Annals of Mathematics **178** (2013), 107–182
http://dx.doi.org/10.4007/annals.2013.178.1.2

Traveling waves for nonlinear Schrödinger equations with nonzero conditions at infinity

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Dedicated to Jean-Claude Saut, who gave me water to cross the desert

Abstract

For a large class of nonlinear Schrödinger equations with nonzero conditions at infinity and for any speed c less than the sound velocity, we prove the existence of nontrivial finite energy traveling waves moving with speed c in any space dimension $N \geq 3$. Our results are valid as well for the Gross-Pitaevskii equation and for NLS with cubic-quintic nonlinearity.

1. Introduction

We consider the nonlinear Schrödinger equation

$$i\frac{\partial \Phi}{\partial t} + \Delta\Phi + F(|\Phi|^2)\Phi = 0 \qquad \text{in } \mathbf{R}^N, \tag{1.1}$$

where $\Phi : \mathbf{R}^N \times \mathbf{R} \longrightarrow \mathbf{C}$ satisfies the "boundary condition" $|\Phi| \longrightarrow r_0$ as $|x| \longrightarrow \infty$, $r_0 > 0$ and F is a real-valued function on $\mathbf{R}_+$ satisfying $F(r_0^2) = 0$.

Equations of the form (1.1), with the considered nonzero conditions at infinity, arise in a large variety of physical problems. They have been used as models for superconductivity, superfluid Helium II and for Bose-Einstein condensation ([2], [3], [4], [16], [25], [28], [31], [33], [32]). In nonlinear optics, they appear in the context of dark solitons, which are localized nonlinear waves (also called "holes") moving on a stable continuous background (see [36], [44]). The boundary condition $|\Phi| \longrightarrow r_0$ at infinity is precisely due to the nonzero background.

Two important particular cases of (1.1) have been extensively studied by physicists and by mathematicians: the Gross-Pitaevskii (GP) equation (where $F(s) = 1 - s$) and the so-called "cubic-quintic" Schrödinger equation (where $F(s) = -\alpha_1 + \alpha_3 s - \alpha_5 s^2$, α_1, α_3, α_5 are positive and F has two positive roots). In both cases we have $F'(r_0^2) < 0$, which ensures that (1.1) is defocusing.

The boundary condition $|\Phi| \longrightarrow r_0 > 0$ at infinity makes the structure of solutions of (1.1) much more complicated than in the usual case of zero boundary conditions (when the associated dynamics is essentially governed by dispersion and scattering). This fact is confirmed by the existence of a remarkable variety of special solutions, such as traveling waves or vortex solutions, and regimes, like the long wave or the transonic limit.

Using the Madelung transformation $\Phi(x,t) = \sqrt{\rho(x,t)}e^{i\theta(x,t)}$ (which is well defined whenever $\Phi \neq 0$), equation (1.1) is equivalent to a system of Euler's equations for a compressible inviscid fluid of density ρ and velocity $2\nabla\theta$. In this context it has been shown that, if F is C^1 near r_0^2 and $F'(r_0^2) < 0$, the sound velocity at infinity associated to (1.1) is $v_s = r_0\sqrt{-2F'(r_0^2)}$; see the introduction of [42].

In the defocusing case $F'(r_0^2) < 0$, we perform a simple scaling ($\Phi(x,t) = r_0\tilde{\Phi}(\tilde{x},\tilde{t})$, where $\tilde{x} = r_0\sqrt{-F'(r_0^2)}x$, $\tilde{t} = -r_0^2F'(r_0^2)t$, and $\tilde{F}(s) = \frac{-1}{r_0^2F'(r_0^2)}F(r_0^2 s)$) and we assume from now on that $r_0 = 1$ and $F'(r_0^2) = -1$. The sound velocity at infinity then becomes $v_s = \sqrt{2}$.

Equation (1.1) is Hamiltonian. Denoting $V(s) = \int_s^1 F(\tau)\,d\tau$, it is easy to see that, at least formally, the "energy"

$$E(\Phi) = \int_{\mathbf{R}^N} |\nabla\Phi|^2\,dx + \int_{\mathbf{R}^N} V(|\Phi|^2)\,dx \tag{1.2}$$

is a conserved quantity.

A second conserved quantity for (1.1) is the momentum

$$P(\Phi) = (P_1(\Phi), \ldots, P_N(\Phi)),$$

which describes the evolution of the center of mass of Φ. Assuming that $\Phi \longrightarrow 1$ at infinity in a suitable sense and denoting by $\langle\cdot,\cdot\rangle$ the scalar product in $\mathbf{C}$, the momentum is formally given by

$$P_k(\Phi) = \int_{\mathbf{R}^N} \langle i\frac{\partial\Phi}{\partial x_k}, \Phi - 1\rangle\,dx. \tag{1.3}$$

Traveling waves and the Roberts programme. In a series of papers (see, e.g., [2], [3], [25], [31], [32], [33]), particular attention has been paid to a special class of solutions of (1.1), namely the traveling waves. These are solutions of the form $\Phi(x,t) = \psi(x - cty)$, where $y \in S^{N-1}$ is the direction of propagation and $c > 0$ is the speed of the traveling wave. We say that ψ has finite energy if $\nabla\psi \in L^2(\mathbf{R}^N)$ and $V(|\psi|^2) \in L^1(\mathbf{R}^N)$. These solutions are supposed to play an important role in the dynamics of (1.1). In view of formal computations and numerical experiments, a list of conjectures, often referred to as *the Roberts programme*, has been formulated about the existence, the qualitative properties, the stability of traveling waves and, more generally, their role in the dynamics of (1.1).

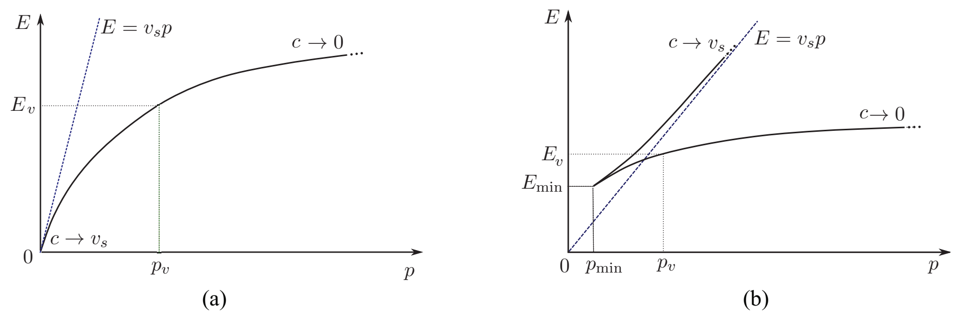

Figure 1. Energy (E) momentum (p) diagrams for (GP): (a) in dimension 2; (b) in dimension 3.

Let ψ be a finite energy traveling wave of (1.1) moving with speed c. Without loss of generality we may assume that $y = (1, 0, \ldots, 0)$. If $N \geq 3$, it follows that $\psi - z_0 \in L^{2^*}(\mathbf{R}^N)$ for some constant $z_0 \in \mathbf{C}$, where $2^* = \frac{2N}{N-2}$ (see, e.g., Lemma 7 and Remark 4.2 in [21, pp. 774–775]). Since $|\psi| \longrightarrow 1$ as $|x| \longrightarrow \infty$, necessarily $|z_0| = 1$. If Φ is a solution of (1.1) and $\alpha \in \mathbf{R}$, then $e^{i\alpha}\Phi$ is also a solution. Hence we may assume that $z_0 = 1$; thus $\psi - 1 \in L^{2^*}(\mathbf{R}^N)$. Let $u = \psi - 1$. We say that u has finite energy if ψ does so. Then u satisfies the equation

$$(1.4) \qquad -ic\frac{\partial u}{\partial x_1} + \Delta u + F(|1+u|^2)(1+u) = 0 \qquad \text{in } \mathbf{R}^N.$$

It is obvious that a function u satisfies (1.4) for some velocity c if and only if $u(-x_1, x')$ satisfies (1.4) with c replaced by $-c$. Hence it suffices to consider the case $c > 0$. This assumption will be made throughout the paper.

For the Gross-Pitaevskii equation, C. A. Jones, C. J. Putterman and P. H. Roberts computed the energy and the momentum of the traveling waves they had found numerically. In space dimension two and three, they obtained the curves given in Figure 1.

Formally, traveling waves are critical points of the energy E when the momentum (with respect to the direction of propagation Ox_1) is fixed, say $P_1 = p$. Equation (1.4) is precisely the Euler-Lagrange equation associated to this variational problem, and the speed c is the Lagrange multiplier. Note also that, formally, $c = \frac{\partial E}{\partial p}$.

The first conjecture in the Roberts programme asserts that finite energy traveling waves of speed c exist if and only if $|c| < v_s$.

In space dimension $N = 1$, in many interesting applications equation (1.4) can be integrated explicitly and one obtains traveling waves for all subsonic speeds. The nonexistence of such solutions for supersonic speeds has also been proven under general conditions (cf. Theorem 5.1 in [42, p. 1099]).

Despite many attempts, a rigorous proof of the existence of traveling waves in higher dimensions has been a long lasting problem. In the particular case of the Gross-Pitaevskii (GP) equation, this problem was considered in a series of papers. In space dimension $N = 2$, the existence of traveling waves was proven in [8] for all speeds in some interval $(0, \varepsilon)$, where ε is small. In space dimension $N \geq 3$, the existence was proven in [7] for a sequence of speeds $c_n \longrightarrow 0$ by using constrained minimization; a similar result was established in [13] for all sufficiently small speeds by using a mountain-pass argument. In [5], the existence of traveling waves for (GP) was proven in space dimension $N = 2$ and $N = 3$ by minimizing the energy at fixed momentum. The propagation speed is then the Lagrange multiplier associated to minimizers. If $N = 2$, this gives solutions for any speed in a set $A \subset (0, v_s)$, where A contains points arbitrarily close to 0 and to v_s (although it is not clear that $A = (0, v_s)$). It was shown later in [14] that the minimization of the energy at fixed momentum can be used in any dimension $N \geq 2$ for general nonlinearities such that the nonlinear potential V appearing in the energy is nonnegative. Moreover, the set of solutions that it gives is orbitally stable. However, this method has two disadvantages. Firstly, it is not clear that the set of speeds of traveling waves constructed in this way form an interval. Secondly, it was proved in [5] and [37] that in space dimension $N \geq 3$ there exists $v_0 \in (0, v_s)$ such that minimizing the energy at fixed momentum cannot give traveling waves of speed $c \in (v_0, v_s)$. In particular, the "upper branch" in Figure 1(b) cannot be obtained in this way.

In the case of cubic-quintic type nonlinearities, it was proved in [41] that traveling waves exist for any sufficiently small speed if $N \geq 4$.

To our knowledge, even for specific nonlinearities there are no existence results in the literature that cover the whole range $(0, v_s)$ of possible speeds.

The nonexistence of traveling waves for supersonic speeds ($c > v_s$) was proven in [26] in the case of the Gross-Pitaevskii equation, respectively in [42] for a large class of nonlinearities.

It is the aim of this paper to prove the existence of nontrivial finite energy traveling waves of (1.1) in space dimension $N \geq 3$, under general conditions on the nonlinearity F and for any speed $c \in (0, v_s)$.

The qualitative properties of traveling waves have been extensively investigated. It turns out that these solutions have the best regularity allowed by the nonlinearity F (see, for instance, [18], [19], [42]). It was proved in [5] that the traveling waves to the (GP) equation are analytic functions. In view of formal computations, Jones, Putterman and Roberts ([32]) predicted the asymptotic behavior of traveling waves as $|x| \longrightarrow \infty$. For the (GP) equation, the asymptotics have been computed by P. Gravejat (see [27] and references therein). It is likely that the proofs of Gravejat can be adapted to general nonlinearities.

Even for specific nonlinearities (such as (GP)), the vortex structure of traveling waves is not yet completely understood. It was conjectured in [33], [32] that there is a critical speed c_v (corresponding to the energy E_v and momentum p_v) such that traveling waves of speed less than c_v present vortices, while those of speed greater than c_v do not. The small velocity solutions to (GP) constructed in [8], [7], [13] have vortices. For other nonlinearities, the behavior may be different. For instance, small speed traveling waves constructed in [41] in the case of nonlinearities of cubic-quintic type do not have vortices. We suspect that traveling waves constructed in the present paper develop vortices in the limit $c \longrightarrow 0$ if and only if (1.1) does not admit finite energy stationary solutions. For general nonlinearities, it was recently proved in [15] that traveling waves do not have vortices if c is close to v_s and $N \in \{2, 3\}$.

The energy-momentum diagrams for (GP) suggest that there are traveling waves of arbitrarily small energy and momentum in dimension two. Such solutions were obtained in [5] by minimizing the energy at fixed (and small) momentum; their velocities are close to v_s. A similar result holds for general nonlinearities; see [14]. A scattering theory for small energy solutions to (1.1) in dimension two is therefore excluded.

The situation is completely different in higher dimensions. It was noticed in [32] that the energy and the momentum of the three-dimensional traveling waves for (GP) are bounded from below by positive constants $E_{\min}$ and $p_{\min}$, respectively. It was proved in [5] that the three-dimensional (GP) equation does not admit small energy traveling waves, and the proof was later extended to higher dimensions in [37]. It turns out that this result is true for general nonlinearities: for any $N \geq 3$, there is $k_N > 0$ such that any traveling wave U to (1.1) satisfying $\|\nabla U\|_{L^2(\mathbf{R}^N)} < k_N$ must be constant (see [14, Prop. 1.4]). This result can be further improved in dimension $N \geq 6$ (see [15, Prop. 18]). Moreover, S. Gustafson, K. Nakanishi and T.-P. Tsai [29], [30] established a scattering theory of small solutions to (GP) in dimension $N \geq 4$ (in the energy space) and $N = 3$ (in some weighted space).

In view of formal computations, Jones, Putterman and Roberts conjectured that after a suitable rescaling, the modulus and the phase of traveling waves converge in the transonic limit $c \longrightarrow v_s$ to the solitary waves of the Kadomtsev-Petviashvili I (KP-I) equation. The present paper is the first to provide finite energy traveling waves to (1.1) of speed close to v_s in dimension $N \geq 3$. Very recently it was proved that for general nonlinearities, the three-dimensional traveling waves found here have modulus close to 1 (thus can be lifted) and their phase and modulus tend, after rescaling, to ground states of the KP-I equation (see [15], Theorem 6). Precise estimates on their energy and momentum have also been established in [15] and are in full agreement with those in [33] and [32]. Hence the conjecture concerning the existence of the "upper branch" of travelling waves has been proven in dimension three (with one exception: it is not clear that we have a continuum of solutions). Quite unexpectedly, a similar asymptotic behavior of traveling waves in the transonic limit cannot be true in dimension $N \geq 4$ (cf. [15, Prop. 19]).

A much more difficult problem is to understand the stability of traveling waves and, more generally, their role in the dynamics of (1.1). The guess formulated in [32] is that the two-dimensional traveling waves represented in the momentum-energy diagram in Figure 1 should be stable. In the three-dimensional case solutions on the "lower branch" should be stable, while those on the "upper branch" should be unstable. It is also suggested in [32, p. 3000] that a solution of (GP) starting in a neighborhood of the upper branch could eventually "collapse" onto the lower branch, generating "sound waves that radiate the excess energy. . . to infinity."

Before even speaking of stability, one has to understand the well-posedness of the Cauchy problem associated to (1.1). Important progress has been achieved in this direction during the last years; we refer to the survey paper [22] (see also [20]). It was proved in [21], [22] that in the subcritical case $N \in \{1, 2, 3\}$, the Cauchy problem for (GP) is globally well-posed for all initial data in the energy space, and that in the critical case $N = 4$ it is globally well-posed for initial data with small energy. The method in [21], [22] adapts to other nonlinearities, including the cubic-quintic case. (Note that the cubic-quintic NLS becomes critical in dimension three.) Global well-posedness of the four-dimensional (GP) and of the three-dimensional cubic-quintic NLS for all initial data in the energy space has been recently proven in [35].

In dimension one, traveling waves to (GP) are known explicitly. Their orbital stability has been studied and proven in [39], [6], [23]. Other nonlinearities are also considered in [39]. The asymptotic stability of these solutions is not known.

If the nonlinear potential V is nonnegative, traveling waves can be obtained by minimizing the energy at fixed momentum. Moreover, all minimizing sequences are precompact, and this gives the orbital stability of the set of solutions constructed in this way (cf. [14, Th. 6.2]). For the (GP) equation, the results in [14] imply the orbital stability of the full branch of traveling waves in dimension 2 and of traveling waves situated on the "lower branch" below the line $E = v_s p$ in dimension 3. If V changes sign, a local minimization of the energy at fixed momentum is still possible in dimension 2 and gives a branch of orbitally stable traveling waves. To our knowledge, the orbital stability/instability of solutions corresponding to the "upper branch" in dimension 3 as well as the asymptotic stability of traveling waves in any dimension $N \geq 2$ are still open problems.

Main results. We will consider the following set of assumptions:

(A1) The function F is continuous on $[0, \infty)$, C^1 in a neighborhood of 1, $F(1) = 0$ and $F'(1) = -1$.

(A2) There exist $C > 0$ and $p_0 < \frac{2}{N-2}$ such that $|F(s)| \leq C(1 + s^{p_0})$ for any $s \geq 0$.

(A3) There exist C, $\alpha_0 > 0$ and $r_* > 1$ such that $F(s) \leq -Cs^{\alpha_0}$ for any $s \geq r_*$.

Our main results can be summarized as follows.

THEOREM 1.1. *Assume that* $N \geq 3$, $0 < c < v_s$ *and conditions* (A1) *and* (A2) *are satisfied. Then equation* (1.4) *admits a nontrivial finite energy solution* u. *Moreover*, $u \in W^{2,p}_{\rm loc}(\mathbf{R}^N)$ *for any* $p \in [1, \infty)$ *and, after a translation,* u *is axially symmetric with respect to* Ox_1.

COROLLARY 1.2. *Suppose that* $N \geq 3$, $0 < c < v_s$ *and conditions* (A1) *and* (A3) *are verified. Then equation* (1.4) *admits a nontrivial finite energy solution* u *such that* $u \in W^{2,p}_{\rm loc}(\mathbf{R}^N)$ *for any* $p \in [1, \infty)$ *and, after a translation,* u *is axially symmetric with respect to* Ox_1.

It is easy to see how Corollary 1.2 follows from Theorem 1.1. Indeed, suppose that Theorem 1.1 holds. Assume that (A1) and (A3) are satisfied. Let C, r_*, α_0 be as in (A3). There exist $\beta \in (0, \frac{2}{N-1})$, $\tilde{r} > r_*$ and $C_1 > 0$ such that

$$Cs^{2\alpha_0} - \frac{1}{2} \geq C_1(s - \tilde{r})^{2\beta} \qquad \text{for any } s \geq \tilde{r}.$$

Let $\tilde{F}$ be a function with the following properties: $F = \tilde{F}$ on $[0, 4\tilde{r}^2]$, $\tilde{F}(s) = -C_2 s^\beta$ for s sufficiently large, and $\tilde{F}(s^2)+\frac{1}{2} \leq -C_3(s-\tilde{r})^{2\beta}$ for any $s \geq \tilde{r}$, where C_2, C_3 are some positive constants. Then $\tilde{F}$ satisfies (A1), (A2), (A3) and Theorem 1.1 implies that equation (1.4) with $\tilde{F}$ instead of F has a nontrivial finite energy solution u. From the proof of Proposition 2.2(i) in [42, pp. 1079–1080] it follows that any such solution satisfies $|1+u|^2 \leq 2\tilde{r}^2$ and, consequently, $F(|1 + u|^2) = \tilde{F}(|1 + u|^2)$. Thus u satisfies (1.4).

We have to mention that the traveling waves in Theorem 1.1 are obtained as minimizers of the functional $E + cP_1$ under a Pohozaev constraint (see below), where E is the energy and P_1 is the momentum with respect to x_1. Of course, if (A1) and (A3) are satisfied but (A2) does not hold, we do not claim that the solutions given by Corollary 1.2 still solve the same minimization problem. In fact, assumptions (A1) and (A3) alone do not imply that E is well defined on a convenient function space.

In particular, for $F(s) = 1 - s$, conditions (A1) and (A3) are satisfied. It follows that the Gross-Pitaevskii equation admits nontrivial traveling waves of finite energy in any space dimension $N \geq 3$ and for any speed $c \in (0, v_s)$ (although (A2) is not true for $N > 3$: the (GP) equation is critical if $N = 4$, and supercritical if $N \geq 5$). A similar result holds for the cubic-quintic NLS.

Notation and function spaces. Throughout the paper, $\mathcal{L}^N$ is the Lebesgue measure on $\mathbf{R}^N$ and $\omega_N = \mathcal{L}^N(B(0, 1))$ is the Lebesgue measure of the unit ball. For $x = (x_1, \ldots, x_N) \in \mathbf{R}^N$, we denote $x' = (x_2, \ldots, x_N) \in \mathbf{R}^{N-1}$. We write $\langle z_1, z_2\rangle$ for the scalar product of two complex numbers z_1, z_2. Given a function f defined on $\mathbf{R}^N$ and λ, $\sigma > 0$, we denote by

$$f_{\lambda,\sigma} = f\left(\frac{x_1}{\lambda}, \frac{x'}{\sigma}\right) \tag{1.5}$$

the dilations of f. The behavior of functions and of functionals with respect to dilations in $\mathbf{R}^N$ will be very important. For $1 \le p < N$, we denote by p^* the Sobolev exponent associated to p; that is, $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{N}$.

If (A1) is satisfied, let $V(s) = \int_s^1 F(\tau)\,d\tau$. Then the sound velocity at infinity associated to (1.1) is $v_s = \sqrt{2}$ and using Taylor's formula for s in a neighborhood of 1, we have

$$V(s) = \frac{1}{2}V''(1)(s-1)^2 + (s-1)^2\varepsilon(s-1) = \frac{1}{2}(s-1)^2 + (s-1)^2\varepsilon(s-1), \tag{1.6}$$

where $\varepsilon(t) \longrightarrow 0$ as $t \longrightarrow 0$. Hence $V(|\psi|^2)$ can be approximated by $\frac{1}{2}(|\psi|^2-1)^2$ for $|\psi|$ close to 1.

We fix an odd function $\varphi \in C^\infty(\mathbf{R})$ such that $\varphi(s) = s$ for $s \in [0,2]$, $0 \le \varphi' \le 1$ on $\mathbf{R}$ and $\varphi(s) = 3$ for $s \ge 4$. If assumptions (A1) and (A2) are satisfied, it is not hard to see that there exists $C_1 > 0$ such that

$$\begin{aligned} &|V(s)| \le C_1(s-1)^2 \quad \text{for any } s \le 9; \\ &\text{in particular, } |V(\varphi^2(\tau))| \le C_1(\varphi^2(\tau)-1)^2 \text{ for any } \tau. \end{aligned} \tag{1.7}$$

Given $u \in H^1_{\mathrm{loc}}(\mathbf{R}^N)$ and an open set $\Omega \subset \mathbf{R}^N$, the modified Ginzburg-Landau energy of u in Ω is defined by

$$E^{\Omega}_{\mathrm{GL}}(u) = \int_\Omega |\nabla u|^2\,dx + \frac{1}{2}\int_\Omega \left(\varphi^2(|1+u|)-1\right)^2\,dx. \tag{1.8}$$

We simply write $E_{\mathrm{GL}}(u)$ instead of $E^{\mathbf{R}^N}_{\mathrm{GL}}(u)$. The modified Ginzburg-Landau energy will play a central role in our analysis. We consider the function space

$$\begin{aligned} \mathcal{X} &= \{u \in \mathcal{D}^{1,2}(\mathbf{R}^N) \mid \varphi^2(|1+u|)-1 \in L^2(\mathbf{R}^N)\} \\ &= \{u \in \dot{H}^1(\mathbf{R}^N) \mid u \in L^{2^*}(\mathbf{R}^N),\ E_{\mathrm{GL}}(u) < \infty\}, \end{aligned} \tag{1.9}$$

where $\mathcal{D}^{1,2}(\mathbf{R}^N)$ is the completion of $C_c^\infty(\mathbf{R}^N)$ for the norm $\|v\| = \|\nabla v\|_{L^2}$.

Since $\varphi^2(|1+u|)-1 = (\varphi(|1+u|)+1)(\varphi(|1+u|)-1)$ and $1 \le \varphi(|1+u|)+1 \le 4$, it is obvious that $\varphi^2(|1+u|)-1 \in L^2(\mathbf{R}^N)$ if and only if $\varphi(|1+u|)-1 \in L^2(\mathbf{R}^N)$. Let $N \ge 3$. We claim that for $u \in \mathcal{D}^{1,2}(\mathbf{R}^N)$, there holds $\varphi(|1+u|)-1 \in L^2(\mathbf{R}^N)$ if and only if $|1+u|-1 \in L^2(\mathbf{R}^N)$. Indeed, if $|1+u| \le 2$, then $\varphi(|1+u|) = |1+u|$. If $|1+u| > 2$, then necessarily $|u| > 1$ and

$$0 \le |1+u| - \varphi(|1+u|) < |1+u| < 2|u| < 2|u|^{\frac{2^*}{2}}.$$

For $N \geq 3$, we have $\mathcal{D}^{1,2}(\mathbf{R}^N) \subset L^{2^*}(\mathbf{R}^N)$ by the Sobolev embedding; hence $|u|^{\frac{2^*}{2}} \in L^2(\mathbf{R}^N)$ and the claim follows. We have thus proved that

$$\mathcal{X} = \{u \in \mathcal{D}^{1,2}(\mathbf{R}^N) \mid |1+u| - 1 \in L^2(\mathbf{R}^N)\}.$$

If $N \geq 3$ and (A1), (A2) are satisfied, it is not hard to see that the function $\psi = 1 + u$ satisfies $\nabla\psi \in L^2(\mathbf{R}^N)$ and $V(|\psi|^2) \in L^1(\mathbf{R}^N)$ if and only if $u \in \mathcal{X}$ (see Lemma 4.1 below). Note that for $N = 3$, $\mathcal{X}$ is *not* a vector space. However, in any space dimension, we have $H^1(\mathbf{R}^N) \subset \mathcal{X}$. If $u \in \mathcal{X}$, it is easy to see that for any $w \in H^1(\mathbf{R}^N)$ with compact support, we have $u + w \in \mathcal{X}$. For $N = 3, 4$, it can be proved that $u \in \mathcal{D}^{1,2}(\mathbf{R}^N)$ belongs to $\mathcal{X}$ if and only if $|1+u|^2 - 1 \in L^2(\mathbf{R}^N)$, and consequently $\mathcal{X}$ coincides with the space F_1 introduced by P. Gérard in [21, §4].

Some ideas in the proofs and outline of the paper. At least formally, the solutions of (1.4) are critical points of the functional

$$E_c(u) = E(u) + cQ(u) = \int_{\mathbf{R}^N} |\nabla u|^2\,dx + cQ(u) + \int_{\mathbf{R}^N} V(|1+u|^2)\,dx, \tag{1.10}$$

where $Q = P_1$ is the momentum with respect to the x_1-direction.

It is the aim of Section 2 to give a convenient definition of the momentum on the whole space $\mathcal{X}$ and to study its basic properties. For now, the formal definition (1.3) is sufficient.

The existence of finite energy traveling waves has been conjectured for all subsonic speeds, and the nonexistence of such solutions is known for all supersonic speeds (at least under some additional technical assumptions; see [42]). Thus it is important to understand what changes in the structure of E_c as c crosses the sound velocity.

If $0 < c < \sqrt{2}$, we may choose ε, $\delta > 0$ such that $c < \sqrt{2}(1-2\varepsilon)(1-\delta)$. Assume that $u \in \mathcal{X}$ satisfies $1 - \delta \leq |1+u| \leq 1 + \delta$. Then there is a lifting $1 + u = \rho e^{i\theta}$, and a simple computation shows that

$$|\nabla u|^2 = |\nabla\rho|^2 + \rho^2|\nabla\theta|^2, \qquad Q(u) = -\int_{\mathbf{R}^N} (\rho^2 - 1)\frac{\partial\theta}{\partial x_1}\,dx$$

and

$$V(|1+u|^2) = V(\rho^2) = \frac{1}{2}(\rho^2-1)^2 + o((\rho^2-1)^2) \geq \frac{1-\varepsilon}{2}(\rho^2-1)^2$$

provided that δ is sufficiently small. Then we have

$$\begin{aligned} |cQ(u)| &\leq \sqrt{2}(1-2\varepsilon)(1-\delta)\int_{\mathbf{R}^N} |\rho^2-1| \cdot \left|\frac{\partial\theta}{\partial x_1}\right| dx \\ &\leq (1-2\varepsilon)\int_{\mathbf{R}^N} (1-\delta)^2\left|\frac{\partial\theta}{\partial x_1}\right|^2 + \frac{1}{2}(\rho^2-1)^2\,dx \\ &\leq \int_{\mathbf{R}^N} (1-2\varepsilon)\rho^2|\nabla\theta|^2 + V(\rho^2) - \frac{\varepsilon}{2}(\rho^2-1)^2\,dx \\ &\leq E(u) - \varepsilon E_{\mathrm{GL}}(u). \end{aligned} \tag{1.11}$$

Thus $E_c(u) \geq \varepsilon E_{\rm GL}(u)$ if $|1+u|$ is sufficiently close to 1 in the L^∞ norm. Since $E_{\rm GL}(u)$ measures, in some sense, the closeness of $1+u$ to 1, we would like to establish a similar estimate for all functions with small Ginzburg-Landau energy. However, $E_{\rm GL}(u)$ does not control $\| \, |1+u|-1\|_{L^\infty}$. Moreover, there are functions with arbitrarily small Ginzburg-Landau energy that present small-scale topological "defects" (e.g., dipoles). To get rid of these difficulties we use a procedure of regularization by minimization, which is introduced and studied in Section 3. Given $u \in \mathcal{X}$, we minimize the functional $v \longmapsto E_{\rm GL}(v) + \frac{1}{h^2}\int_{\mathbf{R}^N} \varphi(|v-u|^2)\,dx$ in the set $\{v \in \mathcal{X} \mid v - u \in H^1(\mathbf{R}^N)\}$. It is shown that minimizers exist (but are perhaps not unique) and any minimizer v_h has remarkable properties. For instance,

- $E_{\rm GL}(v_h) \leq E_{\rm GL}(u)$,
- $\|v_h - u\|_{L^2} \longrightarrow 0$ as $h \longrightarrow 0$, and
- $\| \, |1+v_h| - 1\|_{L^\infty}$ can be estimated in terms of h and $E_{\rm GL}(u)$ and is arbitrarily small if $E_{\rm GL}(u)$ is sufficiently small.

In Section 4 we describe the variational framework. Using the above regularization procedure we prove that for any $\varepsilon \in (0, 1-\frac{c}{v_s})$ and for all $u \in \mathcal{X}$ with $E_{\rm GL}(u)$ sufficiently small, there holds $E_c(u) \geq \varepsilon E_{\rm GL}(u)$. Then we show that for all $k > 0$, the functional E_c is bounded on the set $\{u \in \mathcal{X} \mid E_{\rm GL}(u) \leq k\}$. Let

$$E_{c,\min}(k) = \inf\{E_c(u) \mid u \in \mathcal{X},\ E_{\rm GL}(u) = k\}.$$

We prove that for $0 < c < v_s$, the function $E_{c,\min}$ has the following properties:

(i) For any $\varepsilon \in (0, 1-\frac{c}{v_s})$, there is $k_\varepsilon > 0$ such that $E_{c,\min}(k) > \varepsilon k$ for $k \in (0, k_\varepsilon)$.

(ii) $\lim_{k\to\infty} E_{c,\min}(k) = -\infty$.

(iii) For any $k > 0$, we have $E_{c,\min}(k) < k$.

The situation is very different if $c > v_s$: in that case it can be proved that $E_{c,\min}$ is negative and decreasing on $(0, \infty)$.

In order to get critical points of E_c, it is tempting to minimize $E_c(u)$ under the constraint $E_{\rm GL}(u) = k$ or $Q(u) = k$, where k is a constant, and then to search for those k that give minimizers with the associated Lagrange multiplier equal to zero. However, it is well known that it is hard to control the Lagrange multipliers in minimization problems that do not have appropriate scaling or homogeneity properties. In order to avoid that difficulty we adopt the following strategy. We introduce the functionals:

$$A(u) = \int_{\mathbf{R}^N} \sum_{j=2}^{N} \left| \frac{\partial u}{\partial x_j} \right|^2 dx, \tag{1.12}$$

$$B_c(u) = \int_{\mathbf{R}^N} \left| \frac{\partial u}{\partial x_1} \right|^2 dx + cQ(u) + \int_{\mathbf{R}^N} V(|1+u|^2)\,dx, \tag{1.13}$$

$$P_c(u) = \frac{N-3}{N-1} A(u) + B_c(u). \tag{1.14}$$

It is obvious that $E_c(u) = A(u) + B_c(u) = \frac{2}{N-1}A(u) + P_c(u)$. If assumptions (A1) and (A2) above are satisfied, it can be proved (see Proposition 4.1 in [42, pp. 1091–1092]) that any traveling wave $u \in \mathcal{X}$ of (1.1) must satisfy the Pohozaev-type identity $P_c(u) = 0$. Indeed, it is easy to see that for any $u \in \mathcal{X}$, we have $E_c(u_{1,\sigma}) = \sigma^{N-3}A(u) + \sigma^{N-1}B_c(u)$. Formally, a critical point u of E_c should satisfy $\frac{d}{d\sigma}_{|\sigma=1}(E_c(u_{1,\sigma})) = 0$, which gives precisely $P_c(u) = 0$. We will prove the existence of traveling waves by showing that the problem of minimizing E_c in the set

$$\mathcal{C} = \{u \in \mathcal{X} \mid u \neq 0, P_c(u) = 0\}$$

admits solutions. It turns out that minimizing a functional under a Pohozaev constraint (almost) automatically generates critical points of that functional; that is, the Lagrange multiplier is fixed. This is a very general observation which seems to work in many problems in Calculus of Variations. To our knowledge, it is used here for the first time. Let us explain how it works for E_c in dimension $N \geq 4$. Assume that $u \in \mathcal{C}$ satisfies the Euler-Lagrange equation $E'_c(u) = \alpha P'_c(u)$. Then u is a critical point of the functional $E_c - \alpha P_c$. Formally, we have $\frac{d}{d\sigma}_{|\sigma=1}(E_c(u_{1,\sigma}) - \alpha P_c(u_{1,\sigma})) = 0$, which is equivalent to

$$P_c(u) - \alpha\left[\left(\frac{N-3}{N-1}\right)^2 A(u) + B_c(u)\right] = 0.$$

Since $P_c(u) = 0$, the above identity implies $\alpha\frac{N-3}{N-1}\cdot\left(\frac{N-3}{N-1} - 1\right)A(u) = 0$; thus either $A(u) = 0$ (and u is constant), or $\alpha = 0$.

The next step is to prove that $\mathcal{C}$ is not empty and $\inf\{E_c(u) \mid u \in \mathcal{C}\} > 0$. Let us present here the arguments in dimension $N \geq 4$. If $u \in \mathcal{C}$, we have $B_c(u) = -\frac{N-3}{N-1}A(u) < 0$. Then it is easy to see that the function $\sigma \longmapsto E_c(u_{1,\sigma}) = \sigma^{N-3}A(u) + \sigma^{N-1}B_c(u)$ is increasing on $(0, 1)$ and decreasing on $(1, \infty)$; thus it achieves a maximum at $\sigma = 1$. Hence

$$E_c(u) = E_c(u_{1,1}) \geq E_c(u_{1,\sigma}) \geq E_{c,\min}(E_{\mathrm{GL}}(u_{1,\sigma})) \qquad \text{for all } \sigma > 0.$$

Since $\sigma \longmapsto E_{\mathrm{GL}}(u_{1,\sigma})$ takes all values in $(0, \infty)$, we infer that

$$T_c := \inf\{E_c(u) \mid u \in \mathcal{C}\} \geq \sup\{E_{c,\min}(k) \mid k > 0\} > 0.$$

In Section 5 we consider the case $N \geq 4$ and we prove that the functional E_c has minimizers in $\mathcal{C}$ and these minimizers are solutions of (1.4). To show the existence of minimizers we use the concentration-compactness principle and the regularization procedure developed in Section 3. The most difficult part is to show that minimizing sequences do not "vanish"; that is, their Ginzburg-Landau energy does not spread over $\mathbf{R}^N$. Assume that $N \geq 4$ and $(u_n)_{n\geq 1}$

is a minimizing sequence for E_c on $\mathcal{C}$ that "vanishes." Letting $\sigma_0 = \sqrt{\frac{2(N-1)}{N-3}}$ and $\tilde{u}_n = (u_n)_{1,\sigma_0}$, we see that $(\tilde{u}_n)_{n\geq 1}$ also vanishes and $A(\tilde{u}_n) + E_c(\tilde{u}_n) = \sigma_0^{N-1} P_c(u_n) = 0$. Since $A(\tilde{u}_n) = \sigma_0^{N-3} A(u_n)$ and $A(u_n) = \frac{N-1}{2}(E_c(u_n) - P_c(u_n)) \geq \frac{N-1}{2} T_c > 0$, we get

$$\limsup_{n\to\infty} E_c(\tilde{u}_n) < 0. \tag{1.15}$$

On the other hand, the vanishing of $(\tilde{u}_n)_{n\geq 1}$ implies that

$$\int_{\mathbf{R}^N} V(|1+\tilde{u}_n|^2)\,dx = \frac{1}{2}\int_{\mathbf{R}^N} \left(\varphi^2(|1+\tilde{u}_n|) - 1\right)^2\,dx + o(1).$$

Using the regularization procedure in Section 3 (see Lemma 3.2) we construct a sequence $h_n \longrightarrow 0$, and for each n, we find a minimizer v_n of the functional $E_{\mathrm{GL}}(v) + \frac{1}{h_n^2}\int_{\mathbf{R}^N} \varphi(|v - \tilde{u}_n|^2)\,dx$ such that $\|\,|1+v_n| - 1\|_{L^\infty} \longrightarrow 0$ as $n \longrightarrow \infty$. Then we have $Q(\tilde{u}_n) = Q(v_n) + o(1)$ and

$$E_c(\tilde{u}_n) = E_{\mathrm{GL}}(\tilde{u}_n) + cQ(\tilde{u}_n) + o(1) \geq E_{\mathrm{GL}}(v_n) + cQ(v_n) + o(1) \geq 0 \tag{1.16}$$

for all n sufficiently large by (1.11). It is clear that (1.15) and (1.16) give a contradiction, and this rules out vanishing.

If "dichotomy" occurs, the Ginzburg-Landau energy of u_n is located in two regions that are far away from each other as $n \longrightarrow \infty$. Using again the regularization procedure we show that there are functions $u_{n,1}$, $u_{n,2}$ such that $(E_{\mathrm{GL}}(u_{n,i}))_{n\geq 1}$ is bounded and stays away from zero for $i = 1, 2$, and

$$|A(u_n) - A(u_{n,1}) - A(u_{n,2})| \longrightarrow 0 \quad \text{and} \quad |P_c(u_n) - P_c(u_{n,1}) - P_c(u_{n,2})| \longrightarrow 0 \tag{1.17}$$

as $n \longrightarrow \infty$. It is easy to see that $(P_c(u_{n,i}))_{n\geq 1}$ is bounded for $i = 1, 2$. Passing again to a subsequence, we may assume that $P_c(u_{n,1}) \longrightarrow p_1$ and $P_c(u_{n,2}) \longrightarrow p_2$, where $p_1 + p_2 = 0$.

If $p_1 = p_2 = 0$, we show that $\liminf_{n\to\infty} E_c(u_{n,i}) \geq T_c$ for $i = 1, 2$, and then

$$\liminf_{n\to\infty} E_c(u_n) = \liminf_{n\to\infty} (E_c(u_{n,1}) + E_c(u_{n,2})) \geq 2T_c, \qquad \text{a contradiction.}$$

If $p_1 < 0$, we use Lemma 4.8(ii), which asserts that for any bounded sequence $(v_n)_{n\geq 1} \subset \mathcal{X}$ satisfying $\lim_{n\to\infty} P_c(v_n) < 0$, there holds $\liminf_{n\to\infty} A(v_n) > \frac{N-1}{2} T_c$. Hence,

$$\liminf_{n\to\infty} E_c(u_n) = \frac{2}{N-1} \liminf_{n\to\infty} A(u_n) \geq \frac{2}{N-1} \liminf_{n\to\infty} A(u_{n,1}) > T_c,$$

again a contradiction. A similar argument is valid if $p_2 < 0$.

Since "vanishing" and "dichotomy" are excluded, necessarily "concentration" occurs, and then we show that $(u_n)_{n\geq 1}$ has a subsequence which converges to a minimizer of E_c in $\mathcal{C}$.

There are some important differences in the case $N = 3$ with respect to the case $N \geq 4$, most of them due to different scaling properties. For instance, for any $v \in \mathcal{X}$, we have $A(v_{1,\sigma}) = A(v)$ and $B_c(v_{1,\sigma}) = \sigma^2 B_c(v)$. If $v \in \mathcal{C}$, for all $\sigma > 0$, we have

$$P_c(v_{1,\sigma}) = B_c(v_{1,\sigma}) = \sigma^2 B_c(v) = 0 \quad \text{and} \quad E_c(v_{1,\sigma}) = A(v_{1,\sigma}) = A(v) = E_c(v).$$

It is then clear that one may expect convergence of minimizing sequences for E_c in $\mathcal{C}$ only after scaling. The proofs that vanishing and dichotomy do not occur are also a bit more involved. We treat the case $N = 3$ separately in Section 6.

Next we have to prove that any minimizer u of E_c in $\mathcal{C}$ is nondegenerate and satisfies a Euler-Lagrange equation $E_c'(u) = \alpha P_c'(u)$ (then necessarily $\alpha = 0$, as explained above). This is done in Proposition 5.6 in the case $N \geq 4$, respectively in Lemma 6.4 and Proposition 6.5 in the case $N = 3$.

Finally, we prove that traveling waves found by minimization in Sections 5 and 6 are axially symmetric (as one would expect from physical considerations; see [33]).

In space dimension two the situation is different, mainly because of different scaling properties. Indeed, if $N = 2$, it is easy to see that for any nonconstant function u satisfying $P_c(u) = 0$, the mapping $\sigma \longmapsto E_c(u_{1,\sigma})$ is decreasing on $(0, 1]$ and increasing on $[1, \infty)$; hence it achieves its minimum at $\sigma = 1$. This is exactly the opposite of what happens in the case $N \geq 4$, when $E_c(u_{1,\sigma})$ reaches its maximum at $\sigma = 1$. It can be proved that for $N = 2$, we have $\inf\{E_c(u) \mid u \in \mathcal{X}, u \neq 0, P_c(u) = 0\} = 0$ and there are no minimizers of E_c subject to the constraint $P_c = 0$. By using different approaches, the existence of two-dimensional traveling waves has recently been proven in [14] for a set of speeds that contains elements arbitrarily close to zero and to v_s. The existence for all speeds $c \in (0, v_s)$ is still an open problem. Although some of the results in Sections 2–4 are also valid in space dimension $N = 2$ (with straightforward modifications in the proofs), for simplicity we assume throughout that $N \geq 3$.

If $c = 0$ and assumptions (A1) and (A2) are satisfied, equation (1.4) has finite energy solutions if and only if the nonlinear potential V achieves negative values. The existence follows, for instance, from Theorems 2.1 and 2.2 in [12, pp. 100 and 103] (see also [9]). On the other hand, any finite energy solution ψ of the equation $\Delta\psi + F(|\psi|^2)\psi = 0$ in $\mathbf{R}^N$ satisfies the Pohozaev identity

$$(N-2)\int_{\mathbf{R}^N} |\nabla\psi|^2 \, dx + N \int_{\mathbf{R}^N} V(|\psi|^2) \, dx = 0$$

(see, e.g., Lemma 2.4 in [12, p. 104]), and then it is clear that ψ must be constant if V is nonnegative. In the case $c = 0$, our proofs imply that E_0 has a minimizer in the set $\{u \in \mathcal{X} \mid u \neq 0, P_0(u) = 0\}$ whenever this set is not empty. Then it is not hard to prove that minimizers satisfy (1.4) for $c = 0$ (after a scale change if $N = 3$). However, for simplicity we assume throughout (unless the contrary is explicitly mentioned) that $0 < c < v_s$.

2. The momentum

A good definition of the momentum is essential in any attempt to find solutions of (1.4) by using a variational approach. Roughly speaking, the momentum (with respect to the x_1-direction) should be a functional with derivative $2iu_{x_1}$. Various definitions have been given in the literature (see [8], [5], [7], [41]), any of them having its advantages and its inconveniences. Unfortunately, none of them is valid for all functions in $\mathcal{X}$. We propose a new and more general definition in this section.

It is clear that for functions $u \in H^1(\mathbf{R}^N)$, the momentum should be given by

$$Q_1(u) = \int_{\mathbf{R}^N} \langle iu_{x_1}, u \rangle \, dx, \tag{2.1}$$

and this is indeed a nice functional on $H^1(\mathbf{R}^N)$. The problem is that there are functions $u \in \mathcal{X} \setminus H^1(\mathbf{R}^N)$ such that $\langle iu_{x_1}, u \rangle \notin L^1(\mathbf{R}^N)$.

If $u \in \mathcal{X}$ is such that $1 + u$ admits a lifting $1 + u = \rho e^{i\theta}$, a formal computation gives

$$\int_{\mathbf{R}^N} \langle iu_{x_1}, u \rangle \, dx = -\int_{\mathbf{R}^N} \rho^2 \theta_{x_1} \, dx = -\int_{\mathbf{R}^N} (\rho^2 - 1)\theta_{x_1} \, dx. \tag{2.2}$$

It is not hard to see that if $u \in \mathcal{X}$ is as above, then $(\rho^2 - 1)\theta_{x_1} \in L^1(\mathbf{R}^N)$. However, there are many "interesting" functions $u \in \mathcal{X}$ such that $1 + u$ does not admit a lifting.

Our aim is to define the momentum on $\mathcal{X}$ in such a way that it agrees with (2.1) for functions in $H^1(\mathbf{R}^N)$ and with (2.2) when a lifting as above exists.

LEMMA 2.1. *Let $u \in \mathcal{X}$ be such that $m \leq |1 + u(x)| \leq 2$ a.e. (almost everywhere) on $\mathbf{R}^N$, where $m > 0$. There exist two real-valued functions ρ, θ such that $\rho - 1 \in H^1(\mathbf{R}^N)$, $\theta \in \mathcal{D}^{1,2}(\mathbf{R}^N)$, $1 + u = \rho e^{i\theta}$ a.e. on $\mathbf{R}^N$ and*

$$\langle iu_{x_1}, u \rangle = \frac{\partial}{\partial x_1}(\mathrm{Im}(u) - \theta) - (\rho^2 - 1)\frac{\partial \theta}{\partial x_1} \qquad \text{a.e. on } \mathbf{R}^N. \tag{2.3}$$

Moreover, we have

$$\int_{\mathbf{R}^N} \left| (\rho^2 - 1)\theta_{x_1} \right| dx \leq \frac{1}{\sqrt{2m}} E_{\mathrm{GL}}(u).$$

Proof. Since $1 + u \in H^1_{\mathrm{loc}}(\mathbf{R}^N)$, the fact that there exist $\rho, \theta \in H^1_{\mathrm{loc}}(\mathbf{R}^N)$ such that $1 + u = \rho e^{i\theta}$ a.e. is standard and follows from Theorem 3 in [10,

p. 38]. We have

$$\left|\frac{\partial u}{\partial x_j}\right|^2 = \left|\frac{\partial \rho}{\partial x_j}\right|^2 + \rho^2\left|\frac{\partial \theta}{\partial x_j}\right|^2 \qquad \text{a.e. on } \mathbf{R}^N \text{ for } j = 1, \dots, N. \tag{2.4}$$

Since $\rho = |1+u| \geq m$ a.e., it follows that $\nabla\rho, \nabla\theta \in L^2(\mathbf{R}^N)$. If $N \geq 3$, we infer that there exist $\rho_0, \theta_0 \in \mathbf{R}$ such that $\rho - \rho_0$ and $\theta - \theta_0$ belong to $L^{2^*}(\mathbf{R}^N)$. Then it is not hard to see that $\rho_0 = 1$ and $\theta_0 = 2k_0\pi$, where $k_0 \in \mathbf{Z}$. Replacing θ by $\theta - 2k_0\pi$, we have $\rho - 1, \theta \in \mathcal{D}^{1,2}(\mathbf{R}^N)$. Since $\rho \leq 2$ a.e., we have $\rho^2 - 1 = \varphi(|1+u|)^2 - 1 \in L^2(\mathbf{R}^N)$ because $u \in \mathcal{X}$. Clearly $|\rho - 1| = \frac{|\rho^2-1|}{\rho+1} \leq |\rho^2 - 1|$; hence $\rho - 1 \in L^2(\mathbf{R}^N)$.

A straightforward computation gives

$$\langle iu_{x_1}, u\rangle = \langle iu_{x_1}, -1\rangle - \rho^2\theta_{x_1} = \frac{\partial}{\partial x_1}(\mathrm{Im}(u) - \theta) - (\rho^2 - 1)\frac{\partial \theta}{\partial x_1}.$$

By (2.4), we have $\left|\frac{\partial\theta}{\partial x_j}\right| \leq \frac{1}{\rho}\left|\frac{\partial u}{\partial x_j}\right| \leq \frac{1}{m}\left|\frac{\partial u}{\partial x_j}\right|$, and the Cauchy-Schwarz inequality gives

$$\begin{aligned}\int_{\mathbf{R}^N} \left|(\rho^2-1)\theta_{x_1}\right| dx &\leq \|\rho^2 - 1\|_{L^2}\|\theta_{x_1}\|_{L^2} \\ &\leq \frac{1}{m}\|\rho^2-1\|_{L^2}\|u_{x_1}\|_{L^2} \leq \frac{1}{m\sqrt{2}}E_{\mathrm{GL}}(u).\end{aligned}$$ □

Lemma 2.2. *Let $\chi \in C_c^\infty(\mathbf{C}, \mathbf{R})$ be a function such that $\chi = 1$ on $B(0, \frac{1}{4})$, $0 \leq \chi \leq 1$ and $\mathrm{supp}(\chi) \subset B(0, \frac{1}{2})$. For an arbitrary $u \in \mathcal{X}$, denote $u_1 = \chi(u)u$ and $u_2 = (1-\chi(u))u$. Then $u_1 \in \mathcal{X}$, $u_2 \in H^1(\mathbf{R}^N)$ and the following estimates hold:*

$$|\nabla u_i| \leq C|\nabla u| \qquad \text{a.e. on } \mathbf{R}^N \text{ for } i = 1, 2, \text{ where } C \text{ depends only on } \chi, \tag{2.5}$$

$$\|u_2\|_{L^2(\mathbf{R}^N)} \leq C_1\|\nabla u\|_{L^2(\mathbf{R}^N)}^{\frac{2^*}{2}} \quad \text{and} \quad \|(1-\chi^2(u))u\|_{L^2(\mathbf{R}^N)} \leq C_1\|\nabla u\|_{L^2(\mathbf{R}^N)}^{\frac{2^*}{2}}, \tag{2.6}$$

$$\int_{\mathbf{R}^N} \left(\varphi^2(|1+u_1|) - 1\right)^2 dx \leq \int_{\mathbf{R}^N} \left(\varphi^2(|1+u|) - 1\right)^2 dx + C_2\|\nabla u\|_{L^2(\mathbf{R}^N)}^{2^*}, \tag{2.7}$$

$$\int_{\mathbf{R}^N} \left(\varphi^2(|1+u_2|) - 1\right)^2 dx \leq C_2\|\nabla u\|_{L^2(\mathbf{R}^N)}^{2^*}. \tag{2.8}$$

Let $1 + u_1 = \rho e^{i\theta}$ be the lifting of $1 + u_1$, as given by Lemma 2.1. Then we have

$$\langle iu_{x_1}, u\rangle = (1-\chi^2(u))\langle iu_{x_1}, u\rangle - (\rho^2-1)\frac{\partial\theta}{\partial x_1} + \frac{\partial}{\partial x_1}(\mathrm{Im}(u)) - \frac{\partial\theta}{\partial x_1} \tag{2.9}$$

a.e. on $\mathbf{R}^N$.

Proof. Since $|u_i| \le |u|$, we have $u_i \in L^{2^*}(\mathbf{R}^N)$ for $i = 1, 2$. It is standard to prove that $u_i \in H^1_{\mathrm{loc}}(\mathbf{R}^N)$ (see, e.g., Lemma C1 in [10, p. 66]) and we have

$$\frac{\partial u_1}{\partial x_j} = \left(\partial_1 \chi(u) \frac{\partial(\mathrm{Re}(u))}{\partial x_j} + \partial_2 \chi(u) \frac{\partial(\mathrm{Im}(u))}{\partial x_j}\right) u + \chi(u) \frac{\partial u}{\partial x_j}. \tag{2.10}$$

A similar formula holds for u_2. Since the functions $z \longmapsto \partial_i \chi(z) z$, $i = 1, 2$, are bounded on $\mathbf{C}$, (2.5) follows immediately from (2.10).

Using the Sobolev embedding, we have

$$\|u_2\|^2_{L^2} \le \int_{\mathbf{R}^N} |u|^2 \mathbb{1}_{\{|u| > \frac{1}{4}\}}(x)\, dx \le 4^{2^*-2} \int_{\mathbf{R}^N} |u|^{2^*} \mathbb{1}_{\{|u| > \frac{1}{4}\}}(x)\, dx \le C_1 \|\nabla u\|^{2^*}_{L^2}.$$

This gives the first estimate in (2.6); the second one is similar.

For $|u| \le \frac{1}{4}$, we have $u_1(x) = u(x)$. Hence,

$$\int_{\{|u| \le \frac{1}{4}\}} \left(\varphi^2(|1 + u_1|) - 1\right)^2 dx = \int_{\{|u| \le \frac{1}{4}\}} \left(\varphi^2(|1 + u|) - 1\right)^2 dx.$$

There exists $C' > 0$ such that $\left(\varphi^2(|1 + z|) - 1\right)^2 \le C'|z|^2$ if $|z| \ge \frac{1}{4}$. Proceeding as in the proof of (2.6), for $i = 1, 2$, we have

$$\int_{\{|u| > \frac{1}{4}\}} \left(\varphi^2(|1 + u_i|) - 1\right)^2 dx \le C' \int_{\{|u| > \frac{1}{4}\}} |u_i|^2\, dx \le C_2 \|\nabla u\|^{2^*}_{L^2}.$$

This clearly implies (2.7) and (2.8).

Since $\partial_1 \chi(u) \frac{\partial(\mathrm{Re}(u))}{\partial x_j} + \partial_2 \chi(u) \frac{\partial(\mathrm{Im}(u))}{\partial x_j} \in \mathbf{R}$, using (2.10) we get $\langle i \frac{\partial u_1}{\partial x_1}, u_1 \rangle = \chi^2(u) \langle i u_{x_1}, u \rangle$ a.e. on $\mathbf{R}$. Then (2.9) follows from Lemma 2.1. □

We consider the space $\mathcal{Y} = \{\partial_{x_1} \phi \mid \phi \in \dot{H}^1(\mathbf{R}^N)\}$. It is clear that $\phi_1, \phi_2 \in \dot{H}^1(\mathbf{R}^N)$ and $\partial_{x_1} \phi_1 = \partial_{x_1} \phi_2$ imply that $\phi_1 - \phi_2$ is constant; hence $\nabla \phi_1 = \nabla \phi_2$. Defining

$$\|\partial_{x_1} \phi\|_{\mathcal{Y}} = \|\phi\|_{\dot{H}^1(\mathbf{R}^N)} = \|\nabla \phi\|_{L^2(\mathbf{R}^N)},$$

it is easy to see that $\|\cdot\|_{\mathcal{Y}}$ is a norm on $\mathcal{Y}$ and $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ is a Banach space. The following holds.

LEMMA 2.3. *Let $N \ge 2$. For any $v \in L^1(\mathbf{R}^N) \cap \mathcal{Y}$, we have $\int_{\mathbf{R}^N} v(x)\, dx = 0$.*

Proof. Take $\phi \in \dot{H}^1(\mathbf{R}^N)$ such that $v = \partial_{x_1} \phi$. Then $\phi \in \mathcal{S}'(\mathbf{R}^N)$ and $|\xi| \widehat{\phi} \in L^2(\mathbf{R}^N)$. Hence $\widehat{\phi} \in L^1_{\mathrm{loc}}(\mathbf{R}^N \setminus \{0\})$. On the other hand, we have $v = \partial_{x_1} \phi \in L^1 \cap L^2(\mathbf{R}^N)$ by hypothesis; hence $\widehat{v} = i\xi_1 \widehat{\phi} \in L^2 \cap C^0_b(\mathbf{R}^N)$.

We prove that $\widehat{v}(0) = 0$. We argue by contradiction and assume that $\widehat{v}(0) \ne 0$. By continuity, there exists $m > 0$ and $\varepsilon > 0$ such that $|\widehat{v}(\xi)| \ge m$ for $|\xi| \le \varepsilon$. For $j = 2, \dots, N$, we get

$$|i\xi_j \widehat{\phi}(\xi)| = \frac{|\xi_j|}{|\xi_1|} |\widehat{v}(\xi)| \ge m \frac{|\xi_j|}{|\xi_1|} \qquad \text{for a.e. } \xi \in B(0, \varepsilon).$$

But this contradicts the fact that $i\xi_j\widehat{\phi} \in L^2(\mathbf{R}^N)$. Thus necessarily $\widehat{v}(0) = 0$, and this is exactly the conclusion of Lemma 2.3. □

It is obvious that $L_1(v) = \int_{\mathbf{R}^N} v(x)\,dx$ and $L_2(w) = 0$ are continuous linear functionals on $L^1(\mathbf{R}^N)$ and on $\mathcal{Y}$, respectively. Moreover, by Lemma 2.3, we have $L_1 = L_2$ on $L^1(\mathbf{R}^N) \cap \mathcal{Y}$. Putting

$$L(v + w) = L_1(v) + L_2(w) = \int_{\mathbf{R}^N} v(x)\,dx \tag{2.11}$$

for $v \in L^1(\mathbf{R}^N)$ and $w \in \mathcal{Y}$, we see that L is well defined and is a continuous linear functional on $L^1(\mathbf{R}^N) + \mathcal{Y}$.

It follows from (2.9) and Lemmas 2.1 and 2.2 that for any $u \in \mathcal{X}$, we have $\langle iu_{x_1}, u\rangle \in L^1(\mathbf{R}^N) + \mathcal{Y}$. This enables us to give the following

Definition 2.4. Given $u \in \mathcal{X}$, the momentum of u (with respect to the x_1-direction) is

$$Q(u) = L(\langle iu_{x_1}, u\rangle).$$

If $u \in \mathcal{X}$ and $\chi, u_1, u_2, \rho, \theta$ are as in Lemma 2.2, from (2.9) we get

$$Q(u) = \int_{\mathbf{R}^N} (1 - \chi^2(u))\langle iu_{x_1}, u\rangle - (\rho^2 - 1)\theta_{x_1}\,dx. \tag{2.12}$$

It is easy to check that the right-hand side of (2.12) does not depend on the choice of the cut-off function χ, provided that χ is as in Lemma 2.2.

It follows directly from (2.12) that the functional Q has a nice behavior with respect to dilations in $\mathbf{R}^N$: for any $u \in \mathcal{X}$ and λ, $\sigma > 0$, we have

$$Q(u_{\lambda,\sigma}) = \sigma^{N-1} Q(u). \tag{2.13}$$

The next lemma will enable us to perform "integrations by parts."

LEMMA 2.5. *For any* $u \in \mathcal{X}$ *and* $w \in H^1(\mathbf{R}^N)$, *we have* $\langle iu_{x_1}, w\rangle \in L^1(\mathbf{R}^N)$, $\langle iu, w_{x_1}\rangle \in L^1(\mathbf{R}^N) + \mathcal{Y}$ *and*

$$L(\langle iu_{x_1}, w\rangle + \langle iu, w_{x_1}\rangle) = 0. \tag{2.14}$$

Proof. Since $w, u_{x_1} \in L^2(\mathbf{R}^N)$, the Cauchy-Schwarz inequality implies $\langle iu_{x_1}, w\rangle \in L^1(\mathbf{R}^N)$. Let χ, u_1, u_2 be as in Lemma 2.2. Denote $w_1 = \chi(w)w$, $w_2 = (1 - \chi(w))w$. Then $u = u_1 + u_2$, $w = w_1 + w_2$ and it follows from Lemma 2.2 that $u_1 \in \mathcal{X} \cap L^\infty(\mathbf{R}^N)$ and u_2, w_1, $w_2 \in H^1(\mathbf{R}^N)$.

As above, we have $\langle i\frac{\partial u_2}{\partial x_1}, w\rangle$, $\langle iu_2, \frac{\partial w}{\partial x_1}\rangle \in L^1(\mathbf{R}^N)$ by the Cauchy-Schwarz inequality. The standard integration by parts formula for functions in $H^1(\mathbf{R}^N)$ (see, e.g., [11, p. 197]) gives

$$\int_{\mathbf{R}^N} \left\langle i\frac{\partial u_2}{\partial x_1}, w\right\rangle + \left\langle iu_2, \frac{\partial w}{\partial x_1}\right\rangle\,dx = 0. \tag{2.15}$$

Since $u_1 \in \mathcal{D}^{1,2} \cap L^\infty(\mathbf{R}^N)$ and $w_1 \in H^1 \cap L^\infty(\mathbf{R}^N)$, it is standard to prove that $\langle iu_1, w_1\rangle \in \mathcal{D}^{1,2} \cap L^\infty(\mathbf{R}^N)$ and

$$\left\langle i\frac{\partial u_1}{\partial x_1}, w_1\right\rangle + \left\langle iu_1, \frac{\partial w_1}{\partial x_1}\right\rangle = \frac{\partial}{\partial x_1}\langle iu_1, w_1\rangle \qquad \text{a.e. on } \mathbf{R}^N. \tag{2.16}$$

Let $A_w = \{x \in \mathbf{R}^N \mid |w(x)| \geq \frac{1}{4}\}$. We have $\frac{1}{16}\mathcal{L}^N(A_w) \leq \int_{A_w} |w|^2\,dx \leq \|w\|_{L^2}^2$ and, consequently, A_w has finite measure. It is clear that $w_2 = 0$ and $\nabla w_2 = 0$ a.e. on $\mathbf{R}^N \setminus A_w$. Since $w_2 \in L^{2^*}(\mathbf{R}^N)$ and $\nabla w_2 \in L^2(\mathbf{R}^N)$, we infer that $w_2 \in L^1 \cap L^{2^*}(\mathbf{R}^N)$ and $\nabla w_2 \in L^1 \cap L^2(\mathbf{R}^N)$. Together with the fact that $u_1 \in L^{2^*} \cap L^\infty(\mathbf{R}^N)$ and $\nabla u_1 \in L^2(\mathbf{R}^N)$, this gives $\langle iu_1, w_2\rangle \in L^1 \cap L^{2^*}(\mathbf{R}^N)$ and

$$\left\langle i\frac{\partial u_1}{\partial x_j}, w_2\right\rangle \in L^1 \cap L^{\frac{N}{N-1}}(\mathbf{R}^N), \quad \left\langle iu_1, \frac{\partial w_2}{\partial x_j}\right\rangle \in L^1 \cap L^2(\mathbf{R}^N) \quad \text{for } j = 1, \dots, N.$$

It is easy to see that

$$\frac{\partial}{\partial x_j}\langle iu_1, w_2\rangle = \left\langle i\frac{\partial u_1}{\partial x_j}, w_2\right\rangle + \left\langle iu_1, \frac{\partial w_2}{\partial x_j}\right\rangle \qquad \text{in } \mathcal{D}'(\mathbf{R}^N).$$

From the above we infer that $\langle iu_1, w_2\rangle \in W^{1,1}(\mathbf{R}^N)$. It is obvious that $\int_{\mathbf{R}^N} \frac{\partial \psi}{\partial x_j}\,dx = 0$ for any $\psi \in W^{1,1}(\mathbf{R}^N)$. (Indeed, let $(\psi_n)_{n\geq 1} \subset C_c^\infty(\mathbf{R}^N)$ be a sequence such that $\psi_n \longrightarrow \psi$ in $W^{1,1}(\mathbf{R}^N)$ as $n \longrightarrow \infty$; then $\int_{\mathbf{R}^N} \frac{\partial \psi_n}{\partial x_j}\,dx = 0$ for each n and $\int_{\mathbf{R}^N} \frac{\partial \psi_n}{\partial x_j}\,dx \longrightarrow \int_{\mathbf{R}^N} \frac{\partial \psi}{\partial x_j}\,dx$ as $n \longrightarrow \infty$.) Thus we have $\langle i\frac{\partial u_1}{\partial x_1}, w_2\rangle$, $\langle iu_1, \frac{\partial w_2}{\partial x_1}\rangle \in L^1(\mathbf{R}^N)$ and

$$\int_{\mathbf{R}^N} \left\langle i\frac{\partial u_1}{\partial x_1}, w_2\right\rangle + \left\langle iu_1, \frac{\partial w_2}{\partial x_1}\right\rangle dx = \int_{\mathbf{R}^N} \frac{\partial}{\partial x_1}\langle iu_1, w_2\rangle\,dx = 0. \tag{2.17}$$

Now (2.14) follows from (2.15), (2.16), (2.17), and Lemma 2.5 is proven. □

Corollary 2.6. *Let u, $v \in \mathcal{X}$ be such that $u - v \in L^2(\mathbf{R}^N)$. Then*

$$|Q(u) - Q(v)| \leq \|u - v\|_{L^2(\mathbf{R}^N)} \left(\left\|\frac{\partial u}{\partial x_1}\right\|_{L^2(\mathbf{R}^N)} + \left\|\frac{\partial v}{\partial x_1}\right\|_{L^2(\mathbf{R}^N)} \right). \tag{2.18}$$

Proof. It is clear that $w = u - v \in H^1(\mathbf{R}^N)$. Using (2.14), we get

$$\begin{aligned} Q(u) - Q(v) &= L(\langle i(u-v)_{x_1}, u\rangle + \langle iv_{x_1}, u - v\rangle) \\ &= L(\langle iu_{x_1}, u - v\rangle + \langle iv_{x_1}, u - v\rangle) \\ &= \int_{\mathbf{R}^N} \langle iu_{x_1} + iv_{x_1}, u - v\rangle\,dx. \end{aligned} \tag{2.19}$$

Then (2.18) follows from (2.19) and the Cauchy-Schwarz inequality. □

The next result will be useful to estimate the contribution to the momentum of a domain where the modified Ginzburg-Landau energy is small.

LEMMA 2.7. *Let $M > 0$, and let Ω be an open subset of $\mathbf{R}^N$. Assume that $u \in \mathcal{X}$ satisfies $E_{\mathrm{GL}}(u) \le M$, and let χ, ρ, θ be as in Lemma 2.2. Then we have*

$$\int_\Omega \left|(1-\chi^2(u))\langle iu_{x_1}, u\rangle - (\rho^2-1)\theta_{x_1}\right| dx \le C(M^{\frac12} + M^{\frac{2^*}{4}}) \left(E^\Omega_{\mathrm{GL}}(u)\right)^{\frac12}. \tag{2.20}$$

Proof. Using (2.6) and the Cauchy-Schwarz inequality, we get

$$\begin{aligned}\int_\Omega \left|(1-\chi^2(u))\langle iu_{x_1}, u\rangle\right| dx &\le \|u_{x_1}\|_{L^2(\Omega)} \|(1-\chi^2(u))u\|_{L^2(\Omega)} \\ &\le C_1 \|u_{x_1}\|_{L^2(\Omega)} \|\nabla u\|^{\frac{2^*}{2}}_{L^2(\mathbf{R}^N)}.\end{aligned} \tag{2.21}$$

We have $|u_1| \le \frac12$; hence $|1+u_1| \le \frac32$ and $\varphi(|1+u_1|) = |1+u_1| = \rho$. Then (2.7) gives

$$\|\rho^2 - 1\|^2_{L^2(\mathbf{R}^N)} \le C'(E_{\mathrm{GL}}(u) + E_{\mathrm{GL}}(u)^{\frac{2^*}{2}}) \le C'(M + M^{\frac{2^*}{2}}). \tag{2.22}$$

From (2.4) and (2.5) we have $\left|\frac{\partial\theta}{\partial x_j}\right| \le \frac{1}{\rho}\left|\frac{\partial u_1}{\partial x_j}\right| \le C''\left|\frac{\partial u}{\partial x_j}\right|$ a.e. on $\mathbf{R}^N$. Therefore

$$\begin{aligned}\int_\Omega \left|(\rho^2-1)\theta_{x_1}\right| dx &\le \|\rho^2-1\|_{L^2(\Omega)} \|\theta_{x_1}\|_{L^2(\Omega)} \\ &\le C''\|\rho^2-1\|_{L^2(\mathbf{R}^N)} \|u_{x_1}\|_{L^2(\Omega)} \\ &\le C'''\left(M + M^{\frac{2^*}{2}}\right)^{\frac12} \left(E^\Omega_{\mathrm{GL}}(u)\right)^{\frac12}.\end{aligned} \tag{2.23}$$

Then (2.20) follows from (2.21) and (2.23). □

3. A regularization procedure

Given a function $u \in \mathcal{X}$ and a set $\Omega \subset \mathbf{R}^N$ such that $E^\Omega_{\mathrm{GL}}(u)$ is small, we would like to get a fine estimate of the contribution of Ω to the momentum of u. To do this, we will use a kind of "regularization" procedure for arbitrary functions in $\mathcal{X}$. A similar device has been introduced in [1] to get rid of small-scale topological defects of functions; variants of it have been used for various purposes in [8], [7], [5].

Throughout this section, Ω is an open set in $\mathbf{R}^N$. We do not assume Ω bounded, nor connected. If $\partial\Omega \neq \emptyset$, we assume that $\partial\Omega$ is C^2. Let φ be as in the introduction. Fix $u \in \mathcal{X}$ and $h > 0$. We consider the functional

$$G^u_{h,\Omega}(v) = E^\Omega_{\mathrm{GL}}(v) + \frac{1}{h^2}\int_\Omega \varphi\left(|v-u|^2\right) dx.$$

Note that $G^u_{h,\Omega}(v)$ may equal ∞ for some $v \in \mathcal{X}$; however, $G^u_{h,\Omega}(v)$ is finite whenever $v \in \mathcal{X}$ and $v - u \in L^2(\Omega)$. If there is no risk of confusion, we will simply write $G(v)$ instead of $G^u_{h,\Omega}(v)$. We denote $H^1_0(\Omega) = \{u \in H^1(\mathbf{R}^N) \mid u = 0$ on $\mathbf{R}^N \setminus \Omega\}$ and

$$H^1_u(\Omega) = \{v \in \mathcal{X} \mid v - u \in H^1_0(\Omega)\}.$$

The next lemma gives the properties of functions that minimize G in the space $H^1_u(\Omega)$.

LEMMA 3.1. (i) *The functional G has a minimizer in $H^1_u(\Omega)$.*

(ii) *Let v_h be a minimizer of G in $H^1_u(\Omega)$. There exist constants C_1, $C_2 > 0$, depending only on N, such that v_h satisfies*

$$E^{\Omega}_{\mathrm{GL}}(v_h) \le E^{\Omega}_{\mathrm{GL}}(u), \tag{3.1}$$

$$\|v_h - u\|^2_{L^2(\Omega)} \le h^2 E^{\Omega}_{\mathrm{GL}}(u) + C_1 \left(E^{\Omega}_{\mathrm{GL}}(u)\right)^{1+\frac{2}{N}} h^{\frac{4}{N}}, \tag{3.2}$$

$$\int_{\Omega} \left| \left(\varphi^2(|1+u|) - 1\right)^2 - \left(\varphi^2(|1+v_h|) - 1\right)^2 \right| dx \le 36 h E^{\Omega}_{\mathrm{GL}}(u), \tag{3.3}$$

$$|Q(u) - Q(v_h)| \le C_2 \left(h^2 + \left(E^{\Omega}_{\mathrm{GL}}(u)\right)^{\frac{2}{N}} h^{\frac{4}{N}}\right)^{\frac{1}{2}} E^{\Omega}_{\mathrm{GL}}(u). \tag{3.4}$$

(iii) *For $z \in \mathbf{C}$, denote $H(z) = \left(\varphi^2(|z+1|) - 1\right)\varphi(|z+1|)\varphi'(|z+1|)\frac{z+1}{|z+1|}$ if $z \neq -1$ and $H(-1) = 0$. Then any minimizer v_h of G in $H^1_u(\Omega)$ satisfies the equation*

$$-\Delta v_h + H(v_h) + \frac{1}{h^2}\varphi'\left(|v_h - u|^2\right)(v_h - u) = 0 \qquad \text{in } \mathcal{D}'(\Omega). \tag{3.5}$$

Moreover, for any $\omega \subset\subset \Omega$ we have $v_h \in W^{2,p}(\omega)$ for $p \in [1,\infty)$; thus, in particular, $v_h \in C^{1,\alpha}(\omega)$ for $\alpha \in [0,1)$.

(iv) *For any $h > 0$, $\delta > 0$ and $R > 0$, there exists a constant $K = K(N,h,\delta,R) > 0$ such that for any $u \in \mathcal{X}$ with $E^{\Omega}_{\mathrm{GL}}(u) \le K$ and for any minimizer v_h of G in $H^1_u(\Omega)$, there holds*

$$1 - \delta < |1 + v_h(x)| < 1 + \delta \qquad \text{whenever } x \in \Omega \text{ and } \mathrm{dist}(x, \partial\Omega) \ge 4R. \tag{3.6}$$

Proof. (i) It is obvious that $u \in H^1_u(\Omega)$. Let $(v_n)_{n\ge1}$ be a minimizing sequence for G in $H^1_u(\Omega)$. We may assume that $G(v_n) \le G(u) = E^{\Omega}_{\mathrm{GL}}(u)$. This implies $\int_\Omega |\nabla v_n|^2\,dx \le E^{\Omega}_{\mathrm{GL}}(u)$. It is clear that

$$\int_{\Omega\cap\{|v_n-u|\le\sqrt{2}\}} |v_n - u|^2\,dx \le \int_\Omega \varphi\left(|v_n - u|^2\right)dx \le h^2 E^{\Omega}_{\mathrm{GL}}(u). \tag{3.7}$$

Since $v_n - u \in H^1_0(\Omega) \subset H^1(\mathbf{R}^N)$, by the Sobolev embedding we have

$$\|v_n - u\|_{L^{2^*}(\mathbf{R}^N)} \le C_S \|\nabla v_n - \nabla u\|_{L^2(\mathbf{R}^N)},$$

where C_S depends only on N. Therefore,

$$\begin{aligned}\int_{\{|v_n-u|\ge1\}} |v_n - u|^2\,dx &\le \int_{\{|v_n-u|\ge1\}} |v_n - u|^{2^*}\,dx \le \|v_n - u\|^{2^*}_{L^{2^*}(\mathbf{R}^N)} \\ &\le C'\|\nabla v_n - \nabla u\|^{2^*}_{L^2(\mathbf{R}^N)} \le C\left(E^{\Omega}_{\mathrm{GL}}(u)\right)^{\frac{2^*}{2}}.\end{aligned} \tag{3.8}$$

It follows from (3.7) and (3.8) that $\|v_n - u\|_{L^2(\Omega)}$ is bounded. Hence $v_n - u$ is bounded in $H^1_0(\Omega)$. We infer that there exists a sequence (still denoted

$(v_n)_{n\geq 1})$ and there is $w \in H_0^1(\Omega)$ such that $v_n - u \rightharpoonup w$ weakly in $H_0^1(\Omega)$, $v_n - u \longrightarrow w$ a.e. and $v_n - u \longrightarrow w$ in $L^p_{\rm loc}(\Omega)$ for $1 \leq p < 2^*$. Let $v = u + w$. Then $\nabla v_n \rightharpoonup \nabla v$ weakly in $L^2(\mathbf{R}^N)$, and this implies

$$\int_\Omega |\nabla v|^2\, dx \leq \liminf_{n\to\infty} \int_\Omega |\nabla v_n|^2\, dx.$$

Using the a.e. convergence and Fatou's Lemma we infer that

$$\int_\Omega \left(\varphi^2(|1+v|) - 1\right)^2\, dx \leq \liminf_{n\to\infty} \int_\Omega \left(\varphi^2(|1+v_n|) - 1\right)^2\, dx$$

and

$$\int_\Omega \varphi\left(|v-u|^2\right)\, dx \leq \liminf_{n\to\infty} \int_\Omega \varphi\left(|v_n-u|^2\right)\, dx.$$

Therefore $G(v) \leq \liminf_{n\to\infty} G(v_n)$ and, consequently, v is a minimizer of G in $H_u^1(\Omega)$.

(ii) Since $u \in H_u^1(\Omega)$, we have $E_{\rm GL}^\Omega(v_h) \leq G(v_h) \leq E_{\rm GL}^\Omega(u)$; hence (3.1) holds. It is clear that $\varphi\left(|v_h - u|^2\right) \geq 1$ if $|v_h - u| \geq 1$; thus,

$$\mathcal{L}^N(\{|v_h - u| \geq 1\}) \leq \int_{\mathbf{R}^N} \varphi\left(|v_h - u|^2\right)\, dx \leq h^2 G(v_h) \leq h^2 E_{\rm GL}^\Omega(u).$$

Using Hölder's inequality, the above estimate and the Sobolev inequality we get

(3.9)

$$\begin{aligned}
\int_{\{|v_h-u|\geq 1\}} |v_h - u|^2\, dx &\leq \|v_h - u\|^2_{L^{2^*}(\{|v_h-u|\geq 1\})} \left(\mathcal{L}^N(\{|v_h - u| \geq 1\})\right)^{1-\frac{2}{2^*}} \\
&\leq \|v_h - u\|^2_{L^{2^*}(\mathbf{R}^N)} \left(\mathcal{L}^N(\{|v_h - u| \geq 1\})\right)^{1-\frac{2}{2^*}} \\
&\leq C_S \|\nabla v_h - \nabla u\|^2_{L^2(\mathbf{R}^N)} \left(h^2 E_{\rm GL}^\Omega(u)\right)^{1-\frac{2}{2^*}} \leq 4 C_S h^{\frac{4}{N}} \left(E_{\rm GL}^\Omega(u)\right)^{1+\frac{2}{N}}.
\end{aligned}$$

It is clear that (3.7) holds with v_h instead of v_n and then (3.2) follows from (3.7) and (3.9).

We claim that

$$\text{(3.10)} \qquad \left|\varphi(|z|) - \varphi(|\zeta|)\right| \leq \left[\tfrac{9}{2}\varphi\left(|z-\zeta|^2\right)\right]^{\frac{1}{2}} \qquad \text{for any } z,\ \zeta \in \mathbf{C}.$$

Indeed, let $0 \leq a \leq b$. If $b \in [a, a+\sqrt{2}]$, we have $\varphi((b-a)^2) = (b-a)^2$; hence,

$$0 \leq \varphi(b) - \varphi(a) \leq b - a = \left[\varphi((b-a)^2)\right]^{\frac{1}{2}}.$$

If $b > a + \sqrt{2}$, we have $0 \leq \varphi(b) - \varphi(a) \leq 3$ and $\left[\varphi((b-a)^2)\right]^{\frac{1}{2}} \geq (\varphi(2))^{\frac{1}{2}} = \sqrt{2}$; thus $0 \leq \varphi(b) - \varphi(a) \leq \frac{3}{\sqrt{2}}\left[\varphi((b-a)^2)\right]^{\frac{1}{2}}$. Assuming that $|z| \leq |\zeta|$, we get

$$\left|\varphi(|z|) - \varphi(|\zeta|)\right| = \varphi(|\zeta|) - \varphi(|z|) \leq \left[\frac{9}{2}\varphi\left((|\zeta| - |z|)^2\right)\right]^{\frac{1}{2}} \leq \left[\frac{9}{2}\varphi\left(|\zeta - z|^2\right)\right]^{\frac{1}{2}}.$$

It is obvious that

$$
\begin{aligned}
(3.11)\qquad &\Big|\big(\varphi^2(|1+u|)-1\big)^2-\big(\varphi^2(|1+v_h|)-1\big)^2\Big| \\
&\le 6\Big|\varphi(|1+u|)-\varphi(|1+v_h|)\Big|\cdot\Big|\varphi^2(|1+u|)+\varphi^2(|1+v_h|)-2\Big|.
\end{aligned}
$$

Using (3.11), the Cauchy-Schwarz inequality and (3.10), we get

$$
\begin{aligned}
&\int_\Omega\Big|\big(\varphi^2(|1+u|)-1\big)^2-\big(\varphi^2(|1+v_h|)-1\big)^2\Big|\,dx \\
&\le 6\left(\int_\Omega\Big|\varphi(|1+u|)-\varphi(|1+v_h|)\Big|^2\,dx\right)^{\frac12} \\
&\quad\times\left(\int_\Omega\Big|\varphi^2(|1+u|)+\varphi^2(|1+v_h|)-2\Big|^2\,dx\right)^{\frac12} \\
&\le 6\left(\int_\Omega\frac92\varphi\left(|v_h-u|^2\right)\,dx\right)^{\frac12} \\
&\quad\times\left(2\int_\Omega\big(\varphi^2(|1+u|)-1\big)^2+\big(\varphi^2(|1+v_h|)-1\big)^2\,dx\right)^{\frac12} \\
&\le 18\left(h^2G(v_h)\right)^{\frac12}\left(2E_{\rm GL}^\Omega(u)+2E_{\rm GL}^\Omega(v_h)\right)^{\frac12}\le 36hE_{\rm GL}^\Omega(u),
\end{aligned}
$$

and (3.3) is proven. Finally, (3.4) follows directly from (3.1), (3.2) and Corollary 2.6.

(iii) The proof of (3.5) is standard. For any $\psi\in C_c^\infty(\Omega)$, we have $v+\psi\in H_u^1(\Omega)$, and the function $t\longmapsto G(v+t\psi)$ achieves its minimum at $t=0$. Hence $\frac{d}{dt}_{|t=0}(G(v+t\psi))=0$ for any $\psi\in C_c^\infty(\Omega)$, and this is precisely (3.5).

For any $z\in\mathbf{C}$, we have

$$
(3.12)\qquad |H(z)|\le 3|\varphi^2(|z+1|)-1|\le 24.
$$

Since $v_h\in\mathcal{X}$, we have $\varphi^2(|1+v_h|)-1\in L^2(\mathbf{R}^N)$, and (3.12) gives $H(v_h)\in L^2\cap L^\infty(\mathbf{R}^N)$. We also have $\left|\varphi'\left(|v_h-u|^2\right)(v_h-u)\right|\le|v_h-u|$ and

$$
\left|\varphi'\left(|v_h-u|^2\right)(v_h-u)\right|\le\sup_{s\ge0}\varphi'\left(s^2\right)s<\infty.
$$

Since $v_h-u\in L^2(\mathbf{R}^N)$, we get $\varphi'\left(|v_h-u|^2\right)(v_h-u)\in L^2\cap L^\infty(\mathbf{R}^N)$. Using (3.5) we infer that $\Delta v_h\in L^2\cap L^\infty(\Omega)$. Then (iii) follows from standard elliptic estimates (see, e.g., Theorem 9.11 in [24, p. 235]) and a straightforward bootstrap argument.

(iv) We use (3.5), Sobolev and Gagliardo-Nirenberg inequalities and elliptic regularity theory to prove that there is $r\le R$ such that for all x satisfying $B(x,4R)\subset\Omega$, one may estimate $\|\nabla v_h\|_{L^p(B(x,r))}$ in terms of $E_{\rm GL}^\Omega(u)$ (see (3.26) below). This estimate with $p>N$ and the Morrey inequality imply that v_h is uniformly Hölder continuous on $\{x\in\Omega\mid dist(x,\partial\Omega)\ge 4R\}$. In particular, if $\Big| |1+v_h(x_0)| - 1 \Big| > \delta$ for some x_0 verifying $B(x_0, 4R) \subset \Omega$, then necessarily $\Big| |1+v_h| - 1 \Big| > \frac{\delta}{2}$ on a ball of fixed radius centered at x_0. This implies that $E^{\Omega}_{\mathrm{GL}}(v_h)$ (and, consequently, $E^{\Omega}_{\mathrm{GL}}(u)$) is bounded from below by a positive constant.

We start by estimating the nonlinear terms in (3.5). Using (3.12), we get

$$\int_{\Omega} |H(v_h)|^2 \, dx \le 9 \int_{\Omega} \left(\varphi^2(|1+v_h|) - 1 \right)^2 \, dx \le 18 E^{\Omega}_{\mathrm{GL}}(v_h) \le 18 E^{\Omega}_{\mathrm{GL}}(u);$$

hence $\|H(v_h)\|_{L^2(\Omega)} \le 3\sqrt{2} \left(E^{\Omega}_{\mathrm{GL}}(u) \right)^{\frac{1}{2}}$. By interpolation we find for any $p \in [2, \infty]$,

$$\|H(v_h)\|_{L^p(\Omega)} \le \|H(v_h)\|^{\frac{p-2}{p}}_{L^\infty(\Omega)} \|H(v_h)\|^{\frac{2}{p}}_{L^2(\Omega)} \le C \left(E^{\Omega}_{\mathrm{GL}}(u) \right)^{\frac{1}{p}} . \tag{3.13}$$

It is easy to see that $\Big| \varphi'(s^2) s \Big|^2 \le 2\varphi(s^2)$ and $\Big| \varphi'(s^2) s \Big| \le 2$ for any $s \ge 0$. Then we have

$$\int_{\Omega} \Big| \varphi'\left(|v_h - u|^2 \right) (v_h - u) \Big|^2 \, dx \le 2 \int_{\Omega} \varphi\left(|v_h - u|^2 \right) \, dx \le 2h^2 E^{\Omega}_{\mathrm{GL}}(u);$$

thus $\|\varphi'\left(|v_h - u|^2\right)(v_h - u)\|_{L^2(\Omega)} \le h \left(2E^{\Omega}_{\mathrm{GL}}(u) \right)^{\frac{1}{2}}$. By interpolation, we get

$$\begin{aligned} &\Big\| \varphi'\left(|v_h - u|^2 \right) (v_h - u) \Big\|_{L^p(\Omega)} \\ &\quad \le \Big\| \varphi'\left(|v_h - u|^2 \right) (v_h - u) \Big\|^{\frac{p-2}{p}}_{L^\infty(\Omega)} \Big\| \varphi'\left(|v_h - u|^2 \right) (v_h - u) \Big\|^{\frac{2}{p}}_{L^2(\Omega)} \\ &\quad \le C h^{\frac{2}{p}} \left(E^{\Omega}_{\mathrm{GL}}(u) \right)^{\frac{1}{p}} \end{aligned} \tag{3.14}$$

for any $p \in [2, \infty]$. From (3.5), (3.13) and (3.14), we obtain

$$\|\Delta v_h\|_{L^p(\Omega)} \le C(1 + h^{\frac{2}{p} - 2}) \left(E^{\Omega}_{\mathrm{GL}}(u) \right)^{\frac{1}{p}} \qquad \text{for any } p \ge 2. \tag{3.15}$$

For a measurable set $\omega \subset \mathbf{R}^N$ with $\mathcal{L}^N(\omega) < \infty$ and for any $f \in L^1(\omega)$, we denote by $m(f, \omega) = \frac{1}{\mathcal{L}^N(\omega)} \int_\omega f(x) \, dx$ the mean value of f on ω.

Let x_0 be such that $B(x_0, 4R) \subset \Omega$. Using the Poincaré inequality and (3.1), we have

$$\begin{aligned} \|v_h - m(v_h, B(x_0, 4R))\|_{L^2(B(x_0, 4R))} &\le C_P R \|\nabla v_h\|_{L^2(B(x_0, 4R))} \\ &\le C_P R \left(E^{\Omega}_{\mathrm{GL}}(u) \right)^{\frac{1}{2}} . \end{aligned} \tag{3.16}$$

We claim that there exist $k \in \mathbf{N}$, depending only on N, and $C_* = C_*(N, h, R)$ such that

$$\|v_h - m(v_h, B(x_0, 4R))\|_{W^{2,N}(B(x_0, \frac{R}{2^{k-2}}))} \le C_* \left(\left(E^{\Omega}_{\mathrm{GL}}(u) \right)^{\frac{1}{2}} + \left(E^{\Omega}_{\mathrm{GL}}(u) \right)^{\frac{1}{N}} \right). \tag{3.17}$$

It is well known (see Theorem 9.11 in [24, p. 235]) that for $p \in (1, \infty)$, there exists $C = C(N, r, p) > 0$ such that for any $w \in W^{2,p}(B(a, 2r))$, there holds

$$\|w\|_{W^{2,p}(B(a,r))} \leq C \left(\|w\|_{L^p(B(a,2r))} + \|\Delta w\|_{L^p(B(a,2r))} \right). \tag{3.18}$$

From (3.15), (3.16) and (3.18) we infer that

$$\|v_h - m(v_h, B(x_0, 4R))\|_{W^{2,2}(B(x_0,2R))} \leq C(N, h, R) \left(E^{\Omega}_{\mathrm{GL}}(u) \right)^{\frac{1}{2}}. \tag{3.19}$$

If $\frac{1}{2} - \frac{2}{N} \leq \frac{1}{N}$, from (3.19) and the Sobolev embedding we find

$$\|v_h - m(v_h, B(x_0, 4R))\|_{L^N(B(x_0,2R))} \leq C(N, h, R) \left(E^{\Omega}_{\mathrm{GL}}(u) \right)^{\frac{1}{2}}. \tag{3.20}$$

Then using (3.15) (for $p = N$), (3.20) and (3.18) we infer that (3.17) holds for $k = 2$.

If $\frac{1}{2} - \frac{2}{N} > \frac{1}{N}$, (3.19) and the Sobolev embedding imply

$$\|v_h - m(v_h, B(x_0, 4R))\|_{L^{p_1}(B(x_0,2R))} \leq C(N, h, R) \left(E^{\Omega}_{\mathrm{GL}}(u) \right)^{\frac{1}{2}}, \tag{3.21}$$

where $\frac{1}{p_1} = \frac{1}{2} - \frac{2}{N}$. Then (3.21), (3.15) and (3.18) give

$$\|v_h - m(v_h, B(x_0, 4R))\|_{W^{2,p_1}(B(x_0,R))} \leq C(N, h, R) \left(\left(E^{\Omega}_{\mathrm{GL}}(u) \right)^{\frac{1}{2}} + \left(E^{\Omega}_{\mathrm{GL}}(u) \right)^{\frac{1}{N}} \right). \tag{3.22}$$

If $\frac{1}{p_1} - \frac{2}{N} \leq \frac{1}{N}$, using (3.22), the Sobolev embedding, (3.15) and (3.18) we get

$$\|v_h - m(v_h, B(x_0, 4R))\|_{W^{2,N}(B(x_0,\frac{R}{2}))} \leq C(N, h, R) \left(\left(E^{\Omega}_{\mathrm{GL}}(u) \right)^{\frac{1}{2}} + \left(E^{\Omega}_{\mathrm{GL}}(u) \right)^{\frac{1}{N}} \right);$$

otherwise we repeat the process. After a finite number of steps, we find $k \in \mathbf{N}$ such that (3.17) holds.

We will use the following variant of the Gagliardo-Nirenberg inequality:

$$\|w - m(w, B(a, r))\|_{L^p(B(a,r))} \leq C(p, q, N, r) \|w\|^{\frac{q}{p}}_{L^q(B(a,2r))} \|\nabla w\|^{1-\frac{q}{p}}_{L^N(B(a,2r))} \tag{3.23}$$

for any $w \in W^{1,N}(B(a, 2r))$, where $1 \leq q \leq p < \infty$ (see, e.g., [34, p. 78]).

Using (3.23) with $w = \nabla v_h$ and (3.17), we find

$$\begin{aligned} \left\| \nabla v_h - m\left(\nabla v_h, B\left(x_0, \frac{R}{2^{k-1}} \right) \right) \right\|_{L^p(B(x_0,\frac{R}{2^{k-1}}))} & \\ \leq C \|\nabla v_h\|^{\frac{2}{p}}_{L^2(B(x_0,\frac{R}{2^{k-2}}))} \|\nabla^2 v_h\|^{1-\frac{2}{p}}_{L^N(B(x_0,\frac{R}{2^{k-2}}))} & \\ \leq C \left(E^{\Omega}_{\mathrm{GL}}(u) \right)^{\frac{1}{p}} \left(\left(E^{\Omega}_{\mathrm{GL}}(u) \right)^{\frac{1}{2}} + \left(E^{\Omega}_{\mathrm{GL}}(u) \right)^{\frac{1}{N}} \right)^{1-\frac{2}{p}} & \end{aligned} \tag{3.24}$$

for any $p \in [2, \infty)$, where the constants depend only on N, p, h, R.

Using the Cauchy-Schwarz inequality and (3.1), we have

$$\left|m\left(\nabla v_h, B\left(x_0, \frac{R}{2^{k-1}}\right)\right)\right| \leq \mathcal{L}^N\left(B\left(x_0, \frac{R}{2^{k-1}}\right)\right)^{-\frac{1}{2}} \|\nabla v_h\|_{L^2(B(x_0, \frac{R}{2^{k-1}}))} \leq C\left(E_{\mathrm{GL}}^{\Omega}(u)\right)^{\frac{1}{2}}.$$

We infer that for any $p \in [1, \infty]$, the following estimate holds:

$$\left\|m\left(\nabla v_h, B\left(x_0, \frac{R}{2^{k-1}}\right)\right)\right\|_{L^p(B(x_0, \frac{R}{2^{k-1}}))} \leq \left|m\left(\nabla v_h, B\left(x_0, \frac{R}{2^{k-1}}\right)\right)\right|\left(\mathcal{L}^N\left(B\left(x_0, \frac{R}{2^{k-1}}\right)\right)\right)^{\frac{1}{p}} \leq C(N, p, R)\left(E_{\mathrm{GL}}^{\Omega}(u)\right)^{\frac{1}{2}}. \tag{3.25}$$

From (3.24) and (3.25) we obtain for any $p \in [2, \infty)$,

$$\|\nabla v_h\|_{L^p(B(x_0, \frac{R}{2^{k-1}}))} \leq C(N, p, h, R)\left(\left(E_{\mathrm{GL}}^{\Omega}(u)\right)^{\frac{1}{2}} + \left(E_{\mathrm{GL}}^{\Omega}(u)\right)^{\frac{1}{p}+\frac{1}{N}(1-\frac{2}{p})}\right). \tag{3.26}$$

We will use the Morrey inequality, which asserts that for any $w \in C^0 \cap W^{1,p}(B(x_0, r))$ with $p > N$, there holds

$$|w(x) - w(y)| \leq C(p, N)|x - y|^{1-\frac{N}{p}}\|\nabla w\|_{L^p(B(x_0, r))} \qquad \text{for all } x, y \in B(x_0, r) \tag{3.27}$$

(see, e.g., the proof of Theorem IX.12 in [11, p. 166]). Using (3.26) and the Morrey's inequality (3.27) for $p = 2N$, we get

$$|v_h(x) - v_h(y)| \leq C(N, h, R)|x - y|^{\frac{1}{2}}\left(\left(E_{\mathrm{GL}}^{\Omega}(u)\right)^{\frac{1}{2}} + \left(E_{\mathrm{GL}}^{\Omega}(u)\right)^{\frac{1}{N}(1+\frac{1}{2^*})}\right) \tag{3.28}$$

for any $x, y \in B(x_0, \frac{R}{2^{k-1}})$.

Let $\delta > 0$. Assume that there is $x_0 \in \Omega$ such that $B(x_0, 4R) \subset \Omega$ and $\big||v_h(x_0)+1|-1\big| \geq \delta$. Since $\big||\,|v_h(x)+1|-1| - |\,|v_h(y)+1|-1|\big| \leq |v_h(x) - v_h(y)|$, from (3.28) we infer that

$$\big||v_h(x) + 1| - 1\big| \geq \frac{\delta}{2} \qquad \text{for any } x \in B(x_0, r_\delta),$$

where

$$r_\delta = \min\left(\frac{R}{2^{k-1}}, \left(\frac{\delta}{2C(N, h, R)}\right)^2\left(\left(E_{\mathrm{GL}}^{\Omega}(u)\right)^{\frac{1}{2}} + \left(E_{\mathrm{GL}}^{\Omega}(u)\right)^{\frac{1}{N}(1+\frac{1}{2^*})}\right)^{-2}\right). \tag{3.29}$$

Let

$$\eta(s) = \inf\{(\varphi^2(\tau) - 1)^2 \mid \tau \in (-\infty, 1 - s] \cup [1 + s, \infty)\}. \tag{3.30}$$

It is clear that η is nondecreasing and positive on $(0,\infty)$. We have

$$E^{\Omega}_{\mathrm{GL}}(u) \geq E^{\Omega}_{\mathrm{GL}}(v_h) \geq \frac{1}{2}\int_{B(x_0,r_\delta)} \left(\varphi^2(|1+v_h|)-1\right)^2 dx \tag{3.31}$$
$$\geq \frac{1}{2}\int_{B(x_0,r_\delta)} \eta\left(\frac{\delta}{2}\right) dx = \frac{1}{2}\mathcal{L}^N(B(0,1))\eta(\frac{\delta}{2})r_\delta^N,$$

where r_δ is given by (3.29). It is obvious that there exists a constant $K>0$, depending only on N, h, R, δ such that (3.31) cannot hold for $E^{\Omega}_{\mathrm{GL}}(u) \leq K$. We infer that $\Big| |1+v_h(x_0)|-1\Big| < \delta$ if $B(x_0,4R)\subset\Omega$ and $E^{\Omega}_{\mathrm{GL}}(u)\leq K$. This completes the proof of Lemma 3.1. □

LEMMA 3.2. *Let $(u_n)_{n\geq 1}\subset\mathcal{X}$ be a sequence of functions satisfying*

(a) *$E_{\mathrm{GL}}(u_n)$ is bounded, and*

(b) $\lim_{n\to\infty}\left(\sup_{y\in\mathbf{R}^N} E^{B(y,1)}_{\mathrm{GL}}(u_n)\right)=0.$

There exists a sequence $h_n\longrightarrow 0$ such that for any minimizer v_n of $G^{u_n}_{h_n,\mathbf{R}^N}$ in $H^1_{u_n}(\mathbf{R}^N)$, we have $\|\,|v_n+1|-1\|_{L^\infty(\mathbf{R}^N)}\longrightarrow 0$ as $n\longrightarrow\infty$.

Proof. The proof of Lemma 3.2 is quite tricky, and we split it into four steps. First we explain the choice of the sequence $(h_n)_{n\geq1}$. Then we prove that there is $C>0$ such that for any minimizer v_n of $G^{u_n}_{h_n,\mathbf{R}^N}$ and for all $x\in\mathbf{R}^N$, there holds $\|\Delta v_n\|_{L^N(B(x,1))}\leq C$. To get this estimate we write (3.5) in a convenient form, multiply it by appropriate cut-off functions, then perform integrations by parts and use elliptic regularity and a finite induction to prove that u_n and v_n are locally sufficiently close. (For instance, it follows from (3.36) and (3.51) below that $\|u_n-v_n\|_{L^2(B(x,1))}\leq Ch_n^N$ for all $x\in\mathbf{R}^N$.) Then we use (3.5) again to get the desired bound on Δv_n. In the third step we use Sobolev and Morrey inequalities to prove that v_n is uniformly Hölder continuous. Finally, if δ is fixed and $\Big||1+v_n(x_0)|-1\Big|\geq\delta$ for some $x_0\in\mathbf{R}^N$, we have necessarily $\Big||1+v_n|-1\Big|\geq\frac{\delta}{2}$ on a ball $B(x_0,r)$, where r does not depend on n; thus $\|\varphi^2(|1+v_n|)-1\|_{L^2(B(x_0,1))}$ is bounded from below by a positive constant. This is impossible for large n because $\|\varphi^2(|1+v_n|)-1\|_{L^2(B(x_0,1))}$ is close to $\|\varphi^2(|1+u_n|)-1\|_{L^2(B(x_0,1))}$ and the last quantity tends to zero by assumption (b).

Step 1. *Choice of h_n.* Let $M=\sup_{n\geq1}E_{\mathrm{GL}}(u_n)$. For $n\geq1$ and $x\in\mathbf{R}^N$, we denote

$$m_n(x)=m(u_n,B(x,1))=\frac{1}{\mathcal{L}^N(B(0,1))}\int_{B(x,1)}u_n(y)\,dy.$$

By the Poincaré inequality, there exists $C_0>0$ such that

$$\int_{B(x,1)}|u_n(y)-m_n(x)|^2\,dy\ \leq C_0\int_{B(x,1)}|\nabla u_n(y)|^2\,dy.$$

From (b) and the Poincaré inequality it follows that

$$\sup_{x\in\mathbf{R}^N}\|u_n-m_n(x)\|_{L^2(B(x,1))}\longrightarrow 0 \qquad \text{as } n\longrightarrow\infty. \tag{3.32}$$

Let H be as in Lemma 3.1(iii). From (3.12) and (b), we get

$$\sup_{x\in\mathbf{R}^N}\|H(u_n)\|^2_{L^2(B(x,1))}\le\sup_{x\in\mathbf{R}^N}9\int_{B(x,1)}\left(\varphi^2(|1+u_n(y)|)-1\right)^2\,dy\longrightarrow 0 \tag{3.33}$$

as $n\longrightarrow\infty$. It is obvious that H is Lipschitz on $\mathbf{C}$. Using (3.32), we find

$$\sup_{x\in\mathbf{R}^N}\|H(u_n)-H(m_n(x))\|_{L^2(B(x,1))}\le C_1\sup_{x\in\mathbf{R}^N}\|u_n-m_n(x)\|_{L^2(B(x,1))}\longrightarrow 0 \tag{3.34}$$

as $n\longrightarrow\infty$. From (3.33) and (3.34) we infer that

$$\sup_{x\in\mathbf{R}^N}\|H(m_n(x))\|_{L^2(B(x,1))}\longrightarrow 0 \quad \text{as } n\longrightarrow\infty.$$

Since $\|H(m_n(x))\|_{L^2(B(x,1))} = \left(\mathcal{L}^N(B(0,1))\right)^{\frac12}|H(m_n(x))|$, we have proved that

$$\lim_{n\to\infty}\sup_{x\in\mathbf{R}^N}|H(m_n(x))|=0. \tag{3.35}$$

Let

$$h_n=\max\left(\left(\sup_{x\in\mathbf{R}^N}\|u_n-m_n(x)\|_{L^2(B(x,1))}\right)^{\frac{1}{N+2}},\left(\sup_{x\in\mathbf{R}^N}|H(m_n(x))|\right)^{\frac1N}\right). \tag{3.36}$$

From (3.32) and (3.35) it follows that $h_n\longrightarrow 0$ as $n\longrightarrow\infty$. Thus we may assume that $0<h_n<1$ for any n. (If $h_n=0$, we see that u_n is constant a.e. and there is nothing to prove.)

Let v_n be a minimizer of $G^{u_n}_{h_n,\mathbf{R}^N}$. (Such minimizers exist by Lemma 3.1(i).) It follows from Lemma 3.1(iii) that v_n satisfies (3.5).

Step 2. We prove that there exist $R_N>0$ and $C>0$, independent on n, such that

$$\|\Delta v_n\|_{L^N(B(x,R_N))}\le C \qquad \text{for any } x\in\mathbf{R}^N \text{ and } n\in\mathbf{N}^*. \tag{3.37}$$

Clearly, it suffices to prove (3.37) for $x=0$. Let $m_n=m_n(0)$. Then (3.5) can be written as

$$-\Delta v_n+\frac{1}{h_n^2}\varphi'(|v_n-m_n|^2)(v_n-m_n)=f_n, \tag{3.38}$$

where

$$\begin{aligned} f_n= &-(H(v_n)-H(m_n))-H(m_n)\\ &+\frac{1}{h_n^2}\left(\varphi'(|v_n-m_n|^2)(v_n-m_n)-\varphi'(|v_n-u_n|^2)(v_n-u_n)\right). \end{aligned} \tag{3.39}$$

In view of Lemma 3.1(iii), equality (3.38) holds in $L^p_{\mathrm{loc}}(\mathbf{R}^N)$ (and not only in $\mathcal{D}'(\mathbf{R}^N)$).

The function $z \longmapsto \varphi'(|z|^2)z$ belongs to $C_c^\infty(\mathbf{C})$, and consequently it is Lipschitz. Using (3.36), we see that there exists $C_2 > 0$ such that

$$\|\varphi'(|v_n - m_n|^2)(v_n - m_n) - \varphi'(|v_n - u_n|^2)(v_n - u_n)\|_{L^2(B(0,1))} \leq C_2\|u_n - m_n\|_{L^2(B(0,1))} \leq C_2 h_n^{N+2}. \tag{3.40}$$

By (3.36), we also have

$$\|H(m_n)\|_{L^2(B(0,1))} = \left(\mathcal{L}^N(B(0,1))\right)^{\frac{1}{2}} |H(m_n)| \leq \left(\mathcal{L}^N(B(0,1))\right)^{\frac{1}{2}} h_n^N.$$

From this estimate, (3.39), (3.40) and the fact that H is Lipschitz, we get

$$\|f_n\|_{L^2(B(0,R))} \leq C_3\|v_n - m_n\|_{L^2(B(0,R))} + C_4 h_n^N \qquad \text{for any } R \in (0,1]. \tag{3.41}$$

Let $\chi \in C_c^\infty(\mathbf{R}^N, \mathbf{R})$. Taking the scalar product (in $\mathbf{C}$) of (3.38) by the function $\chi(x)(v_n(x) - m_n)$ and integrating by parts, we find

$$\int_{\mathbf{R}^N} \chi|\nabla v_n|^2\,dx + \frac{1}{h_n^2}\int_{\mathbf{R}^N} \chi\varphi'(|v_n - m_n|^2)|v_n - m_n|^2\,dx = \frac{1}{2}\int_{\mathbf{R}^N}(\Delta\chi)|v_n - m_n|^2\,dx + \int_{\mathbf{R}^N}\langle f_n(x), v_n(x) - m_n\rangle\chi(x)\,dx. \tag{3.42}$$

From (3.2), we have $\|v_n - u_n\|_{L^2(\mathbf{R}^N)} \leq C_5 h_n^{\frac{2}{N}}$; thus,

$$\|v_n - m_n\|_{L^2(B(0,1))} \leq \|v_n - u_n\|_{L^2(B(0,1))} + \|u_n - m_n\|_{L^2(B(0,1))} \leq K_0 h_n^{\frac{2}{N}}. \tag{3.43}$$

We prove that

$$\|v_n - m_n\|_{L^2(B(0,\frac{1}{2^{j-1}}))} \leq K_j h_n^{\frac{2j}{N}} \qquad \text{for } 1 \leq j \leq \left[\frac{N^2}{2}\right] + 1, \tag{3.44}$$

where K_j does not depend on n. We proceed by induction. From (3.43) it follows that (3.44) is true for $j = 1$.

Assume that (3.44) holds for some $j \in \mathbf{N}^*$, $j \leq \left[\frac{N^2}{2}\right]$. Let $\chi_j \in C_c^\infty(\mathbf{R}^N)$ be a real-valued function such that $0 \leq \chi_j \leq 1$, $\mathrm{supp}(\chi_j) \subset B(0, \frac{1}{2^{j-1}})$ and

$\chi_j = 1$ on $B(0, \frac{1}{2^j})$. Replacing χ by χ_j in (3.42) and then using the Cauchy-Schwarz inequality and (3.41), we find

(3.45)
$$\int_{B(0,\frac{1}{2^j})} |\nabla v_n|^2 \, dx + \frac{1}{h_n^2} \int_{B(0,\frac{1}{2^j})} \varphi'(|v_n - m_n|^2)|v_n - m_n|^2 \, dx$$
$$\leq \frac{1}{2} \|\Delta \chi_j\|_{L^\infty(\mathbf{R}^N)} \|v_n - m_n\|^2_{L^2(B(0,\frac{1}{2^{j-1}}))}$$
$$+ \|f_n\|_{L^2(B(0,\frac{1}{2^{j-1}}))} \|v_n - m_n\|_{L^2(B(0,\frac{1}{2^{j-1}}))}$$
$$\leq A_j \|v_n - m_n\|^2_{L^2(B(0,\frac{1}{2^{j-1}}))} + C_4 h_n^N \|v_n - m_n\|_{L^2(B(0,\frac{1}{2^{j-1}}))} \leq A_j' h_n^{\frac{4j}{N}}.$$

From (3.44) and (3.45) we infer that $\|v_n - m_n\|_{H^1(B(0,\frac{1}{2^j}))} \leq B_j h_n^{\frac{2j}{N}}$. Then the Sobolev embedding implies

$$\|v_n - m_n\|_{L^{2^*}(B(0,\frac{1}{2^j}))} \leq D_j h_n^{\frac{2j}{N}}. \tag{3.46}$$

The function $z \longmapsto \varphi(|z|^2)$ is clearly Lipschitz on $\mathbf{C}$; thus, we have

$$\int_{B(0,1)} |\varphi(|v_n - u_n|^2) - \varphi(|v_n - m_n|^2)| \, dx \leq C_6' \int_{B(0,1)} |u_n - m_n| \, dx$$
$$\leq C_6 \|u_n - m_n\|_{L^2(B(0,1))} \leq C_6 h_n^{N+2}.$$

It is clear that $\int_{B(0,1)} \varphi(|v_n - u_n|^2) \, dx \leq h_n^2 G^{u_n}_{h_n,\mathbf{R}^N}(v_n) \leq h_n^2 E_{\mathrm{GL}}(u_n) \leq h_n^2 M$ and we obtain

$$\int_{B(0,1)} \varphi(|v_n - m_n|^2) \, dx \leq C_7 h_n^2. \tag{3.47}$$

If $|v_n(x) - m_n| \geq \sqrt{2}$, we have $\varphi(|v_n(x) - m_n|^2) \geq 2$; hence,

(3.48)
$$\mathcal{L}^N(\{x \in B(0,1) \mid |v_n(x) - m_n| \geq \sqrt{2}\}) \leq \frac{1}{2} \int_{B(0,1)} \varphi\left(|v_n - m_n|^2\right) \, dx \leq \frac{C_7}{2} h_n^2.$$

By Hölder's inequality, (3.46) and (3.48), we have

(3.49)
$$\int_{\{|v_n - m_n| \geq \sqrt{2}\} \cap B(0,\frac{1}{2^j})} |v_n - m_n|^2 \, dx$$
$$\leq \|v_n - m_n\|^2_{L^{2^*}(B(0,\frac{1}{2^j}))} \left(\mathcal{L}^N(\{x \in B(0,1) \mid |v_n(x) - m_n| \geq \sqrt{2}\})\right)^{1-\frac{2}{2^*}}$$
$$\leq \left(D_j h_n^{\frac{2j}{N}}\right)^2 \left(C_7 h_n^2\right)^{1-\frac{2}{2^*}} \leq E_j h_n^{\frac{4j+4}{N}}.$$

From (3.45) it follows that

(3.50)
$$\int_{\{|v_n-m_n|<\sqrt{2}\}\cap B(0,\frac{1}{2^j})} |v_n-m_n|^2\,dx \le \int_{B(0,\frac{1}{2^j})} \varphi'(|v_n-m_n|^2)|v_n-m_n|^2\,dx$$
$$\le A'_j h_n^{2+\frac{4j}{N}} \le A'_j h_n^{\frac{4j+4}{N}}.$$

Then (3.49) and (3.50) imply that (3.44) holds for $j+1$ and the induction is complete. Thus (3.44) is established. Denoting $j_N = \left[\frac{N^2}{2}\right]+1$ and $R_N = \frac{1}{2^{j_N-1}}$, we have proved that

$$\|v_n-m_n\|_{L^2(B(0,R_N))} \le K_{j_N} h_n^{\frac{2j_N}{N}} \le K_{j_N} h_n^N. \tag{3.51}$$

It follows that

$$\int_{B(0,R_N)} \left|\frac{1}{h_n^2}\varphi'(|v_n-m_n|^2)(v_n-m_n)\right|^N dx \tag{3.52}$$
$$\le \frac{1}{h_n^{2N}} \sup_{z\in\mathbf{C}} \left|\varphi'\left(|z|^2\right)z\right|^{N-2} \int_{B(0,R_N)} |v_n-m_n|^2\,dx \le C_8.$$

Arguing as in (3.40) and using (3.36), we get

$$\|\varphi'(|v_n-m_n|^2)(v_n-m_n) - \varphi'(|v_n-u_n|^2)(v_n-u_n)\|^N_{L^N(B(0,1))} \tag{3.53}$$
$$\le C_9 \sup_{z\in\mathbf{C}} \left|\varphi'\left(|z|^2\right)z\right|^{N-2} \|u_n-m_n\|^2_{L^2(B(0,1))} \le C_{10} h_n^{2N+4}.$$

From (3.39), (3.53) and the fact that H is bounded on $\mathbf{C}$, it follows that $\|f_n\|_{L^N(B(0,R_N))} \le C_{11}$, where C_{11} is independent of n. Using this estimate, (3.52) and (3.38), we infer that (3.37) holds.

Since any ball of radius 1 can be covered by a finite number of balls of radius R_N, it follows that there exists $C>0$ such that

$$\|\Delta v_n\|_{L^N(B(x,1))} \le C \qquad \text{for any } x\in\mathbf{R}^N \text{ and } n\in\mathbf{N}^*. \tag{3.54}$$

Step 3. *The functions* v_n *are uniformly Hölder continuous.* We will use (3.18) and (3.54) to prove that there exist $\tilde{R}_N \in (0,1]$ and $C>0$ such that

$$\|v_n - m_n(x)\|_{W^{2,N}(B(x,\tilde{R}_N))} \le C \qquad \text{for any } x\in\mathbf{R}^N \text{ and } n\in\mathbf{N}^*. \tag{3.55}$$

As previously, it suffices to prove (3.55) for $x_0=0$. From (3.54) and Hölder's inequality it follows that for $1\le p\le N$, we have

$$\|\Delta v_n\|_{L^p(B(x,1))} \le \left(\mathcal{L}^N(B(0,1))\right)^{\frac{1}{p}-\frac{1}{N}} \|\Delta v_n\|_{L^N(B(x,1))} \le C(p). \tag{3.56}$$

Using (3.43), (3.56) with $p=2$ and (3.18), we obtain

$$\|v_n - m_n(0)\|_{W^{2,2}(B(x,\frac{1}{2}))} \le C. \tag{3.57}$$

If $\frac{1}{2} - \frac{2}{N} \le \frac{1}{N}$, then (3.57) and the Sobolev embedding give

$$\|v_n - m_n(0)\|_{L^N(B(x,\frac{1}{2}))} \le C.$$

This estimate, together with (3.54) and (3.18), implies that (3.55) holds for $\tilde{R}_N = \frac{1}{4}$.

If $\frac{1}{2} - \frac{2}{N} > \frac{1}{N}$, from (3.57) and the Sobolev embedding we find

$$\|v_n - m_n(0)\|_{L^{p_1}(B(x,\frac{1}{2}))} \le C,$$

where $\frac{1}{p_1} = \frac{1}{2} - \frac{2}{N}$. This estimate, (3.56) and (3.18) imply

$$\|v_n - m_n(0)\|_{W^{2,p_1}(B(x,\frac{1}{4}))} \le C.$$

If $\frac{1}{p_1} - \frac{2}{N} \le \frac{1}{N}$, from the Sobolev embedding we obtain $\|v_n - m_n(0)\|_{L^N(B(x,\frac{1}{4}))} \le C$. Then using (3.54) and (3.18), we infer that (3.55) holds for $\tilde{R}_N = \frac{1}{8}$. Otherwise we repeat the above argument. After a finite number of steps we see that (3.55) holds.

Next we proceed as in the proof of Lemma 3.1(iv). By (3.23) and (3.55) we have for $p \in [2, \infty)$ and any $x_0 \in \mathbf{R}^N$,

$$\begin{aligned}(3.58)\qquad &\|\nabla v_n - m(\nabla v_n, B(x_0, \frac{1}{2}\tilde{R}_N))\|_{L^p(B(x_0,\frac{1}{2}\tilde{R}_N))} \\ &\le C\|\nabla v_n\|^{\frac{2}{p}}_{L^2(B(x_0,\tilde{R}_N))} \|\nabla^2 v_n\|^{1-\frac{2}{p}}_{L^N(B(x_0,\tilde{R}_N))} \le C_1(p).\end{aligned}$$

Arguing as in (3.25), we see $\|m(\nabla v_n, B(x_0, \frac{1}{2}\tilde{R}_N))\|_{L^p(B(x_0,\frac{1}{2}\tilde{R}_N))}$ is bounded independently on n and hence

$$\|\nabla v_n\|_{L^p(B(x_0,\frac{1}{2}\tilde{R}_N))} \le C_2(p) \qquad \text{for any } n \in \mathbf{N}^* \text{ and } x_0 \in \mathbf{R}^N.$$

Using this estimate for $p = 2N$ together with the Morrey inequality (3.27), we see that there exists $C_* > 0$ such that for any $x, y \in \mathbf{R}^N$ with $|x - y| \le \frac{\tilde{R}_N}{2}$ and any $n \in \mathbf{N}^*$, we have

$$(3.59)\qquad |v_n(x) - v_n(y)| \le C_*|x - y|^{\frac{1}{2}}.$$

Step 4. *Conclusion.* Let $\delta_n = \|\, |v_n + 1| - 1\|_{L^\infty(\mathbf{R}^N)}$, and choose $x_n \in \mathbf{R}^N$ such that $\Big|\, |v_n(x_n)+1| - 1\Big| \ge \frac{\delta_n}{2}$. From (3.59) it follows that $\Big|\, |v_n(x)+1| - 1\Big| \ge \frac{\delta_n}{4}$ for all $x \in B(x_n, r_n)$, where

$$r_n = \min\left(\frac{\tilde{R}_N}{2}, \left(\frac{\delta_n}{4C_*}\right)^2\right).$$

Then we have

(3.60)
$$\int_{B(x_n,1)} \left(\varphi^2(|1+v_n(y)|)-1\right)^2 \, dy \geq \int_{B(x_n,r_n)} \left(\varphi^2(|1+v_n(y)|)-1\right)^2 \, dy$$
$$\geq \int_{B(x_n,r_n)} \eta\left(\frac{\delta_n}{4}\right) dy = \mathcal{L}^N(B(0,1))\eta\left(\frac{\delta_n}{4}\right) r_n^N,$$

where η is as in (3.30).

On the other hand, the function $z \longmapsto \left(\varphi^2(|1+z|)-1\right)^2$ is Lipschitz on $\mathbf{C}$. Using this fact, the Cauchy-Schwarz inequality, (3.2) and assumption (a), we get

$$\int_{B(x,1)} \left| \left(\varphi^2(|1+v_n(y)|)-1\right)^2 - \left(\varphi^2(|1+u_n(y)|)-1\right)^2 \right| dy$$
$$\leq C \int_{B(x,1)} |v_n(y)-u_n(y)| \, dy \leq C' \|v_n - u_n\|_{L^2(B(x,1))}$$
$$\leq C' \|v_n - u_n\|_{L^2(\mathbf{R}^N)} \leq C'' h_n^{\frac{2}{N}}.$$

Then using assumption (b), we infer that

$$\sup_{x\in\mathbf{R}^N} \int_{B(x,1)} \left(\varphi^2(|1+v_n(y)|)-1\right)^2 \, dy \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty. \tag{3.61}$$

From (3.60) and (3.61) we get $\lim_{n\to\infty} \eta\left(\frac{\delta_n}{4}\right) r_n^N = 0$, and this clearly implies $\lim_{n\to\infty} \delta_n = 0$. Lemma 3.2 is thus proven. □

The next result is based on Lemma 3.1 and will be very useful in the next sections to prove the "concentration" of minimizing sequences. For $0 < R_1 < R_2$, we denote $\Omega_{R_1,R_2} = B(0,R_2) \setminus \overline{B}(0,R_1)$.

LEMMA 3.3. *Let $A > A_3 > A_2 > 1$. There exist $\varepsilon_0 = \varepsilon_0(N, A, A_2, A_3) > 0$ and $C_i = C_i(N, A, A_2, A_3) > 0$ such that for any $R \geq 1$, $\varepsilon \in (0, \varepsilon_0)$ and $u \in \mathcal{X}$ verifying $E_{\mathrm{GL}}^{\Omega_{R,AR}}(u) \leq \varepsilon$, there exist two functions u_1, $u_2 \in \mathcal{X}$ and a constant $\theta_0 \in [0, 2\pi)$ satisfying the following properties*:

(i) $\mathrm{supp}(u_1) \subset B(0, A_2R)$ *and* $1 + u_1 = e^{-i\theta_0}(1+u)$ *on* $B(0,R)$;

(ii) $u_2 = u$ *on* $\mathbf{R}^N \setminus B(0, AR)$ *and* $1 + u_2 = e^{i\theta_0} = $ *constant on* $B(0, A_3R)$;

(iii) $\displaystyle\int_{\mathbf{R}^N} \left| \left|\frac{\partial u}{\partial x_j}\right|^2 - \left|\frac{\partial u_1}{\partial x_j}\right|^2 - \left|\frac{\partial u_2}{\partial x_j}\right|^2 \right| dx \leq C_1 \varepsilon$ *for* $j = 1, \dots, N$;

(iv) $\displaystyle\int_{\mathbf{R}^N} \left| \left(\varphi^2(|1+u|)-1\right)^2 - \left(\varphi^2(|1+u_1|)-1\right)^2 - \left(\varphi^2(|1+u_2|)-1\right)^2 \right| dx \leq C_2 \varepsilon$;

(v) $|Q(u) - Q(u_1) - Q(u_2)| \leq C_3 \varepsilon$;

(vi) *If assumptions* (A1) *and* (A2) *in the introduction hold*, *then*

$$\int_{\mathbf{R}^N} \Big| V(|1+u|^2) - V(|1+u_1|^2) - V(|1+u_2|^2) \Big| \, dx \le C_4 \varepsilon + C_5 \sqrt{\varepsilon} \, (E_{\mathrm{GL}}(u))^{\frac{2^*-1}{2}} .$$

Proof. Fix $k > 0$, A_1 and A_4 such that $1 + 4k < A_1 < A_2 < A_3 < A_4 < A - 4k$. Let $h = 1$ and $\delta = \frac{1}{2}$. We will prove that Lemma 3.3 holds for $\varepsilon_0 = K(N, h = 1, \delta = \frac{1}{2}, k)$, where $K(N, h, \delta, R)$ is as in Lemma 3.1(iv).

Fix two functions $\eta_1, \eta_2 \in C^\infty(\mathbf{R})$ satisfying the following properties:

$$\eta_1 = 1 \text{ on } (-\infty, A_1], \quad \eta_1 = 0 \text{ on } [A_2, \infty), \quad \eta_1 \text{ is nonincreasing;}$$

$$\eta_2 = 0 \text{ on } (-\infty, A_3], \quad \eta_2 = 1 \text{ on } [A_4, \infty), \quad \eta_2 \text{ is nondecreasing.}$$

Let $\varepsilon < \varepsilon_0$, and let $u \in \mathcal{X}$ be such that $E_{\mathrm{GL}}^{\Omega_{R,AR}}(u) \le \varepsilon$. Let v_1 be a minimizer of $G^u_{1,\Omega_{R,AR}}$ in the space $H^1_u(\Omega_{R,AR})$. The existence of v_1 is guaranteed by Lemma 3.1. We have $v_1 = u$ on $\mathbf{R}^N \setminus \Omega_{R,AR}$. By Lemma 3.1(iii), we know that $v_1 \in W^{2,p}_{\mathrm{loc}}(\Omega_{R,AR})$ for any $p \in [1, \infty)$. Moreover, since $E_{\mathrm{GL}}^{\Omega_{R,AR}}(u) \le K(N, 1, \frac{r_0}{2}, k)$, Lemma 3.1(iv) implies that

$$\frac{1}{2} < |1 + v_1(x)| < \frac{3}{2} \qquad \text{if } R + 4k \le |x| \le AR - 4k. \tag{3.62}$$

Since $N \ge 3$, Ω_{A_1R,A_4R} is simply connected, and it follows directly from Theorem 3 in [10, p. 38] that there exist two real-valued functions ρ, $\theta \in W^{2,p}(\Omega_{A_1R,A_4R})$, $1 \le p < \infty$, such that

$$1 + v_1(x) = \rho(x) e^{i\theta(x)} \qquad \text{in } \Omega_{A_1R,A_4R}. \tag{3.63}$$

For $j = 1, \dots, N$, we have

$$\frac{\partial v_1}{\partial x_j} = \left(\frac{\partial \rho}{\partial x_j} + i\rho \frac{\partial \theta}{\partial x_j} \right) e^{i\theta} \text{ and } \left| \frac{\partial v_1}{\partial x_j} \right|^2 = \left| \frac{\partial \rho}{\partial x_j} \right|^2 + \rho^2 \left| \frac{\partial \theta}{\partial x_j} \right|^2 \tag{3.64}$$

a.e. in Ω_{A_1R,A_4R}. Thus we get the following estimates:

$$\int_{\Omega_{A_1R,\,A_4R}} |\nabla \rho|^2 \, dx \le \int_{\Omega_{A_1R,\,A_4R}} |\nabla v_1|^2 \, dx \le \varepsilon, \tag{3.65}$$

$$\frac{1}{2} \int_{\Omega_{A_1R,\,A_4R}} \left(\rho^2 - 1 \right)^2 \, dx \le E_{\mathrm{GL}}^{\Omega_{A_1R,\,A_4R}}(v_1) \le \varepsilon, \tag{3.66}$$

$$\int_{\Omega_{A_1R,\,A_4R}} |\nabla \theta|^2 \, dx \le 4 \int_{\Omega_{A_1R,\,A_4R}} |\nabla v_1|^2 \, dx \le 4\varepsilon. \tag{3.67}$$

The Poincaré inequality and a scaling argument imply that

$$\int_{\Omega_{A_1R,\,A_4R}} |f - m(f, \Omega_{A_1R,\,A_4R})|^2 \, dx \le C(N, A_1, A_4) R^2 \int_{\Omega_{A_1R,\,A_4R}} |\nabla f|^2 \, dx \tag{3.68}$$

for any $f \in H^1(\Omega_{A_1R,\,A_4R})$, where $C(N, A_1, A_4)$ does not depend on R. Let $\theta_0 = m(\theta, \Omega_{A_1R,\,A_4R})$. We may assume that $\theta_0 \in [0, 2\pi)$. (Otherwise we replace

θ by $\theta - 2\pi \left[\frac{\theta_0}{2\pi}\right]$.) Using (3.67) and (3.68), we get

$$(3.69) \qquad \int_{\Omega_{A_1 R,\, A_4 R}} |\theta - \theta_0|^2 \, dx \le C(N, A_1, A_4) R^2 \int_{\Omega_{A_1 R,\, A_4 R}} |\nabla v_1|^2 \, dx \le C(N, A_1, A_4) R^2 \varepsilon.$$

We define $\tilde{u}_1$ and u_2 by

$$(3.70) \qquad \tilde{u}_1(x) = \begin{cases} u(x) & \text{if } x \in \overline{B}(0, R), \\ v_1(x) & \text{if } x \in B(0, A_1 R) \setminus \overline{B}(0, R), \\ \left(1 + \eta_1\left(\frac{|x|}{R}\right)(\rho(x) - 1)\right) e^{i\left(\theta_0 + \eta_1\left(\frac{|x|}{R}\right)(\theta(x) - \theta_0)\right)} - 1 & \\ \qquad \text{if } x \in B(0, A_4 R) \setminus B(0, A_1 R), & \\ e^{i\theta_0} - 1 & \text{if } x \in \mathbf{R}^N \setminus B(0, A_4 R), \end{cases}$$

$$(3.71) \qquad u_2(x) = \begin{cases} e^{i\theta_0} - 1 & \text{if } x \in \overline{B}(0, A_1 R), \\ \left(1 + \eta_2\left(\frac{|x|}{R}\right)(\rho(x) - 1)\right) e^{i\left(\theta_0 + \eta_2\left(\frac{|x|}{R}\right)(\theta(x) - \theta_0)\right)} - 1 & \\ \qquad \text{if } x \in B(0, A_4 R) \setminus \overline{B}(0, A_1 R), & \\ v_1(x) & \text{if } x \in B(0, AR) \setminus B(0, A_4 R), \\ u(x) & \text{if } x \in \mathbf{R}^N \setminus B(0, AR). \end{cases}$$

Then we define u_1 in such a way that $1 + u_1 = e^{-i\theta_0}(1 + \tilde{u}_1)$. Since $u \in \mathcal{X}$ and $u - v_1 \in H_0^1(\Omega_{R,\, AR})$, it is clear that $u_1 \in H^1(\mathbf{R}^N)$, $u_2 \in \mathcal{X}$ and (i), (ii) hold.

Since $\rho + 1 \ge \frac{3}{2}$ on $\Omega_{A_1 R,\, A_4 R}$, from (3.66) we get

$$(3.72) \qquad \|\rho - 1\|^2_{L^2(\Omega_{A_1 R,\, A_4 R})} \le \frac{8}{9}\varepsilon.$$

Obviously,

$$\nabla\left(1 + \eta_i\left(\frac{|x|}{R}\right)(\rho(x) - 1)\right) = \frac{1}{R}\eta_i'\left(\frac{|x|}{R}\right)(\rho(x) - 1)\frac{x}{|x|} + \eta_i\left(\frac{|x|}{R}\right)\nabla\rho.$$

Using (3.65), (3.72) and the fact that $R \ge 1$, we get

$$(3.73) \qquad \left\|\nabla\left(1 + \eta_i\left(\frac{|x|}{R}\right)(\rho(x) - 1)\right)\right\|_{L^2(\Omega_{A_1 R,\, A_4 R})} \le \frac{1}{R}\sup|\eta_i'| \cdot \|\rho - 1\|_{L^2(\Omega_{A_1 R,\, A_4 R})} + \left\|\eta_i\left(\frac{|\cdot|}{R}\right)\nabla\rho\right\|_{L^2(\Omega_{A_1 R,\, A_4 R})} \le C\sqrt{\varepsilon}.$$

Similarly, using (3.67) and (3.69), we find

(3.74)
$$\left\| \nabla \left(\theta_0 + \eta_i(\frac{|x|}{R})(\theta(x) - \theta_0) \right) \right\|_{L^2(\Omega_{A_1 R,\, A_4 R})} \leq \frac{1}{R} \sup |\eta_i'| \cdot \|\theta - \theta_0\|_{L^2(\Omega_{A_1 R,\, A_4 R})} + \left\| \eta_i \left(\frac{|\cdot|}{R} \right) \nabla \theta \right\|_{L^2(\Omega_{A_1 R,\, A_4 R})} \leq C\sqrt{\varepsilon}.$$

From (3.73), (3.74) and the definition of u_1, u_2 it follows that $\|\nabla u_i\|_{L^2(\Omega_{A_1 R, A_4 R})} \leq C\sqrt{\varepsilon}$ for $i = 1, 2$. Therefore

$$\int_{\mathbf{R}^N} \left| \left| \frac{\partial u}{\partial x_j} \right|^2 - \left| \frac{\partial u_1}{\partial x_j} \right|^2 - \left| \frac{\partial u_2}{\partial x_j} \right|^2 \right| dx = \int_{\Omega_{R,AR}} \left| \left| \frac{\partial u}{\partial x_j} \right|^2 - \left| \frac{\partial u_1}{\partial x_j} \right|^2 - \left| \frac{\partial u_2}{\partial x_j} \right|^2 \right| dx$$
$$\leq \int_{\Omega_{R,A_1 R} \cup \Omega_{A_4 R, AR}} \left| \frac{\partial u}{\partial x_j} \right|^2 + \left| \frac{\partial v_1}{\partial x_j} \right|^2 dx$$
$$+ \int_{\Omega_{A_1 R, A_4 R}} \left| \frac{\partial u}{\partial x_j} \right|^2 + \left| \frac{\partial u_1}{\partial x_j} \right|^2 + \left| \frac{\partial u_2}{\partial x_j} \right|^2 dx \leq C_1 \varepsilon$$

and (iii) is proven.

On $\Omega_{A_1 R, A_4 R}$, we have $\rho \in [\frac{1}{2}, \frac{3}{2}]$; hence $\varphi\left(1 + \eta_i\left(\frac{|x|}{R}\right)(\rho(x) - 1)\right) = 1 + \eta_i\left(\frac{|x|}{R}\right)(\rho(x) - 1)$ and

$$\left(\varphi^2\left(1 + \eta_i\left(\frac{|x|}{R}\right)(\rho(x) - 1)\right) - 1\right)^2 \tag{3.75}$$
$$= (\rho(x) - 1)^2 \eta_i^2 \left(\frac{|x|}{R}\right) \left(2 + \eta_i\left(\frac{|x|}{R}\right)(\rho(x) - 1)\right)^2$$
$$\leq \frac{25}{4}(\rho(x) - 1)^2 \leq \frac{25}{8}|\rho(x) - 1|.$$

From (3.70)–(3.72) and (3.75), it follows that $\|\varphi^2(|1 + u_i|) - 1\|_{L^2(\Omega_{A_1 R,\, A_4 R})} \leq C\sqrt{\varepsilon}$. As above, we get

$$\int_{\mathbf{R}^N} \left| \left(\varphi^2(|1 + u|) - 1\right)^2 - \left(\varphi^2(|1 + u_1|) - 1\right)^2 - \left(\varphi^2(|1 + u_2|) - 1\right)^2 \right| dx$$
$$\leq \int_{\Omega_{R,A_1 R} \cup \Omega_{A_4 R, AR}} \left(\varphi^2(|1 + u|) - 1\right)^2 + \left(\varphi^2(|1 + v_1|) - 1\right)^2 dx$$
$$+ \int_{\Omega_{A_1 R, A_4 R}} \left(\varphi^2(|1 + u|) - 1\right)^2 + \left(\varphi^2(|1 + u_1|) - 1\right)^2$$
$$+ \left(\varphi^2(|1 + u_2|) - 1\right)^2 dx \leq C_2 \varepsilon.$$

This proves (iv).

Next we prove (v). Since $\langle i\frac{\partial \tilde{u}_1}{\partial x_1}, \tilde{u}_1\rangle$ has compact support, a simple computation gives

$$
\begin{aligned}
(3.76)\quad Q(u_1) &= L\left(\left\langle i\frac{\partial u_1}{\partial x_1}, u_1\right\rangle\right)\\
&= L\left(\left\langle ie^{-i\theta_0}\frac{\partial \tilde{u}_1}{\partial x_1}, e^{-i\theta_0} - 1 + e^{-i\theta_0}\tilde{u}_1\right\rangle\right) = \int_{\mathbf{R}^N}\left\langle i\frac{\partial \tilde{u}_1}{\partial x_1}, \tilde{u}_1\right\rangle dx.
\end{aligned}
$$

From the definition of $\tilde{u}_1$ and u_2 and the fact that $u = v_1$ on $\mathbf{R}^N \setminus \Omega_{R,AR}$, we get

$$
\left\langle i\frac{\partial v_1}{\partial x_1}, v_1\right\rangle - \left\langle i\frac{\partial \tilde{u}_1}{\partial x_1}, \tilde{u}_1\right\rangle - \left\langle i\frac{\partial u_2}{\partial x_1}, u_2\right\rangle = 0 \qquad \text{a.e. on } \mathbf{R}^N \setminus \Omega_{A_1R,\, A_4R}.
$$

Using this identity, Definition 2.4, (3.76), then (2.3) and (3.70), (3.71), we obtain
(3.77)

$$
\begin{aligned}
&Q(v_1) - Q(u_1) - Q(u_2)\\
&= \int_{\Omega_{A_1R,\, A_4R}} \left\langle i\frac{\partial v_1}{\partial x_1}, v_1\right\rangle - \left\langle i\frac{\partial \tilde{u}_1}{\partial x_1}, \tilde{u}_1\right\rangle - \left\langle i\frac{\partial u_2}{\partial x_1}, u_2\right\rangle dx\\
&= \int_{\Omega_{A_1R,\, A_4R}} \mathrm{Im}\left(\frac{\partial v_1}{\partial x_1} - \frac{\partial \tilde{u}_1}{\partial x_1} - \frac{\partial u_2}{\partial x_1}\right) dx - \int_{\Omega_{A_1R,\, A_4R}} (\rho^2 - 1)\frac{\partial \theta}{\partial x_1}\, dx\\
&\quad + \int_{\Omega_{A_1R,\, A_4R}} \sum_{i=1}^{2}\left(\left(1 + \eta_i(\frac{|x|}{R})(\rho - 1)\right)^2 - 1\right)\frac{\partial}{\partial x_1}\left(\theta_0 + \eta_i(\frac{|x|}{R})(\theta - \theta_0)\right) dx\\
&\quad - \int_{\Omega_{A_1R,\, A_4R}} \frac{\partial \theta}{\partial x_1} - \sum_{i=1}^{2}\frac{\partial}{\partial x_1}\left(\theta_0 + \eta_i(\frac{|x|}{R})(\theta(x) - \theta_0)\right) dx.
\end{aligned}
$$

The functions $v_1 - \tilde{u}_1 - u_2$ and $\theta^* = \theta - \sum_{i=1}^2 \left(\theta_0 + \eta_i(\frac{|x|}{R})(\theta(x) - \theta_0)\right)$ belong to $C^1(\Omega_{R,AR})$ and $v_1 - \tilde{u}_1 - u_2 = 1 - e^{i\theta_0} = \text{const.}$, $\theta^* = -\theta_0 = \text{const.}$ on $\Omega_{R,AR} \setminus \Omega_{A_1R,A_4R}$. Therefore,
(3.78)

$$
\int_{\Omega_{A_1R,\, A_4R}} \frac{\partial}{\partial x_1}\left(\mathrm{Im}(v_1 - \tilde{u_1} - u_2)\right) dx = 0 \qquad \text{and} \qquad \int_{\Omega_{A_1R,\, A_4R}} \frac{\partial \theta^*}{\partial x_1}\, dx = 0.
$$

Using (3.66), (3.67) and the Cauchy-Schwarz inequality, we have

$$
(3.79)\qquad \left|\int_{\Omega_{A_1R,\, A_4R}} (\rho^2 - 1)\frac{\partial \theta}{\partial x_1}\, dx\right| \leq 2\sqrt{2}\varepsilon.
$$

Similarly, from (3.72), (3.74), (3.75) and the Cauchy-Schwarz inequality, we get
(3.80)

$$
\left|\int_{\Omega_{A_1R,\, A_4R}} \left(\left(1 + \eta_i(\frac{|x|}{R})(\rho - 1)\right)^2 - 1\right)\frac{\partial}{\partial x_1}\left(\theta_0 + \eta_i(\frac{|x|}{R})(\theta - \theta_0)\right) dx\right| \leq C\varepsilon.
$$

From (3.77)–(3.80) we obtain $|Q(v_1) - Q(u_1) - Q(u_2)| \le C\varepsilon$, and (3.4) gives $|Q(u) - Q(v_1)| \le C E_{\rm GL}^{\Omega_{R,AR}}(u) \le C\varepsilon$. These estimates clearly imply (v).

It remains to prove (vi). Assume that assumptions (A1) and (A2) in the introduction are satisfied, and let $W(s) = V(s) - V(\varphi^2(\sqrt{s}))$ so that $W(s) = 0$ for $s \in [0, 4]$. It is not hard to see that there exists $C > 0$ such that

(3.81)
$$|W(b^2) - W(a^2)| \le C|b-a| \left(a^{2p_0+1}\mathbb{1}_{\{a>2\}} + b^{2p_0+1}\mathbb{1}_{\{b>2\}}\right) \quad \text{for any } a, b \ge 0.$$

Using (1.7) and (3.81), then Hölder's inequality, we obtain

(3.82)
$$\begin{aligned}
&\int_{\mathbf{R}^N} \left|V(|1+u|^2) - V(|1+v_1|^2)\right| dx \\
&\quad \le \int_{\Omega_{R,\,AR}} \left|V(\varphi^2(|1{+}u|)) {-} V(\varphi^2(|1{+}v_1|))\right| \\
&\qquad + \left|W(|1{+}u|^2) {-} W(|1{+}v_1|^2)\right| dx \\
&\quad \le C\int_{\Omega_{R,\,AR}} \left(\varphi^2(|1+u|) - 1\right)^2 + \left(\varphi^2(|1+v_1|) - 1\right)^2 \, dx \\
&\qquad + C\int_{\Omega_{R,\,AR}} \Big||1+u| - |1+v_1|\Big| \Big(|1+u|^{2p_0+1}\mathbb{1}_{\{|1+u|>2\}} \\
&\qquad\qquad + |1+v_1|^{2p_0+1}\mathbb{1}_{\{|1+v_1|>2\}}\Big) \, dx \\
&\quad \le C'\varepsilon + C'\int_{\Omega_{R,\,AR}} |u{-}v_1| \left(|1{+}u|^{2^*-1}\mathbb{1}_{\{|1+u|>2\}} + |1{+}v_1|^{2^*-1}\mathbb{1}_{\{|1+v_1|>2\}}\right) dx \\
&\quad \le C'\varepsilon + C'\|u - v_1\|_{L^{2^*}(\Omega_{R,\,AR})} \Big(\| \, |1+u|\mathbb{1}_{\{|1+u|>2\}}\|^{2^*-1}_{L^{2^*}(\Omega_{R,\,AR})} \\
&\qquad\qquad + \| \, |1+v_1|\mathbb{1}_{\{|1+v_1|>2\}}\|^{2^*-1}_{L^{2^*}(\Omega_{R,\,AR})}\Big).
\end{aligned}$$

From the Sobolev embedding, we have

(3.83)
$$\begin{aligned}
\|u - v_1\|_{L^{2^*}(\mathbf{R}^N)} &\le C_S \|\nabla(u - v_1)\|_{L^2(\mathbf{R}^N)} \\
&\le C_S(\|\nabla u\|_{L^2(\Omega_{R,\,AR})} + \|\nabla v_1\|_{L^2(\Omega_{R,\,AR})}) \le 2C_S\sqrt{\varepsilon}.
\end{aligned}$$

It is clear that $|1 + u| > 2$ implies $|u| > 1$ and $|1 + u| < 2|u|$; hence,

(3.84)
$$\begin{aligned}
\| \, |1+u|\mathbb{1}_{\{|1+u|>2\}}\|_{L^{2^*}(\Omega_{R,\,AR})} &\le 2\|u\|_{L^{2^*}(\mathbf{R}^N)} \\
&\le 2C_S\|\nabla u\|_{L^2(\mathbf{R}^N)} \le 2C_S \left(E_{\rm GL}(u)\right)^{\frac{1}{2}}.
\end{aligned}$$

Obviously, a similar estimate holds for v_1. Combining (3.82), (3.83) and (3.84), we find

(3.85)
$$\int_{\Omega_{R,\,AR}} \left|V(|1+u|^2) - V(|1+v_1|^2)\right| dx \le C'\varepsilon + C''\sqrt{\varepsilon}\left(E_{\rm GL}(u)\right)^{\frac{2^*-1}{2}}.$$

From (3.70) and (3.71) it follows that $V(|1+v_1|^2)-V(|1+u_1|^2)-V(|1+u_2|^2)=0$ on $\mathbf{R}^N \setminus \Omega_{A_1R,\,A_4R}$ and $|1+v_1|$, $|1+u_1|$, $|1+u_2| \in \left[\frac{1}{2}, \frac{3}{2}\right]$ on $\Omega_{A_1R,\,A_4R}$. Then using (1.7), (3.66), (3.75) and (3.72), we get

$$\int_{\Omega_{A_1R,\,A_4R}} |V(|1+v_1|^2)|\,dx \le C \int_{\Omega_{A_1R,\,A_4R}} (\rho^2-1)^2\,dx \le C\varepsilon, \tag{3.86}$$

respectively

$$\int_{\Omega_{A_1R,\,A_4R}} |V(|1+u_i|^2)|\,dx \le C \int_{\Omega_{A_1R,\,A_4R}} \left(\left(1+\eta_i\Big(\frac{|x|}{R}\Big)(\rho-1)\right)^2 - 1\right)^2 dx \le C\varepsilon. \tag{3.87}$$

Therefore,

$$\begin{aligned} &\int_{\mathbf{R}^N} \left| V(|1+v_1|^2) - V(|1+u_1|^2) - V(|1+u_2|^2) \right| dx \\ &\quad \le \int_{\Omega_{A_1R,\,A_4R}} |V(|1+v_1|^2)| + |V(|1+u_1|^2)| + |V(|1+u_2|^2)|\,dx \le C\varepsilon. \end{aligned} \tag{3.88}$$

Then (iv) follows from (3.85) and (3.88) and Lemma 3.3 is proven. □

4. **The variational framework**

The aim of this section is to study the properties of the functionals E_c, A, B_c and P_c introduced in (1.10), (1.12), (1.13) and (1.14), respectively. We assume throughout that assumptions (A1) and (A2) in the introduction are satisfied. Let

$$\mathcal{C} = \{u \in \mathcal{X} \mid u \neq 0, P_c(u) = 0\}.$$

In particular, we will prove that $\mathcal{C} \neq \emptyset$ and $\inf\{E_c(u) \mid u \in \mathcal{C}\} > 0$. This will be done in a sequence of lemmas. In the next sections we show that E_c admits a minimizer in $\mathcal{C}$ and this minimizer is a solution of (1.4).

We begin by proving that the above mentioned functionals are well defined on $\mathcal{X}$. Since we have already seen in Section 2 that Q is well defined on $\mathcal{X}$, all we have to do is to prove that $V(|1+u|^2) \in L^1(\mathbf{R}^N)$ for any $u \in \mathcal{X}$. This will be done in the next lemma.

Lemma 4.1. *For any $u \in \mathcal{X}$, we have $V(|1+u|^2) \in L^1(\mathbf{R}^N)$. Moreover, for any $\delta > 0$, there exist $C_1(\delta)$, $C_2(\delta) > 0$ such that for any $u \in \mathcal{X}$, we have*

$$\begin{aligned} &\frac{1-\delta}{2} \int_{\mathbf{R}^N} \left(\varphi^2(|1+u|) - 1\right)^2 dx - C_1(\delta) \|\nabla u\|^{2^*}_{L^2(\mathbf{R}^N)} \\ &\quad \le \int_{\mathbf{R}^N} V(|1+u|^2)\,dx \\ &\quad \le \frac{1+\delta}{2} \int_{\mathbf{R}^N} \left(\varphi^2(|1+u|) - 1\right)^2 dx + C_2(\delta) \|\nabla u\|^{2^*}_{L^2(\mathbf{R}^N)}. \end{aligned} \tag{4.1}$$

Proof. Fix $\delta > 0$. Using (1.6) we see that there exists $\beta = \beta(\delta) \in (0, 1]$ such that

$$\frac{1-\delta}{2}(s-1)^2 \le V(s) \le \frac{1+\delta}{2}(s-1)^2 \quad \text{for any } s \in ((1-\beta)^2, (1+\beta)^2). \tag{4.2}$$

Let $u \in \mathcal{X}$. If $|u(x)| < \beta$ we have $|1+u(x)|^2 \in ((1-\beta)^2, (1+\beta)^2)$ and it follows from (4.2) that $V(|1+u|^2)\mathbb{1}_{\{|u|<\beta\}} \in L^1(\mathbf{R}^N)$ and

$$\begin{aligned} \frac{1-\delta}{2} \int_{\{|u|<\beta\}} \left(\varphi^2(|1+u|) - 1\right)^2 dx &\le \int_{\{|u|<\beta\}} V(|1+u|^2)\, dx \\ &\le \frac{1+\delta}{2} \int_{\{|u|<\beta\}} \left(\varphi^2(|1+u|) - 1\right)^2 dx. \end{aligned} \tag{4.3}$$

Assumption (A2) implies that there exists $C_1'(\delta) > 0$ such that

$$\left| V(|1+z|^2) - \frac{1-\delta}{2}(\varphi^2(|1+z|) - 1)^2 \right| \le C_1'(\delta)|z|^{2p_0+2} \le C_1''(\delta)|z|^{2^*}$$

for any $z \in \mathbf{C}$ satisfying $|z| \ge \beta$. Using the Sobolev embedding, we obtain

$$\begin{aligned} &\int_{\{|u|\ge\beta\}} \left| V(|1+u|^2) - \frac{1-\delta}{2}(\varphi^2(|1+u|) - 1)^2 \right| dx \\ &\quad \le C_1''(\delta) \int_{\{|u|\ge\beta\}} |u|^{2^*} dx \le C_1''(\delta) \int_{\mathbf{R}^N} |u|^{2^*} dx \le C_1(\delta) \|\nabla u\|_{L^2(\mathbf{R}^N)}^{2^*}. \end{aligned} \tag{4.4}$$

Consequently, $V(|1+u|^2)\mathbb{1}_{\{|u|\ge\beta\}} \in L^1(\mathbf{R}^N)$ and it follows from (4.3) and (4.4) that the first inequality in (4.1) holds; the proof of the second inequality is similar. □

LEMMA 4.2. *Let $\delta \in (0,1)$ and let $u \in \mathcal{X}$ such that $1-\delta \le |1+u| \le 1+\delta$ a.e. on $\mathbf{R}^N$. Then*

$$|Q(u)| \le \frac{1}{\sqrt{2}(1-\delta)} E_{\mathrm{GL}}(u).$$

Proof. From Lemma 2.1 we know that there are two real-valued functions ρ, θ such that $\rho - 1 \in H^1(\mathbf{R}^N)$, $\theta \in \mathcal{D}^{1,2}(\mathbf{R}^N)$ and $1 + u = \rho e^{i\theta}$ a.e. on $\mathbf{R}^N$. Moreover, from (2.3) and Definition 2.4 we infer that

$$Q(u) = -\int_{\mathbf{R}^N} (\rho^2 - 1)\theta_{x_1}\, dx.$$

Using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} \sqrt{2}(1-\delta)|Q(u)| &\le \sqrt{2}(1-\delta)\|\theta_{x_1}\|_{L^2(\mathbf{R}^N)}\|\rho^2 - 1\|_{L^2(\mathbf{R}^N)} \\ &\le (1-\delta)^2 \int_{\mathbf{R}^N} |\theta_{x_1}|^2\, dx + \frac{1}{2}\int_{\mathbf{R}^N} \left(\rho^2 - 1\right)^2 dx \\ &\le \int_{\mathbf{R}^N} \rho^2|\nabla\theta|^2 + \frac{1}{2}\left(\rho^2 - 1\right)^2 dx \le E_{\mathrm{GL}}(u). \qquad \square \end{aligned}$$

LEMMA 4.3. *Assume that* $0 \le c < v_s$, *and let* $\varepsilon \in (0, 1 - \frac{c}{v_s})$. *There exists a constant* $K_1 = K_1(F, N, c, \varepsilon) > 0$ *such that for any* $u \in \mathcal{X}$ *satisfying* $E_{\rm GL}(u) < K_1$, *we have*

$$\int_{\mathbf{R}^N} |\nabla u|^2 \, dx + \int_{\mathbf{R}^N} V(|1+u|^2) \, dx - c|Q(u)| \ge \varepsilon E_{\rm GL}(u).$$

Proof. Fix ε_1 such that $\varepsilon < \varepsilon_1 < 1 - \frac{c}{v_s}$. Then fix $\delta_1 \in (0, \varepsilon_1 - \varepsilon)$. By Lemma 4.1, there exists $C_1(\delta_1) > 0$ such that for any $u \in \mathcal{X}$, there holds

$$\int_{\mathbf{R}^N} V(|1+u|^2) \, dx \ge \frac{1-\delta_1}{2} \int_{\mathbf{R}^N} \left(\varphi^2(|1+u|) - 1\right)^2 \, dx - C_1(\delta_1) \left(E_{\rm GL}(u)\right)^{\frac{2^*}{2}}. \tag{4.5}$$

Using (3.4) we see that there exists $A > 0$ such that for any $w \in \mathcal{X}$ with $E_{\rm GL}(w) \le 1$, for any $h \in (0, 1]$ and for any minimizer v_h of $G^w_{h,\mathbf{R}^N}$ in $H^1_w(\mathbf{R}^N)$, we have

$$|Q(w) - Q(v_h)| \le A h^{\frac{2}{N}} E_{\rm GL}(w). \tag{4.6}$$

Choose $h \in (0, 1]$ such that $\varepsilon_1 - \delta_1 - cAh^{\frac{2}{N}} > \varepsilon$. (This choice is possible because $\varepsilon_1 - \delta_1 - \varepsilon > 0$.) Then fix $\delta > 0$ such that $\frac{c}{\sqrt{2}(1-\delta)} < 1 - \varepsilon_1$. (Such δ exist because $\varepsilon_1 < 1 - \frac{c}{v_s} = 1 - \frac{c}{\sqrt{2}}$.)

Let $K = K(N, h, \delta, 1)$ be as in Lemma 3.1(iv). Consider $u \in \mathcal{X}$ such that $E_{\rm GL}(u) \le \min(K, 1)$. Let v_h be a minimizer of $G^u_{h,\mathbf{R}^N}$ in $H^1_u(\mathbf{R}^N)$. The existence of v_h follows from Lemma 3.1(i). By Lemma 3.1(iv), we have $1 - \delta < |1 + v_h| < 1 + \delta$ a.e. on $\mathbf{R}^N$. Then Lemma 4.2 implies

$$c|Q(v_h)| \le \frac{c}{\sqrt{2}(1-\delta)} E_{\rm GL}(v_h) \le (1-\varepsilon_1) E_{\rm GL}(v_h) \le (1-\varepsilon_1) E_{\rm GL}(u). \tag{4.7}$$

We have

$$\begin{aligned}
&\int_{\mathbf{R}^N} |\nabla u|^2 \, dx + \int_{\mathbf{R}^N} V(|1+u|^2) \, dx - c|Q(u)| \\
&\quad \ge (1-\delta_1) E_{\rm GL}(u) - C_1(\delta_1) \left(E_{\rm GL}(u)\right)^{\frac{2^*}{2}} - c|Q(u)| \qquad \text{by (4.5)} \\
&\quad \ge (1-\delta_1) E_{\rm GL}(u) - C_1(\delta_1) \left(E_{\rm GL}(u)\right)^{\frac{2^*}{2}} - c|Q(u) - Q(v_h)| - c|Q(v_h)| \\
&\quad \ge (1-\delta_1) E_{\rm GL}(u) - C_1(\delta_1) \left(E_{\rm GL}(u)\right)^{\frac{2^*}{2}} - cAh^{\frac{2}{N}} E_{\rm GL}(u) - (1-\varepsilon_1) E_{\rm GL}(u) \\
&\qquad \qquad \text{by (4.6) and (4.7)} \\
&\quad = \left(\varepsilon_1 - \delta_1 - cAh^{\frac{2}{N}} - C_1(\delta_1) \left(E_{\rm GL}(u)\right)^{\frac{2^*}{2} - 1}\right) E_{\rm GL}(u).
\end{aligned} \tag{4.8}$$

Note that (4.8) holds for any $u \in \mathcal{X}$ with $E_{\rm GL}(u) \le \min(K, 1)$. Since $\varepsilon_1 - \delta_1 - cAh^{\frac{2}{N}} > \varepsilon$, it is obvious that $\varepsilon_1 - \delta_1 - cAh^{\frac{2}{N}} - C_1(\delta_1) \left(E_{\rm GL}(u)\right)^{\frac{2^*}{2} - 1} > \varepsilon$ if $E_{\rm GL}(u)$ is sufficiently small, and the conclusion of Lemma 4.3 follows. □

An obvious consequence of Lemma 4.3 is that $E_c(u) > 0$ if $u \in \mathcal{X} \setminus \{0\}$ and $E_{\rm GL}(u)$ is sufficiently small. The next lemma implies that there are functions $v \in \mathcal{X}$ such that $E_c(v) < 0$.

LEMMA 4.4. *Let $N \geq 2$. There exists a continuous map from $[2, \infty)$ to $H^1(\mathbf{R}^N)$, $R \longmapsto v_R$ such that $v_R \in C_c(\mathbf{R}^N)$ for any $R \geq 2$, and the following estimates hold*:

(i) $\int_{\mathbf{R}^N} |\nabla v_R|^2 \, dx \leq C_1 R^{N-2} + C_2 R^{N-2} \ln R$,

(ii) $\left| \int_{\mathbf{R}^N} V(|1 + v_R|^2) \, dx \right| \leq C_3 R^{N-2}$,

(iii) $\left| \int_{\mathbf{R}^N} \left(\varphi^2(|1 + v_R|) - 1\right)^2 \, dx \right| \leq C_4 R^{N-2}$,

(iv) $-2\pi\omega_{N-1} R^{N-1} \leq Q(v_R) \leq -2\pi\omega_{N-1}(R-2)^{N-1}$,

where the constants C_1–C_4 depend only on N and $\omega_{N-1} = \mathcal{L}^{N-1}(B_{\mathbf{R}^{N-1}}(0,1))$.

Proof. Let

$$T_R = \{x = (x_1, x') \in \mathbf{R}^N \;\big|\; 0 \leq |x'| \leq R \text{ and } -R + |x'| < x_1 < R - |x'|\}.$$

We define $\theta_R : \mathbf{R}^N \longrightarrow \mathbf{R}$ in the following way: if $|x'| \geq R$, we put $\theta_R(x) = 0$ and if $|x'| < R$, we define

$$\theta_R(x) = \begin{cases} 0 & \text{if } x_1 \leq -R + |x'|, \\ \frac{\pi}{R - |x'|} x_1 + \pi & \text{if } x \in T_R, \\ 2\pi & \text{if } x_1 \geq R - |x'|. \end{cases} \tag{4.9}$$

It is easy to see that $x \longmapsto e^{i\theta_R(x)}$ is continuous on $\mathbf{R}^N \setminus \{x \mid x_1 = 0, \ |x'| = R\}$ and equals 1 on $\mathbf{R}^N \setminus T_R$.

Fix $\psi \in C^\infty(\mathbf{R})$ such that $\psi = 0$ on $(-\infty, 1]$, $\psi = 1$ on $[2, \infty)$ and $0 \leq \psi' \leq 2$. Let

$$\psi_R(x) = \psi\left(\sqrt{x_1^2 + (|x'| - R)^2}\right) \quad \text{and} \quad v_R(x) = \psi_R(x) e^{i\theta_R(x)} - 1. \tag{4.10}$$

It is obvious that $v_R \in C_c(\mathbf{R}^N)$. (In fact, v_R is C^∞ on $\mathbf{R}^N \setminus B$, where $B = \partial T_R \cup \{(x_1, 0, \ldots, 0) \mid x_1 \in [-R, R]\}$.) On $\mathbf{R}^N \setminus B$, we have

$$\frac{\partial \theta_R}{\partial x_1} = \begin{cases} \frac{\pi}{R - |x'|} & \text{if } x \in T_R, \\ 0 & \text{otherwise}, \end{cases} \qquad \frac{\partial \theta_R}{\partial x_j} = \begin{cases} \frac{\pi x_1}{(R - |x'|)^2} \frac{x_j}{|x'|} & \text{if } x \in T_{A,R}, \\ 0 & \text{otherwise}, \end{cases} \tag{4.11}$$

$$\frac{\partial \psi_R}{\partial x_1}(x) = \psi'\left(\sqrt{x_1^2 + (|x'| - R)^2}\right) \frac{x_1}{\sqrt{x_1^2 + (|x'| - R)^2}}, \tag{4.12}$$

(4.13)
$$\frac{\partial \psi_R}{\partial x_j}(x) = \psi'\left(\sqrt{x_1^2 + (|x'| - R)^2}\right) \frac{|x'| - R}{\sqrt{x_1^2 + (|x'| - R)^2}} \frac{x_j}{|x'|} \text{ for } j \geq 2 \text{ and } x' \neq 0.$$

Then a simple computation gives $\langle i\frac{\partial v_R}{\partial x_1}, v_R\rangle = -\psi_R^2 \frac{\partial \theta_R}{\partial x_1} + \frac{\partial}{\partial x_1}(\mathrm{Im}(v_R))$ on $\mathbf{R}^N \setminus B$. Thus, we have

$$Q(v_R) = -\int_{\mathbf{R}^N} \psi_R^2 \frac{\partial \theta_R}{\partial x_1}\, dx.$$

It is obvious that

(4.14)

$$\int_{-\infty}^{\infty} \frac{\partial \theta_R}{\partial x_1}\, dx_1 = 0 \quad \text{if } |x'| > R \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{\partial \theta_R}{\partial x_1}\, dx_1 = 2\pi \quad \text{if } 0 < |x'| < R.$$

Since $\frac{\partial \theta_R}{\partial x_1} \geq 0$ a.e. on $\mathbf{R}^N$ and $0 \leq \psi_R \leq 1$, we get

$$\int_{\{|R-|x'||\geq 2\}} \frac{\partial \theta_R}{\partial x_1}\, dx \leq \int_{\mathbf{R}^N} \psi_R^2 \frac{\partial \theta_R}{\partial x_1}\, dx \leq \int_{\mathbf{R}^N} \frac{\partial \theta_R}{\partial x_1}\, dx,$$

and using Fubini's theorem and (4.14) we obtain that v_R satisfies (iv).

Using cylindrical coordinates (x_1, r, ζ) in $\mathbf{R}^N$, where $r = |x'|$ and $\zeta = \frac{x'}{|x'|} \in S^{N-2}$, we get

(4.15)

$$\int_{\mathbf{R}^N} V(|1+v_R|^2)\, dx = \left|S^{N-2}\right| \int_{-\infty}^{\infty}\int_0^{\infty} V\Big(\psi^2\left(\sqrt{x_1^2+(r-R)^2}\right)\Big)\, r^{N-2}\, dr\, dx_1.$$

Next we use polar coordinates in the (x_1, r) plane, that is we write $x_1 = \tau\cos\alpha$, $r = R + \tau\sin\alpha$ (thus $\tau = \sqrt{x_1^2 + (R-r)^2}$). Since $V(\psi^2(s)) = 0$ for $s \geq 2$, we get

$$\begin{aligned}(4.16)\qquad &\int_{-\infty}^{\infty}\int_0^{\infty} V\Big(\psi^2\Big(\sqrt{x_1^2+(r-R)^2}\Big)\Big) r^{N-2} dr\, dx_1 \\ &= \int_0^2\int_0^{2\pi} V(\psi^2(\tau))(R+\tau\sin\alpha)^{N-2}\tau\, d\alpha\, d\tau.\end{aligned}$$

It is obvious that $\left|\int_0^{2\pi}(R+\tau\sin\alpha)^{N-2} d\alpha\right| \leq 2\pi(R+2)^{N-2}$ for any $\tau \in [0,2]$. Then using (4.15) and (4.16), we infer that v_R satisfies (ii). The proof of (iii) is similar.

It is clear that on $\mathbf{R}^N \setminus B$, we have

$$|\nabla v_R|^2 = |\nabla \psi_R|^2 + |\psi_R|^2 |\nabla \theta_R|^2. \tag{4.17}$$

From (4.12) and (4.13) we see that $|\nabla \psi_R(x)|^2 = \left|\psi'\Big(\sqrt{x_1^2 + (|x'|-R)^2}\Big)\right|^2$. Proceeding as above and using cylindrical coordinates (x_1, r, ζ) in $\mathbf{R}^N$, then passing to polar coordinates $x_1 = \tau\cos\alpha$, $r = R + \tau\sin\alpha$, we obtain

(4.18)

$$\int_{\mathbf{R}^N} \left|\psi'\left(\sqrt{x_1^2+(|x'|-R)^2}\right)\right|^2 dx \leq 2\pi|S^{N-2}|(R+2)^{N-2}\int_0^2 s|\psi'(s)|^2\, ds.$$

It is easily seen from (4.11) that $|\nabla \theta_R(x)|^2 = \frac{\pi^2}{(R-|x'|)^2}\left(1 + \frac{x_1^2}{(R-|x'|)^2}\right)$ if $x \in T_R$, $|x'| \neq 0$, and $\nabla \theta_R(x) = 0$ a.e. on $\mathbf{R}^N \setminus \overline{T}_R$. Moreover, if $(x_1, x') \in T_R$

and $|x'| \geq R - \frac{1}{\sqrt{2}}$, we have $\psi_R(x_1, x') = 0$. Therefore,

(4.19)

$$\int_{\mathbf{R}^N} |\psi_R|^2 |\nabla \theta_R|^2 \, dx \leq \int_{T_R \cap \{|x'| < R - \frac{1}{\sqrt{2}}\}} |\nabla \theta_R|^2 \, dx$$

$$= \int_{\{|x'| < R - \frac{1}{\sqrt{2}}\}} \int_{-R+|x'|}^{R-|x'|} |\nabla \theta_R|^2 \, dx_1 \, dx'$$

$$= \int_{\{|x'| < R - \frac{1}{\sqrt{2}}\}} \frac{2\pi^2}{R - |x'|} + \frac{2\pi^2}{3} \frac{1}{R - |x'|} \, dx' = \frac{8}{3}\pi^2 |S^{N-2}| \int_0^{R - \frac{1}{\sqrt{2}}} \frac{r^{N-2}}{R - r} \, dr$$

$$= \frac{8}{3}\pi^2 |S^{N-2}| R^{N-2} \left(- \sum_{k=1}^{N-2} \frac{1}{k} \left(1 - \frac{1}{R\sqrt{2}}\right)^k + \ln\left(R\sqrt{2}\right)\right).$$

From (4.17), (4.18) and (4.19) it follows that v_R satisfies (i). It is not hard to see that the mapping $R \longmapsto v_R$ is continuous from $[2, \infty)$ to $H^1(\mathbf{R}^N)$ and Lemma 4.4 is proven. □

LEMMA 4.5. *For any $k > 0$, the functional Q is bounded on the set*

$$\{u \in \mathcal{X} \mid E_{\mathrm{GL}}(u) \leq k\}.$$

Proof. Let $c \in (0, v_s)$, and let $\varepsilon \in (0, 1 - \frac{c}{v_s})$. From Lemmas 4.1 and 4.3 it follows that there exist two positive constants $C_2(\frac{\varepsilon}{2})$ and K_1 such that for any $u \in \mathcal{X}$ satisfying $E_{\mathrm{GL}}(u) < K_1$, we have

$$\left(1 + \frac{\varepsilon}{2}\right) E_{\mathrm{GL}}(u) + C_2\left(\frac{\varepsilon}{2}\right) \left(E_{\mathrm{GL}}(u)\right)^{\frac{2^*}{2}} - c|Q(u)|$$
$$\geq \int_{\mathbf{R}^N} |\nabla u|^2 \, dx + \int_{\mathbf{R}^N} V(|1+u|^2) \, dx - c|Q(u)| \geq \varepsilon E_{\mathrm{GL}}(u).$$

This inequality implies that there exists $K_2 \leq K_1$ such that for any $u \in \mathcal{X}$ satisfying $E_{\mathrm{GL}}(u) \leq K_2$, we have

$$c|Q(u)| \leq E_{\mathrm{GL}}(u). \tag{4.20}$$

Hence Lemma 4.5 is proven if $k \leq K_2$.

Now let $u \in \mathcal{X}$ be such that $E_{\mathrm{GL}}(u) > K_2$. Using the notation (1.5), it is clear that for $\sigma > 0$, we have $Q(u_{\sigma,\sigma}) = \sigma^{N-1} Q(u)$ (see (2.13)) and

$$E_{\mathrm{GL}}(u_{\sigma,\sigma}) = \sigma^{N-2} \int_{\mathbf{R}^N} |\nabla u|^2 \, dx + \frac{\sigma^N}{2} \int_{\mathbf{R}^N} \left(\varphi^2(|1+u|) - 1\right)^2 \, dx.$$

Let $\sigma_0 = \left(\frac{K_2}{E_{\mathrm{GL}}(u)}\right)^{\frac{1}{N-2}}$. Then $\sigma_0 \in (0,1)$ and we have $E_{\mathrm{GL}}(u_{\sigma_0,\sigma_0}) \leq \sigma_0^{N-2} E_{\mathrm{GL}}(u)$ $= K_2$. Using (4.20) we infer that $c|Q(u_{\sigma_0,\sigma_0})| \leq E_{\mathrm{GL}}(u_{\sigma_0,\sigma_0})$, and this implies

$c\sigma_0^{N-1}|Q(u)| \le \sigma_0^{N-2} E_{\rm GL}(u)$, or equivalently

$$|Q(u)| \le \frac{1}{c\sigma_0} E_{\rm GL}(u) = \frac{1}{c} K_2^{-\frac{1}{N-2}} \left(E_{\rm GL}(u)\right)^{\frac{N-1}{N-2}}. \tag{4.21}$$

Since (4.21) holds for any $u \in \mathcal{X}$ with $E_{\rm GL}(u) > K_2$, Lemma 4.5 is proven. □

From Lemma 4.1 and Lemma 4.5 it follows that for any $k > 0$, the functional E_c is bounded on the set $\{u \in \mathcal{X} \mid E_{\rm GL}(u) = k\}$. For $k > 0$, we define

$$E_{c,\min}(k) = \inf\{E_c(u) \mid u \in \mathcal{X},\ E_{\rm GL}(u) = k\}.$$

Clearly, the function $E_{c,\min}$ is bounded on any bounded interval of $\mathbf{R}_+$. The next result gives some basic properties of $E_{c,\min}$ which will be important for our variational argument.

Lemma 4.6. *Assume that $N \ge 3$ and $0 < c < v_s$. The function $E_{c,\min}$ has the following properties*:

(i) *There exists $k_0 > 0$ such that $E_{c,\min}(k) > 0$ for any $k \in (0, k_0)$.*
(ii) *We have $\lim_{k\to\infty} E_{c,\min}(k) = -\infty$.*
(iii) *For any $k > 0$, we have $E_{c,\min}(k) < k$.*

Proof. (i) is an easy consequence of Lemma 4.3.

(ii) It is obvious that $H^1(\mathbf{R}^N) \subset \mathcal{X}$ and the functionals $E_{\rm GL}$, E_c and Q are continuous on $H^1(\mathbf{R}^N)$. For $\varepsilon = 1$ and $R > 2$, consider the functions v_R constructed in Lemma 4.4. Clearly, $R \longmapsto v_R$ is a continuous curve in $H^1(\mathbf{R}^N)$. Lemma 4.4 implies $E_c(v_R) \longrightarrow -\infty$ as $R \longrightarrow \infty$. From Lemma 4.5 we infer that $E_{\rm GL}(v_R) \longrightarrow \infty$ as $R \longrightarrow \infty$, and then it is not hard to see that (ii) holds.

(iii) Fix $k > 0$. Let v_R be as above, and let $u = v_R$ for some R sufficiently large, so that

$$E_{\rm GL}(u) > k, \quad Q(u) < 0 \quad \text{ and } E_c(u) < 0.$$

In particular, we have

$$E_c(u) - E_{\rm GL}(u) = cQ(u) + \int_{\mathbf{R}^N} V(|1+u|^2) - \frac{1}{2}\left(\varphi^2(|1+u|^2) - 1\right)^2 \, dx < 0.$$

It is obvious that $E_{\rm GL}(u_{\sigma,\sigma}) \longrightarrow 0$ as $\sigma \longrightarrow 0$; hence there exists $\sigma_0 \in (0,1)$ such that $E_{\rm GL}(u_{\sigma_0,\sigma_0}) = k$. Using the fact that $E_{\rm GL}(u) - E_c(u) < 0$ and $Q(u) < 0$, we get

$$\begin{aligned}
&E_c(u_{\sigma_0,\sigma_0}) - E_{\rm GL}(u_{\sigma_0,\sigma_0}) \\
&\quad = \sigma_0^{N-1} cQ(u) + \sigma_0^N \int_{\mathbf{R}^N} V(|1+u|^2) - \frac{1}{2}\left(\varphi^2(|1+u|^2) - 1\right)^2 \, dx \\
&\quad = (\sigma_0^{N-1} - \sigma_0^N) cQ(u) + \sigma_0^N (E_c(u) - E_{\rm GL}(u)) < 0.
\end{aligned}$$

Thus $E_c(u_{\sigma_0,\sigma_0}) < E_{\rm GL}(u_{\sigma_0,\sigma_0})$. Since $E_{\rm GL}(u_{\sigma_0,\sigma_0}) = k$, we necessarily have $E_{c,\min}(k) \le E_c(u_{\sigma_0,\sigma_0}) < k$. □

From Lemma 4.6(i) and (ii) it follows that

$$0 < S_c := \sup\{E_{c,\min}(k) \mid k > 0\} < \infty. \tag{4.22}$$

LEMMA 4.7. *The set* $\mathcal{C} = \{u \in \mathcal{X} \mid u \neq 0,\ P_c(u) = 0\}$ *is not empty, and we have*

$$T_c := \inf\{E_c(u) \mid u \in \mathcal{C}\} \geq S_c > 0.$$

Proof. Let $w \in \mathcal{X} \setminus \{0\}$ be such that $E_c(w) < 0$. (We have seen in the proof of Lemma 4.6 that such functions w exist.) It is obvious that $A(w) > 0$ and $\int_{\mathbf{R}^N} \left|\frac{\partial w}{\partial x_1}\right|^2 dx > 0$; therefore, $B_c(w) = E_c(w) - A(w) < 0$ and $P_c(w) = E_c(w) - \frac{2}{N-1}A(w) < 0$. Clearly,

$$P_c(w_{\sigma,1}) = \frac{1}{\sigma}\int_{\mathbf{R}^N}\left|\frac{\partial w}{\partial x_1}\right|^2 dx + \frac{N-3}{N-1}\sigma A(w) + cQ(w) + \sigma\int_{\mathbf{R}^N} V(|1+w|^2)\,dx. \tag{4.23}$$

Since $P_c(w_{1,1}) = P_c(w) < 0$ and $\lim_{\sigma\to 0} P_c(w_{\sigma,1}) = \infty$, there exists $\sigma_0 \in (0,1)$ such that $P_c(w_{\sigma_0,1}) = 0$; that is, $w_{\sigma_0,1} \in \mathcal{C}$. Thus $\mathcal{C} \neq \emptyset$.

To prove the second part of Lemma 4.7, consider first the case $N \geq 4$. Let $u \in \mathcal{C}$. It is clear that $A(u) > 0$, $B_c(u) = -\frac{N-3}{N-1}A(u) < 0$ and for any $\sigma > 0$, we have $E_c(u_{1,\sigma}) = A(u_{1,\sigma}) + B_c(u_{1,\sigma}) = \sigma^{N-3}A(u) + \sigma^{N-1}B_c(u)$. Hence,

$$\frac{d}{d\sigma}(E_c(u_{1,\sigma})) = (N-3)\sigma^{N-4}A(u) + (N-1)\sigma^{N-2}B_c(u)$$

is positive on $(0,1)$ and negative on $(1,\infty)$. Consequently the function $\sigma \longmapsto E_c(u_{1,\sigma})$ achieves its maximum at $\sigma = 1$.

On the other hand, we have

$$E_{\mathrm{GL}}(u_{1,\sigma}) = \sigma^{N-3}A(u) + \sigma^{N-1}\left(\int_{\mathbf{R}^N}\left|\frac{\partial u}{\partial x_1}\right|^2 + \frac{1}{2}\left(\varphi^2(|1+u|) - 1\right)^2 dx\right).$$

It is easy to see that the mapping $\sigma \longmapsto E_{\mathrm{GL}}(u_{1,\sigma})$ is strictly increasing and one-to-one from $(0,\infty)$ to $(0,\infty)$. Hence for any $k > 0$, there is a unique $\sigma(k,u) > 0$ such that $E_{\mathrm{GL}}(u_{1,\sigma(k,u)}) = k$. Then we have

$$E_{c,\min}(k) \leq E_c(u_{1,\sigma(k,u)}) \leq E_c(u_{1,1}) = E_c(u).$$

Since this is true for any $k > 0$ and any $u \in \mathcal{C}$, the conclusion follows.

Next we consider the case $N = 3$. Let $u \in \mathcal{C}$. We have $P_c(u) = B_c(u) = 0$ and $E_c(u) = A(u) > 0$. For $\sigma > 0$, we get

$$E_c(u_{1,\sigma}) = A(u) + \sigma^2 B_c(u) = A(u)$$

and

$$E_{\mathrm{GL}}(u_{1,\sigma}) = A(u) + \sigma^2\left(\int_{\mathbf{R}^3}\left|\frac{\partial u}{\partial x_1}\right|^2 + \frac{1}{2}\left(\varphi^2(|1+u|) - 1\right)^2 dx\right).$$

Clearly, $\sigma \longmapsto E_{\mathrm{GL}}(u_{1,\sigma})$ is increasing on $(0,\infty)$ and is one-to-one from $(0,\infty)$ to $(A(u),\infty)$.

Fix $\varepsilon > 0$. Consider $k_\varepsilon > 0$ such that $E_{c,\min}(k_\varepsilon) > S_c - \varepsilon$. If $A(u) \geq k_\varepsilon$, from Lemma 4.6(iii) we have $E_{c,\min}(k_\varepsilon) < k_\varepsilon$; hence,

$$E_c(u) = A(u) \geq k_\varepsilon > E_{c,\min}(k_\varepsilon) > S_c - \varepsilon.$$

If $A(u) < k_\varepsilon$, there exists $\sigma(k_\varepsilon, u) > 0$ such that $E_{\mathrm{GL}}(u_{1,\sigma(k_\varepsilon,u)}) = k_\varepsilon$. Then we get

$$E_c(u) = A(u) = E_c(u_{1,\sigma(k_\varepsilon,u)}) \geq E_{c,\min}(k_\varepsilon) > S_c - \varepsilon.$$

So far we have proved that for any $u \in \mathcal{C}$ and any $\varepsilon > 0$, we have $E_c(u) > S_c - \varepsilon$. The conclusion follows letting $\varepsilon \longrightarrow 0$, then taking the infimum for $u \in \mathcal{C}$. □

We do not know whether $T_c = S_c$ in Lemma 4.7.

Lemma 4.8. *Let T_c be as in Lemma 4.7. The following assertions hold*:

(i) *For any $u \in \mathcal{X}$ with $P_c(u) < 0$, we have $A(u) > \frac{N-1}{2} T_c$.*

(ii) *Let $(u_n)_{n\geq 1} \subset \mathcal{X}$ be a sequence such that $(E_{\mathrm{GL}}(u_n))_{n\geq 1}$ is bounded and $\lim_{n\to\infty} P_c(u_n) = \mu < 0$. Then $\liminf_{n\to\infty} A(u_n) > \frac{N-1}{2} T_c$.*

Proof. (i) Since $P_c(u) < 0$, it is clear that $u \neq 0$ and thus $\int_{\mathbf{R}^N} \left| \frac{\partial u}{\partial x_1} \right|^2 dx > 0$. As in the proof of Lemma 4.7, we have $P_c(u_{1,1}) = P_c(u) < 0$, and (4.23) implies that $\lim_{\sigma\to 0} P_c(u_{\sigma,1}) = \infty$. Hence there exists $\sigma_0 \in (0,1)$ such that $P_c(u_{\sigma_0,1}) = 0$. From Lemma 4.7 we get $E_c(u_{\sigma_0,1}) \geq T_c$, and this implies $E_c(u_{\sigma_0,1}) - P_c(u_{\sigma_0,1}) \geq T_c$; that is, $\frac{2}{N-1} A(u_{\sigma_0,1}) \geq T_c$. From the last inequality we find

$$A(u) \geq \frac{N-1}{2} \frac{1}{\sigma_0} T_c > \frac{N-1}{2} T_c. \tag{4.24}$$

(ii) For n sufficiently large (so that $P_c(u_n) < 0$), we have $u_n \neq 0$ and $\int_{\mathbf{R}^N} \left| \frac{\partial u_n}{\partial x_1} \right|^2 dx > 0$. As in the proof of part (i), using (4.23) we see that for each n sufficiently big, there exists $\sigma_n \in (0,1)$ such that

$$P_c((u_n)_{\sigma_n,1}) = 0, \tag{4.25}$$

and we infer that $A(u_n) \geq \frac{N-1}{2} \frac{1}{\sigma_n} T_c$. We claim that

$$\limsup_{n\to\infty} \sigma_n < 1. \tag{4.26}$$

Notice that if (4.26) holds, we have

$$\liminf_{n\to\infty} A(u_n) \geq \frac{N-1}{2} \frac{1}{\limsup_{n\to\infty} \sigma_n} T_c > \frac{N-1}{2} T_c$$

and Lemma 4.8 is proven.

To prove (4.26) we argue by contradiction and assume that there is a subsequence $(\sigma_{n_k})_{k\geq 1}$ such that $\sigma_{n_k} \longrightarrow 1$ as $k \longrightarrow \infty$. Since $(E_{\rm GL}(u_n))_{n\geq 1}$ is bounded, using Lemmas 4.1 and 4.5 we infer that

$$\left(\int_{\mathbf{R}^N} \left|\frac{\partial u_n}{\partial x_1}\right|^2 dx\right)_{n\geq 1}, \quad \left(\int_{\mathbf{R}^N} V(|1+u_n|^2)\, dx\right)_{n\geq 1}, \quad (A(u_n))_{n\geq 1}, \quad (Q(u_n))_{n\geq 1}$$

are bounded. Consequently, there is a subsequence $(n_{k_\ell})_{\ell\geq 1}$ and there are α_1, α_2, β, $\gamma \in \mathbf{R}$ such that

$$\int_{\mathbf{R}^N} \left|\frac{\partial u_{n_{k_\ell}}}{\partial x_1}\right|^2 dx \longrightarrow \alpha_1, \qquad \int_{\mathbf{R}^N} V(|1+u_{n_{k_\ell}}|^2)\, dx \longrightarrow \gamma,$$
$$A(u_{n_{k_\ell}}) \longrightarrow \alpha_2, \qquad Q(u_{n_{k_\ell}}) \longrightarrow \beta \quad \text{as } \ell \longrightarrow \infty.$$

Writing (4.25) and (4.23) (with $(u_{n_{k_\ell}})_{\sigma_{n_{k_\ell}},1}$ instead of $(u_n)_{\sigma_n,1}$ and $w_{\sigma,1}$, respectively), then passing to the limit as $\ell \longrightarrow \infty$ and using the fact that $\sigma_{n_k} \longrightarrow 1$, we find $\alpha_1 + \frac{N-3}{N-1}\alpha_2 + c\beta + \gamma = 0$. On the other hand, we have $\lim_{\ell\to\infty} P_c(u_{n_{k_\ell}}) = \mu < 0$, and this gives $\alpha_1 + \frac{N-3}{N-1}\alpha_2 + c\beta + \gamma = \mu < 0$. This contradiction proves that (4.26) holds, and the proof of Lemma 4.8 is complete. □

5. The case $N \geq 4$

Throughout this section we assume that $N \geq 4$, $0 < c < v_s$ and assumptions (A1) and (A2) in the introduction are satisfied. Most of the results below do *not* hold for $c > v_s$. Some of them may not hold for $c = 0$ and some particular nonlinearities F.

LEMMA 5.1. *Let $(u_n)_{n\geq 1} \subset \mathcal{X}$ be a sequence such that $(E_c(u_n))_{n\geq 1}$ is bounded and $P_c(u_n) \longrightarrow 0$ as $n \longrightarrow \infty$. Then $(E_{\rm GL}(u_n))_{n\geq 1}$ is bounded.*

Proof. We have $\frac{2}{N-1}A(u_n) = E_c(u_n) - P_c(u_n)$; hence $(A(u_n))_{n\geq 1}$ is bounded. It remains to prove that $\int_{\mathbf{R}^N} \left|\frac{\partial u_n}{\partial x_1}\right| + \frac{1}{2}\left(\varphi^2(|1+u_n|) - 1\right)^2 dx$ is bounded. We argue by contradiction, and we assume that there is a subsequence, still denoted $(u_n)_{n\geq 1}$, such that

$$\int_{\mathbf{R}^N} \left|\frac{\partial u_n}{\partial x_1}\right| + \frac{1}{2}\left(\varphi^2(|1+u_n|) - 1\right)^2 dx \longrightarrow \infty \qquad \text{as } n \longrightarrow \infty. \tag{5.1}$$

Fix $k_0 > 0$ such that $E_{c,\min}(k_0) > 0$. Arguing as in the proof of Lemma 4.7, it is easy to see that there exists a sequence $(\sigma_n)_{n\geq 1}$ such that

$$E_{\rm GL}((u_n)_{1,\sigma_n}) = \sigma_n^{N-3} A(u_n) + \sigma_n^{N-1} \int_{\mathbf{R}^N} \left|\frac{\partial u_n}{\partial x_1}\right| + \frac{1}{2}\left(\varphi^2(|1+u_n|) - 1\right)^2 dx = k_0. \tag{5.2}$$

From (5.1) and (5.2) we have $\sigma_n \longrightarrow 0$ as $n \longrightarrow \infty$. Since $B_c(u_n) = -\frac{N-3}{N-1}A(u_n) + P_c(u_n)$, it is clear that $(B_c(u_n))_{n\geq 1}$ is bounded and we obtain

$$E_c((u_n)_{1,\sigma_n}) = \sigma_n^{N-3}A(u_n) + \sigma_n^{N-1}B_c(u_n) \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty.$$

But this contradicts the fact that $E_{c,\min}(k_0) > 0$, and the proof of Lemma 5.1 is complete. □

LEMMA 5.2. *Let $(u_n)_{n\geq 1} \subset \mathcal{X}$ be a sequence satisfying the following properties*:

(a) *There exist C_1, $C_2 > 0$ such that $C_1 \leq E_{\mathrm{GL}}(u_n)$ and $A(u_n) \leq C_2$ for any $n \geq 1$.*
(b) *$P_c(u_n) \longrightarrow 0$ as $n \longrightarrow \infty$.*

Then $\liminf_{n\to\infty} E_c(u_n) \geq T_c$, *where T_c is as in Lemma* 4.7.

Notice that in Lemma 5.2 the assumption $E_{\mathrm{GL}}(u_n) \geq C_1 > 0$ is necessary. To see this, consider a sequence $(u_n)_{n\geq 1} \subset H^1(\mathbf{R}^N)$ such that $u_n \neq 0$ and $u_n \longrightarrow 0$ as $n \longrightarrow \infty$. It is clear that $P_c(u_n) \longrightarrow 0$ and $E_c(u_n) \longrightarrow 0$ as $n \longrightarrow \infty$.

Proof. First we prove that

$$C_3 := \liminf_{n\to\infty} A(u_n) > 0. \tag{5.3}$$

To see this, fix $k_0 > 0$ such that $E_{c,\min}(k_0) > 0$. Exactly as in the proof of Lemma 4.7, it is easy to see that for each n, there exists a unique $\sigma_n > 0$ such that (5.2) holds. Since $k_0 = E_{\mathrm{GL}}((u_n)_{1,\sigma_n}) \geq \min(\sigma_n^{N-3}, \sigma_n^{N-1})E_{\mathrm{GL}}((u_n)) \geq \min(\sigma_n^{N-3}, \sigma_n^{N-1})C_1$, it follows that $(\sigma_n)_{n\geq 1}$ is bounded. On the other hand, we have $E_c((u_n)_{1,\sigma_n}) = \sigma_n^{N-3}A(u_n) + \sigma_n^{N-1}B_c(u_n) \geq E_{c,\min}(k_0) > 0$; that is,

$$\sigma_n^{N-3}A(u_n) + \sigma_n^{N-1}\left(P_c(u_n) - \frac{N-3}{N-1}A(u_n)\right) \geq E_{c,\min}(k_0) > 0. \tag{5.4}$$

If there is a subsequence $(u_{n_k})_{k\geq 1}$ such that $A(u_{n_k}) \longrightarrow 0$, putting u_{n_k} in (5.4) and letting $k \longrightarrow \infty$ we would get $0 \geq E_{c,\min}(k_0) > 0$, a contradiction. Thus (5.3) is proven.

We have $B_c(u_n) = P_c(u_n) - \frac{N-3}{N-1}A(u_n)$, and using (b) and (5.3), we obtain

$$\limsup_{n\to\infty} B_c(u_n) = -\frac{N-3}{N-1}C_3 < 0. \tag{5.5}$$

Clearly, for any $\sigma > 0$, we have

$$\begin{aligned} P_c((u_n)_{1,\sigma}) &= \sigma^{N-3}\frac{N-3}{N-1}A(u_n) + \sigma^{N-1}B_c(u_n) \\ &= \sigma^{N-3}\left(\frac{N-3}{N-1}A(u_n) + \sigma^2 B_c(u_n)\right). \end{aligned}$$

For n sufficiently large (so that $B_c(u_n) < 0$), let $\tilde{\sigma}_n = \left(\frac{\frac{N-3}{N-1} A(u_n)}{-B_c(u_n)} \right)^{\frac{1}{2}}$. Then $P_c((u_n)_{1,\tilde{\sigma}_n}) = 0$, or equivalently $(u_n)_{1,\tilde{\sigma}_n} \in \mathcal{C}$. From Lemma 4.7, we obtain

$$E_c((u_n)_{1,\tilde{\sigma}_n}) = \tilde{\sigma}_n^{N-3} A(u_n) + \tilde{\sigma}_n^{N-1} B_c(u_n) \geq T_c;$$

that is,

$$E_c(u_n) + \left(\tilde{\sigma}_n^{N-3} - 1\right) A(u_n) + \left(\tilde{\sigma}_n^{N-1} - 1\right) \left(P_c(u_n) - \frac{N-3}{N-1} A(u_n) \right) \geq T_c. \tag{5.6}$$

Clearly, $\tilde{\sigma}_n$ can be written as $\tilde{\sigma}_n = \left(\frac{P_c(u_n)}{-B_c(u_n)} + 1 \right)^{\frac{1}{2}}$. Using (b) and (5.5) it follows that $\lim_{n\to\infty} \tilde{\sigma}_n = 1$. Then passing to the limit as $n \longrightarrow \infty$ in (5.6) and using the fact that $(A(u_n))_{n\geq 1}$ and $(P_c(u_n))_{n\geq 1}$ are bounded, we obtain $\liminf_{n\to\infty} E_c(u_n) \geq T_c$. □

We can now state the main result of this section.

THEOREM 5.3. *Let $(u_n)_{n\geq 1} \subset \mathcal{X} \setminus \{0\}$ be a sequence such that*

$$P_c(u_n) \longrightarrow 0 \qquad \text{and} \qquad E_c(u_n) \longrightarrow T_c \qquad \text{as } n \longrightarrow \infty.$$

There exist a subsequence $(u_{n_k})_{k\geq 1}$, a sequence $(x_k)_{k\geq 1} \subset \mathbf{R}^N$ and $u \in \mathcal{C}$ such that

$$\nabla u_{n_k}(\cdot + x_k) \longrightarrow \nabla u \quad \text{and} \quad |1 + u_{n_k}(\cdot + x_k)| - 1 \longrightarrow |1+u| - 1 \quad \text{in } L^2(\mathbf{R}^N).$$

Moreover, we have $E_c(u) = T_c$; that is, u minimizes E_c in $\mathcal{C}$.

Proof. From Lemma 5.1 we know that $E_{\mathrm{GL}}(u_n)$ is bounded. We have $\frac{2}{N-1} A(u_n) = E_c(u_n) - P_c(u_n) \longrightarrow T_c$ as $n \longrightarrow \infty$. Therefore,

$$\lim_{n\to\infty} A(u_n) = \frac{N-1}{2} T_c \ \text{ and } \ \liminf_{n\to\infty} E_{\mathrm{GL}}(u_n) \geq \lim_{n\to\infty} A(u_n) = \frac{N-1}{2} T_c. \tag{5.7}$$

Passing to a subsequence if necessary, we may assume that there exists $\alpha_0 \geq \frac{N-1}{2} T_c$ such that

$$E_{\mathrm{GL}}(u_n) \longrightarrow \alpha_0 \qquad \text{as } n \longrightarrow \infty. \tag{5.8}$$

We will use the concentration-compactness principle ([40]). We denote by $q_n(t)$ the concentration function of $E_{\mathrm{GL}}(u_n)$; that is,

$$q_n(t) = \sup_{y\in\mathbf{R}^N} \int_{B(y,t)} \left\{ |\nabla u_n|^2 + \frac{1}{2} \left(\varphi^2(|1+u_n|) - 1 \right)^2 \right\} dx. \tag{5.9}$$

As in [40], it follows that there exists a subsequence of $((u_n, q_n))_{n\geq 1}$, still denoted $((u_n, q_n))_{n\geq 1}$, there exists a nondecreasing function $q : [0, \infty) \longrightarrow \mathbf{R}$ and there is $\alpha \in [0, \alpha_0]$ such that

$$q_n(t) \longrightarrow q(t) \ \text{ a.e on } [0,\infty) \ \text{ as } n \longrightarrow \infty \qquad \text{and} \qquad q(t) \longrightarrow \alpha \text{ as } t \longrightarrow \infty. \tag{5.10}$$

We claim that

(5.11)

$$\text{there is a nondecreasing sequence } t_n \longrightarrow \infty \text{ such that } \lim_{n\to\infty} q_n(t_n) = \alpha.$$

To prove the claim, fix an increasing sequence $x_k \longrightarrow \infty$ such that $q_n(x_k) \longrightarrow q(x_k)$ as $n \longrightarrow \infty$ for any k. Then there exists $n_k \in \mathbf{N}$ such that $|q_n(x_k) - q(x_k)| < \frac{1}{k}$ for any $n \geq n_k$; clearly, we may assume that $n_k < n_{k+1}$ for all k. If $n_k \leq n < n_{k+1}$, put $t_n = x_k$. Then for $n_k \leq n < n_{k+1}$, we have

$$|q_n(t_n)-\alpha| = |q_n(x_k)-\alpha| \leq |q_n(x_k)-q(x_k)|+|q(x_k)-\alpha| \leq \frac{1}{k}+|q(x_k)-\alpha| \longrightarrow 0$$

as $k \longrightarrow \infty$, and (5.11) is proved.

Next we claim that

$$q_n(t_n) - q_n\left(\frac{t_n}{2}\right) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{5.12}$$

To see this, fix $\varepsilon > 0$. Take $y > 0$ such that $q(y) > \alpha - \frac{\varepsilon}{4}$ and $q_n(y) \longrightarrow q(y)$ as $n \longrightarrow \infty$. There is some $\tilde{n} \geq 1$ such that $q_n(y) > \alpha - \frac{\varepsilon}{2}$ for $n \geq \tilde{n}$. Then we can find $n_* \geq \tilde{n}$ such that $t_n > 2y$ for $n \geq n_*$, and consequently we have $q_n(\frac{t_n}{2}) \geq q_n(y) > \alpha - \frac{\varepsilon}{2}$. Therefore $\limsup_{n\to\infty} \left(q_n(t_n) - q_n(\frac{t_n}{2})\right) = \lim_{n\to\infty} q_n(t_n) - \liminf_{n\to\infty} q_n(\frac{t_n}{2}) < \varepsilon$. Since ε was arbitrary, (5.12) follows.

Our aim is to show that $\alpha = \alpha_0$ in (5.10). It follows from the next lemma that $\alpha > 0$.

Lemma 5.4. *Let $(u_n)_{n\geq 1} \subset \mathcal{X}$ be a sequence satisfying*

(a) *$M_1 \leq E_{\mathrm{GL}}(u_n) \leq M_2$ for some positive constants M_1, M_2.*

(b) *$\lim_{n\to\infty} P_c(u_n) = 0$.*

There exists $k > 0$ such that

$$\sup_{y\in\mathbf{R}^N} \int_{B(y,1)} \left\{|\nabla u_n|^2 + \frac{1}{2}\left(\varphi^2(|1+u_n|) - 1\right)^2\right\} dx \geq k$$

for all sufficiently large n.

Proof. We argue by contradiction, and we suppose that the conclusion is false. Then there exists a subsequence (still denoted $(u_n)_{n\geq 1}$) such that

$$\lim_{n\to\infty} \sup_{y\in\mathbf{R}^N} \int_{B(y,1)} \left\{|\nabla u_n|^2 + \frac{1}{2}\left(\varphi^2(|1+u_n|) - 1\right)^2\right\} dx = 0. \tag{5.13}$$

In order to get a contradiction, we proceed in four steps.

Step 1. *We show that $|E(u_n) - E_{\mathrm{GL}}(u_n)| \longrightarrow 0$ as $n \longrightarrow \infty$. More precisely, we prove that*

$$\lim_{n\to\infty} \int_{\mathbf{R}^N} \left| V(|1+u_n|^2) - \frac{1}{2}\left(\varphi^2(|1+u_n|) - 1\right)^2 \right| dx = 0. \tag{5.14}$$

Fix $\varepsilon > 0$. Assumptions (A1) and (A2) imply that there exists $\delta(\varepsilon) > 0$ such that

$$\left| V(|1+z|^2) - \frac{1}{2}\left(\varphi^2(|1+z|) - 1\right)^2 \right| \leq \frac{\varepsilon}{2}\left(\varphi^2(|1+z|) - 1\right)^2 \tag{5.15}$$

for any $z \in \mathbf{C}$ satisfying $\Big| |1+z| - 1 \Big| \leq \delta(\varepsilon)$ (see (4.2)). Therefore,

$$\begin{aligned} &\int_{\{| \, |1+u_n|-1| \leq \delta(\varepsilon)\}} \left| V(|1+u_n|^2) - \frac{1}{2}\left(\varphi^2(|1+u_n|) - 1\right)^2 \right| dx \\ &\qquad \leq \frac{\varepsilon}{2} \int_{\{| \, |1+u_n|-1| \leq \delta(\varepsilon)\}} \left(\varphi^2(|1+u_n|) - 1\right)^2 \, dx \leq \varepsilon M_2. \end{aligned} \tag{5.16}$$

Assumption (A2) implies that there exists $C(\varepsilon) > 0$ such that

$$\left| V(|1+z|^2) - \frac{1}{2}\left(\varphi^2(|1+z|) - 1\right)^2 \right| \leq C(\varepsilon) \big| \, |1+z| - 1 \big|^{2p_0+2} \tag{5.17}$$

for any $z \in \mathbf{C}$ verifying $\Big| |1+z| - 1 \Big| \geq \delta(\varepsilon)$.

Let $w_n = \big| \, |1+u_n| - 1 \big|$. It is clear that $|w_n| \leq |u_n|$. Using the inequality $|\nabla |v| \, | \leq |\nabla v|$ almost everywhere for $v \in H^1_{\mathrm{loc}}(\mathbf{R}^N)$, we infer that $w_n \in \mathcal{D}^{1,2}(\mathbf{R}^N)$ and

$$\int_{\mathbf{R}^N} |\nabla w_n|^2 \, dx \leq M_2 \qquad \text{for any } n. \tag{5.18}$$

Using (5.17), Hölder's inequality, the Sobolev embedding and (5.18) we find

$$\begin{aligned} &\int_{\{|1+u_n|-1| > \delta(\varepsilon)\}} \left| V(|1+u_n|^2) - \frac{1}{2}\left(\varphi^2(|1+u_n|) - 1\right)^2 \right| dx \\ &\quad \leq C(\varepsilon) \int_{\{w_n > \delta(\varepsilon)\}} |w_n|^{2p_0+2} \, dx \\ &\quad \leq C(\varepsilon) \left(\int_{\{w_n > \delta(\varepsilon)\}} |w_n|^{2^*} \, dx \right)^{\frac{2p_0+2}{2^*}} \left(\mathcal{L}^N(\{w_n > \delta(\varepsilon)\}) \right)^{1 - \frac{2p_0+2}{2^*}} \\ &\quad \leq C(\varepsilon) C_S^{2p_0+2} \|\nabla w_n\|^{2p_0+2}_{L^2(\mathbf{R}^N)} \left(\mathcal{L}^N(\{w_n > \delta(\varepsilon)\}) \right)^{1 - \frac{2p_0+2}{2^*}} \\ &\quad \leq C(\varepsilon) C_S^{2p_0+2} M_2^{p_0+1} \left(\mathcal{L}^N(\{w_n > \delta(\varepsilon)\}) \right)^{1 - \frac{2p_0+2}{2^*}}. \end{aligned} \tag{5.19}$$

We claim that for any $\delta > 0$, we have

$$\lim_{n \to \infty} \mathcal{L}^N(\{w_n > \delta\}) = 0. \tag{5.20}$$

To prove the claim, we argue by contradiction and we assume that there exist $\delta_0 > 0$, a subsequence $(w_{n_k})_{k \geq 1}$ and $\gamma > 0$ such that $\mathcal{L}^N\left(\{w_{n_k} > \delta_0\}\right) \geq \gamma > 0$ for any $k \geq 1$. Since $\|\nabla w_n\|_{L^2(\mathbf{R}^N)}$ is bounded, using Lieb's lemma (see Lemma 6 in [38, p. 447] or Lemma 2.2 in [12, p. 101]), we infer that there exists $\beta > 0$

and $y_k \in \mathbf{R}^N$ such that $\mathcal{L}^N\left(\{w_{n_k} > \frac{\delta_0}{2}\} \cap B(y_k,1)\right) \geq \beta$. Let η be as in (3.30). Then $w_{n_k}(x) \geq \frac{\delta_0}{2}$ implies $\left(\varphi^2(|1+u_{n_k}(x)|)-1\right)^2 \geq \eta\left(\frac{\delta_0}{2}\right) > 0$. Therefore,

$$\int_{B(y_k,1)} \left(\varphi^2(|1+u_{n_k}(x)|)-1\right)^2 dx \geq \eta\left(\frac{\delta_0}{2}\right)\beta > 0$$

for any $k \geq 1$, and this clearly contradicts (5.13). Thus we have proved that (5.20) holds.

From (5.16), (5.19) and (5.20) it follows that

$$\int_{\mathbf{R}^N} \left| V(|1+u_n|^2) - \frac{1}{2}\left(\varphi^2(|1+u_n|)-1\right)^2 \right| dx \leq 2\varepsilon M_2$$

for all sufficiently large n. Thus (5.14) holds and the proof of Step 1 is complete.

Step 2. *We find a convenient scaling of* u_n. From Lemma 5.2 we know that $\liminf_{n\to\infty} E_c(u_n) \geq T_c$. Combined with (b), this implies that $\liminf_{n\to\infty} \frac{2}{N-1}A(u_n) \geq T_c$. Let $\sigma_0 = \sqrt{\frac{2(N-1)}{N-3}}$, and let $\tilde{u}_n = (u_n)_{1,\sigma_0}$. It is obvious that

$$\liminf_{n\to\infty} A(\tilde{u}_n) = \sigma_0^{N-3} \liminf_{n\to\infty} A(u_n) \geq \frac{N-1}{2}\sigma_0^{N-3} T_c. \tag{5.21}$$

Using assumption (a), (5.13) and (5.14) it is easy to see that

(5.22) there exist $\tilde{M}_1$, $\tilde{M}_2 > 0$ such that $\tilde{M}_1 \leq E_{\mathrm{GL}}(\tilde{u}_n) \leq \tilde{M}_2$ for any n,

$$\lim_{n\to\infty} \sup_{y\in\mathbf{R}^N} \int_{B(y,1)} \left\{ |\nabla \tilde{u}_n|^2 + \frac{1}{2}\left(\varphi^2(|1+\tilde{u}_n|)-1\right)^2 \right\} dx = 0 \quad \text{and} \tag{5.23}$$

$$\lim_{n\to\infty} \int_{\mathbf{R}^N} \left| V(|1+\tilde{u}_n|^2) - \frac{1}{2}\left(\varphi^2(|1+\tilde{u}_n|)-1\right)^2 \right| dx = 0. \tag{5.24}$$

It is clear that $P_c(u_n) = \frac{N-3}{N-1}\sigma_0^{3-N} A(\tilde{u}_n) + \sigma_0^{1-N} B_c(\tilde{u}_n)$, and then assumption (b) implies

$$\lim_{n\to\infty} \left(\frac{N-3}{N-1}\sigma_0^2 A(\tilde{u}_n) + B_c(\tilde{u}_n) \right) = \lim_{n\to\infty} \left(A(\tilde{u}_n) + E_c(\tilde{u}_n) \right) = 0. \tag{5.25}$$

Step 3. *Regularization of* $\tilde{u}_n$. Using (5.22), (5.23) and Lemma 3.2 we infer that there is a sequence $h_n \longrightarrow 0$ and for each n, there exists a minimizer v_n of $G^{\tilde{u}_n}_{h_n,\mathbf{R}^N}$ in $H^1_{\tilde{u}_n}(\mathbf{R}^N)$ such that $\delta_n := \| \, |1+v_n| - 1 \|_{L^\infty(\mathbf{R}^N)} \longrightarrow 0$ as $n \longrightarrow \infty$. Then using Lemma 4.2 and the fact that $|c| < v_s = \sqrt{2}$, we obtain

$$E_{\mathrm{GL}}(v_n) + cQ(v_n) \geq 0 \qquad \text{for all sufficiently large } n. \tag{5.26}$$

From (5.22) and (3.4) we get

$$|Q(\tilde{u}_n) - Q(v_n)| \leq C\left(h_n^2 + h_n^{\frac{4}{N}} \tilde{M}_2^{\frac{2}{N}}\right)^{\frac{1}{2}} \tilde{M}_2 \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty. \tag{5.27}$$

Step 4. *Conclusion.* Since $E_{\rm GL}(v_n) \le E_{\rm GL}(\tilde u_n)$, it is clear that

$$\begin{aligned} E_c(\tilde u_n) &= E_{\rm GL}(\tilde u_n) \\ &\quad + cQ(\tilde u_n) + \int_{\mathbf{R}^N} \left\{ V(|1+\tilde u_n|^2) - \frac12 \left(\varphi^2(|1+\tilde u_n|) - 1\right)^2 \right\} dx \\ &\ge E_{\rm GL}(v_n) + cQ(v_n) + c(Q(\tilde u_n) - Q(v_n)) \\ &\quad - \int_{\mathbf{R}^N} \left| V(|1+\tilde u_n|^2) - \frac12 \left(\varphi^2(|1+\tilde u_n|) - 1\right)^2 \right| dx. \end{aligned}$$

Using the last inequality and (5.24), (5.26), (5.27), we get $\liminf_{n\to\infty} E_c(\tilde u_n) \ge 0$. Combined with (5.25), this gives $\limsup_{n\to\infty} A(\tilde u_n) \le 0$, which clearly contradicts (5.21). This completes the proof of Lemma 5.4. □

Next we prove that we cannot have $\alpha \in (0, \alpha_0)$. To do this we argue again by contradiction and we assume that $0 < \alpha < \alpha_0$. Let t_n be as in (5.11), and let $R_n = \frac{t_n}{2}$. For each $n \ge 1$, fix $y_n \in \mathbf{R}^N$ such that $E_{\rm GL}^{B(y_n,R_n)}(u_n) \ge q_n(R_n) - \frac1n$. Using (5.12), we have

$$\begin{aligned} (5.28) \qquad \varepsilon_n &:= \int_{B(y_n,2R_n)\setminus B(y_n,R_n)} |\nabla u_n|^2 + \frac12 \left(\varphi^2(|1+u_n|) - 1\right)^2 dx \\ &\le q_n(2R_n) - \left(q_n(R_n) - \frac1n \right) \longrightarrow 0 \text{ as } n \longrightarrow \infty. \end{aligned}$$

After a translation, we may assume that $y_n = 0$. Using Lemma 3.3 with $A = 2$, $R = R_n$, $\varepsilon = \varepsilon_n$, we infer that for all n sufficiently large, there exist two functions $u_{n,1}$, $u_{n,2}$ having the properties (i)–(vi) in Lemma 3.3.

From Lemma 3.3(iii) and (iv), we get

$$|E_{\rm GL}(u_n) - E_{\rm GL}(u_{n,1}) - E_{\rm GL}(u_{n,2})| \le C\varepsilon_n,$$

while Lemma 3.3(i) and (ii) imply $E_{\rm GL}(u_{n,1}) \ge E_{\rm GL}^{B(0,R_n)}(u_n) > q_n(R_n) - \frac1n$, respectively $E_{\rm GL}(u_{n,2}) \ge E_{\rm GL}^{\mathbf{R}^N\setminus B(0,2R_n)}(u_n) \ge E_{\rm GL}(u_n) - q_n(2R_n)$. Taking into account (5.8), (5.11), (5.12) and (5.28), we infer that

$$(5.29) \quad E_{\rm GL}(u_{n,1}) \longrightarrow \alpha \qquad \text{and} \qquad E_{\rm GL}(u_{n,2}) \longrightarrow \alpha_0 - \alpha \qquad \text{as } n \longrightarrow \infty.$$

By (5.28) and Lemma 3.3(iii)−(vi), we obtain

$$(5.30) \qquad |A(u_n) - A(u_{n,1}) - A(u_{n,2})| \longrightarrow 0,$$

$$(5.31) \qquad |E_c(u_n) - E_c(u_{n,1}) - E_c(u_{n,2})| \longrightarrow 0,$$

and

$$(5.32) \qquad |P_c(u_n) - P_c(u_{n,1}) - P_c(u_{n,2})| \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty.$$

From (5.32) and the fact that $P_c(u_n) \longrightarrow 0$, we infer that $P_c(u_{n,1}) + P_c(u_{n,2}) \longrightarrow 0$ as $n \longrightarrow \infty$. Moreover, Lemmas 4.1, 4.5 and 5.1 imply that the sequences $(P_c(u_{n,i}))_{n\ge1}$ and $(E_c(u_{n,i}))_{n\ge1}$ are bounded for $i = 1, 2$. Passing again to a subsequence (still denoted $(u_n)_{n\ge1}$), we may assume that

$\lim_{n\to\infty} P_c(u_{n,1}) = p_1$ and $\lim_{n\to\infty} P_c(u_{n,2}) = p_2$, where $p_1, p_2 \in \mathbf{R}$ and $p_1 + p_2 = 0$. There are only two possibilities: either $p_1 = p_2 = 0$, or one element of $\{p_1, p_2\}$ is negative.

If $p_1 = p_2 = 0$, then (5.29) and Lemma 5.2 imply that $\liminf_{n\to\infty} E_c(u_{n,i}) \geq T_c$ for $i = 1, 2$. Using (5.31), we obtain $\liminf_{n\to\infty} E_c(u_n) \geq 2T_c$, and this clearly contradicts the assumption $E_c(u_n) \longrightarrow T_c$ in Theorem 5.3.

If $p_i < 0$, it follows from (5.29) and Lemma 4.8(ii) that $\liminf_{n\to\infty} A(u_{n,i}) > \frac{N-1}{2} T_c$. Using (5.30) and the fact that $A \geq 0$, we obtain $\liminf_{n\to\infty} A(u_n) > \frac{N-1}{2} T_c$, which is in contradiction with (5.7).

We conclude that we cannot have $\alpha \in (0, \alpha_0)$.

So far we have proved that $\lim_{t\to\infty} q(t) = \alpha_0$. Proceeding as in [40], it follows that for each $n \geq 1$, there exists $x_n \in \mathbf{R}^N$ such that for any $\varepsilon > 0$, there are $R_\varepsilon > 0$ and $n_\varepsilon \in \mathbf{N}$ satisfying

$$E_{\mathrm{GL}}^{B(x_n, R_\varepsilon)}(u_n) > \alpha_0 - \varepsilon \qquad \text{for any } n \geq n_\varepsilon. \tag{5.33}$$

Let $\tilde{u}_n = u_n(\cdot + x_n)$, so that $\tilde{u}_n$ satisfies (5.33) with $B(0, R_\varepsilon)$ instead of $B(x_n, R_\varepsilon)$. Let $\chi \in C_c^\infty(\mathbf{C}, \mathbf{R})$ be as in Lemma 2.2, and denote $\tilde{u}_{n,1} = \chi(\tilde{u}_n)\tilde{u}_n$, $\tilde{u}_{n,1} = (1 - \chi(\tilde{u}_n))\tilde{u}_n$. Since $E_{\mathrm{GL}}(\tilde{u}_n) = E_{\mathrm{GL}}(u_n)$ is bounded, we infer from Lemma 2.2 that $(\tilde{u}_{n,1})_{n\geq 1}$ is bounded in $\mathcal{D}^{1,2}(\mathbf{R}^N)$, $(\tilde{u}_{n,2})_{n\geq 1}$ is bounded in $H^1(\mathbf{R}^N)$ and $(E_{\mathrm{GL}}(\tilde{u}_{n,i}))_{n\geq 1}$ is bounded for $i = 1, 2$.

Using Lemma 2.1 we may write $1 + \tilde{u}_{n,1} = \rho_n e^{i\theta_n}$, where $\frac{1}{2} \leq \rho_n \leq \frac{3}{2}$ and $\theta_n \in \mathcal{D}^{1,2}(\mathbf{R}^N)$. From (2.4) and (2.7) we find that $(\rho_n - 1)_{n\geq 1}$ is bounded in $H^1(\mathbf{R}^N)$ and $(\theta_n)_{n\geq 1}$ is bounded in $\mathcal{D}^{1,2}(\mathbf{R}^N)$.

We infer that there exists a subsequence $(n_k)_{k\geq 1}$ and there are functions $u_1 \in \mathcal{D}^{1,2}(\mathbf{R}^N)$, $u_2 \in H^1(\mathbf{R}^N)$, $\theta \in \mathcal{D}^{1,2}(\mathbf{R}^N)$, $\rho \in 1 + H^1(\mathbf{R}^N)$ such that

$$\begin{aligned}
&\tilde{u}_{n_k,1} \rightharpoonup u_1 \quad \text{and} \quad \theta_{n_k} \rightharpoonup \theta && \text{weakly in } \mathcal{D}^{1,2}(\mathbf{R}^N),\\
&\tilde{u}_{n_k,2} \rightharpoonup u_2 \quad \text{and} \quad \rho_{n_k} - 1 \rightharpoonup \rho - 1 && \text{weakly in } H^1(\mathbf{R}^N),\\
&\tilde{u}_{n_k,1} \longrightarrow u_1, \qquad \tilde{u}_{n_k,2} \longrightarrow u_2, \qquad \theta_{n_k} \longrightarrow \theta, \qquad \rho_{n_k} - 1 \longrightarrow \rho - 1
\end{aligned}$$

strongly in $L^p(K)$, $1 \leq p < 2^*$ for any compact set $K \subset \mathbf{R}^N$ and a.e. on $\mathbf{R}^N$. Since $\tilde{u}_{n_k,1} = \rho_{n_k} e^{i\theta_{n_k}} - 1 \longrightarrow \rho e^{i\theta} - 1$ a.e., we have $u_1 = \rho e^{i\theta} - 1$ a.e. on $\mathbf{R}^N$.

Denoting $u = u_1 + u_2$, we see that $\tilde{u}_{n_k} \rightharpoonup u$ weakly in $\mathcal{D}^{1,2}(\mathbf{R}^N)$, $\tilde{u}_{n_k} \longrightarrow u$ a.e. on $\mathbf{R}^N$ and strongly in $L^p(K)$, $1 \leq p < 2^*$ for any compact set $K \subset \mathbf{R}^N$.

The weak convergence $\tilde{u}_{n_k} \rightharpoonup u$ in $\mathcal{D}^{1,2}(\mathbf{R}^N)$ implies

$$\int_{\mathbf{R}^N} \left| \frac{\partial u}{\partial x_j} \right|^2 dx \leq \liminf_{k\to\infty} \int_{\mathbf{R}^N} \left| \frac{\partial \tilde{u}_{n_k}}{\partial x_j} \right|^2 dx < \infty \qquad \text{for } j = 1, \dots, N. \tag{5.34}$$

Using the a.e. convergence $\tilde{u}_{n_k} \longrightarrow u$ and Fatou's Lemma, we obtain

$$\int_{\mathbf{R}^N} \left(\varphi^2(|1 + u|) - 1 \right)^2 dx \leq \liminf_{k\to\infty} \int_{\mathbf{R}^N} \left(\varphi^2(|1 + \tilde{u}_{n_k}|) - 1 \right)^2 dx. \tag{5.35}$$

From (5.34) and(5.35) it follows that $u \in \mathcal{X}$ and $E_{\mathrm{GL}}(u) \leq \liminf_{k\to\infty} E_{\mathrm{GL}}(\tilde{u}_{n_k})$.

We will prove that

$$\lim_{k\to\infty}\int_{\mathbf{R}^N} V(|1+\tilde{u}_{n_k}|^2)\,dx = \int_{\mathbf{R}^N} V(|1+u|^2)\,dx, \tag{5.36}$$

$$\lim_{k\to\infty} \|\,|1+\tilde{u}_{n_k}| - |1+u|\,\|_{L^2(\mathbf{R}^N)} = 0 \qquad \text{and} \tag{5.37}$$

$$\lim_{k\to\infty} Q(\tilde{u}_{n_k}) = Q(u). \tag{5.38}$$

Fix $\varepsilon > 0$. Let R_ε be as in (5.33). Since $E_{\mathrm{GL}}(\tilde{u}_{n_k}) \longrightarrow \alpha_0$ as $k \longrightarrow \infty$, it follows from (5.33) that there exists $k_\varepsilon \geq 1$ such that

$$E_{\mathrm{GL}}^{\mathbf{R}^N\setminus B(0,R_\varepsilon)}(\tilde{u}_{n_k}) < 2\varepsilon \qquad \text{for any } k \geq k_\varepsilon. \tag{5.39}$$

As in (5.34)–(5.35), the weak convergence $\nabla\tilde{u}_{n_k} \rightharpoonup \nabla u$ in $L^2(\mathbf{R}^N \setminus B(0,R_\varepsilon))$ implies

$$\int_{\mathbf{R}^N\setminus B(0,R_\varepsilon)} |\nabla u|^2\,dx \leq \liminf_{k\to\infty} \int_{\mathbf{R}^N\setminus B(0,R_\varepsilon)} |\nabla \tilde{u}_{n_k}|^2\,dx,$$

while the fact that $\tilde{u}_{n_k} \longrightarrow u$ a.e. on $\mathbf{R}^N$ and Fatou's Lemma imply

$$\int_{\mathbf{R}^N\setminus B(0,R_\varepsilon)} \left(\varphi^2(|1+u|)-1\right)^2\,dx \leq \liminf_{k\to\infty} \int_{\mathbf{R}^N\setminus B(0,R_\varepsilon)} \left(\varphi^2(|1+\tilde{u}_{n_k}|)-1\right)^2\,dx.$$

Therefore,

$$E_{\mathrm{GL}}^{\mathbf{R}^N\setminus B(0,R_\varepsilon)}(u) \leq \liminf_{k\to\infty} E_{\mathrm{GL}}^{\mathbf{R}^N\setminus B(0,R_\varepsilon)}(\tilde{u}_{n_k}) \leq 2\varepsilon. \tag{5.40}$$

Let $v \in \mathcal{X}$ be a function satisfying $E_{\mathrm{GL}}^{\mathbf{R}^N\setminus B(0,R_\varepsilon)}(v) \leq 2\varepsilon$. Since $\varphi(|1+v|) = |1+v|$ and $\Big|\,|1+v|-1\Big|^2 \leq \left(\varphi^2(|1+v|)-1\right)^2$ if $|1+v| \leq 2$, using (1.7) we find

(5.41)

$$\begin{aligned}\int_{\{|1+v|\leq 2\}\setminus B(0,R_\varepsilon)} |V(|1+v|^2)|\,dx &\leq C_1 \int_{\{|1+v|\leq 2\}\setminus B(0,R_\varepsilon)} \left(\varphi^2(|1+v|)-1\right)^2\,dx \\ &\leq 2C_1 E_{\mathrm{GL}}^{\mathbf{R}^N\setminus B(0,R_\varepsilon)}(v) \leq 4C_1\varepsilon\end{aligned}$$

and

$$\begin{aligned}\int_{\{|1+v|\leq 2\}\setminus B(0,R_\varepsilon)} &(|1+v|-1)^2\,dx \\ &\leq \int_{\{|1+v|\leq 2\}\setminus B(0,R_\varepsilon)} \left(\varphi^2(|1+v|)-1\right)^2\,dx \leq 4\varepsilon.\end{aligned} \tag{5.42}$$

On the other hand, $|1+v(x)| > 2$ implies $\left(\varphi^2(|1+v(x)|)-1\right)^2 > 9$, consequently

$$\begin{aligned}9\mathcal{L}^N(\{x \in \mathbf{R}^N \setminus B(0,R_\varepsilon) \mid |1+v(x)| > 2\}) \\ \leq \int_{\mathbf{R}^N\setminus B(0,R_\varepsilon)} \left(\varphi^2(|1+v|)-1\right)^2\,dx \leq 4\varepsilon.\end{aligned}$$

Using the fact that $\left|V(|1+s|^2)\right| \leq C\left(|1+s|^2-1\right)^{p_0+1} \leq C_2|s|^{2p_0+2}$ if $|1+s| \geq 2$, Hölder's inequality, the above estimate and the Sobolev embedding we find

(5.43)

$$\begin{aligned}\int_{\{|1+v|>2\}\setminus B(0,R_\varepsilon)} |V(|1+v|^2)|\,dx &\leq C_2 \int_{\{|1+v|>2\}\setminus B(0,R_\varepsilon)} |v|^{2p_0+2}\,dx \\ &\leq C\left(\int_{\mathbf{R}^N}|v|^{2^*}dx\right)^{\frac{2p_0+2}{2^*}}\left(\mathcal{L}^N(\{x\in\mathbf{R}^N\setminus B(0,R_\varepsilon)\;\big|\;|1+v(x)|>2\})\right)^{1-\frac{2p_0+2}{2^*}} \\ &\leq C_3\|\nabla v\|_{L^2(\mathbf{R}^N)}^{2p_0+2}\varepsilon^{1-\frac{2p_0+2}{2^*}} \leq C_3\left(E_{\mathrm{GL}}(v)\right)^{p_0+1}\varepsilon^{1-\frac{2p_0+2}{2^*}}.\end{aligned}$$

Similarly, we get

$$\begin{aligned}(5.44)\qquad \int_{\{|1+v|>2\}\setminus B(0,R_\varepsilon)} (|1+v|-1)^2\,dx &\leq \int_{\{|1+v|>2\}\setminus B(0,R_\varepsilon)} |v|^2\,dx \\ &\leq \left(\mathcal{L}^N(\{x\in\mathbf{R}^N\setminus B(0,R_\varepsilon)\;\big|\;|1+v(x)|>2\})\right)^{1-\frac{2}{2^*}}\|v\|^2_{L^{2^*}(\mathbf{R}^N)} \\ &\leq C\varepsilon^{1-\frac{2}{2^*}}\|\nabla v\|^2_{L^2(\mathbf{R}^N)} \leq CE_{\mathrm{GL}}(v)\varepsilon^{1-\frac{2}{2^*}}.\end{aligned}$$

It is obvious that u and $\tilde{u}_{n_k}$ (with $k \geq k_\varepsilon$) satisfy (5.41) and (5.43). If $M > 0$ is such that $E_{\mathrm{GL}}(u_n) \leq M$ for all n, from (5.41) and (5.43) we infer that

(5.45)

$$\begin{aligned}\int_{\mathbf{R}^N\setminus B(0,R_\varepsilon)} &\left|V(|1+\tilde{u}_{n_k}|^2)-V(|1+u|^2)\right|dx \\ &\leq \int_{\mathbf{R}^N\setminus B(0,R_\varepsilon)} \left|V(|1+\tilde{u}_{n_k}|^2)\right| + \left|V(|1+u|^2)\right|dx \leq C\varepsilon + CM^{p_0+1}\varepsilon^{1-\frac{2p_0+2}{2^*}},\end{aligned}$$

while (5.42) and (5.44) give

$$(5.46)\qquad \|\,|1+\tilde{u}_{n_k}|-1\|^2_{L^2(\mathbf{R}^N\setminus B(0,R_\varepsilon))} \leq 4\varepsilon + CM\varepsilon^{1-\frac{2}{2^*}}.$$

Of course, a similar estimate is valid for u.

The mapping $z \longmapsto V(|1+z|^2)$ is obviously C^1. Since $|V(|1+z|^2)| \leq C(1+|z|^{2p_0+2})$ and $\tilde{u}_{n_k} \longrightarrow u$ in $L^2 \cap L^{2p_0+2}(B(0,R_\varepsilon))$ and almost everywhere, it follows that $|1+\tilde{u}_{n_k}| \longrightarrow |1+u|$ in $L^2(B(0,R_\varepsilon))$ and $V(|1+\tilde{u}_{n_k}|^2) \longrightarrow V(|1+u|^2)$ in $L^1(B(0,R_\varepsilon))$. (See, e.g., Theorem A2 in [46, p. 133].) Hence,

$$(5.47)\qquad \int_{B(0,R_\varepsilon)} \left|V(|1+\tilde{u}_{n_k}|^2)-V(|1+u|^2)\right|dx \leq \varepsilon$$

and

$$(5.48)\qquad \|\,|1+\tilde{u}_{n_k}|-|1+u|\,\|_{L^2(B(0,R_\varepsilon))} \leq \varepsilon \quad \text{for all } k \text{ sufficiently large.}$$

Since $\varepsilon > 0$ is arbitrary, (5.36) follows from (5.45) and (5.47), while (5.37) is a consequence of (5.46) and (5.48).

Next we prove (5.38). Fix $\varepsilon > 0$, and let R_ε and k_ε be as in (5.33) and (5.39), respectively. From (2.6) we obtain

$$\|(1-\chi^2(\tilde{u}_n))\tilde{u}_n\|_{L^2(\mathbf{R}^N)} \le C\|\nabla \tilde{u}_n\|_{L^2(\mathbf{R}^N)}^{\frac{2^*}{2}} \le C\left(E_{\mathrm{GL}}(u_n)\right)^{\frac{2^*}{4}}.$$

Using the Cauchy-Schwarz inequality and (5.39), we get

$$\begin{aligned}(5.49)\qquad &\int_{\mathbf{R}^N\setminus B(0,R_\varepsilon)} \left|(1-\chi^2(\tilde{u}_{n_k}))\left\langle i\frac{\partial \tilde{u}_{n_k}}{\partial x_1}, \tilde{u}_{n_k}\right\rangle\right| dx \\ &\qquad \le \|(1-\chi^2(\tilde{u}_{n_k}))\tilde{u}_{n_k}\|_{L^2(\mathbf{R}^N)} \left\|\frac{\partial \tilde{u}_{n_k}}{\partial x_1}\right\|_{L^2(\mathbf{R}^N\setminus B(0,R_\varepsilon))} \le CM^{\frac{2^*}{4}}\sqrt{\varepsilon}\end{aligned}$$

for any $k \ge k_\varepsilon$.

From (2.7) we infer that

$$\|\rho_n^2 - 1\|_{L^2(\mathbf{R}^N)} \le C\left(E_{\mathrm{GL}}(\tilde{u}_n) + \|\nabla \tilde{u}_n\|_{L^2(\mathbf{R}^N)}^{2^*}\right)^{\frac{1}{2}} \le C\left(M + M^{\frac{2^*}{2}}\right)^{\frac{1}{2}}.$$

Using (2.4) and (2.5) we obtain $\left|\frac{\partial \theta_n}{\partial x_1}\right| \le 2\left|\frac{\partial(\chi(\tilde{u}_n)\tilde{u}_n)}{\partial x_1}\right| \le C\left|\frac{\partial \tilde{u}_n}{\partial x_1}\right|$ a.e. on $\mathbf{R}^N$, and then (5.39) implies $\left\|\frac{\partial \theta_{n_k}}{\partial x_1}\right\|_{L^2(\mathbf{R}^N\setminus B(0,R_\varepsilon))} \le C\sqrt{\varepsilon}$ for any $k \ge k_\varepsilon$. Using again the Cauchy-Schwarz inequality, we find

$$\begin{aligned}(5.50)\qquad &\int_{\mathbf{R}^N\setminus B(0,R_\varepsilon)} \left|\left(\rho_{n_k}^2 - 1\right)\frac{\partial \theta_{n_k}}{\partial x_1}\right| dx \\ &\qquad \le \|\rho_{n_k}^2 - 1\|_{L^2(\mathbf{R}^N)} \left\|\frac{\partial \theta_{n_k}}{\partial x_1}\right\|_{L^2(\mathbf{R}^N\setminus B(0,R_\varepsilon))} \le C(M)\sqrt{\varepsilon}\end{aligned}$$

for any $k \ge k_\varepsilon$. It is obvious that the estimates (5.49) and (5.50) also hold with u, ρ and θ instead of $\tilde{u}_{n_k}$, ρ_{n_k} and θ_{n_k}, respectively.

Using the fact that $\tilde{u}_{n_k} \longrightarrow u$ and $\rho_{n_k} - 1 \longrightarrow \rho - 1$ in $L^2(B(0,R_\varepsilon))$ and a.e. and the dominated convergence theorem, we infer that

$$(1-\chi^2(\tilde{u}_{n_k}))\tilde{u}_{n_k} \longrightarrow (1-\chi^2(u))u \quad \text{and} \quad \rho_{n_k}^2 - 1 \longrightarrow \rho^2 - 1 \quad \text{in } L^2(B(0,R_\varepsilon)).$$

This information and the fact that $\frac{\partial \tilde{u}_{n_k}}{\partial x_1} \rightharpoonup \frac{\partial u}{\partial x_1}$ and $\frac{\partial \theta_{n_k}}{\partial x_1} \rightharpoonup \frac{\partial \theta}{\partial x_1}$ weakly in $L^2(B(0,R_\varepsilon))$ imply

(5.51)

$$\int_{B(0,R_\varepsilon)} \left\langle i\frac{\partial \tilde{u}_{n_k}}{\partial x_1}, (1-\chi^2(\tilde{u}_{n_k}))\tilde{u}_{n_k}\right\rangle dx \longrightarrow \int_{B(0,R_\varepsilon)} \left\langle i\frac{\partial u}{\partial x_1}, (1-\chi^2(u))u\right\rangle dx$$

and

$$(5.52)\qquad \int_{B(0,R_\varepsilon)} \left(\rho_{n_k}^2 - 1\right)\frac{\partial \theta_{n_k}}{\partial x_1}\, dx \longrightarrow \int_{B(0,R_\varepsilon)} \left(\rho^2 - 1\right)\frac{\partial \theta}{\partial x_1}\, dx.$$

Using (5.49)−(5.52) and the representation formula (2.12), we infer that there is some $k_1(\varepsilon) \ge k_\varepsilon$ such that for any $k \ge k_1(\varepsilon)$, we have

$$|Q(\tilde{u}_{n_k}) - Q(u)| \le C\sqrt{\varepsilon},$$

where C does not depend on $k \geq k_1(\varepsilon)$ and ε. Since $\varepsilon > 0$ is arbitrary, (5.38) is proven.

Notice that the proofs of (5.36)–(5.38) above are also valid if $N = 3$.

It is obvious that

$$
\begin{aligned}
-cQ(\tilde{u}_{n_k}) - \int_{\mathbf{R}^N} V(|1+\tilde{u}_{n_k}|^2)\,dx &= \frac{N-3}{N-1}A(\tilde{u}_{n_k}) + \int_{\mathbf{R}^N} \left|\frac{\partial \tilde{u}_{n_k}}{\partial x_1}\right|^2 dx - P_c(\tilde{u}_{n_k}) \\
&\geq \frac{N-3}{N-1}A(\tilde{u}_{n_k}) - P_c(\tilde{u}_{n_k}).
\end{aligned} \tag{5.53}
$$

Passing to the limit as $k \longrightarrow \infty$ in this inequality and using (5.36), (5.38) and the fact that $A(\tilde{u}_n) \longrightarrow \frac{N-1}{2}T_c$, $P_c(\tilde{u}_n) \longrightarrow 0$ as $n \longrightarrow \infty$, we find

$$
-cQ(u) - \int_{\mathbf{R}^N} V(|1+u|^2)\,dx \geq \frac{N-3}{2}T_c > 0. \tag{5.54}
$$

In particular, (5.54) implies that $u \neq 0$.

From (5.34) we get

$$
A(u) \leq \liminf_{k\to\infty} A(\tilde{u}_{n_k}) = \frac{N-1}{2}T_c. \tag{5.55}
$$

Using (5.34), (5.36) and (5.38), we find

$$
P_c(u) \leq \liminf_{k\to\infty} P_c(\tilde{u}_{n_k}) = 0. \tag{5.56}
$$

If $P_c(u) < 0$, from Lemma 4.8(i) we get $A(u) > \frac{N-1}{2}T_c$, contradicting (5.55). Thus necessarily $P_c(u) = 0$; that is, $u \in \mathcal{C}$. Since $A(v) \geq \frac{N-1}{2}T_c$ for any $v \in \mathcal{C}$, we infer from (5.55) that $A(u) = \frac{N-1}{2}T_c$. Therefore $E_c(u) = T_c$ and u is a minimizer of E_c in $\mathcal{C}$.

It follows from the above that

$$
A(u) = \frac{N-1}{2}T_c = \lim_{k\to\infty} A(\tilde{u}_{n_k}). \tag{5.57}
$$

Since $P_c(u) = 0$, $\lim_{k\to\infty} P_c(\tilde{u}_{n_k}) = 0$ and (5.36), (5.38) and (5.57) hold, it is obvious that

$$
\int_{\mathbf{R}^N} \left|\frac{\partial u}{\partial x_1}\right|^2 dx = \lim_{k\to\infty} \int_{\mathbf{R}^N} \left|\frac{\partial \tilde{u}_{n_k}}{\partial x_1}\right|^2 dx. \tag{5.58}
$$

Now (5.57) and (5.58) imply $\lim_{k\to\infty} \|\nabla \tilde{u}_{n_k}\|^2_{L^2(\mathbf{R}^N)} = \|\nabla u\|^2_{L^2(\mathbf{R}^N)}$. Together with the fact that $\nabla \tilde{u}_{n_k} \rightharpoonup \nabla u$ weakly in $L^2(\mathbf{R}^N)$, this implies $\nabla \tilde{u}_{n_k} \longrightarrow \nabla u$ strongly in $L^2(\mathbf{R}^N)$ (that is, $\tilde{u}_{n_k} \longrightarrow u$ in $\mathcal{D}^{1,2}(\mathbf{R}^N)$), and the proof of Theorem 5.3 is complete. □

In order to prove that the minimizers provided by Theorem 5.3 solve equation (1.4), we need the following regularity result.

LEMMA 5.5. *Let $N \geq 3$. Assume that conditions* (A1) *and* (A2) *in the introduction hold and that $u \in \mathcal{X}$ satisfies* (1.4) *in $\mathcal{D}'(\mathbf{R}^N)$. Then $u \in W^{2,p}_{\text{loc}}(\mathbf{R}^N)$ for any $p \in [1, \infty)$, $\nabla u \in W^{1,p}(\mathbf{R}^N)$ for $p \in [2, \infty)$, $u \in C^{1,\alpha}(\mathbf{R}^N)$ for $\alpha \in [0, 1)$ and $u(x) \longrightarrow 0$ as $|x| \longrightarrow \infty$.*

Proof. First we prove that for any $R > 0$ and $p \in [2, \infty)$, there exists $C(R, p) > 0$ (depending on u, but not on $x \in \mathbf{R}^N$) such that

$$\|u\|_{W^{2,p}(B(x,R))} \leq C(R, p) \qquad \text{for any } x \in \mathbf{R}^N. \tag{5.59}$$

We write $u = u_1 + u_2$, where u_1 and u_2 are as in Lemma 2.2. Then $|u_1| \leq \frac{1}{2}$, $\nabla u_1 \in L^2(\mathbf{R}^N)$ and $u_2 \in H^1(\mathbf{R}^N)$. Hence for any $R > 0$, there exists $C(R) > 0$ such that

$$\|u\|_{H^1(B(x,R))} \leq C(R) \qquad \text{for any } x \in \mathbf{R}^N. \tag{5.60}$$

Let $\phi(x) = e^{-\frac{icx_1}{2}}(1 + u(x))$. It is easy to see that ϕ satisfies

$$\Delta\phi + \left(F(|\phi|^2) + \frac{c^2}{4}\right)\phi = 0 \qquad \text{in } \mathcal{D}'(\mathbf{R}^N). \tag{5.61}$$

Moreover, (5.60) holds for ϕ instead of u. From (5.60), (5.61), (3.18) and a standard bootstrap argument, we infer that ϕ satisfies (5.59). (Note that assumption (A2) is needed for this bootstrap argument.) It is then clear that (5.59) also holds for u.

From (5.59), the Sobolev embeddings and Morrey's inequality (3.27), we find that u and ∇u are continuous and bounded on $\mathbf{R}^N$ and $u \in C^{1,\alpha}(\mathbf{R}^N)$ for $\alpha \in [0, 1)$. In particular, u is Lipschitz; since $u \in L^{2^*}(\mathbf{R}^N)$, we necessarily have $u(x) \longrightarrow 0$ as $|x| \longrightarrow \infty$.

The boundedness of u implies that there is some $C > 0$ such that

$$\left|F(|1 + u|^2)(1 + u)\right| \leq C\left|\varphi^2(|1 + u|) - 1\right|$$

on $\mathbf{R}^N$. Therefore $F(|1 + u|^2)(1 + u) \in L^2 \cap L^\infty(\mathbf{R}^N)$. Since $\nabla u \in L^2(\mathbf{R}^N)$, from (1.4) we find $\Delta u \in L^2(\mathbf{R}^N)$. It is well known that $\Delta u \in L^p(\mathbf{R}^N)$ with $1 < p < \infty$ implies $\frac{\partial^2 u}{\partial x_i \partial x_j} \in L^p(\mathbf{R}^N)$ for any i, j. (see, e.g., Theorem 3 in [45, p. 96].) Thus we get $\nabla u \in W^{1,2}(\mathbf{R}^N)$. Then the Sobolev embedding implies $\nabla u \in L^p(\mathbf{R}^N)$ for $p \in [2, 2^*]$. Repeating the previous argument, after an easy induction we find $\nabla u \in W^{1,p}(\mathbf{R}^N)$ for any $p \in [2, \infty)$. □

PROPOSITION 5.6. *Assume that conditions* (A1) *and* (A2) *in the introduction are satisfied. Let $u \in \mathcal{C}$ be a minimizer of E_c in $\mathcal{C}$. Then $u \in W^{2,p}_{\text{loc}}(\mathbf{R}^N)$ for any $p \in [1, \infty)$, $\nabla u \in W^{1,p}(\mathbf{R}^N)$ for $p \in [2, \infty)$ and u is a solution of* (1.4).

Proof. It is standard to prove that for any $R > 0$,

$$J_u(v) = \int_{\mathbf{R}^N} V(|1 + u + v|^2)\, dx$$

is a C^1 functional on $H_0^1(B(0,R))$ and

$$J_u'(v).w = -2\int_{\mathbf{R}^N} F(|1+u+v|^2)\langle 1+u+v, w\rangle\, dx.$$

(see, e.g., Lemma 17.1 in [34, p. 64] or Appendix A in [46].) It follows easily that for any $R>0$, the functionals $\tilde{P}_c(v) = P_c(u+v)$ and $\tilde{E}_c(v) = E_c(u+v)$ are C^1 on $H_0^1(B(0,R))$. The differentiability of Q follows, for instance, from (2.19). We divide the proof of Proposition 5.6 into several steps.

Step 1. *There exists a function* $w \in C_c^1(\mathbf{R}^N)$ *such that* $\tilde{P}_c'(0).w \neq 0$. To prove this, we argue by contradiction and we assume that the above statement is false. Then u satisfies

(5.62)
$$-\frac{\partial^2 u}{\partial x_1^2} - \frac{N-3}{N-1}\left(\sum_{k=2}^{N}\frac{\partial^2 u}{\partial x_k^2}\right) + icu_{x_1} - F(|1+u|^2)(1+u) = 0 \qquad \text{in } \mathcal{D}'(\mathbf{R}^N).$$

Let $\sigma = \sqrt{\frac{N-1}{N-3}}$. It is not hard to see that $u_{1,\sigma}$ satisfies (1.4) in $\mathcal{D}'(\mathbf{R}^N)$. Hence the conclusion of Lemma 5.5 holds for $u_{1,\sigma}$ (and thus for u). This regularity is enough to prove that $u_{1,\sigma}$ satisfies the Pohozaev identity

$$\int_{\mathbf{R}^N}\left|\frac{\partial u_{1,\sigma}}{\partial x_1}\right|^2 dx + \frac{N-3}{N-1}\int_{\mathbf{R}^N}\sum_{k=2}^{N}\left|\frac{\partial u_{1,\sigma}}{\partial x_k}\right|^2 dx + cQ(u_{1,\sigma}) + \int_{\mathbf{R}^N} V(|1+u_{1,\sigma}|^2)\, dx = 0. \tag{5.63}$$

To prove (5.63), we multiply (1.4) by $\sum_{k=2}^N \tilde{\chi}(\frac{x}{n})\frac{\partial u_{1,\sigma}}{\partial x_k}$, where $\tilde{\chi} \in C_c^\infty(\mathbf{R}^N)$ is a cut-off function such that $\tilde{\chi} = 1$ on $B(0,1)$ and $\operatorname{supp}(\tilde{\chi}) \subset B(0,2)$, we integrate by parts, then we let $n \longrightarrow \infty$; see the proof of Proposition 4.1 and equation (4.13) in [42, p. 1094] for details.

Since $\sigma = \sqrt{\frac{N-1}{N-3}}$, (5.63) is equivalent to $\left(\frac{N-3}{N-1}\right)^2 A(u) + B_c(u) = 0$. On the other hand, we have $P_c(u) = \frac{N-3}{N-1}A(u) + B_c(u) = 0$ and we infer that $A(u) = 0$. But this contradicts the fact that $A(u) = \frac{N-1}{2}T_c > 0$, and the proof of Step 1 is complete.

Step 2. *Existence of a Lagrange multiplier.* Let w be as above, and let $v \in H^1(\mathbf{R}^N)$ be a function with compact support such that $\tilde{P}_c'(0).v = 0$. For $s, t \in \mathbf{R}$, put $\Phi(t,s) = P_c(u+tv+sw) = \tilde{P}_c(tv+sw)$, so that $\Phi(0,0)=0$, $\frac{\partial\Phi}{\partial t}(0,0) = \tilde{P}_c'(0).v = 0$ and $\frac{\partial\Phi}{\partial s}(0,0) = \tilde{P}_c'(0).w \neq 0$. The implicit function theorem implies that there exist $\delta > 0$ and a C^1 function $\eta : (-\delta,\delta) \longrightarrow \mathbf{R}$ such that $\eta(0)=0$, $\eta'(0)=0$ and $P_c(u+tv+\eta(t)w) = P_c(u) = 0$ for $t \in (-\delta,\delta)$. Since u is a minimizer of A in $\mathcal{C}$, the function $t \longmapsto A(u+tv+\eta(t)w)$ achieves a minimum at $t=0$. Differentiating at $t=0$, we get $A'(u).v = 0$.

Hence $A'(u).v = 0$ for any $v \in H^1(\mathbf{R}^N)$ with compact support satisfying $\tilde{P}'_c(0).v = 0$. Taking $\alpha = \frac{A'(u).w}{\tilde{P}'_c(0).w}$ (where w is as in Step 1), we see that

$$A'(u).v = \alpha P'_c(u).v \quad \text{for any } v \in H^1(\mathbf{R}^N) \text{ with compact support.} \tag{5.64}$$

Step 3. *We have* $\alpha < 0$. To see this, we argue by contradiction. Suppose that $\alpha > 0$. Let w be as in Step 1. We may assume that $P'_c(u).w > 0$. From (5.64) we obtain $A'(u).w > 0$. Since $A'(u).w = \lim_{t\to 0} \frac{A(u+tw)-A(u)}{t}$ and $P'_c(u).w = \lim_{t\to 0} \frac{P_c(u+tw)-P_c(u)}{t}$, we see that for $t < 0$, t sufficiently close to 0, we have $u + tw \neq 0$, $P_c(u+tw) < P_c(u) = 0$ and $A(u+tw) < A(u) = \frac{N-1}{2}T_c$. But this contradicts Lemma 4.8(i). Therefore $\alpha \leq 0$.

Assume that $\alpha = 0$. Then (5.64) implies

$$\int_{\mathbf{R}^N} \sum_{k=2}^{N} \langle \frac{\partial u}{\partial x_j}, \frac{\partial v}{\partial x_j} \rangle \, dx = 0 \quad \text{for any } v \in H^1(\mathbf{R}^N) \text{ with compact support.} \tag{5.65}$$

Let $\tilde{\chi} \in C_c^\infty(\mathbf{R}^N)$ be such that $\chi = 1$ on $B(0,1)$ and $\text{supp}(\tilde{\chi}) \subset B(0,2)$. Put $v_n(x) = \chi(\frac{x}{n})u(x)$, so that $\nabla v_n(x) = \frac{1}{n}\nabla\tilde{\chi}(\frac{x}{n})u + \tilde{\chi}(\frac{x}{n})\nabla u$. It is easy to see that $\tilde{\chi}(\frac{\cdot}{n})\nabla u \longrightarrow \nabla u$ in $L^2(\mathbf{R}^N)$ and $\frac{1}{n}\nabla\tilde{\chi}(\frac{\cdot}{n})u \rightharpoonup 0$ weakly in $L^2(\mathbf{R}^N)$. Replacing v by v_n in (5.65) and passing to the limit as $n \longrightarrow \infty$, we get $A(u) = 0$, which contradicts the fact that $A(u) = \frac{N-1}{2}T_c$. Hence we cannot have $\alpha = 0$. Thus, necessarily, $\alpha < 0$.

Step 4. *Conclusion.* Since $\alpha < 0$, it follows from (5.64) that u satisfies

$$-\frac{\partial^2 u}{\partial x_1^2} - \Big(\frac{N-3}{N-1} - \frac{1}{\alpha}\Big)\sum_{k=2}^{N}\frac{\partial^2 u}{\partial x_k^2} + icu_{x_1} - F(|1+u|^2)(1+u) = 0 \qquad \text{in } \mathcal{D}'(\mathbf{R}^N). \tag{5.66}$$

Let $\sigma_0 = \left(\frac{N-3}{N-1} - \frac{1}{\alpha}\right)^{-\frac{1}{2}}$. It is easy to see that u_{1,σ_0} satisfies (1.4) in $\mathcal{D}'(\mathbf{R}^N)$. Therefore the conclusion of Lemma 5.5 holds for u_{1,σ_0} (and consequently for u). Then Proposition 4.1 in [42] implies that u_{1,σ_0} satisfies the Pohozaev identity $\frac{N-3}{N-1}A(u_{1,\sigma_0}) + B_c(u_{1,\sigma_0}) = 0$, or equivalently $\frac{N-3}{N-1}\sigma_0^{N-3}A(u) + \sigma_0^{N-1}B_c(u) = 0$, which implies

$$\frac{N-3}{N-1}\Big(\frac{N-3}{N-1} - \frac{1}{\alpha}\Big)A(u) + B_c(u) = 0.$$

On the other hand, we have $P_c(u) = \frac{N-3}{N-1}A(u) + B_c(u) = 0$. Since $A(u) > 0$, we get $\frac{N-3}{N-1} - \frac{1}{\alpha} = 1$. Then coming back to (5.66) we see that u satisfies (1.4). □

6. **The case** $N = 3$

This section is devoted to the proof of Theorem 1.1 in space dimension $N = 3$. We only indicate the differences with respect to the case $N \geq 4$.

Clearly, if $N = 3$, we have $P_c = B_c$. For $v \in \mathcal{X}$, we denote

$$D(v) = \int_{\mathbf{R}^3} \left| \frac{\partial v}{\partial x_1} \right|^2 dx + \frac{1}{2} \int_{\mathbf{R}^3} \left(\varphi^2(|1 + v|) - 1 \right)^2 dx.$$

For any $v \in \mathcal{X}$ and $\sigma > 0$ we have

(6.1)

$$A(v_{1,\sigma}) = A(v), \qquad B_c(v_{1,\sigma}) = \sigma^2 B_c(v) \qquad \text{and} \qquad D(v_{1,\sigma}) = \sigma^2 D(v).$$

If $N = 3$, we cannot have a result similar to Lemma 5.1. To see this consider $u \in \mathcal{C}$, so that $B_c(u) = 0$. Using (6.1) we see that $u_{1,\sigma} \in \mathcal{C}$ for any $\sigma > 0$, and we have $E_c(u_{1,\sigma}) = A(u) + \sigma^2 B_c(u) = A(u)$, while $E_{\mathrm{GL}}(u_{1,\sigma}) = A(u) + \sigma^2 D(u) \longrightarrow \infty$ as $\sigma \longrightarrow \infty$.

However, for any $u \in \mathcal{C}$, there exists $\sigma > 0$ such that $D(u_{1,\sigma}) = 1$ (and obviously $u_{1,\sigma} \in \mathcal{C}$, $E_c(u_{1,\sigma}) = E_c(u)$). Since $\mathcal{C} \neq \emptyset$ and $T_c = \inf\{E_c(u) \mid u \in \mathcal{C}\}$, we see that there exists a sequence $(u_n)_{n\geq 1} \subset \mathcal{C}$ such that

$$D(u_n) = 1 \qquad \text{and} \qquad E_c(u_n) = A(u_n) \longrightarrow T_c \quad \text{as } n \longrightarrow \infty. \tag{6.2}$$

In particular, (6.2) implies $E_{\mathrm{GL}}(u_n) \longrightarrow T_c + 1$ as $n \longrightarrow \infty$.

The following result is the equivalent of Lemma 5.2 in the case $N = 3$.

Lemma 6.1. *Let $N = 3$, and let $(u_n)_{n\geq 1} \subset \mathcal{X}$ be a sequence satisfying*

(a) *there exists $C > 0$ such that $D(u_n) \geq C$ for any n, and*

(b) *$B_c(u_n) \longrightarrow 0$ as $n \longrightarrow \infty$.*

Then $\liminf_{n\to\infty} E_c(u_n) = \liminf_{n\to\infty} A(u_n) \geq S_c$, where S_c is given by (4.22).

Proof. It suffices to prove that for any $k > 0$, there holds

$$\liminf_{n\to\infty} A(u_n) \geq E_{c,\min}(k). \tag{6.3}$$

Fix $k > 0$. Let $n \geq 1$. If $A(u_n) \geq k$, by Lemma 4.6(iii) we have $A(u_n) \geq k > E_{c,\min}(k)$. If $A(u_n) < k$, since $E_{\mathrm{GL}}((u_n)_{1,\sigma}) = A(u_n) + \sigma^2 D(u_n)$, we see that there exists $\sigma_n > 0$ such that $E_{\mathrm{GL}}((u_n)_{1,\sigma_n}) = k$. Obviously, we have $\sigma_n^2 D(u_n) < k$; hence, $\sigma_n^2 \leq \frac{k}{C}$ by (a). It is clear that $E_c((u_n)_{1,\sigma_n}) = A(u_n) + \sigma_n^2 B_c(u_n) \geq E_{c,\min}(k)$, therefore $A(u_n) \geq E_{c,\min}(k) - \sigma_n^2 |B_c(u_n)| \geq E_{c,\min}(k) - \frac{k}{C}|B_c(u_n)|$. Passing to the limit as $n \longrightarrow \infty$, we obtain (6.3). Since $k > 0$ is arbitrary, Lemma 6.1 is proven. □

Let

$$\Lambda_c = \{\lambda \in \mathbf{R} \mid \text{ there exists a sequence } (u_n)_{n\geq 1} \subset \mathcal{X} \text{ such that } D(u_n) \geq 1,\ B_c(u_n) \longrightarrow 0 \text{ and } A(u_n) \longrightarrow \lambda \text{ as } n \longrightarrow \infty\}.$$

Using a scaling argument, we see that

$$\Lambda_c = \{\lambda \in \mathbf{R} \mid \text{ there exist a sequence } (u_n)_{n\geq 1} \subset \mathcal{X} \text{ and } C > 0 \text{ such that } D(u_n) \geq C,\ B_c(u_n) \longrightarrow 0 \text{ and } A(u_n) \longrightarrow \lambda \text{ as } n \longrightarrow \infty\}.$$

Let $\lambda_c = \inf \Lambda_c$. From (6.2) it follows that $T_c \in \Lambda_c$. It is standard to prove that Λ_c is closed in $\mathbf{R}$; hence $\lambda_c \in \Lambda_c$. From Lemma 6.1 we obtain

$$S_c \leq \lambda_c \leq T_c. \tag{6.4}$$

The main result of this section is as follows.

THEOREM 6.2. *Let $N = 3$, and let $(u_n)_{n\geq 1} \subset \mathcal{X}$ be a sequence such that*

$$D(u_n) \longrightarrow 1, \quad B_c(u_n) \longrightarrow 0 \quad \text{and} \quad A(u_n) \longrightarrow \lambda_c \quad \text{as } n \longrightarrow \infty. \tag{6.5}$$

There exist a subsequence $(u_{n_k})_{k\geq 1}$, a sequence $(x_k)_{k\geq 1} \subset \mathbf{R}^3$ and $u \in \mathcal{C}$ such that

$$\nabla u_{n_k}(\cdot + x_k) \longrightarrow \nabla u \quad \text{and} \quad |1 + u_{n_k}(\cdot + x_k)| - 1 \longrightarrow |1 + u| - 1 \quad \text{in } L^2(\mathbf{R}^3).$$

Moreover, we have $E_c(u) = A(u) = T_c = \lambda_c$ and u minimizes E_c in $\mathcal{C}$.

Proof. By (6.5), we have $E_{\mathrm{GL}}(u_n) = A(u_n) + D(u_n) \longrightarrow \lambda_c + 1$ as $n \longrightarrow \infty$. Let $q_n(t)$ be the concentration function of $E_{\mathrm{GL}}(u_n)$, as in (5.9). Proceeding as in the proof of Theorem 5.3, we infer that there exist a subsequence of $(u_n, q_n)_{n\geq 1}$, still denoted $(u_n, q_n)_{n\geq 1}$, a nondecreasing function $q : [0, \infty) \longrightarrow [0, \infty)$ and $\alpha \in [0, \lambda_c + 1]$ such that (5.10) holds. We see also that there exists a sequence $t_n \longrightarrow \infty$ satisfying (5.11) and (5.12).

Clearly, our aim is to prove that $\alpha = \lambda_c + 1$. The next result implies that $\alpha > 0$.

LEMMA 6.3. *Assume that $N=3$, $0 \leq c < v_s$ and $(u_n)_{n\geq 1} \subset \mathcal{X}$ is a sequence satisfying $D(u_n) \longrightarrow 1$, $B_c(u_n) \longrightarrow 0$ as $n \longrightarrow \infty$ and $\sup_{n\geq 1} E_{\mathrm{GL}}(u_n) = M < \infty$. There exists $k > 0$ such that*

$$\sup_{y\in\mathbf{R}^3} \int_{B(y,1)} \left\{ |\nabla u_n|^2 + \frac{1}{2}\left(\varphi^2(|1+u_n|) - 1\right)^2 \right\} dx \geq k$$

for all sufficiently large n.

Proof. We argue by contradiction and assume that the conclusion of Lemma 6.3 is false. Then there exists a subsequence, still denoted $(u_n)_{n\geq 1}$, such that

$$\sup_{y\in\mathbf{R}^3} E_{\mathrm{GL}}^{B(y,1)}(u_n) \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty. \tag{6.6}$$

Exactly as in Lemma 5.4 we prove that (5.14) holds; that is,

$$\lim_{n\to\infty} \int_{\mathbf{R}^3} \left| V(|1+u_n|^2) - \frac{1}{2}\left(\varphi^2(|1+u_n|) - 1\right)^2 \right| dx = 0. \tag{6.7}$$

Using (6.7) and the assumptions of Lemma 6.3, we find

$$cQ(u_n) = B_c(u_n) - D(u_n) - \int_{\mathbf{R}^3} \left\{ V(|1+u_n|^2) - \frac{1}{2}\left(\varphi^2(|1+u_n|) - 1\right)^2 \right\} dx \longrightarrow -1 \tag{6.8}$$

as $n \longrightarrow \infty$. If $c = 0$, then (6.8) gives a contradiction and Lemma 6.3 is proven. From now on we assume that $0 < c < v_s$.

Fix $c_1 \in (c, v_s)$, then fix $\sigma > 0$ such that

$$\sigma^2 > \frac{Mc}{c_1 - c}. \tag{6.9}$$

A simple change of variables shows that $\tilde{M} := \sup_{n\geq 1} E_{\mathrm{GL}}((u_n)_{1,\sigma}) < \infty$ and (6.7) holds with $(u_n)_{1,\sigma}$ instead of u_n. It is easy to see that $((u_n)_{1,\sigma})_{n\geq 1}$ also satisfies (6.6). Using Lemma 3.2 we infer that there exists a sequence $h_n \longrightarrow 0$ and for each n there exists a minimizer v_n of $G^{(u_n)_{1,\sigma}}_{h_n,\mathbf{R}^3}$ in $H^1_{(u_n)_{1,\sigma}}(\mathbf{R}^3)$ such that

$$\| \, |1+v_n| - 1 \|_{L^\infty(\mathbf{R}^3)} \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty. \tag{6.10}$$

From (3.4) we obtain

$$|Q((u_n)_{1,\sigma}) - Q(v_n)| \leq C\left(h_n^2 + h_n^{\frac{4}{3}}\tilde{M}^{\frac{2}{3}}\right)^{\frac{1}{2}} \tilde{M} \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty. \tag{6.11}$$

Using (6.10), the fact that $0 < c_1 < v_s$ and Lemma 4.2 we infer that for all sufficiently large n, there holds

$$E_{\mathrm{GL}}(v_n) + c_1 Q(v_n) \geq 0. \tag{6.12}$$

Since $E_{\mathrm{GL}}(v_n) \leq E_{\mathrm{GL}}((u_n)_{1,\sigma})$, for large n we have

(6.13)

$$\begin{aligned}
0 &\leq E_{\mathrm{GL}}(v_n) + c_1 Q(v_n) \\
&\leq E_{\mathrm{GL}}((u_n)_{1,\sigma}) + c_1 Q((u_n)_{1,\sigma}) + c_1 |Q((u_n)_{1,\sigma}) - Q(v_n)| \\
&= A(u_n) + B_c((u_n)_{1,\sigma}) + (c_1 - c) Q((u_n)_{1,\sigma}) + c_1 |Q((u_n)_{1,\sigma}) - Q(v_n)| \\
&\quad + \int_{\mathbf{R}^3} \left\{ \frac{1}{2}\left(\varphi^2(|1+(u_n)_{1,\sigma}|) - 1\right)^2 - V(|1+(u_n)_{1,\sigma}|^2) \right\} dx \\
&= A(u_n) + \sigma^2 B_c(u_n) + \sigma^2 (c_1 - c) Q(u_n) \\
&\leq M + \sigma^2 B_c(u_n) + \sigma^2(c_1 - c) Q(u_n) + a_n,
\end{aligned}$$

where

$$\begin{aligned}
a_n &= c_1 |Q((u_n)_{1,\sigma}) - Q(v_n)| \\
&\quad + \int_{\mathbf{R}^3} \left\{ \frac{1}{2}\left(\varphi^2(|1+(u_n)_{1,\sigma}|) - 1\right)^2 - V(|1+(u_n)_{1,\sigma}|^2) \right\} dx.
\end{aligned}$$

From (6.7) and (6.11) we infer that $\lim_{n\to\infty} a_n = 0$. Then passing to the limit as $n \longrightarrow \infty$ in (6.13), using (6.8) and the fact that $\lim_{n\to\infty} B_c(u_n) = 0$ we find $0 \leq M - \sigma^2 \frac{c_1 - c}{c}$. The last inequality clearly contradicts the choice of σ in (6.9). This contradiction shows that (6.6) cannot hold, and Lemma 6.3 is proven. □

Next we show that we cannot have $\alpha \in (0, \lambda_c + 1)$. We argue again by contradiction, and we assume that $\alpha \in (0, \lambda_c + 1)$. Proceeding exactly as in the proof of Theorem 5.3 and using Lemma 3.3, we infer that for each n sufficiently large, there exist two functions $u_{n,1}$, $u_{n,2}$ having the following properties:

$$E_{\mathrm{GL}}(u_{n,1}) \longrightarrow \alpha, \qquad E_{\mathrm{GL}}(u_{n,2}) \longrightarrow \lambda_c + 1 - \alpha, \tag{6.14}$$

$$|A(u_n) - A(u_{n,1}) - A(u_{n,2})| \longrightarrow 0, \tag{6.15}$$

$$|B_c(u_n) - B_c(u_{n,1}) - B_c(u_{n,2})| \longrightarrow 0, \tag{6.16}$$

$$|D(u_n) - D(u_{n,1}) - D(u_{n,2})| \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty. \tag{6.17}$$

Since $(E_{\mathrm{GL}}(u_{n,i}))_{n\geq 1}$ are bounded, from Lemmas 4.1 and 4.5 we see that $B_c(u_{n,i}))_{n\geq 1}$ are bounded. Moreover, by (6.16), we have

$$\lim_{n\to\infty} (B_c(u_{n,1}) + B_c(u_{n,2})) = \lim_{n\to\infty} B_c(u_n) = 0.$$

Similarly, $(D(u_{n,i}))_{n\geq 1}$ are bounded and

$$\lim_{n\to\infty} (D(u_{n,1}) + D(u_{n,2})) = \lim_{n\to\infty} D(u_n) = 1.$$

Passing again to a subsequence (still denoted $(u_n)_{n\geq 1}$), we may assume that

$$\lim_{n\to\infty} B_c(u_{n,1}) = b_1, \ \lim_{n\to\infty} B_c(u_{n,2}) = b_2, \quad \text{where } b_i \in \mathbf{R},\ b_1 + b_2 = 0, \tag{6.18}$$

$$\lim_{n\to\infty} D(u_{n,1}) = d_1, \ \lim_{n\to\infty} D(u_{n,2}) = d_2, \quad \text{where } d_i \geq 0,\ d_1 + d_2 = 1. \tag{6.19}$$

From (6.18) it follows that either $b_1 = b_2 = 0$, or one of b_1 or b_2 is negative.

Case 1. If $b_1 = b_2 = 0$, we distinguish two subcases.

Subcase 1a. We have $d_1 > 0$ and $d_2 > 0$. Let $\sigma_i = \frac{2}{\sqrt{d_i}}$, $i = 1, 2$. Then $D((u_{n,i})_{1,\sigma_i}) = \sigma_i^2 D(u_{n,i}) \longrightarrow 4$ and $B_c((u_{n,i})_{1,\sigma_i}) = \sigma_i^2 B_c(u_{n,i}) \longrightarrow 0$ as $n \longrightarrow \infty$. From (6.1) and the definition of λ_c, it follows that

$$\liminf_{n\to\infty} A(u_{n,i}) = \liminf_{n\to\infty} A((u_{n,i})_{1,\sigma_i}) \geq \lambda_c, \quad i = 1, 2.$$

Then (6.15) implies

$$\liminf_{n\to\infty} A(u_n) \geq \liminf_{n\to\infty} A(u_{n,1}) + \liminf_{n\to\infty} A(u_{n,2}) \geq 2\lambda_c,$$

and this is a contradiction because by (6.5), we have $\lim_{n\to\infty} A(u_n) = \lambda_c$.

Subcase 1b. One of the d_i's is zero, say $d_1 = 0$. Then necessarily $d_2 = 1$; that is, $\lim_{n\to\infty} D(u_{n,2}) = 1$. Since $E_{\mathrm{GL}}(u_{n,2}) = A(u_{n,2}) + D(u_{n,2}) \longrightarrow 1 + \lambda_c - \alpha$ as $n \longrightarrow \infty$, we infer that $\lim_{n\to\infty} A(u_{n,2}) = \lambda_c - \alpha$. Hence $D(u_{n,2}) \longrightarrow 1$,

$B_c(u_{n,2}) \longrightarrow 0$ and $A(u_{n,2}) \longrightarrow \lambda_c - \alpha$ as $n \longrightarrow \infty$, which implies $\lambda_c - \alpha \in \Lambda_c$. Since $\alpha > 0$, this contradicts the definition of λ_c.

Case 2. One of b_i's is negative, say $b_1 < 0$. From Lemma 4.8(ii) we get $\liminf_{n\to\infty} A(u_{n,1}) > T_c \geq \lambda_c$, and then using (6.15) we find $\liminf_{n\to\infty} A(u_n) > \lambda_c$, in contradiction with (6.5).

Consequently, in all cases we get a contradiction, and this proves that we cannot have $\alpha \in (0, \lambda_c + 1)$.

Up to now we have proved that $\lim_{t\to\infty} q(t) = \lambda_c + 1$; that is, "concentration" occurs.

Proceeding as in the case $N \geq 4$, we see that there exist a subsequence $(u_{n_k})_{k\geq 1}$, a sequence of points $(x_k)_{k\geq 1} \subset \mathbf{R}^3$ and $u \in \mathcal{X}$ such that, denoting $\tilde{u}_{n_k}(x) = u_{n_k}(x + x_k)$, we have

$$\text{(6.20)} \qquad \nabla \tilde{u}_{n_k} \rightharpoonup \nabla u \text{ in } L^2(\mathbf{R}^3)$$
$$\text{and } \tilde{u}_{n_k} \longrightarrow u \text{ in } L^p_{\text{loc}}(\mathbf{R}^3) \text{ for } 1 \leq p < 6 \text{ and a.e. on } \mathbf{R}^3,$$

$$\text{(6.21)} \qquad |1 + \tilde{u}_{n_k}| - 1 \longrightarrow |1 + u| - 1 \qquad \text{in } L^2(\mathbf{R}^3),$$

$$\text{(6.22)} \qquad \int_{\mathbf{R}^3} V(|1 + \tilde{u}_{n_k}|^2)\, dx \longrightarrow \int_{\mathbf{R}^3} V(|1 + u|^2)\, dx,$$

$$\text{(6.23)} \qquad Q(\tilde{u}_{n_k}) \longrightarrow Q(u) \qquad \text{as } k \longrightarrow \infty.$$

Since $\left| \left(\varphi^2(s) - 1\right)^2 - \left(\varphi^2(t) - 1\right)^2 \right| \leq 24|s - t|\,(|s - 1| + |t - 1|)$, from (6.21) we get

$$\text{(6.24)} \qquad \int_{\mathbf{R}^3} \left(\varphi^2(|1 + \tilde{u}_{n_k}|) - 1\right)^2 dx \longrightarrow \int_{\mathbf{R}^3} \left(\varphi^2(|1 + u|) - 1\right)^2 dx.$$

Passing to the limit as $k \longrightarrow \infty$ in the identity

$$\int_{\mathbf{R}^3} \left\{ V(|1+\tilde{u}_{n_k}|^2) - \frac{1}{2}\left(\varphi^2(|1 + \tilde{u}_{n_k}|) - 1\right)^2 \right\} dx + cQ(\tilde{u}_{n_k}) = B_c(\tilde{u}_{n_k}) - D(\tilde{u}_{n_k}),$$

using (6.22)–(6.24) and the fact that $B_c(\tilde{u}_{n_k}) \longrightarrow 0$, $D(\tilde{u}_{n_k}) \longrightarrow 1$, we obtain

$$\int_{\mathbf{R}^3} \left\{ V(|1 + u|^2) - \frac{1}{2}\left(\varphi^2(|1 + u|) - 1\right)^2 \right\} dx + cQ(u) = -1.$$

Thus $u \neq 0$.

From the weak convergence $\nabla \tilde{u}_{n_k} \rightharpoonup \nabla u$ in $L^2(\mathbf{R}^3)$, we get

$$\text{(6.25)} \qquad \int_{\mathbf{R}^3} \left| \frac{\partial u}{\partial x_j} \right|^2 dx \leq \liminf_{k\to\infty} \int_{\mathbf{R}^3} \left| \frac{\partial \tilde{u}_{n_k}}{\partial x_j} \right|^2 dx \qquad \text{for } j = 1, 2, 3.$$

In particular, we have

$$\text{(6.26)} \qquad A(u) \leq \lim_{k\to\infty} A(\tilde{u}_{n_k}) = \lambda_c.$$

From (6.22), (6.23) and (6.25), we obtain

$$\text{(6.27)} \qquad B_c(u) \leq \lim_{k\to\infty} B_c(\tilde{u}_{n_k}) = 0.$$

Since $u \neq 0$, (6.27) and Lemma 4.8(i) imply $A(u) \geq T_c$. Then using (6.26) and the fact that $\lambda_c \leq T_c$, we infer that necessarily

$$A(u) = T_c = \lambda_c = \lim_{k\to\infty} A(\tilde{u}_{n_k}). \tag{6.28}$$

The fact that $B_c(\tilde{u}_{n_k}) \longrightarrow 0$, (6.22) and (6.23) imply that $\left(\int_{\mathbf{R}^3} \left|\frac{\partial \tilde{u}_{n_k}}{\partial x_1}\right|^2 dx\right)_{k\geq 1}$ converges. If $\int_{\mathbf{R}^3} \left|\frac{\partial u}{\partial x_1}\right|^2 dx < \lim_{k\to\infty} \int_{\mathbf{R}^3} \left|\frac{\partial \tilde{u}_{n_k}}{\partial x_1}\right|^2 dx$, we get

$$B_c(u) < \lim_{k\to\infty} B_c(\tilde{u}_{n_k}) = 0$$

in (6.27), and then Lemma 4.8(i) implies $A(u) > T_c$, a contradiction. Taking (6.25) into account, we see that necessarily

$$\int_{\mathbf{R}^3} \left|\frac{\partial u}{\partial x_1}\right|^2 dx = \lim_{k\to\infty} \int_{\mathbf{R}^3} \left|\frac{\partial \tilde{u}_{n_k}}{\partial x_1}\right|^2 dx \qquad \text{and} \qquad B_c(u) = 0. \tag{6.29}$$

Thus we have proved that $u \in \mathcal{C}$ and $\|\nabla u\|_{L^2(\mathbf{R}^3)} = \lim_{k\to\infty} \|\nabla \tilde{u}_{n_k}\|_{L^2(\mathbf{R}^3)}$. Combined with the weak convergence $\nabla \tilde{u}_{n_k} \rightharpoonup \nabla u$ in $L^2(\mathbf{R}^3)$, this implies the strong convergence $\nabla \tilde{u}_{n_k} \longrightarrow \nabla u$ in $L^2(\mathbf{R}^3)$, and the proof of Theorem 6.2 is complete. □

To prove that any minimizer provided by Theorem 6.2 satisfies a Euler-Lagrange equation, we will need the next lemma. It is clear that for any $v \in \mathcal{X}$ and any $R > 0$, the functional $\tilde{B}_c^v(w) := B_c(v + w)$ is C^1 on $H_0^1(B(0,R))$. We denote by $(\tilde{B}_c^v)'(0).w = \lim_{t\to 0} \frac{B_c(v+tw)-B_c(v)}{t}$ its derivative at the origin.

LEMMA 6.4. *Assume that $N \geq 3$ and conditions* (A1) *and* (A2) *are satisfied. Let $v \in \mathcal{X}$ be such that $(\tilde{B}_c^v)'(0).w = 0$ for any $w \in C_c^1(\mathbf{R}^N)$. Then $v = 0$ almost everywhere in $\mathbf{R}^N$.*

Proof. We denote by v^* the precise representative of v; that is, $v^*(x) = \lim_{r\to 0} m(v, B(x,r))$ if this limit exists, and 0 otherwise. Since $v \in L^1_{\text{loc}}(\mathbf{R}^N)$, it is well known that $v = v^*$ almost everywhere on $\mathbf{R}^N$. (See, e.g., Corollary 1 in [17, p. 44].) Throughout the proof of Lemma 6.4, we replace v by v^*. We proceed in three steps.

Step 1. There exists a set $S \subset \mathbf{R}^{N-1}$ such that $\mathcal{L}^{N-1}(S) = 0$, and for any $x' \in \mathbf{R}^{N-1} \setminus S$, the function $v_{x'} := v(\cdot, x')$ belongs to $C^2(\mathbf{R})$ and solves the differential equation

$$-(v_{x'})''(s) + ic(v_{x'})'(s) - F(|1 + v_{x'}(s)|^2)(1 + v_{x'}(s)) = 0 \qquad \text{for any } s \in \mathbf{R}. \tag{6.30}$$

Moreover, we have $|v_{x'}(s)| \longrightarrow 0$ as $s \longrightarrow \pm\infty$, and $v_{x'}$ satisfies the following properties:

(6.31)
$$v_{x'} \in L^{2^*}(\mathbf{R}), \;\; \varphi^2(|1+v_{x'}|) - 1 \in L^2(\mathbf{R}) \quad \text{and} \quad (v_{x'})' = \frac{\partial v}{\partial x_1}(\cdot, x') \in L^2(\mathbf{R}),$$

$$F(|1+v_{x'}|^2)(1+v_{x'}) \in L^2(\mathbf{R}) + L^{\frac{2^*}{2p_0+1}}(\mathbf{R}). \tag{6.32}$$

It is easy to see that $F(|1+v|^2)(1+v) \in L^2(\mathbf{R}^N) + L^{\frac{2^*}{2p_0+1}}(\mathbf{R}^N)$. Since $v \in H^1_{\text{loc}}(\mathbf{R}^3)$, using Theorem 2 in [17, p. 164] and Fubini's Theorem, respectively, we see that there exists a set $\tilde{S} \subset \mathbf{R}^{N-1}$ such that $\mathcal{L}^{N-1}(\tilde{S}) = 0$, and for any $x' \in \mathbf{R}^{N-1} \setminus \tilde{S}$, the function $v_{x'}$ is absolutely continuous, $v_{x'} \in H^1_{\text{loc}}(\mathbf{R})$ and (6.31)$-$(6.32) hold.

Given $\phi \in C^1_c(\mathbf{R})$, we denote

$$\begin{aligned}\Lambda_\phi(x_1, x') = \Big\langle \frac{\partial v}{\partial x_1}(x_1, x'), \phi'(x_1) \Big\rangle + c \Big\langle i \frac{\partial v}{\partial x_1}(x_1, x'), \phi(x_1) \Big\rangle \\ - \langle F(|1+v|^2)(1+v)(x_1, x'), \phi(x_1) \rangle,\end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product of two complex numbers. From (6.31) and (6.32) it follows that $\Lambda_\phi(\cdot, x') \in L^1(\mathbf{R})$ for $x' \in \mathbf{R}^{N-1} \setminus \tilde{S}$. We define $\lambda_\phi(x') = \int_{\mathbf{R}} \Lambda_\phi(x_1, x') dx_1$ if $x' \in \mathbf{R}^{N-1} \setminus \tilde{S}$ and $\lambda_\phi(x') = 0$ if $x' \in \tilde{S}$. Let $\psi \in C^1_c(\mathbf{R}^{N-1})$. It is obvious that the function $(x_1, x') \longmapsto \Lambda_\phi(x_1, x')\psi(x')$ belongs to $L^1(\mathbf{R}^N)$, and using Fubini's Theorem, we get

$$\int_{\mathbf{R}^N} \Lambda_\phi(x_1, x')\psi(x')\, dx = \int_{\mathbf{R}^{N-1}} \lambda_\phi(x')\psi(x')\, dx'.$$

On the other hand, using the assumption of Lemma 6.4, we obtain

$$2 \int_{\mathbf{R}^N} \Lambda_\phi(x_1, x')\psi(x')\, dx = \left(\tilde{B}^v_c\right)'(0).(\phi(x_1)\psi(x')) = 0.$$

Hence we have $\int_{\mathbf{R}^{N-1}} \lambda_\phi(x')\psi(x')\, dx' = 0$ for any $\psi \in C^1_c(\mathbf{R}^{N-1})$, and this implies that there exists a set $S_\phi \subset \mathbf{R}^{N-1} \setminus \tilde{S}$ such that $\mathcal{L}^{N-1}(S_\phi) = 0$ and $\lambda_\phi = 0$ on $\mathbf{R}^{N-1} \setminus S_\phi$.

Denote $q_0 = \frac{2^*}{2p_0+1} \in (1, \infty)$. There exists a countable set $\{\phi_n \in C^1_c(\mathbf{R}) \mid n \in \mathbf{N}\}$ which is dense in $H^1(\mathbf{R}) \cap L^{q_0'}(\mathbf{R})$. For each n, consider the set $S_{\phi_n} \subset \mathbf{R}^{N-1}$ as above. Let $S = \tilde{S} \cup \bigcup_{n \in \mathbf{N}} S_{\phi_n}$. It is clear that $\mathcal{L}^{N-1}(S) = 0$.

Let $x' \in \mathbf{R}^{N-1} \setminus S$. Fix $\phi \in C^1_c(\mathbf{R})$. There is a sequence $(\phi_{n_k})_{k \geq 1}$ such that $\phi_{n_k} \longrightarrow \phi$ in $H^1(\mathbf{R})$ and in $L^{q_0'}(\mathbf{R})$. Then $\lambda_{\phi_{n_k}}(x') = 0$ for each k and (6.31)$-$(6.32) imply that $\lambda_{\phi_{n_k}}(x') \longrightarrow \lambda_\phi(x')$. Consequently, $\lambda_\phi(x') = 0$ for any $\phi \in C^1_c(\mathbf{R})$ and this implies that $v_{x'}$ satisfies equation (6.30) in $\mathcal{D}'(\mathbf{R})$. Using (6.30) we infer that $(v_{x'})''$ (the *weak* second derivative of $v_{x'}$) belongs to $L^1_{\text{loc}}(\mathbf{R})$. Then it follows that $(v_{x'})'$ is continuous on $\mathbf{R}$. (See, e.g., Lemma VIII.2 in [11, p. 123].) In particular, we have $v_{x'} \in C^1(\mathbf{R})$. Coming back to

(6.30) we see that $(v_{x'})''$ is continuous. Hence, $v_{x'} \in C^2(\mathbf{R})$ and (6.30) holds at each point of $\mathbf{R}$. Finally, we have $|v_{x'}(s_2) - v_{x'}(s_1)| \le |s_2 - s_1|^{\frac{1}{2}} \| (v_{x'})' \|_{L^2}$; this estimate and the fact that $v_{x'} \in L^{2^*}(\mathbf{R})$ imply that $v_{x'}(s) \longrightarrow 0$ as $s \longrightarrow \pm\infty$.

Step 2. There exist two positive constants k_1, k_2 (depending only on F and c) such that for any $x' \in \mathbf{R}^{N-1} \setminus S$, we have either $v_{x'} = 0$ on $\mathbf{R}$ or there exists an interval $I_{x'} \subset \mathbf{R}$ with $\mathcal{L}^1(I_{x'}) \ge k_1$ and $\Big| |1 + v_{x'}| - 1 \Big| \ge k_2$ on $I_{x'}$.

To see this, fix $x' \in \mathbf{R}^{N-1} \setminus S$ and denote $g = |1 + v_{x'}|^2 - 1$. Then $g \in C^2(\mathbf{R}, \mathbf{R})$ and g tends to zero at $\pm\infty$. Proceeding exactly as in [42, pp. 1100–1101], we integrate (6.30) and we see that g satisfies

$$(g')^2(s) + c^2 g^2(s) - 4(g(s) + 1)V(g(s) + 1) = 0 \qquad \text{in } \mathbf{R}. \tag{6.33}$$

Using (1.6) we have $c^2t^2 - 4(t+1)V(t+1) = t^2(c^2 - v_s^2 + \varepsilon_1(t))$, where $\varepsilon_1(t) \longrightarrow 0$ as $t \longrightarrow 0$. In particular, there exists $k_0 > 0$ such that

$$c^2t^2 - 4(t + 1)V(t + 1) < 0 \qquad \text{for } t \in [-2k_0, 0) \cup (0, 2k_0]. \tag{6.34}$$

If $g = 0$ on $\mathbf{R}$, then $|1 + v_{x'}| = 1$. Consequently, there exists a lifting $1 + v_{x'}(s) = e^{i\theta(s)}$ with $\theta \in C^2(\mathbf{R}, \mathbf{R})$. Using equation (6.30) and proceeding as in [42, p. 1101], we see that either $1 + v_{x'}(s) = e^{i\theta_0}$ or $1 + v_{x'}(s) = e^{ics+\theta_0}$, where $\theta_0 \in \mathbf{R}$ is a constant. Since $v_{x'} \in L^{2^*}(\mathbf{R})$, we must have $v_{x'} = 0$.

If $g \not\equiv 0$, the function g achieves a negative minimum or a positive maximum at some $s_0 \in \mathbf{R}$. Then $g'(s_0) = 0$, and using (6.33) and (6.34) we infer that $|g(s_0)| > 2k_0$. Let

$$s_2 = \inf\{s < s_0 \;\Big|\; |g(s)| \ge 2k_0\}, \qquad s_1 = \sup\{s < s_2 \;\Big|\; |g(s)| \le k_0\},$$

so that $s_1 < s_2$, $|g(s_1)| = k_0$, $|g(s_2)| = 2k_0$ and $k_0 \le |g(s)| \le 2k_0$ for $s \in [s_1, s_2]$. Denote $M = \sup\{4(t + 1)V(t + 1) - c^2t^2 \mid t \in [-2k_0, 2k_0]\}$. From (6.33) we obtain $|g'(s)| \le \sqrt{M}$ if $g(s) \in [-2k_0, 2k_0]$, and we infer that

$$k_0 = |g(s_2)| - |g(s_1)| \le \left| \int_{s_1}^{s_2} g'(s)\, ds \right| \le \sqrt{M}(s_2 - s_1);$$

hence $s_2 - s_1 \ge \frac{k_0}{\sqrt{M}}$. Obviously, there exists $k_2 > 0$ such that $\big| |1+z|^2 - 1 \big| \ge k_0$ implies $\big| |1 + z| - 1 \big| \ge k_2$. Taking $k_1 = \frac{k_0}{\sqrt{M}}$ and $I_{x'} = [s_1, s_2]$, the proof of Step 2 is complete.

Step 3. *Conclusion.* Let $K = \{x' \in \mathbf{R}^{N-1} \setminus S \mid v_{x'} \not\equiv 0\}$. It is standard to prove that K is $\mathcal{L}^{N-1}$-measurable. The conclusion of Lemma 6.4 follows if we prove that $\mathcal{L}^{N-1}(K) = 0$. We argue by contradiction and we assume that $\mathcal{L}^{N-1}(K) > 0$.

If $x' \in K$, it follows from Step 2 that there exists an interval $I_{x'}$ of length at least k_1 such that $\big(\varphi^2(|1 + v_{x'}|) - 1\big)^2 \ge \eta(k_2)$ on $I_{x'}$, where η is as in (3.30).

This implies $\int_{\mathbf{R}} \left(\varphi^2(|1+v(x_1,x')|)-1\right)^2 dx_1 \ge k_1\eta(k_2)$. Using Fubini's theorem, we get

$$\int_{\mathbf{R}^N} \left(\varphi^2(|1+v(x)|)-1\right)^2 dx = \int_K \left(\int_{\mathbf{R}} \left(\varphi^2(|1+v(x_1,x')|)-1\right)^2 dx_1\right) dx' \ge k_1\eta(k_2)\mathcal{L}^{N-1}(K).$$

Since $v \in \mathcal{X}$, we infer that $\mathcal{L}^{N-1}(K)$ is finite.

It is obvious that there exist $x_1' \in K$ and $x_2' \in \mathbf{R}^{N-1} \setminus (K \cup S)$ arbitrarily close to each other. Then $|v_{x_1'}| \ge k_2$ on an interval $I_{x_1'}$ of length k_1, while $v_{x_2'} \equiv 0$. If we knew that v is uniformly continuous, this would lead to a contradiction. However, equation (6.30) satisfied by v involves only derivatives with respect to x_1 and does not imply any regularity properties of v with respect to the transverse variables. (Notice that if v is a solution of (6.30), then $v(x_1+\delta(x'),x')$ is also a solution, even if δ is discontinuous.) For instance, for the Gross-Pitaevskii nonlinearity $F(s)=1-s$, it is possible to construct bounded, C^∞ functions v such that $v \in L^{2^*}(\mathbf{R}^N)$, (6.30) is satisfied for almost every x' and the set K constructed as above is a nontrivial ball in $\mathbf{R}^{N-1}$. (Of course, these functions do not tend uniformly to zero at infinity, are not uniformly continuous and their gradient is not in $L^2(\mathbf{R}^N)$.)

We use the fact that one transverse derivative of v (for instance, $\frac{\partial v}{\partial x_2}$) is in $L^2(\mathbf{R}^N)$ to get a contradiction.

For $x' = (x_2, x_3, \dots, x_N) \in \mathbf{R}^{N-1}$, we denote $x'' = (x_3, \dots, x_N)$. Since $v \in H^1_{\text{loc}}(\mathbf{R}^N)$, from Theorem 2 in [17, p. 164] it follows that there exists $J \subset \mathbf{R}^{N-1}$ such that $\mathcal{L}^{N-1}(J)=0$ and $u(x_1,\cdot,x'') \in H^1_{\text{loc}}(\mathbf{R}^N)$ for any $(x_1,x'') \in \mathbf{R}^{N-1}\setminus J$. Given $x'' \in \mathbf{R}^{N-2}$, we denote

$$K_{x''} = \{x_2 \in \mathbf{R} \mid (x_2,x'') \in K\},$$
$$S_{x''} = \{x_2 \in \mathbf{R} \mid (x_2,x'') \in S\},$$
$$J_{x''} = \{x_1 \in \mathbf{R} \mid (x_1,x'') \in J\}.$$

Fubini's Theorem implies that for almost all $x'' \in \mathbf{R}^{N-2}$, the sets $K_{x''}$, $S_{x''}$, $J_{x''}$ are $\mathcal{L}^1$-measurable, $\mathcal{L}^1(K_{x''}) < \infty$ and $\mathcal{L}^1(S_{x''}) = \mathcal{L}^1(J_{x''}) = 0$. Let

$$G = \{x'' \in \mathbf{R}^{N-2} \mid K_{x''}, S_{x''}, J_{x''} \text{ are } \mathcal{L}^1\text{-measurable}, \mathcal{L}^1(S_{x''}) = \mathcal{L}^1(J_{x''}) = 0 \text{ and } 0 < \mathcal{L}^1(K_{x''}) < \infty\}. \tag{6.35}$$

Clearly, G is $\mathcal{L}^{N-2}$-measurable and $\int_G \mathcal{L}^1(K_{x''})\,dx'' = \mathcal{L}^{N-1}(K) > 0$; thus $\mathcal{L}^{N-2}(G) > 0$. We claim that

$$\int_{\mathbf{R}^2} \left|\frac{\partial v}{\partial x_2}(x_1,x_2,x'')\right|^2 dx_1\,dx_2 = \infty \qquad \text{for any } x'' \in G. \tag{6.36}$$

Indeed, let $x'' \in G$. Fix $\varepsilon > 0$. Using (6.35) we infer that there exist s_1, $s_2 \in \mathbf{R}$ such that $(s_1,x'') \in \mathbf{R}^{N-1} \setminus (K \cup S)$, $(s_2,x'') \in K$ and $|s_2 - s_1| < \varepsilon$.

Then $v(t, s_1, x'') = 0$ for any $t \in \mathbf{R}$. From Step 2 it follows that there exists an interval I with $\mathcal{L}^1(I) \geq k_1$ such that $|v(t, s_2, x'')| \geq ||1 + v(t, s_2, x'')| - 1| \geq k_2$ for $t \in I$. Assume $s_1 < s_2$. If $t \in I \setminus J_{x''}$, we have $v(t, \cdot, x'') \in H^1_{\mathrm{loc}}(\mathbf{R})$. Hence,

$$k_2 \leq |v(t, s_2, x'') - v(t, s_1, x'')| = \left| \int_{s_1}^{s_2} \frac{\partial v}{\partial x_2}(t, \tau, x'')\, d\tau \right| \leq (s_2 - s_1)^{\frac{1}{2}} \left(\int_{s_1}^{s_2} \left| \frac{\partial v}{\partial x_2}(t, \tau, x'') \right|^2 d\tau \right)^{\frac{1}{2}}.$$

Clearly, this implies $\int_{s_1}^{s_2} \left| \frac{\partial v}{\partial x_2}(t, \tau, x'') \right|^2 d\tau \geq \frac{k_2^2}{\varepsilon}$. Consequently,

$$\int_{\mathbf{R}^2} \left| \frac{\partial v}{\partial x_2}(x_1, x_2, x'') \right|^2 dx_1\, dx_2 \geq \int_I \int_{s_1}^{s_2} \left| \frac{\partial v}{\partial x_2}(t, \tau, x'') \right|^2 d\tau\, dt \geq \frac{k_1 k_2^2}{\varepsilon}.$$

Since the last inequality holds for any $\varepsilon > 0$, (6.36) is proven. Using (6.36), the fact that $\mathcal{L}^{N-2}(G) > 0$ and Fubini's Theorem, we get $\int_{\mathbf{R}^N} \left| \frac{\partial v}{\partial x_2} \right|^2 dx = \infty$, contradicting the fact that $v \in \mathcal{X}$. Thus necessarily $\mathcal{L}^{N-1}(K) = 0$, and the proof of Lemma 6.4 is complete. □

PROPOSITION 6.5. *Assume that* $N = 3$ *and conditions* (A1) *and* (A2) *are satisfied. Let* $u \in \mathcal{C}$ *be a minimizer of* E_c *in* $\mathcal{C}$. *Then* $u \in W^{2,p}_{\mathrm{loc}}(\mathbf{R}^3)$ *for any* $p \in [1, \infty)$, $\nabla u \in W^{1,p}(\mathbf{R}^3)$ *for* $p \in [2, \infty)$ *and there exists* $\sigma > 0$ *such that* $u_{1,\sigma}$ *is a solution of* (1.4).

Proof. The proof is very similar to the proof of Proposition 5.6. It is clear that $A(u) = E_c(u) = T_c$ and u is a minimizer of A in $\mathcal{C}$. For any $R > 0$, the functionals $\tilde{B}^u_c$ and $\tilde{A}(v) := A(u + v)$ are C^1 on $H^1_0(B(0, R))$. We proceed in four steps.

Step 1. There exists $w \in C^1_c(\mathbf{R}^3)$ such that $(\tilde{B}^u_c)'(0).w \neq 0$. This follows from Lemma 6.4.

Step 2. There exists a Lagrange multiplier $\alpha \in \mathbf{R}$ such that

$$\tilde{A}'(0).v = \alpha(\tilde{B}^u_c)'(0).v \quad \text{for any } v \in H^1(\mathbf{R}^3),\ v \text{ with compact support.} \tag{6.37}$$

Step 3. We have $\alpha < 0$.

The proof of Steps 2 and 3 is the same as the proof of Steps 2 and 3 in Proposition 5.6.

Step 4. *Conclusion.* Let $\beta = -\frac{1}{\alpha}$. Then (6.37) implies that u satisfies

$$-\frac{\partial^2 u}{\partial x_1^2} - \beta \left(\frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2} \right) + icu_{x_1} - F(|1 + u|^2)(1 + u) = 0 \text{ in } \mathcal{D}'(\mathbf{R}^3).$$

For $\sigma^2 = \frac{1}{\beta}$, we see that $u_{1,\sigma}$ satisfies (1.4). It is clear that $u_{1,\sigma} \in \mathcal{C}$ and $u_{1,\sigma}$ minimizes A (respectively E_c) in $\mathcal{C}$. Finally, the regularity of $u_{1,\sigma}$ (thus the regularity of u) follows from Lemma 5.5. □

7. Further properties of traveling waves

By Propositions 5.6 and 6.5 we already know that the solutions of (1.4) found there are in $W^{2,p}_{\mathrm{loc}}(\mathbf{R}^N)$ for any $p \in [1,\infty)$ and in $C^{1,\alpha}(\mathbf{R}^N)$ for any $\alpha \in (0,1)$. In general, a straightforward bootstrap argument shows that the finite energy traveling waves of (1.1) have the best regularity allowed by the nonlinearity F. For instance, if $F \in C^k([0,\infty))$ for some $k \in \mathbf{N}^*$, it can be proved that all finite energy solutions of (1.4) are in $W^{k+2,p}_{\mathrm{loc}}(\mathbf{R}^N)$ for any $p \in [1,\infty)$ (see, for instance, Proposition 2.2 (ii) in [42]). If F is analytic, it can be proved that finite energy traveling waves are also analytic. In the case of the Gross-Pitaevskii equation, this has been done in [5].

Our next result concerns the symmetry of those solutions of (1.4) that minimize E_c in $\mathcal{C}$.

Proposition 7.1. *Assume that $N \geq 3$ and conditions* (A1) *and* (A2) *in the introduction hold. Let $u \in \mathcal{C}$ be a minimizer of E_c in $\mathcal{C}$. Then, after a translation in the variables $(x_2, \dots, x_N)$, u is axially symmetric with respect to Ox_1.*

Proof. Let T_c be as in Lemma 4.7. We know that any minimizer u of E_c in $\mathcal{C}$ satisfies $A(u) = \frac{N-1}{2} T_c > 0$. Using Lemma 4.8(i), it is easy to prove that a function $u \in \mathcal{X}$ is a minimizer of E_c in $\mathcal{C}$ if and only if

$$\text{(7.1)} \quad u \text{ minimizes the functional } P_c \text{ in the set } \left\{ v \in \mathcal{X} \mid A(v) = \frac{N-1}{2} T_c \right\}.$$

The minimization problem (7.1) is of the type studied in [43]. All we have to do is to verify that assumptions ($\mathbf{A1}_c$) and ($\mathbf{A2}_c$) in [43, p. 329] are satisfied and then to apply the general theory developed there.

Let Π be an affine hyperplane in $\mathbf{R}^N$ parallel to Ox_1. We denote by s_Π the symmetry of $\mathbf{R}^N$ with respect to Π and by Π^+, Π^- the two half-spaces determined by Π. Given a function $v \in \mathcal{X}$, we denote

$$v_{\Pi^+}(x) = \begin{cases} v(x) & \text{if } x \in \Pi^+ \cup \Pi, \\ v(s_\Pi(x)) & \text{if } x \in \Pi^- \end{cases}$$

and

$$v_{\Pi^-}(x) = \begin{cases} v(x) & \text{if } x \in \Pi^- \cup \Pi, \\ v(s_\Pi(x)) & \text{if } x \in \Pi^+. \end{cases}$$

It is easy to see that $v_{\Pi^+}, v_{\Pi^-} \in \mathcal{X}$. Moreover, for any $v \in \mathcal{X}$, we have

$$A(v_{\Pi^+}) + A(v_{\Pi^-}) = 2A(v) \quad \text{and} \quad P_c(v_{\Pi^+}) + P_c(v_{\Pi^-}) = 2P_c(v).$$

This implies that assumption ($\mathbf{A1}_c$) in [43] is satisfied.

By Propositions 5.6 and 6.5 and Lemma 5.5 we know that any minimizer of (7.1) is C^1 on $\mathbf{R}^N$, hence assumption ($\mathbf{A2}_c$) in [43] holds. Then the axial symmetry of solutions of (7.1) follows directly from Theorem 2′ in [43, p. 329]. □

Acknowledgements. I am indebted to Jean-Claude Saut, who brought this subject to my attention almost ten years ago, for many useful and stimulating conversations. I am also very grateful to Radu Ignat, Clément Gallo, Alberto Farina, David Chiron and Mihai Bostan for interesting discussions and to the anonymous referees who suggested improvements in the exposition.

This paper was written when I was a member of the Department of Mathematics at Université de Franche-Comté in Besançon, France. I would like to thank the people at the Department of Mathematics in Besançon for their hospitality.

References

[1] L. Almeida and F. Bethuel, Topological methods for the Ginzburg-Landau equations, *J. Math. Pures Appl.* **77** (1998), 1–49. MR 1617594. Zbl 0904.35023. http://dx.doi.org/10.1016/S0021-7824(98)80064-0.

[2] I. V. Barashenkov, A. D. Gocheva, V. G. Makhan′kov, and I. V. Puzynin, Stability of the soliton-like "bubbles", *Phys. D* **34** (1989), 240–254. MR 0982390. Zbl 0697.35127. http://dx.doi.org/10.1016/0167-2789(89)90237-6.

[3] I. V. Barashenkov and V. G. Makhan′kov, Soliton-like "bubbles" in a system of interacting bosons, *Phys. Lett. A* **128** (1988), 52–56. MR 0935918. http://dx.doi.org/10.1016/0375-9601(88)91042-0.

[4] N. G. Berloff, Quantised vortices, travelling coherent structures and superfluid turbulence, in *Stationary and Time Dependent Gross-Pitaevskii Equations*, *Contemp. Math.* **473**, Amer. Math. Soc., Providence, RI, 2008, pp. 27–54. MR 2522013. Zbl 1166.35360. http://dx.doi.org/10.1090/conm/473/09223.

[5] F. Béthuel, P. Gravejat, and J.-C. Saut, Travelling waves for the Gross-Pitaevskii equation. II, *Comm. Math. Phys.* **285** (2009), 567–651. MR 2461988. Zbl 1190.35196. http://dx.doi.org/10.1007/s00220-008-0614-2.

[6] F. Béthuel, P. Gravejat, J.-C. Saut, and D. Smets, Orbital stability of the black soliton for the Gross-Pitaevskii equation, *Indiana Univ. Math. J.* **57** (2008), 2611–2642. MR 2482993. Zbl 1171.35012. http://dx.doi.org/10.1512/iumj.2008.57.3632.

[7] F. Béthuel, G. Orlandi, and D. Smets, Vortex rings for the Gross-Pitaevskii equation, *J. Eur. Math. Soc.* (*JEMS*) **6** (2004), 17–94. MR 2041006. Zbl 1091.35085. Available at http://link.springer.de/cgi/linkref?issn=1435-9855&year=04&volume=6&page=17.

[8] F. Béthuel and J.-C. Saut, Travelling waves for the Gross-Pitaevskii equation. I, *Ann. Inst. H. Poincaré Phys. Théor.* **70** (1999), 147–238. MR 1669387.

Zbl 0933.35177. Available at http://www.numdam.org/item?id=AIHPA_1999__70_2_147_0.

[9] A. de Bouard, Instability of stationary bubbles, *SIAM J. Math. Anal.* **26** (1995), 566–582. MR 1325903. Zbl 0823.35017. http://dx.doi.org/10.1137/S0036141092237029.

[10] J. Bourgain, H. Brezis, and P. Mironescu, Lifting in Sobolev spaces, *J. Anal. Math.* **80** (2000), 37–86. MR 1771523. Zbl 0967.46026. http://dx.doi.org/10.1007/BF02791533.

[11] H. Brezis, *Analyse Fonctionnelle*, *Collect. Math. Appl. pour la Maîtrise.* [*Collect. Appl. Math. for the Master's Degree*], Masson, Paris, 1983. MR 0697382. Zbl 0511.46001.

[12] H. Brezis and E. H. Lieb, Minimum action solutions of some vector field equations, *Comm. Math. Phys.* **96** (1984), 97–113. MR 0765961. Zbl 0579.35025. http://dx.doi.org/10.1007/BF01217349.

[13] D. Chiron, Travelling waves for the Gross-Pitaevskii equation in dimension larger than two, *Nonlinear Anal.* **58** (2004), 175–204. MR 2070812. Zbl 1054.35091. http://dx.doi.org/10.1016/j.na.2003.10.028.

[14] D. Chiron and M. Mariş, Traveling waves for nonlinear Schrödinger equations with nonzero conditions at infinity, II. arXiv 1203.1912.

[15] D. Chiron and M. Mariş, Rarefaction pulses for the nonlinear Schrödinger equation in the transonic limit. arXiv 1210.1315.

[16] C. Coste, Nonlinear Schrödinger equation and superfluid hydrodynamics, *Eur. Phys. J. B Condens. Matter Phys.* **1** (1998), 245–253. MR 1646182. http://dx.doi.org/10.1007/s100510050178.

[17] L. C. Evans and R. F. Gariepy, *Measure Theory and Fine Properties of Functions*, *Stud. Adv. Math.*, CRC Press, Boca Raton, FL, 1992. MR 1158660. Zbl 0804.28001.

[18] A. Farina, Finite-energy solutions, quantization effects and Liouville-type results for a variant of the Ginzburg-Landau systems in $\mathbf{R}^K$, *Differential Integral Equations* **11** (1998), 875–893. MR 1659256. Zbl 1074.35504.

[19] A. Farina, From Ginzburg-Landau to Gross-Pitaevskii, *Monatsh. Math.* **139** (2003), 265–269. MR 2001707. Zbl 1126.35063. http://dx.doi.org/10.1007/s00605-002-0514-z.

[20] C. Gallo, The Cauchy problem for defocusing nonlinear Schrödinger equations with non-vanishing initial data at infinity, *Comm. Partial Differential Equations* **33** (2008), 729–771. MR 2424376. Zbl 1156.35086. http://dx.doi.org/10.1080/03605300802031614.

[21] P. Gérard, The Cauchy problem for the Gross-Pitaevskii equation, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **23** (2006), 765–779. MR 2259616. Zbl 1122.35133. http://dx.doi.org/10.1016/j.anihpc.2005.09.004.

[22] P. Gérard, The Gross-Pitaevskii equation in the energy space, in *Stationary and Time Dependent Gross-Pitaevskii Equations*, *Contemp. Math.* **473**, Amer. Math. Soc., Providence, RI, 2008, pp. 129–148. MR 2522016. Zbl 1166.35373. http://dx.doi.org/10.1090/conm/473/09226.

[23] P. Gérard and Z. Zhang, Orbital stability of traveling waves for the one-dimensional Gross-Pitaevskii equation, *J. Math. Pures Appl.* **91** (2009), 178–210. MR 2498754. Zbl 1232.35152. http://dx.doi.org/10.1016/j.matpur.2008.09.009.

[24] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, 3rd ed., *Classics in Math.*, Springer-Verlag, New York, 2001. MR 1814364. Zbl 1042.35002.

[25] J. Grant and P. H. Roberts, Motions in a Bose condensate. III. The structure and effective masses of charged and uncharged impurities, *J. Phys. A: Math., Nucl. Gen.* **7** (1974), 260–279. http://dx.doi.org/10.1088/0305-4470/7/2/012.

[26] P. Gravejat, A non-existence result for supersonic travelling waves in the Gross-Pitaevskii equation, *Comm. Math. Phys.* **243** (2003), 93–103. MR 2020221. Zbl 1044.35087. http://dx.doi.org/10.1007/s00220-003-0961-y.

[27] P. Gravejat, Asymptotics for the travelling waves in the Gross-Pitaevskii equation, *Asymptot. Anal.* **45** (2005), 227–299. MR 2191764. Zbl 1092.35103.

[28] E. P. Gross, Hydrodynamics of a superfluid condensate, *J. Math. Phys.* **4** (1963), 195–207. http://dx.doi.org/10.1063/1.1703944.

[29] S. Gustafson, K. Nakanishi, and T.-P. Tsai, Scattering for the Gross-Pitaevskii equation, *Math. Res. Lett.* **13** (2006), 273–285. MR 2231117. Zbl 1119.35084.

[30] S. Gustafson, K. Nakanishi, and T.-P. Tsai, Scattering theory for the Gross-Pitaevskii equation in three dimensions, *Commun. Contemp. Math.* **11** (2009), 657–707. MR 2559713. Zbl 1180.35481. http://dx.doi.org/10.1142/S0219199709003491.

[31] S. V. Iordanskii and A. V. Smirnov, Three-dimensional solitons in He II, *JETP Lett.* **27** (1978), 535–538. Available at http://www.jetpletters.ac.ru/ps/1553/article_23773.shtml.

[32] C. A. Jones, S. J. Putterman, and P. H. Roberts, Stability of wave solutions of nonlinear Schrödinger equations in two and three dimensions, *J. Phys A: Math. Gen.* **19** (1986), 2991–3011.

[33] C. A. Jones and P. H. Roberts, Motions in a Bose condensate IV, Axisymmetric solitary waves, *J. Phys A: Math. Gen.* **15** (1982), 2599–2619. http://dx.doi.org/10.1088/0305-4470/15/8/036.

[34] O. Kavian, *Introduction à la Théorie des Points Critiques et Applications aux Problèmes Elliptiques*, *Math. Appl. (Berlin)* **13**, Springer-Verlag, New York, 1993. MR 1276944. Zbl 0797.58005.

[35] R. Killip, T. Oh, O. Pocovnicu, and M. Vişan, Global well-posedness of the Gross–Pitaevskii and cubic-quintic nonlinear Schrödinger equations with non-vanishing boundary conditions, *Math. Res. Lett.* **19** (2012), 969–986.

[36] Y. S. Kivshar and B. Luther-Davies, Dark optical solitons: physics and applications, *Phys. Rep.* **298** (1998), 81–197. http://dx.doi.org/10.1016/S0370-1573(97)00073-2.

[37] A. de Laire, Non-existence for travelling waves with small energy for the Gross-Pitaevskii equation in dimension $N \geq 3$, *C. R. Math. Acad. Sci. Paris* **347** (2009),

375–380. MR 2537233. Zbl 1165.35047. http://dx.doi.org/10.1016/j.crma.2009.02.006.

[38] E. H. Lieb, On the lowest eigenvalue of the Laplacian for the intersection of two domains, *Invent. Math.* **74** (1983), 441–448. MR 0724014. Zbl 0538.35058. http://dx.doi.org/10.1007/BF01394245.

[39] Z. Lin, Stability and instability of traveling solitonic bubbles, *Adv. Differential Equations* **7** (2002), 897–918. MR 1895111. Zbl 1033.35117.

[40] P.-L. Lions, The concentration-compactness principle in the calculus of variations. The locally compact case. I, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **1** (1984), 109–145. MR 0778970. Zbl 0541.49009. Available at http://www.numdam.org/item?id=AIHPC_1984__1_2_109_0.

[41] M. Mariş, Existence of nonstationary bubbles in higher dimensions, *J. Math. Pures Appl.* **81** (2002), 1207–1239. MR 1952162. Zbl 1040.35116. http://dx.doi.org/10.1016/S0021-7824(02)01274-6.

[42] M. Mariş, Nonexistence of supersonic traveling waves for nonlinear Schrödinger equations with nonzero conditions at infinity, *SIAM J. Math. Anal.* **40** (2008), 1076–1103. MR 2452881. http://dx.doi.org/10.1137/070711189.

[43] M. Mariş, On the symmetry of minimizers, *Arch. Ration. Mech. Anal.* **192** (2009), 311–330. MR 2486598. Zbl 1159.49005. http://dx.doi.org/10.1007/s00205-008-0136-2.

[44] D. E. Pelinovsky, Y. A. Stepanyants, and Y. S. Kivshar, Self-focusing of plane dark solitons in nonlinear defocusing media, *Phys. Rev. E* **51** (1995), 5016–5026. MR 1383087. http://dx.doi.org/10.1103/PhysRevE.51.5016.

[45] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, *Princeton Math. Series* **30**, Princeton Univ. Press, Princeton, N.J., 1970. MR 0290095. Zbl 0207.13501.

[46] M. Willem, *Minimax Theorems*, *Progr. Nonlinear Differential Equations Appl.* **24**, Birkhäuser, Boston, MA, 1996. MR 1400007. Zbl 0856.49001. http://dx.doi.org/10.1007/978-1-4612-4146-1.

(Received: March 2, 2009)
(Revised: November 27, 2012)

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