonzero element* $u \in M$ *satisfies* $\Delta(u) = u \otimes u$ *if and only if* $u = u_g$ *for some* $g \in \Lambda$. *A unital von Neumann subalgebra* $A \subset M$ *satisfies* $\Delta(A) \subset A \mathbin{\overline{\otimes}} A$ *if and only if* A *is of the form* $A = \mathrm{L}\Sigma$ *for some subgroup* $\Sigma < \Lambda$.

Proof. Observe that $(\mathrm{id} \otimes \tau u_g^*)\Delta(x) = \tau(xu_g^*)u_g$ for all $g \in \Lambda$, $x \in M$. Let $u \in M$ be a nonzero element satisfying $\Delta(u) = u \otimes u$. Take $g \in \Lambda$ such that $\tau(uu_g^*) \neq 0$. It follows that u is a nonzero multiple of u_g. Since $\Delta(u) = u \otimes u$, this multiple must be 1.

Let $A \subset M$ be a von Neumann subalgebra satisfying $\Delta(A) \subset A \mathbin{\overline{\otimes}} A$. Define the subset $\Sigma \subset \Lambda$ consisting of the elements $g \in \Lambda$ for which there exists $a \in A$ with $\tau(au_g^*) \neq 0$. Since $A \ni (\mathrm{id} \otimes \tau u_g^*)\Delta(a) = \tau(au_g^*)u_g$, it follows that $u_g \in A$ for all $g \in \Sigma$. Conversely, it is obvious that $g \in \Sigma$ whenever $u_g \in A$. Since A is a von Neumann subalgebra, it follows that Σ is a subgroup of Λ and that $A = \mathrm{L}\Sigma$. □

Recall from Section 2.4 the notion of relative amenablity for von Neumann subalgebras.

Proposition 7.2. *Let $P \subset M$ be a von Neumann subalgebra.*

(1) *If P is diffuse, then $\Delta(P) \not\prec M \otimes 1$ and $\Delta(P) \not\prec 1 \otimes M$.*

(2) *If $\Delta(M) \prec M \mathbin{\overline{\otimes}} P$, there exists a nonzero projection $p \in P' \cap M$ such that $Pp \subset pMp$ has finite index.*

(3) *Denote by* $\operatorname{Centr} g$ *the centralizer of $g \in \Lambda$, and assume that for all $g \neq e$ we have $P \not\prec \mathrm{L}(\operatorname{Centr} g)$. If $\mathcal{H} \subset \mathrm{L}^2(M \mathbin{\overline{\otimes}} M)$ is a $\Delta(P)$-$\Delta(M)$-subbimodule that is finitely generated as a right $\Delta(M)$-module, then $\mathcal{H} \subset \Delta(\mathrm{L}^2(M))$.*

In particular, the quasi-normalizer of $\Delta(P)$ inside $M \mathbin{\overline{\otimes}} M$ is contained in $\Delta(M)$. So, if Λ is an icc *group, the quasi-normalizer of $\Delta(M)$ inside $M \mathbin{\overline{\otimes}} M$ equals $\Delta(M)$.*

(4) *If P has no amenable direct summand, then $\Delta(P)$ is strongly nonamenable relative to $M \otimes 1$. In particular, if $N \subset M$ is an amenable von Neumann subalgebra, we have $\Delta(P) \not\prec M \mathbin{\overline{\otimes}} N$.*

Proof. (1) Let P be diffuse. Take a sequence $v_n \in \mathcal{U}(P)$ tending to 0 weakly. We claim that $\|E_{M\otimes 1}(x\Delta(v_n)y^*)\|_2 \to 0$ for all $x, y \in M \mathbin{\overline{\otimes}} M$. It suffices to prove this claim for $x = 1 \otimes u_g$ and $y = 1 \otimes u_h$, $g, h \in \Lambda$. Then,

$$\|E_{M\otimes 1}((1 \otimes u_g)\Delta(v_n)(1 \otimes u_h)^*)\|_2 = \|\tau(u_g v_n u_h^*)u_{g^{-1}h}\|_2 = |\tau(u_g v_n u_h^*)| \to 0$$

and the claim follows. By Definition 2.1, $\Delta(P) \not\prec M \otimes 1$. The statement $\Delta(P) \not\prec 1 \otimes M$ follows similarly.

(2) Assume that $\Delta(M) \prec M \mathbin{\overline{\otimes}} P$. Definition 2.1 provides elements $h_1, \ldots, h_n \in \Lambda$ and $\delta > 0$ such that

$$\sum_{i,j=1}^{n} \|E_{M\overline{\otimes}P}((1 \otimes u_{h_i})\Delta(u_g)(1 \otimes u_{h_j})^*)\|_2^2 \geqslant \delta \quad \forall g \in \Lambda.$$

This precisely means that

$$\sum_{i,j=1}^{n} \|E_P(u_{h_i} u_g u_{h_j}^*)\|_2^2 \geqslant \delta \quad \forall g \in \Lambda.$$

So, $M \prec_M P$. This means that $Pp \subset pMp$ has finite index for some nonzero projection $p \in P' \cap M$.

(3) Assume that $P \not\prec \mathrm{L}(\operatorname{Centr} g)$ for all $g \neq e$. By Definition 2.1, we find a sequence of unitaries $v_n \in \mathcal{U}(P)$ such that $\|E_{\mathrm{L}(\operatorname{Centr} g)}(u_h v_n u_k^*)\|_2 \to 0$ for all $h, k \in \Lambda$ and all $g \neq e$. To conclude the proof of the proposition, it suffices to prove the following (see, e.g., [Vae07, Lemma D.3], based on [Pop06c, Th. 3.1]):

$$\|E_{\Delta(M)}(x\Delta(v_n)y^*)\|_2 \to 0 \quad \forall x, y \in (M \mathbin{\overline{\otimes}} M) \ominus \Delta(M).$$

It is sufficient to prove this statement for $x = u_h \otimes u_k$ and $y = u_{h'} \otimes u_{k'}$ with $h \neq k$ and $h' \neq k'$. In that case,

$$\|E_{\Delta(M)}((u_h \otimes u_k)\Delta(v_n)(u_{h'} \otimes u_{k'})^*)\|_2^2 = \sum_{g \in \Lambda, hg(h')^{-1} = kg(k')^{-1}} |\tau(v_n u_g^*)|^2.$$

If for all $g \in \Gamma$ we have $hg(h')^{-1} \neq kg(k')^{-1}$, this last expression is zero. If there is at least one $g_0 \in \Lambda$ such that $hg_0(h')^{-1} = kg_0(k')^{-1}$, this last expression equals

$$\sum_{g \in \operatorname{Centr} k^{-1}h} |\tau(v_n u_{gg_0}^*)|^2 = \|E_{\mathrm{L}(\operatorname{Centr} k^{-1}h)}(v_n u_{g_0}^*)\|_2^2 \to 0.$$

(4) Note that the M-$(M \mathbin{\overline{\otimes}} M)$-bimodule

$${}_{(\Delta(M) \otimes 1)}\mathrm{L}^2(M \mathbin{\overline{\otimes}} M \mathbin{\overline{\otimes}} M)_{(M \mathbin{\overline{\otimes}} 1 \mathbin{\overline{\otimes}} M)}$$

is isomorphic with the coarse M-$(M \mathbin{\overline{\otimes}} M)$-bimodule $\mathrm{L}^2(M) \mathbin{\overline{\otimes}} \mathrm{L}^2(M \mathbin{\overline{\otimes}} M)$. Assume that $\Delta(P)$ is not strongly nonamenable relative to $M \otimes 1$. We get a nonzero projection $p \in \Delta(P)' \cap (M \mathbin{\overline{\otimes}} M)$ such that ${}_{\Delta(P)}\mathrm{L}^2(p(M \mathbin{\overline{\otimes}} M))_{M \mathbin{\overline{\otimes}} M}$ is weakly contained in ${}_{(\Delta(P) \otimes 1)}\mathrm{L}^2(M \mathbin{\overline{\otimes}} M \mathbin{\overline{\otimes}} M)_{(M \mathbin{\overline{\otimes}} 1 \mathbin{\overline{\otimes}} M)}$ and hence, weakly contained in the coarse P-$(M \mathbin{\overline{\otimes}} M)$-bimodule. Take $z \in P$ such that $\Delta(z)$ is the support projection of $E_{\Delta(P)}(p)$. Note that z is a nonzero central projection in P and that Δ embeds the trivial Pz-Pz-bimodule into

$${}_{\Delta(Pz)}\mathrm{L}^2(\Delta(z)(M \mathbin{\overline{\otimes}} M)\Delta(z))_{\Delta(Pz)}.$$

It follows that the trivial Pz-Pz-bimodule is weakly contained in the coarse Pz-Pz-bimodule so that Pz is amenable.

If $N \subset M$ is an amenable von Neumann subalgebra, then $M \mathbin{\overline{\otimes}} N$ is amenable relative to $M \otimes 1$. If $\Delta(P) \prec M \mathbin{\overline{\otimes}} N$, it follows that $\Delta(P)p$ is amenable relative to $M \otimes 1$ for some nonzero projection $p \in \Delta(P)' \cap (M \mathbin{\overline{\otimes}} M)$. So, $\Delta(P)$ is not strongly nonamenable relative to $M \otimes 1$. The previous paragraph implies that P has an amenable direct summand. $\square$

8. **Proof of Theorem 1.1: superrigidity of group von Neumann algebras**

Theorem 1.1 is a specific instance of a general superrigidity theorem for group factors $\mathrm{L}G$ where G arises as a generalized wreath product $G = H_0 \wr_I \Gamma$ for certain group actions $\Gamma \curvearrowright I$. The class of actions $\Gamma \curvearrowright I$ that we are able to treat is defined as follows.

Condition 8.1. We say that $\Gamma \curvearrowright I$ satisfies Condition 8.1 if the following two sets of conditions hold.

Conditions on the group. The group Γ is icc and admits a chain of infinite subgroups $\Gamma_0 < \Gamma_1 < \cdots < \Gamma_n = \Gamma$ such that Γ_{k-1} is almost normal in Γ_k for all $k = 1, \dots, n$. Moreover, at least one of the following rigidity properties hold:

- $\Gamma_0 < \Gamma_1$ has the relative property (T).
- The centralizer of Γ_0 inside Γ_1 is nonamenable.

Conditions on the action.

- There exists $\kappa \in \mathbb{N}$ such that $\operatorname{Stab} J$ is finite whenever $J \subset I$ and $|J| \geqslant \kappa$.
- $\operatorname{Stab} i$ is amenable for all $i \in I$.

The conditions on the group Γ in 8.1 are satisfied whenever Γ is an icc group with property (T), whenever Γ is the direct product of two icc groups with at least one of them being nonamenable or whenever Γ is itself a wreath product $\Gamma = \Gamma_0 \wr S$ with Γ_0 being nonamenable and S nontrivial. Indeed, in this last case, we consider the chain of subgroups $\Gamma_0 < \Gamma_0^{(S)} < \Gamma$.

The conditions on the action in 8.1 are automatically satisfied when we let Γ act on itself by multiplication. They are also satisfied when $\Gamma \curvearrowright \Gamma/S$, where $S < \Gamma$ is an amenable subgroup that is almost malnormal: $gSg^{-1} \cap S$ is finite for all $g \in \Gamma - S$.

Whenever $\Gamma \curvearrowright I$ satisfies Condition 8.1, we consider the generalized wreath product $G = H_0 \wr_I \Gamma$ and describe all countable groups Λ such that $\mathrm{L}\Lambda \cong \mathrm{L}G$. The main result is the following Theorem 8.2. The conclusions of Theorem 8.2 can be made significantly more precise if moreover we assume that $\operatorname{Stab} i \cdot j$ is infinite for all $i \neq j$. This excludes plain wreath products and will lead to Theorem 8.3 below, of which Theorem 1.1 is a special case.

THEOREM 8.2. *Assume that $\Gamma \curvearrowright I$ satisfies Condition* 8.1. *Let H_0 be a nontrivial abelian group, and define the generalized wreath product group $G := H_0 \wr_I \Gamma := H_0^{(I)} \rtimes \Gamma$. Denote by A the abelian von Neumann algebra $A = \mathrm{L}(H_0^{(I)})$, and denote by $(\sigma_g)_{g \in \Gamma}$ the corresponding generalized Bernoulli action of Γ on A.*

If Λ is any countable group and $\pi : \mathrm{L}\Lambda \to \mathrm{L}(G)^t$ is a $*$-isomorphism for some $t > 0$, then $t = 1$ and $\Lambda \cong \Sigma \rtimes \Gamma$ for some infinite abelian group Σ and some action $\Gamma \overset{\alpha}{\curvearrowright} \Sigma$ by automorphisms.

More precisely, there exists a group isomorphism $\delta : \Lambda \to \Sigma \rtimes \Gamma$, a $*$-isomorphism $\theta : \mathrm{L}\Sigma \to A$ satisfying $\theta \circ \alpha_g = \sigma_g \circ \theta$ for all $g \in \Gamma$, a character $\omega : G \to \mathbb{T}$ and a unitary $w \in \mathrm{L}G$ such that $\pi = \operatorname{Ad} w \circ \pi_\omega \circ \pi_\theta \circ \pi_\delta$, where

- *$\pi_\delta : \mathrm{L}\Lambda \to \mathrm{L}(\Sigma \rtimes \Gamma)$ is the isomorphism given by $\pi_\delta(v_s) = u_{\delta(s)}$ for all $s \in \Lambda$,*
- *$\pi_\theta : \mathrm{L}(\Sigma) \rtimes \Gamma \to A \rtimes \Gamma$ is given by $\pi_\theta(a u_g) = \theta(a) u_g$ for all $a \in \mathrm{L}(\Sigma)$ and all $g \in \Gamma$,*

- *π_ω is the automorphism of* $\mathrm{L}G$ *given by* $\pi_\omega(u_g) = \omega(g)\, u_g$ *for all* $g \in G$.

In order to fully understand all groups Λ for which $\mathrm{L}\Lambda \cong \mathrm{L}G$, we need to classify all actions of Γ by group automorphisms of a countable abelian group Σ such that the corresponding measure preserving action $\Gamma \curvearrowright \widehat{\Sigma}$ is conjugate with the given generalized Bernoulli action $\Gamma \curvearrowright X_0^I$ with base space $X_0 = \widehat{H_0}$. As we illustrate in Section 9, such a classification is untractable for plain wreath products $H_0 \wr \Gamma$. However, if we specialize to the case where moreover $\operatorname{Stab} i \cdot j$ is infinite for all $i \neq j$, we get the following full superrigidity theorem.

THEOREM 8.3. *Assume that* $\Gamma \curvearrowright I$ *satisfies Condition* 8.1 *and that* $\operatorname{Stab} i \cdot j$ *is infinite for all* $i \neq j$. *Let* H_0 *be a nontrivial abelian group, and define the generalized wreath product group* $G := H_0 \wr_I \Gamma = H_0^{(I)} \rtimes \Gamma$. *Let* Λ *be any countable group, and let* $\pi : \mathrm{L}\Lambda \to \mathrm{L}(G)^t$ *be a* $*$*-isomorphism for some* $t > 0$.

- *In the case where* $|H_0|$ *is a square-free integer, we must have* $t = 1$ *and* $\Lambda \cong G$.
- *In the general case, but assuming that* $\Gamma \curvearrowright I$ *is transitive, we must have* $t = 1$ *and* $\Lambda \cong H_1 \wr_I \Gamma$ *for some abelian group* H_1 *with* $|H_1| = |H_0|$.
- *In the case where* $H_0 = \mathbb{Z}/2\mathbb{Z}$ *or* $H_0 = \mathbb{Z}/3\mathbb{Z}$, *we must have* $t = 1$ *and there exist an isomorphism of groups* $\delta : \Lambda \to G$, *a character* $\omega : \Lambda \to \mathbb{T}$ *and a unitary* $w \in \mathrm{L}G$ *such that*

$$\pi(v_s) = \omega(s)\, w\, u_{\delta(s)}\, w^* \quad \forall s \in \Lambda.$$

Example 8.4. If $\Gamma \curvearrowright I$ is defined as in Theorem 1.1, it is easy to check that all conditions of Theorem 8.3 are indeed satisfied, using the subgroup $\Gamma_0 < \Gamma$ (which we put in an arbitrary position of $\Gamma_0^{(S)}$) and the chain of normal subgroups $\Gamma_0 \lhd \Gamma_0^{(S)} \lhd \Gamma$.

Define $\Gamma = \mathrm{SL}(2,\mathbb{Z}) \ltimes \mathbb{Z}^2$. Let $A \in \mathrm{SL}(2,\mathbb{Z})$ be any matrix whose eigenvalues have modulus different from 1. Define the subgroup $\Gamma_A < \mathrm{SL}(2,\mathbb{Z})$ consisting of the matrices B such that $BAB^{-1} = A^{\pm 1}$. View Γ_A as a subgroup of Γ. Then, the action $\Gamma \curvearrowright \Gamma/\Gamma_A$ satisfies all conditions of Theorem 8.3 with $\kappa = 2$.

More generally, whenever the icc group Γ admits an infinite almost normal subgroup with the relative property (T) and $S < \Gamma$ is an infinite amenable almost malnormal subgroup, then $\Gamma \curvearrowright \Gamma/S$ satisfies the conditions of Theorem 8.3 with $\kappa = 2$. Examples of infinite amenable almost malnormal subgroups of $\mathrm{PSL}(n,\mathbb{Z})$ are provided in [PV08, Example 7.4].

Proof of Theorem 8.2. Fix $\Gamma \curvearrowright I$ satisfying Condition 8.1. Choose a nontrivial abelian group H_0, and put $A_0 := \mathrm{L}(H_0)$, $A := \mathrm{L}(H_0^{(I)})$. Denote $M = \mathrm{L}(H_0 \wr_I \Gamma) = A \rtimes \Gamma$.

We first prove that the action $\Gamma \curvearrowright A$ is essentially free and ergodic. It suffices to prove that every $g \in \Gamma - \{e\}$ moves infinitely many $i \in I$. Choose $n \in \mathbb{N}$. For every $g \in \Gamma$, denote $\operatorname{Fix} g := \{i \in I \mid g \cdot i = i\}$. It suffices to prove that $\mathcal{G}_n := \{g \in \Gamma \mid |I - \operatorname{Fix} g| \leqslant n\}$ equals $\{e\}$. Since $h\mathcal{G}_n h^{-1} = \mathcal{G}_n$ for all $h \in \Gamma$ and since Γ is icc, it suffices to prove that $\mathcal{G}_n$ is finite. Choose a finite subset $\mathcal{F} \subset I$ such that $|\mathcal{F}| = \kappa + n$. Then,

$$\mathcal{G}_n \subset \bigcup_{\mathcal{F}_0 \subset \mathcal{F},\, |\mathcal{F}_0| = \kappa} \operatorname{Stab} \mathcal{F}_0.$$

Since all $\operatorname{Stab} \mathcal{F}_0$ are finite, $\mathcal{G}_n$ is finite as well. We have proven that $\Gamma \curvearrowright A$ is essentially free. Because $\Gamma \cdot i$ is infinite for all $i \in I$, the action $\Gamma \curvearrowright A$ is ergodic as well.

Assume that $\mathrm{L}\Lambda = M^t$ for some countable group Λ. The amplification of the comultiplication on $\mathrm{L}\Lambda$ yields a unital $*$-homomorphism $\Delta : M \to (M \mathbin{\overline{\otimes}} M)^t$. To avoid unnecessary notational complexity, in the first steps of the proof, through Step 3, we will proceed as if $t \leqslant 1$ and consider $\Delta : M \to p(M \mathbin{\overline{\otimes}} M)p$ for some projection $p \in M \mathbin{\overline{\otimes}} M$. The reader can check easily that this notational simplification does not hide any essential steps of the argument.

Step 1. There exists $v \in M \mathbin{\overline{\otimes}} M$ with $v^*v = p$ and $v\Delta(\mathrm{L}\Gamma)v^* \subset \mathrm{L}(\Gamma \times \Gamma)$.

Proof. Take a chain of subgroups $\Gamma_0 < \Gamma_1 < \cdots < \Gamma_n = \Gamma$ as in Condition 8.1. Note that Γ_1 is nonamenable. Put $Q = \Delta(\mathrm{L}\Gamma_0)$, and denote by P the quasi-normalizer of Q inside $p(M \mathbin{\overline{\otimes}} M)p$. Note that $\Delta(\mathrm{L}\Gamma_1) \subset P$. In the case where $\Gamma_0 < \Gamma_1$ has the relative property (T), $Q \subset P$ has the relative property (T). In the case where the centralizer of Γ_0 inside Γ_1 is nonamenable, Proposition 7.2 implies that the relative commutant $Q' \cap P$ is strongly nonamenable relative to $M \otimes 1$.

By Proposition 7.2(1), $Q \not\prec M \otimes 1$. By Proposition 7.2(4) and because $\Delta(\Gamma_1) \subset P$, we have $P \not\prec M \mathbin{\overline{\otimes}} (A \rtimes \operatorname{Stab} i)$ for all $i \in I$. So, Corollary 4.3 yields $v \in M \mathbin{\overline{\otimes}} M$ with $v^*v = p$ and $vPv^* \subset M \mathbin{\overline{\otimes}} \mathrm{L}\Gamma$.

Repeating the same argument and applying Corollary 4.3 with $N = \mathrm{L}\Gamma$, we find $w \in M \mathbin{\overline{\otimes}} \mathrm{L}\Gamma$ such that $w^*w = vv^*$ and $wvPv^*w^* \subset \mathrm{L}\Gamma \mathbin{\overline{\otimes}} \mathrm{L}\Gamma$.

We write v instead of wv, so that $v^*v = p$ and $vPv^* \subset \mathrm{L}(\Gamma \times \Gamma)$. In particular, $v\Delta(\mathrm{L}\Gamma_1)v^* \subset \mathrm{L}(\Gamma \times \Gamma)$. Write $P_k := v\Delta(\mathrm{L}\Gamma_k)v^*$. We prove by induction on k that automatically $P_k \subset \mathrm{L}(\Gamma \times \Gamma)$. For $k = 1$, the statement is already proven. Assume that $P_k \subset \mathrm{L}(\Gamma \times \Gamma)$ for some $1 \leqslant k \leqslant n-1$. We already observed that $P_1 \not\prec \mathrm{L}(\Gamma \times \operatorname{Stab} i)$ and $P_1 \not\prec \mathrm{L}(\operatorname{Stab} i \times \Gamma)$ so that, *a fortiori*, the same holds for P_k instead of P_1. By Lemma 4.1(1) and because $P_k \subset P_{k+1}$ is quasi-regular, it follows that $P_{k+1} \subset \mathrm{L}(\Gamma \times \Gamma)$.

Since $\Gamma = \Gamma_n$, we have proven that $v\Delta(\mathrm{L}\Gamma)v^* \subset \mathrm{L}(\Gamma \times \Gamma)$. □

From now on, we replace $\Delta : M \to p(M \mathbin{\overline{\otimes}} M)p$ by $v\Delta(\,\cdot\,)v^*$ and p by $vv^* \in \mathrm{L}(\Gamma \times \Gamma)$, so that $\Delta(\mathrm{L}\Gamma) \subset p\mathrm{L}(\Gamma \times \Gamma)p$.

Denote $C := \Delta(A)' \cap p(M \mathbin{\overline{\otimes}} M)p$.

Step 2. We have $C \prec A \mathbin{\overline{\otimes}} A$.

Proof. We apply Theorem 5.1 to the abelian von Neumann subalgebra $D := \Delta(A)$ of $p(M \mathbin{\overline{\otimes}} M)p$ that is normalized by the unitaries $(\Delta(u_g))_{g\in\Gamma}$ that belong to $p\mathrm{L}(\Gamma \times \Gamma)p$. So we have to check the four assumptions (1)–(4) of Theorem 5.1.

Since A is diffuse, Proposition 7.2(1) says that $\Delta(A) \not\prec M \otimes 1$ and $\Delta(A) \not\prec 1 \otimes M$. So assumption (1) holds.

The quasi-normalizer of $\Delta(A)$ inside $p(M \mathbin{\overline{\otimes}} M)p$ contains $\Delta(M)$. Since for every $i \in I$ we have that $\operatorname{Stab} i \subset \Gamma$ has infinite index, Proposition 7.2(2) implies that $\Delta(M) \not\prec M \mathbin{\overline{\otimes}} (A \rtimes \operatorname{Stab} i)$ and $\Delta(M) \not\prec (A \rtimes \operatorname{Stab} i) \mathbin{\overline{\otimes}} M$. So assumption (2) holds. Since also $\mathrm{L}\Gamma \subset M$ has infinite index, for the same reason assumption (3) holds.

Finally, since Γ is nonamenable and $\operatorname{Stab} i$ is amenable for every $i \in I$, Proposition 7.2(4) implies that $\Delta(\mathrm{L}\Gamma) \not\prec \mathrm{L}(\Gamma \times \operatorname{Stab} i)$ and $\Delta(\mathrm{L}\Gamma) \not\prec \mathrm{L}(\operatorname{Stab} i \times \Gamma)$. So also assumption (4) holds.

The conclusion of Step 2 now follows from Theorem 5.1. $\square$

Since $C = p(M \mathbin{\overline{\otimes}} M)p \cap \Delta(A)'$, the unitaries $\Delta(u_g)$ normalize C and define an action $(\beta_g)_{g\in\Gamma}$ of Γ on C given by $\beta_g(d) = \Delta(u_g) d \Delta(u_g)^*$ for all $g \in \Gamma$, $d \in C$.

Step 3. If $\mathcal{H} \subset \mathrm{L}^2(C)$ is a finite dimensional $(\beta_g)_{g\in\Gamma}$-invariant subspace, we have $\mathcal{H} \subset \mathbb{C}1$.

Proof. Define $\mathcal{K} \subset p\mathrm{L}^2(M \mathbin{\overline{\otimes}} M)p$ as the norm closed linear span of $\mathcal{H}\Delta(M)$. Then, $\Delta(A)\mathcal{K} \subset \mathcal{K}$ because $\mathcal{H}$ and $\Delta(A)$ commute. Also, $\Delta(u_g)\mathcal{K} = \mathcal{K}$ for all $g \in \Gamma$ because $\mathcal{H}$ is globally invariant under $(\beta_g)_{g\in\Gamma}$. So, $\mathcal{K}$ is a $\Delta(M)$-$\Delta(M)$-bimodule which, by construction, is finitely generated as a right $\Delta(M)$-module. By Proposition 7.2(3), we have $\mathcal{K} \subset \Delta(\mathrm{L}^2(M))$ and hence $\mathcal{H} \subset \Delta(\mathrm{L}^2(M))$. Since elements of $\mathcal{H}$ commute with $\Delta(A)$, we have $\mathcal{H} \subset \Delta(\mathrm{L}^2(A))$. Since the action of Γ on A is weakly mixing, the global invariance under $(\beta_g)_{g\in\Gamma}$ forces $\mathcal{H} \subset \mathbb{C}1$. $\square$

Step 4. We have $t = 1$, and there exist a unitary $\Omega \in M \mathbin{\overline{\otimes}} M$, a group homomorphism $\delta : \Gamma \to \Gamma \times \Gamma$ and a character $\omega : \Gamma \to \mathbb{T}$ such that

$$\Omega^* \Delta(u_g) \Omega = \omega(g)\, u_{\delta(g)} \quad \forall g \in \Gamma \quad \text{and} \quad \Omega^* \Delta(A) \Omega \subset A \mathbin{\overline{\otimes}} A. \tag{8.1}$$

Proof. We apply Corollary 6.2 to the crossed product $M \mathbin{\overline{\otimes}} M = (A \mathbin{\overline{\otimes}} A) \rtimes (\Gamma \times \Gamma)$. We no longer make the simplifying assumption that $t \leqslant 1$. So, take a projection $p \in \mathrm{M}_n(\mathbb{C}) \mathbin{\overline{\otimes}} M \mathbin{\overline{\otimes}} M$ with $(\mathrm{Tr} \otimes \tau \otimes \tau)(p) = t$. The amplified comultiplication is a unital $*$-homomorphism $\Delta : M \to p(\mathrm{M}_n(\mathbb{C}) \mathbin{\overline{\otimes}} M \mathbin{\overline{\otimes}} M)p$, and by Step 1 we may assume, after a unitary conjugacy, that $p \in \mathrm{M}_n(\mathbb{C}) \otimes \mathrm{L}(\Gamma \times \Gamma)$ and $\Delta(\mathrm{L}\Gamma) \subset p(\mathrm{M}_n(\mathbb{C}) \otimes \mathrm{L}(\Gamma \times \Gamma))p$. Put $C = \Delta(A)' \cap p(\mathrm{M}_n(\mathbb{C}) \overline{\otimes} M \overline{\otimes} M)p$. Since A is abelian, $C' \cap p(\mathrm{M}_n(\mathbb{C}) \overline{\otimes} M \overline{\otimes} M)p = \mathcal{Z}(C)$. By Step 2, $C \prec A \overline{\otimes} A$. By Step 3, the action $(\operatorname{Ad} \Delta(u_g))_{g \in \Gamma}$ is weakly mixing on $\mathcal{Z}(C)$. Even more so, $\mathbb{C}1$ is the only finite-dimensional globally $(\operatorname{Ad} \Delta(u_g))_{g \in \Gamma}$-invariant subspace of C. Since Γ is an icc group, Proposition 7.2(2) implies that $\Delta(M) \not\prec M \overline{\otimes} (A \rtimes \operatorname{Centr} g)$ and $\Delta(M) \not\prec (A \rtimes \operatorname{Centr} g) \overline{\otimes} M$. Because the von Neumann algebra generated by C and $\Delta(u_g), g \in \Gamma$ contains $\Delta(M)$, all conditions of Theorem 6.1, together with the extra condition 1 in Corollary 6.2, are satisfied.

By Corollary 6.2 we get that $t = 1$ and that there exist a unitary $\Omega \in M \overline{\otimes} M$, a group homomorphism $\delta : \Gamma \to \Gamma \times \Gamma$ and a character $\omega : \Gamma \to \mathbb{T}$ such that $\Omega^* \Delta(u_g) \Omega = \omega(g)\, u_{\delta(g)}$ and $\Omega^* C \Omega = A \overline{\otimes} A$. In particular, $\Omega^* \Delta(A) \Omega \subset A \overline{\otimes} A$. $\square$

Step 5. End of the proof of Theorem 8.2.

Proof. Take Ω, δ, ω as in Step 4. After Step 1, we decided to replace Δ by $\operatorname{Ad} v \circ \Delta$. From now on, $\Delta : \mathrm{L}\Lambda \to \mathrm{L}\Lambda \overline{\otimes} \mathrm{L}\Lambda$ is again the comultiplication. The conclusion of Step 4 remains of course true, replacing Ω by $v^* \Omega$.

Write $\delta(g) = (\delta_1(g), \delta_2(g))$. By Proposition 7.2(2), $\Delta(M) \not\prec M \overline{\otimes} (A \rtimes S)$ whenever $S < \Gamma$ is of infinite index. Hence, the subgroups $\delta_i(\Gamma)$, $i = 1, 2$, are of finite index in Γ.

Applying the flip to (8.1), it follows that $u_{\delta_1(g)} \otimes u_{\delta_2(g)}$ and $u_{\delta_2(g)} \otimes u_{\delta_1(g)}$ are unitarily conjugate inside $M \overline{\otimes} M$. Since $\delta_i(\Gamma) \subset \Gamma$ has finite index, there must exist $h \in \Gamma$ such that $\delta_2(g) = h \delta_1(g) h^{-1}$ for all $g \in \Gamma$. Replacing Ω by $\Omega(1 \otimes u_h)$, we may assume that $\delta_1 = \delta_2$ and we write δ instead of δ_1, δ_2.

Define $\Gamma' = \delta(\Gamma)$. The co-associativity of Δ implies that the representations $(u_{\delta(g)} \otimes u_{\delta(g)} \otimes u_g)_{g \in \Gamma'}$ and $(u_g \otimes u_{\delta(g)} \otimes u_{\delta(g)})_{g \in \Gamma'}$ are unitarily conjugate in $M \overline{\otimes} M \overline{\otimes} M$. Since $\Gamma' < \Gamma$ has finite index, it follows that there exists $h \in \Gamma$ such that $\delta(g) = hgh^{-1}$ for all $g \in \Gamma'$. Then automatically, $\delta(g) = hgh^{-1}$ for all $g \in \Gamma$. Replacing Ω by $\Omega(u_h \otimes u_h)$, we may assume that

$$\Omega^* \Delta(u_g) \Omega = \omega(g)\; u_g \otimes u_g \;\; \forall g \in \Gamma.$$

If $\sigma(a \otimes b) = b \otimes a$ denotes the flip map, it follows that $\Omega^* \sigma(\Omega)$ commutes with all $u_g \otimes u_g$, $g \in \Gamma$ and hence, is scalar. Similarly,

$$(\Omega \otimes 1)^* (\Delta \otimes \mathrm{id})(\Omega)^* (\mathrm{id} \otimes \Delta)(\Omega)(1 \otimes \Omega) \quad \text{commutes with all} \quad u_g \otimes u_g \otimes u_g,\; g \in \Gamma$$

and hence, is scalar. By Theorem 3.3, there exists a unitary $w \in M$ such that $\Omega = \Delta(w^*)(w \otimes w)$.

To make the end of the argument more clear, again we write explicitly the isomorphism $\pi : \mathrm{L}\Lambda \to \mathrm{L}(G)^t$, instead of the implicit identification $\mathrm{L}\Lambda = \mathrm{L}(G)^t$. So far, we have shown that $t = 1$ and we have found a unitary $w \in \mathrm{L}G$

and a character $\omega : \Gamma \to \mathbb{T}$ such that after replacing π by $\pi_\omega^{-1} \circ \operatorname{Ad} w^* \circ \pi$, we have

$$(\pi \otimes \pi)\Delta(\pi^{-1}(A)) = A \mathbin{\overline{\otimes}} A \quad \text{and} \quad (\pi \otimes \pi)\Delta(\pi^{-1}(u_g)) = u_g \otimes u_g$$

for all $g \in \Gamma$. By Lemma 7.1 we find an abelian subgroup $\Sigma < \Lambda$ such that $\pi^{-1}(A) = \mathrm{L}\Sigma$ and an injective group homomorphism $\rho : \Gamma \to \Lambda$ such that $\pi^{-1}(u_g) = v_{\rho(g)}$. By construction, $\operatorname{Ad} v_{\rho(g)}$ normalizes $\mathrm{L}\Sigma$ and hence, $\operatorname{Ad} \rho(g)$ normalizes Σ. We have found an action of Γ by automorphisms of Σ and an isomorphism of groups $\delta : \Lambda \to \Sigma \rtimes \Gamma$ satisfying $\delta(s\rho(g)) = (s, g)$ for all $s \in \Sigma$, $g \in \Gamma$. Moreover, the $*$-isomorphism $\pi \circ \pi_{\delta^{-1}} : \mathrm{L}(\Sigma) \rtimes \Gamma \to A \rtimes \Gamma$ maps $\mathrm{L}\Sigma$ onto A and is the identity on $u_g, g \in \Gamma$. We define $\theta : \mathrm{L}\Sigma \to A$ as the restriction of $\pi \circ \pi_{\delta^{-1}}$ to $\mathrm{L}\Sigma$, ending the proof of the theorem. □

This ends the proof of Theorem 8.2. □

Proof of Theorem 8.3. By Theorem 8.2, we have $t = 1$ and $\pi = \operatorname{Ad} w \circ \pi_\omega \circ \pi_\theta \circ \pi_\delta$, where $w \in \mathrm{L}G$ is a unitary, $\delta : \Lambda \to \Sigma \rtimes \Gamma$ is a group isomorphism, $\omega : \Lambda \to \mathbb{T}$ is a character and $\theta : \mathrm{L}\Sigma \to A$ is a $*$-isomorphism satisfying $\theta \circ \alpha_g = \sigma_g \circ \theta$ for all $g \in \Gamma$.

For all $i \in I$, put $\Gamma_i := \operatorname{Stab} i$. Recall that $A = \mathrm{L}(H_0^{(I)})$. Denote by $H_0^i < H_0^{(I)}$ the copy of H_0 in position $i \in I$. Define the subalgebra $B_i \subset \mathrm{L}\Sigma$ given by $B_i := \theta^{-1}\bigl(\mathrm{L}H_0^i\bigr)$. We claim that $\Delta(B_i) \subset B_i \mathbin{\overline{\otimes}} B_i$. If $b \in B_i$, the element $(\theta\otimes\theta)\Delta(b)$ is fixed under the automorphisms $\sigma_g \otimes \sigma_g$, $g \in \operatorname{Stab} i$. Since $\operatorname{Stab} i \cdot j$ is infinite for all $j \neq i$, this implies that $(\theta \otimes \theta)\Delta(b) \in \mathrm{L}(H_0^i \times H_0^i)$. Hence, $\Delta(b) \in B_i \mathbin{\overline{\otimes}} B_i$. By Lemma 7.1 we find subgroups $\Sigma_i < \Sigma$ such that $B_i = \mathrm{L}\Sigma_i$. By construction, the subalgebras $B_i \subset \mathrm{L}\Sigma$ are independent and generate $\mathrm{L}\Sigma$. Hence $\Sigma = \bigoplus_{i\in I} \Sigma_i$. Denote by θ_i the restriction of θ to $\mathrm{L}\Sigma_i$. So, $\theta_i : \mathrm{L}\Sigma_i \to \mathrm{L}H_0^i$ is a $*$-isomorphism.

In particular, Σ_i is an abelian group of order $|H_0|$. So, if $|H_0|$ is a square-free integer, then necessarily $\Sigma_i \cong H_0$ for every $i \in I$ and we easily conclude that $\Lambda \cong G$. For general nontrivial abelian groups H_0, but assuming that $\Gamma \curvearrowright I$ is transitive, choose $i_0 \in I$ and put $H_1 := \Sigma_{i_0}$. We have proven that $\Lambda \cong H_1 \wr_I \Gamma$.

In the specific case where $H_0 = \frac{\mathbb{Z}}{2\mathbb{Z}}$ or $\frac{\mathbb{Z}}{3\mathbb{Z}}$, every algebra isomorphism $\mathrm{L}\Sigma_i \to \mathrm{L}H_0^i$ is group-like. So, we find characters $\gamma_i : H_0^i \to \mathbb{T}$ and group isomorphisms $\rho_i : \Sigma_i \to H_0^i$ such that $\theta_i = \pi_{\gamma_i} \circ \pi_{\rho_i}$.

By construction, $\gamma_{g\cdot i} = \gamma_i \circ \sigma_g^{-1}$, $\alpha_g(\Sigma_i) = \Sigma_{g\cdot i}$ and $\sigma_g \circ \rho_i = \rho_{g\cdot i} \circ \alpha_g$. So, all γ_i combine into a $(\sigma_g)_{g\in\Gamma}$-invariant character $\gamma : H_0^{(I)} \to \mathbb{T}$ and all ρ_i combine into an group isomorphism $\rho : \Sigma \to H_0^{(I)}$ satisfying $\rho \circ \alpha_g = \sigma_g \circ \rho$ for all $g \in \Gamma$. By construction, $\theta = \pi_\gamma \circ \pi_\rho$. We extend γ to a character $\gamma : G \to \mathbb{T}$ by putting $\gamma(g) = 1$ for all $g \in \Gamma$. We extend ρ to a group isomorphism $\rho : \Sigma \rtimes \Gamma \to G$ by putting $\rho(g) = g$ for all $g \in \Gamma$. By construction, $\pi_\theta = \pi_\gamma \circ \pi_\rho$.

We have proven that

$$\pi = \operatorname{Ad} w \circ \pi_{\omega\,\gamma} \circ \pi_{\rho\circ\delta}.$$

This ends the proof of Theorem 8.3. □

9. Counterexamples for plain wreath products: proof of Theorem 1.2

Assume that Γ is a countable group and $\mathbb{Z} \hookrightarrow \Gamma$ is an embedding. Let H_0 be a nontrivial finite abelian group. Using the *co-induction construction*, we construct a new group Λ such that $\mathrm{L}(\Lambda) \cong \mathrm{L}(H_0 \wr \Gamma)$.

Define the countable abelian group $\Sigma_0 := \mathbb{Z}[|H_0|^{-1}]$, and denote by α the automorphism of Σ_0 given through multiplication by $|H_0|$. We also denote by $(\alpha_k)_{k\in\mathbb{Z}}$ the corresponding action of $\mathbb{Z}$ by group automorphisms of Σ_0 and then by automorphisms of $\mathrm{L}(\Sigma_0)$. We claim that α is conjugate with a Bernoulli action with base space $\{1, \dots, |H_0|\}$ equipped with the normalized counting measure. View $\mathrm{L}^\infty(\mathbb{T}) \cong \mathrm{L}\mathbb{Z} \subset \mathrm{L}(\Sigma_0)$. Identify $\mathrm{L}(H_0) \cong \ell^\infty(\{1, \dots, |H_0|\})$ with the subalgebra of $\mathrm{L}^\infty(\mathbb{T})$ that consists of the functions that are constant on the intervals $\{\exp(2\pi i t) \mid t \in [(j-1)/|H_0|, j/|H_0|)\}$. After all these identifications, one checks that the subalgebras $\alpha_k(\mathrm{L}(H_0))$, $k \in \mathbb{Z}$ of $\mathrm{L}(\Sigma_0)$ are independent and generate $\mathrm{L}(\Sigma_0)$. This results into a $*$-isomorphism

$$\theta_0 : \mathrm{L}(\Sigma_0) \to \mathrm{L}(H_0^{(\mathbb{Z})}) \quad \text{satisfying} \quad \theta_0 \circ \alpha_k = \sigma_k \circ \theta_0 \;\; \forall k \in \mathbb{Z}.$$

Here, $(\sigma_k)_{k\in\mathbb{Z}}$ denotes the Bernoulli shift on $\mathrm{L}(H_0^{(\mathbb{Z})})$.

We now perform the co-induction construction. Choose representatives $I \subset \Gamma$ for the coset space $\Gamma/\mathbb{Z}$. So, the multiplication map $I \times \mathbb{Z} \to \Gamma$ is a bijection. We get an action $\Gamma \curvearrowright I : (g, i) \mapsto g \cdot i$ and a map $\omega : \Gamma \times I \to \mathbb{Z}$ such that $gi = (g \cdot i)\omega(g, i)$ for all $g \in \Gamma$, $i \in I$. The map ω is a 1-cocycle: $\omega(gh, i) = \omega(g, h \cdot i)\omega(h, i)$ for all $g, h \in \Gamma$ and $i \in I$. Define $\Sigma := \Sigma_0^{(I)}$ and denote by $\pi_i : \Sigma_0 \to \Sigma$ the embedding in position i. Then, Γ acts on Σ by group automorphisms $(\beta_g)_{g\in\Gamma}$ defined as $\beta_g \circ \pi_i = \pi_{g\cdot i} \circ \alpha_{\omega(g,i)}$.

Put $\Lambda = \Sigma \rtimes \Gamma$. Observe that Λ is torsion free whenever Γ is torsion free. We claim that $\mathrm{L}\Lambda \cong \mathrm{L}(H_0 \wr \Gamma)$.

Identifying $(\mathrm{L}(H_0^{(\mathbb{Z})}))^I \cong \mathrm{L}(H_0^{(\Gamma)})$ through the multiplication map $I \times \mathbb{Z} \to \Gamma$, the formula $\theta = \bigotimes_{i\in I} \theta_0$ defines a $*$-isomorphism

$$\theta : \mathrm{L}(\Sigma) \to \mathrm{L}(H_0^{(\Gamma)}) \quad \text{satisfying} \quad \theta \circ \beta_g = \sigma_g \circ \theta \;\; \forall g \in \Gamma.$$

But then, θ extends to an isomorphism of the corresponding crossed product II_1 factors that are isomorphic with $\mathrm{L}(\Lambda)$ and $\mathrm{L}(H_0 \wr \Gamma)$ respectively. This proves the claim.

We have already proven that for Γ torsion free, there exists a torsion free group Λ satisfying $\mathrm{L}\Lambda \cong \mathrm{L}(H_0 \wr \Gamma)$. To conclude the proof of Theorem 1.2 we show that by varying the initial embedding $\mathbb{Z} \hookrightarrow \Gamma := \mathrm{PSL}(n, \mathbb{Z})$, the above construction provides infinitely many nonisomorphic groups Λ. Assume that $\Lambda = \Sigma \rtimes \Gamma$ and $\Lambda' = \Sigma' \rtimes \Gamma$ are constructed as above from embeddings $\eta : \mathbb{Z} \to \Gamma$ and $\eta' : \mathbb{Z} \to \Gamma$. It suffices to prove the following.

Claim. If for every automorphism $\delta \in \mathrm{Aut}(\Gamma)$, the intersection $\delta(\eta(\mathbb{Z})) \cap \eta'(\mathbb{Z})$ is reduced to $\{1\}$, then $\Lambda \not\cong \Lambda'$.

Assume that $\lambda : \Lambda \to \Lambda'$ is an isomorphism of groups. Since $\mathrm{PSL}(n, \mathbb{Z})$ has no normal abelian subgroups except from $\{1\}$, it follows that $\lambda(\Sigma) = \Sigma'$. Hence, λ is of the form $\lambda(x, g) = (\dots, \delta(g))$ for some automorphism δ of $\Gamma = \mathrm{PSL}(n, \mathbb{Z})$. Since Σ and Σ' are abelian groups, it follows that $\lambda|_\Sigma \circ \beta_g = \beta_{\delta(g)} \circ \lambda|_\Sigma$ for all $g \in \Gamma$. Denote by $i \in I$ the coset $\eta(\mathbb{Z})$ of the identity element. Take a nontrivial element $x \in \Sigma_0$. Take a finite subset $\mathcal{F} \subset \Gamma/\eta'(\mathbb{Z})$ such that $\lambda(\pi_i(x)) \subset \Sigma_0^{\mathcal{F}}$. We prove that $\lambda(\pi_i(\Sigma_0)) \subset \Sigma_0^{\mathcal{F}}$. Choose $y \in \Sigma_0$. By construction, we can find $z \in \Sigma_0$ such that both x and y are a multiple of z. So, $\lambda(\pi_i(x))$ is a multiple of $\lambda(\pi_i(z))$. Since Σ_0 is torsion free and $\lambda(\pi_i(x)) \in \Sigma_0^{\mathcal{F}}$, it follows that $\lambda(\pi_i(z)) \in \Sigma_0^{\mathcal{F}}$. But then, $\lambda(\pi_i(y)) \in \Sigma_0^{\mathcal{F}}$ as well.

Since the subgroup $\pi_i(\Sigma_0)$ is globally invariant under $\eta(\mathbb{Z})$, it follows that $\lambda(\pi_i(\Sigma_0))$ is globally invariant under $\delta(\eta(\mathbb{Z}))$. But $\lambda(\pi_i(\Sigma_0)) \subset \Sigma_0^{\mathcal{F}}$. Hence, the action of $\delta(\eta(\mathbb{Z}))$ on $\Gamma/\eta'(\mathbb{Z})$ has at least one finite orbit. Applying the assumption to the automorphism $\mathrm{Ad}\, g \circ \delta$, we have $g\delta(\eta(\mathbb{Z}))g^{-1} \cap \eta'(\mathbb{Z}) = \{1\}$ for all $g \in \Gamma$, so that $\delta(\eta(\mathbb{Z}))$ acts freely on $\Gamma/\eta'(\mathbb{Z})$. We have reached a contradiction.

Remark 9.1. There are essentially two sources of unexpected isomorphisms between II_1 factors. The first one is Connes' uniqueness theorem for amenable II_1 factors [Con76] implying that all $\mathrm{L}\Gamma$ for Γ amenable icc are isomorphic. Secondly, Voiculescu's free probability theory leads to striking isomorphisms between von Neumann algebras constructed as free products; see, e.g., [Voi90] and the later developments in [Dyk94], [Dyk93], [DR00]. As an illustration we provide the following list of isomorphic group factors $\mathrm{L}G$.

(1) Since infinite tensor products of II_1 factors are McDuff, it follows that whenever $G = \bigoplus_{i=1}^{\infty} \Lambda_i$ is the infinite direct sum of icc groups Λ_i, then $\mathrm{L}G \cong \mathrm{L}(\Gamma \times G)$ for all icc amenable groups Γ.

(2) We have that $\mathrm{L}(\Gamma_1 * \cdots * \Gamma_n) \cong \mathrm{L}\mathbb{F}_n$ whenever $\Gamma_1, \dots, \Gamma_n$ are infinite amenable groups and $n \geqslant 2$. By [Dyk93, Cor. 5.3], the statement holds for $n = 2$ and next, by induction,

$$\begin{aligned} \mathrm{L}(\Gamma_1 * \cdots * \Gamma_n) &\cong \mathrm{L}(\Gamma_1 * \cdots * \Gamma_{n-2}) * \mathrm{L}(\Gamma_{n-1} * \Gamma_n) \\ &\cong \mathrm{L}(\Gamma_1 * \cdots * \Gamma_{n-2}) * \mathrm{L}(\mathbb{F}_2) \\ &\cong \mathrm{L}(\Gamma_1 * \cdots * \Gamma_{n-2} * \mathbb{Z}) * \mathrm{L}(\mathbb{Z}) \cong \mathrm{L}(\mathbb{F}_{n-1}) * \mathrm{L}(\mathbb{Z}) \cong \mathrm{L}(\mathbb{F}_n). \end{aligned}$$

In the same vein, by [DR00, Th. 1.5] it follows that whenever $G = \Lambda_1 * \Lambda_2 * \cdots$ is the infinite free product of nontrivial groups Λ_i, then $\mathrm{L}(G) \cong \mathrm{L}(\mathbb{F}_\infty * G)$.

(3) The subtlety of how $\mathrm{L}G$ depends on G is nicely illustrated by the following remark due to Ozawa [Oza06]. Fix a nonamenable group Γ and an infinite group Λ. Consider $G_n := \mathbb{F}_\infty * (\Gamma \times \Lambda)^{*n}$.

- If Λ is abelian and $\mathrm{L}\Gamma \cong \mathrm{M}_2(\mathbb{C}) \overline{\otimes} \mathrm{L}(\Gamma)$, then all $\mathrm{L}(G_n)$ are isomorphic, though nonisomorphic with $\mathrm{L}\mathbb{F}_\infty$.
- If Γ, Λ are icc (and still Γ nonamenable), then all $\mathrm{L}(G_n)$ are nonisomorphic.

The reason for this is the following. Fix arbitrary von Neumann algebras P, Q equipped with faithful normal tracial states. Consider the II_1 factors $N_n := \mathrm{L}\mathbb{F}_\infty * (P \overline{\otimes} Q)^{*n}$. If $P \cong \mathrm{M}_2(\mathbb{C}) \otimes P$ and if Q is diffuse abelian, then all N_n are isomorphic. Indeed, applying [Dyk94, Th. 3.5(iii)] to $A = \mathrm{L}\mathbb{F}_\infty$, $B = P \overline{\otimes} Q$ and using the fact that 2 belongs to the fundamental group of $\mathrm{L}\mathbb{F}_\infty$, it follows that 2 belongs to the fundamental group of N_1. Applying [Dyk94, Th. 3.5(ii)] to the same algebras A, B and using the obvious isomorphism $Q \cong Q \overline{\otimes} \mathrm{L}(\mathbb{Z}/2\mathbb{Z})$, it follows that $N_1 \cong \mathrm{M}_2(\mathbb{C}) \otimes N_2$. Since $1/2$ belongs to the fundamental group of N_1, we conclude that $N_1 \cong N_2$. But then, $N_1 \cong N_n$ for all n. On the other hand, if P is a nonamenable factor and Q is a diffuse factor, then the II_1 factors M_n are nonisomorphic. When P and Q are semi-exact, this follows from [Oza06, Cor. 3.5]. In the general case, the methods of [IPP08] can be used; see [Pet09, Th. 1.4] and [CH10, Th. 1.1].

(4) As observed in [Ioa07, Prop. 6.4], if $\mathrm{L}(H_1)$ and $\mathrm{L}(H_2)$ are stably isomorphic, then $\mathrm{L}(H_1 \wr \mathbb{Z}) \cong \mathrm{L}(H_2 \wr \mathbb{Z})$. In particular, all group von Neumann algebras $\mathrm{L}(\mathbb{F}_n \wr \mathbb{Z})$, $n \geqslant 2$, are isomorphic. This is in sharp contrast with our Theorem 1.1 saying that the group $(\mathbb{Z}/2\mathbb{Z})^{(I)} \rtimes (\mathbb{F}_n \wr \mathbb{Z})$ is superrigid, where $I = (\mathbb{F}_n \wr \mathbb{Z})/\mathbb{Z}$.

(5) In [Bow11a, Cor. 1.2] it is shown that the Bernoulli actions $\mathbb{F}_2 \curvearrowright (X_0, \mu_0)^{\mathbb{F}_2}$ are orbit equivalent for different choices of the base probability space (X_0, μ_0). It follows that all $\mathrm{L}(H \wr \mathbb{F}_2)$, H a nontrivial abelian group, are isomorphic. In [Bow11b, Th. 1.1] it is shown that for different values of n, the Bernoulli actions $\mathbb{F}_n \curvearrowright (X_0, \mu_0)^{\mathbb{F}_n}$ are stably orbit equivalent. Hence, for all choices of n, m and all nontrivial abelian groups H_1, H_2, the II_1 factors $\mathrm{L}(H_1 \wr \mathbb{F}_n)$ and $\mathrm{L}(H_2 \wr \mathbb{F}_m)$ are stably isomorphic. In particular, $\mathrm{L}((H_1 \wr \mathbb{F}_n) \times \Lambda_1) \cong \mathrm{L}((H_2 \wr \mathbb{F}_m) \times \Lambda_2)$ when Λ_1, Λ_2 are icc amenable.

10. W*-superrigidity for Bernoulli actions of product groups

THEOREM 10.1. *Let Γ be an* icc *group that admits a chain of infinite subgroups $\Gamma_0 < \Gamma_1 < \cdots < \Gamma_n = \Gamma$ such that Γ_{k-1} is almost normal in Γ_k, for every $k = 1, \ldots, n$ and the centralizer of Γ_0 inside Γ_1 is nonamenable. Let (X_0, μ_0) be a nontrivial standard probability space. Then the Bernoulli action $\Gamma \curvearrowright (X, \mu) := (X_0, \mu_0)^\Gamma$ is W*-superrigid. If $\Lambda \curvearrowright (Y, \eta)$ is an arbitrary free* *ergodic probability measure-preserving action and* $\pi : \mathrm{L}^\infty(Y) \rtimes \Lambda \to \mathrm{L}^\infty(X) \rtimes \Gamma$ *a* $*$*-isomorphism, then* $\Lambda \cong \Gamma$ *and the actions are conjugate.*

More precisely, there exist an isomorphism of groups $\delta : \Lambda \to \Gamma$, *an isomorphism of probability spaces* $\Psi : Y \to X$, *a character* $\omega : \Gamma \to \mathbb{T}$ *and a unitary* $w \in \mathrm{L}^\infty(X) \rtimes \Gamma$ *such that* $\Psi(s \cdot y) = \delta(s) \cdot \Psi(y)$ *for all* $s \in \Lambda$ *and almost every* $y \in Y$ *and such that*

$$\pi = (\mathrm{Ad}\, w) \circ \pi_\omega \circ \pi_{\Psi,\delta},$$

where $\pi_{\Psi,\delta}(bv_s) = (b \circ \Psi^{-1})u_{\delta(s)}$ *for all* $b \in \mathrm{L}^\infty(Y)$, $s \in \Lambda$ *and* $\pi_\omega(au_g) = \omega(g)au_g$ *for all* $a \in \mathrm{L}^\infty(X)$, $g \in \Gamma$.

Denote $A = \mathrm{L}^\infty(X)$ and $M = A \rtimes \Gamma$. Put $B = \mathrm{L}^\infty(Y)$, and identify $M = B \rtimes \Lambda$ through π. Let $\Delta : M \to M \overline{\otimes} M$ be the unital $*$-homomorphism defined as $\Delta(bv_s) = bv_s \otimes v_s$ for all $b \in B$ and $s \in \Lambda$. Before continuing, let us record a few useful properties of Δ.

Lemma 10.2. *Let* $P \subset M$ *be a von Neumann subalgebra.*

- *If* $P \not\prec B$, *then* $\Delta(P) \not\prec M \otimes 1$.
- *If* P *is diffuse, then* $\Delta(P) \not\prec 1 \otimes M$.
- *If* $\Delta(M) \prec M \overline{\otimes} P$, *then* $\mathrm{L}\Lambda \prec M$.
- *If* $\Delta(M) \prec P \overline{\otimes} M$, *there exists a nonzero projection* $p \in P' \cap M$ *such that* $Pp \subset pMp$ *has finite index.*
- *If* P *has no amenable direct summand, then* $\Delta(P)$ *is strongly nonamenable relative to* $M \otimes 1$ *and* $1 \otimes M$. *In particular, if* $N \subset M$ *is an amenable von Neumann subalgebra, then* $\Delta(P) \not\prec M \overline{\otimes} N$ *and* $\Delta(P) \not\prec N \overline{\otimes} M$.

Proof. To prove (1)–(4), see [Ioa11, Lemma 9.2] or adapt the proof of Proposition 7.2. Since B is amenable, the M-$(M \overline{\otimes} M)$-bimodule

$${}_{(\Delta(M) \otimes 1)}\mathrm{L}^2(M \overline{\otimes} M \overline{\otimes} M)_{(M \overline{\otimes} 1 \overline{\otimes} M)}$$

is weakly contained in the coarse M-$(M \overline{\otimes} M)$-bimodule $\mathrm{L}^2(M) \overline{\otimes} \mathrm{L}^2(M \overline{\otimes} M)$. Continuing exactly as in the proof of Proposition 7.2(4) yields (5). $\square$

To prove Theorem 10.1 it is sufficient to show that $B \prec A$ or $\mathrm{L}\Lambda \prec \mathrm{L}\Gamma$. First, if $\mathrm{L}\Lambda \prec \mathrm{L}\Gamma$, then [Ioa11, Case (5) in the proof of Theorem 9.1] shows that automatically $B \prec A$. If $B \prec A$ then, by [Pop06a, Th. A.1], B and A are unitarily conjugate, so that after such a unitary conjugacy, π is implemented by an orbit equivalence between $\Lambda \curvearrowright Y$ and $\Gamma \curvearrowright X$, together with an $\mathbb{T}$-valued cocycle for the action $\Gamma \curvearrowright X$. By the cocycle superrigidity theorem [Pop08, Th. 1.1], we can assume that the orbit equivalence is a conjugacy and that the $\mathbb{T}$-valued cocycle is a character.

We prove Theorem 10.1 by contradiction, assuming that $B \not\prec A$ and $\mathrm{L}\Lambda \not\prec \mathrm{L}\Gamma$. The proof consists of several steps.

Step 1. There exists a unitary $v \in M\overline{\otimes}M$ such that $v\Delta(\mathrm{L}\Gamma)v^* \subset \mathrm{L}(\Gamma\times\Gamma)$.

Proof. Let $Q = \Delta(\mathrm{L}\Gamma_0)$, and denote by P the quasi-normalizer of Q inside $M\overline{\otimes}M$. Since $\Delta(\mathrm{L}\Gamma_1) \subset P$ and the centralizer of Γ_0 inside Γ_1 is nonamenable, Lemma 10.2.5 implies that $Q' \cap P$ is strongly nonamenable relative to $M \otimes 1$ and $1 \otimes M$. Since Γ_1 is nonamenable and $\Delta(\mathrm{L}\Gamma_1) \subset P$, Lemma 10.2.5 gives that $P \not\prec M\overline{\otimes}A$ and $P \not\prec A\overline{\otimes}M$.

We claim that $Q \not\prec M \otimes 1$ and $Q \not\prec 1 \otimes M$. Indeed, by Lemma 10.2 it suffices to prove that $\mathrm{L}\Gamma_0 \not\prec B$. If we assume that $\mathrm{L}\Gamma_0 \prec B$, then [Va07, Lemma 3.5.] implies that $B \prec (\mathrm{L}\Gamma_0)' \cap M$. Since Γ_0 is infinite, we get that $(\mathrm{L}\Gamma_0)' \cap M \subset \mathrm{L}\Gamma$ and therefore $B \prec \mathrm{L}\Gamma$, contradicting the fact that B is regular in M (see [Pop06c, Th. 3.1]).

Applying Corollary 4.3(3) we get a unitary $v \in M\overline{\otimes}M$ such that $vPv^* \subset M\overline{\otimes}\mathrm{L}\Gamma$. Repeating the last part of Step 1 in the proof of Theorem 8.2 yields the conclusion. □

From now we replace Δ by $(\mathrm{Ad}\, v)\circ\Delta$ and assume that $\Delta(\mathrm{L}\Gamma) \subset \mathrm{L}(\Gamma\times\Gamma)$. Let $C = \Delta(A)' \cap (M\overline{\otimes}M)$.

Step 2. For every projection $p \in \mathcal{Z}(C)$ we have $Cp \prec A\overline{\otimes}A$. Moreover there exists a unitary $u \in M\overline{\otimes}M$ such that $u\mathcal{Z}(C)u^* \subset A\overline{\otimes}A$.

Proof. Since Γ is nonamenable, by Lemma 10.2 we have that $\Delta(\mathrm{L}\Gamma) \not\prec \mathrm{L}\Gamma \otimes 1$ and $\Delta(\mathrm{L}\Gamma) \not\prec 1 \otimes \mathrm{L}\Gamma$. We claim that $\Delta(A) \not\prec \mathrm{L}\Gamma\overline{\otimes}M$ and $\Delta(A) \not\prec M\overline{\otimes}\mathrm{L}\Gamma$. If we assume the contrary, since $\Delta(M)$ is contained in the quasi-normalizer of $\Delta(A)$ inside $M\overline{\otimes}M$, [Ioa11, Prop. 3.5] implies that one of the following holds: $\Delta(A) \prec 1 \otimes M$, $\Delta(M) \prec \mathrm{L}\Gamma\overline{\otimes}M$, $\Delta(A) \prec M \otimes 1$ or $\Delta(M) \prec M\overline{\otimes}\mathrm{L}\Gamma$. Applying Lemma 10.2 we get that either A is not diffuse, $\mathrm{L}(\Gamma)$ has finite index in M, $A \prec B$ or $\mathrm{L}\Lambda \prec \mathrm{L}\Gamma$, all of which give a contradiction.

Since $\{\Delta(u_g)\}_{g\in\Gamma}$ normalize $\Delta(A)$, the previous paragraph allows us to apply [Ioa11, Th. 6.1] and the conclusion follows. □

Note that the unitaries $\{\Delta(u_g)\}_{g\in\Gamma}$ normalize C, and denote by $(\beta_g)_{g\in\Gamma}$ the action of Γ on C given by $\beta_g(x) = \Delta(u_g)x\Delta(u_g)^*$ for $g \in \Gamma$ and $x \in C$. Step 2 implies that the algebra $\mathcal{Z}_0 := \mathcal{Z}(C) \cap \mathrm{L}(\Gamma \times \Gamma)$ is completely atomic. Let $p \in \mathcal{Z}_0$ be a minimal projection, and let $G \subset \Gamma$ be a finite index subgroup such that p is $(\beta_g)_{g\in G}$-invariant.

We claim that the action $(\beta_g)_{g\in G}$ on $\mathcal{Z}(C)p$ is weakly mixing. To prove this claim, let $\mathcal{H} \subset \mathcal{Z}(C)p$ be a finite dimensional $(\beta_g)_{g\in G}$-invariant subspace. Then $\mathcal{H}$ is contained in the quasi-normalizer of $\Delta(\mathrm{L}G)p$ inside $p(M\overline{\otimes}M)p$. Since $\Delta(\mathrm{L}G) \not\prec \mathrm{L}\Gamma\otimes 1$ and $\Delta(\mathrm{L}G) \not\prec 1\otimes\mathrm{L}\Gamma$, we get from [Vae08, Lemma 4.2] that $\mathcal{H} \subset p\mathrm{L}(\Gamma \times \Gamma)p$. Thus $\mathcal{H} \subset \mathcal{Z}_0 p = \mathbb{C}p$, proving the claim.

For $d \geqslant 1$, we denote by $\mathcal{G}_d$ the group $\{u \otimes u_g | u \in \mathcal{U}(\mathrm{M}_d(\mathbb{C})), g \in \Gamma \times \Gamma\}$.

Step 3. There exist $d \geqslant 1$, two groups $K \subset \mathcal{G} \subset \mathcal{G}_d$ with K finite and normal in $\mathcal{G}$, a group homomorphism $\delta : G \to \mathcal{G}/K$, a partial isometry $w \in \mathrm{M}_{1,d}(\mathbb{C}) \otimes \mathrm{L}(\Gamma \times \Gamma)$ with $ww^* = p$ and $w^*w = p_K := |K|^{-1} \sum_{k \in K} k$ such that $w^*Cw = (\mathrm{M}_d(\mathbb{C}) \otimes (A \overline{\otimes} A))^{\mathrm{Ad}\, K} p_K$ and $w^*\Delta(u_g)w = \delta(g)p_K$ for all $g \in G$.

Proof. We apply Theorem 6.1 and Corollary 6.2(2) to the crossed product $M \overline{\otimes} M = (A \overline{\otimes} A) \rtimes (\Gamma \times \Gamma)$. Since the action $(\mathrm{Ad}(\Delta(u_g)p))_{g \in G}$ on $\mathcal{Z}(Cp)$ is weakly mixing, $Cp \prec A \overline{\otimes} A$ by Step 2, and $(Cp)' \cap p(M \overline{\otimes} M)p = \mathcal{Z}(Cp)$, all conditions of Theorem 6.1 are indeed satisfied.

Moreover, also the extra condition (2) in Corollary 6.2 holds. Indeed, if a subgroup H of $\Gamma \times \Gamma$ acts nonergodically on $A \overline{\otimes} A$, then $H \subset H_0 \times \Gamma$ or $H \subset \Gamma \times H_0$ for some finite subgroup H_0 of Γ. Since G is nonamenable, from Lemma 10.2 we know that $\Delta(\mathrm{L}G) \not\prec (A \overline{\otimes} A) \rtimes H$ for every such subgroup H of $\Gamma \times \Gamma$. Thus, we also have that $N \not\prec (A \overline{\otimes} A) \rtimes H$, where N denotes the von Neumann algebra generated by Cp and $\Delta(\mathrm{L}G)p$.

Theorem 6.1 and Corollary 6.2(2) provide the conclusion of Step 3. □

Step 4. End of the proof of Theorem 10.1.

Proof. Denote by $\gamma : \mathcal{U}(\mathrm{M}_d(\mathbb{C})) \times \Gamma \times \Gamma \to \Gamma$ the group morphism $\gamma(u, g, h) = h$. Put $\mathcal{G}_0 := \gamma(\mathcal{G})$ and $K_0 := \gamma(K)$. By construction, K_0 is a finite normal subgroup of $\mathcal{G}_0$, and we still denote by γ the natural group homomorphism $\gamma : \mathcal{G}/K \to \mathcal{G}_0/K_0$. Denote by $G_1 < G$ the kernel of the homomorphism $\gamma \circ \delta$. By construction, $w^*\Delta(\mathrm{L}G_1)w \prec M \otimes 1$ and hence, $\Delta(\mathrm{L}G_1) \prec M \otimes 1$. By Lemma 10.2 we have $\mathrm{L}G_1 \prec B$, and the proof of Step 1 implies that G_1 cannot be infinite.

We consider the Fourier decomposition of elements in $\mathrm{M}_d(\mathbb{C}) \overline{\otimes} M \overline{\otimes} M$ with respect to the crossed product $\mathrm{M}_d(\mathbb{C}) \overline{\otimes} M \overline{\otimes} M = (\mathrm{M}_d(\mathbb{C}) \overline{\otimes} M \overline{\otimes} A) \rtimes \Gamma$, where Γ only acts on A. Note that the Fourier coefficients of a bounded sequence $x_n \in \mathrm{M}_d(\mathbb{C}) \overline{\otimes} M \overline{\otimes} M$ tend to zero pointwise in $\| \cdot \|_2$ if and only if

$$\|E_{\mathrm{M}_d(\mathbb{C}) \overline{\otimes} M \overline{\otimes} A}(a x_n b)\|_2 \to 0 \;\; \forall a, b \in \mathrm{M}_d(\mathbb{C}) \overline{\otimes} M \overline{\otimes} M.$$

It follows that for all $a, b \in \mathrm{M}_d(\mathbb{C}) \overline{\otimes} M \overline{\otimes} M$, the Fourier coefficients of $a x_n b$ also tend to zero pointwise. We also consider the Fourier decomposition of elements in M with respect to the crossed product $M = A \rtimes \Gamma$. In both situations, we denote the Fourier coefficients of an element x as $(x)_g$, $g \in \Gamma$. When $x \in \mathrm{M}_d(\mathbb{C}) \overline{\otimes} M \overline{\otimes} M$, then $(x)_g \in \mathrm{M}_d(\mathbb{C}) \overline{\otimes} M \overline{\otimes} A$.

Define the normal $*$-homomorphism $\theta : A \to (\mathrm{M}_d(\mathbb{C}) \overline{\otimes} A \overline{\otimes} A)^{\mathrm{Ad}\, K}$ such that $w^*\Delta(a)w = \theta(a)p_K$ for all $a \in A$. By Step 3 we get that for all $x \in A \rtimes G$ and all $h \in \Gamma$, we have

$$\sum_{k \in K_0} (w^*\Delta(x)w)_{hk}(1 \otimes 1 \otimes u_{hk}) = \sum_{g \in G, \gamma(\delta(g)) = hK_0} \theta((x)_g)\delta(g)p_K.$$

Recall that for a fixed $h \in \Gamma$, there are only finitely many $g \in G$ satisfying $\gamma(\delta(g)) = hK_0$. So, if x_n is a bounded sequence in $A \rtimes G$ whose Fourier coefficients tend to zero pointwise, the same is true for $w^*\Delta(x_n)w$. By the remarks in the previous paragraph, the Fourier coefficients of $\Delta(x_n)p$ then also tend to zero pointwise.

Next, let x_n be a bounded sequence in $A \rtimes \Gamma$ whose Fourier coefficients tend to zero pointwise. Let $g_1, \ldots, g_s \in \Gamma$ be representatives for Γ/G. Define for $j = 1, \ldots, s$,

$$x_n^j := E_{A \rtimes G}(u_{g_j}^* x_n).$$

Then for every j, we have that $(x_n^j)_n$ is a bounded sequence in $A \rtimes G$ whose Fourier coefficients tend to zero pointwise. The previous paragraph, together with the formula

$$\Delta(x_n)p = \sum_{j=1}^{s} \Delta(u_{g_j})\Delta(x_n^j)p,$$

imply that the Fourier coefficients of $\Delta(x_n)p$ tend to zero pointwise.

Since $B \not\prec A$, Definition 2.1 provides a sequence of unitaries b_n whose Fourier coefficients tend to zero pointwise. By the previous paragraph the same is true for $\Delta(b_n)p$. But for $b \in B$, we have $\Delta(b) = v(b \otimes 1)v^*$ (recall that the unitary $v \in M \mathbin{\overline{\otimes}} M$ was given by Step 1 and that we conjugated the initial comultiplication Δ by v). So, $\Delta(b_n)p = v(b_n \otimes 1)vp$ and it follows that the Fourier coefficients of $(b_n \otimes 1)vp$ tend to zero pointwise. Taking the g-th Fourier coefficient we get that

$$\|(vp)_g\|_2 = \|b_n(vp)_g\|_2 = \|((b_n \otimes 1)vp)_g\|_2 \to 0$$

for all $g \in \Gamma$. We reached the contradiction that p must be 0. □

This ends the proof of Theorem 10.1. □

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