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A bounded linear extension operator for $L^{2,p}(\mathbb{R}^2)$

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Abstract

Let $L^{2,p}(\mathbb{R}^2)$ be the Sobolev space of real-valued functions on the plane whose Hessian belongs to L^p. For any finite subset $E \subset \mathbb{R}^2$ and $p > 2$, let $L^{2,p}(\mathbb{R}^2)|_E$ be the space of real-valued functions on E, equipped with the trace seminorm. In this paper we construct a bounded linear extension operator $T : L^{2,p}(\mathbb{R}^2)|_E \to L^{2,p}(\mathbb{R}^2)$. We also provide an explicit formula that approximates the $L^{2,p}(\mathbb{R}^2)|_E$ trace seminorm.

1. Introduction

Given an arbitrary subset $E \subset \mathbb{R}^n$ and function $f : E \to \mathbb{R}$, how can we tell whether there exists a smooth function $F : \mathbb{R}^n \to \mathbb{R}$ such that $F = f$ on E? If a smooth extension exists, can we take it to depend linearly on f? To investigate these questions in more detail we introduce some notation. We work with the following spaces of smooth functions on $\mathbb{R}^n$:

- $\mathbb{X} = C^m(\mathbb{R}^n)$ (functions with continuous m'th derivatives), defined by the norm
$$\|F\|_{\mathbb{X}} := \sup_{x\in\mathbb{R}^n} \max_{k\leq m} |\nabla^k F(x)|;$$
- $\mathbb{X} = C^{m,1}(\mathbb{R}^n)$ (functions with Lipschitz continuous m'th derivatives), defined by the norm
$$\|F\|_{\mathbb{X}} := \sup_{x\in\mathbb{R}^n} \max_{k\leq m} |\nabla^k F(x)| + \sup_{x,y\in\mathbb{R}^n} \frac{|\nabla^m F(x) - \nabla^m F(y)|}{|x-y|};$$
- $\mathbb{X} = L^{m,p}(\mathbb{R}^n)$ (functions with m'th derivatives belonging to L^p) in the range $n < p < \infty$, defined by the seminorm
$$\|F\|_{\mathbb{X}} := \left(\int_{\mathbb{R}^n} |\nabla^m F(x)|^p dx\right)^{1/p}.$$

For any of these spaces,[1] we define the trace space $\mathbb{X}|_E := \{F|_E : F \in \mathbb{X}\}$. This vector space carries the natural trace seminorm

$$\|f\|_{\mathbb{X}|_E} := \inf\{\|F\|_{\mathbb{X}} : \ F \in \mathbb{X},\ F|_E = f\} \quad \text{for } f \in \mathbb{X}|_E.$$

An extension of $f \in \mathbb{X}|_E$ is a function $F \in \mathbb{X}$ such that $F = f$ on E. A bounded linear extension operator with norm $A \geq 1$ is a linear map $T : \mathbb{X}|_E \to \mathbb{X}$ such that

$$Tf = f \text{ on } E, \quad \text{and} \quad \|Tf\|_{\mathbb{X}} \leq A\|f\|_{\mathbb{X}|_E} \text{ for all } f \in \mathbb{X}|_E. \tag{1.1}$$

We can now formulate the Whitney extension problem.

Problem 1. Let $E \subseteq \mathbb{R}^n$ (arbitrary) be given.

(a) Provide necessary and sufficient conditions for membership in the trace space $\mathbb{X}|_E$.
(b) Construct a bounded linear extension operator $T : \mathbb{X}|_E \to \mathbb{X}$.

For spaces $\mathbb{X}$ with suitable compactness properties, Problem 1 reduces to the finite, quantitative problem stated below.

Problem 2. Let $E \subseteq \mathbb{R}^n$ be finite.

(a) Find an expression $M : \mathbb{X}|_E \to \mathbb{R}$ that satisfies

$$C^{-1}M(f) \leq \|f\|_{\mathbb{X}|_E} \leq CM(f) \quad \text{for all } f : E \to \mathbb{R}.$$

(b) Construct a bounded linear extension operator $T : \mathbb{X}|_E \to \mathbb{X}$ with norm C.

Here, the constant $C = C(\mathbb{X}) \geq 1$ should not depend on E.

In this paper we answer Problem 2 for the Sobolev space $\mathbb{X} = L^{2,p}(\mathbb{R}^2)$ through the following extension theorem.

THEOREM 1. *Let $2 < p < \infty$. Suppose that $E \subseteq \mathbb{R}^2$ has cardinality $\#E = N < \infty$. Then there exists a bounded linear extension operator $T : L^{2,p}(\mathbb{R}^2)|_E \to L^{2,p}(\mathbb{R}^2)$ with norm C. Moreover, there exist linear functionals $\lambda_1, \ldots, \lambda_K$, where $K \leq CN^2$, such that*

$$M(f) := \left(\sum_{k=1}^{K} |\lambda_k(f)|^p\right)^{1/p} \textit{ satisfies } C^{-1}M(f) \leq \|f\|_{L^{2,p}(\mathbb{R}^2)|_E} \leq CM(f) \textit{ for all } f : E \to \mathbb{R}.$$

Here, the constant $C \geq 1$ depends only on p.

[1]The Sobolev embedding $L^{m,p}(\mathbb{R}^n) \subseteq C^{m-1}_{\text{loc}}(\mathbb{R}^n)$ holds, due to the assumption $n < p < \infty$. In particular, each $F \in L^{m,p}(\mathbb{R}^n)$ can be identified with a continuous function after modification on a measure-zero subset. Due to this identification, we can meaningfully restrict an $L^{m,p}(\mathbb{R}^n)$ function to an arbitrary subset of euclidean space.

Remark 1.1. In future joint work [11] it is shown that the functionals $\lambda_1, \dots, \lambda_K$ can be taken to have an additional structural property called assisted bounded depth and that one may take $K \leq CN$. This improvement is discussed later in the introduction.

Remark 1.2. A compactness argument involving Banach limits and Theorem 1 proves the existence of a bounded linear extension operator $T : L^{2,p}(\mathbb{R}^2)|_E \to L^{2,p}(\mathbb{R}^2)$ for infinite subsets $E \subseteq \mathbb{R}^2$; this argument solves Problem 1 and is presented in [11].

We now describe some related results in the literature. Whitney introduced and studied the extension problem for the space $\mathbb{X} = C^m(\mathbb{R}^n)$. In [26], he solved Problems 1 and 2 for $C^m(\mathbb{R}^1)$ through the method of divided difference quotients. In addition, Whitney laid the groundwork for future progress on the higher-dimensional case through introducing his classical extension operator for polynomial jets; see [22], [25]. Next came the conjecture of Brudnyi and Shvartsman termed the finiteness principle for $C^{m,1}(\mathbb{R}^n)$; they proved their conjecture for $m = 1$, thereby solving the Whitney extension problem for $C^{1,1}(\mathbb{R}^n)$; see [1], [2], [3], [4], [16], [17], [18]. The next breakthrough was the proof by C. Fefferman of the finiteness principle and the solution to Problems 1 and 2 for the spaces $C^{m,1}(\mathbb{R}^n)$ and $C^m(\mathbb{R}^n)$ for all $m, n \geq 1$; see [7], [6], [8], [9], [10].

On the other hand, most cases of the extension problem for $L^{m,p}(\mathbb{R}^n)$ are open despite the progress for $C^m(\mathbb{R}^n)$. In [15], Luli constructs a bounded linear extension operator for $\mathbb{X} = L^{m,p}(\mathbb{R})$ and finite E (solving Problem 2(b)), while Shvartsman [20] treats the more general case when E is infinite, solving Problems 1 and 2 for $L^{m,p}(\mathbb{R})$. Recently, Shvartsman has shown that the classical Whitney extension operator for $(m-1)$-jets produces a function with near-minimal $L^{m,p}(\mathbb{R}^n)$-seminorm for $p > n$; see [20]. In particular, this work resolves Problems 1 and 2 for the space $L^{1,p}(\mathbb{R}^n)$ when $p > n$; see [19].

The inherent gap in difficulty between $L^{1,p}(\mathbb{R}^2)$ and $L^{2,p}(\mathbb{R}^2)$ comes from the fact that pointwise evaluation of the gradient of an $L^{1,p}(\mathbb{R}^2)$ function is not well defined, while for $F \in L^{2,p}(\mathbb{R}^2)$, $p > 2$, the Sobolev embedding theorem provides a simple notion of consistency:

$$|\nabla F(x) - \nabla F(y)| \leq C|x-y|^{1-2/p}\|F\|_{L^{2,p}(\mathbb{R}^2)} \quad \text{for } x, y \in \mathbb{R}^2. \tag{1.2}$$

Thus, when choosing an extension we must ensure that its gradient vectors are consistent enough so that (1.2) does not force it to have large seminorm; making an intelligent choice for the gradient of our extension at certain points of the plane is a key aspect of our solution.

We now provide a sketch of the proof of Theorem 1. For certain subsets $S \subseteq \mathbb{R}^2$ that appear "flat," the extension problem can be reduced to an easier one-dimensional problem, and we construct a bounded linear extension operator $T_S : L^{2,p}(\mathbb{R}^2)|_S \to L^{2,p}(\mathbb{R}^2)$ in this case. In this manner, given $f : S \to \mathbb{R}$ we obtain an extension $F = T_S f$ having near-optimal seminorm. This near-optimal extension may be far from unique. For instance, suppose that S is contained in a line. Then $F + L$ extends f and has the same seminorm as F for any affine function L that vanishes on S. An analogous phenomenon occurs more generally for flat subsets $S \subseteq \mathbb{R}^2$. Indeed, there may exist two near-optimal extensions $F_1, F_2 \in L^{2,p}(\mathbb{R}^2)$ of the given $f : S \to \mathbb{R}^2$ such that $F_1 - F_2$ has nonnegligible size.

Now consider an arbitrary finite set $E \subseteq \mathbb{R}^2$. Using a Calderón-Zygmund decomposition we split E into an almost-disjoint partition $\{E_\nu\}_{\nu=1}^K$ formed by intersecting E with a collection of CZ squares. The decomposition is defined in such a way that each localized subset E_ν is flat. Given $f : E \to \mathbb{R}$ we may then construct a near-optimal local extension F_ν of each function $f|_{E_\nu}$ by previous remarks. As mentioned before, there may be considerable freedom in the choice of each F_ν. This complicates our approach since we cannot ensure that the local extensions are related to one another. To resolve these issues, we must place additional constraints on the local extensions that enforce their mutual consistency. To this end, we introduce the *keystone squares*, which are a subcollection of the CZ squares. We can accurately determine the first order Taylor polynomial of the desired extension near each of the keystone squares. Then, using this information we specify the Taylor polynomial of every other local extension at an appropriately chosen basepoint, providing the aforementioned constraints. These carefully chosen local extensions are mutually consistent. Thanks to this consistency, the local extensions can be patched together using a partition of unity to produce a near-optimal extension of the function f.

Calderón-Zygmund decompositions were used in [7], [6] to solve the Whitney extension problem for $C^{m-1,1}(\mathbb{R}^n)$. However, in that work the Calderón-Zygmund squares are treated on equal footing (in some sense). This is in contrast to our proof of Theorem 1, where the keystone squares play a special rôle. This completes the overview of the proof of Theorem 1.

We switch settings for the moment and consider $\mathbb{X} = C^{m-1,1}(\mathbb{R}^n)$. Suppose that $E \subseteq \mathbb{R}^n$ has cardinality $N < \infty$.

In [13], Fefferman-Klartag construct linear functionals $\lambda_1, \ldots, \lambda_L$ on the trace space $\mathbb{X}|_E$ such that

$$(1.3) \qquad \|f\|_{\mathbb{X}|_E} \approx \max\{|\lambda_\ell(f)| : \ell = 1, \ldots, L\} \quad \text{for every } f \in \mathbb{X}|_E.$$

This formula has several interesting properties. First, not many functionals are used; in fact, $L \leq CN$. Moreover, each functional $\lambda_\ell(f)$ depends on the restriction of f to a subset $S_\ell \subseteq E$ with cardinality at most $k = k(m, n)$

(independent of E). In [10], Fefferman constructs a bounded linear extension operator $T : \mathbb{X}|_E \to \mathbb{X}$ with the following structural property: there exists $d = d(m,n)$ such that, for each $x \in \mathbb{R}^n$, there exist $y_1, \ldots, y_d \in E$ and $a_1, \ldots, a_d \in \mathbb{R}$ such that

$$Tf(x) = \sum_{j=1}^{d} a_j f(y_j) \quad \text{for every } f \in \mathbb{X}|_E. \tag{1.4}$$

The *depth* of an extension operator is defined to be the smallest d as above. In [14], Luli constructs bounded depth extension operators in the case when $E \subseteq \mathbb{R}^n$ is an arbitrary closed subset. These results are essentially the best possible. Motivated by them, we pose several open problems.

Open Problem 1. Does there exist $d = d(p)$ such that for every finite subset $E \subseteq \mathbb{R}^2$ there exists a bounded linear extension operator $T : L^{2,p}(\mathbb{R}^2)|_E \to L^{2,p}(\mathbb{R}^2)$ with depth d?

Open Problem 2. Do there exist $k = k(p)$ and $C = C(p)$ such that for every finite subset $E \subseteq \mathbb{R}^2$ of cardinality $\#E = N$, there exist linear functionals $\lambda_\ell : L^{2,p}(\mathbb{R}^2)|_E \to \mathbb{R}$ and subsets $S_\ell \subseteq E$ such that $\#S_\ell \leq k$ and such that $\lambda_\ell(f)$ depends only on $f|_{S_\ell}$, for each $\ell = 1, \ldots, L$, where $L \leq CN$, and

$$C^{-1} \cdot \left(\sum_{\ell=1}^{L} |\lambda_\ell(f)|^p \right)^{1/p} \leq \|f\|_{L^{2,p}(\mathbb{R}^2)|_E} \leq C \cdot \left(\sum_{\ell=1}^{L} |\lambda_\ell(f)|^p \right)^{1/p} \qquad \forall f : E \to \mathbb{R}?$$

Unfortunately, the first open problem is answered negatively in the forthcoming article [12]. Indeed, for each $N \in \mathbb{N}$, there exists $E_N \subseteq \mathbb{R}^2$ such that $\#(E_N) = N$, and such that any sequence of linear extension operators $T_N : L^{2,p}(\mathbb{R}^2)|_{E_N} \to L^{2,p}(\mathbb{R}^2)$ with uniformly bounded norm must have depth approaching infinity. This shows that the Sobolev and $C^{m,1}$ versions of the extension problem have some fundamental differences.

In recent work, Shvartsman [21] has given an alternate proof of Theorem 1. This paper also answers the second open problem in the affirmative, with the constant $k = 6$. The methods in [21] are somewhat different from those presented here. It would be interesting to understand better the relationship between these two seemingly different approaches.

We now mention some further results on the structure of linear extension operators and formulas for the trace norm. In fact, certain (weaker) variants of the above open problems have been answered positively in [11]. In this paper, the more general extension problem for $L^{m,p}(\mathbb{R}^n)$ is solved, provided $p > n$. We conclude the introduction by stating these improvements in the special case $m = n = 2$ of current interest. Given linear functionals $\omega_1, \ldots, \omega_L : L^{2,p}(\mathbb{R}^2)|_E \to \mathbb{R}$, we say that $\lambda : L^{2,p}(\mathbb{R}^2)|_E \to \mathbb{R}$ has *assisted bounded depth* with *assists* $\omega_1, \dots, \omega_L$ provided that

$$\sum_{\ell=1}^{L} (\#\text{ of nonzero coefficients of } \omega_\ell) \leq CN$$

and

$$\lambda(f) = \sum_{x \in E} \alpha_x f(x) + \sum_{\ell=1}^{L} \beta_\ell \omega_\ell(f),$$

where

$$\#\big\{x \in E : \ \alpha_x \neq 0\big\} + \#\big\{1 \leq \ell \leq L : \ \beta_\ell \neq 0\big\} \leq C$$

for some constant $C = C(p)$.

The improvement to Theorem 1 from [11] states the following:

- There exist a bounded linear extension operator $T : L^{2,p}(\mathbb{R}^2)|_E \to L^{2,p}(\mathbb{R}^2)$ and linear functionals $\omega_1, \dots, \omega_L : L^{2,p}(\mathbb{R}^2)|_E \to \mathbb{R}$ such that, for each $y \in \mathbb{R}^2$, the linear functional $f \mapsto Tf(y)$ has assisted bounded depth with assists $\omega_1, \dots, \omega_L$.
- There exist linear functionals $\lambda_1, \dots, \lambda_{L'}$ that each have assisted bounded depth with assists $\omega_1, \dots, \omega_L$ such that $L' \leq CN$ and

$$C^{-1} \cdot \left(\sum_{\ell=1}^{L'} |\lambda_\ell(f)|^p \right)^{1/p} \leq \|f\|_{L^{2,p}(\mathbb{R}^2)|_E} \leq C \cdot \left(\sum_{\ell=1}^{L'} |\lambda_\ell(f)|^p \right)^{1/p}.$$

This concludes the introduction. In the next section we start on the proof of Theorem 1.

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2. Notation and definitions

Let $2 < p < \infty$ be fixed throughout.

Smooth function spaces. For a domain $\Omega \subseteq \mathbb{R}^2$, the Sobolev spaces $L^{2,p}(\Omega)$ and $W^{2,p}(\Omega)$ consist of functions $F : \Omega \to \mathbb{R}$ with finite seminorm, as defined below:

$$\|F\|_{L^{2,p}(\Omega)} := \|\nabla^2 F\|_{L^p(\Omega)},$$
$$\|F\|_{W^{2,p}(\Omega)} := \|F\|_{L^p(\Omega)} + \|\nabla F\|_{L^p(\Omega)} + \|\nabla^2 F\|_{L^p(\Omega)}.$$

These seminorms are extended to vector valued mappings $\Phi : \Omega \to \mathbb{R}^2$ in the obvious way. As before, since $p > 2$, the Sobolev theorem states that we may regard each $F \in L^{2,p}(\Omega)$ as belonging to $C^1_{\text{loc}}(\Omega)$.

For an interval $I \subseteq \mathbb{R}$ (perhaps unbounded), the standard Besov spaces $\dot{B}^{2-\frac{1}{p},p}(I)$ and $B^{2-\frac{1}{p},p}(I)$ are written $\dot{B}_p(I)$ and $B_p(I)$ respectively; these spaces consist of smooth functions $\varphi : I \to \mathbb{R}$ with finite seminorm, as defined below:

$$\|\varphi\|_{\dot{B}_p(I)} := \left(\int_I \int_I \frac{|\varphi'(x) - \varphi'(y)|^p}{|x-y|^p} dxdy \right)^{1/p},$$
$$\|\varphi\|_{B_p(I)} := \|\varphi\|_{L^p(I)} + \|\varphi'\|_{L^p(I)} + \|\varphi\|_{\dot{B}_p(I)}.$$

For a domain $\Omega \subseteq \mathbb{R}^d$, $d \in \{1,2\}$, and $\alpha \in (0,1]$, the Hölder space $\dot{C}^{1,\alpha}(\Omega)$ consists of differentiable functions $F : \Omega \to \mathbb{R}$ with finite seminorm:

$$\|F\|_{\dot{C}^{1,\alpha}(\Omega)} := \sup_{x,y \in \Omega} \frac{|\nabla F(x) - \nabla F(y)|}{|x-y|^\alpha}.$$

Coordinates. To specify euclidean coordinates (u, v) means to choose $x_0 \in \mathbb{R}^2$ and an orthonormal basis $e_1, e_2 \in \mathbb{R}^2$ and set $(s,t)_{uv} := se_1 + te_2 + x_0 \in \mathbb{R}^2$ for $s, t \in \mathbb{R}$. We may drop the subscript uv when the coordinate system being used is clear from the context.

Unless stated otherwise, coordinate expansions are taken with respect to some base coordinate system that remains fixed throughout the paper.

Geometry. A *square* $Q \subseteq \mathbb{R}^2$ takes the form $Q = (c_1, c_2) + [-h, h]^2$. The sidelength and center of Q are denoted $\delta_Q := 2h$ and $c_Q := (c_1, c_2)$, respectively.

Write $|\cdot|$ for the standard euclidean norm in $\mathbb{R}^2$, and denote the open ball $\{y \in \mathbb{R}^2 : |x-y| < r\}$ by $B(x,r)$. For $S, S' \subseteq \mathbb{R}^2$, we put $\text{dist}(S, S') := \inf\{|x-y| : x \in S,\ y \in S'\}$.

For any line $l \subseteq \mathbb{R}^2$ and subset $S \subseteq \mathbb{R}^2$, the orthogonal projection of S onto l is denoted by $\text{proj}_l S$.

Jets and Whitney fields. Let $\mathcal{P}$ denote the space of affine polynomials $\{A \cdot x + b : A \in \mathbb{R}^2,\ b \in \mathbb{R}\}$. For $F \in C^1(\mathbb{R}^2)$ and $y \in \mathbb{R}^2$, we define $J_y F \in \mathcal{P}$ (the jet of F at y) by

$$(J_y F)(x) = F(y) + \nabla F(y) \cdot (x - y).$$

Given a finite set $S \subseteq \mathbb{R}^2$, the space of *Whitney fields* on S is denoted by

$$\text{Wh}(S) := \{(L_y)_{y \in S} : L_y \in \mathcal{P} \text{ for each } y \in S\}.$$

We define $J_S F \in \text{Wh}(S)$ (the jet of F on S) by $J_S F = (J_y F)_{y \in S}$. Similarly, for $g \in C^1(\mathbb{R})$ and $y \in \mathbb{R}$, we set $J_y g(x) = g(y) + g'(y) \cdot (x - y)$. Denote the order of a multi-index $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}^2_+$ by $|\alpha| := \alpha_1 + \alpha_2$.

Trace space. Let $\mathbb{X}$ be a space of functions on a domain $\Omega \subset \mathbb{R}^d$, $d \in \{1,2\}$, with seminorm $\|\cdot\|_{\mathbb{X}}$. We assume that the functions in $\mathbb{X}$ are continuous.

For $F \in \mathbb{X}$ and $C \geq 1$, we say that $\|F\|_{\mathbb{X}}$ is C-optimal (rather, $F \in \mathbb{X}$ is C-optimal) with respect to some properties $p_1, p_2, \ldots, p_k$ provided that F satisfies $p_1, p_2, \ldots, p_k$, and

$$\|F\|_{\mathbb{X}} \leq C \cdot \inf\{\|G\|_{\mathbb{X}} : G \in \mathbb{X},\ G \text{ satisfies } p_1, p_2, \ldots, p_k\}.$$

For $E \subseteq \Omega$, we define the trace space $\mathbb{X}|_E := \{F|_E : F \in \mathbb{X}\}$ and trace seminorm

$$\|f\|_{\mathbb{X}|_E} = \inf\{\|F\|_{\mathbb{X}} : F \in \mathbb{X},\ \ F = f \ \text{ on } E\} \qquad \text{for } \ f \in \mathbb{X}|_E.$$

Flat subsets. We now define a geometric quantity that measures the flatness of euclidean subsets; this concept was mentioned briefly in the introduction. The *Besov seminorm* of a subset $S \subseteq \mathbb{R}^2$ is

$$\|S\|_{\dot{B}_p} := \inf\Big\{\|\varphi\|_{\dot{B}_p(\mathbb{R})} : \exists\, \varphi \in \dot{B}_p(\mathbb{R}),\ \exists \text{ coordinates } (u,v),$$
$$S \subseteq \{(t, \varphi(t))_{uv} : t \in \mathbb{R}\}\Big\}.$$

Note that subsets of line segments have Besov seminorm zero and that finite subsets have finite Besov seminorm.

Gamma. For a square $Q \subseteq \mathbb{R}^2$, finite subset $S \subseteq Q$, function $f : S \to \mathbb{R}$, point $x \in Q$, and $M \geq 0$, we set

(2.1)
$$\Gamma_Q(f, x, M) := \Big\{L \in \mathcal{P} :\ \exists F \in L^{2,p}(Q),\ F|_S = f,\ J_x F = L,\ \ \|F\|_{L^{2,p}(Q)} \leq M\Big\}.$$

Conventions. Unless stated otherwise, all constants are positive and may depend only on p; they are called *universal constants.* We write $A \lesssim B$ to mean $A \leq CB$ for some universal constant C. Similarly, we write $A \approx B$ to mean $C^{-1}B \leq A \leq CB$ for some universal constant C.

Let $c_1, c_2, c_3, c_4 \in (0,1)$ be small universal constants whose values remain fixed throughout the paper. We use $c, C, \widetilde{c}, \widetilde{C}, C_1, C_2, \ldots$ to denote other universal constants that may change value from one occurrence to the next.

Suppose that an object $\mathcal{O}$ has been constructed with designated properties, numbered, e.g., (1), (2), (3). These properties are referenced within the body of text where $\mathcal{O}$ is defined (i.e., Section, Lemma, etc.) as (P1) of $\mathcal{O}$, (P2) of $\mathcal{O}$, and (P3) of $\mathcal{O}$.

3. Background

In this section we establish preliminary results. These results include an implicit function theorem for the Sobolev space and an extension theorem for the one-dimensional Besov space. The reader may wish to start reading Section 4, referencing this section as necessary. We now state the two important embedding theorems; see [5], [24].

PROPOSITION 3.1 (Sobolev embedding theorem). *Let $Q \subseteq \mathbb{R}^2$ be a square. For $F \in L^{2,p}(Q)$ and $x, y \in Q$, the following hold*:

$$\begin{aligned}
|\nabla F(x) - \nabla F(y)| &\le C|x-y|^{1-2/p}\|F\|_{L^{2,p}(Q)},\\
|F(x) - J_yF(x)| &\le C|x-y|^{2-2/p}\|F\|_{L^{2,p}(Q)},\\
\|\nabla F - \nabla F(y)\|_{L^p(Q)} &\le C\delta_Q\|F\|_{L^{2,p}(Q)},\\
\|F - J_yF\|_{L^p(Q)} &\le C\delta_Q^2\|F\|_{L^{2,p}(Q)}.
\end{aligned}$$

Remark 3.1. The Sobolev theorem also holds for the space $L^{2,p}(\Omega)$, where $\Omega = Q_1 \cup Q_2$ and the squares Q_1, Q_2 have nonempty intersection. Here, one should replace δ_Q with $\mathrm{diam}(\Omega)$ in the above inequalities. Indeed, if x and y belong to the same square, the initial two inequalities extend directly. In the alternate case, pick $z \in Q_1 \cap Q_2$ such that $|z-y| \le |x-y|$ and $|z-x| \le |x-y|$. Then deduce the inequality for the pair (x, y) by summing the corresponding inequalities for (x, z) and (y, z). The third and fourth inequalities are derived by integrating p'th powers of the initial inequalities over Ω.

PROPOSITION 3.2 (Besov embedding theorem). *Let $I \subseteq \mathbb{R}$ be an interval. For $\varphi \in \dot{B}_p(I)$ and $r, s \in I$, the following hold*:

$$\begin{aligned}
|\varphi'(r) - \varphi'(s)| &\le C|r-s|^{1-\frac{2}{p}}\|\varphi\|_{\dot{B}_p(I)},\\
|\varphi(r) - J_s\varphi(r)| &\le C|r-s|^{2-\frac{2}{p}}\|\varphi\|_{\dot{B}_p(I)}.
\end{aligned}$$

The Sobolev space and Besov space are related through the following trace/extension theorem; see [22], [23].

PROPOSITION 3.3. *For $G : \mathbb{R}^2 \to \mathbb{R}$, let $g := G|_{\mathbb{R}\times\{0\}}$. Then,*

$$\begin{aligned}
G \in L^{2,p}(\mathbb{R}^2) &\implies g \in \dot{B}_p(\mathbb{R}) \text{ and } \|g\|_{\dot{B}_p(\mathbb{R})} \le C\|G\|_{L^{2,p}(\mathbb{R}^2)},\\
G \in W^{2,p}(\mathbb{R}^2) &\implies g \in B_p(\mathbb{R}) \text{ and } \|g\|_{B_p(\mathbb{R})} \le C\|G\|_{W^{2,p}(\mathbb{R}^2)}.
\end{aligned}$$

Conversely, there exists a linear extension operator $T_1 : \dot{B}_p(\mathbb{R}) \to L^{2,p}(\mathbb{R}^2)$ that satisfies

(1) $T_1 g = g$ *on* $\mathbb{R} \times \{0\}$,
(2) $\|T_1 g\|_{L^{2,p}(\mathbb{R}^2)} \le C\|g\|_{\dot{B}_p(\mathbb{R})}$ *if* $g \in \dot{B}_p(\mathbb{R})$,
(3) $\|T_1 g\|_{W^{2,p}(\mathbb{R}^2)} \le C\|g\|_{B_p(\mathbb{R})}$ *if* $g \in B_p(\mathbb{R})$.

Here, the constant C depends only on p.

We now prove a technical lemma stating that, in an appropriate coordinate system, a subset of $\mathbb{R}^2$ with controlled Besov seminorm must lie on the graph of a function with controlled Besov norm.

LEMMA 3.1. *Let $D \geq 1$ be given. There exist constants $a = a(D,p) > 0$ and $A = A(D,p)$ such that the following hold. Let (u,v) be euclidean coordinates, and let $S \subseteq \mathbb{R}^2$ be given. Suppose that for some $\kappa_1, \kappa_2 \in (0,a)$, the following hold*:

$$\operatorname{diam}(S) \leq D \quad \textit{and} \quad \|S\|_{\dot{B}_p} \leq \kappa_1; \tag{3.1}$$

$$\textit{if } \#S \geq 1, \textit{ then there exists } x = (u_x, v_x)_{uv} \in S \textit{ with } |v_x| \leq \kappa_2; \tag{3.2}$$

$$\textit{if } \#S \geq 2, \textit{ then there exists } y = (u_y, v_y)_{uv} \in S \textit{ with } y \neq x \textit{ and} \tag{3.3}$$

$$\left|\frac{v_y - v_x}{u_y - u_x}\right| \leq \kappa_2.$$

Then there exists $\varphi \in B_p(\mathbb{R})$ such that $S \subseteq \left\{(t,\varphi(t))_{uv} : t \in \mathbb{R}\right\}$ and $\|\varphi\|_{B_p(\mathbb{R})} \leq A \cdot (\kappa_1 + \kappa_2)$.

Proof. By rescaling, we may assume that $D = 1$. By adding points to S, we may assume that S is closed and that $\operatorname{diam}(S) = 1$ (in particular, $\#S \geq 2$).

Applying (3.1), we produce euclidean coordinates $(\overline{u}, \overline{v})$, a curve $\vec{\phi}(s) := (s, \widehat{\varphi}(s))_{\overline{uv}}$ such that $\|\widehat{\varphi}\|_{\dot{B}_p(\mathbb{R})} \leq 2\kappa_1$, and an interval $I = [a,b]$ such that $\vec{\phi}(a), \vec{\phi}(b) \in S$ and $S \subseteq \vec{\phi}(I)$. Note that

$$|I| = b - a \leq |\vec{\phi}(b) - \vec{\phi}(a)| \leq \operatorname{diam}(S) = 1. \tag{3.4}$$

We change coordinates and write $\vec{\phi}(s) = (\phi_1(s), \phi_2(s))_{uv}$. Notice that $\|\phi_\ell\|_{\dot{B}_p(\mathbb{R})} \leq C\kappa_1$ for $\ell = 1,2$. Subsequently, we work with (u,v) coordinates and drop the uv subscript for notational convenience.

Take $x, y \in S$ as in (3.2), (3.3). Choose $s_x, s_y \in I$ with

$$x = (u_x, v_x) = (\phi_1(s_x), \phi_2(s_x)) \ \text{ and } \ y = (u_y, v_y) = (\phi_1(s_y), \phi_2(s_y)).$$

Applying the mean-value theorem and Besov embedding theorem, for $\vec{b} := \left(\frac{u_x - u_y}{s_x - s_y}, \frac{v_x - v_y}{s_x - s_y}\right)$, we obtain

$$\left|\frac{d}{ds}\vec{\phi}(s) - \vec{b}\right| \leq C\kappa_1 \qquad \text{for } \ s \in 2I. \tag{3.5}$$

(Here, $2I$ denotes the interval with the same center and twice the length of I.) Integrating (3.5),

$$|\vec{b}| \cdot |I| \leq \left|\int_I \frac{d}{ds}\vec{\phi}(s) ds\right| + C\kappa_1 |I| \leq |\vec{\phi}(a) - \vec{\phi}(b)| + C\kappa_1 \leq \operatorname{diam}(S) + C\kappa_1,$$

$$|\vec{b}| \cdot |I| \geq \int_I \left|\frac{d}{ds}\vec{\phi}(s)\right| ds - C\kappa_1 |I| \geq \operatorname{diam}(S) - C\kappa_1.$$

Thus, for small enough κ_1, we have

$$c|I|^{-1} \leq |\vec{b}| \leq C|I|^{-1}, \quad \text{for some universal constants } c, C > 0. \tag{3.6}$$

The second coordinate of $\vec{b}$ is bounded by κ_2 times the first coordinate (see (3.3)). Thus, for sufficiently small κ_1 and κ_2, by (3.4), (3.5), and (3.6) we have

$$c|I|^{-1} \leq |\phi_1'(s)| \leq C|I|^{-1} \quad \text{and} \quad \left|\frac{\phi_2'(s)}{\phi_1'(s)}\right| \leq C \cdot (\kappa_1 + \kappa_2) \quad \text{for } s \in 2I. \tag{3.7}$$

We set $\overline{\varphi} := \phi_2 \circ \phi_1^{-1}$. Since $S \subseteq \vec{\phi}(I)$,

$$S \subseteq \{(t, \overline{\varphi}(t)) : t \in \phi_1(I)\}. \tag{3.8}$$

Note that $|\phi_1'| \geq c$ on $2I$, thanks to (3.4) and (3.7). Thus, for $\overline{I} := \phi_1(2I)$, we have

$$\begin{aligned} \|\overline{\varphi}\|^p_{\dot{B}_p(\overline{I})} &= \int_{\overline{I}}\int_{\overline{I}} \frac{\left|\frac{\phi_2'(\phi_1^{-1}(\bar{s}))}{\phi_1'(\phi_1^{-1}(\bar{s}))} - \frac{\phi_2'(\phi_1^{-1}(\bar{t}))}{\phi_1'(\phi_1^{-1}(\bar{t}))}\right|^p}{|\bar{s} - \bar{t}|^p} d\bar{s}d\bar{t} \\ &= \int_{2I}\int_{2I} \frac{\left|\frac{\phi_2'(s)}{\phi_1'(s)} - \frac{\phi_2'(t)}{\phi_1'(t)}\right|^p}{|\phi_1(s) - \phi_1(t)|^p} \cdot |\phi_1'(s)| \cdot |\phi_1'(t)| ds dt \\ &\lesssim \int_{2I}\int_{2I} \frac{|\phi_2'(s) - \phi_2'(t)|^p}{|s-t|^p} ds dt + \int_{2I}\int_{2I} \frac{|\phi_1'(s) - \phi_1'(t)|^p}{|s-t|^p} ds dt \\ &\lesssim \|\phi_2\|^p_{\dot{B}_p(2I)} + \|\phi_1\|^p_{\dot{B}_p(2I)} \lesssim \kappa_1^p. \end{aligned} \tag{3.9}$$

Invoking (3.7) provides the estimates

- $\mathrm{dist}(\mathbb{R} \setminus \phi_1(2I), \phi_1(I)) \geq \frac{1}{2}|I| \cdot \min_{s \in 2I}|\phi_1'(s)| \geq \frac{c}{2}$,
- $|\phi_1(2I)| \leq 2|I| \cdot \max_{s\in 2I}|\phi_1'(s)| \leq 2C$,
- $\|\overline{\varphi}'\|_{L^\infty(\phi_1(2I))} = \|\phi_2'/\phi_1'\|_{L^\infty(2I)} \leq C \cdot (\kappa_1 + \kappa_2)$,
- $\|\overline{\varphi}\|_{L^\infty(\phi_1(2I))} \leq |\overline{\varphi}(\phi_1(s_x))| + |\phi_1(2I)| \cdot \|\overline{\varphi}'\|_{L^\infty(\phi_1(2I))} \leq |v_x| + C\cdot(\kappa_1+\kappa_2) \leq C' \cdot (\kappa_1 + \kappa_2)$ (see (3.2)).

Thanks to the last three bullet points and (3.9), we have $\|\overline{\varphi}\|_{B_p(\overline{I})} \lesssim \kappa_1 + \kappa_2$. Thanks to the first bullet point, we may choose $\theta \in C_c^\infty(\overline{I})$ such that

(1) $\theta \equiv 1$ on $\phi_1(I)$,
(2) $\|\theta\|_{C^2} \lesssim 1$.

Finally, we set $\varphi := \theta \cdot \overline{\varphi}$. Note that $S \subseteq \{(t, \varphi(t)) : t \in \mathbb{R}\}$, thanks to (3.8), and that

$$\|\varphi\|_{B_p(\mathbb{R})} \leq C \cdot \|\theta\|_{C^2} \cdot \|\overline{\varphi}\|_{B_p(\overline{I})} \lesssim \kappa_1 + \kappa_2,$$

as desired. □

For a matrix $M = (m_{ij})$, we define $|M| := \max|m_{ij}|$.

Lemma 3.2. *Let $\Phi : \Omega_1 \to \Omega_2$ be a diffeomorphism. Suppose that $\Phi \in L^{2,p}(\Omega_1)$,*

$$\|\nabla\Phi\|_{L^\infty(\Omega_1)} \le A \;\; \textit{and} \;\; \|(\nabla\Phi)^{-1}\|_{L^\infty(\Omega_1)} \le A \;\; \textit{for some} \;\; A \ge 1.$$

Then $\Phi^{-1} \in L^{2,p}(\Omega_2)$ and $\|\Phi^{-1}\|_{L^{2,p}(\Omega_2)} \le C \cdot A^{3+2/p} \cdot \|\Phi\|_{L^{2,p}(\Omega_1)}$.

Proof. We expand $\Phi = (\Phi_1, \Phi_2)$ and $\Phi^{-1} = \Psi = (\Psi_1, \Psi_2)$ in coordinates. Let $y = \Phi(x)$. Differentiating the identity $\Psi(y) = x$, we obtain

$$\nabla\Phi(x)\nabla^2\Psi_k(y)\nabla\Phi(x) + \sum_{i=1}^{2} \partial_i\Psi_k(y)\nabla^2\Phi_i(x) = 0 \quad \text{for } k = 1, 2.$$

Thus, using the identity $\nabla\Psi(y) = [\nabla\Phi(x)]^{-1}$, we see that

$$\nabla^2\Psi_k(y) = -\sum_{i=1}^{2} \partial_i\Psi_k(y)\nabla\Psi(y)\nabla^2\Phi_i(\Psi(y))\nabla\Psi(y) \quad \text{for } k = 1, 2.$$

Now, take p'th powers and integrate over $y \in \Omega_2$, apply the change of variables $x = \Psi(y)$, and use the estimates $\|\nabla\Psi\|_{L^\infty(\Omega_2)} \le A$ and $\|\det(\nabla\Phi)\|_{L^\infty(\Omega_1)} \le 2A^2$. This completes the proof of Lemma 3.2. □

Now we present an implicit function theorem for Sobolev functions.

Proposition 3.4. *There exists $\bar{c} > 0$ such that the following hold. Let $Q \subseteq \mathbb{R}^2$ with $\delta_Q = 1$, and let $\bar{x} \in 0.9Q$ be given.*

- *Suppose that $h \in L^{2,p}(Q)$ satisfies $\|h\|_{L^{2,p}(Q)} \le \bar{c}$ and $|\nabla h(\bar{x})| \ge 1$. Then*

$$\gamma := \{x \in 0.9Q : h(x) = 0\} \quad \textit{satisfies} \quad \|\gamma\|_{\dot{B}_p} \lesssim \|h\|_{L^{2,p}(Q)}.$$

- *Conversely, suppose that $S \subseteq 0.9Q$ satisfies $\|S\|_{\dot{B}_p} \le \bar{c}$. Then there exists $H \in L^{2,p}(Q)$ with*

$$H = 0 \textit{ on } S, \quad \|H\|_{L^{2,p}(Q)} \lesssim \|S\|_{\dot{B}_p} \textit{ and } |\nabla H(\bar{x})| \ge 1.$$

Proof. Let $\bar{c} > 0$ be some small constant, determined later in the proof. We start by proving the first bullet point. Without loss of generality, we may assume that

(1) $A := \|h\|_{L^{2,p}(Q)} \le \bar{c}$,
(2) $|\nabla h(\bar{x})| = 1$.

Let $\gamma := \big\{x \in 0.9Q : h(x) = 0\big\}$.

Pick $\theta \in C_c^\infty(Q)$ such that

(1) $\theta \equiv 1$ on $0.9Q$,
(2) $|\partial^\alpha\theta| \lesssim 1$ for $|\alpha| \le 2$.

Put

$$g := J_{\overline{x}}h + \theta \cdot (h - J_{\overline{x}}h).$$

We shall establish three properties of this function:

(1) $\|g\|_{L^{2,p}(\mathbb{R}^2)} \lesssim A$,
(2) $\|\nabla g - \nabla g(\overline{x})\|_{L^\infty(\mathbb{R}^2)} \leq \frac{1}{10}$,
(3) $\gamma = \big\{x \in 0.9Q : g(x) = 0\big\}$.

We obtain the estimate $\|g\|_{L^{2,p}(Q)} \lesssim \|h - J_{\overline{x}}h\|_{W^{2,p}(Q)}$ through the Leibniz rule and (P2) of θ. Thus, we have $\|g\|_{L^{2,p}(Q)} \lesssim \|h\|_{L^{2,p}(Q)} = A$ by the Sobolev theorem. Moreover, the Sobolev theorem implies that

$$\|\nabla g - \nabla g(\overline{x})\|_{L^\infty(Q)} \lesssim \|g\|_{L^{2,p}(Q)} \lesssim A.$$

Note that g equals an affine function on $\mathbb{R}^2 \setminus Q$ since $\operatorname{supp}(\theta) \subseteq Q$. Hence, $\|g\|_{L^{2,p}(\mathbb{R}^2)} \lesssim A$ and $\|\nabla g - \nabla g(\overline{x})\|_{L^\infty(\mathbb{R}^2)} \lesssim A \leq \overline{c}$. This proves the first two properties of g provided that $\overline{c}$ is sufficiently small. The third property of g holds because $g = h$ on $0.9Q$.

We set $e_2 := \nabla g(\overline{x}) = \nabla h(\overline{x})$. We choose $e_1 \in \mathbb{R}^2$ such that (e_1, e_2) forms an orthonormal basis. We work with euclidean coordinates $(s,t) := \overline{x} + se_1 + te_2$ until the first bullet point is proven.

Define the mapping $\Phi(s,t) := (s, g(s,t))$. Thus,

$$|\nabla\Phi(x) - \mathrm{Id}| = |\nabla\Phi(x) - \nabla\Phi(\overline{x})| \leq |\nabla g(x) - \nabla g(\overline{x})| \leq 1/10 \quad \text{for } x \in \mathbb{R}^2,$$
$$\|\Phi\|_{L^{2,p}(\mathbb{R}^2)} = \|g\|_{L^{2,p}(\mathbb{R}^2)} \lesssim A.$$

From these properties we learn that $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ is bijective, $\|\nabla\Phi\|_{L^\infty(\mathbb{R}^2)} \leq 10$, and $\|(\nabla\Phi)^{-1}\|_{L^\infty(R^2)} \leq 10$. Using Lemma 3.2, we thus have $\|\Phi^{-1}\|_{L^{2,p}(\mathbb{R}^2)} \lesssim A$.

We define $\varphi(s)$ to be $(\Phi^{-1})_2(s,0)$; i.e., the t-coordinate function of $\Phi^{-1}(s,0)$. From (P3) of g, we have

$$\gamma \subseteq \Big\{(s,t) \in \mathbb{R}^2 : \Phi(s,t) = (s,0)\Big\} = \Big\{\Phi^{-1}(s,0) : s \in \mathbb{R}\Big\} = \Big\{(s, \varphi(s)) : s \in \mathbb{R}\Big\}.$$

Proposition 3.3 implies that $\|\varphi\|_{\dot{B}_p(\mathbb{R})} \lesssim \|\Phi^{-1}\|_{L^{2,p}(\mathbb{R}^2)} \lesssim A$. Thus, $\|\gamma\|_{\dot{B}_p} \leq \|\varphi\|_{\dot{B}_p} \lesssim A$, which proves the first bullet point.

We now turn attention to the second bullet point. Suppose that $S \subseteq 0.9Q$ and $B := \|S\|_{\dot{B}_p} \leq \overline{c}$. The second bullet-point clearly holds when $\#S \leq 1$; thus, we may assume that $\#S \geq 2$.

Fix distinct points $x_0, y_0 \in S$ and choose euclidean coordinates (u,v) such that $x_0 = (0,0)$ and $y_0 = (q,0)$ for some $q \in \mathbb{R}$. Lemma 3.1 implies that $S \subseteq \{(u, \varphi(u)) : u \in \mathbb{R}\}$ for some $\varphi \in B_p(\mathbb{R})$ with $\|\varphi\|_{B_p(\mathbb{R})} \lesssim B$.

Invoking Proposition 3.3, we obtain $G \in W^{2,p}(\mathbb{R}^2)$ such that $G = \varphi$ on $\mathbb{R} \times \{0\}$ and $\|G\|_{W^{2,p}(\mathbb{R}^2)} \lesssim B$. The Sobolev theorem implies that

$$\|\nabla G\|_{L^\infty(\mathbb{R}^2)} \lesssim \|G\|_{W^{2,p}(\mathbb{R}^2)} \lesssim B \leq \overline{c}.$$

Taking $\overline{c}$ sufficiently small, we ensure that $\|\nabla G\|_{L^\infty(\mathbb{R}^2)} \leq 10^{-1}$. Now define the mapping $\Psi(u,v) = (u, v + G(u,v))$. The following properties are immediate:

(1) $S \subseteq \{(u,\varphi(u)) : u \in \mathbb{R}\} = \{(u, G(u,0)) : u \in \mathbb{R}\} = \{\Psi(u,0) : u \in \mathbb{R}\}$;
(2) $\|\nabla\Psi - \mathrm{Id}\|_{L^\infty(\mathbb{R}^2)} \leq 10^{-1}$, and thus $\|\nabla\Psi\|_{L^\infty(\mathbb{R}^2)} \leq 10$ and $\|(\nabla\Psi)^{-1}\|_{L^\infty(\mathbb{R}^2)} \leq 10$;
(3) $\|\Psi\|_{L^{2,p}(\mathbb{R}^2)} = \|G\|_{L^{2,p}(\mathbb{R}^2)} \lesssim B$;
(4) $\|\Psi^{-1}\|_{L^{2,p}(\mathbb{R}^2)} \lesssim B$ (see Lemma 3.2).

Define $\phi(u,v) := (\Psi^{-1})_2(u,v)$. Note that $\phi = 0$ on S, $|\nabla\phi(\overline{x})| \geq 1/2$, and $\|\phi\|_{L^{2,p}(Q)} \lesssim B$, thanks to (P1), (P2) and (P4) of Ψ, respectively. We take $H := 2\phi$ to complete the proof of the second bullet point.

The constant $\overline{c} > 0$ is chosen small enough so that the previous arguments hold. This completes the proof of Proposition 3.4. □

Lemma 3.3 (Straightening lemma). *There exists $\overline{c} > 0$ such that the following hold. Let $Q \subseteq \mathbb{R}^2$ satisfy $\delta_Q = 1$, and suppose that $S \subseteq 0.9Q$ satisfies $\|S\|_{\dot{B}_p} \leq \overline{c}$. Then there exists a diffeomorphism $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ such that*

(1) $\Phi(S) \subseteq \mathbb{R} \times \{0\}$,
(2) $\|\Phi\|_{L^{2,p}(\mathbb{R}^2)} \lesssim \|S\|_{\dot{B}_p}$ *and* $\|\Phi^{-1}\|_{L^{2,p}(\mathbb{R}^2)} \lesssim \|S\|_{\dot{B}_p}$,
(3) $\|\nabla\Phi\|_{L^\infty(\mathbb{R}^2)} \leq 10$ *and* $\|\nabla\Phi^{-1}\|_{L^\infty(\mathbb{R}^2)} \leq 10$.

Proof. Take Ψ as in the proof of Proposition 3.4, and set $\Phi := \Psi^{-1}$. □

Lemma 3.4. *Let $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ be a diffeomorphism. Suppose $\|\Phi\|_{L^{2,p}(\mathbb{R}^2)} \leq A$, $\|\nabla\Phi\|_{L^\infty(\mathbb{R}^2)} \leq A$ and $\|\nabla\Phi^{-1}\|_{L^\infty(\mathbb{R}^2)} \leq A$. Then*

$$\widehat{C}^{-1}\|F\|_{W^{2,p}(\mathbb{R}^2)} \leq \|F\circ\Phi\|_{W^{2,p}(\mathbb{R}^2)} \leq \widehat{C}\|F\|_{W^{2,p}(\mathbb{R}^2)} \text{ for every } F \in W^{2,p}(\mathbb{R}^2).$$

Here, the constant $\widehat{C}$ depends only on A and p.

Proof. We expand $\Phi = (\Phi_1, \Phi_2)$ in coordinates. Then

$$\partial_{ij}(F\circ\Phi) = \sum_{k,l\in\{1,2\}} c_{k,l}\cdot\partial_i\Phi_k\cdot\partial_j\Phi_l\cdot\partial_{kl}F\circ\Phi + \sum_{k\in\{1,2\}} \partial_{ij}\Phi_k\cdot\partial_k F\circ\Phi. \tag{3.10}$$

Note the estimate $\|\nabla F\|_{L^\infty(\mathbb{R}^2)} \lesssim \|F\|_{W^{2,p}(\mathbb{R}^2)}$ following from the Sobolev theorem. Thus, from (3.10) we obtain

$$\|\nabla^2(F\circ\Phi)\|^p_{L^p(\mathbb{R}^2)} \leq \widetilde{C}(A,p)\cdot\left[\|(\nabla^2F)\circ\Phi\|^p_{L^p(\mathbb{R}^2)} + \|F\|^p_{W^{2,p}(\mathbb{R}^2)}\right].$$

Since the coordinate change $y = \Phi(x)$ has bounded Jacobian, we have

$$\|\nabla^2(F\circ\Phi)\|_{L^p(\mathbb{R}^2)} \leq \overline{C}(A,p)\cdot\|F\|_{W^{2,p}(\mathbb{R}^2)}.$$

In the same fashion, we show that $\|\nabla(F\circ\Phi)\|_{L^p(\mathbb{R}^2)} + \|F\circ\Phi\|_{L^p(\mathbb{R}^2)} \leq \overline{C}(A,p)\cdot\|F\|_{W^{2,p}(\mathbb{R}^2)}$; hence $\|F\circ\Phi\|_{W^{2,p}(\mathbb{R}^2)} \leq C'(A,p)\cdot\|F\|_{W^{2,p}(\mathbb{R}^2)}$.

Now observe that $\|\Phi^{-1}\|_{L^{2,p}(\mathbb{R}^2)} \le C'$, thanks to Lemma 3.2. Thus, applying the above reasoning with Φ^{-1} instead of Φ, we see that $\|F\|_{W^{2,p}(\mathbb{R}^2)} \lesssim \|F \circ \Phi\|_{W^{2,p}(\mathbb{R}^2)}$. This completes the proof of Lemma 3.4. $\square$

We reduce the two-dimensional Sobolev extension problem on certain "flat" subsets to a one-dimensional Besov extension problem, which we solve in the next result. Our construction is closely related to Whitney's extension theorem for the space $C^2(\mathbb{R})$.

Proposition 3.5. *Let $E_1 \subseteq \mathbb{R}$ satisfy $\operatorname{diam}(E_1) \lesssim 1$. Then there exist a linear map $T_b : B_p(\mathbb{R})|_{E_1} \to B_p(\mathbb{R})$ and linear functionals $\lambda_1, \dots, \lambda_{N_0}$, where $N_0 \le C \cdot (\#E_1)^2$, such that*

- *T_b is a bounded linear extension operator: $T_b g = g$ on E_1 and $\|T_b g\|_{B_p(\mathbb{R})} \le C\|g\|_{B_p(\mathbb{R})|_{E_1}}$ for any $g : E_1 \to \mathbb{R}$.*
- *An approximate formula holds for the Besov trace norm:*

$$C^{-1}\left(\sum_{i=1}^{N_0} |\lambda_i(g)|^p\right)^{1/p} \le \|g\|_{B_p(\mathbb{R})|_{E_1}} \le C\left(\sum_{i=1}^{N_0} |\lambda_i(g)|^p\right)^{1/p} \qquad \forall g : E_1 \to \mathbb{R}.$$

Proof. If $\#E_1 \le 1$, then the proposition clearly holds; hence, we may assume that $\#E_1 \ge 2$. We write $E_1 = \{x_1, \dots, x_N\}$, where $x_1 < \cdots < x_N$. It is convenient to set $x_0 = -\infty$ and $x_{N+1} = \infty$. Let $g : E_1 \to \mathbb{R}$ be given. We define the following objects:

- For $1 \le k \le N$, let $\nu(k) \in \{k-1, k+1\}$ be chosen so that $x_{\nu(k)} \in E_1$ is nearest to x_k.
- For $1 \le k \le N$, set

(3.11)
$$m_k := \frac{g(x_k) - g(x_{\nu(k)})}{x_k - x_{\nu(k)}} \text{ and the affine function } L_k(x) := g(x_k) + m_k(x - x_k).$$

- For $0 \le k \le N$, set $I_k := [x_k, x_{k+1}] \setminus \{-\infty, \infty\}$ and $\Delta_k := |x_k - x_{k+1}|$.

By the classical Whitney extension theorem (see [22]), there exists $F_k \in \dot{C}^{1,1}(I_k)$ such that

(1) for $k = 1, \dots, N-1$, we have $F_k \equiv L_k$ on $[x_k, x_k + \frac{\Delta_k}{10}]$ and $F_k \equiv L_{k+1}$ on $[x_{k+1} - \frac{\Delta_k}{10}, x_{k+1}]$;
(2) $\|F_k\|_{\dot{C}^{1,1}(I_k)}$ is C-optimal with respect to the above property;
(3) F_k depends linearly on the polynomials L_k and L_{k+1}, and hence also on g;
(4) $F_0 = L_1$ and $F_N = L_N$.

Moreover, the classical Whitney extension theorem provides certain expressions:

(3.12)
$$M_k := |m_{k+1} - m_k|\Delta_k^{-1} + |L_k(x_{k+1}) - g(x_{k+1})|\Delta_k^{-2} \quad \text{for } k = 1, \dots, N-1,$$
$$M_0 := 0 \ \text{ and } \ M_N := 0,$$

which satisfy

$$M_k \approx \|F_k\|_{\dot{C}^{1,1}(\mathbb{R}^n)}. \tag{3.13}$$

We define $F : \mathbb{R} \to \mathbb{R}$ by $F(x) = F_k(x)$ when $x \in I_k$. Let

$$A_{kl} := \left(\int_{I_k} \int_{I_l} \frac{1}{|x-y|^p} dxdy \right)^{1/p} \quad \text{for every } k, l \in \{0, \dots, N\} \text{ with } k < l,$$

and set

$$M := \left(\sum_{k=1}^{N-1} M_k^p \Delta_k^2 + \sum_{\substack{0 \le k, l \le N \\ k+2 \le l}} |m_{k+1} - m_l|^p A_{kl}^p \right)^{1/p}. \tag{3.14}$$

Note that $A_{k(k+1)} = \infty$, by definition.

We prove several claims before returning to the task of constructing T_b and estimating the Besov trace norm.

Claim 1. $F = g$ on E_1 and $\|F\|_{\dot{B}_p(\mathbb{R})} \lesssim M$.

First, we have $F = g$ on E_1, by (3.11) and the fact that $F_k(x_k) = L_k(x_k)$ for each k. Next, we write

$$\|F\|_{\dot{B}_p(\mathbb{R})}^p = \sum_{k=0}^{N} \int_{I_k} \int_{I_k} \frac{|F'(x) - F'(y)|^p}{|x-y|^p} dxdy + 2 \sum_{0 \le k < l \le N} \int_{I_k} \int_{I_l} \frac{|F'(x) - F'(y)|^p}{|x-y|^p} dxdy. \tag{3.15}$$

By the Lipschitz bound on the derivative of $F = F_k$ on I_k (see (3.13)) and since F' is constant on I_0 and on I_N, we have

$$\sum_{k=0}^{N} \int_{I_k} \int_{I_k} \frac{|F'(x) - F'(y)|^p}{|x-y|^p} dxdy \lesssim \sum_{k=1}^{N-1} M_k^p \Delta_k^2. \tag{3.16}$$

For $0 \le k < l \le N$, we estimate

$$\begin{aligned}
&\int_{I_k} \int_{I_l} \frac{|F'(x) - F'(y)|^p}{|x-y|^p} dydx \\
&\quad \lesssim \int_{I_k} \int_{I_l} \frac{|F'(x_{k+1}) - F'(x_l)|^p}{|x-y|^p} dydx \\
&\qquad + \int_{I_k} \int_{I_l} \frac{|F'(x) - F'(x_{k+1})|^p}{|x-y|^p} dydx + \int_{I_k} \int_{I_l} \frac{|F'(y) - F'(x_l)|^p}{|x-y|^p} dydx
\end{aligned}$$

$$\lesssim |m_{k+1} - m_l|^p A_{kl}^p + M_k^p \Delta_k^p \int_{x_k}^{x_{k+1}-\frac{\Delta_k}{10}} \int_{I_l} \frac{1}{|x-y|^p} dy dx + M_l^p \Delta_l^p \int_{x_l+\frac{\Delta_l}{10}}^{x_{l+1}} \int_{I_k} \frac{1}{|x-y|^p} dx dy.$$

Here, in the second inequality we use that $F' \equiv m_{k+1}$ on $[x_{k+1} - \frac{\Delta_k}{10}, x_{k+1}]$ and $F' \equiv m_l$ on $[x_l, x_l + \frac{\Delta_l}{10}]$. Our convention here is that $0 \cdot \infty = 0$, which arises when $k+1 = l$ in the first term, when $k = 0$ in the second term, or when $l = N$ in the last term. After making the straightforward estimate on the above integrals, we sum and obtain

(3.17)
$$\sum_{0 \le k < l \le N} \int_{I_k} \int_{I_l} \frac{|F'(x) - F'(y)|^p}{|x-y|^p} dy dx \lesssim \sum_{\substack{0 \le k,l \le N \\ k+2 \le l}} |m_{k+1} - m_l|^p A_{kl}^p + \sum_{k=1}^{N-1} M_k^p \Delta_k^2.$$

At this time we can derive Claim 1 from (3.14)–(3.17).

Claim 2. Let $F \in \dot{B}_p(I)$ for some interval $I = [a, b]$. Then

$$\int_a^b \frac{|F'(x) - F'(a)|^p}{|x-a|^{p-1}} dx \lesssim \|F\|_{\dot{B}_p(I)}^p.$$

In proving the above claim, we may assume that $I = [0, 1]$ through scale invariance. For $k \ge 0$, we set $I_k' := (2^{-k-1}, 2^{-k}]$.

Note that $|F'(x) - F'(2^{-k})|^p \lesssim \|F\|_{\dot{B}_p(I_k')}^p 2^{-k(p-2)}$ for $x \in I_k'$, by the Besov embedding theorem. Therefore,

$$\begin{aligned}
(3.18) \qquad & \int_0^1 \frac{|F'(x) - F'(0)|^p}{x^{p-1}} dx \\
& \lesssim \sum_{k \ge 0} \int_{I_k'} \frac{|F'(2^{-k}) - F'(0)|^p}{x^{p-1}} dx + \sum_{k \ge 0} \int_{I_k'} \frac{|F'(x) - F'(2^{-k})|^p}{x^{p-1}} dx \\
& \lesssim \sum_{k \ge 0} |F'(2^{-k}) - F'(0)|^p 2^{-k(2-p)} + \sum_{k \ge 0} \|F\|_{\dot{B}_p(I_k')}^p \\
& \lesssim \sum_{k \ge 0} |F'(2^{-k}) - F'(0)|^p 2^{-k(2-p)} + \|F\|_{\dot{B}_p([0,1])}^p.
\end{aligned}$$

By the Besov embedding theorem, F belongs to $C^1([0,1])$; hence $\lim_{x \to 0} F'(x) = F'(0)$. Thus,

$$F'(2^{-k}) - F'(0) = \sum_{l=k}^{\infty} \left[F'(2^{-l}) - F'(2^{-l-1}) \right].$$

Now, fix $0 < \varepsilon < \frac{p-2}{p}$ and apply Hölder's inequality, using the above formula:

$$\begin{aligned}
&\sum_{k\geq 0} |F'(2^{-k}) - F'(0)|^p 2^{k(p-2)} \\
&\quad = \sum_{k\geq 0} \left| \sum_{l\geq k} (F'(2^{-l}) - F'(2^{-l-1})) 2^{l\varepsilon} 2^{-l\varepsilon} \right|^p 2^{k(p-2)} \\
&\quad \leq \sum_{k\geq 0} 2^{k(p-2)} \sum_{l\geq k} |F'(2^{-l}) - F'(2^{-l-1})|^p 2^{l\varepsilon p} \left[\sum_{l\geq k} 2^{-lp'\varepsilon} \right]^{p/p'} \\
&\quad \lesssim \sum_{k\geq 0} 2^{k(p-2)} \sum_{l\geq k} |F'(2^{-l}) - F'(2^{-l-1})|^p 2^{l\varepsilon p} 2^{-kp\varepsilon} \\
&\quad = \sum_{l\geq 0} |F'(2^{-l}) - F'(2^{-l-1})|^p 2^{l\varepsilon p} \sum_{0\leq k\leq l} 2^{k(p-2)-kp\varepsilon} \\
&\quad \lesssim \sum_{l\geq 0} |F'(2^{-l}) - F'(2^{-l-1})|^p 2^{l(p-2)} \lesssim \sum_{l\geq 0} \|F\|^p_{\dot{B}_p(I'_l)} \leq \|F\|^p_{\dot{B}_p([0,1])}.
\end{aligned}$$

(Here, p' denotes the dual exponent to p, and thus $\frac{1}{p'} + \frac{1}{p} = 1$.) This estimate and (3.18) complete the proof of Claim 2.

Claim 3. Let $\widetilde{F} \in \dot{B}_p(\mathbb{R})$ satisfy $\widetilde{F} = g$ on E_1. Then $M \lesssim \|\widetilde{F}\|_{\dot{B}_p(\mathbb{R})}$.

We now prove Claim 3. For $1 \leq k \leq N$, let J_k denote the interval with endpoints x_k and $x_{\nu(k)}$, and set $\delta_k := |J_k| = |x_k - x_{\nu(k)}|$. Note that $\delta_k, \delta_{k+1} \leq \Delta_k$ for $1 \leq k \leq N-1$ because δ_k equals the distance from x_k to its nearest neighbor in E_1 and $\Delta_k = |x_k - x_{k+1}|$.

Recall that $m_k = \frac{\widetilde{F}(x_k) - \widetilde{F}(x_{\nu(k)})}{x_k - x_{\nu(k)}}$ (since $\widetilde{F} = g$ on E_1 and by (3.11)). Pick $x_k^* \in J_k$ such that $m_k = \widetilde{F}'(x_k^*)$. Set $\alpha := 1 - 2/p$. By the Besov embedding theorem,

$$(3.19)\quad |\widetilde{F}'(x_k) - m_k| = |\widetilde{F}'(x_k) - \widetilde{F}'(x_k^*)| \lesssim \|\widetilde{F}\|_{\dot{B}_p(J_k)} |x_k - x_k^*|^\alpha \leq \|\widetilde{F}\|_{\dot{B}_p(J_k)} \delta_k^\alpha.$$

We consider the first sum in (3.14). For $k \in \{1, \dots, N\}$, we write

$$\begin{aligned}
M_k &= |m_{k+1} - m_k| \Delta_k^{-1} \quad + \quad |L_k(x_{k+1}) - g(x_{k+1})| \Delta_k^{-2} \\
&\leq |m_{k+1} - \widetilde{F}'(x_{k+1})| \Delta_k^{-1} \\
&\quad + |\widetilde{F}'(x_k) - \widetilde{F}'(x_{k+1})| \Delta_k^{-1} \quad + \quad |m_k - \widetilde{F}'(x_k)| \Delta_k^{-1} \\
&\quad + |J_{x_k} \widetilde{F}(x_{k+1}) - g(x_{k+1})| \Delta_k^{-2} \quad + \quad |J_{x_k} \widetilde{F}(x_{k+1}) - L_k(x_{k+1})| \Delta_k^{-2}
\end{aligned}$$

(see (3.12)). Since $|x_k - x_{k+1}| = \Delta_k$,

$$|J_{x_k} \widetilde{F}(x_{k+1}) - L_k(x_{k+1})| \Delta_k^{-2} = |\widetilde{F}'(x_k) - m_k| \Delta_k^{-1}$$

(see (3.11)). From this equality, the Besov embedding theorem (to help estimate the second and fourth terms above), and (3.19), we obtain

$$M_k \lesssim \|\widetilde{F}\|_{\dot{B}_p(J_{k+1})}\delta_{k+1}^{\alpha}\Delta_k^{-1} + \|\widetilde{F}\|_{\dot{B}_p(I_k)}\Delta_k^{\alpha-1} + \|\widetilde{F}\|_{\dot{B}_p(J_k)}\delta_k^{\alpha}\Delta_k^{-1}.$$

Thus, because $\delta_k, \delta_{k+1} \leq \Delta_k$,

$$M_k^p \lesssim \Delta_k^{p(\alpha-1)}\left[\|\widetilde{F}\|^p_{\dot{B}_p(J_{k+1})} + \|\widetilde{F}\|^p_{\dot{B}_p(I_k)} + \|\widetilde{F}\|^p_{\dot{B}_p(J_k)}\right].$$

Summing over k and using that $p(\alpha-1) = -2$, we obtain

$$\sum_{k=1}^{N-1} M_k^p \Delta_k^2 \lesssim \|\widetilde{F}\|^p_{\dot{B}_p(\mathbb{R})}. \tag{3.20}$$

We consider the second sum in (3.14). For $k, l \in \{0, \dots, N\}$ with $k+2 \leq l$, we write

$$\begin{aligned}|m_{k+1} - m_l|^p \lesssim |m_{k+1} - \widetilde{F}'(x_{k+1})|^p + |\widetilde{F}'(x_{k+1}) - \widetilde{F}'(x)|^p + |\widetilde{F}'(x) - \widetilde{F}'(y)|^p \\ + |\widetilde{F}'(y) - \widetilde{F}'(x_l)|^p + |\widetilde{F}'(x_l) - m_l|^p.\end{aligned}$$

Now divide the previous inequality by $|x-y|^p$, integrate over $x \in I_k$, $y \in I_l$, and sum over k and l:

$$\begin{aligned}\sum_{\substack{0\leq k,l\leq N\\ k+2\leq l}} |m_{k+1} - m_l|^p A_{kl}^p \lesssim &\sum_{k=0}^{N-2}\int_{x_k}^{x_{k+1}}\int_{x_{k+2}}^{\infty}\frac{|m_{k+1} - \widetilde{F}'(x_{k+1})|^p}{|x-y|^p}dydx \\ &+ \sum_{k=0}^{N-2}\int_{x_k}^{x_{k+1}}\int_{x_{k+2}}^{\infty}\frac{|\widetilde{F}'(x_{k+1}) - \widetilde{F}'(x)|^p}{|x-y|^p}dydx \\ &+ \int_{\mathbb{R}}\int_{\mathbb{R}}\frac{|\widetilde{F}'(x) - \widetilde{F}'(y)|^p}{|x-y|^p}dydx \\ &+ \sum_{l=2}^{N}\int_{x_l}^{x_{l+1}}\int_{-\infty}^{x_{l-1}}\frac{|\widetilde{F}'(y) - \widetilde{F}'(x_l)|^p}{|x-y|^p}dxdy \\ &+ \sum_{l=2}^{N}\int_{x_l}^{x_{l+1}}\int_{-\infty}^{x_{l-1}}\frac{|\widetilde{F}'(x_l) - m_l|^p}{|x-y|^p}dxdy.\end{aligned} \tag{3.21}$$

From (3.19), we have

$$\begin{aligned}&\int_{x_k}^{x_{k+1}}\int_{x_{k+2}}^{\infty}\frac{|m_{k+1} - \widetilde{F}'(x_{k+1})|^p}{|x-y|^p}dydx \\ &\qquad \lesssim \|\widetilde{F}\|^p_{\dot{B}_p(J_{k+1})}\delta_{k+1}^{\alpha p}\cdot|x_{k+1} - x_{k+2}|^{2-p} \leq \|\widetilde{F}\|^p_{\dot{B}_p(J_{k+1})}.\end{aligned} \tag{3.22}$$

The last inequality follows because $\delta_{k+1} \leq \Delta_{k+1} = |x_{k+1} - x_{k+2}|$, and $\alpha \cdot p = p - 2 > 0$. Moreover, Claim 2 implies that

(3.23)
$$\int_{x_k}^{x_{k+1}} \int_{x_{k+2}}^{\infty} \frac{|\widetilde{F}'(x_{k+1}) - \widetilde{F}'(x)|^p}{|x-y|^p} dy dx = C(p) \int_{x_k}^{x_{k+1}} \frac{|\widetilde{F}'(x) - \widetilde{F}'(x_{k+1})|^p}{|x - x_{k+2}|^{p-1}} dx$$
$$\leq C(p) \int_{x_k}^{x_{k+1}} \frac{|\widetilde{F}'(x) - \widetilde{F}'(x_{k+1})|^p}{|x - x_{k+1}|^{p-1}} dx \lesssim \|\widetilde{F}\|^p_{\dot{B}_p(I_k)}.$$

From (3.21), (3.22), (3.23), and the mirrored analogues of (3.22) and (3.23), we obtain

$$\sum_{\substack{0 \leq k,l \leq N \\ k+2 \leq l}} |m_{k+1} - m_l|^p A^p_{kl} \lesssim \sum_{k=0}^{N-2} \|\widetilde{F}\|^p_{\dot{B}_p(J_{k+1})} + \sum_{k=0}^{N-2} \|\widetilde{F}\|^p_{\dot{B}_p(I_k)} + \|\widetilde{F}\|^p_{\dot{B}_p(\mathbb{R})}$$
$$+ \sum_{l=2}^{N} \|\widetilde{F}\|^p_{\dot{B}_p(I_l)} + \sum_{l=2}^{N} \|\widetilde{F}\|^p_{\dot{B}_p(J_l)} \lesssim \|\widetilde{F}\|^p_{\dot{B}_p(\mathbb{R})}.$$

The above inequality and (3.20) immediately imply Claim 3.

We return to the task of constructing the extension operator T_b and estimating the Besov trace norm. We have constructed $F \in \dot{B}_p(\mathbb{R})$ that satisfies

(1) $F = g$ on E_1,
(2) $\|F\|_{\dot{B}_p(\mathbb{R})} \lesssim M$

(see Claim 1). Note that F depends linearly on g.

Choose $\theta \in C_c^\infty(\mathbb{R})$ such that

(1) $\theta = 1$ on E_1,
(2) $\|\theta\|_{C^2} \lesssim 1$,
(3) $\mathrm{supp}(\theta) \subseteq [a_1, b_1]$ with $|a_1 - b_1| \lesssim 1$.

This cutoff function exists because $\mathrm{diam}(E_1) \lesssim 1$.

We define $\widehat{F} := \theta F$. Note that (P1) of F and (P1) of θ imply that $\widehat{F} = g$ on E_1.

Let $L(x) := g(x_1) + \frac{g(x_2) - g(x_1)}{x_2 - x_1}(x - x_1)$. By the mean value theorem, the Besov embedding theorem, and since $F(x_1) = L(x_1)$ and $F(x_2) = L(x_2)$, we have

$$|F'(x) - L'(x)| \lesssim \|F\|_{\dot{B}_p(\mathbb{R})} \text{ and } |F(x) - L(x)| \lesssim \|F\|_{\dot{B}_p(\mathbb{R})} \text{ for } x \in [a_1, b_1].$$

Since $|a_1 - b_1| \lesssim 1$, we may integrate p-th powers in the above inequality and obtain

$$\|F'\|_{L^p([a_1,b_1])} + \|F\|_{L^p([a_1,b_1])} \lesssim \|F\|_{\dot{B}_p(\mathbb{R})} + \|L'\|_{L^p([a_1,b_1])} + \|L\|_{L^p([a_1,b_1])}$$
$$\lesssim M^p + \frac{|g(x_1) - g(x_2)|^p}{|x_1 - x_2|^p} + |g(x_1)|^p.$$

Thus, by (P2)–(P3) of θ, we have

$$\begin{aligned}\|\widehat{F}\|^p_{B_p(\mathbb{R})} &\lesssim \|\theta\|_{C^2}\cdot\left[\|F\|_{\dot{B}_p([a_1,b_1])}+\|F'\|_{L^p([a_1,b_1])}+\|F\|_{L^p([a_1,b_1])}\right]\\ &\lesssim M^p+\frac{|g(x_1)-g(x_2)|^p}{|x_1-x_2|^p}+|g(x_1)|^p.\end{aligned}$$

Using Claim 3, the mean value theorem and the Besov embedding theorem, we bound each term on the right-hand side above by $C\|g\|^p_{B_p(\mathbb{R})|_{E_1}}$. Therefore, $\|\widehat{F}\|_{B_p(\mathbb{R})}\lesssim\|g\|_{B_p(\mathbb{R})|_{E_1}}$. Since $\widehat{F}=g$ on E_1 we have $\|g\|_{B_p(\mathbb{R})|_{E_1}}\lesssim\|\widehat{F}\|_{B_p(\mathbb{R})}$. In particular, the above inequalities are approximate equalities, and hence

$$\|g\|^p_{B_p(\mathbb{R})|_{E_1}}\approx M^p+\frac{|g(x_1)-g(x_2)|^p}{|x_1-x_2|^p}+|g(x_1)|^p.$$

By definition of M in (3.14), the desired estimate for $\|g\|_{B_p(\mathbb{R})|_{E_1}}$ holds. Since F depends linearly on g, so does $\widehat{F}$. We set $T_b(g):=\widehat{F}$ to complete the proof of Proposition 3.5. □

4. **The Calderón-Zygmund decomposition**

4.1. CZ *squares.* Fix a finite set $E\subseteq\mathbb{R}^2$. Choose some square $Q^\circ\subseteq\mathbb{R}^2$, centered at the origin, such that $E\subseteq\frac{1}{10}Q^\circ$.

To bisect some given square $Q\subseteq\mathbb{R}^2$ means to write $Q=Q_1\cup\cdots\cup Q_4$, where $\delta_{Q_j}=\frac{1}{2}\delta_Q$. We call $Q_1,\dots,Q_4$ the *children* of Q.

A dyadic square is one that arises from Q° by repeated bisection. Every dyadic square $Q\subsetneq Q^\circ$ is the child of another dyadic square called the *dyadic parent* of Q, which we denote by Q^+.

Two dyadic squares Q,Q' are called *neighbors* provided that $Q\cap Q'\neq\emptyset$ and $\mathrm{int}(Q)\cap\mathrm{int}(Q')=\emptyset$, or $Q=Q'$; we denote this property by $Q\leftrightarrow Q'$.

Let $c_1>0$ be a small universal constant whose value is determined later. We assume that c_1 is less than than other more explicit universal constants that arise in our proof.

Definition 4.1. A dyadic square $Q\subseteq Q^\circ$ is OK provided that

$$\|3Q\cap E\|_{\dot{B}_p}\le c_1\delta_Q^{2/p-1}.$$

Remark 4.1. By the above definition, $3Q\cap E$ lies on the graph of a Besov function with controlled seminorm; we later use this property to construct an extension operator for functions on $3Q\cap E$.

We partition Q° into a collection of OK squares with pairwise disjoint interiors using a *Calderón-Zygmund decomposition.*

Cutting procedure: Given a dyadic square $Q\subseteq Q^\circ$, proceed as follows: If Q is OK, then terminate and return the singleton collection $\Lambda_Q=\{Q\}$.

Otherwise, return the collection

$$\Lambda_Q = \bigcup \{\Lambda_{Q'} : Q' \text{ dyadic and } (Q')^+ = Q\}.$$

LEMMA 4.1. *The collection Λ_{Q° contains finitely many squares.*

Proof. We set $\varepsilon := 100^{-1} \inf\left\{|x-y| : x, y \in E, x \neq y\right\}$. If $Q \subseteq Q^\circ$ is dyadic and $\delta_Q \leq \varepsilon$, then $\#(3Q \cap E) \leq 1$; hence $\|3Q \cap E\|_{\dot{B}_p} = 0$. Thus, dyadic squares with sidelength at most ε are OK. Therefore, the cutting procedure terminates after finitely many steps. The lemma now follows. □

Denote $\Lambda := \Lambda_{Q^\circ}$. The squares in Λ are called CZ squares; we index them by $\Lambda = \{Q_\nu\}_{\nu=1}^K$.

Denote $\delta_\nu = \delta_{Q_\nu}$. We write $\nu \leftrightarrow \nu'$ whenever $Q_\nu \leftrightarrow Q_{\nu'}$. For any square Q, we write $\widetilde{Q} := 1.3Q$.

We say that a collection of sets Π has the *bounded intersection property* provided that

$$\#\left\{S' \in \Pi : S \cap S' \neq \emptyset\right\} \leq C \text{ for each } S \in \Pi, \quad \text{for some universal constant } C.$$

LEMMA 4.2 (Good Geometry). *For every $Q, Q' \in \Lambda$, the following properties hold:*

(1) *If $Q \leftrightarrow Q'$, then $\frac{1}{2}\delta_{Q'} \leq \delta_Q \leq 2\delta_{Q'}$.*
(2) *If $Q \cap Q' = \emptyset$, then $\widetilde{Q} \cap \widetilde{Q'} = \emptyset$.*
(3) *If $Q \cap Q' = \emptyset$, then $\operatorname{dist}(Q, Q') \geq \frac{1}{10}\max\{\delta_Q, \delta_{Q'}\}$.*
(4) *The collection $\{\widetilde{Q}_\nu\}_{\nu=1}^K$ satisfies the bounded intersection property (with constant $C = 13$).*

Proof. The last three properties are easily deduced from the first property. We now prove the first property. For the sake of contradiction, suppose that there exist $Q, Q' \in \Lambda$ such that $Q \leftrightarrow Q'$ and $\delta_Q \leq \frac{1}{4}\delta_{Q'}$. Therefore, $\delta_{Q^+} \leq \frac{1}{2}\delta_{Q'}$ and $3Q^+ \subseteq 3Q'$.

Since Q^+ is not OK and $2/p - 1 < 0$,

$$\|3Q' \cap E\|_{\dot{B}_p} \geq \|3Q^+ \cap E\|_{\dot{B}_p} > c_1 \delta_{Q^+}^{2/p-1} \geq c_1 \delta_{Q'}^{2/p-1}.$$

Therefore, Q' is not OK. This contradicts that $Q' \in \Lambda$ and completes the proof of Lemma 4.2. □

4.2. *Keystone squares.* The keystone squares are simply the CZ squares that have locally minimal sidelength.

Definition 4.2. The collection of *keystone squares* $\Lambda^\sharp$ consists of all $Q^\sharp \in \Lambda$ such that

$$Q \in \Lambda \text{ and } Q \cap 100Q^\sharp \neq \emptyset \implies \delta_Q \geq \delta_{Q^\sharp}.$$

Remark 4.2. For $Q^\sharp \in \Lambda^\sharp$, notice that $10Q^\sharp$ intersects a bounded number of squares, ensuring that $10Q^\sharp \cap E$ has simple geometry; in fact, $10Q^\sharp \cap E$ is contained in the union of several Besov curves that each have controlled norm.

We use the next definition momentarily.

Definition 4.3. Let $a_1, a_2, a_3 > 0$ be given. Let $Q \subseteq \mathbb{R}^2$ be a square, and let $S \subseteq \mathbb{R}^2$ be finite. We say that Q satisfies $\mathrm{R}(a_1, a_2, a_3)$ (rough with constants a_1, a_2, a_3) relative to S, provided that either of the following conditions hold:

(R1) There exist $x_1, x_2, y_1, y_2 \in Q \cap S$ with $x_1 \neq x_2$ and $y_1 \neq y_2$ such that

$$\min\left\{\left|\frac{x_1 - x_2}{|x_1 - x_2|} - \frac{y_1 - y_2}{|y_1 - y_2|}\right|, \left|\frac{x_1 - x_2}{|x_1 - x_2|} + \frac{y_1 - y_2}{|y_1 - y_2|}\right|\right\} > a_2.$$

(R2) $a_1 \delta_Q^{2/p-1} \leq \|Q \cap S\|_{\dot{B}_p} \leq a_3 \delta_Q^{2/p-1}$.

We record the following small lemma concerning rough squares.

Lemma 4.3. *Suppose that Q satisfies $\mathrm{R}(a_1, a_2, a_3)$ relative to S. Then $\|Q \cap S\|_{\dot{B}_p} \geq \hat{c} \cdot \delta_Q^{2/p-1}$ for some constant $\hat{c} = \hat{c}(a_1, a_2, p) > 0$.*

Proof. If (R2) holds, then the conclusion follows immediately.

Suppose that (R1) holds. Take $\varphi \in \dot{B}_p(\mathbb{R})$ and euclidean coordinates (u, v) such that $Q \cap S \subseteq \{(u, \varphi(u)) : u \in \mathbb{R}\}$. From (R1) and the mean value theorem, we find that $|\varphi'(u_1) - \varphi'(u_2)| \gtrsim a_2$ for some $u_1, u_2 \in \mathbb{R}$ with $|u_1 - u_2| \lesssim \delta_Q$. Thus, by the Besov embedding theorem,

$$\|\varphi\|_{\dot{B}_p(\mathbb{R})} \gtrsim |\varphi'(u_1) - \varphi'(u_2)| \cdot |u_1 - u_2|^{2/p-1} \gtrsim a_2 \delta_Q^{2/p-1}.$$

Therefore, $\|Q \cap S\|_{\dot{B}_p} = \inf\left\{\|\varphi\|_{\dot{B}_p(\mathbb{R})} : (u, v) \text{ and } \varphi \text{ as above}\right\} \gtrsim a_2 \delta_Q^{2/p-1}$, as desired. □

Let $c_2, c_3 > 0$ be small universal constants, to be determined later in the paper. We make the following assumption.

Order Remark (OR). We may assume inequalities $c_1 \leq c \cdot c_3$ and $c_2 \leq c \cdot c_3$, where c is another universal constant arising in the paper.

A sequence of squares $\{Q_i'\}_{i=1}^k$ is called a *path* provided that $Q_i' \leftrightarrow Q_{i+1}'$ for each $i = 1, \ldots k - 1$.

We now present the main result of the section.

Proposition 4.1. *The following properties hold:*

(K1) *For each $Q \in \Lambda$, there exists $Q^\sharp \in \Lambda^\sharp$ and there exists a path*

$$Q = Q_1' \leftrightarrow Q_2' \leftrightarrow \cdots \leftrightarrow Q_m' = Q^\sharp$$

such that $Q_k' \in \Lambda$ for $1 \leq k \leq m$ and such that

$$\delta_{Q_{k_2}'} \leq C \cdot (1 - c)^{k_2 - k_1} \delta_{Q_{k_1}'} \quad \textit{for } 1 \leq k_1 \leq k_2 \leq m,$$

where $C > 0$ and $0 < c < 1$ are universal constants.

(K2) *If* $\Lambda \neq \{Q^\circ\}$, *then* $9Q^\sharp$ *satisfies* $\mathrm{R}(c_1, c_2, c_3)$ *relative to* E *for every* $Q^\sharp \in \Lambda^\sharp$.
(K3) *The collection* $\{10Q^\sharp : Q^\sharp \in \Lambda^\sharp\}$ *has the bounded intersection property.*

Proof. We first construct the paths from (K1). Fix $Q \in \Lambda$, and set $Q_1' = \overline{Q}_0 = Q$. The path is determined iteratively starting from $\overline{Q}_0 \in \Lambda$. By definition of keystone squares, one of the following conditions holds:

Case 1: $\overline{Q}_0 \in \Lambda^\sharp$.
Case 2: There exists $\overline{Q} \in \Lambda$ such that $\overline{Q} \cap 100\overline{Q}_0 \neq \emptyset$ and $\delta_{\overline{Q}} \leq \frac{1}{2}\delta_{\overline{Q}_0}$.

In the first case, the trivial path $Q \leftrightarrow Q^\sharp := Q$ satisfies (K1). Alternatively, suppose that the second case holds. Choose $\overline{Q}_1 \in \Lambda$ that satisfies

(1) $\overline{Q}_1 \cap 100\overline{Q}_0 \neq \emptyset$ and $\delta_{\overline{Q}_1} \leq \frac{1}{2}\delta_{\overline{Q}_0}$,
(2) $\mathrm{dist}(\overline{Q}_1, \overline{Q}_0) \leq \mathrm{dist}(\overline{Q}, \overline{Q}_0)$ for every $\overline{Q} \in \Lambda$ such that $\overline{Q} \cap 100\overline{Q}_0 \neq \emptyset$ and $\delta_{\overline{Q}} \leq \frac{1}{2}\delta_{\overline{Q}_0}$.

Pick a line segment $\mathfrak{l} \subseteq \mathbb{R}^2$ that intersects both $\overline{Q}_0$ and $\overline{Q}_1$ and that has minimal length. Denote by $\{Q_2', \dots, Q_{k_1-1}'\}$ the collection of CZ squares that intersect $\mathfrak{l}$ on an interval with nonempty interior. We also set $Q_1' := \overline{Q}_0$ and $Q_{k_1}' := \overline{Q}_1$. We assume that these squares are indexed to form a path:

$$Q_1' \leftrightarrow Q_2' \leftrightarrow \cdots \leftrightarrow Q_{k_1}'.$$

By definition of $\mathfrak{l}$ and since $Q_k' \cap \mathfrak{l}$ has nonempty interior, we have $\mathrm{dist}(Q_k', \overline{Q}_0) < \mathrm{dist}(\overline{Q}_1, \overline{Q}_0)$ for each $2 \leq k \leq k_1 - 1$. Also note that $\mathfrak{l} \subseteq 100\overline{Q}_0$ thanks to (P1) of $\overline{Q}_1$, and thus $Q_k' \cap 100\overline{Q}_0 \neq \emptyset$. Therefore, by (P2) of $\overline{Q}_1$, we have

$$\delta_{Q_k'} \geq \delta_{\overline{Q}_0} \quad \text{for } 2 \leq k \leq k_1 - 1. \tag{4.1}$$

Since each Q_k' intersects $100\overline{Q}_0$, by Good Geometry of the CZ squares we have $\delta_{Q_k'} \leq C\delta_{\overline{Q}_0}$ for $2 \leq k \leq k_1 - 1$. Therefore, $k_1 \leq C'$ for some universal constant C'.

We iterate the above procedure, starting next with $\overline{Q}_1$ instead of $\overline{Q}_0$. In this manner we obtain a sequence of *marker squares* $\overline{Q}_j \in \Lambda$, $j \geq 0$ such that

$$\delta_{\overline{Q}_j} \leq 2^{i-j}\delta_{\overline{Q}_i} \quad \text{for } j > i \geq 0, \tag{4.2}$$

and a sequence of *intermediate squares* $Q_k' \in \Lambda$, $k \geq 1$ such that

(4.3)
$$Q = \boxed{\overline{Q}_0} = Q_1' \leftrightarrow \cdots \leftrightarrow Q_{k_1}' = \boxed{\overline{Q}_1} \leftrightarrow Q_{k_1+1}' \leftrightarrow \cdots \leftrightarrow Q_{k_2}' = \boxed{\overline{Q}_2} \leftrightarrow \cdots$$

(4.4)
$$k_{j+1} - k_j \leq C \text{ and } C^{-1}\delta_{\overline{Q}_j} \leq \delta_{Q_k'} \leq C\delta_{\overline{Q}_j} \quad \text{for all } j \geq 0,\ k_j \leq k \leq k_{j+1}.$$

By (4.2) and the finiteness of Λ, the above path must terminate. Thus, the iteration eventually enters Case 1; hence, $\overline{Q}_n \in \Lambda^\sharp$ for some n. We set $Q^\sharp := \overline{Q}_n$. By (4.2) and (4.4), the path in (4.3) satisfies the conditions from (K1).

We now prove (K2). Let $Q^\sharp \in \Lambda^\sharp$, and set $E_0 := 9Q^\sharp \cap E$. To prove that $9Q^\sharp$ is Rough relative to E, we may assume that (R1) fails. Thus for every $x_1, x_2, x_3, x_4 \in E_0$ with $x_1 \neq x_2$ and $x_3 \neq x_4$,

$$\min\left\{\left|\frac{x_1-x_2}{|x_1-x_2|} - \frac{x_3-x_4}{|x_3-x_4|}\right|, \left|\frac{x_1-x_2}{|x_1-x_2|} + \frac{x_3-x_4}{|x_3-x_4|}\right|\right\} \leq c_2. \tag{4.5}$$

If $\Lambda \neq \{Q^\circ\}$, then every CZ square has a parent. By definition of the CZ squares, $(Q^\sharp)^+$ is not OK. Therefore, since $p > 2$, by definition of OK squares, we have

$$\|9Q^\sharp \cap E\|_{\dot{B}_p} \geq \|3(Q^\sharp)^+ \cap E\|_{\dot{B}_p} \geq c_1 \delta_{(Q^\sharp)^+}^{2/p-1} \geq c_1 \delta_{9Q^\sharp}^{2/p-1}. \tag{4.6}$$

This proves one inequality from (R2). We now prove the second inequality: $\|9Q^\sharp \cap E\|_{\dot{B}_p} \leq c_3 \delta_{9Q^\sharp}^{2/p-1}$. To accomplish this task, we build an interpolating curve through $9Q^\sharp \cap E$. This will complete the proof of (K2).

Let $Q \in \Lambda$ with $Q \cap 9Q^\sharp \neq \emptyset$ be given. By definition of the keystone squares, we have $\delta_Q \geq \delta_{Q^\sharp}$.

Now we show that $\delta_Q \leq 100\delta_{Q^\sharp}$. For the sake of contradiction, suppose that $\delta_Q \geq 100\delta_{Q^\sharp}$. Since $Q \cap 9Q^\sharp \neq \emptyset$, we have $\widetilde{Q} \cap \widetilde{Q}^\sharp \neq \emptyset$. (Recall that $\widetilde{Q} = 1.3Q$.) Thus, by Good Geometry, we have $\delta_Q \leq 2\delta_{Q^\sharp}$, yielding the desired contradiction.

Let $\{Q^1, \ldots, Q^m\}$ be the collection of CZ squares that intersect E_0. Hence, $Q^i \cap 9Q^\sharp \neq \emptyset$ and $\delta_{Q^i}/\delta_{Q^\sharp} \in [1, 100]$ for each i. Therefore, $m \leq 200$. Since Q^i is OK, we have

$$\|3Q^i \cap E_0\|_{\dot{B}_p} \leq \|3Q^i \cap E\|_{\dot{B}_p} \leq c_1 \delta_{Q^i}^{2/p-1} \leq c_1 \delta_{Q^\sharp}^{2/p-1} \quad \text{for } 1 \leq i \leq m.$$

We recall the current setup.

- (A1) $\min\{|\frac{x_1-x_2}{|x_1-x_2|} - \frac{x_3-x_4}{|x_3-x_4|}|, |\frac{x_1-x_2}{|x_1-x_2|} + \frac{x_3-x_4}{|x_3-x_4|}|\} \leq c_2$ for all $x_1, x_2, x_3, x_4 \in E_0$ with $x_1 \neq x_2$, and $x_3 \neq x_4$;
- (A2) $E_0 = \bigcup_{i=1}^m Q^i \cap E_0$ and $m \leq 200$;
- (A3) $\|3Q^i \cap E_0\|_{\dot{B}_p} \leq c_1 \delta_{Q^\sharp}^{2/p-1}$ for $i = 1, \ldots, m$;
- (A4) $\delta_{Q^\sharp} \leq \delta_{Q^i} \leq 100\delta_{Q^\sharp}$ for $i = 1, \ldots, m$.

For any collection of squares $Q^\sharp, Q^1, \ldots, Q^m$ and any subset $E_0 \subseteq 9Q^\sharp$ that satisfy (A1)–(A4), we now prove that

$$\|E_0\|_{\dot{B}_p} \leq c_3 \delta_{9Q^\sharp}^{2/p-1}. \tag{4.7}$$

By rescaling assumptions (A1)–(A4) and conclusion (4.7), we may assume that $\delta_{Q^\sharp} = 1$; we drop this assumption after proving (4.7).

If $\#E_0 \le 1$, then (4.7) holds trivially; thus, we may assume that $\#E_0 \ge 2$.

Fix distinct $y_1, y_2 \in E_0$. We specify euclidean coordinates (u, v) such that $y_1 = (0, 0)$ and $y_2 = (q, 0)$ for some $q \in \mathbb{R}$. We continue working in these coordinates throughout the proof of Proposition 4.1. For $j = 1, \dots, m$, we set

$$\begin{aligned} &E'_j := 3Q^j \cap E_0 \ \text{ and } \ \overline{E}'_j := \mathrm{proj}_{\{v=0\}}(E'_j), \\ &E_j := Q^j \cap E_0 \ \text{ and } \ \overline{E}_j := \mathrm{proj}_{\{v=0\}} E_j, \\ &\overline{E}_0 := \mathrm{proj}_{\{v=0\}} E_0. \end{aligned}$$

Since $y_1, y_2 \in E_0$ lie on the u-axis, (A1) and (A3) imply that the hypotheses of Lemma 3.1 hold for E'_j and $\kappa_2 := 100c_2$. Thus, for small enough c_1,c_2, there exists $\varphi_j \in B_p(\mathbb{R})$ such that

(1) $E'_j \subseteq \{(u, \varphi_j(u)) : u \in \mathbb{R}\}$,
(2) $\|\varphi_j\|_{B_p(\mathbb{R})} \lesssim c_1 + c_2$.

Let I_j be the convex hull of $\overline{E}_j$. We consider I_j, $\overline{E}_j$, and $\overline{E}_0$ as subsets of the u-axis. Write $I_j = [a_j, b_j]$, and set $\tilde{I}_j := [a_j - \frac{1}{10}\delta_{Q^j}, b_j + \frac{1}{10}\delta_{Q^j}]$.

The distance from I_j to the endpoints of $\tilde{I}_j$ equals $\frac{1}{10}\delta_{Q^j} \ge \frac{1}{10}$ (see (A4)). By taking projections in (A2), we have

$$\overline{E}_0 = \bigcup_{j=1}^m \overline{E}_j \subseteq \bigcup_{j=1}^m I_j.$$

Thus we may choose cutoff functions with the following properties:

(1) $\widetilde{\theta}_j \in C^\infty_c(\tilde{I}_j)$,
(2) $\widetilde{\theta}_j \equiv 1$ on I_j,
(3) $|\frac{d^k}{dx^k}\widetilde{\theta}_j| \lesssim 1$ for $k \le 2$.

Notice that $\psi = \sum_{j=1}^m \widetilde{\theta}_j$ satisfies $|\psi| \lesssim 1$ on $\mathbb{R}$ and $\psi \ge 1$ on $\bigcup_j I_j$. Fix $\eta \in C^\infty(\mathbb{R})$ with $\eta(w) \equiv w$ for $w \ge 1$ and $\eta(w) \ge \frac{1}{2}$ for $w \le 1$. We define $\theta_j := \widetilde{\theta}_j \cdot (\eta \circ \psi)^{-1}$, which satisfies

(1) $\sum_{j=1}^m \theta_j = 1$ on $\overline{E}_0$,
(2) $|\frac{d^k}{dx^k}\theta_j| \lesssim 1$ for $k \le 2$,
(3) $\theta_j \in C^\infty_c(\tilde{I}_j)$.

Let $\varphi := \displaystyle\sum_{j=1}^m \theta_j \varphi_j$. From (P2) of θ_j and the bound $m \le 200$, we estimate

$$\|\varphi\|_{\dot{B}_p(\mathbb{R})} \le \sum_{j=1}^m \|\theta_j\varphi_j\|_{\dot{B}_p(\mathbb{R})} \le \sum_{j=1}^m \|\theta_j\varphi_j\|_{B_p(\mathbb{R})} \lesssim \sum_{j=1}^m \|\varphi_j\|_{B_p(\mathbb{R})} \lesssim c_1 + c_2. \tag{4.8}$$

Given $x_0 \in E_0$, we write $x_0 = (\overline{x}_0, \overline{y}_0)$ (in (u, v) coordinates). We now prove the following

Claim. If $\overline{x}_0 \in \tilde{I}_j$, then $x_0 \in 3Q^j$.

Recall that $\tilde{I}_j = [a_j - \frac{1}{10}\delta_{Q^j}, b_j + \frac{1}{10}\delta_{Q^j}]$ and that $I_j = [a_j, b_j]$ is the convex hull of the projection of $E_j \subseteq Q^j$ onto the u-axis. Thus, $\mathrm{diam}(I_j) \leq \mathrm{diam}(Q^j) = \sqrt{2}\delta_{Q^j}$. Let $\overline{x}_j \in \overline{E}_j$ be an endpoint of I_j that is closest to $\overline{x}_0$. Therefore,

$$|\overline{x}_0 - \overline{x}_j| \leq \max\left\{\frac{1}{2}\mathrm{diam}(I_j), \frac{1}{10}\delta_{Q^j}\right\} \leq \frac{\sqrt{2}}{2}\delta_{Q^j}.$$

Since $\overline{E}_j = \mathrm{proj}_{\{v=0\}} E_j$, there exists $\overline{y}_j \in \mathbb{R}$ with $x_j = (\overline{x}_j, \overline{y}_j) \in E_j = E_0 \cap Q^j$. For small enough c_2, because two points of E_0 lie on the u-axis, (A1) implies that $|\overline{y}_0 - \overline{y}_j| \leq |\overline{x}_0 - \overline{x}_j|$; hence,

$$|x_0 - x_j|^2 = |\overline{x}_0 - \overline{x}_j|^2 + |\overline{y}_0 - \overline{y}_j|^2 \leq 2|\overline{x}_0 - \overline{x}_j|^2 \leq \delta_{Q^j}^2.$$

Since $x_j \in Q^j$, the above inequality implies that $x_0 \in 3Q^j$. This proves the desired claim.

We now return to proving (K2). We have

$$\varphi(\overline{x}_0) = \sum_{j=1}^{m} \theta_j(\overline{x}_0)\varphi_j(\overline{x}_0);$$

moreover, the sum may be taken over j such that $\overline{x}_0 \in \tilde{I}_j$ (see (P3) of θ_j). For each such j, our claim implies that $x_0 \in 3Q^j \cap E_0 = E_j'$; hence, $\varphi_j(\overline{x}_0) = \overline{y}_0$ by (P1) of φ_j. Consequently,

$$\varphi(\overline{x}_0) = \sum_j \theta_j(\overline{x}_0)\overline{y}_0 = \overline{y}_0,$$

where the last equality follows from (P1) of θ_j. Since $x_0 \in E_0$ was arbitrary, we have shown that

$$E_0 \subseteq \{(u, \varphi(u)) : u \in \mathbb{R}\}.$$

Thus, (4.8) implies that $\|E_0\|_{\dot{B}_p} \lesssim c_1 + c_2$. For small enough c_1 and c_2 depending on c_3, we then have $\|E_0\|_{\dot{B}_p} \leq 9^{2/p-1}c_3$. This proves (4.7) under the assumption that $\delta_{Q^\sharp} = 1$. By rescaling, we may drop this assumption. Together with (4.6), this completes the proof of (K2).

Finally, we prove (K3). Let $Q_1^\sharp, Q_2^\sharp \in \Lambda^\sharp$ satisfy $10Q_1^\sharp \cap 10Q_2^\sharp \neq \emptyset$ and $\delta_{Q_1^\sharp} \geq \delta_{Q_2^\sharp}$. Hence, $Q_2^\sharp \cap 100Q_1^\sharp \neq \emptyset$. Thus, by definition of the keystone squares, $\delta_{Q_2^\sharp} \geq \delta_{Q_1^\sharp}$. Because the CZ squares have disjoint interiors, for each $Q_1^\sharp$ there can be at most C distinct choices of $Q_2^\sharp$ that satisfy the above conditions. This proves (K3) and completes the proof of Proposition 4.1. $\square$

In Section 7 we prove Theorem 1 in the easier case when $\Lambda = \{Q^\circ\}$. Until then, we suppose that $\Lambda \neq \{Q^\circ\}$. Thus, the conclusion of (K2) holds.

The keystone squares are indexed by $\Lambda^\sharp = \{Q^\sharp_\mu : \mu = 1, \ldots, K^\sharp\}$. Let $\delta^\sharp_\mu := \delta_{Q^\sharp_\mu}$. Define the index map $\mu : \{1, \ldots, K\} \to \{1, \ldots, K^\sharp\}$ so that the path from (K1) connects $Q_\nu \in \Lambda$ and $Q^\sharp_{\mu(\nu)} \in \Lambda^\sharp$. Next we state an easy corollary of (K1).

COROLLARY 4.1. *For some universal constant C, we have* $\mathrm{dist}(Q_\nu, Q^\sharp_{\mu(\nu)}) \leq C\delta_{Q_\nu}$ *and* $\delta_{Q^\sharp_{\mu(\nu)}} \leq C\delta_{Q_\nu}$ *for each* $\nu = 1, \ldots, K$.

4.3. *Representative points.* Recall that $\|3Q_\nu \cap E\|_{\dot{B}_p} \leq c_1 \delta_\nu^{2/p-1}$ for each CZ square Q_ν. By definition this means that

$$3Q_\nu \cap E \subseteq \{(s, \varphi_\nu(s)) : s \in \mathbb{R}\} \quad \text{(in some euclidean coordinate system } (s,t)),$$
$$\text{where } \varphi_\nu \in \dot{B}_p(\mathbb{R}), \text{ and } \|\varphi_\nu\|_{\dot{B}_p(\mathbb{R})} \leq 2c_1\delta_\nu^{2/p-1}.$$

The Besov embedding theorem implies $\frac{d}{ds}\varphi_\nu$ varies by at most $Cc_1\delta_\nu^{2/p-1}\delta_\nu^{1-2/p}$ on the projection of $3Q_\nu$ onto the s-axis. (Recall that $\delta_\nu = \delta_{Q_\nu}$.) We assume that $c_1 < \frac{1}{200C}$; hence, $\frac{d}{ds}\varphi_\nu$ varies by at most $\frac{1}{100}$ on this interval. This allows us to choose $x_\nu \in \frac{1}{2}Q_\nu$ with $\mathrm{dist}(x_\nu, E) \geq \frac{1}{5}\delta_\nu$. These points are called the CZ *representative points*, and we set $E' := \{x_\nu : \nu = 1, \ldots, K\}$.

For each $1 \leq \mu \leq K^\sharp$, there exists ν such that $Q^\sharp_\mu = Q_\nu$. We set $x^\sharp_\mu := x_\nu$ for this ν. These points are called the *keystone representative points*, and we set $E^\sharp := \{x^\sharp_\mu : \mu = 1, \ldots, K^\sharp\}$.

LEMMA 4.4. *The* CZ *representative points satisfy* $E' \subseteq 0.99Q^\circ$.

Proof. Let $1 \leq \nu \leq K$. Note that either

(A) $Q_\nu \subseteq Q^\circ$ and Q_ν intersects the boundary of Q°, or
(B) $Q_\nu \subseteq \mathrm{int}(Q^\circ)$.

If (A) holds, then we claim that $\delta_\nu \geq \frac{1}{32}\delta_{Q^\circ}$. Suppose for the sake of contradiction that $\delta_\nu \leq \frac{1}{64}Q^\circ$. Since $E \subseteq \frac{1}{10}Q^\circ$, we have $3Q^+_\nu \cap E \subseteq 9Q_\nu \cap E = \emptyset$; hence Q^+_ν is OK. However, this contradicts the fact that Q_ν is a CZ square. This shows that $\delta_\nu \geq \frac{1}{32}\delta_{Q^\circ}$. Consequently, $x_\nu \in \frac{1}{2}Q_\nu \subseteq 0.99Q^\circ$ as desired.

Alternatively, suppose that (B) holds. By the above analysis, the distance between ∂Q° and Q_ν is at least $\frac{1}{32}\delta_{Q^\circ}$. Therefore, $x_\nu \in Q_\nu \subseteq 0.99Q^\circ$ as desired. $\square$

5. The modified extension problem

5.1. *A Sobolev-type inequality.* We start by proving an inequality related to the Sobolev theorem and Proposition 4.1.

PROPOSITION 5.1 (Sobolev-type inequality). *Let $F \in L^{2,p}(\mathbb{R}^2)$. Then*

$$\sum_{\nu=1}^{K}\left[|\nabla F(x_\nu) - \nabla F(x^\sharp_{\mu(\nu)})|^p \delta_\nu^{2-p} + |F(x_\nu) - J_{x^\sharp_{\mu(\nu)}}F(x_\nu)|^p\delta_\nu^{2-2p}\right] \lesssim \|F\|^p_{L^{2,p}(\mathbb{R}^2)}.$$

Proof. Recall (K1) from Proposition 4.1, where we defined the map $\mu : \{1,\dots,K\} \to \{1,\dots,K^\sharp\}$ such that for each $\nu \in \{1,2,\dots,K\}$, there exists a sequence of CZ squares $Q_{k_1^\nu},\dots,Q_{k_{N_\nu}^\nu}$ with

$$(5.1)\quad \begin{gathered} Q_\nu = Q_{k_1^\nu} \leftrightarrow \cdots \leftrightarrow Q_{k_{N_\nu}^\nu} = Q^\sharp_{\mu(\nu)}, \text{ and} \\ \delta_{Q_{k_m^\nu}} \lesssim (1-c)^{m-n}\delta_{Q_{k_n^\nu}} \text{ for } 1 \le n < m \le N_\nu \quad (\text{here } c = c(p) \in (0,1)). \end{gathered}$$

Let $\varepsilon \in (0, 1-2/p)$ be fixed, and set

$$X := \sum_{\nu=1}^{K} \left[|\nabla F(x_\nu) - \nabla F(x^\sharp_{\mu(\nu)})|^p \delta_\nu^{2-p} + |F(x_\nu) - J_{x^\sharp_{\mu(\nu)}} F(x_\nu)|^p \delta_\nu^{2-2p} \right].$$

Note that

$$\begin{aligned} F(x_\nu) - J_{x^\sharp_{\mu(\nu)}} F(x_\nu) &= \left(J_{x_\nu}F - J_{x^\sharp_{\mu(\nu)}} F\right)(x^\sharp_{\mu(\nu)}) \\ &\quad + \left(\nabla F(x_\nu) - \nabla F(x^\sharp_{\mu(\nu)})\right) \cdot (x_\nu - x^\sharp_{\mu(\nu)}). \end{aligned}$$

Since $x^\sharp_{\mu(\nu)} \in Q^\sharp_{\mu(\nu)}$ and $x_\nu \in Q_\nu$, we have $|x_\nu - x^\sharp_{\mu(\nu)}| \le C\delta_\nu$ (see Corollary 4.1). Thus, through Hölder's inequality,

(5.2)

$$\begin{aligned} X &\lesssim \sum_{\nu=1}^{K} \sum_{|\alpha|\le 1} \left| \partial^\alpha \left(J_{x_\nu}F - J_{x^\sharp_{\mu(\nu)}} F\right)(x^\sharp_{\mu(\nu)}) \right|^p \delta_\nu^{2-(2-|\alpha|)p} \\ &= \sum_{\nu=1}^{K} \sum_{|\alpha|\le 1} \left| \sum_{n=1}^{N_\nu - 1} \partial^\alpha \left(J_{x_{k_n^\nu}}F - J_{x_{k_{n+1}^\nu}} F\right)(x^\sharp_{\mu(\nu)}) \cdot \left[\delta_{Q_{k_n^\nu}}\right]^{-\varepsilon} \left[\delta_{Q_{k_n^\nu}}\right]^{+\varepsilon} \right|^p \delta_\nu^{2-(2-|\alpha|)p} \\ &\le \sum_{\nu=1}^{K} \sum_{|\alpha|\le 1} \left[\sum_{n=1}^{N_\nu - 1} \left| \partial^\alpha \left(J_{x_{k_n^\nu}}F - J_{x_{k_{n+1}^\nu}} F\right)(x^\sharp_{\mu(\nu)}) \right|^p \left[\delta_{Q_{k_n^\nu}}\right]^{-\varepsilon p} \right] \\ &\quad \cdot \left[\sum_{n=1}^{N_\nu - 1} \left[\delta_{Q_{k_n^\nu}}\right]^{\varepsilon p'} \right]^{p/p'} \delta_\nu^{2-(2-|\alpha|)p}. \end{aligned}$$

(Here, p' denotes the dual exponent to p; thus $\frac{1}{p'} + \frac{1}{p} = 1$.)

From (5.1), we have

$$(5.3)\quad \operatorname{dist}(Q_\nu, Q_{k_n^\nu}) \le \sum_{m=1}^{n} \operatorname{diam}(Q_{k_m^\nu}) \lesssim \sum_{m=1}^{n} (1-c)^{m-1} \delta_{Q_{k_1^\nu}} \lesssim \delta_{Q_{k_1^\nu}} = \delta_\nu.$$

Moreover, since $x_{k_n^\nu} \in Q_{k_n^\nu}$ and $x^\sharp_{\mu(\nu)} \in Q^\sharp_{\mu(\nu)}$, we have $|x_{k_n^\nu} - x^\sharp_{\mu(\nu)}| \le C\delta_\nu$. Thus,

$$\sum_{|\alpha|\le 1} \left| \partial^\alpha P(x^\sharp_{\mu(\nu)}) \right|^p \delta_\nu^{2-(2-|\alpha|)p} \lesssim \sum_{|\alpha|\le 1} \left| \partial^\alpha P(x_{k_n^\nu}) \right|^p \delta_\nu^{2-(2-|\alpha|)p} \quad \text{for any } P \in \mathcal{P}.$$

Applying (5.1) and the previous estimate in (5.2), we obtain

$$(5.4)\quad X \lesssim \sum_{\nu=1}^{K} \sum_{|\alpha|\le 1} \sum_{n=1}^{N_\nu - 1} \left|\partial^\alpha \left[J_{x_{k_n^\nu}} F - J_{x_{k_{n+1}^\nu}} F\right](x_{k_n^\nu})\right|^p \left[\delta_{Q_{k_n^\nu}}\right]^{-\varepsilon p} \delta_{Q_\nu}^{2-(2-|\alpha|)p+\varepsilon p}.$$

For each $\nu \in \{1,\dots,K\}$ and $n \in \{1,\dots,N_\nu\}$, we have $Q_{k_n^\nu} \subseteq CQ_\nu$ for some universal constant C, thanks to (5.3). Note that the sequence $k_1^\nu, \dots, k_{N_\nu}^\nu$ repeats at most $C = C(p)$ many times for each fixed ν, thanks to (5.1). Note also that $k_n^\nu \leftrightarrow k_{n+1}^\nu$. Thus, by switching the order of summation in (5.4),

$$\begin{aligned} X &\lesssim \sum_{k \leftrightarrow k'} \sum_{|\alpha|\le 1} \left|\partial^\alpha \left[J_{x_k} F - J_{x_{k'}} F\right](x_k)\right|^p [\delta_{Q_k}]^{-\varepsilon p} \\ &\qquad \cdot \sum \left\{ \delta_Q^{2-(2-|\alpha|)p+\varepsilon p} : \ Q \text{ dyadic}, \ Q_k \subseteq CQ \right\} \\ &\lesssim \sum_{k \leftrightarrow k'} \sum_{|\alpha|\le 1} \left|\partial^\alpha \left[J_{x_k} F - J_{x_{k'}} F\right](x_k)\right|^p [\delta_{Q_k}]^{2-(2-|\alpha|)p}. \end{aligned}$$

(Since $\varepsilon < 1-2/p$, the exponent in the dyadic sum above is negative.) Applying the Sobolev theorem (see Remark 3.1) and the bounded intersection property of the squares $\{\widetilde{Q}_\nu\}$, we see that

$$X \lesssim \sum_{k \leftrightarrow k'} \|F\|^p_{L^{2,p}(\widetilde{Q}_k \cup \widetilde{Q}_{k'})} \lesssim \sum_k \|F\|^p_{L^{2,p}(\widetilde{Q}_k)} \lesssim \|F\|^p_{L^{2,p}(\mathbb{R}^2)}.$$

This completes the proof of Proposition 5.1. □

5.2. *The constant-path property.* We start by introducing some notation. Given $L \in \mathrm{Wh}(E')$ we denote $L_\nu = L_{x_\nu}$ for $\nu = 1,\dots,K$. Similarly, given $L^\sharp \in \mathrm{Wh}(E^\sharp)$ we denote $L_\mu^\sharp = L^\sharp_{x_\mu^\sharp}$ for $\mu = 1,\dots,K^\sharp$.

Let $L^\sharp \in \mathrm{Wh}(E^\sharp)$ be given. We denote the *constant-path extension* of $L^\sharp$ by $\mathrm{ext}(L^\sharp) \in \mathrm{Wh}(E')$ which is defined by

$$\mathrm{ext}(L^\sharp) = L, \ \text{ where } L_\nu = L^\sharp_{\mu(\nu)} \quad \text{for } \ 1 \le \nu \le K.$$

(Recall that we defined the index map μ at the end of Section 4.2.)

The main important idea in the proof of Theorem 1 is the manner in which we place additional linear constraints on the sought extension of f, which help eliminate undesirable degrees of freedom. These constraints form the *constant-path property* defined below.

We say that $F \in L^{2,p}(\mathbb{R}^2)$ satisfies the constant-path property (CPP) provided that $J_{x_\nu} F = J_{x^\sharp_{\mu(\nu)}} F$ for $\nu = 1,\dots,K$. (Equivalently, this means that $J_{E'} F = \mathrm{ext}(J_{E^\sharp} F)$.) The constant-path property is natural because there always exist C-optimal extensions that satisfy it, as we demonstrate in the next result.

LEMMA 5.1. *Let $f : E \to \mathbb{R}$ be given. Then there exists $\widehat{F} \in L^{2,p}(\mathbb{R}^2)$ such that*

(1) *$\widehat{F} = f$ on E and $\|\widehat{F}\|_{L^{2,p}(\mathbb{R}^2)} \lesssim \|f\|_{L^{2,p}(\mathbb{R}^2)|_E}$,*
(2) *$\widehat{F}$ satisfies the constant-path property.*

Proof. Choose any function $F \in L^{2,p}(\mathbb{R}^2)$ such that

(1) $F = f$ on E,
(2) $\|F\|_{L^{2,p}(\mathbb{R}^2)} \leq 2\|f\|_{L^{2,p}(\mathbb{R}^2)|_E}$.

We now modify this function near the representative points E' yielding an $\widehat{F} \in L^{2,p}(\mathbb{R}^2)$ that satisfies the constant-path property and extends f while maintaining the near-minimal seminorm.

We set $L^\sharp := J_{E^\sharp}F \in \mathrm{Wh}(E^\sharp)$ and $L := \mathrm{ext}(L^\sharp) \in \mathrm{Wh}(E')$. Pick cutoff functions θ_ν for $\nu = 1, \dots, K$ that satisfy

(1) $\theta_\nu = 1$ on a neighborhood of x_ν,
(2) $|\partial^\alpha \theta_\nu| \leq C\delta_\nu^{-|\alpha|}$ for $|\alpha| \leq 2$,
(3) $\mathrm{supp}(\theta_\nu) \subseteq B(x_\nu, \frac{1}{20}\delta_\nu)$.

Because $x_\nu \in \frac{1}{2}Q_\nu$ and $\mathrm{dist}(x_\nu, E) \geq \delta_\nu/5$, we have

$$\theta_\nu = 0 \text{ on } E, \tag{5.5}$$

and

$$\mathrm{supp}(\theta_\nu) \subseteq 0.9Q_\nu \quad \text{for each } \nu. \tag{5.6}$$

In particular, the cutoff functions θ_ν have disjoint supports. We now define

$$\widehat{F} := F + \sum_{\nu=1}^{K} \theta_\nu \cdot (L_\nu - J_{x_\nu}F).$$

By (5.5), (5.6), and the first property of θ_ν, we see that

$$\widehat{F} = f \text{ on } E, \quad \text{and} \quad J_{x_\nu}\widehat{F} = L_\nu \quad \text{for each } \nu. \tag{5.7}$$

In particular, $\widehat{F}$ satisfies the constant-path property, thanks to the definition $L = \mathrm{ext}(L^\sharp)$. It remains to estimate $\|\widehat{F}\|_{L^{2,p}(\mathbb{R}^2)}$. From (5.6), we obtain

$$\begin{aligned} \|\widehat{F}\|^p_{L^{2,p}(\mathbb{R}^2)} &\lesssim \|F\|^p_{L^{2,p}(\mathbb{R}^2)} + \sum_\nu \|\theta_\nu \cdot (L_\nu - J_{x_\nu}F)\|^p_{L^{2,p}(Q_\nu)} \lesssim \|F\|^p_{L^{2,p}(\mathbb{R}^2)} \\ &\quad + \sum_\nu \left[|(L_\nu - J_{x_\nu}F)(x_\nu)|^p \delta_\nu^{2-2p} + |\nabla L_\nu - \nabla F(x_\nu)|^p \delta_\nu^{2-p} \right]. \end{aligned} \tag{5.8}$$

The second inequality is a consequence of the following fact: If $\theta \in C^2(Q)$ satisfies $|\partial^\alpha \theta| \leq C\delta_Q^{-|\alpha|}$ for $|\alpha| \leq 2$, then $\|\theta \cdot P\|_{L^{2,p}(Q)} \lesssim |P(x)| \cdot \delta_Q^{2/p-2} + |\nabla P| \cdot \delta_Q^{2/p-1}$ for every $P \in \mathcal{P}$ and $x \in Q$. To prove this fact one may assume that $\delta_Q = 1$ by scale-invariance, which reduces the problem to an easy computation.

Since $L_\nu = L^\sharp_{\mu(\nu)} = J_{x^\sharp_{\mu(\nu)}} F$, from Proposition 5.1 and (5.8) we conclude that

$$\|\widehat{F}\|_{L^{2,p}(\mathbb{R}^2)} \lesssim \|F\|_{L^{2,p}(\mathbb{R}^2)} \leq 2\|f\|_{L^{2,p}(\mathbb{R}^2)|_E}.$$

This completes the proof of Lemma 5.1 □

5.3. *The local extension problem.* The problem of extending $f : E \to \mathbb{R}$ will be decomposed into numerous smaller extension problems, each localized to some square Q. The geometry of the local subsets $E \cap Q$ are quite simple — in fact, each subset lies on a Besov curve with controlled norm — making these problems easy to solve.

In this section we make the following

Geometric Assumptions. The following objects are given:

- a constant $\kappa > 0$,
- a square $Q \subseteq \mathbb{R}^2$ and a finite subset $S \subseteq 0.9Q$ such that $\|S\|_{\dot{B}_p} \leq \kappa \delta_Q^{2/p-1}$,
- a point $z \in \frac{1}{2}Q$ such that $\operatorname{dist}(z, S) \geq \frac{1}{100}\delta_Q$.

Proposition 5.2. *There exists $\widehat{a} > 0$ depending only on p such that if $\kappa \leq \widehat{a}$ in the* Geometric Assumptions, *then there exists a linear map $T_Q : L^{2,p}(Q)|_S \times \mathcal{P} \to L^{2,p}(Q)$ such that*

(1) *$T_Q(f, P) = f$ on S, and $J_z T_Q(f, P) = P$ for every $f : S \to \mathbb{R}$ and $P \in \mathcal{P}$;*
(2) $\|T_Q(f, P)\|_{L^{2,p}(Q)} \approx \inf\{\|F\|_{L^{2,p}(Q)} : F = f$ *on* S *and* $J_z F = P\}$.

Moreover, there exist linear functionals $\lambda_1, \dots, \lambda_J$, where $J \lesssim \#(S)^2$, such that

$$M_Q(f, P) := \left(\sum_{j=1}^{J} |\lambda_j(f, P)|^p\right)^{1/p} \quad \textit{satisfies} \quad M_Q(f, P) \approx \|T_Q(f, P)\|_{L^{2,p}(Q)}.$$

Remark 5.1. Note that $M_Q(f, P)$ is within a constant factor of the semi-norm

$$\|(f, P)\| := \inf\left\{\|F\|_{L^{2,p}(Q)} : F = f \text{ on } S \text{ and } J_z F = P\right\}.$$

Hence, M_Q is essentially subadditive in the sense that

$$M_Q(f_0 + f_1, P_0 + P_1) \leq C \cdot [M_Q(f_0, P_0) + M_Q(f_1, P_1)]$$

for every $f_0, f_1 : S \to \mathbb{R}$ and $P_0, P_1 \in \mathcal{P}$.

Proof. By a standard rescaling argument, we may assume that $\delta_Q = 1$. Let $0 < \widehat{a} < 1$ be some small universal constant, to be determined later. We assume that $\kappa \leq \widehat{a}$.

Suppose we can find a linear map $\overline{T} : L^{2,p}(Q)|_S \to L^{2,p}(Q)$ that satisfies the conclusion of the proposition, but with $P = 0$. Then the map $T_Q : L^{2,p}(Q)|_S \times \mathcal{P} \to L^{2,p}(Q)$ defined by

$$T_Q(f, P) := \overline{T}(f - P|_S) + P$$

satisfies the conclusion of the proposition. Thus, it suffices to construct a linear map $\overline{T} : L^{2,p}(Q)|_S \to L^{2,p}(Q)$ such that $\overline{T}f = f$ on S, $J_z[\overline{T}f] = 0$ and

$$\|\overline{T}f\|_{L^{2,p}(Q)} \le C \inf\Big\{\|F\|_{L^{2,p}(Q)} : F = f \text{ on } S \text{ and } J_z F = 0\Big\}.$$

We momentarily consider the spaces $\mathbb{X} = W^{2,p}$ and $\mathbb{X} = L^{2,p}$. For each $F \in \mathbb{X}(Q)$, we claim that there exists $F_0 \in \mathbb{X}(\mathbb{R}^2)$ such that $F_0 = F$ on $0.9Q$ and $\|F_0\|_{\mathbb{X}(\mathbb{R}^2)} \le C\|F\|_{\mathbb{X}(Q)}$. This follows from a standard cutoff function argument. Thus, since $S \subseteq 0.9Q$, we have

$$\|f\|_{L^{2,p}(Q)|_S} \approx \|f\|_{L^{2,p}(\mathbb{R}^2)|_S} \quad \text{and} \quad \|f\|_{W^{2,p}(Q)|_S} \approx \|f\|_{W^{2,p}(\mathbb{R}^2)|_S}. \tag{5.9}$$

Recall that $S \subseteq 0.9Q$ satisfies $\|S\|_{\dot{B}_p} \le \widehat{a}$. For small enough $\widehat{a}$, Lemma 3.3 provides a diffeomorphism $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ such that

(1) $\Phi(S) \subseteq \mathbb{R} \times \{0\}$,
(2) $\|\Phi\|_{L^{2,p}(\mathbb{R}^2)} \le C$,
(3) $\|\nabla\Phi\|_{L^\infty(R^2)} \le C$ and $\|\nabla\Phi^{-1}\|_{L^\infty(\mathbb{R}^2)} \le C$.

Let $\overline{S} := \Phi(S)$, and define $\overline{f} : \overline{S} \to \mathbb{R}$ by $\overline{f} = f \circ \Phi^{-1}|_{\overline{S}}$. We subsequently identify $\overline{S} \subseteq \mathbb{R} \times \{0\}$ with a subset of $\mathbb{R}$ through the natural projection. Now, from Lemma 3.4 we obtain

$$\begin{aligned}\|f\|_{W^{2,p}(\mathbb{R}^2)|_S} &= \inf\Big\{\|F\|_{W^{2,p}(\mathbb{R}^2)} : F = f \text{ on } S\Big\} \\ &= \inf\Big\{\|\overline{F} \circ \Phi\|_{W^{2,p}(\mathbb{R}^2)} : \overline{F} \circ \Phi = f \text{ on } S\Big\} \\ &\approx \inf\Big\{\|F\|_{W^{2,p}(\mathbb{R}^2)} : F = \overline{f} \text{ on } \overline{S}\Big\} = \|\overline{f}\|_{W^{2,p}(\mathbb{R}^2)|_{\overline{S}}}.\end{aligned} \tag{5.10}$$

Moreover, Proposition 3.3 implies that

$$\begin{aligned}\|\overline{f}\|_{W^{2,p}(\mathbb{R}^2)|_{\overline{S}}} &= \inf\Big\{\|F\|_{W^{2,p}(\mathbb{R}^2)} : F \in W^{2,p}(\mathbb{R}^2),\ F = \overline{f} \text{ on } \overline{S}\Big\} \\ &\approx \inf\Big\{\|g\|_{B_p(\mathbb{R})} : g \in B_p(\mathbb{R}),\ g = \overline{f} \text{ on } \overline{S}\Big\} = \|\overline{f}\|_{B_p(\mathbb{R})|_{\overline{S}}}.\end{aligned} \tag{5.11}$$

We invoke Proposition 3.5 to choose $g \in B_p(\mathbb{R})$ depending linearly on $\overline{f}$ such that

(1) $g = \overline{f}$ on $\overline{S}$,
(2) $\|g\|_{B_p(\mathbb{R})} \lesssim \|\overline{f}\|_{B_p(\mathbb{R})|_{\overline{S}}}$.

We invoke Proposition 3.3 to choose $G \in W^{2,p}(\mathbb{R}^2)$ depending linearly on g such that

(1) $G = g$ on $\mathbb{R} \times \{0\}$,
(2) $\|G\|_{W^{2,p}(\mathbb{R}^2)} \lesssim \|g\|_{B_p(\mathbb{R})}$.

Therefore, $G = \overline{f}$ on $\overline{S}$ and

$$\|G\|_{W^{2,p}(\mathbb{R}^2)} \lesssim \|\overline{f}\|_{B_p(\mathbb{R})|_{\overline{S}}} \approx \|\overline{f}\|_{W^{2,p}(\mathbb{R}^2)|_{\overline{S}}}. \tag{5.12}$$

We define $F := G \circ \Phi$. Firstly, note that $F = \overline{f} \circ \Phi = f$ on S. Secondly, we have $\|F\|_{W^{2,p}(\mathbb{R}^2)} \approx \|G\|_{W^{2,p}(\mathbb{R}^2)}$; hence, (5.10) and (5.12) imply that

$$\|F\|_{W^{2,p}(\mathbb{R}^2)} \lesssim \|f\|_{W^{2,p}(\mathbb{R}^2)|_S}. \tag{5.13}$$

Pick $\theta \in C_c^\infty(B(z, \frac{1}{150}))$ such that

(1) $\theta \equiv 1$ on $B(z, \frac{1}{200})$,
(2) $\|\theta\|_{C^2} \lesssim 1$.

Note that $\theta = 0$ on S, because $\mathrm{dist}(z, S) \geq \frac{1}{100}$.

We set $\widehat{F} := (F - \theta \cdot J_z F)|_Q$. Note that $\widehat{F} = F - 0 = f$ on S, and $J_z\widehat{F} = J_z F - J_z F = 0$. From the Sobolev theorem, (P2) of θ, (5.13), and (5.9), we obtain

$$\|\widehat{F}\|_{W^{2,p}(Q)} = \|F - \theta \cdot J_z F\|_{W^{2,p}(Q)} \lesssim \|F\|_{W^{2,p}(\mathbb{R}^2)} \lesssim \|f\|_{W^{2,p}(\mathbb{R}^2)|_S} \approx \|f\|_{W^{2,p}(Q)|_S}.$$

Because $\widehat{F} = f$ on S, the previous inequality implies that

$$\|\widehat{F}\|_{W^{2,p}(Q)} \approx \|f\|_{W^{2,p}(Q)|_S}. \tag{5.14}$$

We also have

$$\begin{aligned} \|\widehat{F}\|_{W^{2,p}(Q)} &\lesssim \inf\left\{\|H\|_{W^{2,p}(Q)} : H = f \text{ on } S,\ J_z H = 0\right\} \\ &\lesssim \inf\left\{\|H\|_{L^{2,p}(Q)} : H = f \text{ on } S,\ J_z H = 0\right\}. \end{aligned}$$

In the last inequality above we applied the Sobolev theorem to show that $\|H\|_{W^{2,p}(Q)} \approx \|H\|_{L^{2,p}(Q)}$ whenever $J_z H = 0$. In particular, we have

$$\|\widehat{F}\|_{W^{2,p}(Q)} \approx \|\widehat{F}\|_{L^{2,p}(Q)}. \tag{5.15}$$

We define $\overline{T}_Q(f) = \widehat{F}$, which satisfies the desired extension/optimality properties from the proposition (with $P = 0$). Applying (5.14), (5.15), (5.10), (5.11), and (5.9), we see that

$$\|\widehat{F}\|_{L^{2,p}(Q)} \approx \|\widehat{F}\|_{W^{2,p}(Q)} \approx \|f\|_{W^{2,p}(\mathbb{R}^2)|_S} \approx \|\overline{f}\|_{W^{2,p}(\mathbb{R}^2)|_{\overline{S}}} \approx \|\overline{f}\|_{B_p(\mathbb{R})|_{\S}}.$$

Proposition 3.5 yields the formula

$$\|\overline{f}\|^p_{B_p(\mathbb{R})|_{\overline{S}}} \approx \sum_{j=1}^{J} |\lambda_j(\overline{f})|^p,$$

for certain linear functionals $\lambda_j : B_p(\mathbb{R})|_{\overline{S}} \to \mathbb{R}$, where $J \lesssim (\#E_1)^2 = (\#S)^2$. Since $\overline{f} = f \circ \Phi^{-1}$ depends linearly on f, the last two equations complete the proof of Proposition 5.2. $\square$

5.4. *Patching the local extensions.* For each $f : E \to \mathbb{R}$ and $\nu \in \{1, \dots, K\}$, we set $E_\nu := 1.1Q_\nu \cap E$ and $f_\nu := f|_{E_\nu}$. Recall that Q_ν is OK. That is,

$$\|E_\nu\|_{\dot{B}_p} \le \|3Q_\nu \cap E\|_{\dot{B}_p} \le c_1 \delta_\nu^{2/p-1} \quad \text{and} \quad E_\nu \subseteq 1.1Q_\nu.$$

We have already determined $x_\nu \in \frac{1}{2}Q_\nu$ such that $\mathrm{dist}(x_\nu, E_\nu) \ge \mathrm{dist}(x_\nu, E) \ge \frac{1}{5}\delta_\nu$. Recall that $\widetilde{Q}_\nu = 1.3Q_\nu$. Thus, for small enough c_1, by Proposition 5.2 we obtain a linear map $T_\nu : L^{2,p}(\widetilde{Q}_\nu)|_{E_\nu} \times \mathcal{P} \to L^{2,p}(\widetilde{Q}_\nu)$ such that $F_\nu = T_\nu(f_\nu, L_\nu)$ satisfies

(1) $F_\nu = f_\nu$ on E_ν and $J_{x_\nu} F_\nu = L_\nu$ for each $f : E \to \mathbb{R}$ and $L_\nu \in \mathcal{P}$;
(2) $\|F_\nu\|_{L^{2,p}(\widetilde{Q}_\nu)} \approx \inf\big\{\|F\|_{L^{2,p}(\widetilde{Q}_\nu)} : F = f_\nu \text{ on } E_\nu, \text{ and } J_{x_\nu} F = L_\nu\big\}$.

Moreover, there exist linear functionals $\lambda_1^\nu, \dots, \lambda_{N_\nu}^\nu$ with $N_\nu \lesssim (\#E_\nu)^2$ such that

$$M_\nu(f_\nu, L_\nu) := \left(\sum_{i=1}^{N_\nu} |\lambda_i^\nu(f_\nu, L_\nu)|^p\right)^{1/p} \quad \text{satisfies} \quad M_\nu(f_\nu, L_\nu) \approx \|F_\nu\|_{L^{2,p}(\widetilde{Q}_\nu)}.$$

Using these extension operators and functionals, we prove the next result.

PROPOSITION 5.3. *There exists a linear map $T : L^{2,p}(\mathbb{R}^2)|_E \times \mathrm{Wh}(E^\sharp) \to L^{2,p}(\mathbb{R}^2)$ such that the following holds. Given $f : E \to \mathbb{R}$ and $L^\sharp \in \mathrm{Wh}(E^\sharp)$, set $L := \mathrm{ext}(L^\sharp)$ and $\widehat{F} := T(f, L^\sharp)$. Then*

(1) $\widehat{F} = f$ *on E and* $J_{E'}\widehat{F} = L$.
(2) $\|\widehat{F}\|_{L^{2,p}(\mathbb{R}^2)} \approx \inf\left\{\|F\|_{L^{2,p}(\mathbb{R}^2)} : F = f \text{ on } E,\ J_{E'}F = L\right\}$.
(3) $\|\widehat{F}\|_{L^{2,p}(\mathbb{R}^2)} \approx M(f, L^\sharp)$, *where*

$$M(f, L^\sharp)^p := \sum_{\nu=1}^K M_\nu(f_\nu, L_\nu)^p + \sum_{\nu \leftrightarrow \nu'} \left[|\nabla L_\nu - \nabla L_{\nu'}|^p \delta_\nu^{2-p} + |(L_\nu - L_{\nu'})(x^\sharp_{\mu(\nu)})|^p \delta_\nu^{2-2p}\right].$$

Proof. Let $f : E \to \mathbb{R}$ and $L^\sharp \in \mathrm{Wh}(E^\sharp)$ be given. We set $L := \mathrm{ext}(L^\sharp)$. We define $F_\nu := T_\nu(f_\nu, L_\nu)$ and $M_\nu := M_\nu(f_\nu, L_\nu)$ for $\nu = 1, \dots, K$. By definition,

(1) $F_\nu = f_\nu$ on E_ν,
(2) $J_{x_\nu} F_\nu = L_\nu$,
(3) $M_\nu \approx \|F_\nu\|_{L^{2,p}(\widetilde{Q}_\nu)} \approx \inf\left\{\|F\|_{L^{2,p}(\widetilde{Q}_\nu)} : F = f_\nu \text{ on } E_\nu,\ J_{x_\nu} F = L_\nu\right\}$.

Pick a partition of unity:

(1) $\sum_{\nu=1}^K \theta_\nu \equiv 1$ on Q°,

where

(2) $0 \le \theta_\nu \le 1$,
(3) $\theta_\nu \equiv 1$ on $0.9Q_\nu$,

(4) $\operatorname{supp}(\theta_\nu) \subseteq 1.1Q_\nu$,
(5) $|\partial^\alpha \theta_\nu| \leq C\delta_\nu^{-|\alpha|}$ for $|\alpha| \leq 2$.

We set

$$\overline{F} := \sum_\nu \theta_\nu F_\nu \in L^{2,p}(Q^\circ).$$

Applying (P1)–(P4) of θ_ν, (P1)–(P2) of F_ν, $E_\nu \subseteq 1.1Q_\nu$, and $x_\nu \in 0.9Q_\nu$, we see that

$$\overline{F} = f \text{ on } E, \text{ and } J_{E'}\overline{F} = L. \tag{5.16}$$

Moreover, $\overline{F}$ depends linearly on $(f, L^\sharp)$.

Now we estimate $\|\overline{F}\|_{L^{2,p}(Q^\circ)}$. Note that

$$\sum_\nu \partial^\alpha \theta_\nu(x) = \partial^\alpha \left(\sum_\nu \theta_\nu\right)(x) = \frac{\partial^\alpha}{\partial x^\alpha} 1 = 0$$

for $x \in Q^\circ$ and any nonzero multi-index α.

Let $\nu' \in \{1, \ldots, K\}$. By the previous identity,

$$\nabla^2 \overline{F} = \sum_\nu \nabla^2 F_\nu \cdot \theta_\nu + 2\sum_\nu \nabla(F_\nu - F_{\nu'}) \otimes \nabla\theta_\nu + \sum_\nu (F_\nu - F_{\nu'}) \cdot \nabla^2\theta_\nu, \tag{5.17}$$

where $(v^1 \otimes v^2)_{ij} := (v_i^1 v_j^2 + v_j^1 v_i^2)/2$ denotes the symmetrized tensor product.

If $x \in \operatorname{supp}(\theta_\nu) \cap Q_{\nu'}$, then $x \in \widetilde{Q}_\nu \cap Q_{\nu'}$, due to (P4) of θ_ν, and thus $\nu \leftrightarrow \nu'$ by Good Geometry; moreover, this may occur for at most $C(p)$ distinct indices ν. Thus, from (5.17),

$$\begin{aligned} \|\nabla^2 \overline{F}\|^p_{L^p(Q_{\nu'})} \lesssim & \sum_{\nu : \nu \leftrightarrow \nu'} \|\nabla^2(F_\nu)\theta_\nu\|^p_{L^p(Q_{\nu'})} \\ & + \sum_{\nu : \nu \leftrightarrow \nu'} \|\nabla(F_\nu - F_{\nu'}) \otimes \nabla(\theta_\nu)\|^p_{L^p(Q_{\nu'})} \\ & + \sum_{\nu : \nu \leftrightarrow \nu'} \|(F_\nu - F_{\nu'})\nabla^2(\theta_\nu)\|^p_{L^p(Q_{\nu'})}. \end{aligned} \tag{5.18}$$

Let ν be such that $\nu \leftrightarrow \nu'$. From (P4), (P5) of θ_ν and (P3) of F_ν, we have

$$\|\nabla^2(F_\nu)\theta_\nu\|_{L^p(Q_{\nu'})} \leq C\|\nabla^2(F_\nu)\|_{L^p(\widetilde{Q}_\nu)} \approx M_\nu. \tag{5.19}$$

From the Sobolev theorem, (P2), (P3) of F_ν and (P4), (P5) of θ_ν, we estimate

$$\begin{aligned} \|\nabla(F_\nu - F_{\nu'}) \otimes \nabla\theta_\nu\|^p_{L^p(Q_{\nu'})} & \lesssim \|\nabla(L_\nu - L_{\nu'}) \otimes \nabla\theta_\nu\|^p_{L^p(Q_{\nu'})} \\ & \quad + \|\nabla(F_\nu - L_\nu) \otimes \nabla\theta_\nu\|^p_{L^p(Q_{\nu'})} + \|\nabla(L_{\nu'} - F_{\nu'}) \otimes \nabla\theta_\nu\|^p_{L^p(Q_{\nu'})} \\ & \lesssim \delta_\nu^{2-p}|\nabla L_\nu - \nabla L_{\nu'}|^p \\ & \quad + \delta_\nu^{-p}\|\nabla F_\nu - \nabla L_\nu\|^p_{L^p(\widetilde{Q}_\nu)} + \delta_\nu^{-p}\|\nabla L_{\nu'} - \nabla F_{\nu'}\|^p_{L^p(\widetilde{Q}_{\nu'})} \\ & \lesssim \delta_\nu^{2-p}|\nabla L_\nu - \nabla L_{\nu'}|^p + \|F_\nu\|^p_{L^{2,p}(\widetilde{Q}_\nu)} + \|F_{\nu'}\|^p_{L^{2,p}(\widetilde{Q}_{\nu'})} \\ & \approx \delta_\nu^{2-p}|\nabla L_\nu - \nabla L_{\nu'}|^p + M_\nu^p + M_{\nu'}^p. \end{aligned} \tag{5.20}$$

As in (5.20), we estimate

$$\begin{aligned}(5.21)\qquad &\|(F_\nu - F_{\nu'})\nabla^2(\theta_\nu)\|^p_{L^p(Q_{\nu'})}\\ &\lesssim \delta_\nu^{-2p}\|L_\nu - L_{\nu'}\|^p_{L^p(Q_{\nu'})} + \|F_\nu\|^p_{L^{2,p}(\widetilde{Q}_\nu)} + \|F_{\nu'}\|^p_{L^{2,p}(\widetilde{Q}_{\nu'})}\\ &\lesssim \delta_\nu^{2-2p}|L_\nu(x_\nu) - L_{\nu'}(x_\nu)|^p + \delta_\nu^{2-p}|\nabla L_\nu - \nabla L_{\nu'}|^p + M_\nu^p + M_{\nu'}^p,\end{aligned}$$

where the second inequality is a consequence of the estimate

$$\|L_\nu - L_{\nu'}\|^p_{L^p(Q_{\nu'})} \lesssim \delta_\nu^2|L_\nu(x_\nu) - L_{\nu'}(x_\nu)|^p + \delta_\nu^{2+p}|\nabla L_\nu - \nabla L_{\nu'}|^p.$$

Combining (5.18), (5.19), (5.20) and (5.21) yields

$$\begin{aligned}\|\nabla^2\overline{F}\|^p_{L^p(Q_{\nu'})} \lesssim &\sum_{\nu:\nu\leftrightarrow\nu'} M_\nu^p\\ &+ \sum_{\nu:\nu\leftrightarrow\nu'}\left[|\nabla L_\nu - \nabla L_{\nu'}|^p\delta_\nu^{2-p} + |L_\nu(x_\nu) - L_{\nu'}(x_\nu)|^p\delta_\nu^{2-2p}\right].\end{aligned}$$

We now sum the above inequality over $\nu' \in \{1, 2, \ldots, K\}$ and use that $\{Q_{\nu'}\}$ partitions Q°. Thus, $\|\overline{F}\|_{L^{2,p}(Q^\circ)} \lesssim \Xi$, where

(5.22)

$$\Xi^p := \sum_\nu M_\nu^p + \sum_{\nu\leftrightarrow\nu'}\left[|\nabla L_\nu - \nabla L_{\nu'}|^p\delta_\nu^{2-p} + |L_\nu(x_\nu) - L_{\nu'}(x_\nu)|^p\delta_\nu^{2-2p}\right].$$

Here, we use that for each fixed $\nu' \in \{1, \ldots, K\}$, there are at most C indices $\nu \in \{1, \ldots, K\}$ such that $\nu \leftrightarrow \nu'$. Thus, each pair (ν, ν') with $\nu \leftrightarrow \nu'$ is counted at most $2C$ times in the sum $\sum_{\nu'}\sum_{\nu:\nu\leftrightarrow\nu'}$.

Now we extend $\overline{F} \in L^{2,p}(Q^\circ)$ to a function $\widehat{F} \in L^{2,p}(\mathbb{R}^2)$ without changing the function values on $0.99Q^\circ$ or increasing the seminorm by more than a constant factor.

We pick $\theta \in C_c^\infty(Q^\circ)$ that satisfies

(1) $\theta \equiv 1$ on $0.99Q^\circ$,
(2) $|\partial^\alpha\theta| \lesssim \delta_{Q^\circ}^{-|\alpha|}$ for $|\alpha| \leq 2$.

Fix an arbitrary point $x_0 \in \frac{1}{2}Q^\circ$, and set $L_0 := J_{x_0}\overline{F}$. We define

$$\widehat{F} := \theta\overline{F} + (1-\theta)L_0 = \theta(\overline{F} - L_0) + L_0.$$

Clearly, $\widehat{F}$ depends linearly on $(f, L^\sharp)$. Since $E \subseteq 0.99Q^\circ$ and $E' \subseteq 0.99Q^\circ$ (recall Lemma 4.4), from (5.16) and (P1) of θ we have

$$\widehat{F} = f \text{ on } E \quad\text{and}\quad J_{E'}\widehat{F} = L. \tag{5.23}$$

Since $\mathrm{supp}(\theta) \subseteq Q^\circ$, from (P2) of θ and the Sobolev theorem, we see that

$$\begin{aligned}(5.24)\qquad \|\widehat{F}\|_{L^{2,p}(\mathbb{R}^2)} &= \|\theta(\overline{F} - L_0)\|_{L^{2,p}(Q^\circ)} \lesssim \|(\overline{F} - L_0)\nabla^2\theta\|_{L^p(Q^\circ)} \\ &\quad + \|\nabla(\overline{F} - L_0) \otimes \nabla\theta\|_{L^p(Q^\circ)} + \|\nabla^2(\overline{F} - L_0)\theta\|_{L^p(Q^\circ)} \\ &\lesssim \|\overline{F}\|_{L^{2,p}(Q^\circ)} \lesssim \Xi.\end{aligned}$$

Take an arbitrary $F \in L^{2,p}(\mathbb{R}^2)$ such that $F = f$ on E and $J_{E'}F = L$. By applying (P3) of F_ν and then the bounded intersection property of $\{\widetilde{Q}_\nu\}$, we see that

$$\sum_\nu M_\nu^p \approx \sum_\nu \|F_\nu\|^p_{L^{2,p}(\widetilde{Q}_\nu)} \lesssim \sum_\nu \|F\|^p_{L^{2,p}(\widetilde{Q}_\nu)} \lesssim \|F\|^p_{L^{2,p}(\mathbb{R}^2)}.$$

In addition, the Sobolev theorem (see Remark 3.1) implies that

$$\begin{aligned}&\sum_{\nu \leftrightarrow \nu'} \left[|\nabla L_\nu - \nabla L_{\nu'}|^p \delta_\nu^{2-p} + |L_\nu(x_\nu) - L_{\nu'}(x_\nu)|^p \delta_\nu^{2-2p}\right] \\ &\qquad \lesssim \sum_{\nu \leftrightarrow \nu'} \|F\|^p_{L^{2,p}(\widetilde{Q}_\nu \cup \widetilde{Q}_{\nu'})} \lesssim \sum_\nu \|F\|^p_{L^{2,p}(\widetilde{Q}_\nu)} \lesssim \|F\|^p_{L^{2,p}(\mathbb{R}^2)}.\end{aligned}$$

Here, the second and third "$\lesssim$" follow from the bounded intersection property of $\{\widetilde{Q}_\nu\}$. Thus we have shown that $\Xi \lesssim \|F\|_{L^{2,p}(\mathbb{R}^2)}$. Therefore, in conjunction with (5.23) and (5.24), we have that

$$(5.25)\qquad \|\widehat{F}\|_{L^{2,p}(\mathbb{R}^2)} \approx \inf\left\{\|F\|_{L^{2,p}(\mathbb{R}^2)} : F = f \text{ on } E,\ \ J_{E'}F = L\right\} \approx \Xi.$$

Notice that $|x_\nu - x^\sharp_{\mu(\nu)}| \lesssim \delta_\nu$, thanks to Corollary 4.1. Thus, for each $\nu \in \{1, \dots, K\}$, we have

$$|[L_\nu - L_{\nu'}](x_\nu)| \approx |[L_\nu - L_{\nu'}](x^\sharp_{\mu(\nu)})| + |\nabla L_\nu - \nabla L_{\nu'}| \cdot \delta_\nu.$$

Inserting this equation in the definition for Ξ in (5.22) completes the proof of Proposition 5.3. □

Corollary 5.1. *The functional M from the conclusion of Proposition 5.3 satisfies*

$$\|f\|_{L^{2,p}(\mathbb{R}^2)|_E} \approx \inf\left\{M(f, L^\sharp) : L^\sharp \in \mathrm{Wh}(E^\sharp)\right\} \quad \textit{for any } f : E \to \mathbb{R}.$$

Proof. Note that

$$\begin{aligned}&\|f\|_{L^{2,p}(\mathbb{R}^2)|_E} \\ &\quad \approx \inf\left\{\inf\left\{\|F\|_{L^{2,p}(\mathbb{R}^2)} : F = f \text{ on } E,\ J_{E'}F = \mathrm{ext}(L^\sharp)\right\} : L^\sharp \in \mathrm{Wh}(E^\sharp)\right\}.\end{aligned}$$

Indeed, the $\lesssim$ direction is trivial. To verify the $\gtrsim$ direction, we take $F = \widehat{F}$ from Lemma 5.1 and $L^\sharp = J_{E^\sharp}\widehat{F}$. Then $J_{E'}F = \mathrm{ext}(L^\sharp)$ because $\widehat{F}$ satisfies the constant-path property.

Thanks to the conditions satisfied by M in Proposition 5.3, the result follows immediately. □

6. Determining the optimal Whitney field

Let $f : E \to \mathbb{R}$ be given. In this section we solve for $L^\sharp \in \mathrm{Wh}(E^\sharp)$ that depends linearly on f and nearly minimizes the expression $M(f, L^\sharp)$ from Proposition 5.3. This minimization problem relates to the trace seminorm through Corollary 5.1.

LEMMA 6.1. *There exists $\overline{a} = \overline{a}(p) > 0$ such that the following holds. Let $S \subseteq 0.9Q$ and $z \in \frac{1}{2}Q$ be given with $\mathrm{dist}(z, S) \geq \frac{1}{100}\delta_Q$. Suppose that Q satisfies $\mathrm{R}(c, c', c'')$ relative to S for some constants $c, c' > 0$ and $0 < c'' < \overline{a}$. Then there exists a linear map $T_1 : L^{2,p}(Q)|_S \to \mathcal{P}$ such that*

$$T_1(\overline{f}) \in \Gamma_Q\big(\overline{f}, z, \widehat{C} \cdot \|\overline{f}\|_{L^{2,p}(Q)|_S}\big) \quad \textit{for all } \overline{f} : S \to \mathbb{R},$$

where $\widehat{C}$ depends only on c' and p.

Proof. By rescaling we may assume that $\delta_Q = 1$. We choose the universal constant $\overline{a} > 0$ later in the proof. Let $\overline{f} : S \to \mathbb{R}$ be given. Since Q satisfies $\mathrm{R}(c, c', c'')$ relative to S, there are two cases to consider.

Case 1. There exist $x_1, x_2 \in S$ and $y_1, y_2 \in S$, with $x_1 \neq x_2$ and $y_1 \neq y_2$, such that the vectors $v_1 = \frac{x_1 - x_2}{|x_1 - x_2|}$ and $v_2 = \frac{y_1 - y_2}{|y_1 - y_2|}$ satisfy

$$\min\big\{|v_1 - v_2|, |v_1 + v_2|\big\} > c'.$$

In this case, take the matrix $M_1 \in \mathbb{R}^{2\times 2}$ whose columns are v_1 and v_2. The above condition implies that the entries of $M_2 := M_1^{-1}$ are bounded by some constant $\overline{C} = \overline{C}(c') \geq 1$.

We set $m_1 := \frac{\overline{f}(x_1) - \overline{f}(x_2)}{|x_1 - x_2|}$ and $m_2 := \frac{\overline{f}(y_1) - \overline{f}(y_2)}{|y_1 - y_2|}$, and we define the vector $A := (m_1, m_2)M_2$.

Define the affine polynomial $L_1(x) := A \cdot (x - x_1) + \overline{f}(x_1)$. We set $T_1(\overline{f}) := L_1$. We now show that L_1 belongs to the appropriate $\Gamma_Q(\cdots)$.

Pick some function $F \in L^{2,p}(Q)$ with

(1) $F = \overline{f}$ on S,
(2) $\|F\|_{L^{2,p}(Q)} \leq 2\|\overline{f}\|_{L^{2,p}(Q)|_S}$.

By the mean value theorem, there exist $x^*, y^* \in Q$ such that $v_1 \cdot \nabla F(x^*) = m_1$ and $v_2 \cdot \nabla F(y^*) = m_2$. Since $z \in Q$ and $\delta_Q = 1$, the Sobolev theorem implies that

$$|v_1 \cdot \nabla F(z) - m_1| \lesssim \|F\|_{L^{2,p}(Q)} \text{ and } |v_2 \cdot \nabla F(z) - m_2| \lesssim \|F\|_{L^{2,p}(Q)}.$$

In other terms, $|\nabla F(z) \cdot M_1 - (m_1, m_2)| \lesssim \|F\|_{L^{2,p}(Q)}$. Multiplying by the matrix M_2, we obtain

$$|\nabla F(z) - \nabla L_1| = |\nabla F(z) - A| = |\nabla F(z) - (m_1, m_2)M_2| \lesssim \overline{C}\|F\|_{L^{2,p}(Q)}.$$

Thus, since $F(x_1) = \overline{f}(x_1)$, the Sobolev theorem implies that

$$\begin{aligned}|L_1(z) - F(z)| &= |F(x_1) + A \cdot (z - x_1) - F(z)| \\ &\leq |A \cdot (z - x_1) - \nabla F(z) \cdot (z - x_1)| + |J_z F(x_1) - F(x_1)| \\ &\lesssim \overline{C} \|F\|_{L^{2,p}(Q)}.\end{aligned}$$

Therefore,

$$\begin{aligned}\|J_z F - L_1\|_{W^{2,p}(Q)} &\approx |(J_z F - L_1)(z)| \cdot \delta_Q^{2/p-2} + |\nabla F(z) - \nabla L_1| \cdot \delta_Q^{2/p-1} \\ &\lesssim \overline{C} \|F\|_{L^{2,p}(Q)}.\end{aligned}$$

Now choose $\theta_1 \in C_c^\infty(B(z, \frac{1}{100}))$ such that

(1) $\theta_1 \equiv 1$ on $B(z, \frac{1}{200})$,
(2) $\|\theta_1\|_{C^2} \lesssim 1$.

Note that $\theta_1 = 0$ on S (because $\operatorname{dist}(z, S) \geq \frac{1}{100}$).

Define $G := F + \theta_1 \cdot (L_1 - J_z F)$, which satisfies

(1) $G = \overline{f} + 0 = \overline{f}$ on S, $J_z G = J_z F + (L_1 - J_z F) = L_1$;
(2) $\|G\|_{L^{2,p}(Q)} \lesssim \|F\|_{L^{2,p}(Q)} + \|J_z F - L_1\|_{W^{2,p}(Q)} \lesssim \overline{C} \|F\|_{L^{2,p}(Q)}$ $\leq 2\overline{C} \|\overline{f}\|_{L^{2,p}(Q)|_S}$.

Thus,

$$L_1 \in \Gamma_Q(\overline{f}, z, \widehat{C} \|\overline{f}\|_{L^{2,p}(Q)|_S}) \quad \text{for some } \widehat{C} = \widehat{C}(c', p).$$

This completes the proof of the lemma in the first case.

Case 2: $\|S\|_{\dot{B}_p} \leq c''$, where $0 < c'' < \overline{a}$.

We take $\overline{a} = \widehat{a}$, where $\widehat{a}$ comes from Proposition 5.2. Therefore, Q, S and z satisfy the hypotheses of Proposition 5.2; hence, there exist linear functionals $\lambda_1, \ldots, \lambda_J$ such that

$$\sum_{j=1}^{J} |\lambda_j(\overline{f}, P)|^p \approx \inf\left\{\|F\|_{L^{2,p}(Q)}^p : F = \overline{f} \text{ on } S, \ \ J_z F = P\right\} \tag{6.1}$$

for any $\overline{f} : S \to \mathbb{R}$ and $P \in \mathcal{P}$. We now consider the problem of minimizing the expression from (6.1) with respect to $P \in \mathcal{P}$ for fixed $\overline{f} : S \to \mathbb{R}$.

Expand $P \in \mathcal{P}$ in coordinates as $P(u, v) = qu + rv + s$ for $q, r, s \in \mathbb{R}$. We abuse notation and write $\lambda_j(\overline{f}, q, r, s)$ for $\lambda_j(\overline{f}, P)$. Let $K(\overline{f}, q, r, s) := \sum_{j=1}^J |\lambda_j(\overline{f}, q, r, s)|^p$.

We apply the following claim, which is an elementary consequence of Hölder's inequality.

Main Claim. *Let* $\vec{\beta} = (\beta_i)_{j=1}^J \in \mathbb{R}^J$. *Define the linear functional* $\omega : \mathbb{R}^J \to \mathbb{R}$ *by*

$$\omega(z) := \frac{1}{\|\vec{\beta}\|_{l^p}^p} \sum_{j=1}^{J} \frac{|\beta_j|^p}{\beta_j} z_j \quad \textit{when } \vec{\beta} \neq 0,$$

and $\omega(z) := 0$ *when* $\vec{\beta} = 0$. *Then*

(6.2)
$$\sum_{j=1}^{J} |z_j - \beta_j \omega(z)|^p \leq C \cdot \inf\Big\{\sum_{j=1}^{J} |z_j - \beta_j a|^p : a \in \mathbb{R}\Big\} \quad \textit{for some constant } C = C(p).$$

We apply the Main Claim to compute $\overline{s} \in \mathbb{R}$ (depending linear on $(\overline{f}, q, r)$) such that $K_1(\overline{f}, q, r) := K(\overline{f}, q, r, \overline{s})$ is near-minimal for fixed $(\overline{f}, q, r)$. Then we apply the Main Claim twice again, to compute $\overline{r} \in \mathbb{R}$ such that $K_2(\overline{f}, q) := K_1(\overline{f}, q, \overline{r})$ is near-minimal for fixed $(\overline{f}, q)$ and to compute $\overline{q} \in \mathbb{R}$ such that $K_2(\overline{f}, \overline{q})$ is near-minimal for fixed $\overline{f}$.

Thus, the polynomial $P_1(u, v) := \overline{q}u + \overline{r}v + \overline{s}$ satisfies

$$\sum_{j=1}^{J} |\lambda_j(\overline{f}, P_1)|^p \approx \inf\left\{\sum_{j=1}^{J} |\lambda_j(\overline{f}, P)|^p : P \in \mathcal{P}\right\}.$$

Moreover, P_1 depends linearly on $\overline{f}$. Inserting (6.1) on both sides above, we see that

$$\inf\Big\{\|F\|_{L^{2,p}(Q)} : F = \overline{f} \text{ on } S \ \text{ and } \ J_z F = P_1\Big\} \approx \|\overline{f}\|_{L^{2,p}(Q)|_S}.$$

In other terms,

$$P_1 \in \Gamma_Q\left(\overline{f}, z, C_0 \|\overline{f}\|_{L^{2,p}(Q)|_S}\right) \quad \text{for some } C_0 = C_0(p).$$

This completes the proof of Lemma 6.1. $\square$

LEMMA 6.2. *Let* $Q \subseteq \mathbb{R}^2$, $S \subseteq 0.9Q$ *and* $z \in \frac{1}{2}Q$ *satisfy the hypotheses of Proposition* 5.2. *Then*

$$M_Q(0, L)^p \leq C \cdot (|L(z)|^p \delta_Q^{2-2p} + |\nabla L|^p \delta_Q^{2-p}) \quad \textit{for any } \ L \in \mathcal{P}.$$

Moreover, for any $a > 0$, *if* $\|S\|_{\dot{B}_p} \geq a\delta_Q^{1-2/p}$, *then*

$$|L(z)|^p \delta_Q^{2-2p} + |\nabla L|^p \delta_Q^{2-p} \leq C_0 \cdot M_Q(0, L)^p \quad \textit{for any } \ L \in \mathcal{P}.$$

Here, the constant C *depends only on* p, *while* C_0 *depends only on* a *and* p.

Proof. By rescaling, we may assume that $\delta_Q = 1$. Fix $L \in \mathcal{P}$ for the remainder of the proof. Recall that

$$M_Q(0, L) \approx \inf\Big\{\|h\|_{L^{2,p}(Q)} : h = 0 \text{ on } S, \ J_z h = L\Big\} \quad \text{(see Remark 5.1)}. \tag{6.3}$$

Choose $\theta \in C_c^\infty(B(z, \frac{1}{150}))$ such that $\theta \equiv 1$ on $B(z, \frac{1}{200})$ and $\|\theta\|_{C^2} \lesssim 1$. Note that $\theta = 0$ on S, because $\operatorname{dist}(z, S) \geq \frac{1}{100}$.

We set $h := \theta \cdot L$. Note that $J_z h = L$ and $h = 0$ on S. A straightforward computation shows that $\|h\|_{L^{2,p}(Q)}^p \lesssim |L(z)|^p + |\nabla L|^p$. Thus, from (6.3) we obtain

$$M_Q(0, L)^p \lesssim |L(z)|^p + |\nabla L|^p. \tag{6.4}$$

This proves the first inequality.

For the second conclusion, let $a > 0$ be given and suppose that $\|S\|_{\dot{B}_p} \geq a$. Applying (6.3), we find $h \in L^{2,p}(Q)$ such that

(1) $J_z h = L$,
(2) $h = 0$ on S,
(3) $\|h\|_{L^{2,p}(Q)} \lesssim M_Q(0, L)$.

We now show that $M_Q(0, L) \neq 0$. For the sake of contradiction, suppose that $M_Q(0, L) = 0$. Thus, h is affine (thanks to (P3) of h); hence, S is contained in a line (thanks to (P2) of h). However, this contradicts that $\|S\|_{\dot{B}_p} > 0$.

Pick an arbitrary point $\overline{z} \in S$. The Sobolev theorem implies that

$$|L(\overline{z})| = |J_z h(\overline{z}) - h(\overline{z})| \lesssim \|h\|_{L^{2,p}(Q)} \lesssim M_Q(0, L).$$

Thus, since $z, \overline{z} \in Q$ and $\delta_Q = 1$,

$$|L(z)| \lesssim |L(\overline{z})| + |\nabla L| \lesssim M_Q(0, L) + |\nabla L|. \tag{6.5}$$

Next, we prove that $|\nabla L| \leq Z(a, p) M_Q(0, L)$. Assume for the sake of contradiction that

$$|\nabla L| > Z \cdot M_Q(0, L) \quad \text{for some large parameter } Z. \tag{6.6}$$

By the Sobolev theorem and (P1), (P3) of h, we see that

$$|\nabla h(\overline{z})| \geq |\nabla h(z)| - C\|h\|_{L^{2,p}(Q)} \geq |\nabla L| - C_1 M_Q(0, L) > (Z - C_1) \cdot M_Q(0, L).$$

Here, C_1 is some universal constant. Therefore, the function $\overline{h}(x) := h(x) \cdot [(Z - C_1) \cdot M_Q(0, L)]^{-1}$ satisfies

(1) $\overline{h} = 0$ on S,
(2) $|\nabla \overline{h}(\overline{z})| \geq 1$,
(3) $\|\overline{h}\|_{L^{2,p}(Q)} \lesssim (Z - C_1)^{-1}$.

Applying Proposition 3.4 for some large enough choice of Z, we see that the curve $\gamma := \left\{x \in 0.9Q : \overline{h}(x) = 0\right\}$ satisfies $\|\gamma\|_{\dot{B}_p} \leq a/2$. Since $S \subseteq \gamma$, we have $\|S\|_{\dot{B}_p} \leq \|\gamma\|_{\dot{B}_p} \leq a/2$. This contradicts $\|S\|_{\dot{B}_p} \geq a$, proving that (6.6) cannot hold. Together with (6.5), this proves the second conclusion of the lemma. □

We denote $E_\mu^\sharp := 9Q_\mu^\sharp \cap E$ for each keystone square $Q_\mu^\sharp$. By (K2) from Proposition 4.1, we see that $9Q_\mu^\sharp$ satisfies $\mathrm{R}(c_1, c_2, c_3)$ relative to the subset $E_\mu^\sharp$. Therefore,

$$10Q_\mu^\sharp \text{ satisfies } \mathrm{R}\left(c_1 \cdot (9/10)^{2/p-1}, c_2, c_3 \cdot (9/10)^{2/p-1}\right) \text{ relative to } E_\mu^\sharp. \tag{6.7}$$

The keystone representative point $x_\mu^\sharp$ defined in Section 4.3 satisfies

$$x_\mu^\sharp \in \frac{1}{2} Q_\mu^\sharp \text{ and } \mathrm{dist}(x_\mu^\sharp, E_\mu^\sharp) \geq \mathrm{dist}(x_\mu^\sharp, E) \geq \frac{1}{5}\delta_{Q_\mu^\sharp} \geq \frac{1}{100}\delta_{10Q_\mu^\sharp}.$$

We apply Lemma 6.1 (for small enough c_3), which provides a linear map $T^\sharp_\mu : L^{2,p}(10Q^\sharp_\mu)|_{E^\sharp_\mu} \to \mathcal{P}$ such that $\widetilde{L}^\sharp_\mu := T^\sharp_\mu(f|_{E^\sharp_\mu})$ satisfies

$$\widetilde{L}^\sharp_\mu \in \Gamma_{10Q^\sharp_\mu}\left(f|_{E^\sharp_\mu}, x^\sharp_\mu, \widehat{C}\cdot \|f|_{E^\sharp_\mu}\|_{L^{2,p}(10Q^\sharp_\mu)|_{E^\sharp_\mu}}\right) \text{ for some constant } \widehat{C} = \widehat{C}(c_2, p).$$

Finally, we set $\widetilde{L}^\sharp = (\widetilde{L}^\sharp_\mu)_{\mu=1}^{K^\sharp}$ and $\widetilde{L} = \mathrm{ext}(\widetilde{L}^\sharp)$.

Lemma 6.3. *There exists $\overline{C} = \overline{C}(c_1, c_2, c_3, p)$ such that*

$$M(f, \widetilde{L}^\sharp) \le \overline{C}\cdot M(f, L^\sharp) \ \textit{for every}\ L^\sharp \in \mathrm{Wh}(E^\sharp).$$

Proof. For an arbitrary $L^\sharp \in \mathrm{Wh}(E^\sharp)$, we set $L := \mathrm{ext}(L^\sharp)$. We manipulate the expression for $M(f, \widetilde{L}^\sharp)$ from Proposition 5.3. We first apply the approximate subadditivity of $M_\nu = M_{\widetilde{Q}_\nu}$ (see Remark 5.1) and the triangle inequality to obtain

$$\begin{aligned} M(f, \widetilde{L}^\sharp)^p \lesssim & \sum_\nu M_\nu(f_\nu, L_\nu)^p \\ & + \sum_{\nu\leftrightarrow\nu'} \left[|\nabla L_\nu - \nabla L_{\nu'}|^p \delta_\nu^{2-p} + |(L_\nu - L_{\nu'})(x^\sharp_{\mu(\nu)})|^p \delta_\nu^{2-2p}\right] \\ & + \sum_\nu M_\nu(0, L_\nu - \widetilde{L}_\nu)^p \\ & + \sum_{\nu\leftrightarrow\nu'} \left[|\nabla L_\nu - \nabla \widetilde{L}_\nu|^p \delta_\nu^{2-p} + |(L_\nu - \widetilde{L}_\nu)(x^\sharp_{\mu(\nu)})|^p \delta_\nu^{2-2p}\right] \\ & + \sum_{\nu\leftrightarrow\nu'} |(L_{\nu'} - \widetilde{L}_{\nu'})(x^\sharp_{\mu(\nu)})|^p \delta_\nu^{2-2p}. \end{aligned}$$

Next, substitute the expression for $M(f, L^\sharp)$ and apply Lemma 6.2. Thus,

(6.8)
$$\begin{aligned} M(f, \widetilde{L}^\sharp)^p \lesssim M(f, L^\sharp)^p + & \sum_{\nu\leftrightarrow\nu'} \left[|\nabla L_\nu - \nabla \widetilde{L}_\nu|^p \delta_\nu^{2-p} + |(L_\nu - \widetilde{L}_\nu)(x^\sharp_{\mu(\nu)})|^p \delta_\nu^{2-2p}\right] \\ & + \sum_\nu |(L_\nu - \widetilde{L}_\nu)(x_\nu)|^p \delta_\nu^{2-2p} + \sum_{\nu\leftrightarrow\nu'} |(L_{\nu'} - \widetilde{L}_{\nu'})(x^\sharp_{\mu(\nu)})|^p \delta_\nu^{2-2p}. \end{aligned}$$

Note that the last two sums in (6.8) are bounded by the first sum. This follows because $|x_\nu - x^\sharp_{\mu(\nu)}| \le C\delta_\nu$ and $|x^\sharp_{\mu(\nu')} - x^\sharp_{\mu(\nu)}| \le C\delta_\nu$ whenever $\nu \leftrightarrow \nu'$ (thanks to Corollary 4.1 and the Good Geometry of the CZ squares). Thus, $M(f, \widetilde{L}^\sharp)^p \lesssim M(f, L^\sharp)^p + X$, where

$$X := \sum_\nu \left[|\nabla L_\nu - \nabla \widetilde{L}_\nu|^p \delta_\nu^{2-p} + |(L_\nu - \widetilde{L}_\nu)(x^\sharp_{\mu(\nu)})|^p \delta_\nu^{2-2p}\right].$$

Since $L = \mathrm{ext}(L^\sharp)$,

$$X = \sum_\mu \left[|\nabla L^\sharp_\mu - \nabla \widetilde{L}^\sharp_\mu|^p \sum_{\nu:\mu(\nu)=\mu} \delta_\nu^{2-p} + |L^\sharp_\mu(x^\sharp_\mu) - \widetilde{L}^\sharp_\mu(x^\sharp_\mu)|^p \sum_{\nu:\mu(\nu)=\mu} \delta_\nu^{2-2p}\right]. \tag{6.9}$$

From (5.1), we have $Q^\sharp_\mu \subset CQ_\nu$ for $\mu = \mu(\nu)$. Thus, for any $\varepsilon > 0$,

$$\sum_{\nu:\mu(\nu)=\mu} \delta_\nu^{-\varepsilon} \leq \sum \left\{\delta_Q^{-\varepsilon} : Q \text{ dyadic}, \ Q^\sharp_\mu \subseteq CQ\right\} \lesssim (\delta^\sharp_\mu)^{-\varepsilon}.$$

Hence, in particular for $\varepsilon = p-2$ and $\varepsilon = 2p-2$, from (6.9) we have

$$X \lesssim \sum_\mu \left[|\nabla L^\sharp_\mu - \nabla \widetilde{L}^\sharp_\mu|^p (\delta^\sharp_\mu)^{2-p} + |L^\sharp_\mu(x^\sharp_\mu) - \widetilde{L}^\sharp_\mu(x^\sharp_\mu)|^p (\delta^\sharp_\mu)^{2-2p}\right]. \tag{6.10}$$

Note that $\|E^\sharp_\mu\|_{\dot{B}_p} \geq \hat{c}(c_1,c_2,c_3,p)\delta^{2/p-1}_{Q^\sharp_\mu}$, thanks to (6.7) and Lemma 4.3. By the second part of Lemma 6.2 and the approximate subadditivity of $M_{10Q^\sharp_\mu}$ (see Remark 5.1), following (6.10) we see that

$$\begin{aligned} X &\lesssim C' \sum_\mu M_{10Q^\sharp_\mu}(0|_{E^\sharp_\mu}, L^\sharp_\mu - \widetilde{L}^\sharp_\mu)^p \\ &\lesssim C' \sum_\mu \left[M_{10Q^\sharp_\mu}(f|_{E^\sharp_\mu}, L^\sharp_\mu)^p + M_{10Q^\sharp_\mu}(f|_{E^\sharp_\mu}, \widetilde{L}^\sharp_\mu)^p\right] \end{aligned} \tag{6.11}$$

for some constant $C' = C'(c_1,c_2,c_3,p)$.

Recall that

$$\widetilde{L}^\sharp_\mu \in \Gamma_{10Q^\sharp_\mu}(f|_{E^\sharp_\mu}, x^\sharp_\mu, \widehat{C}\|f|_{E^\sharp_\mu}\|) \quad \text{for some } \widehat{C} = \widehat{C}(c_2,p).$$

Thus, by Proposition 5.2,

$$\begin{aligned} M_{10Q^\sharp_\mu}(f|_{E^\sharp_\mu}, \widetilde{L}^\sharp_\mu) &\approx \inf\left\{\|F\|_{L^{2,p}(10Q^\sharp_\mu)} : F = f \text{ on } E^\sharp_\mu, \ \ J_{x^\sharp_\mu} F = \widetilde{L}^\sharp_\mu\right\} \\ &\leq \widehat{C} \cdot \inf\left\{\|F\|_{L^{2,p}(10Q^\sharp_\mu)} : F = f \text{ on } E^\sharp_\mu, \ \ J_{x^\sharp_\mu} F = L^\sharp_\mu\right\} \\ &\approx \widehat{C} \cdot M_{10Q^\sharp_\mu}(f|_{E^\sharp_\mu}, L^\sharp_\mu). \end{aligned}$$

Therefore, from (6.11) we obtain

$$X \lesssim C'\widehat{C} \sum_\mu \inf\left\{\|F\|^p_{L^{2,p}(10Q^\sharp_\mu)} : F = f \text{ on } E^\sharp_\mu, \ \ J_{x^\sharp_\mu} F = L^\sharp_\mu\right\}.$$

Since $\{10Q^\sharp_\mu\}_{\mu=1}^{K^\sharp}$ have the bounded intersection property (recall (K3)), we thus have

$$X \leq \overline{C} \inf\{\|F\|_{L^{2,p}(\mathbb{R}^2)} : F = f \text{ on } E \ \text{ and } J_{E'}F = \text{ext}(L^\sharp)\} \approx \overline{C} \cdot M(f, L^\sharp)^p$$

for some constant $\overline{C} = \overline{C}(c_1,c_2,c_3,p)$. This completes the proof of Lemma 6.3. □

7. Proof of Theorem 1

The CZ decomposition Λ is either the trivial decomposition $\{Q^\circ\}$ or some nontrivial decomposition. We consider these cases separately below.

Case 1. $\Lambda \neq \{Q^\circ\}$.

Let $f : E \to \mathbb{R}$ be given. We apply Lemma 6.3 (and Corollary 5.1) to choose $\widetilde{L}^\sharp \in \mathrm{Wh}(E^\sharp)$ that depends linearly on f and satisfies

$$M(f, \widetilde{L}^\sharp) \leq C \inf\left\{M(f, L^\sharp) : L^\sharp \in \mathrm{Wh}(E^\sharp)\right\} \approx \|f\|_{L^{2,p}(\mathbb{R}^2)|_E}. \tag{7.1}$$

We set $\widetilde{L} := \mathrm{ext}(\widetilde{L}^\sharp)$.

Take T and M as in Proposition 5.3. We define $T(f) := T(f, \widetilde{L}^\sharp)$ and $M(f) := M(f, \widetilde{L}^\sharp)$. From (7.1) and Proposition 5.3, we have

(1) $T(f) = f$ on E,
(2) $\|T(f)\|_{L^{2,p}(\mathbb{R}^2)} \approx M(f) \approx \|f\|_{L^{2,p}(\mathbb{R}^2)|_E}$.

We now estimate the number of terms used in the expression that defines $M(f)$. From Proposition 5.3 and because $\widetilde{L} = \mathrm{ext}(\widetilde{L}^\sharp)$, we have

$$\begin{aligned} M(f, \widetilde{L}^\sharp)^p = & \sum_{\nu=1}^{K} M_\nu(f_\nu, \widetilde{L}_\nu)^p \\ & + \sum_{\mu,\mu'=1}^{K^\sharp} \left[|\nabla \widetilde{L}^\sharp_\mu - \nabla \widetilde{L}^\sharp_{\mu'}|^p \Delta_{1,\mu\mu'} + |\widetilde{L}^\sharp_\mu(x^\sharp_\mu) - \widetilde{L}^\sharp_{\mu'}(x^\sharp_\mu)|^p \Delta_{2,\mu\mu'}\right], \end{aligned}$$

where

$$\begin{aligned} \Delta_{k,\mu\mu'} := \sum \Big\{\delta_\nu^{2-kp} : \exists\, \nu, \nu' \in \{1, \dots, K\} & \\ \text{such that } \nu' \leftrightarrow \nu,\ \mu(\nu) = \mu,\ \mu(\nu') = \mu' \Big\} & \text{ for } k = 1, 2. \end{aligned}$$

Since $\widetilde{L}_\nu$ depends linearly on f, from Proposition 5.2 we have

$$M_\nu(f_\nu, \widetilde{L}_\nu)^p = \sum_{k=1}^{N_\nu} |\lambda_k^\nu(f)|^p \qquad \text{for each } \nu,$$

for linear functionals λ_k^ν, where $N_\nu \lesssim (\#E_\nu)^2$. Therefore,

$$\begin{aligned} M(f, \widetilde{L}^\sharp)^p = & \sum_{\nu=1}^{K} \sum_{k=1}^{N_\nu} |\lambda_k^\nu(f)|^p \\ & + \sum_{\mu,\mu'=1}^{K^\sharp} \left[|\nabla \widetilde{L}^\sharp_\mu - \nabla \widetilde{L}^\sharp_{\mu'}|^p \Delta_{1,\mu\mu'} + |\widetilde{L}^\sharp_\mu(x^\sharp_\mu) - \widetilde{L}^\sharp_{\mu'}(x^\sharp_\mu)|^p \Delta_{2,\mu\mu'}\right]. \end{aligned} \tag{7.2}$$

The number of terms appearing in the first sum in (7.2) is

$$\sum_\nu N_\nu \lesssim \sum_\nu \#(E_\nu)^2 = \sum_\nu \#(E \cap 1.1Q_\nu)^2 \lesssim \#(E)^2,$$

where we have used that $\{1.1Q_\nu\}$ has the bounded intersection property.

Note that $E \cap 9Q^\sharp$ is nonempty for each keystone square $Q^\sharp$, thanks to (K2) and the definition of property $\mathrm{R}(\cdots)$. Thus we may assign to each keystone square $Q^\sharp$ some point $y^\sharp \in E \cap 9Q^\sharp$. Since $\{10Q^\sharp : Q^\sharp \in \Lambda^\sharp\}$ has the bounded intersection property (see (K3)), the preimage of each $y \in E$ has bounded cardinality. Therefore, $K^\sharp = \#(\Lambda^\sharp) \lesssim \#(E) = N$. It follows that the second sum from (7.2) contains at most CN^2 terms.

We have shown that the sum that defines $M(f)^p$ contains at most CN^2 terms. This completes the proof of Theorem 1 in the case that $\Lambda \neq \{Q^\circ\}$.

Case 2. $\Lambda = \{Q^\circ\}$.

In this case Q° is OK, meaning that $\|E\|_{\dot{B}_p} \leq c_1 \delta_{Q^\circ}^{2/p-1}$. Since $E \subseteq \frac{1}{10}Q^\circ$, we can pick $z \in 0.9Q^\circ$ such that $\mathrm{dist}(z, E) \geq \frac{1}{10}\delta_{Q^\circ}$.

Let $f : E \to \mathbb{R}$ and $P \in \mathcal{P}$ be given. For small enough c_1, Proposition 5.2 implies the existence of a linear map $T : L^{2,p}(Q^\circ)|_E \times \mathcal{P} \to L^{2,p}(Q^\circ)$ such that

(1) $T(f, P) = f$ on E;
(2) $J_z T(f, P) = P$;
(3) $\|T(f, P)\|_{L^{2,p}(Q^\circ)} \approx \inf\{\|F\|_{L^{2,p}(Q^\circ)} : F = f$ on E, $J_z F = P$;
(4) $\|T(f, P)\|_{L^{2,p}(Q^\circ)} \approx M(f, P)$, where

$$M(f, P)^p := \sum_{i=1}^{N_1} |\lambda_i(f, P)|^p$$

for linear functionals $\lambda_1, \ldots, \lambda_{N_1}$, where $N_1 \lesssim (\#E)^2$.

We reason as in the paragraph containing (5.23) from the proof of Proposition 5.3, and extend $T(f, P) \in L^{2,p}(Q^\circ)$ to a function $\overline{T}(f, P) \in L^{2,p}(\mathbb{R}^2)$ without increasing the seminorm by more than a constant factor or disturbing the function values on $0.9Q^\circ$. Thus,

(1) $\overline{T}(f, P) = f$ on E;
(2) $J_z \overline{T}(f, P) = P$;
(3) $\|\overline{T}(f, P)\|_{L^{2,p}(\mathbb{R}^2)} \approx \inf\{\|F\|_{L^{2,p}(\mathbb{R}^2)} : F = f$ on E, $J_z F = P\}$;
(4) $\|\overline{T}(f, P)\|_{L^{2,p}(\mathbb{R}^2)} \approx M(f, P)$.

We apply the Main Claim from the proof of Lemma 6.1 to the expression $M(f, P)$. Thus, there exists $\overline{P} \in \mathcal{P}$ depending linearly on f such that $M(f, \overline{P}) \lesssim M(f, P)$ for all $P \in P$. Define $Tf := \overline{T}(f, \overline{P})$, $M(f) := M(f, \overline{P})$, and $\lambda_i(f) := \lambda_i(f, \overline{P})$. Then

(1) $Tf = f$ on E,
(2) $\|Tf\|_{L^{2,p}(\mathbb{R}^2)} \approx \inf_{P \in \mathcal{P}} \inf\{\|F\|_{L^{2,p}(\mathbb{R}^2)} : F = f$ on E, $J_z F = P\} = \|f\|_{L^{2,p}(\mathbb{R}^2)|_E}$,
(3) $\|f\|_{L^{2,p}(\mathbb{R}^2)|_E} \approx M(f)$, where $M(f)^p = \sum_{i=1}^{N_1} |\lambda_i(f)|^p$ and $N_1 \lesssim (\#E)^2$.

This proves Theorem 1 in the case that $\Lambda = \{Q^\circ\}$. $\square$

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