The Witten equation, mirror symmetry, and quantum singularity theory

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Abstract

For any nondegenerate, quasi-homogeneous hypersurface singularity, we describe a family of moduli spaces, a virtual cycle, and a corresponding cohomological field theory associated to the singularity. This theory is analogous to Gromov-Witten theory and generalizes the theory of r-spin curves, which corresponds to the simple singularity A_{r-1}.

We also resolve two outstanding conjectures of Witten. The first conjecture is that ADE-singularities are self-dual, and the second conjecture is that the total potential functions of ADE-singularities satisfy corresponding ADE-integrable hierarchies. Other cases of integrable hierarchies are also discussed.

Contents

1. Introduction

The study of singularities has a long history in mathematics. For example, in algebraic geometry it is often necessary to study algebraic varieties with singularities even if the initial goal was to work only with smooth varieties. Many important surgery operations, such as flops and flips, are closely associated with singularities. In lower-dimensional topology, links of singularities give rise to many important examples of 3-manifolds. Singularity theory is also an important subject in its own right. In fact, singularity theory has been well established for many decades (see [AGZV85]). One of the most famous examples is the ADE-classification of hypersurface singularities of zero modality. We will refer to this part of singularity theory as *classical singularity theory* and review some aspects of the classical theory later. Even though we are primarily interested in the quantum aspects of singularity theory, the classical theory always serves as a source of inspiration.

Singularity theory also appears in physics. Given a polynomial $W :$ $\mathbb{C}^n \longrightarrow \mathbb{C}$ with only isolated critical (singular) points, one can associate to it the so-called *Landau-Ginzburg model*. In the early days of quantum cohomology, the Landau-Ginzburg model and singularity theory gave some of the first examples of Frobenius manifolds. It is surprising that although the Landau-Ginzburg model is one of the best understood models in physics, there has been no construction of Gromov-Witten type invariants for it until now. However, our initial motivation was not about singularities and the Landau-Ginzburg model. Instead, we wanted to solve the *Witten equation*

$$\bar{\partial}u_i + \overline{\frac{\partial W}{\partial u_i}} = 0,$$

where W is a quasi-homogeneous polynomial and u_i is interpreted as a section of an appropriate orbifold line bundle on a Riemann surface $\mathscr{C}$.

The simplest Witten equation is the A_{r-1} case. This is of the form

$$\bar{\partial}u + r\bar{u}^{r-1} = 0.$$

It was introduced by Witten [Wit93a] more than fifteen years ago as a generalization of topological gravity. Somehow, it was buried in the literature without attracting much attention. Several years ago, Witten generalized his equation for an arbitrary quasi-homogeneous polynomial [Wit] and coined it the "Landau-Ginzburg A-model." Let us briefly recall the motivation behind Witten's equation. Around 1990, Witten proposed a remarkable conjecture relating the intersection numbers of the Deligne-Mumford moduli space of stable curves with the KdV hierarchy [Wit91]. His conjecture was soon proved by Kontsevich [Kon92]. About the same time, Witten also proposed a generalization of his conjecture. In his generalization, the stable curve is replaced by a curve with a root of the canonical bundle (r-spin curve), and the KdV-hierarchy was replaced by more general KP-hierarchies called nKdV, or Gelfand-Dikii, hierarchies. The r-spin curve can be thought of as the background data to be used to set up the Witten equation in the A_{r-1}-case. Since then, the moduli space of r-spin curves has been rigorously constructed by Abramovich, Kimura, Vaintrob and the second author [AJ03], [Jar00], [Jar98], [JKV01]. The more general Witten conjecture was proved in genus zero several years ago [JKV01], in genus one and two by Y.-P. Lee [Lee06], and recently in higher genus by Faber, Shadrin, and Zvonkine [FSZ10].

The theory of r-spin curves (corresponding to the A_{r-1}-case of our theory) does not need the Witten equation at all. This partially explains the fact that the Witten equation has been neglected in the literature for more than ten years. In the r-spin case, the algebro-geometric data is an orbifold line bundle $\mathscr{L}$ satisfying the equation $\mathscr{L}^r = K_{\log}$. Assume that all the orbifold points are marked points. A marked point with trivial orbifold structure is called a *broad* (or *Ramond* in our old notation) marked point, and a marked point with nontrivial orbifold structure is called a *narrow* (or *Neveu-Schwarz* in our old notation) marked point. Contrary to intuition, broad marked points are much harder to study than narrow marked points. If there is no broad marked point, a simple lemma of Witten's shows that the Witten equation has only the zero solution. Therefore, our moduli problem becomes an algebraic geometry problem. In the r-spin case the contribution from the broad marked point to the corresponding field theory is zero (the decoupling of the broad sector). This was conjectured by Witten and proved true for genus zero in [JKV01] and for higher genus in [Pol04]. This means that in the r-spin case, there is no need for the Witten equation, which partly explains why the moduli space of

higher spin curves has been around for a long time while the Witten equation seems to have been lost in the literature.

In the course of our investigation, we discovered that in the D_n-case the broad sector gives a nonzero contribution. Hence, we had to develop a theory that accounts for the contribution of the solution of the Witten equation in the presence of broad marked points.

It has taken us a while to understand the general picture, as well as various technical issues surrounding our current theory. In fact, an announcement was made in 2001 by the last two authors for some special cases coupled with an orbifold target. We apologize for the long delay because we realized later that (1) the theory admits a vast generalization to an arbitrary quasi-homogeneous singularity and (2) the broad sector has to be investigated. We would like to mention that the need to investigate the broad sector led us to the space of Lefschetz thimbles and other interesting aspects of the Landau-Ginzburg model, including Seidel's work on the Landau-Ginzburg A-model derived category [Sei08]. In many ways, we are happy to have waited for several years to arrive at a much more complete and more interesting theory!

To describe our theory, let us first review some classical singularity theory. Let $W : \mathbb{C}^N \longrightarrow \mathbb{C}$ be a quasi-homogeneous polynomial. Recall that W is a quasi-homogeneous polynomial if there are positive integers $d, n_1, \ldots, n_n$ such that $W(\lambda^{n_1} x_1, \ldots, \lambda^{n_N} x_n) = \lambda^d w(x_1, \ldots, x_N)$. We define the *weight* (or *charge*), of x_i to be $q_i := \frac{n_i}{d}$. We say W is *nondegenerate* if (1) the choices of weights q_i are unique and (2) W has a singularity only at zero. There are many examples of nondegenerate quasi-homogeneous singularities, including all the nondegenerate homogeneous polynomials and the famous ADE-examples.

Example 1.0.1.

A_n: $W = x^{n+1}$, $n \geq 1$;
D_n: $W = x^{n-1} + xy^2$, $n \geq 4$;
E_6: $W = x^3 + y^4$;
E_7: $W = x^3 + xy^3$;
E_8: $W = x^3 + y^5$.

The simple singularities (A, D, and E) are the only examples with so-called *central charge* $\hat{c}_W < 1$. There are many more examples with $\hat{c}_W \geq 1$.

In addition to the choice of a nondegenerate singularity W, our theory also depends on a choice of subgroup G of the group $\mathrm{Aut}(W)$ of diagonal matrices γ such that $W(\gamma x) = W(x)$. We often use the notation $G_W := \mathrm{Aut}(W)$, and we call this group the *maximal diagonal symmetry group* of W. The group G_W always contains the *exponential grading* (or *total monodromy*) *element* $J = \mathrm{diag}(e^{2\pi i q_1}, \ldots, e^{2\pi i q_N})$, and hence it is always nontrivial.

Given a choice of nondegenerate W and a choice of admissible (see Section 2.3) subgroup $G \leq G_W$ with $\langle J \rangle \leq G \leq \mathrm{Aut}(W)$, we construct a cohomological field theory whose state space is defined as follows. For each $\gamma \in G$, let $\mathbb{C}^{N_\gamma}$ be the fixed point set of γ and $W_\gamma = W|_{\mathbb{C}^{N_\gamma}}$. Let $\mathscr{H}_{\gamma,G}$ be the G-invariants of the middle-dimensional relative cohomology

$$\mathscr{H}_{\gamma,G} = H^{\mathrm{mid}}(\mathbb{C}^{N_\gamma}, (\mathfrak{Re} W)^{-1}(M, \infty), \mathbb{C})^G$$

of $\mathbb{C}^{N_\gamma}$ for $M \gg 0$, as described in Section 3. The *state space* of our theory is the sum

$$\mathscr{H}_{W,G} = \bigoplus_{\gamma \in G} \mathscr{H}_{\gamma,G}.$$

The state space $\mathscr{H}_{W,G}$ admits a grading and a natural nondegenerate pairing.

For $\alpha_1, \ldots, \alpha_k \in \mathscr{H}_{W,G}$ and a sequence of nonnegative integers $l_1, \ldots, l_k$, we define (see Definition 4.2.6) the genus-g correlator

$$\langle \tau_{l_1}(\alpha_1), \ldots, \tau_{l_k}(\alpha_k) \rangle_g^{W,G}$$

by integrating over a certain virtual fundamental cycle. In this paper we describe the axioms that this cycle satisfies and the consequences of those axioms. In a separate paper [FJR] we construct the cycle and prove that it satisfies the axioms.

THEOREM 1.0.2. *The correlators* $\langle \tau_{l_1}(\alpha_1), \ldots, \tau_{l_k}(\alpha_k) \rangle_g^{W,G}$ *satisfy the usual axioms of Gromov-Witten theory (see Section 4.2), but where the divisor axiom is replaced with another axiom that facilitates computation.*

In particular, the three-point correlator together with the pairing defines a Frobenius algebra structure on $\mathscr{H}_{W,G}$ by the formula

$$\langle \alpha \star \beta, \gamma \rangle = \langle \tau_0(\alpha), \tau_0(\beta), \tau_0(\gamma) \rangle_0^{W,G}.$$

One important point is the fact that our construction depends crucially on the Abelian automorphism group G. Although there are at least two choices of group that might be considered canonical (the group generated by the exponential grading operator J or the maximal diagonal symmetry group G_W), we do not know how to construct a Landau-Ginzburg A-model defined by W alone. In this sense, the orbifold LG-model W/G is more natural than the LG-model for W itself.

We also remark that our theory is also new in physics. Until now there has been no description of the closed-string sector of the Landau-Ginzburg model.

Let us come back to the Witten-Kontsevich theorem regarding the KdV hierarchy in geometry. Roughly speaking, an integrable hierarchy is a system of differential equations for a function of infinitely many time variables $F(x, t_1, t_2, \ldots)$ where x is a spatial variable and $t_1, t_2, \ldots$, are time variables.

The PDE is a system of evolution equations of the form

$$\frac{\partial F}{\partial t_n} = R_n(x, F_x, F_{xx}, \ldots),$$

where R_n is a polynomial. Usually, R_n is constructed recursively. There is an alternative formulation in terms of the so-called Hirota bilinear equation which e^F will satisfy. We often say that e^F is a τ-*function* of hierarchy. It is well known that KdV is the A_1-case of more general ADE-hierarchies. As far as we know, there are two versions of ADE-integrable hierarchies: the first constructed by Drinfeld-Sokolov [DS84] and the second constructed by Kac-Wakimoto [KW89]. Both of them are constructed from integrable representations of affine Kac-Moody algebras. These two constructions are equivalent by the work of Hollowood and Miramontes [HM93].

Witten's original motivation was to generalize the geometry of Deligne-Mumford space to realize ADE-integrable hierarchies. Now, we can state his integrable hierarchy conjecture rigorously. Choose a basis α_i $(i \leq s)$ of $\mathscr{H}_{W,G}$. Define the genus-g generating function

$$\mathscr{F}_{g,W,G} = \sum_{k \geq 0} \langle \tau_{l_1}(\alpha_{i_1}), \ldots, \tau_{l_n}(\alpha_{i_n}) \rangle_g^{W,G} \frac{t_{i_1}^{l_1} \cdots t_{i_s}^{l_n}}{n!}.$$

Define the total potential function

$$\mathscr{D}_{W,G} = \exp\left(\sum_{g \geq 0} h^{2g-2} \mathscr{F}_{g,W,G} \right).$$

CONJECTURE 1.0.3 (Witten's ADE-integrable hierarchy conjecture). *The total potential functions of the* A, D, *and* E *singularities with the symmetry group* $\langle J \rangle$ *generated by the exponential grading operator, are* τ-*functions of the corresponding* A, D, *and* E *integrable hierarchies.*

In the A_n case, this conjecture is often referred as *the generalized Witten conjecture*, as compared to the original Witten conjecture proved by Kontsevich [Kon92]. As mentioned earlier, the conjecture for the A_n-case has been established recently by Faber, Shadrin, and Zvonkine [FSZ10]. The original Witten conjecture also inspired a great deal of activity related to Gromov-Witten theory of more general spaces. Those cases are 2-Toda for $\mathbb{CP}^1$ by Okounkov-Pandharipande [OP06a] and the Virasoro constraints for toric manifolds by Givental [Giv01], and Riemann surfaces by Okounkov-Pandharipande [OP06b]. In some sense, the ADE-integrable hierarchy conjecture is analogous to these lines of research but where the targets are singularities.

The main application of our theory is the resolution of the ADE-integrable hierarchy conjecture, as manifested by the following two theorems.

THEOREM 1.0.4. *The total potential functions of the singularities D_n with even $n \geq 6$, and E_6, E_7, and E_8, with the group $\langle J \rangle$ are τ-functions of the corresponding Kac-Wakimoto/Drinfeld-Sokolov hierarchies.*

We expect the conjecture for D_4 to be true as well. However, our calculational tools are not strong enough to prove it at this moment. We hope to come back to it at another occasion.

Surprisingly, the Witten conjecture for D_n with n odd is false. Note that in the case of n even, the subgroup $\langle J \rangle$ has index two in the maximal group G_{D_n} of diagonal symmetries, but in the case that n is odd, $\langle J \rangle$ is equal to G_{D_n}. In this paper we prove

THEOREM 1.0.5.   (1) *For all $n > 4$, the total potential function of the D_n-singularity with the maximal diagonal symmetry group G_{D_n} is a τ-function of the A_{2n-3}-Kac-Wakimoto/Drinfeld-Sokolov hierarchies.*
  (2) *For all $n > 4$, the total potential function of $W = x^{n-1}y + y^2$ $(n \geq 4)$ with the maximal diagonal symmetry group is a τ-function of the D_n-Kac-Wakimoto/Drinfeld-Sokolov hierarchy.*

The above two theorems realize the ADE-hierarchies completely in our theory. Moreover, it illustrates the important role that the group of symmetries plays in our constructions: When the symmetry group is G_{D_n}, we have the A_{2n-3}-hierarchy, but when the symmetry group is $\langle J \rangle$, and when $\langle J \rangle$ is a proper subgroup of G_{D_n}, we have the D_n-hierarchy.

Readers may wonder about the singularity $W = x^{n-1}y + y^2$ (which is isomorphic to A_{2n-3}). Its appearance reveals a deep connection between integrable hierarchies and mirror symmetry. (See more in Section 6.)

Although the simple singularities are the only singularities with central charge $\hat{c}_W < 1$, there are many more examples of singularities. It would be an extremely interesting problem to find other integrable hierarchies corresponding to singularities with $\hat{c}_W \geq 1$.

Witten's second conjecture is the following ADE self-mirror conjecture which interchanges the A-model with the B-model.

CONJECTURE 1.0.6 (ADE self-mirror conjecture). *If W is a simple singularity, then for the symmetry group $\langle J \rangle$, generated by the exponential grading operator, the ring $\mathscr{H}_{W,\langle J \rangle}$ is isomorphic to the Milnor ring of W.*

The second main theorem of this paper is the following.

THEOREM 1.0.7.   (1) *Except for D_n with n odd, the ring $\mathscr{H}_{W,\langle J \rangle}$ of any simple (ADE) singularity W with group $\langle J \rangle$ is isomorphic to the Milnor ring $\mathscr{Q}_W$ of the same singularity.*

(2) *The ring $\mathscr{H}_{D_n, G_{D_n}}$ of D_n with the maximal diagonal symmetry group G_{D_n} is isomorphic to the Milnor ring $\mathscr{Q}_{A_{2n-3}}$ of $W = x^{n-1}y + y^2$.*

(3) *The ring $\mathscr{H}_{W, G_W}$ of $W = x^{n-1}y + y^2$ ($n \geq 4$) with the maximal diagonal symmetry group G_W is isomorphic to the Milnor ring $\mathscr{Q}_{D_n}$ of D_n.*

The readers may note the similarities between the statements of the above mirror symmetry theorem and our integrable hierarchies theorems. In fact, the mirror symmetry theorem is the first step towards the proof of integrable hierarchies theorems.

Of course we cannot expect that most singularities will be self-mirror, but we can hope for mirror symmetry beyond just the simple singularities. Since the initial draft of this paper, much progress has been made [FJJS12], [Kra10], [KPA+10] for invertible singularities. An invertible singularity has the property that the number of monomials is equal to the number of variables. This is a large class of quasi-homogeneous singularities.

In general, it is a very difficult problem to compute Gromov-Witten invariants of compact Calabi-Yau manifolds. While there are many results for low genus cases [Giv98], [LLY97], [Zin08], there are only a very few compact examples [MP06], [OP06b] where one knows how to compute Gromov-Witten invariants in all genera by either mathematical or physical methods. (For some recent advances, see [HKQ09].)

Note that a Calabi-Yau hypersurface of weighted projective space defines a quasi-homogenenous singularity and hence an LG-theory. This type of singularity has $\sum_i q_i = 1$. In the early 1990s, Martinec-Vafa-Warner-Witten proposed a famous conjecture [Mar90], [VW89], [Wit93b] connecting these two points of view.

CONJECTURE 1.0.8 (Landau-Ginzburg/Calabi-Yau correspondence). *The LG-theory of a generic quasi-homogeneous singularity $W/\langle J \rangle$ and the corresponding Calabi-Yau theory are isomorphic in a certain sense.*

This is certainly one of the most important conjectures in the subject. The importance of the conjecture comes from the physical indication that the LG theory and singularity theory is much easier to compute than the Calabi-Yau geometry. The precise mathematical statement of the above conjecture is still lacking at this moment (see [CR10] also). We hope to come back to it on another occasion.

We conclude by noting that it would be a very interesting problem to explore how to extend our results to a setting like that treated by Guffin and Sharpe in [GS09a], [GS09b]. They have considered twisted Landau-Ginzburg models without coupling to topological gravity, but over more general orbifolds,

whereas our model couples to topological gravity, but we work exclusively with orbifold vector bundles.

1.1. **Organization of the paper.** A complete construction of our theory will be carried out in a series of papers. In this paper, we give a complete description of the algebro-geometric aspects of our theory. The information missing is the analytic construction of the moduli space of solutions of the Witten equation and its virtual fundamental cycle, which is done in a separate paper [FJR]. Here, we summarize the main properties or axioms of the cycle and their consequences. The main application is the proof of Witten's self-mirror conjecture and integrable hierarchies conjecture for ADE-singularities.

The paper is organized as follows. In Section 2, we will set up the theory of W-structures. This is the background data for the Witten equation and a generalization of the well-known theory of r-spin curves. The analog of quantum cohomology groups and the state space of the theory will be described in Section 3. In Section 4, we formulate a list of axioms of our theory. The proof of Witten's mirror symmetry conjecture is in Section 5. The proof of his integrable hierarchies conjecture is in Section 6.

Finally, the third author would like to thank the University of Wisconsin-Madison, where much of the current work has taken place, for warm support and fond memories.

2. W-curves and their moduli

2.1. W-structures on orbicurves.

2.1.1. *Orbicurves and line bundles.* Recall that an orbicurve $\mathscr{C}$ with marked points $p_1, \ldots, p_k$ is a (possibly nodal) Riemann surface C with orbifold structure at each p_i and each node. That is to say, for each marked point p_i, there is a local group G_{p_i} and (since we are working over $\mathbb{C}$) a canonical isomorphism $G_{p_i} \cong \mathbb{Z}/m_i$ for some positive integer m_i. A neighborhood of p_i is uniformized by the branched covering map $z \longrightarrow z^{m_i}$. For each node p, there is again a local group $G_p \cong \mathbb{Z}/n_j$ whose action is complementary on the two different branches. That is to say, a neighborhood of a nodal point (viewed as a neighborhood of the origin of $\{zw = 0\} \subset \mathbb{C}^2$) is uniformized by a branched covering map $(z, w) \longrightarrow (z^{n_j}, w^{n_j})$, with $n_j \geq 1$, and with group action $e^{2\pi i/n_j}(z, w) = (e^{2\pi i/n_j}z, e^{-2\pi i/n_j}w)$.

Definition 2.1.1. We will call the orbicurve $\mathscr{C}$ *smooth* if the underlying curve C is smooth, and we will call the orbicurve *nodal* if the underlying curve C is nodal.

Note that this definition agrees with that of algebraic geometers for smooth Deligne-Mumford stacks, but it differs from that of many topologists (e.g., [CR04]) since orbicurves with nontrivial orbifold structure at a point will still be called smooth when the underlying curve is smooth.

We denote by $\varrho : \mathscr{C} \longrightarrow C$ the natural projection to the underlying (coarse, or nonorbifold) Riemann surface C. If $\mathscr{L}$ is a line bundle on C, it can be uniquely lifted to an orbifold line bundle $\varrho^* \mathscr{L}$ over $\mathscr{C}$. When there is no danger of confusion, we use the same symbol $\mathscr{L}$ to denote its lifting.

Definition 2.1.2. Let K_C be the canonical bundle of C. We define the *log-canonical bundle of C* to be the line bundle

$$K_{C,\log} := K \otimes \mathscr{O}(p_1) \otimes \cdots \otimes \mathscr{O}(p_k),$$

where $\mathscr{O}(p)$ is the holomorphic line bundle of degree one whose sections may have a simple pole at p. This bundle $K_{C,\log}$ can be thought of as the canonical bundle of the punctured Riemann surface $C - \{p_1, \ldots, p_k\}$.

The *log-canonical bundle of $\mathscr{C}$* is defined to be the pullback to $\mathscr{C}$ of the log-canonical bundle of C:

$$(1) \qquad\qquad\qquad K_{\mathscr{C},\log} := \varrho^* K_{C,\log}.$$

Near a marked point p of C with local coordinate x, the bundle $K_{C,\log}$ is locally generated by the meromorphic one-form dx/x. If the local coordinate near p on $\mathscr{C}$ is z, with $z^m = x$, then the lift $K_{\mathscr{C},\log} := \varrho^*(K_{C,\log})$ is still locally generated by $m\,dz/z = dx/x$. When there is no risk of confusion, we will denote both $K_{C,\log}$ and $K_{\mathscr{C},\log}$ by $K_{\log}$. Near a node with coordinates z and w satisfying $zw = 0$, both K and $K_{\log}$ are locally generated by the one-form $dz/z = -dw/w$.

Note that although $\varrho^* K_{C,\log} = K_{\mathscr{C},\log}$, the usual canonical bundle does not pull back to itself:

$$(2) \qquad \varrho^* K_C = K_{\mathscr{C}} \otimes \mathscr{O}\left(-\sum_{i=1}^{k}(m_i - 1)p_i \right) \neq K_{\mathscr{C}},$$

where m_i is the order of the local group at p_i. This can be seen from the fact that when $x = z^m$, we have

$$(3) \qquad dx = mz^{m-1}dz.$$

2.1.2. *Pushforward to the underlying curve.* If $\mathscr{L}$ is an orbifold line bundle on a smooth orbicurve $\mathscr{C}$, then the sheaf of locally invariant sections of $\mathscr{L}$ is locally free of rank one and hence dual to a unique line bundle $|\mathscr{L}|$ on $\mathscr{C}$. We also denote $|\mathscr{L}|$ by $\varrho_*\mathscr{L}$, and it is called the "desingularization" of $\mathscr{L}$ in [CR04, Prop. 4.1.2]. It can be constructed explicitly as follows.

We keep the local trivialization at nonorbifold points and change it at each orbifold point p. If $\mathscr{L}$ has a local chart $\Delta \times \mathbb{C}$ with coordinates (z, s) and if the generator $1 \in \mathbb{Z}/m \cong G_p$ acts locally on $\mathscr{L}$ by

$$(z, s) \mapsto (\exp(2\pi i/m)z, \exp(2\pi i v/m)s),$$

then we use the $\mathbb{Z}/m$-equivariant map $\Psi : (\Delta - \{0\}) \times \mathbb{C} \longrightarrow \Delta \times \mathbb{C}$ given by

$$(4) \qquad (z, s) \longrightarrow (z^m, z^{-v}s),$$

where $\mathbb{Z}/m$ acts trivially on the second $\Delta \times \mathbb{C}$. Since $\mathbb{Z}/m$ acts trivially, this gives a line bundle over C, which is $|\mathscr{L}|$.

If the orbicurve $\mathscr{C}$ is nodal, then the pushforward $\varrho_*\mathscr{L}$ of a line bundle $\mathscr{L}$ may not be a line bundle on C. In fact, if the local group G_p at a node acts nontrivially on $\mathscr{L}$, then the invariant sections of $\mathscr{L}$ form a rank-one torsion-free sheaf on C (see [AJ03]). However, we may take the normalizations $\widetilde{\mathscr{C}}$ and $\widetilde{C}$ to get (possibly disconnected) smooth curves, and the pushforward of $\mathscr{L}$ from $\widetilde{\mathscr{C}}$ will give a line bundle on $\widetilde{C}$. Thus $|\mathscr{L}|$ is a line bundle away from the nodes of C, but its fiber at a node is two-dimensional; that is, there is (usually) no gluing condition on $|\mathscr{L}|$ at the nodal points. The situation is slightly more subtle than this (see [AJ03]), but for our purposes, it will be enough to consider the pushforward $|\mathscr{L}|$ as a line bundle on the normalization $\widetilde{C}$ where the local group acts trivially on $\mathscr{L}$.

It is also important to understand more about the sections of the push-forward $\varrho_*\mathscr{L}$. Suppose that s is a section of $|\mathscr{L}|$ having local representative $g(u)$. Then $(z, z^v g(z^m))$ is a local section of $\mathscr{L}$. Therefore, we obtain a section $\varrho^*(s) \in \Omega^0(\mathscr{L})$ which equals s away from orbifold points under the identification given by equation 4. It is clear that if s is holomorphic, so is $\varrho^*(s)$. If we start from an analytic section of $\mathscr{L}$, we can reverse the above process to obtain a section of $|\mathscr{L}|$. In particular, $\mathscr{L}$ and $|\mathscr{L}|$ have isomorphic spaces of holomorphic sections:

$$\varrho^* : H^0(C, |\mathscr{L}|) \overset{\sim}{\longrightarrow} H^0(\mathscr{C}, \mathscr{L}).$$

In the same way, there is a map $\varrho^* : \Omega^{0,1}(|\mathscr{L}|) \longrightarrow \Omega^{0,1}(\mathscr{L})$, where $\Omega^{0,1}(\mathscr{L})$ is the space of orbifold $(0,1)$-forms with values in $\mathscr{L}$. Suppose that $g(u)d\bar{u}$ is a local representative of a section of $t \in \Omega^{0,1}(|\mathscr{L}|)$. Then $\varrho^*(t)$ has a local representative $z^v g(z^m) m \bar{z}^{m-1} d\bar{z}$. Moreover, ϱ induces an isomorphism

$$\varrho^* : H^1(C, |\mathscr{L}|) \overset{\sim}{\longrightarrow} H^1(\mathscr{C}, \mathscr{L}).$$

Example 2.1.3. The pushforward $|K_{\mathscr{C}}|$ of the log-canonical bundle of any orbicurve $\mathscr{C}$ is again the log-canonical bundle of C, because at a point p with local group $G_p \cong \mathbb{Z}/m$, the one-form $m\, dz/z = dx/x$ is invariant under the local group action.

Similarly, the pushforward $|K_{\mathscr{C}}|$ of the canonical bundle of $\mathscr{C}$ is just the canonical bundle of C:

(5) $$|K_{\mathscr{C}}| = \varrho_* K_{\mathscr{C}} = K_C,$$

because the local group $\mathbb{Z}/m$ acts on the one-form dz by $\exp(2\pi i/m)dz$, and the invariant holomorphic one-forms are precisely those generated by $mz^{m-1}dz = dx$.

2.1.3. *Quasi-homogeneous polynomials and their Abelian automorphisms.*

Definition 2.1.4. A *quasi-homogeneous* (or *weighted homogeneous*) *polynomial* $W \in \mathbb{C}[x_1, \ldots, x_N]$ is a polynomial for which there exist positive rational numbers $q_1, \ldots, q_N \in \mathbb{Q}^{>0}$, such that for any $\lambda \in \mathbb{C}^*$,

(6) $$W(\lambda^{q_1} x_1, \ldots, \lambda^{q_N} x_N) = \lambda W(x_1, \ldots, x_N).$$

We will call q_j the *weight* of x_j. We define d and n_i for $i \in \{1, \ldots, N\}$ to be the unique positive integers such that $(q_1, \ldots, q_N) = (n_1/d, \ldots, n_N/d)$ with $\gcd(d, n_1, \ldots, n_N) = 1$.

Throughout this paper we will need a certain nondegeneracy condition on W.

Definition 2.1.5. We call W *nondegenerate* if
(1) W contains no monomial of the form $x_i x_j$ for $i \neq j$;

(2) the hypersurface defined by W in weighted projective space is nonsingular or, equivalently, the affine hypersurface defined by W has an isolated singularity at the origin.

The following proposition was pointed out to us by N. Priddis and follows from [HK, Thm. 3.7(b)].

PROPOSITION 2.1.6. *If W is a nondegenerate, quasi-homogeneous polynomial, then the weights q_i are bounded by $q_i \leq \frac{1}{2}$ and are unique.*

From now on, when we speak of a quasi-homogeneous polynomial W, we will assume it to be nondegenerate.

Definition 2.1.7. Write the polynomial $W = \sum_{j=1}^{s} W_j$ as a sum of monomials $W_j = c_j \prod_{\ell=1}^{N} x_\ell^{b_{j\ell}}$, with $b_{j\ell} \in \mathbb{Z}^{\geq 0}$ and with $c_j \neq 0$. Define the $s \times N$ matrix

(7) $$B_W := (b_{j\ell}),$$

and let $B_W = VTQ$ be the *Smith normal form* of B_W [Art91, §12, Thm. 4.3]. That is, V is an $s \times s$ invertible integer matrix and Q is an $N \times N$ invertible integer matrix. The matrix $T = (t_{j\ell})$ is an $s \times N$ integer matrix with $t_{j\ell} = 0$ unless $\ell = j$, and $t_{\ell,\ell}$ divides $t_{\ell+1,\ell+1}$ for each $\ell \in \{1, \ldots, N-1\}$.

LEMMA 2.1.8. *If W is nondegenerate, then the group*

$$G_W := \{(\alpha_1, \ldots, \alpha_N) \in (\mathbb{C}^*)^N \mid W(\alpha_1 x_1, \ldots, \alpha_N x_N) = W(x_1, \ldots, x_N)\}$$

of diagonal symmetries of W is finite.

Proof. The uniqueness of the weights q_i is equivalent to saying that the matrix B_W has rank N. We may as well assume that B_W is invertible. Now write $\gamma = (\alpha_1, \ldots, \alpha_N) \in G_W$, as $\alpha_j = \exp(u_j + v_j i)$ for $u_j \in \mathbb{R}$ uniquely determined, and $v_j \in \mathbb{R}$ determined up to integral multiple of $2\pi i$. The equation $W(\alpha_1 x_1, \ldots, \alpha_N x_N) = W(x_1, \ldots, x_N)$ can be written as $B_W(\mathbf{u} + \mathbf{v}i) \equiv \mathbf{0}$ (mod $2\pi i$), where $\mathbf{u} + \mathbf{v}i = (u_1 + v_1 i, \ldots, u_N + v_N i)$ and $\mathbf{0}$ is the zero vector. Invertibility of B_W shows that $u_\ell = 0$ for all ℓ. Thus G_W is a subgroup of $U(1)^N$, and a straightforward argument shows that the number of solutions (modulo $2\pi i$) to the equation $B_W(\mathbf{v}i) \equiv \mathbf{0}$ (mod $2\pi i$) is also finite. $\square$

Definition 2.1.9. We write each element $\gamma \in G_W$ (uniquely) as

$$\gamma = (\exp(2\pi i \Theta_1^\gamma), \ldots, \exp(2\pi i \Theta_N^\gamma)),$$

with $\Theta_i^\gamma \in [0, 1) \cap \mathbb{Q}$.

There is a special element J of the group G_W, which is defined to be

$$J := (\exp(2\pi i q_1), \ldots, \exp(2\pi i q_N)),$$

where the q_i are the weights defined in Definition 2.1.5. Since $q_i \neq 0$ for all i, we have $\Theta_i^J = q_i$. By definition, the order of the element J is d. The element J will play an important role in the remainder of the paper.

For any $\gamma \in G_W$, let $\mathbb{C}^{N_\gamma} := (\mathbb{C}^N)^\gamma$ be the set of fixed points of γ in $\mathbb{C}^N$, let N_γ denote its complex dimension, and let $W_\gamma := W|_{\mathbb{C}^{N_\gamma}}$ be the quasi-homogeneous singularity restricted to the fixed point locus of γ. The polynomial W_γ defines a quasi-homogeneous singularity of its own in $\mathbb{C}_\gamma^{N_\gamma}$, and W_γ has its own Abelian automorphism group. However, we prefer to think of the original group G_W acting on $\mathbb{C}^{N_\gamma}$. Note that G_W preserves the subspace $\mathbb{C}^{N_\gamma} \subseteq \mathbb{C}^N$.

LEMMA 2.1.10. *If W is a nondegenerate, quasi-homogeneous polynomial, then for any $\gamma \in G_W$, the polynomial W_γ has no nontrivial critical points. Therefore, W_γ is itself a nondegenerate, quasi-homogeneous polynomial in the variables fixed by γ.*

Proof. Let $\mathfrak{m} \subset \mathbb{C}[x_1, \dots, x_N]$ be the ideal generated by the variables not fixed by γ, and write W as $W = W_\gamma + W_{\mathrm{moved}}$, where $W_{\mathrm{moved}} \in \mathfrak{m}$. In fact, we have $W_{\mathrm{moved}} \in \mathfrak{m}^2$ because if any monomial in W_{moved} does not lie in $\mathfrak{m}^2$, it can be written as $x_m M$, where M is a monomial fixed by γ. However, $\gamma \in G_W$ acts diagonally, and it must fix W, and hence it must fix every monomial of W, including $x_m M$. Since it fixes M and $x_m M$, it must also fix x_m—a contradiction. This shows that $W_{\mathrm{moved}} \in \mathfrak{m}^2$.

Now we can show that there are no nontrivial critical points of W_γ. For simplicity, re-order the variables so that $x_1, \dots, x_\ell$ are the fixed variables and $x_{\ell+1}, \dots, x_N$ are the remaining variables. If there were a nontrivial critical point of W_γ, say $(\alpha_1, \dots, \alpha_\ell) \in \mathbb{C}^\ell$, then the point $(\alpha_1, \dots, \alpha_\ell, 0, \dots, 0) \in \mathbb{C}^N$ would be a nontrivial critical point of W. To see this, note that for any $i \in \{1, \dots, N\}$, we have

$$\frac{\partial W_{\mathrm{moved}}}{\partial x_i}\bigg|_{(\alpha_1, \dots, \alpha_\ell, 0, \dots, 0)} = 0$$

since $W_{\mathrm{moved}} \in \mathfrak{m}^2$. This gives

$$\frac{\partial W}{\partial x_i}\bigg|_{(\alpha_1, \dots, \alpha_\ell, 0, \dots, 0)} = \frac{\partial W_\gamma}{\partial x_i}\bigg|_{(\alpha_1, \dots, \alpha_\ell)} + \frac{\partial W_{\mathrm{moved}}}{\partial x_i}\bigg|_{(\alpha_1, \dots, \alpha_\ell, 0, \dots, 0)} = 0,$$

which shows that $(\alpha_1, \dots, \alpha_\ell, 0, \dots, 0)$ is a nontrivial critical point of W—a contradiction. $\square$

2.1.4. *W-structures on an orbicurve.* A W-structure on an orbicurve $\mathscr{C}$ is essentially a choice of N line bundles $\mathscr{L}_1, \dots, \mathscr{L}_N$ so that for each monomial

$W_j = x_1^{b_{j,1}} \cdots x_N^{b_{j,N}}$, we have an isomorphism of line bundles

$$\varphi_j : \mathscr{L}_1^{\otimes b_{j,1}} \cdots \mathscr{L}_N^{\otimes b_{j,N}} \longrightarrow K_{\log}.$$

However, the isomorphisms φ_j need to be compatible, in the sense that at any point p there exists a trivialization $\mathscr{L}_i|_p \cong \mathbb{C}$ for every i and $K_{\log}|_p \cong \mathbb{C} \cdot dz/z$ such that for all $j \in \{1, \ldots, s\}$, we have $\varphi_j(1, \ldots, 1) = 1 \cdot dz/z \in \mathbb{C}$. If $s = N$, we can choose such trivializations for any choice of maps $\{\varphi_j\}$, but if $s > N$, then the choices of $\{\varphi_j\}$ need to be related. To do this we use the Smith normal form to give us a sort of minimal generating set of isomorphisms that will determine all the maps $\{\varphi_j\}$.

Definition 2.1.11. For any nondegenerate, quasi-homogeneous polynomial $W \in \mathbb{C}[x_1, \ldots, x_N]$, with matrix of exponents $B_W = (b_{\ell j})$ and Smith normal form $B_W = VTQ$, let $A := (a_{j\ell}) := V^{-1}B = TQ$, and let u_ℓ be the sum of the entries in the ℓ-th row of V^{-1} (i.e., the ℓ-th term in the product $V^{-1}(1, 1, \ldots, 1)^T$).

For any $\ell \in \{1, \ldots, N\}$, denote by $A_\ell(\mathscr{L}_1, \ldots, \mathscr{L}_N)$ the tensor product

$$A_\ell(\mathscr{L}_1, \ldots, \mathscr{L}_N) := \mathscr{L}_1^{\otimes a_{\ell 1}} \otimes \cdots \otimes \mathscr{L}_N^{\otimes a_{\ell N}}.$$

We define a W-*structure* on an orbicurve $\mathscr{C}$ to be the data of an N-tuple $(\mathscr{L}_1, \ldots, \mathscr{L}_N)$ of orbifold line bundles on $\mathscr{C}$ and isomorphisms

$$\tilde{\varphi}_\ell : A_\ell(\mathscr{L}_1, \ldots, \mathscr{L}_N) \overset{\sim}{\longrightarrow} K_{\mathscr{C}, \log}^{u_\ell}$$

for every $\ell \in \{1, \ldots, N\}$.

Note that for each point $p \in \mathscr{C}$, an orbifold line bundle $\mathscr{L}$ on $\mathscr{C}$ induces a representation $G_p \longrightarrow \mathrm{Aut}(\mathscr{L})$. Moreover, a W-structure on $\mathscr{C}$ will induce a representation $r_p : G_p \longrightarrow U(1)^N$. For all our W-structures, we require that this representation r_p be faithful at every point.

The next two propositions follow immediately from the definitions.

PROPOSITION 2.1.12. *The Smith normal form is not necessarily unique, but for any two choices of Smith normal form $B = VTQ = V'T'Q'$, a W-structure $(\mathscr{L}_1, \ldots, \mathscr{L}_N, \tilde{\varphi}_1, \ldots, \tilde{\varphi}_N)$ with respect to VTQ induces a canonical W-structure $(\mathscr{L}_1, \ldots, \mathscr{L}_N, \tilde{\varphi}'_1, \ldots, \tilde{\varphi}'_N)$ with respect to $V'T'Q'$, where the isomorphism $\tilde{\varphi}'_i$ is given by*

$$\tilde{\varphi}'_i = \tilde{\varphi}_1^{z_{i1}} \otimes \cdots \otimes \tilde{\varphi}_N^{z_{iN}}$$

and where $Z = (z_{ij}) := (V')^{-1}V$.

PROPOSITION 2.1.13. *For each $j \in \{1, \ldots, s\}$, the maps $\{\tilde{\varphi}_\ell\}$ induce an isomorphism*

$$(8) \quad \varphi_j := \tilde{\varphi}_1^{v_{j1}} \otimes \cdots \otimes \tilde{\varphi}_N^{v_{jN}} : W_j(\mathscr{L}_1, \ldots, \mathscr{L}_N)$$

$$= \mathscr{L}_1^{\otimes b_{j1}} \otimes \cdots \otimes \mathscr{L}_N^{\otimes b_{j,N}} = \mathscr{L}_1^{\sum_\ell v_{j\ell} a_{\ell 1}} \otimes \cdots \otimes \mathscr{L}_N^{\sum_\ell v_{j\ell} a_{\ell N}} \longrightarrow K_{\mathscr{C}, \log},$$

where $V = (v_{j\ell})$.

Moreover, *if B is square (and hence invertible), then a choice of isomorphisms $\varphi_j : \mathscr{L}_1^{\otimes b_{j1}} \otimes \cdots \otimes \mathscr{L}_N^{\otimes b_{j,N}} \longrightarrow K_{\mathscr{C}, \log}$ for every $j \in \{1, \ldots, N\}$ is equivalent to a choice of isomorphisms $\tilde{\varphi}_\ell : \mathscr{L}_1^{\otimes a_{\ell 1}} \otimes \cdots \otimes \mathscr{L}_N^{\otimes a_{\ell N}} \longrightarrow K_{\mathscr{C}, \log}^{u_\ell}$ for every $\ell \in \{1, \ldots, N\}$.*

Finally, *the induced maps $\varphi_j : \mathscr{L}_1^{b_{j,1}} \otimes \cdots \otimes \mathscr{L}_N^{b_{j,N}} \longrightarrow K_{\mathscr{C}, \log}$ are independent of the choice of Smith normal form VTQ*

For the rest of this paper, we will assume that a choice of Smith normal form $B_W = VTQ$ has been fixed for each W.

Definition 2.1.14. Given any two W-structures

$$\mathfrak{L} := (\mathscr{L}_1, \ldots, \mathscr{L}_N, \tilde{\varphi}_1, \ldots, \tilde{\varphi}_N) \quad \text{and} \quad \mathfrak{L}' := (\mathscr{L}_1', \ldots, \mathscr{L}_N', \tilde{\varphi}_1', \ldots, \tilde{\varphi}_N')$$

on $\mathscr{C}$, any set of morphisms $\xi_j : \mathscr{L}_j \longrightarrow \mathscr{L}_j'$ of orbifold line bundles for $j \in \{1, \ldots, N\}$ will induce a morphism

$$\Xi_l : \mathscr{L}_1^{a_{\ell 1}} \otimes \cdots \otimes \mathscr{L}_N^{a_{\ell N}} \longrightarrow \mathscr{L'}_1^{a_{\ell 1}} \otimes \cdots \otimes \mathscr{L'}_N^{a_{\ell N}}$$

for every $l \in \{1, \ldots, s\}$.

An *isomorphism of W-structures* $\Upsilon : \mathfrak{L} \longrightarrow \mathfrak{L}'$ on $\mathscr{C}$ is defined to be a collection of isomorphisms $\xi_j : \mathscr{L}_j \longrightarrow \mathscr{L}_j'$ such that for every $\ell \in \{1, \ldots, N\}$, we have $\tilde{\varphi}_\ell = \tilde{\varphi}_\ell' \circ \Xi_\ell$.

It will be important later to know that different choices of maps $\{\tilde{\varphi}_j\}$ all give isomorphic W-structures.

PROPOSITION 2.1.15. *For a given orbicurve $\mathscr{C}$, any two W-structures $\mathfrak{L}_1 := (\mathscr{L}_1, \ldots, \mathscr{L}_N, \tilde{\varphi}_1, \ldots, \tilde{\varphi}_N)$ and $\mathfrak{L}_2 := (\mathscr{L}_1, \ldots, \mathscr{L}_N, \tilde{\varphi}_1', \ldots, \tilde{\varphi}_N')$ on $\mathscr{C}$ that have identical bundles $\mathscr{L}_1, \ldots, \mathscr{L}_N$ are isomorphic.*

Proof. For each $j \in \{1, \ldots, N\}$, the composition $\tilde{\varphi}_j^{-1} \circ \tilde{\varphi}_j'$ is an automorphism of $K_{\log}^{u_j}$ and hence defined by an element $\exp(\alpha_j) \in \mathbb{C}^*$.

Since $B := B_W$ is of maximal rank, the product TQ in the Smith normal form decomposition of B consists of a nonsingular $N \times N$ block C on top, with all remaining rows identically equal to zero.

$$V^{-1}B = TQ = \begin{pmatrix} C \\ - \\ \mathbf{0} \end{pmatrix}.$$

Let $(\beta_1, \ldots, \beta_N)^T := C^{-1}(\alpha_1, \ldots, \alpha^N)^T \in \mathbb{Q}^N$. For every $\ell \in \{1, \ldots, N\}$, the collection of automorphisms $\{\exp(\beta_j) : \mathscr{L}_j \longrightarrow \mathscr{L}_j\}$ induces the automorphism $\exp(\sum_{i=1}^N a_{\ell i}\beta_i) = \exp(\alpha_\ell)$ on $\mathscr{L}_1^{\otimes a_{\ell 1}} \otimes \cdots \otimes \mathscr{L}_N^{\otimes a_{\ell N}}$ and hence takes $\tilde{\varphi}_\ell$ to $\exp(\alpha_\ell)\tilde{\varphi}_\ell = \tilde{\varphi}_\ell'$. Thus the collection $\{\exp(\beta_j)\}$ induces an isomorphism of W-structures $\mathfrak{L}_1 \widetilde{\longrightarrow} \mathfrak{L}_2$. $\qquad\square$

Example 2.1.16. In the case where $W = x^r$ (the A_{r-1} singularity), a W-structure is an *r-spin structure* (see [AJ03]).

Definition 2.1.17. For each orbifold marked point p_i, we will denote the image $r_{p_i}(1)$ of the canonical generator $1 \in \mathbb{Z}/m_i \cong G_{p_i}$ in $U(1)^N$ by

$$\gamma_i := \gamma_{p_i} := r_{p_i}(1) = (\exp(2\pi i\Theta_1^\gamma), \ldots, \exp(2\pi i\Theta_N^\gamma)).$$

The choices of orbifold structure for the line bundles in the W-structure is severely restricted by W.

LEMMA 2.1.18. *Let $(\mathscr{L}_1, \ldots, \mathscr{L}_N, \tilde{\varphi}_1, \ldots, \tilde{\varphi}_N)$ be a W-structure on an orbicurve $\mathscr{C}$ at an orbifold point $p \in \mathscr{C}$. The faithful representation $r_p : G_p \longrightarrow U(1)^N$ factors through G_W, so $\gamma_i \in G_W$ for all $i \in \{1, \ldots, k\}$.*

Proof. Recall that for each $j \in \{1, \ldots, s\}$, the bundle $W_j(\mathscr{L}_1, \ldots, \mathscr{L}_N) = \mathscr{L}_1^{\otimes b_{1,j}} \otimes \cdots \otimes \mathscr{L}_N^{\otimes b_{N,j}}$ is isomorphic to $K_{\log}$, and so the local group acts trivially on it. However, the generator $\gamma_p \in G_p$ acts on $W_j(\mathscr{L}_1, \ldots, \mathscr{L}_N)$ as $\exp(2\pi i \sum_i b_{ij}\Theta_i^\gamma)$. Therefore $\sum_i b_{ij}\Theta_i^\gamma \in \mathbb{Z}$, and γ fixes W_j. $\qquad\square$

Definition 2.1.19. A marked point p of a W-curve is called *narrow* if the fixed point locus $\mathrm{Fix}(\gamma) \subseteq \mathbb{C}^N$ is just $\{0\}$. The point p is called *broad* otherwise.

Remark 2.1.20. Note that for any given orbicurve $\mathscr{C}$, any two W-structures on $\mathscr{C}$ differ by line bundles $\mathscr{N}_1, \ldots, \mathscr{N}_N$ with isomorphisms $\xi_j : \mathscr{N}_1^{\otimes a_{\ell 1}} \otimes \cdots \otimes \mathscr{N}_N^{\otimes a_{\ell N}} \widetilde{\longrightarrow} \mathscr{O}_\mathscr{C}$. The set of such tuples $(\mathscr{N}_1, \ldots, \mathscr{N}_N, \xi_1, \ldots, \xi_s)$, up to isomorphism, is a group under tensor product and is isomorphic to the (finite) cohomology group $H^1(\mathscr{C}, G_W)$. Thus the set of W-structures on $\mathscr{C}$ is an $H^1(\mathscr{C}, G_W)$-torsor.

An automorphism of a W-curve $\mathfrak{L}$ induces an automorphism of the orbicurve $\mathscr{C}$ and underlying (coarse) curve C. It is easy to see that the group of automorphisms of $\mathfrak{L}$ that fix the underlying (coarse) curve C consists of all elements in the group G_W, acting by multiplication of the fibers of $\mathscr{L}_1, \ldots, \mathscr{L}_N$. This gives the exact sequence

$$1 \longrightarrow \mathrm{Aut}_C(\mathfrak{L}) = G_W \longrightarrow \mathrm{Aut}(\mathfrak{L}) \longrightarrow \mathrm{Aut}(C).$$

More generally, if the stable curve $\mathscr{C}$ has irreducible components $\mathscr{C}_l$ for $l \in \{1, \ldots, t\}$ and nodes $\nu \in E$, we denote by $\mathfrak{L}_i$ the restriction to $\mathscr{C}_i$ of the W-structure. To describe the automorphisms of the W-structure in this case,

it will be convenient to choose an orientation on the edges of the dual graph of C. This amounts to choosing, for each node $\nu \in E$, one of the components passing through ν to be designated as C_{ν_+}. The other component passing through ν is designated C_{ν_-}. If the same irreducible component C_i passes through ν twice, then that component will be designated both C_{ν_+} and C_{ν_-}. The final result will be independent of these choices.

Let G_ν denote the local group at the node ν. Any element $g \in \mathrm{Aut}_{C_i}(\mathfrak{L}_i)$ induces (by restriction) elements g_{ν_+} and g_{ν_-} in G_ν. We define $\delta : \prod_i \mathrm{Aut}_{C_i}(\mathfrak{L}_i) \longrightarrow \prod_\nu G_\nu$ to be the homomorphism defined as $(\delta(g))_\nu = g_{\nu_+} g_{\nu_-}^{-1}$. We have an exact sequence

$$(9) \qquad 1 \longrightarrow \mathrm{Aut}_C \mathfrak{L} \longrightarrow \prod_i \mathrm{Aut}_{C_i}(\mathfrak{L}_i) \longrightarrow \prod_{\nu \in E} G_\nu.$$

Example 2.1.21. Consider a W-curve with two irreducible components $\mathscr{C}_1$ and $\mathscr{C}_2$ with marked points $\{p_i | i \in I_1\} \cup \{q_+\} \subset \mathscr{C}_1$ and $\{p_i | i \in I_2\} \cup \{q_-\} \subset \mathscr{C}_2$, such that the components meet at a single node $q = q_+ = q_-$ and such that $I_1 \sqcup I_2 = \{1, \ldots, k\}$. Denote the local group at $q_\pm$ by $\langle \gamma_\pm \rangle$. Note that $\gamma_- = \gamma_+^{-1}$. In this case we have

$$(10) \qquad \mathrm{Aut}_C(\mathfrak{L}) = G_W \times_{G/\langle \gamma_+ \rangle} G_W,$$

where $G_W \times_{G_W/\langle \gamma_+ \rangle} G_W$ denotes the group of pairs (g_1, g_2) such that the images of g_1 and g_2 are equal in $G_W/\langle \gamma_+ \rangle$.

Example 2.1.22. If $\mathscr{C}$ consists of a single (possibly nodal) irreducible component, then we have

$$(11) \qquad \mathrm{Aut}_C(\mathfrak{L}) = G_W.$$

2.1.5. *Pushforward of W-structures.* We need to understand the behavior of W-structures when forgetting the orbifold structure at marked points; that is, when they are pushed down to the underlying (coarse) curve.

Consider, as an initial example, the case of $W = x^r$, so that a W-structure consists of a line bundle $\mathscr{L}^r \cong K_{\log}$. Near an orbifold point p with local coordinate z, the canonical generator $1 \in \mathbb{Z}/m \cong G_p$ of the local group G_p acts on $\mathscr{L}$ by $(z, s) \mapsto (\exp(2\pi i/m)z, \exp(2\pi i(v/m))s)$ for some $v \in \{0, \ldots, m-1\}$. Since $K_{\log}$ is invariant under the local action of G_p, we must have $rv = \ell m$ for some $\ell \in \{0, \ldots, r-1\}$, and $\frac{v}{m} = \frac{\ell}{r}$. Denote the (invariant) local coordinate on the underlying curve C by $u = z^m$. Any section in $\sigma \in \Omega^0(|\mathscr{L}|)$ must locally be of the form $\sigma = g(u)z^v s$, in order to be $\mathbb{Z}/m$-invariant. So σ^r has local representative $z^{rv} g^r(u) \frac{dz}{z} = u^\ell g^r(u) \frac{du}{mu}$. Hence, $\sigma^r \in \Omega^0(K_{\log} \otimes \mathscr{O}((-\ell)p))$, and thus when $\ell \neq 0$, we have $\sigma^r \in \Omega^0(K)$.

Remark 2.1.23. More generally, if $\mathscr{L}^r \cong K_{\log}$ on a *smooth* orbicurve with action of the local group on L defined by $\ell_i > 0$ (as above) at each marked

point p_i, then we have

$$(\varrho_* \mathscr{L})^r = |\mathscr{L}|^r = K_{\log} \otimes \left(\bigotimes_i \mathscr{O}((-\ell_i)p_i) \right) = K_{\log} \otimes \left(\bigotimes_i \mathscr{O}((-r(v/m))p_i) \right).$$

PROPOSITION 2.1.24. *Let* $(\mathscr{L}_1, \ldots, \mathscr{L}_N, \tilde{\varphi}_1, \ldots, \tilde{\varphi}_N)$ *be a* W*-structure on an orbicurve* $\mathscr{C}$ *that is smooth* (*the underlying curve* C *is nonsingular*) *at an orbifold point* $p \in \mathscr{C}$. *Suppose also that the local group* $G_p \cong \mathbb{Z}/m$ *of* p *acts on* $\mathscr{L}_j$ *by*

$$\gamma = (\exp(2\pi i \Theta_1^\gamma), \ldots, \exp(2\pi i \Theta_N^\gamma));$$

that is, $\exp(2\pi i/m)(z, w_j) = (\exp(2\pi i/m)z, \exp(2\pi i \Theta_j^\gamma)w_j)$ *with* $1 > \Theta_j^\gamma \geq 0$.

Let $\overline{\mathscr{C}}$ *denote the orbicurve obtained from* $\mathscr{C}$ *by making the orbifold structure at* p *trivial* (*but retaining the orbifold structure at all other points*). *Let* $\overline{\varrho} : \mathscr{C} \longrightarrow \overline{\mathscr{C}}$ *be the obvious induced morphism, and let* $\overline{\varrho}_*(\mathscr{L})$ *denote the pushforward to* $\overline{\mathscr{C}}$ *of an orbifold line bundle* $\mathscr{L}$ *on* $\mathscr{C}$.

For any isomorphism $\psi : \mathscr{L}_1^{e_1} \otimes \cdots \otimes \mathscr{L}_N^{e_N} \longrightarrow K_{\log}$, *we have an induced isomorphism on the pushforward*

$$(12) \qquad \overline{\varrho}_*(\mathscr{L}_1)^{e_1} \otimes \cdots \otimes \overline{\varrho}_*(\mathscr{L}_N)^{e_N} \longrightarrow K_{\overline{\mathscr{C}}, \log} \otimes \mathscr{O}\left(-\sum_{j=1}^N e_j \Theta_j^\gamma p \right).$$

If $\mathscr{C}$ *is a smooth orbicurve* (*i.e.,* C *is a smooth curve*), *let* γ_ℓ *define the action of the local group* G_{p_ℓ} *near* p_ℓ. *For any isomorphism* $\psi : \mathscr{L}_1^{e_1} \otimes \cdots \otimes \mathscr{L}_N^{e_N} \longrightarrow K_{\log}$, *we have a* (*global*) *induced isomorphism*

$$|\psi| : |\mathscr{L}_1|^{e_1} \otimes \cdots \otimes |\mathscr{L}_N|^{e_N} \longrightarrow K_{\overline{\mathscr{C}}, \log} \otimes \mathscr{O}\left(-\sum_{\ell=1}^k \sum_{j=1}^N e_j \Theta_j^{\gamma_\ell} p_\ell \right).$$

In particular, for every monomial W_i, *the isomorphism of equation* (8) *induces an isomorphism*

$$(13) \qquad W_i(|\mathscr{L}_1|, \ldots, |\mathscr{L}_N|) \cong K_{C, \log} \otimes \mathscr{O}\left(-\sum_{\ell=1}^k \sum_{j=1}^N b_{ij} \Theta_j^{\gamma_\ell} p_\ell \right)$$

Proof. Equation (12) is a straightforward generalization of the argument given above when $W = x^r$, the description of γ as

$$\gamma = (\exp(2\pi i \Theta_1^\gamma), \ldots, \exp(2\pi i \Theta_N^\gamma))$$

and the description of $|\mathscr{L}_j|$ in terms of the action of the local group G_p given above. $\square$

2.2. **Moduli of stable W-orbicurves.**

Definition 2.2.1. A pair $\mathfrak{C} = (\mathscr{C}, \mathfrak{L})$ consisting of an orbicurve $\mathscr{C}$ with k marked points and with W-structure $\mathfrak{L}$ is called a *stable W-orbicurve* if the underlying curve C is a stable curve and if for each point p of $\mathscr{C}$, the representation $r_p : G_p \longrightarrow G_W$ is faithful.

Definition 2.2.2. A *genus-g, stable W-orbicurve with k marked points over a base T* is given by a flat family of genus-g, k-pointed orbicurves $\mathscr{C} \longrightarrow T$ with (gerbe) markings $\mathscr{S}_i \subset \mathscr{C}$ and sections $\sigma_i : T \longrightarrow \mathscr{S}_i$, and the data $(\mathscr{L}_1, \ldots, \mathscr{L}_N, \tilde{\varphi}_1, \ldots, \tilde{\varphi}_N)$. The sections σ_i are required to induce isomorphisms between T and the coarse moduli of $\mathscr{S}_i$ for $i \in \{1, \ldots, k\}$. The $\mathscr{L}_i$ are orbifold line bundles on $\mathscr{C}$. And the $\tilde{\varphi}_j : A_j(\mathscr{L}_1, \ldots, \mathscr{L}_N) \widetilde{\longrightarrow} K^{u_j}_{\mathscr{C}/T,log} :=$ $(K_{\mathscr{C}/T}(\sum \mathscr{S}_i))^{u_j}$ are isomorphisms to the u_j-fold tensor power of the relative log-canonical bundle which, together with the $\mathscr{L}_i$, induce a W-structure on every fiber $\mathscr{C}_t$.

Definition 2.2.3. A *morphism of stable W-orbicurves*

$$(\mathscr{C}/T, \mathscr{S}_1, \ldots, \mathscr{S}_k, \mathscr{L}_1, \ldots, \mathscr{L}_N, \tilde{\varphi}_1, \ldots, \tilde{\varphi}_N)$$

and

$$(\mathscr{C}'/T', \mathscr{S}'_1, \ldots, \mathscr{S}'_k, \mathscr{L}'_1, \ldots, \mathscr{L}'_N, \tilde{\varphi}'_1, \ldots, \tilde{\varphi}'_s)$$

is a tuple of morphisms $(\tau, \mu, \alpha_1, \ldots, \alpha_N)$ such that the pair (τ, μ) forms a morphism of pointed orbicurves:

$$
\begin{array}{ccc}
\mathscr{C} & \xrightarrow{\mu} & \mathscr{C}' \\
\downarrow & & \downarrow \\
T & \xrightarrow{\tau} & T'
\end{array}
$$

and the $\alpha_j : \mathscr{L}_j \widetilde{\longrightarrow} \mu^* \mathscr{L}'_j$ are isomorphisms of line bundles that form an isomorphism of W-structures on $\mathscr{C}$ (see Definition 2.1.14).

Definition 2.2.4. For a given choice of nondegenerate W, we denote the stack of stable W-orbicurves by $\overline{\mathscr{W}}_{g,k}(W)$. If the choice of W is either clear or is unimportant, we simply write $\overline{\mathscr{W}}_{g,k}$.

Remark 2.2.5. This definition depends on the choice of Smith normal form $B = VTQ$, but by Proposition 2.1.12, any other choice of Smith normal form for the same polynomial W will give a canonically isomorphic stack.

Forgetting the W-structure and the orbifold structure gives a morphism

$$\mathrm{st} : \overline{\mathscr{W}}_{g,k} \longrightarrow \overline{\mathscr{M}}_{g,k}.$$

The morphism st plays a role similar to that played by the stabilization morphism of stable maps. It is quasi-finite by Remark 2.1.20.

THEOREM 2.2.6. *For any nondegenerate, quasi-homogeneous polynomial W, the stack $\overline{\mathscr{W}}_{g,k}$ is a smooth, compact orbifold (Deligne-Mumford stack) with projective coarse moduli. In particular, the morphism* st $: \overline{\mathscr{W}}_{g,k} \longrightarrow \overline{\mathscr{M}}_{g,k}$ *is flat, proper, and quasi-finite (but not representable).*

Proof. Denote the classifying stack of $\mathbb{C}^*$ by $\mathscr{B}\mathbb{C}^*$. For each orbicurve $\mathscr{C}$, the line bundle $K_{\log}$ corresponds to a 1-morphism

$$\mathscr{C} \longrightarrow \mathscr{B}\mathbb{C}^*,$$

and composing with the diagonal embedding $\Delta : \mathscr{B}\mathbb{C}^* \longrightarrow (\mathscr{B}\mathbb{C}^*)^N$, we have

$$(14) \qquad\qquad \delta := \Delta \circ K_{\log} : \mathscr{C} \longrightarrow (\mathscr{B}\mathbb{C}^*)^N.$$

Furthermore, each isomorphism $\tilde{\varphi}_i$ induces a 1-morphism $(\mathscr{B}\mathbb{C}^*)^N \longrightarrow \mathscr{B}\mathbb{C}^*$, and together they yield a morphism

$$(15) \qquad\qquad \Phi_W : (\mathscr{B}\mathbb{C}^*)^N \longrightarrow (\mathscr{B}\mathbb{C}^*)^N.$$

It is easy to see that the data of a W-structure on $\mathscr{C}$ is equivalent to the data of a representable 1-morphism

$$\mathfrak{L} : \mathscr{C} \longrightarrow (\mathscr{B}\mathbb{C}^*)^N,$$

which makes the diagram

$$
\begin{array}{ccc}
 & & (\mathscr{B}\mathbb{C}^*)^N \\
 & \stackrel{\mathfrak{L}}{\nearrow} & \downarrow {\scriptstyle \Phi_W} \\
\mathscr{C} & \stackrel{\delta}{\longrightarrow} & (\mathscr{B}\mathbb{C}^*)^N
\end{array}
$$

commute.

As in [AJ03, §1.5] we let $C_{g,k} \longrightarrow \overline{\mathscr{M}}_{g,k}$ denote the universal curve, and we consider the stack

$$C_{g,k,W} := C_{g,k} \underset{(\mathscr{B}\mathbb{C}^*)^N}{\times} (\mathscr{B}\mathbb{C}^*)^N,$$

where the fiber product is taken with respect to δ on the left and Φ_W on the right. The stack $C_{g,k,W}$ is an étale gerbe over $C_{g,k}$ banded by G_W. In particular, it is a Deligne-Mumford stack.

Any W-curve $(\mathscr{C}/S, p_1, \ldots, p_k, \mathscr{L}_1, \ldots, \mathscr{L}_N, \tilde{\varphi}_1, \ldots, \tilde{\varphi}_s)$ induces a representable map $\mathscr{C} \longrightarrow C_{g,k,W}$, which is a balanced twisted stable map. The homology class of the image of the coarse curve C is the class F of a fiber of the universal curve $C_{g,k} \longrightarrow \overline{\mathscr{M}}_{g,k}$. The family of coarse curves $C \longrightarrow S$ gives rise to a morphism $S \longrightarrow \overline{\mathscr{M}}_{g,k}$, and we have an isomorphism $C \cong S \times_{\overline{\mathscr{M}}_{g,k}} C_{g,k}$.

We thus have a base-preserving functor from the stack $\overline{\mathscr{W}}_{g,k}$ of W-curves to the stack $\mathscr{K}_{g,k}(C_{g,k,W}/\overline{\mathscr{M}}_{g,k}, F)$ of balanced, k-pointed twisted stable maps of genus g and class F into $C_{g,n,W}$ relative to the base stack $\overline{\mathscr{M}}_{g,k}$ (see [AV02, §8.3]). The image lies in the closed substack where the markings of C line up over the markings of $C_{g,n}$. It is easy to see that the resulting functor is an equivalence. Thus $\overline{\mathscr{W}}_{g,k}$ is a proper Deligne-Mumford stack admitting a projective coarse moduli space.

Smoothness of the stack $\overline{\mathscr{W}}_{g,k}$ follows, as in the A_n case (see [AJ03, Prop. 2.1.1]), from the fact that the relative cotangent complex $\mathbb{L}_{\Phi_W}$ of $\Phi_W :$ $(\mathscr{B}\mathbb{C}^*)^N \longrightarrow (\mathscr{B}\mathbb{C}^*)^N$ is trivial. That means that the deformations and obstructions of a W-curve are identical to those of the underlying orbicurves, but these are known to be unobstructed (see [AJ03, §2.1]). $\qquad\square$

2.2.1. *Decomposition of $\overline{\mathscr{W}}_{g,k}$ into components.* The orbifold structure and the image $\gamma_i = r_{p_i}(1)$ of the canonical generator $1 \in \mathbb{Z}/m_i \cong G_{p_i}$ at each marked point p_i are locally constant and hence are constant for each component of $\overline{\mathscr{W}}_{g,k}$. Therefore, we can use these decorations to decompose the moduli space into components.

Definition 2.2.7. For any choice $\boldsymbol{\gamma} := (\gamma_1, \ldots, \gamma_k) \in G_W^k$, we define $\overline{\mathscr{W}}_{g,k}(\boldsymbol{\gamma})$ $\subseteq \overline{\mathscr{W}}_{g,k}$ to be the open and closed substack with orbifold decoration $\boldsymbol{\gamma}$. We call $\boldsymbol{\gamma}$ the *type* of any W-orbicurve in $\overline{\mathscr{W}}_{g,k}(\boldsymbol{\gamma})$.

We have the decomposition

$$\overline{\mathscr{W}}_{g,k} = \sum_{\boldsymbol{\gamma}} \overline{\mathscr{W}}_{g,k}(\boldsymbol{\gamma}).$$

Note that by applying the degree map to equation (13) we gain an important selection rule.

PROPOSITION 2.2.8. *A necessary and sufficient condition for $\overline{\mathscr{W}}_{g,k}(\boldsymbol{\gamma})$ to be nonempty is*

$$(16) \qquad q_j(2g - 2 + k) - \sum_{l=1}^{k} \Theta_j^{\gamma_l} \in \mathbb{Z}.$$

Proof. Although the degree of an orbifold bundle on $\mathscr{C}$ may be a rational number, the degree of the pushforward $\varrho_*\mathscr{L}_j = |\mathscr{L}_j|$ on the underlying curve C must be an integer, so for all $i \in \{1, \ldots, s\}$, the following equations must hold for integral values of $\deg(|\mathscr{L}_j|)$:

$$(17) \qquad \sum_{j=1}^{N} b_{ij} \deg(|\mathscr{L}_j|) = 2g - 2 + k - \sum_{l=1}^{k} \sum_{j=1}^{N} b_{ij} \Theta_j^{\gamma_l}.$$

Moreover, because W is nondegenerate, the weights q_j are uniquely determined by the requirement that they satisfy the equations $\sum_{j=1}^{N} b_{ij} q_j = 1$ for all $i \in \{1, \ldots, s\}$, so we find that for every $j \in \{1, \ldots, N\}$, we have

$$(18) \qquad \deg(|\mathscr{L}_j|) = \left(q_j(2g - 2 + k) - \sum_{l=1}^{k} \Theta_j^{\gamma_l} \right) \in \mathbb{Z}.$$

Conversely, if the degree condition (16) holds, then for any smooth curve C (not orbifolded), we may choose line bundles $E_1, \ldots, E_N$ on C with $\deg(E_j) = q_j(2g - 2 + k) - \sum_{l=1}^{k} \Theta_j^{\gamma_l}$ for each l. If we take $A = (a_{ij}) = V^{-1}B$ and $u = (u_i) = V^{-1}(1, \ldots, 1)^T$ as in Definition 2.1.11, then for each $i \in \{1, \ldots, s\}$, we have a line bundle

$$X_i := E_1^{a_{i,1}} \otimes \cdots \otimes E_N^{a_{i,N}} \otimes K_{C,\log}^{-u_i} \otimes \mathscr{O}\left(\sum_{l=1}^{k} \sum_{j=1}^{N} a_{ij} \Theta_j^{\gamma_l} p_l \right)$$

and $\deg(X_i)$ satisfies

$$\begin{pmatrix} \deg(X_1) \\ \vdots \\ \deg(X_N) \end{pmatrix} = A \begin{pmatrix} q_1 \\ \vdots \\ q_N \end{pmatrix} (2g - 2 + k) - \sum_{l=1}^{k} A \begin{pmatrix} \Theta_1^{\gamma_l} \\ \vdots \\ \Theta_N^{\gamma_l} \end{pmatrix}$$
$$- V^{-1} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} (2g - 2 + k) + \sum_{l=1}^{k} A \begin{pmatrix} \Theta_1^{\gamma_l} \\ \vdots \\ \Theta_N^{\gamma_l} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

on C. Since the Jacobian $\mathrm{Pic}^0(C)$ of any smooth curve C is a divisible group, and since the matrix A is of rank N, there is at least one solution $(Y_1, \ldots, Y_N) \in \mathrm{Pic}^0(C)^N$ to the system of equations

$$Y_1^{a_{1,1}} \otimes \cdots \otimes Y_N^{a_{1,N}} = X_1$$
$$\vdots = \vdots$$
$$Y_1^{a_{N,1}} \otimes \cdots \otimes Y_N^{a_{N,N}} = X_N.$$

This means that the (un-orbifolded) line bundles $L_j := Y_j^{-1} E_j$ satisfy $L_1^{a_{i,1}} \otimes \cdots \otimes L_N^{a_{i,N}} \cong K_{C,\log}^{u_i} \otimes \mathscr{O}\left(-\sum_{l=1}^{k} \sum_{j=1}^{N} a_{ij} \Theta_j^{\gamma_l} p_l \right)$ for each $i \in \{1, \ldots, N\}$.

Now we may construct an orbicurve $\mathscr{C}$ on C with local group at p_l generated by γ_l for each $l \in \{1, \ldots, k\}$, and we can construct the desired orbifold line bundles $\mathscr{L}_j$ on $\mathscr{C}$ from L_j by inverting the map described in Section 2.1.2 at each marked point. It is easy to see that these line bundles form a W-structure on $\mathscr{C}$, and therefore $\overline{\mathscr{W}}_{g,k}(\gamma)$ is not empty. $\square$

Example 2.2.9. For three-pointed, genus-zero W-curves, the choice of orbifold line bundles $\mathscr{L}_1, \ldots, \mathscr{L}_N$ providing the W-structure is unique, if it exists at all. Hence, if the selection rule is satisfied, $\overline{\mathscr{W}}_{0,3}(\boldsymbol{\gamma})$ is isomorphic to $\mathscr{B}G$.

2.2.2. *Dual graphs.* We must generalize the concept of a decorated dual graph, given for r-spin curves in [JKV01], to the case of a general W-orbicurve.

Definition 2.2.10. Let Γ be a dual graph of a stable curve $(\mathscr{C}, p_1, \ldots, p_k)$ as in [JKV01]. A half-edge of a graph Γ is either a tail or one of the two ends of a "real" edge of Γ.

Let $V(\Gamma)$ be the set of vertices of Γ, let $T(\Gamma)$ denote the tails of Γ, and let $E(\Gamma)$ be the set of "real" edges. For each $\nu \in V(\Gamma)$, let g_ν be the (geometric) genus of the component of $\mathscr{C}$ corresponding to ν, let $T(\nu)$ denote the set of all half-edges of Γ at the vertex ν, and let k_ν be the number of elements of $T(\nu)$.

Definition 2.2.11. Let Γ be a dual graph. The *genus* of Γ is defined as

$$g(\Gamma) = \dim H^1(\Gamma) + \sum_{\nu \in V(\Gamma)} g_\nu.$$

A graph Γ is called *stable* if $2g_\nu + k_\nu \geq 3$ for every $\nu \in V(\Gamma)$.

Definition 2.2.12. A G_W-*decorated stable graph* is a stable graph Γ with a decoration of each tail $\tau \in T(\Gamma)$ by a choice of $\gamma_\tau \in G_W$.

It is often useful to decorate all the half-edges—not just the tails. In that case, we will require that for any edge $e \in E(\Gamma)$ consisting of two half edges τ_+ and τ_-, the corresponding decorations γ_+ and γ_- satisfy

(19) $$\gamma_- = (\gamma_+)^{-1},$$

and we call such a graph a *fully G_W-decorated stable graph*.

Definition 2.2.13. Given a W-curve

$$\mathfrak{C} := (\mathscr{C}, p_1, \ldots, p_k, \mathscr{L}_1, \ldots, \mathscr{L}_N, \tilde{\varphi}_1, \ldots, \tilde{\varphi}_N),$$

the underlying (coarse) curve C defines a dual graph Γ. Each half-edge τ of Γ corresponds to an orbifold point p_τ of the normalization of $\mathscr{C}$, and thus has a corresponding choice of $\gamma_\tau \in G_W$, as given in Proposition 2.1.24.

We define the *fully G_W-decorated dual graph of $\mathfrak{C}$* to be the graph Γ where each half-edge τ is decorated with the group element γ_τ.

Remark 2.2.14. If a fully G_W-decorated graph Γ is to correspond to an actual W-orbicurve, the selection rules of equation (77) must be satisfied on every vertex of Γ; namely, for each $\nu \in V(\Gamma)$ and for each $j \in \{1, \ldots, N\}$, the degree of $|\mathscr{L}_j|$ on the component of the underlying curve associated to ν must

be integral:

$$(20) \qquad \deg(|\mathscr{L}_j|_\nu) = \left(q_j(2g_\nu - 2 + k_\nu) - \left(\sum_{\tau \in T(\nu)} \Theta_j^{\gamma_\tau} \right) \right) \in \mathbb{Z}.$$

Definition 2.2.15. For any G_W-decorated stable W-graph Γ, we define $\overline{\mathscr{W}}(\Gamma)$ to be the closure in $\overline{\mathscr{W}}_{g,k}$ of the stack of stable W-curves with G_W-decorated dual graph equal to Γ.

Remark 2.2.16. Note that no deformation of a nodal orbicurve will deform a node with one orbifold structure to a node with a different orbifold structure—the only possibility for change is to smooth the node away. This means that if Γ is G_W-decorated only on the tails and not on its edges, then the space $\overline{\mathscr{W}}(\Gamma)$ is a disjoint union of closed subspaces $\overline{\mathscr{W}}(\widetilde{\Gamma})$ where the $\widetilde{\Gamma}$ run through all the choices of fully G_W-decorated graphs obtained by decorating all edges of Γ with elements of G_W.

When a graph is a tree with only two vertices and one (separating) edge, then the rules of equation (20) imply that the decorations on the tails uniquely determine the decoration in the edge: each Θ_i for the edge is completely determined by the integrality condition.

However, if the graph is a loop, with only one vertex and one edge, then the rules of equation (20) provide no restriction on the decoration γ_+ at the node.

Let the genus of Γ be $g = g(\Gamma)$, let the number of tails of Γ be k, and let the ordered k-tuple of the decorations associated to those tails be $\boldsymbol{\gamma} := (\gamma_1, \ldots, \gamma_k)$. In this case it is clear that $\overline{\mathscr{W}}(\Gamma) \subseteq \overline{\mathscr{W}}_{g,k}(\boldsymbol{\gamma})$ is a closed substack.

2.2.3. *Morphisms.* We have already discussed the morphism

$$\mathrm{st} : \overline{\mathscr{W}}_{g,k} \longrightarrow \overline{\mathscr{M}}_{g,k}.$$

In this subsection we define several other important morphisms.

Forgetting tails. If $\boldsymbol{\gamma} = (\gamma_1, \ldots, J, \ldots, \gamma_k)$ is such that $\gamma_i = J$ for some $i \in \{1, \ldots, k\}$ (that is, $\Theta_l^{\gamma_i} = q_l$ for every $l \in \{1, \ldots, N\}$) and if $\boldsymbol{\gamma}' = (\gamma_1, \ldots, \hat{\gamma}_i, \ldots, \gamma_k) \in G_W^{k-1}$ is the $k-1$-tuple obtained by omitting the i-th component of $\boldsymbol{\gamma}$, then the *forgetting tails morphism*

$$\vartheta : \overline{\mathscr{W}}_{g,k}(\boldsymbol{\gamma}) \longrightarrow \overline{\mathscr{W}}_{g,k-1}(\boldsymbol{\gamma}')$$

is obtained by forgetting the orbifold structure at the point p_i.

We describe the morphism more explicitly as follows. Let $\overline{\mathscr{C}}$ denote the orbicurve obtained by forgetting the marked point p_i and its orbifold structure, but leaving the rest of the marked points of the orbicurve $\mathscr{C}$ unchanged. Let $\overline{\varrho} : \mathscr{C} \longrightarrow \overline{\mathscr{C}}$ be the obvious morphism. By Proposition 2.1.24, the pushforwards

$\overline{\varrho}_*(\mathscr{L}_j)$ for $j \in \{1, \ldots, N\}$ satisfy

$$(\overline{\varrho}_*(\mathscr{L}_1))^{a_{j,1}} \otimes \cdots \otimes (\overline{\varrho}_*(\mathscr{L}_N))^{a_{j,N}} \xrightarrow{\sim} K_{\mathscr{C},\log}^{u_j} \otimes \mathscr{O}\left(-\sum_{\ell=1}^{N} a_{j,\ell}\Theta_j^J p_i\right)$$

$$= K_{\mathscr{C},\log}^{u_j} \otimes \mathscr{O}\left(-\sum_{j=1}^{N} a_{j\ell}q_\ell p_i\right)$$

$$= K_{\mathscr{C},\log}^{u_j} \otimes \mathscr{O}(-u_j p_i) = K_{\overline{\mathscr{C}},\log}$$

since $\sum_{j=1}^{N} a_{j\ell}\Theta_j^J = \sum_{j=1}^{N} a_{j\ell}q_\ell = u_j$ (because $Aq = V^{-1}Bq = V^{-1}(1, \ldots, 1)^T = u$). We denote the induced isomorphisms by

$$\tilde{\varphi}_j' : (\overline{\varrho}_*(\mathscr{L}_1))^{a_{j,1}} \otimes \cdots \otimes (\overline{\varrho}_*(\mathscr{L}_N))^{a_{j,N}} \xrightarrow{\sim} K_{\overline{\mathscr{C}},\log}^{u_j}.$$

The tuple $(\overline{\mathscr{C}}, p_1, \ldots, \hat{p}_i, \ldots, p_k, \overline{\varrho}_*(\mathscr{L}_1), \ldots, \overline{\varrho}_*(\mathscr{L}_N), \tilde{\varphi}_1', \ldots, \tilde{\varphi}_N')$ is a W-orbicurve of type $\boldsymbol{\gamma}'$. This procedure induces the desired morphism

$$\vartheta : \overline{\mathscr{W}}_{g,k}(\boldsymbol{\gamma}) \longrightarrow \overline{\mathscr{W}}_{g,k-1}(\boldsymbol{\gamma}').$$

Note that the essential property of γ_i that allows the forgetting tails morphism to exist is the fact that $\sum_{j=1}^{N} a_{lj}\Theta_j^{\gamma_i} = u_l$ for every $l \in \{1, \ldots, s\}$. Since the weights q_j are uniquely determined by this property (since B and A are of rank N), this means that a marked point p_i may not be forgotten unless $\gamma_i = J \in G_W$.

Gluing and cutting. Gluing two marked points on a stable curve or on a pair of stable curves defines a Riemann surface with a node. This procedure defines two well-known morphisms

(21) $$\rho_{\text{tree}} : \overline{\mathscr{M}}_{g_1,k_1+1} \times \overline{\mathscr{M}}_{g_2,k_2+1} \longrightarrow \overline{\mathscr{M}}_{g_1+g_2,k_1+k_2},$$

(22) $$\rho_{\text{loop}} : \overline{\mathscr{M}}_{g-1,k+2} \longrightarrow \overline{\mathscr{M}}_{g,k}.$$

More generally, if Γ is a dual graph, then we can cut an edge to form $\widehat{\Gamma}$, and there is a gluing map

$$\rho : \overline{\mathscr{M}}(\widehat{\Gamma}) \longrightarrow \overline{\mathscr{M}}(\Gamma) \subseteq \overline{\mathscr{M}},$$

where $\overline{\mathscr{M}}(\Gamma)$ denotes the closure in $\overline{\mathscr{M}}_{g,k}$ of the locus of stable curves with dual graph Γ.

Unfortunately, there is no direct lift of ρ to the moduli stack of W-curves because there is no canonical way to glue the fibers of the line bundles $\mathscr{L}_i$ on the two points that map to a node. In fact, if anything, the morphism goes the other way; that is, restricting a W-structure on a nodal (i.e., glued) curve to the normalization (i.e., cutting) of that curve will induce a W-structure on the normalization. Unfortunately, this does not induce a morphism from $\overline{\mathscr{W}}(\Gamma)$ to $\overline{\mathscr{W}}(\widehat{\Gamma})$ because for many curves, the normalization of the curve does not have a well-defined choice of a marking (section) for the two points that map to the node.

Nevertheless, we can use this restriction property to create a pair of morphisms that will serve our purposes just as well as a gluing morphism would. To do this, we first consider the fiber product

$$F := \overline{\mathscr{M}}(\widehat{\Gamma}) \times_{\overline{\mathscr{M}}(\Gamma)} \overline{\mathscr{W}}(\Gamma).$$

F is the stack of triples $(\tilde{\mathscr{C}}, (\mathscr{C}, \mathfrak{L}), \beta)$, where $\tilde{\mathscr{C}}$ is a pointed stable orbicurve with dual graph $\widehat{\Gamma}$ and $\mathscr{C}$ is a pointed stable orbicurve with dual graph Γ; also, $\mathfrak{L}$ is a W-structure on $\mathscr{C}$ and $\beta : \rho[\tilde{\mathscr{C}}] \longrightarrow \mathscr{C}$ is an isomorphism of the glued curve $\rho[\tilde{\mathscr{C}}]$ with the orbicurve $\mathscr{C}$.

Instead of a lifted gluing (or cutting) map, we will use the following pair of maps:

$$\overline{\mathscr{W}}(\widehat{\Gamma}) \xleftarrow{\ q\ } F \xrightarrow{\ \mathrm{pr}_2\ } \overline{\mathscr{W}}(\Gamma),$$

where the morphism q simply takes the triple to the W-curve $(\tilde{\mathscr{C}}, \beta^*(\mathfrak{L}))$ by pulling back the W-structure to $\tilde{\mathscr{C}}$. This is well defined because the fiber product has well-defined choices of sections of $\tilde{\mathscr{C}}$ mapping to the node of $\mathscr{C}$.

Alternatively, we could also describe a gluing process in terms of an additional structure that we call *rigidification*. Let p be a marked point. Let $j_p : \mathscr{B}G_p \longrightarrow \mathscr{C}$ be the corresponding gerbe section of $\mathscr{C}$. A *rigidification at* p is an isomorphism

$$\psi : j_p^*(\mathscr{L}_1 \oplus \cdots \oplus \mathscr{L}_N) \longrightarrow [\mathbb{C}^N/G_p]$$

such that for every $\ell \in \{1, \ldots, N\}$, the following diagram commutes:

$$
(23) \qquad
\begin{array}{ccc}
j_p^*\left(\displaystyle\bigoplus_{m=1}^{N} \mathscr{L}_m\right) & \xrightarrow{\ \psi\ } & [\mathbb{C}^N/G_p] \\
{\scriptstyle \tilde{\varphi}_\ell \circ A_\ell}\Big\downarrow & & \Big\downarrow{\scriptstyle A_\ell} \\
j_p^*(K_{\log}^{u_\ell}) & \xrightarrow{\ \mathrm{res}^{u_\ell}\ } & \mathbb{C}
\end{array}
$$

where the map res^{u_ℓ} takes $(dz/z)^{u_\ell}$ to 1. Note that the two terms in the bottom of the diagram have trivial orbifold structure. Since each monomial W_ℓ of W is fixed by G_W, we also have that each monomial A_ℓ is fixed by G_W and hence by G_p. This means that the vertical maps are both well defined.

One can define the equivalence class of W-structures with rigidification in an obvious fashion. The notion of rigidification is also important for constructing the perturbed Witten equation, but we will not use it in any essential way in this paper.

A more geometric way to understand the rigidification is as follows. Suppose the fiber of the W-structure at the marked point is $[(\mathscr{L}_1 \oplus \cdots \oplus \mathscr{L}_N)/G_p]$. The rigidification can be thought as a G_p-equivariant map $\psi : \bigoplus_i \mathscr{L}_i \longrightarrow \mathbb{C}^N$ commuting with the W-structure. For any element $g \in G_p$, the rigidification

$g\psi$ is considered to be an equivalent rigidification. The choice of ψ is equivalent to a choice of basis $e_i \in \mathscr{L}_i$ such that $A_j(e_1, \ldots, e_N) = (dz/z)^{u_j}$, and the basis $g(e_1), \ldots g(e_N)$ is considered to be an equivalent choice. In particular, if $\mathscr{L}_{i_1}, \ldots, \mathscr{L}_{i_m}$ are the line bundles fixed by G_p (we call the corresponding variables x_{i_j} the *broad variables*), then in each equivalance class of rigidifications, the basis elements $e_{i_1}, \ldots, e_{i_m}$ for the subspace $\bigoplus_{j=1}^m \mathscr{L}_{i_j}\big|_p$ are unique, but the basis elements for the terms not fixed by G_p (the *narrow variables*) are only unique up to the action of G_p.

It is clear that the group G_W/G_p acts transitively on the set of rigidifications within a single orbit. Let $\overline{\mathscr{W}}^{\mathrm{rig}_p}(\Gamma)$ be the closure of the substack of equivalence classes of W-curves with dual graph Γ and a rigidification at p. The group G_W/G_p acts on $\overline{\mathscr{W}}^{\mathrm{rig}_p}(\Gamma)$ by interchanging the rigidifications. The stack $\overline{\mathscr{W}}^{\mathrm{rig}_p}(\Gamma)$ is a principal G_W/G_p-bundle over $\overline{\mathscr{W}}(\Gamma)$ and

$$\left[\overline{\mathscr{W}}^{\mathrm{rig}_p}(\Gamma)/(G_W/G_p)\right] = \overline{\mathscr{W}}(\Gamma).$$

Now we describe the gluing. To simplify notation, we ignore the orbifold structures at other marked points and denote the type of the marked points p_+, p_- being glued by γ_+, γ_-. Recall that the resulting orbicurve must be balanced, which means that $\gamma_- = \gamma_+^{-1}$. Let

$$\psi_\pm : j_{p_\pm}^*(\mathscr{L}_1 \oplus \cdots \oplus \mathscr{L}_N) \longrightarrow [\mathbb{C}^N/G_{p_\pm}]$$

be the rigidifications. However, the residues at p_+, p_- are opposite to each other. The obvious identification will not preserve the rigidifications. Here, we fix once and for all an isomorphism

$$I : \mathbb{C}^N \longrightarrow \mathbb{C}^N$$

such that $W(I(x)) = -W(x)$. I can be explicitly constructed as follows. Suppose that $q_i = n_i/d$ for common denominator d. Choose $\xi^d = -1$, and set $I(x_1, \ldots, x_N) = (\xi^{n_1} x_1, \ldots, \xi^{n_N} x_N)$. If I' is another choice, then $I^{-1}I' \in \langle J \rangle \leq G_W$. Furthermore, $I^2 \in \langle J \rangle \leq G_W$ as well. The identification by I induces a W-structure on the nodal orbifold Riemann surface with a rigidification at the nodal point. Forgetting the rigidification at the node yields the lifted gluing morphisms

(24) $\widetilde{\rho}_{\mathrm{tree}, \gamma} : \overline{\mathscr{W}}^{\mathrm{rig}}_{g_1, k_1+1}(\gamma) \times \overline{\mathscr{W}}^{\mathrm{rig}}_{g_2, k_2+1}(\gamma^{-1}) \longrightarrow \overline{\mathscr{W}}_{g_1+g_2, k_1+k_2},$

(25) $\widetilde{\rho}_{\mathrm{loop}, \gamma} : \overline{\mathscr{W}}^{\mathrm{rig}}_{g, k+2}(\gamma, \gamma^{-1}) \longrightarrow \overline{\mathscr{W}}_{g+1, k},$

where $\widetilde{\rho}$ is defined by gluing the rigidifications at the extra tails and forgetting the rigidification at the node.

Degree of st. There are various subtle factors in our theory arising from the orbifold degrees of the maps. These factors can be a major source of confusion. The degree of the stabilization morphism $\mathrm{st}_{\boldsymbol{\gamma}} : \overline{\mathscr{W}}_{g,k}(\boldsymbol{\gamma}) \longrightarrow \overline{\mathscr{M}}_{g,k}$ is especially important in this paper.

As described in Remark 2.1.20, for a given choice of $\boldsymbol{\gamma} \in G_W^k$, the set of all W-structures of type $\boldsymbol{\gamma}$ on a given orbicurve $\mathscr{C}$ with underlying curve C is either empty or is an $H^1(C, G_W)$-torsor; therefore, $H^1(C, G_W)$ acts on the nonempty $\overline{\mathscr{W}}_{g,k}(\boldsymbol{\gamma})$ and the coarse quotient is $\overline{\mathscr{M}}_{g,k}$. One might think that $\deg(\mathrm{st}_{\boldsymbol{\gamma}}) = |H^1(C, G_W)|$, but further examination shows that this is not the case because $\overline{\mathscr{M}}_{g,k}$ is not isomorphic to $\left[\overline{\mathscr{W}}_{g,k}/H^1(C, G_W)\right]$ as a stack. This is particularly evident because $\overline{\mathscr{W}}_{g,k}$ has a nontrivial isotropy group at each point, while the generic point of $\mathscr{M}_{g,k}$ has no isotropy group. The key point is that the automorphism group of any W-structure over a fixed, smooth orbicurve $\mathscr{C}$ is all of G_W. Therefore, we have

$$(26) \qquad\qquad \deg(\mathrm{st}_{\boldsymbol{\gamma}}) = |G_W|^{2g-1}.$$

Since there are $|G_W|^{k-1}$ choices of $\boldsymbol{\gamma}$ that produce a nonempty $W_{g,n}(\boldsymbol{\gamma})$, this shows that the total degree of st $: \overline{\mathscr{W}}_{g,k} \longrightarrow \overline{\mathscr{M}}_{g,k}$ is $|G_W|^{2g-2+k}$.

For any decorated graph Γ, we also have a stabilization map

$$\mathrm{st}_{\Gamma} : \overline{\mathscr{W}}(\Gamma) \longrightarrow \overline{\mathscr{M}}(\Gamma),$$

but the degree of st_{Γ} is not the same as that of st. For example, if Γ is a graph with two vertices and one (separating) edge labeled with the element γ_+, then the number of W-structures over a generic point of $\overline{\mathscr{M}}(\Gamma)$ is still $|H^1(C, G_W)| = |G_W|^{2g}$, but, by equation (10), the automorphism group of a generic point of $\overline{\mathscr{W}}(\Gamma)$ is $G_W \times_{G_W/\langle\gamma_+\rangle} G_W$.

For a tree, the selection rules uniquely determine the choice of γ_+; therefore, we have the following.

PROPOSITION 2.2.17. *For a tree Γ with two vertices and one edge, with tails decorated with $\boldsymbol{\gamma} := (\gamma_1, \ldots, \gamma_k) \in G_W^k$ and edge decorated with γ_+, the map $\mathrm{st}_{\boldsymbol{\gamma}}$ is ramified along $\overline{\mathscr{W}}(\Gamma)$, and*

$$(27) \qquad\qquad \deg(\mathrm{st}_{\boldsymbol{\gamma}}) = |\langle\gamma_+\rangle| \deg(\mathrm{st}_{\Gamma}).$$

If Γ is a loop with one vertex and one (nonseparating) edge, such that the edge is labeled with the element γ, then we have the following proposition.

PROPOSITION 2.2.18. *Let $\boldsymbol{\gamma} := (\gamma_1, \ldots, \gamma_k) \in G_W^k$ be chosen so that $\overline{\mathscr{W}}_{g,k}(\boldsymbol{\gamma})$ is nonempty. For the loop Γ with a single vertex and a single edge decorated with γ_+ and tails decorated with $\boldsymbol{\gamma}$, the stack $\overline{\mathscr{W}}(\Gamma)$ is nonempty. Moreover, the morphism $\mathrm{st}_{\boldsymbol{\gamma}}$ is ramified along st_{Γ} and*

$$(28) \qquad\qquad \deg(\mathrm{st}_{\Gamma}) = \frac{|G_W|^{2g-2}}{|\langle\gamma_+\rangle|}.$$

Proof. First, we claim that the number of W-structures over a generic point of $\overline{\mathscr{M}}_{g,k}$ that degenerate to a given W-structure over Γ is $|\langle\gamma\rangle|$. To see this, note that for any W-structure $\mathfrak{L}$ on a smooth orbicurve $\mathscr{C}$ with underlying curve C, all other W-structures on $\mathscr{C}$ differ from it by an element of $H^1(C, G_W)$. Consider any fixed 1-parameter family of W-curves such that the W-curve $(\mathscr{C}, \mathfrak{L})$ degenerates to a W-curve $(\mathscr{C}', \mathfrak{L}')$ with dual graph Γ, corresponding to the contraction of a cycle $\alpha \in H_1(C, \mathbb{Z})$. In this case, we may choose a basis of $H^1(C, G_W)$ such that the first basis element is dual to α and the second basis element is dual to a cycle β such that $\alpha \cdot \beta = 1$, and $\beta \cdot \sigma = 0$ for any remaining basis element σ of $H_1(C, G_W)$.

In this case, the W-structure obtained by multiplying $\mathfrak{L}$ by an element of the form $(1, \varepsilon_2, \ldots, \varepsilon_{2g}) \in H^1(C, G_W)$ will again degenerate (over the same family of stable underlying curves) to $\mathfrak{L}'$ if and only if $\varepsilon_2 \in \langle\gamma\rangle$.

Second, by equation (11), the automorphism groups for both smooth W-curves and these degenerate W-curves are isomorphic to G_W. This, combined with the previous degeneration count, proves that the ramification is $|\langle\gamma\rangle|$.

More generally, the pair $(\mathscr{C}, \mathfrak{L} \cdot (1, \varepsilon_2, \ldots, \varepsilon_{2g}))$ will always degenerate to a W-curve with dual graph Γ, and $(\mathscr{C}, \mathfrak{L} \cdot (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{2g}))$ for $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{2g}) \in H^1(C, G_W)$ will degenerate to a W-curve with dual graph labeled by $\gamma\varepsilon_1$ instead of by γ. Thus the moduli $\overline{\mathscr{W}}(\Gamma)$ is nonempty for every choice of decoration $\gamma \in G_W$ of the edge of Γ.   □

2.3. **Admissible groups G and $\overline{\mathscr{W}}_{g,k,G}$.** The constructions of this paper depend quite heavily on the group of diagonal symmetries G_W of the singularity W. It is useful to generalize these constructions to the case of a subgroup G of G_W. First, the isomorphism I is only well defined up to an element of $\langle J \rangle$. Therefore, we will always require that $J \in G$. The problem is that it is not *a priori* obvious that the stack of W-curves with markings only coming from a subgroup G is a proper stack. Namely, the orbifold structure at nodes may not be in G.

However, we note[*] that for any Laurent polynomial $Z = \sum_j \prod_{i=1}^N x_i^{a_{ij}}$ of weighted total degree 1, with $a_{ij} \in \mathbb{Z}$ for all i and j, the diagonal symmetry group $G_{\widetilde{W}}$ of $\widetilde{W} := W + Z$ is clearly a subgroup of G_W containing $\langle J \rangle$, and the stack $\overline{\mathscr{W}}_{g,k}(\widetilde{W})$ of $\widetilde{W}$-curves is a proper substack of $\overline{\mathscr{W}}_{g,k}(W)$.

PROPOSITION 2.3.1. *For every quasi-homogeneous Laurent polynomial $\widetilde{W} = W + Z$, where Z has no monomials in common with W, there is a natural morphism* adm $: \overline{\mathscr{W}}_{g,k}(W + Z) \longrightarrow \overline{\mathscr{W}}_{g,k}(W)$ *from the stack $\overline{\mathscr{W}}_{g,k}(\widetilde{W})$ to an*

---

[*]We are grateful to H. Tracy Hall for suggesting this approach to us.

open and closed substack of $\overline{\mathscr{W}}_{g,k}(W)$. *Moreover, this morphism is finite of degree equal to the index of* $G_{\widetilde{W}}$ *in* G_W.

Proof. It suffices to consider the case of $\widetilde{W} = W + M$, where $M = \prod_{i=1}^{N} x_i^{\beta_i}$ is a single monomial of degree 1 (i.e., $\sum_{i=1}^{N} \beta_i q_i = 1$), distinct from the monomials $W_j = c_j \prod_{l=1}^{s} x_i^{b_{l,j}}$ of W.

The morphism adm is simply the functor that forgets the additional conditions arising from the monomial in M.

Given a W-structure $(\mathscr{L}_1, \ldots, \mathscr{L}_N, \phi_1, \ldots, \phi_s)$ on an orbicurve $\mathscr{C}$, we can produce s choices of a d-th root of $\mathscr{O}$—one for each monomial of the original polynomial W—as follows. For each $j \in \{1, \ldots, s\}$, let

$$(29) \qquad \mathscr{N}_j := \bigotimes_{i=1}^{N} \mathscr{L}_i^{\otimes(b_{i,j} - \beta_i)}.$$

Using the fact that $B(q_1, \ldots, q_N)^T = (1, \ldots, 1)^T$ and $\beta \cdot (q_1, \ldots, q_N) = 1$, we see that $\mathscr{N}_j^{\otimes d} \cong \mathscr{O}$, where d is defined (as in Definition 2.1.4) to be the smallest positive integer such that $(dq_1, \ldots, dq_N) \in \mathbb{Z}$. This gives s morphisms

$$(30) \qquad \overline{\mathscr{W}}_{g,k}(W) \xrightarrow{\Phi_j} \mathscr{J}_{g,k,d},$$

where $\mathscr{J}_{g,k,d} := \{(\mathscr{C}, p_1, \ldots, p_k, \mathscr{L}, \psi : \mathscr{L}^d \longrightarrow \mathscr{O}_{\mathscr{C}})\}$ denotes the stack of k-pointed, genus-g orbicurves with a d-th root $\mathscr{N}$ of the trivial bundle. It is easy to see that the stack $\mathscr{J}_{g,k,d}$ has a connected component $\mathscr{J}^0_{g,k,d}$ corresponding to the trivial d-th root of $\mathscr{O}$. The inverse image $\overline{\mathscr{W}}^0_{g,k}(W) := \Phi_j^{-1}(\mathscr{J}^0_{g,k,d})$ of the trivial component for each $j \in \{1, \ldots, s\}$ is independent of j, is open and closed, and is the image of the forgetful morphism adm:

$$\overline{\mathscr{W}}_{g,k}(\widetilde{W}) \xrightarrow{\mathrm{adm}} \overline{\mathscr{W}}^0_{g,k}(W) \subseteq \overline{\mathscr{W}}_{g,k}(W).$$

The objects of the stack $\overline{\mathscr{W}}_{g,k}(\widetilde{W})$ are $\widetilde{W}$-curves

$$(\mathscr{C}, \mathscr{L}_1, \ldots, \mathscr{L}_N, \phi_1, \ldots, \phi_{s+1}),$$

where ϕ_{s+1} is an isomorphism $\phi_{s+1} : M(\mathscr{L}_1, \ldots, \mathscr{L}_N) \longrightarrow K_{\log}$, whereas the objects of the stack $\overline{\mathscr{W}}^0_{g,k}(W)$ are W-curves $(\mathscr{C}, \mathscr{L}_1, \ldots, \mathscr{L}_N, \phi_1, \ldots, \phi_s)$ such that there exists some isomorphism $\psi : M(\mathscr{L}_1, \ldots, \mathscr{L}_N) \longrightarrow K_{\log}$ that is compatible with the isomorphisms ϕ_i of the W-structure. Any W-curve with such a ψ is isomorphic to the image of some $\widetilde{W}$-curve, but since an automorphism of a W-curve in $\overline{\mathscr{W}}^0_{g,k}(W)$ need not fix the isomorphism ψ, the automorphism group of a generic W-curve in $\overline{\mathscr{W}}^0_{g,k}(W)$ is G_W, while the automorphism group of a generic $\widetilde{W}$-curve is $G_{\widetilde{W}}$. $\qquad \square$

Definition 2.3.2. We say that a subgroup $G \leq G_W$ is *admissible* or is an *admissible group of Abelian symmetries of* W if there exists a Laurent

polynomial Z, quasi-homogeneous with the same weights q_i as W, but with no monomials in common with W, such that $G = G_{W+Z}$.

Definition 2.3.3. Suppose that G is admissible. We define the stack $\overline{\mathscr{W}}_{g,k,G}(W) := \overline{\mathscr{W}}_{g,k}(\widetilde{W})$ for any $\widetilde{W} = W + Z$ with $G_{\widetilde{W}} = G$.

The most important consequence of Proposition 2.3.1 is that we may restrict (pull back) the virtual cycle $\left[\overline{\mathscr{W}}_{g,k}(W)\right]^{\mathrm{vir}}$ to the substack $\overline{\mathscr{W}}_{g,k,G}(W)$ (see Section 4.1).

Remark 2.3.4. An admissible group G may have more than one Z such that $G = G_{W+Z}$. One can show (see [CR10]) that $\overline{\mathscr{W}}_{g,k,G} := \overline{\mathscr{W}}_{g,k}(W + Z)$ is independent of Z and depends only on G.

It is immediate that every admissible group contains J. Marc Krawitz [Kra10, Prop. 3.4] has proved the converse. For the reader's convenience we repeat his proof here.

PROPOSITION 2.3.5 (Krawitz). *For any nondegenerate $W \in \mathbb{C}[x_1, \ldots, x_N]$, any group of diagonal symmetries of W containing J is admissible.*

Proof. The subring of G-invariants in $A := \mathbb{C}[x_1, \ldots, x_N]$ is finitely generated by monomials. Let Z be the sum of all G-invariant monomials in A not divisible by monomials in W.

We claim that G is the maximal diagonal symmetry group of $W + Z$. If it were not, there would be a diagonal symmetry group H, with $G \leq H$ and $A^G \subseteq A^H$. The actions of G and H on A extend to actions on the fraction field $E := \mathbb{C}(x_1, \ldots, x_N)$. Since the action is diagonal, it is easy to show that this implies that the fraction field of A^G equals E^G and the fraction field of A^H equals E^H. Since $A^G = A^H$, we have $E^G = E^H$. Since G and H are finite, we have, by [Mil, Cor 3.5], that

$$G = \mathrm{Aut}(E/E^G) = \mathrm{Aut}(E/E^H) = H.$$

Therefore G is the maximal symmetry group of $W + Z$.

Now, since J preserves each of the constituent monomials of Z, each of these monomials has integral quasi-homogeneous degree. We may correct each of these monomials by a (negative) power of any monomial in W to ensure that each of the monomials has quasi-homogeneous degree equal to 1, and since we are correcting by G-invariants not dividing the monomials of Z, we do not change the maximal symmetry group of $W + Z$. □

2.4. **The tautological ring of $\overline{\mathscr{W}}_{g,k}$.** A major topic in Gromov-Witten theory is the tautological ring of $\overline{\mathscr{M}}_{g,k}$. The stack $\overline{\mathscr{W}}_{g,k}$ is similar to $\overline{\mathscr{M}}_{g,k}$ in many ways, and we can readily generalize the notion of the tautological ring to $\overline{\mathscr{W}}_{g,k}$. We expect that the study of the tautological ring of $\overline{\mathscr{W}}_{g,k}$ will

be important to the calculation of our invariants. It is not unreasonable to conjecture that the virtual cycle constructed in the next section is, in fact, tautological.

Throughout this section, we will refer to the following diagram:

$$(31)$$

$$
\begin{array}{ccc}
\mathscr{C}_{g,k} & \xrightarrow{\;\varrho\;} & C_{g,k} \\
{\scriptstyle\sigma_i}\Big\uparrow\Big\downarrow{\scriptstyle\pi} & \swarrow{\scriptstyle\varpi} & \\
\overline{\mathscr{W}}_{g,k}. & &
\end{array}
$$

Here, $\mathscr{C}_{g,k} \xrightarrow{\;\pi\;} \overline{\mathscr{W}}_{g,k}$ is the universal orbicurve and $\varpi : C_{g,k} \longrightarrow \overline{\mathscr{W}}_{g,k}$ is the universal underlying stable curve. The map σ_i is the i-th section of π, and we denote by $\bar{\sigma}_i$ the i-th section of ϖ. The map ϱ forgets the local orbifold structure and takes a point to its counterpart in $C_{g,k}$. On $\mathscr{C}_{g,k}$, we also have the universal W-structure $\bigoplus \mathscr{L}_i$ and the line bundles $K_{\mathscr{C},\log}$ and $K_{\mathscr{C}}$.

2.4.1. ψ-*classes.* As in the case of the moduli of stable maps, we denote by $\widetilde{\psi}_i$ the first Chern class of the $\mathscr{C}$-cotangent line bundle on $\overline{\mathscr{W}}_{g,k}$. That is,

$$(32) \qquad\qquad \widetilde{\psi}_i := c_1(\sigma_i^*(K_{\mathscr{C}})).$$

We note that since $C_{g,k}$ is the pullback of the universal stable curve from $\overline{\mathscr{M}}_{g,k}$, replacing the $\mathscr{C}$-cotangent bundle by the C-cotangent bundle would give the pullback of the usual ψ-class, which we also denote by ψ:

$$(33) \qquad\qquad \psi_i := c_1(\bar{\sigma}_i^*(K_C)) = \mathrm{st}^*(\psi_i).$$

These classes are related as follows.

PROPOSITION 2.4.1. *If the orbifold structure along the marking σ_i is of type γ_i, with $|\langle\gamma_i\rangle| = m_i$, then we have the relation*

$$(34) \qquad\qquad m_i\widetilde{\psi}_i = \mathrm{st}^*\psi_i.$$

Proof. Let D_i denote the image of the section σ_i in $\mathscr{C}_{g,k}$. Note that since $\bar{\sigma}_i = \varrho \circ \sigma_i$, then by equation (3), we have

$$(35) \qquad \bar{\sigma}_i^* K_{C_{g,k}} = \sigma_i^*\left(\varrho^* K_{C_{g,k}}\right) = \sigma_i^*\left(K_{\mathscr{C}_{g,k}} \otimes \mathscr{O}(-(m_i - 1)D_i)\right),$$

and the residue map shows that

$$(36) \qquad\qquad \sigma^* K_{\log} = \mathscr{O},$$

hence

$$(37) \qquad\qquad \sigma_i^*(\mathscr{O}(-D_i)) = \sigma_i^*(K_{\mathscr{C}}),$$

which gives the relation (34). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

2.4.2. ψ_{ij}-classes. It seems natural to use the W-structure to try to define the following tautological classes:

$$\psi_{ij} := c_1(\sigma_i^*(\mathscr{L}_j)).$$

However, these are all zero. To see this, note that for every monomial $W_\ell = \prod x_j^{b_{\ell,j}}$ and for every $i \in \{1, \ldots, k\}$, we have, by the definition of the W-structure and by equation (36),

$$(38) \qquad \sum_{j=1}^{N} b_{\ell,j}\psi_{ij} = 0.$$

Coupled with the nondegeneracy condition (Definition 2.1.5) on W, this implies that every ψ_{ij} is torsion in $H^*(\overline{\mathscr{W}}_{g,k}, \mathbb{Z})$ and thus vanishes in $H^*(\overline{\mathscr{W}}_{g,k}, \mathbb{Q})$.

2.4.3. κ-classes. The traditional definition of the κ-classes on $\overline{\mathscr{M}}_{g,k}$ is

$$\kappa_a := \varpi_*(c_1(K_{C,\log})^{a+1}).$$

We will define the analogue of these classes for W-curves as follows:

$$\widetilde{\kappa}_a := \pi_*(c_1(K_{\mathscr{C},\log})^{a+1}).$$

Note that since $K_{\mathscr{C},\log} = \varrho^* K_{C,\log}$, and since $\deg(\varrho) = 1$, we have

$$(39) \qquad \widetilde{\kappa}_a = \pi_*(c_1(K_{\mathscr{C},\log})^{a+1}) = \varpi_*\varrho_*\varrho^*(c_1(K_{\mathscr{C},\log})^{a+1}) = \kappa_a.$$

2.4.4. μ-classes. The Hodge classes λ_i for the usual stack of stable curves are defined to be the Chern classes of the K-theoretic pushforward $R\varpi_* K_C$. We could also work on the universal orbicurve $\mathscr{C}_{g,k} \xrightarrow{\pi} \overline{\mathscr{M}}_{g,k}$, but ϱ is finite, so by equation (5) we have

$$R\pi_* K_{\mathscr{C}} = R\varpi_*(\varrho_* K_{\mathscr{C}}) = R\varpi_* K_C.$$

Therefore, the two definitions of lambda classes agree. Moreover, it is known that the λ-classes can be expressed in terms of κ-classes, so they need not be included in the definition of the tautological ring.

A more interesting Hodge-like variant comes from pushing down the W-structure bundles $\mathscr{L}_j$. We also find it more convenient to work with the components of the Chern *character* rather than the Chern classes. We define μ-classes to be the components of the Chern character of the W-structure line bundles:

$$\mu_{ij} := \mathrm{Ch}_i(R\pi_*\mathscr{L}_j).$$

By the orbifold Grothendieck-Riemann-Roch theorem, these can be expressed in terms of the kappa, psi, and boundary classes. (See, for example, the proof of Theorem 6.3.3.)

2.4.5. *Tautological ring of $\overline{\mathscr{W}}_{g,k}$.*

Definition 2.4.2. We define the *tautological ring of $\overline{\mathscr{W}}_{g,k}$* to be the subring of $H^*(\overline{\mathscr{W}}_{g,k}, \mathbb{Q})$ generated by $\widetilde{\psi}_i$, $\widetilde{\kappa}_a$, and the obvious boundary classes.

We would like to propose the following conjecture.

CONJECTURE 2.4.3 (Tautological virtual cycle conjecture). *The virtual cycle (constructed in the next section) is tautological in the sense that its Poincaré dual lies in the tensor product of the tautological ring of $\overline{\mathscr{W}}_{g,k}$ and relative cohomology.*

3. The state space associated to a singularity

Ordinary Gromov-Witten invariants take their inputs from the cohomology of a symplectic manifold—the state space. In this section, we describe the analogue of that state space for singularity theory. As mentioned above, however, our theory depends heavily on the choice of symmetry group G and not just on the singularity W. In this sense, it should be thought of as an orbifold singularity or orbifold Landau-Ginzburg theory of W/G.

We have mirror symmetry in mind when we develop our theory. Some of the choices, such as degree shifting number, are partially motivated by a physics paper by Intriligator-Vafa [IV90] and a mathematical paper by Kaufmann [Kau06] where they studied orbifolded B-model Chiral rings. The third author's previous work on Chen-Ruan orbifold cohomology also plays an important role in our understanding.

3.1. **Lefschetz thimble.** Suppose that a quasi-homogeneous polynomial $W : \mathbb{C}^N \longrightarrow \mathbb{C}$ defines a nondegenerate singularity at zero and that for each $i \in \{1, \ldots, N\}$, the weight of the variable x_i is q_i. An important classical invariant of the singularity is the *local algebra*, also known as the *Chiral ring* or the *Milnor ring*:

$$(40) \qquad \mathscr{Q}_W := \mathbb{C}[x_1, \ldots, x_N]/\operatorname{Jac}(W),$$

where $\operatorname{Jac}(W)$ is the Jacobian ideal, generated by partial derivatives

$$\operatorname{Jac}(W) := \left(\frac{\partial W}{\partial x_1}, \ldots, \frac{\partial W}{\partial x_N} \right).$$

Let us review some of the basic facts about the local algebra. It is clear that the local algebra is generated by monomials. The degree of a monomial allows us to make the local algebra into a graded algebra. There is a unique highest-degree element $\det \left(\frac{\partial^2 W}{\partial x_i \partial x_j} \right)$ with degree

$$(41) \qquad \qquad \hat{c}_W = \sum_i (1 - 2q_i).$$

The degree $\hat{c}_W$ is called the *central charge* and is a fundamental invariant of W.

The singularities with $\hat{c}_W < 1$ are called *simple singularities* and have been completely classified into the famous ADE-sequence. Quasi-homogeneous singularities with $\hat{c}_W = N - 2$ correspond to Calabi-Yau hypersurfaces in weighted projective space. Here, the singularity/LG-theory makes contact with Calabi-Yau geometry. There are many examples with fractional value $\hat{c}_W > 1$. These can be viewed as "fractional dimension Calabi-Yau manifolds."

The dimension of the local algebra is given by the formula

$$\mu = \prod_i \left(\frac{1}{q_i} - 1 \right).$$

From the modern point of view, the local algebra is considered to be part of the B-model theory of singularities. The A-model theory considers the relative cohomology $H^N(\mathbb{C}^N, W^\infty, \mathbb{C})$, where $W^\infty = (\mathfrak{Re}W)^{-1}(M, \infty)$ for $M \gg 0$. Similarly, let $W^{-\infty} = (\mathfrak{Re}W)^{-1}(-\infty, -M)$ for $M \gg 0$. The above space is the dual space of the relative homology $H_N(\mathbb{C}^N, W^\infty, \mathbb{Z})$. The latter is often referred as the space of *Lefschetz thimbles*.

There is a natural pairing

$$\langle \, , \rangle : H_N(\mathbb{C}^N, W^{-\infty}, \mathbb{Z}) \otimes H_N(\mathbb{C}^N, W^\infty, \mathbb{Z}) \to \mathbb{Z}$$

defined by intersecting the relative homology cycles. This pairing is a perfect pairing for the following reason. Consider a family of perturbations W_λ such that W_λ is a holomorphic Morse function for $\lambda \neq 0$. We can construct a basis of $H_N(\mathbb{C}^N, W^{\pm\infty}, \mathbb{Z})$ by choosing a system of *virtually horizontal paths*. A system of virtually horizonal paths $u_i^\pm : [0, \pm\infty) \longrightarrow \mathbb{C}$ emitting from critical values z_i has the properties

(i) $u_i^\pm$; has no self-intersection,
(ii) $u_i^\pm$ is horizonal for large t and extends to $\pm\infty$;
(iii) the paths $u_1^\pm, \ldots, u_\mu^\pm$ are ordered by their imaginary values for large t.

For each $u_i^\pm$, we can associate a Lefschetz thimble $\Delta_i^\pm \in H_N(\mathbb{C}^N, W^{\pm\infty}, \mathbb{Z})$ as follows. The neighborhood of the critical point of z_i contains a local vanishing cycle. Using the homotopy lifting property, we can transport the local vanishing cycle along $u_i^\pm$ to $\pm\infty$. Define $\Delta_i^\pm$ as the union of the vanishing cycles along the corresponding path $u_i^\pm$. The cycles $\Delta_i^\pm$ define a basis of $H^N(\mathbb{C}^N, W^{\pm\infty}, \mathbb{Z})$, and it is clear that

$$\Delta_i^+ \cap \Delta_j^- = \delta_{ij}.$$

Hence, the pairing is perfect for $\lambda \neq 0$.

On the other hand, the complex relative homology $H_N(\mathbb{C}^N, W_\lambda^{\pm\infty}, \mathbb{C})$ defines a vector bundle over the space of λ's. The integral homology classes define a so-called Gauss-Manin connection. The intersection pairing is clearly preserved by the Gauss-Manin connection; hence, it is also perfect at $\lambda = 0$.

We wish to define a pairing on $H^N(\mathbb{C}^N, W_\lambda^\infty, \mathbb{C})$ alone. As we have done in the last section, write $q_i = n_i/d$ for a common denominator d, and choose ξ such that $\xi^d = -1$. Multiplication by the diagonal matrix $(\xi^{n_1}, \ldots, \xi^{n_N})$ defines a map $I : \mathbb{C}^N \longrightarrow \mathbb{C}^N$ sending $W^\infty \longrightarrow W^{-\infty}$. Hence, it induces an isomorphism

$$I_* : H_N(\mathbb{C}^N, W^\infty, \mathbb{C}) \longrightarrow H_N(\mathbb{C}^N, W^{-\infty}, \mathbb{C}).$$

Definition 3.1.1. We define a pairing on $H_N(\mathbb{C}^N, W^\infty, \mathbb{Z})$ by

$$\langle \Delta_i, \Delta_j \rangle = \langle \Delta_i, I_*(\Delta_j) \rangle.$$

It induces a pairing (still denoted by $\langle , \rangle$) on the dual space $H^N(\mathbb{C}^N, W^\infty, \mathbb{C})$ that is equivalent to the residue pairing on the Milnor ring (see Section 5.1). As noted earlier, changing the choice of ξ will change the isomorphism I by an element of the group $\langle J \rangle$, and $I^2 \in \langle J \rangle$. Therefore, the pairing is independent of the choice of I on the invariant subspace $H_N(\mathbb{C}^N, W^\infty, \mathbb{Z})^{\langle J \rangle}$ or on $H_N(\mathbb{C}^N, W^\infty, \mathbb{Z})^G$ for any admissible group G.

3.2. **Orbifolding and state space.** Now we shall "orbifold" the previous construction. Suppose that G is an admissible subgroup. For each $\gamma \in G$, W_γ is again nondegenerate.

Definition 3.2.1. We define the γ-*twisted sector* $\mathscr{H}_\gamma$ of the state space to be the G-invariant part of the middle-dimensional relative cohomology for W_γ; that is,

$$(42) \qquad \mathscr{H}_\gamma := H^{N_\gamma}(\mathbb{C}_\gamma^{N_\gamma}, W_\gamma^\infty, \mathbb{C})^G.$$

The central charge of the singularity W_γ is denoted $\hat{c}_\gamma$:

$$(43) \qquad \hat{c}_\gamma := \sum_{i:\Theta_i^\gamma = 0} (1 - 2q_i).$$

As in Chen-Ruan orbifold cohomology theory, it is important to shift the degree.

Definition 3.2.2. Suppose that $\gamma = (e^{2\pi i \Theta_1^\gamma}, \ldots, e^{2\pi i \Theta_N^\gamma})$ for rational numbers $0 \le \Theta_i^\gamma < 1$.

We define the *degree shifting number*

$$(44) \qquad \iota_\gamma = \sum_i (\Theta_i^\gamma - q_i)$$

$$(45) \qquad = \frac{\hat{c}_W - N_\gamma}{2} + \sum_{i:\Theta_i^\gamma \ne 0} (\Theta_i^\gamma - 1/2)$$

$$(46) \qquad = \frac{\hat{c}_\gamma - N_\gamma}{2} + \sum_{i:\Theta_i^\gamma \ne 0} (\Theta_i^\gamma - q_i).$$

For a class $\alpha \in \mathscr{H}_\gamma$, we define

$$\deg_W(\alpha) = \deg(\alpha) + 2\iota_\gamma. \tag{47}$$

PROPOSITION 3.2.3. *For any $\gamma \in G_W$, we have*

$$\iota_\gamma + \iota_{\gamma^{-1}} = \hat{c}_W - N_\gamma, \tag{48}$$

and for any $\alpha \in \mathscr{H}_\gamma$ and $\beta \in \mathscr{H}_{\gamma^{-1}}$, we have

$$\deg_W(\alpha) + \deg_W(\beta) = 2\hat{c}_W. \tag{49}$$

Proof. The first relation (equation (48)) follows immediately from equation (45) and from the fact that if $\Theta_i^\gamma \neq 0$, then $\Theta_i^{\gamma^{-1}} = 1 - \Theta_i^\gamma$ and, otherwise, $\Theta_i^{\gamma^{-1}} = \Theta_i^\gamma = 0$.

The second relation (equation (49)) follows from the first relation and from the fact that every class in $\mathscr{H}_\gamma$ has degree N_γ. $\square$

Remark 3.2.4. $H^N(\mathbb{C}^N, W^\infty, \mathbb{C})$ also carries an internal Hodge grading due to its mixed Hodge structure. This defines a bi-grading for $\mathscr{H}_\gamma$.

Definition 3.2.5. The *state space* (or *quantum cohomology group*) of the singularity W/G is defined as

$$\mathscr{H}_W = \bigoplus_{\gamma \in G} \mathscr{H}_\gamma. \tag{50}$$

Definition 3.2.6. The J-sector $\mathscr{H}_J$ is always one-dimensional, and the constant function 1 defines a generator $\mathbf{e}_1 := 1 \in \mathscr{H}_J$ of degree 0. This element is the unit in the ring $\mathscr{H}_W$, and because of this, we often denote it by $\mathbf{1}$ instead of $\mathbf{e}_1$.

Definition 3.2.7. For any $\gamma \in G$, we say that the γ-sector is *narrow* if the fixed point locus is trivial (i.e., $N_\gamma = 0$). If the fixed point locus is nontrivial, we say that the γ-sector is *broad*.

Since γ and γ^{-1} have the same fixed point set, there is an obvious isomorphism

$$\varepsilon : \mathscr{H}_\gamma \longrightarrow \mathscr{H}_{\gamma^{-1}}.$$

We define a pairing on $\mathscr{H}_W$ as the direct sum of the pairings

$$\langle\,,\rangle_\gamma : \mathscr{H}_\gamma \otimes \mathscr{H}_{\gamma^{-1}} \longrightarrow \mathbb{C}$$

by $\langle f, g\rangle_\gamma = \langle f, \varepsilon^* g\rangle$, where the second pairing is the pairing of the space of relative cohomology. The above pairing is obviously symmetric and nondegenerate.

Now the pairing on $\mathscr{H}_W$ is defined as the direct sum of the pairings $\langle\,,\rangle_\gamma$.

LEMMA 3.2.8. *The above pairing preserves* $\deg_W$. *Namely, if* $\mathscr{H}_\gamma^a$ *denotes the elements* $\varkappa \in \mathscr{H}_\gamma$ *with* $\deg_W(\varkappa) = a$, *then* $\langle \, , \rangle$ *gives a pairing of* $\mathscr{H}_W^a$ *with* $\mathscr{H}_W^{2\hat{c}-a}$:

$$\mathscr{H}_W^a \otimes \mathscr{H}_W^{2c\hat{w}-a} \longrightarrow \mathbb{C}.$$

Proof. This is a direct consequence of Proposition 3.2.3. $\square$

Remark 3.2.9. The lemma indicates that one can view W/G as an object of complex dimension $\hat{c}_W$. Under the shift, $\mathscr{H}_J$ has degree 0. On the other hand, the untwisted sector has degree $\hat{c}_W$ and the sector $\mathscr{H}_{J^{-1}}$ has degree $2\hat{c}_W$.

Remark 3.2.10. In the usual orbifold theory, the unit comes from the untwisted sector. In our case, the unit element is from $\mathscr{H}_J$. In this sense, our theory is quite different from usual orbifold theory and instead corresponds to the so-called (a,c)-*ring* in physics.

4. Virtual cycles and axioms

In this section, we will discuss the main properties of the virtual cycles $\left[\mathscr{W}_{g,k}(W;\boldsymbol{\gamma})\right]^{\mathrm{vir}}$. These are the key ingredients in the definition of our invariants. We formulate the main properties of the virtual cycle as axioms similar to those of the virtual fundamental cycle of stable maps [CR04] and generalizing the axioms of r-spin curves listed in [JKV01, §4.1].

In the special case of the A_{r-1} singularity, an algebraic virtual class satisfying the axioms of [JKV01, §4.1] has been constructed for the twisted sectors (often called narrow sectors) by Polishchuk and Vaintrob [Pol04], [PV01]. Chiodo has lifted this class to K-theory [Chi06b], [Chi06a], and an analytic class has been proposed by T. Mochizuki [Moc06]—modeled after Witten's original sketch.

4.1. $\left[\overline{\mathscr{W}}(\Gamma)\right]^{\mathrm{vir}}$ and its axioms.

4.1.1. *Review of the construction.* The construction of the virtual cycle $\left[\mathscr{W}_{g,k}(W)\right]^{\mathrm{vir}}$ is highly nontrivial. The details of the construction and the proof of the axioms are presented in [FJR], but we will outline the main ideas here and then focus the rest of this paper on the consequences of the axioms.

The heart of our construction is the analytic problem of solving the moduli problem for the Witten equation. The Witten equation is a first order elliptic PDE of the form

$$\bar{\partial} S_i + \overline{\frac{\partial W}{\partial s_i}} = 0,$$

where S_i is a C^∞-section of $\mathscr{L}_i$.

Our goal is to construct a virtual cycle of the moduli space of solutions of the Witten equation. Let us briefly outline the construction. Let

$\overline{\mathscr{W}}_{g,k}(\gamma_1, \ldots, \gamma_k)$ be the moduli space of W-structures decorated with the orbifold structure defined by γ_i at the i-th marked point. It can be considered as the background data to set up the Witten equation.

To make this construction requires that we leave the algebraic world and enter the world of differential geometry and analysis. The stack (orbifold) $\overline{\mathscr{W}}_{g,k}(\gamma_1, \ldots, \gamma_k)$ has a geometric structure similar to $\overline{\mathscr{M}}_{g,k}$, including a stratification described by dual graphs and something like the gluing structure at a node. Our starting point is to give a differential geometric structure of $\overline{\mathscr{W}}_{g,k}(\gamma_1, \ldots, \gamma_k)$. This can be done in a fashion similar to that for $\overline{\mathscr{M}}_{g,k}$ [FJR, §2.2]. The variable in the Witten equation is a smooth section of the W-structure $\oplus_i L_i$, while the target of the Witten equation is the space of its $(0, 1)$-forms. Formally, the Witten equation can be phrased as a Fredholm section of a Banach bundle over a fiber-wise Banach manifold. Unfortunately, it is rather difficult to solve the Witten equation due to the fact that the singularity of W has high multiplicity. It is much easier to solve a perturbed equation of the form $W + W_0$, where W_0 is a linear perturbation term such that $W_\gamma + W_{0\gamma}$ is a holomorphic Morse function for every γ. Here W_γ and $W_{0\gamma}$ are the restrictions of W, and W_0, respectively, to the fixed point set $\mathbb{C}^{N_\gamma}$. The background data for the perturbed Witten equation is naturally the moduli space (stack) of *rigidified W-structures* $\overline{\mathscr{W}}_{g,k}^{\mathrm{rig}}(\gamma_1, \ldots, \gamma_k)$.

The crucial part of the analysis is to show that a solution of the Witten equation converges to a critical point of $W_{\gamma_i} + W_{0\gamma_i}$. This enables us to construct a moduli space (stack) $\overline{\mathscr{W}}_{g,k}^s(\kappa_{j_1}, \ldots, \kappa_{j_k})$ of solutions of the perturbed Witten equation converging to the critical point κ_i at the marked point x_i. We call W_0 *strongly regular* if (i) $W_{\gamma_i} + W_{0\gamma_i}$ is holomorphic Morse; (ii) the critical values of $W_{\gamma_i} + W_{0\gamma_i}$ have distinct imaginary parts. The first important result is

THEOREM 4.1.1. *If W_0 is strongly regular, then $\overline{\mathscr{W}}_{g,k}^s(\kappa_{j_1}, \ldots, \kappa_{j_k})$ is compact and has a virtual fundamental cycle $[\overline{\mathscr{W}}_{g,k}^s(\kappa_{j_1}, \ldots, \kappa_{j_k})]^{\mathrm{vir}}$ of degree*

$$2\left((c_W - 3)(1 - g) + k - \sum_i \iota_{\gamma_i}\right) - \sum_i N_{\gamma_i}.$$

Here, ι_{γ_i} is the degree-shifting number defined previously.

It is convenient to map the above virtual cycle into $H_*(\overline{\mathscr{W}}_{g,k}^{\mathrm{rig}}, \mathbb{Q})$, even though it is not an element of the latter in any way.

The state space of the theory (or rather its dual) enters in a surprising new way, as we now describe. It turns out that the above virtual cycle *does* depend on the perturbation. It will change whenever W_0 fails to be strongly regular. We observe that for a strongly regular perturbation, we can construct

a canonical system of *horizontal* paths $u_i^{\pm}$'s and the associated Lefschetz thimble $\Delta_i^{\pm}$. When we perturb W_0 crossing the "wall" (where the imaginary parts of critical values happen to be the same), we arrive at another canonical system of paths and its Lefschetz thimble $\Delta_i'^{\pm}$. The relation between $\Delta_i^{\pm}$ and $\Delta_i'^{\pm}$ is determined by the well-known *Picard-Lefschetz formula.* The "wall crossing formula" for virtual fundamental cycles can be summarized in the following *quantum Picard-Lefschetz theorem.* For a more precise statement of this theorem, see [FJR, §6.1, esp. Thm. 6.1.6].

THEOREM 4.1.2. *When W_0 varies, $[\overline{\mathscr{W}}_{g,k}^s(\kappa_{j_1}, \ldots, \kappa_{j_k})]^{\mathrm{vir}}$ transforms in the same way as the Lefschetz thimble $\Delta_{j_i}^-$ attached to the critical point κ_{j_i}.*

The Δ_i^+'s transform in the opposite way as Δ_i^-'s. It is well known that the "diagonal class" $\sum_i \Delta_i^- \otimes \Delta_i^+$ is independent of perturbation, and this suggests the following definition of an "extended virtual class." To simplify the notation, we assume that there is only one marked point with the orbifold decoration γ. Then, the wall crossing formula of $[\overline{\mathscr{W}}_{g,1}^s(\kappa_i)]^{\mathrm{vir}}$ shows precisely that $\sum_j [\overline{\mathscr{W}}_{g,1}^s(\kappa_j)]^{\mathrm{vir}} \otimes \Delta_j^+$, viewed as a class in $H_*(\overline{\mathscr{W}}_{g,1}^{\mathrm{rig}}(\gamma), \mathbb{Q}) \otimes H_{N_\gamma}(\mathbb{C}^{N_\gamma}, W_\gamma^\infty, \mathbb{Q})$, is independent of the perturbation. Now, we define

$$[\overline{\mathscr{W}}_{g,1}^s(\gamma)]^{\mathrm{vir}} := \sum_j [\overline{\mathscr{W}}_{g,1}^s(\kappa_j)]^{\mathrm{vir}} \otimes \Delta_j^+.$$

The above definition can be generalized to multiple marked points in an obvious way so that

$$[\overline{\mathscr{W}}_{g,k}^s(\gamma_1, \ldots, \gamma_k)]^{\mathrm{vir}} \in H_*(\overline{\mathscr{W}}_{g,k}^{\mathrm{rig}}(\gamma_1, \ldots, \gamma_k), \mathbb{Q}) \otimes \prod_i H_{N_{\gamma_i}}(\mathbb{C}^{N_{\gamma_i}}, W_{\gamma_i}^\infty, \mathbb{Q})$$

has degree

$$2\left((c_W - 3)(1 - g) + k - \sum_i \iota_{\gamma_i}\right).$$

COROLLARY 4.1.3. *$[\overline{\mathscr{W}}_{g,k}^s(\gamma_1, \ldots, \gamma_k)]^{\mathrm{vir}}$ is independent of the perturbation W_0.*

Of course, W_0 is only part of the perturbation data. Eventually, we want to work on the stack $\overline{\mathscr{W}}_{g,k}$. It is known that the map so : $\overline{\mathscr{W}}_{g,k}^{\mathrm{rig}} \to \overline{\mathscr{W}}_{g,k}$, defined by forgetting all the rigidifications, is quasi-finite and proper, so we can define

$$(51) \qquad [\overline{\mathscr{W}}_{g,k}(\gamma_1, \ldots, \gamma_k)]^{\mathrm{vir}} := \frac{(-1)^D}{\deg(\mathrm{so})}(\mathrm{so})_*[\overline{\mathscr{W}}_{g,k}^s(\gamma_1, \ldots, \gamma_k)]^{\mathrm{vir}},$$

where $-D$ is the sum of the indices of the W-structure bundles:

$$(52) \qquad D := -\sum_{i=1}^N \mathrm{index}(\mathscr{L}_i) = \hat{c}_W(g - 1) + \sum_{j=1}^k \iota_{\gamma_j}.$$

Remark 4.1.4. The sign $(-1)^D$ is put here to match the older definition in the r-spin case.

The fact that the above virtual cycle is independent of the rigidification implies that

$$[\overline{\mathscr{W}}_{g,k}(\gamma_1,\ldots,\gamma_k)]^{\mathrm{vir}} \in H_*(\overline{\mathscr{W}}_{g,k}(\gamma_1,\ldots,\gamma_k),\mathbb{Q}) \otimes \prod_i H_{N_{\gamma_i}}(\mathbb{C}^{N_{\gamma_i}}, W_{\gamma_i}^\infty, \mathbb{Q})^{G_W}.$$

More generally, we have the following definition.

Definition 4.1.5. Let Γ be a decorated stable W-graph (not necessarily connected) with each tail $\tau \in T(\Gamma)$ decorated by an element $\gamma_\tau \in G_W$. Denote by $k := |T(\Gamma)|$ the number of tails of Γ. We define the *virtual cycle*

$$\left[\overline{\mathscr{W}}(\Gamma)\right]^{\mathrm{vir}} \in H_*(\overline{\mathscr{W}}(\Gamma),\mathbb{Q}) \otimes \prod_{\tau \in T(\Gamma)} H_{N_{\gamma_\tau}}(\mathbb{C}^{N_{\gamma_\tau}}, W_{\gamma_\tau}^\infty, \mathbb{Q})^{G_W}$$

as given in equation (51).

When Γ has a single vertex of genus g, k tails, and no edges (i.e, Γ is a corolla), we denote the virtual cycle by $\left[\overline{\mathscr{W}}(\boldsymbol{\gamma})\right]^{\mathrm{vir}}$, where $\boldsymbol{\gamma} := (\gamma_1,\ldots,\gamma_k)$.

4.1.2. *The virtual cycle for admissible subgroups.* We now wish to consider the more general case of admissible subgroups. Recall that G is admissible if $G = G_{\widetilde{W}}$ for some $\widetilde{W} = W + Z$. One can show [CR10, Rem 2.3.11] that the stack $\overline{\mathscr{W}}_{g,k,G} := \overline{\mathscr{W}}_{g,k}(\widetilde{W})$ is independent of the choice of Z, provided $G = G_{\widetilde{W}}$.

Denote by adm and adm$^{\mathrm{rig}}$ the natural morphisms of stacks

$$\mathrm{adm} : \overline{\mathscr{W}}_{g,k,G} = \overline{\mathscr{W}}_{g,k}(\widetilde{W}_t) \longrightarrow \overline{\mathscr{W}}_{g,k}(W)$$

and

$$\mathrm{adm}^{\mathrm{rig}} : \overline{\mathscr{W}}_{g,k,G}^{\mathrm{rig}} = \overline{\mathscr{W}}_{g,k}(\widetilde{W}_t) \longrightarrow \overline{\mathscr{W}}_{g,k}^{\mathrm{rig}}(W),$$

respectively. And denote by so$_G$ the restriction of so to $\overline{\mathscr{W}}_{g,k,G}^{\mathrm{rig}}$

Definition 4.1.6. Define

$$[\overline{\mathscr{W}}_{g,k,G}^{\mathrm{rig}}(W;\boldsymbol{\gamma})]^{\mathrm{vir}} := \mathrm{adm}^{rig,*}\left([\overline{\mathscr{W}}_{g,k}^{\mathrm{rig}}(W;\boldsymbol{\gamma})]^{\mathrm{vir}}\right)$$

$$\in H_*(\overline{\mathscr{W}}_{g,k,G}^{\mathrm{rig}}(W;\gamma_1,\ldots,\gamma_k),\mathbb{Q}) \otimes \prod_i H_{N_{\gamma_i}}(\mathbb{C}^{N_{\gamma_i}}, W_{\gamma_i}^\infty, \mathbb{Q})$$

and

$$[\overline{\mathscr{W}}_{g,k,G}(W;\gamma_1,\ldots,\gamma_k)]^{\mathrm{vir}} := \frac{(-1)^D}{\deg(\mathrm{so}_G)}(\mathrm{so}_G)_*[\overline{\mathscr{W}}_{g,k,G}^{\mathrm{rig}}(W;\gamma_1,\ldots,\gamma_k)]^{\mathrm{vir}}$$

so that

$$[\overline{\mathscr{W}}_{g,k,G}(W;\gamma_1,\ldots,\gamma_k)]^{\mathrm{vir}} \in H_*(\overline{\mathscr{W}}_{g,k,G}(W;\gamma_1,\ldots,\gamma_k),\mathbb{Q})$$
$$\otimes \prod_i H_{N_{\gamma_i}}(\mathbb{C}^{N_{\gamma_i}}, W^{\infty}_{\gamma_i}, \mathbb{Q})^G.$$

On the other hand, for any $\widetilde{W}$ with $G_{\widetilde{W}} = G$, one may consider the virtual cycle

$$[\overline{\mathscr{W}}^{\mathrm{rig}}_{g,k}(\widetilde{W};\boldsymbol{\gamma})]^{\mathrm{vir}} := \mathrm{adm}^{rig,*}\left([\overline{\mathscr{W}}^{\mathrm{rig}}_{g,k}(W;\boldsymbol{\gamma})]^{\mathrm{vir}}\right)$$
$$\in H_*(\overline{\mathscr{W}}^{\mathrm{rig}}_{g,k}(\widetilde{W};\gamma_1,\ldots,\gamma_k),\mathbb{Q}) \otimes \prod_i H_{N_{\gamma_i}}(\mathbb{C}^{N_{\gamma_i}}, \widetilde{W}^{\infty}_{\gamma_i}, \mathbb{Q})$$

and the pushforward

$$[\overline{\mathscr{W}}_{g,k}(\widetilde{W};\gamma_1,\ldots,\gamma_k)]^{\mathrm{vir}} := \frac{(-1)^D}{\deg(\mathrm{so}_G)}(\mathrm{so}_G)_*[\overline{\mathscr{W}}^{\mathrm{rig}}_{g,k,G}(\widetilde{W};\gamma_1,\ldots,\gamma_k)]^{\mathrm{vir}}$$

in $H_*(\overline{\mathscr{W}}_{g,k}(\widetilde{W};\gamma_1,\ldots,\gamma_k),\mathbb{Q}) \otimes \prod_i H_{N_{\gamma_i}}(\mathbb{C}^{N_{\gamma_i}}, \widetilde{W}^{\infty}_{\gamma_i}, \mathbb{Q})^G$.

Note that we have a canonical isomorphism of G-representations:

$$H_{N_\gamma}\left(\mathbb{C}^{N_\gamma}, \left(\widetilde{W}\right)^{\infty}_{\gamma}, \mathbb{Q}\right) = H_{N_\gamma}\left(\mathbb{C}^{N_\gamma}, W^{\infty}_{\gamma}, \mathbb{Q}\right).$$

PROPOSITION 4.1.7. *The cycles* $[\overline{\mathscr{W}}^{\mathrm{rig}}_{g,k}(\widetilde{W};\boldsymbol{\gamma})]^{\mathrm{vir}}$ *and* $[\overline{\mathscr{W}}^{\mathrm{rig}}_{g,k,G}(W;\boldsymbol{\gamma})]^{\mathrm{vir}}$ *are equal in* $H_*(\overline{\mathscr{W}}^{\mathrm{rig}}_{g,k}(\widetilde{W};\gamma_1,\ldots,\gamma_k),\mathbb{Q}) \otimes \prod_i H_{N_{\gamma_i}}(\mathbb{C}^{N_{\gamma_i}}, \widetilde{W}^{\infty}_{\gamma_i}, \mathbb{Q})$, *and thus the pushforwards also agree:*

$$[\overline{\mathscr{W}}_{g,k,G}(W;\gamma_1,\ldots,\gamma_k)]^{\mathrm{vir}} = [\overline{\mathscr{W}}_{g,k}(\widetilde{W};\gamma_1,\ldots,\gamma_k)]^{\mathrm{vir}}.$$

Proof. This follows from the deformation invariance axiom of [FJR, Thm. 6.2.1(9)]. Namely, if we let $t \in [0,1] \subset \mathbb{R}$ be a parameter and let $\widetilde{W}_t$ denote the family of quasi-homogeneous polynomials, then

$$\widetilde{W}_t := W + tZ.$$

Since W is nondegenerate, so is $\widetilde{W}_t$ for every $t \in [0,1]$. The definition of the stack $\overline{\mathscr{W}}_{g,k}(\widetilde{W}_t)$ is independent of t, provided $t \neq 0$, and for notational convenience, we also define $\overline{\mathscr{W}}_{g,k}(\widetilde{W}_0)$ to be equal to $\overline{\mathscr{W}}_{g,k}(\widetilde{W}_{t\neq0})$. It is clear that the cycles $[\overline{\mathscr{W}}^{\mathrm{rig}}_{g,k}(\widetilde{W}_0;\boldsymbol{\gamma})]^{\mathrm{vir}}$ and $[\overline{\mathscr{W}}^{\mathrm{rig}}_{g,k,G}(W;\boldsymbol{\gamma})]^{\mathrm{vir}}$ are equal, and the deformation invariance axiom of [FJR, Thm. 6.2.1(9)] shows that for all $t \in [0,1]$, the cycles $[\overline{\mathscr{W}}^{\mathrm{rig}}_{g,k}(\widetilde{W}_t;\boldsymbol{\gamma})]^{\mathrm{vir}}$ are all equal. $\square$

The following theorem now follows immediately from [FJR, Thms. 1.2.5 and 6.2.1].

THEOREM 4.1.8. *For any admissible group G and any W-graph Γ, the following axioms are satisfied for* $\left[\overline{\mathscr{W}}(\Gamma)\right]^{\mathrm{vir}}$:

(1) **Dimension**: If D is not an integer, then $\left[\overline{\mathscr{W}}(\Gamma)\right]^{\mathrm{vir}} = 0$. Otherwise, the cycle $\left[\overline{\mathscr{W}}(\Gamma)\right]^{\mathrm{vir}}$ has degree

(53)
$$6g - 6 + 2k - 2\#E(\Gamma) - 2D = 2\left((\hat{c} - 3)(1 - g) + k - \#E(\Gamma) - \sum_{\tau \in T(\Gamma)} \iota_\tau\right).$$

So the cycle lies in $H_r(\overline{\mathscr{W}}(\Gamma), \mathbb{Q}) \otimes \prod_{\tau \in T(\Gamma)} H_{N_{\gamma_\tau}}(\mathbb{C}^{N_{\gamma_\tau}}, W_{\gamma_\tau}^\infty, \mathbb{Q})$, where

$$r := 6g - 6 + 2k - 2\#E(\Gamma) - 2D - \sum_{\tau \in T(\Gamma)} N_{\gamma_\tau}$$

$$= 2\left((\hat{c} - 3)(1 - g) + k - \#E(\Gamma) - \sum_{\tau \in T(\Gamma)} \iota(\gamma_\tau) - \sum_{\tau \in T(\Gamma)} \frac{N_{\gamma_\tau}}{2}\right).$$

(2) **Symmetric group invariance**: There is a natural S_k-action on $\overline{\mathscr{W}}_{g,k}$ obtained by permuting the tails. This action induces an action on homology. That is, for any $\sigma \in S_k$, we have

$$\sigma_* : H_*(\overline{\mathscr{W}}_{g,k}, \mathbb{Q}) \otimes \prod_i H_{N_{\gamma_i}}(\mathbb{C}^{N_{\gamma_i}}, W_{\gamma_i}^\infty, \mathbb{Q})^G$$

$$\longrightarrow H_*(\overline{\mathscr{W}}_{g,k}, \mathbb{Q}) \otimes \prod_i H_{N_{\gamma_{\sigma(i)}}}(\mathbb{C}^{N_{\gamma_{\sigma(i)}}}, W_{\gamma_{\sigma(i)}}^\infty, \mathbb{Q})^G.$$

For any decorated graph Γ, let $\sigma\Gamma$ denote the graph obtained by applying σ to the tails of Γ.

We have

(54)
$$\sigma_* \left[\overline{\mathscr{W}}(\Gamma)\right]^{\mathrm{vir}} = \left[\overline{\mathscr{W}}(\sigma\Gamma)\right]^{\mathrm{vir}}.$$

(3) **Degenerating connected graphs**: Let Γ be a connected, genus-g, stable, decorated W-graph.

The cycles $\left[\overline{\mathscr{W}}(\Gamma)\right]^{\mathrm{vir}}$ and $\left[\overline{\mathscr{W}}_{g,k}(\boldsymbol{\gamma})\right]^{\mathrm{vir}}$ are related by

(55)
$$\left[\overline{\mathscr{W}}(\Gamma)\right]^{\mathrm{vir}} = \tilde{i}^* \left[\overline{\mathscr{W}}_{g,k}(\boldsymbol{\gamma})\right]^{\mathrm{vir}},$$

where $\tilde{i} : \overline{\mathscr{W}}(\Gamma) \longrightarrow \overline{\mathscr{W}}_{g,k}(\boldsymbol{\gamma})$ is the canonical inclusion map.

(4) **Disconnected graphs**: Let $\Gamma = \coprod_i \Gamma_i$ be a stable, decorated W-graph that is the disjoint union of connected W-graphs Γ_i. The classes $\left[\overline{\mathscr{W}}(\Gamma)\right]^{\mathrm{vir}}$ and $\left[\overline{\mathscr{W}}(\Gamma_i)\right]^{\mathrm{vir}}$ are related by

(56)
$$\left[\overline{\mathscr{W}}(\Gamma)\right]^{\mathrm{vir}} = \left[\overline{\mathscr{W}}(\Gamma_1)\right]^{\mathrm{vir}} \times \cdots \times \left[\overline{\mathscr{W}}(\Gamma_d)\right]^{\mathrm{vir}}.$$

(5) **Topological Euler class for the narrow sector**: Suppose that all the decorations on tails of Γ are narrow, meaning that $\mathbb{C}^{N_{\gamma_i}} = \{0\}$, and so we can omit $H_{N_{\gamma_i}}(\mathbb{C}^{N_{\gamma_i}}, W_{\gamma_i}^\infty, \mathbb{Q}) = \mathbb{Q}$ from our notation.

Consider the universal W-structure $(\mathscr{L}_1, \ldots, \mathscr{L}_N)$ on the universal curve $\pi : \mathscr{C} \longrightarrow \overline{\mathscr{W}}(\Gamma)$ and the two-term complex of sheaves

$$\pi_*(|\mathscr{L}_i|) \longrightarrow R^1\pi_*(|\mathscr{L}_i|).$$

There is a family of maps

$$W_i = \frac{\partial W}{\partial x_i} : \pi_*(\bigoplus_j |\mathscr{L}_j|) \longrightarrow \pi_*(K \otimes |\mathscr{L}_i|^*) \cong R^1\pi_*(|\mathscr{L}_i|)^*.$$

The above two-term complex is quasi-isomorphic to a complex of vector bundles [PV01]

$$E_i^0 \xrightarrow{\ d_i\ } E_i^1$$

such that

$$\ker(d_i) \longrightarrow \mathrm{coker}(d_i)$$

is isomorphic to the original two-term complex. W_i is naturally extended (denoted by the same notation) to

$$\bigoplus_i E_i^0 \longrightarrow (E_i^1)^*.$$

Choosing an Hermitian metric on E_i^1 defines an isomorphism $\bar{E}_i^{1*} \cong E_i^1$. Define the Witten map to be the following:

$$\mathscr{D} = \bigoplus(d_i + \bar{W}_i) : \bigoplus_i E_i^0 \longrightarrow \bigoplus_i \bar{E}_i^{1*} \cong \bigoplus_i E_i^1.$$

Let $\pi_j : \bigoplus_i E_i^j \longrightarrow \overline{\mathscr{M}}$ be the projection map. The Witten map defines a proper section (also denoted $\mathscr{D}$) $\mathscr{D} : \bigoplus_i E_i^0 \longrightarrow \pi_0^* \bigoplus_i E_i^1$ of the bundle $\pi_0^* \bigoplus_i E_i^1$ over $\bigoplus_i E_i^0$. The above data defines a topological Euler class $e\left(\mathscr{D} : \bigoplus_i E_i^0 \longrightarrow \pi_0^* \bigoplus_i E_i^1\right)$. Then,

$$[\overline{\mathscr{W}}_\Gamma]^{\mathrm{vir}} = (-1)^D e\left(\mathscr{D} : \bigoplus_i E_i^0 \longrightarrow \pi_0^* \bigoplus_i E_i^1\right) \cap [\overline{\mathscr{M}}_\Gamma].$$

The above axiom implies two subcases.
(a) **Concavity:**[†] Suppose that all tails of Γ are narrow. If $\pi_*\left(\bigoplus_{i=1}^t \mathscr{L}_i\right) = 0$, then the virtual cycle is given by capping the top Chern class of the

---

[†]This axiom was called *convexity* in [JKV01] because the original form of the construction outlined by Witten in the A_{r-1} case involved the Serre dual of $\mathscr{L}$, which is convex precisely when our $\mathscr{L}$ is concave.

dual $\left(R^1\pi_*\left(\bigoplus_{i=1}^t \mathscr{L}_i\right)\right)^*$ of the pushforward with the usual fundamental cycle of the moduli space:

$$
(57) \qquad \left[\overline{\mathscr{W}}(\Gamma)\right]^{\mathrm{vir}} = c_{\mathrm{top}}\left(\left(R^1\pi_*\bigoplus_{i=1}^t \mathscr{L}_i\right)^*\right) \cap \left[\overline{\mathscr{W}}(\Gamma)\right]
$$

$$
= (-1)^D c_D\left(R^1\pi_*\bigoplus_{i=1}^t \mathscr{L}_i\right) \cap \left[\overline{\mathscr{W}}(\Gamma)\right].
$$

(b) **Index zero:** *Suppose that* $\dim(\overline{\mathscr{W}}(\Gamma)) = 0$ *and all the decorations on tails are narrow. If the pushforwards* $\pi_*\left(\bigoplus \mathscr{L}_i\right)$ *and* $R^1\pi_*\left(\bigoplus \mathscr{L}_i\right)$ *are both vector bundles of the same rank, then the virtual cycle is just the degree* $\deg(\mathscr{D})$ *of the Witten map times the fundamental cycle:*

$$
\left[\overline{\mathscr{W}}(\Gamma)\right]^{\mathrm{vir}} = \deg(\mathscr{D})\left[\overline{\mathscr{W}}(\Gamma)\right],
$$

(6) ***Composition law**: Given any genus-g decorated stable W-graph Γ with k tails, and given any edge e of Γ, let $\widehat{\Gamma}$ denote the graph obtained by "cutting" the edge e and replacing it with two unjoined tails τ_+ and τ_- decorated with γ_+ and γ_-, respectively.*

The fiber product

$$
F := \overline{\mathscr{W}}(\widehat{\Gamma}) \times_{\overline{\mathscr{W}}(\Gamma)} \overline{\mathscr{W}}(\Gamma)
$$

has morphisms

$$
\overline{\mathscr{W}}(\widehat{\Gamma}) \xleftarrow{\ q\ } F \xrightarrow{\ \mathrm{pr}_2\ } \overline{\mathscr{W}}(\Gamma).
$$

We have

$$
(58) \qquad \left\langle\left[\overline{\mathscr{W}}(\widehat{\Gamma})\right]^{\mathrm{vir}}\right\rangle_{\pm} = \frac{1}{\deg(q)} q_*\mathrm{pr}_2^*\left(\left[\overline{\mathscr{W}}(\Gamma)\right]^{\mathrm{vir}}\right),
$$

where $\langle\,\rangle_{\pm}$ *is the map from*

$$
H_*(\overline{\mathscr{W}}(\widehat{\Gamma})) \otimes \prod_{\tau \in T(\Gamma)} H_{N_{\gamma_\tau}}(\mathbb{C}^{N_{\gamma_\tau}}, W^\infty_{\gamma_\tau}, \mathbb{Q})^G \otimes H_{N_{\gamma_+}}(\mathbb{C}^{N_{\gamma_+}}, W^\infty_{\gamma_+}, \mathbb{Q})^G
$$

$$
\otimes H_{N_{\gamma_-}}(\mathbb{C}^{N_{\gamma_-}}, W^\infty_{\gamma_-}, \mathbb{Q})^G
$$

to

$$
H_*(\overline{\mathscr{W}}(\widehat{\Gamma})) \otimes \prod_{\tau \in T(\Gamma)} H_{N_{\gamma_\tau}}(\mathbb{C}^{N_{\gamma_\tau}}, W^\infty_{\gamma_\tau}, \mathbb{Q})^G
$$

obtained by contracting the last two factors via the pairing

$$
\langle\,,\rangle : H_{N_{\gamma_+}}(\mathbb{C}^{N_{\gamma_+}}, W^\infty_{\gamma_+}, \mathbb{Q})^G \otimes H_{N_{\gamma_-}}(\mathbb{C}^{N_{\gamma_-}}, W^\infty_{\gamma_-}, \mathbb{Q})^G \longrightarrow \mathbb{Q}.
$$

(7) **Forgetting tails**:

(a) Let Γ have its i-th tail decorated with J, where J is the exponential grading element of G. Further, let Γ' be the decorated W-graph obtained from Γ by forgetting the i-th tail and its decoration. Assume that Γ' is stable, and denote the forgetting tails morphism by

$$\vartheta : \overline{\mathscr{W}}(\Gamma) \longrightarrow \overline{\mathscr{W}}(\Gamma').$$

We have

$$(59) \qquad \left[\overline{\mathscr{W}}(\Gamma)\right]^{\mathrm{vir}} = \vartheta^* \left[\overline{\mathscr{W}}(\Gamma')\right]^{\mathrm{vir}}.$$

(b) In the case of $g = 0$ and $k = 3$, the space $\overline{\mathscr{W}}(\gamma_1, \gamma_2, J)$ is empty if $\gamma_1 \gamma_2 \neq 1$ and $\overline{\mathscr{W}}_{0,3}(\gamma, \gamma^{-1}, J) = \mathscr{B}G_W$. We omit $H_{N_J}(\mathbb{C}^{N_J}, W_J^\infty, \mathbb{Q})^{G_W} = \mathbb{Q}$ from the notation. In this case, the cycle

$$\left[\overline{\mathscr{W}}_{0,3}(\gamma, \gamma^{-1}, J)\right]^{\mathrm{vir}} \in H_*(\mathscr{B}G_W, \mathbb{Q}) \otimes H_{N_\gamma}(\mathbb{C}^{N_\gamma}, W_\gamma^\infty, \mathbb{Q})^G$$
$$\otimes H_{N_{\gamma^{-1}}}(\mathbb{C}^{N_{\gamma^{-1}}}, W_{\gamma^{-1}}^\infty, \mathbb{Q})^G$$

is the fundamental cycle of $\mathscr{B}G_W$ times the Casimir element. Here the Casimir element is defined as follows. Choose a basis $\{\alpha_i\}$ of $H_{N_\gamma}(\mathbb{C}^{N_\gamma}, W_\gamma^\infty, \mathbb{Q})^G$ and a basis $\{\beta_j\}$ of $H_{N_{\gamma^{-1}}}(\mathbb{C}^{N_{\gamma^{-1}}}, W_{\gamma^{-1}}^\infty, \mathbb{Q})^G$. Let $\eta_{ij} = \langle \alpha_i, \beta_j \rangle$ and (η^{ij}) be the inverse matrix of (η_{ij}). The Casimir element is defined as $\sum_{ij} \alpha_i \eta^{ij} \otimes \beta_j$.

(8) **Sums of singularities**: If $W_1 \in \mathbb{C}[z_1, \ldots, z_t]$ and $W_2 \in C[z_{t+1}, \ldots, z_{t+t'}]$ are two quasi-homogeneous polynomials with admissible groups G_1 and G_2, respectively, then $G_1 \times G_2$ is an admissible group of automorphisms of $W_1 + W_2$ whose state space $\mathscr{H}_{W_1+W_2,G_1\times G_2}$ is naturally isomorphic to the tensor product

$$(60) \qquad \mathscr{H}_{W_1+W_2,G_1\times G_2} = \mathscr{H}_{W_1,G_1} \otimes \mathscr{H}_{W_2,G_2},$$

and the stack $\overline{\mathscr{W}}_{g,k,G_1\times G_2}$ has a natural map to the fiber product

$$\overline{\mathscr{W}}_{g,k,G_1\times G_2}(W_1 + W_2) \xrightarrow{\omega} \overline{\mathscr{W}}_{g,k,G_1}(W_1) \times_{\overline{\mathscr{M}}_{g,k}} \overline{\mathscr{W}}_{g,k,G_2}(W_2).$$

Indeed, since any $G_1 \times G_2$-decorated stable graph Γ induces a G_1-decorated graph Γ_1 and G_2-decorated graph Γ_2 with the same underlying graph $\overline{\Gamma}$, we have

$$(61) \qquad \overline{\mathscr{W}}(W_1 + W_2, \Gamma) \xrightarrow{\omega} \overline{\mathscr{W}}(W_1, \Gamma_1) \times_{\overline{\mathscr{M}}(\overline{\Gamma})} \overline{\mathscr{W}}(W_2, \Gamma_2).$$

Composing with the natural inclusion

$$\overline{\mathscr{W}}_{g,k,G_1}(W_1) \times_{\overline{\mathscr{M}}_{g,k}} \overline{\mathscr{W}}_{g,k,G_2}(W_2) \xhookrightarrow{\Delta} \overline{\mathscr{W}}_{g,k,G_1}(W_1) \times \overline{\mathscr{W}}_{g,k,G_2}(W_2),$$

and using the isomorphism of middle homology gives a homomorphism

$$\omega^* \circ \Delta^* : \left(H_*(\overline{\mathscr{W}}_{g,k,G_1}(W_1), \mathbb{Q}) \otimes \prod_{i=1}^k H_{N_{\gamma_{i,1}}}(\mathbb{C}^{N_{\gamma_{i,1}}}, (W_1)^\infty_{\gamma_{i,1}}, \mathbb{Q})^{G_1} \right)$$

$$\otimes \left(H_*(\overline{\mathscr{W}}_{g,k,G_2}(W_2), \mathbb{Q}) \otimes \prod_{i=1}^k H_{N_{\gamma_{i,2}}}(\mathbb{C}^{N_{\gamma_{i,2}}}, (W_2)^\infty_{\gamma_{i,2}}, \mathbb{Q})^{G_2} \right)$$

$$\longrightarrow H_*(\overline{\mathscr{W}}_{g,k,G_1 \times G_2}(W_1 + W_2), \mathbb{Q})$$

$$\otimes \prod_{i=1}^k H_{N_{(\gamma_{i,1},\gamma_{i,2})}}(\mathbb{C}^{N_{(\gamma_{i,1},\gamma_{i,2})}}, W^\infty_{(\gamma_{i,1},\gamma_{i,2})}, \mathbb{Q})^{G_1 \times G_2}.$$

The virtual cycle satisfies

$$(62) \qquad \omega^* \circ \Delta^* \left(\left[\overline{\mathscr{W}}_{g,k,G_1}(W_1) \right]^{\mathrm{vir}} \otimes \left[\overline{\mathscr{W}}_{g,k,G_2}(W_2) \right]^{\mathrm{vir}} \right)$$

$$= \left[\overline{\mathscr{W}}_{g,k,G_1 \times G_2}(W_1 + W_2) \right]^{\mathrm{vir}}.$$

(9) **Deformation Invariance**: *Let $W_t \in \mathbb{C}[z_1, \ldots, z_N]$ be a family of nondegenerate quasi-homogeneous polynomials depending smoothly on a parameter $t \in [a,b] \subset \mathbb{R}$. Suppose that G is the common automorphism group of W_t. The corresponding stacks $\overline{\mathscr{W}}(\Gamma_t)$ are all naturally isomorphic. We denote this generic stack by $\overline{\mathscr{W}}(\Gamma)$. The virtual cycle $[\overline{\mathscr{W}}(\Gamma)]^{\mathrm{vir}}$ associated to (W_t, G) is independent of t.*

(10) G_W**-Invariance**: *For any admissible G and any G-decorated graph Γ, the homology $H_*(\overline{\mathscr{W}}_{g,k,G}(\Gamma), \mathbb{Q})$ as well as the homology groups*

$$H_{N_{\gamma_\tau}}(\mathbb{C}^{N_{\gamma_\tau}}, (W)^\infty_{\gamma_\tau}, \mathbb{Q})^G$$

each have a natural G_W-action, which induces a G_W action on

$$H_*(\overline{\mathscr{W}}_{g,k,G}(\Gamma), \mathbb{Q}) \otimes \prod_{\tau \in T(\Gamma)} H_{N_{\gamma_\tau}}(\mathbb{C}^{N_{\gamma_\tau}}, (W)^\infty_{\gamma_\tau}, \mathbb{Q})^G.$$

The virtual cycle $[\overline{\mathscr{W}}(\Gamma)]^{\mathrm{vir}}$ is invariant under this G_W-action.

Remark 4.1.9. In the case of A_{r-1} our virtual cycle can be used to construct an r-spin virtual class in the sense of [JKV01, §4.1]. The details of this construction are given in [FJR11]

Remark 4.1.10. As usual, we can define Gromov-Witten type correlators by integrating tautological classes such as ψ_i and μ_{ij} over the $\left[\overline{\mathscr{W}}_{g,k,G} \right]^{\mathrm{vir}}$.

A direct consequence of the above axioms is the fact that the above correlators defined by ψ_i, together with the rescaled pairing $(\,,\,)_\gamma := \frac{|\langle\gamma\rangle|}{|G|} \langle\,,\,\rangle_\gamma$, satisfy the usual axioms of Gromov-Witten theory (without the divisor axiom) and a modified version of the unit axiom

$$\langle \alpha_1, \alpha_2, \mathbf{e}_J \rangle_0^W = |\langle\gamma\rangle| \langle \alpha_1, \alpha_2 \rangle$$

for $\alpha_1 \in \mathscr{H}_\gamma$ and for $\alpha_2 \in \mathscr{H}_{\gamma^{-1}}$. In this paper, we favor a slightly different version, which we now explain.

4.2. **Cohomological field theory.** One gets a cleaner formula by pushing $\left[\overline{\mathscr{W}}_{g,k,G}(\boldsymbol{\gamma})\right]^{\mathrm{vir}}$ down to $\overline{\mathscr{M}}_{g,k}$.

Definition 4.2.1. Let $\Lambda^W_{g,k} \in \mathrm{Hom}(\mathscr{H}^{\otimes k}_W, H^*(\overline{\mathscr{M}}_{g,k}))$ be given for homogeneous elements $\boldsymbol{\alpha} := (\alpha_1, \ldots, \alpha_k)$ with $\alpha_i \in \mathscr{H}_{\gamma_i}$ by

$$(63) \qquad \Lambda^{W,G}_{g,k}(\boldsymbol{\alpha}) := \frac{|G|^g}{\deg(\mathrm{st})} \mathrm{PD}\,\mathrm{st}_* \left(\left[\overline{\mathscr{W}}_{g,k}(W, \boldsymbol{\gamma})\right]^{\mathrm{vir}} \cap \prod_{i=1}^k \alpha_i\right),$$

and then extend linearly to general elements of $\mathscr{H}^{\otimes k}_{W,G}$. Here, PD is the Poincare duality map.

Let $\mathbf{e}_1 := \mathbf{1}$ be the distinguished generator of $\mathscr{H}_J$, and let $\langle\,,\rangle^{W,G}$ denote the pairing on the state space $\mathscr{H}_{W,G}$.

THEOREM 4.2.2. *The collection $(\mathscr{H}_{W,G}, \langle\,,\rangle^{W,G}, \{\Lambda^{W,G}_{g,k}\}, \mathbf{e}_1)$ is a cohomological field theory with flat identity.*

Moreover, if W_1 and W_2 are two singularities in distinct variables with admissible groups G_1 and G_2, respectively, then the cohomological field theory arising from $W_1 + W_2, G_1 \times G_2$ is the tensor product of the cohomological field theories arising from W_1, G_1 and W_2, G_2:

$$(\mathscr{H}_{W_1+W_2,G_1\times G_2}, \{\Lambda^{W_1+W_2,G_1\times G_2}_{g,k}\})$$
$$= (\mathscr{H}_{W_1,G_1} \otimes \mathscr{H}_{W_2,G_2}, \{\Lambda^{W_1,G_1}_{g,k} \otimes \Lambda^{W_2,G_2}_{g,k}\}).$$

Proof. To show that the classes form a cohomological field theory, we must show that the following properties hold (see, for example, [JKV01, §3.1]):

C1. The element $\Lambda^{W,G}_{g,n}$ is invariant under the action of the symmetric group S_k.
C2. Let $g = g_1 + g_2$; let $k = k_1 + k_2$; and let

$$\rho_{\mathrm{tree}} : \overline{\mathscr{M}}_{g_1,k_1+1} \times \overline{\mathscr{M}}_{g_2,k_2+1} \longrightarrow \overline{\mathscr{M}}_{g,k}$$

be the gluing trees morphism (21). Then the forms $\Lambda^{W,G}_{g,n}$ satisfy the composition property

$$(64) \quad \rho^*_{\mathrm{tree}}\Lambda^{W,G}_{g_1+g_2,k}(\alpha_1, \alpha_2 \ldots, \alpha_k)$$
$$= \sum_{\mu,\nu} \Lambda^{W,G}_{g_1,k_1+1}(\alpha_{i_1}, \ldots, \alpha_{i_{k_1}}, \mu)\, \eta^{\mu\nu} \otimes \Lambda^{W,G}_{g_2,k_2+1}(\nu, \alpha_{i_{k_1+1}}, \ldots, \alpha_{i_{k_1+k_2}})$$

for all $\alpha_i \in \mathscr{H}_W$, where μ and ν run through a basis of $\mathscr{H}_W$, and $\eta^{\mu\nu}$ denotes the inverse of the pairing $\langle\,,\rangle$ with respect to that basis.

C3. Let

$$(65) \qquad \rho_{\text{loop}} : \overline{\mathscr{M}}_{g-1,k+2} \longrightarrow \overline{\mathscr{M}}_{g,k}$$

be the gluing loops morphism (22). Then

$$(66) \qquad \rho_{\text{loop}}^* \Lambda_{g,k}^{W,G}(\alpha_1, \alpha_2, \ldots, \alpha_k) = \sum_{\mu,\nu} \Lambda_{g-1,k+2}^{W,G}(\alpha_1, \alpha_2, \ldots, \alpha_n, \mu, \nu) \, \eta^{\mu\nu},$$

where α_i, μ, ν, and η are as in C2.

C4a. For all α_i in $\mathscr{H}_W$, we have

$$(67) \qquad \Lambda_{g,k+1}^{W,G}(\alpha_1, \ldots, \alpha_k, \mathbf{e}_1) = \vartheta^* \Lambda_{g,k}^{W,G}(\alpha_1, \ldots, \alpha_k),$$

where $\vartheta : \overline{\mathscr{M}}_{g,n+1} \longrightarrow \overline{\mathscr{M}}_{g,n}$ is the universal curve.

C4b.

$$(68) \qquad \int_{\overline{\mathscr{M}}_{0,3}} \Lambda_{0,3}^{W,G}(\alpha_1, \alpha_2, \mathbf{e}_1) = \langle \alpha_1, \alpha_2 \rangle^W.$$

Axiom C1 follows immediately from the symmetric group invariance (Axiom 2) of the virtual cycle.

To prove Axioms C2 and C3 we first need a simple lemma: that the Casimir element is Poincaré dual to the pairing. This is well known, but we include it for completeness because we use it often.

LEMMA 4.2.3. *Let $\alpha_i \in \mathscr{H}_W$ be a basis. Consider the Casimir element $\sum_{ij} \eta^{ij} \alpha_i \otimes \alpha_j$ of its pairing. For any $u, v \in \mathscr{H}_W^*$, we have*

$$\langle u, v \rangle = u \otimes v \cap \sum_{ij} \eta^{ij} \alpha_i \otimes \alpha_j.$$

Proof. Let α_i^* be the dual basis, and let $u := \sum_i \langle u, \alpha_i \rangle \alpha_i^*$ and $v := \sum_j \langle v, \alpha_j \rangle \alpha_j^*$. Therefore,

$$\langle u, v \rangle = \sum_{ij} \langle u, \alpha_i \rangle \langle v, \alpha_j \rangle \langle \alpha_i^*, \alpha_j^* \rangle.$$

Notice that $\eta_{ij} = \langle \alpha_i, \alpha_j \rangle$ and $\eta^{ij} = \langle \alpha_i^*, \alpha_j^* \rangle$. The right-hand side is precisely $u \otimes v \cap \sum_{ij} \eta^{ij} \alpha_i \otimes \alpha_j$. $\qquad \square$

Let $\alpha_i \in \mathscr{H}_{\gamma_i}$, and let Γ denote the W-graph of either the tree (two vertices, of genus g_1 and g_2, respectively, with k_1 and k_2 tails, respectively, and one separating edge) or of the loop (one vertex of genus $g-1$ with k tails and one edge) where the i-th tail is decorated with the group element γ_i.

Let $\widehat{\Gamma}$ denote the "cut" version of the graph Γ. Note that the data given do not determine a decoration of the edge, so Γ and $\widehat{\Gamma}$ are really sums over all choices Γ_ε or $\widehat{\Gamma}_\varepsilon$ decorated with $\varepsilon \in G$ on the edge.

Using the notation of the Composition Axiom (6), we have the following commutative diagram for each ε:

$$(69)$$

$$
\begin{array}{ccc}
& F_\varepsilon & \xrightarrow{\ \mathrm{pr}_2\ } & \overline{\mathscr{W}}(\Gamma_\varepsilon) \\
& \swarrow{\scriptstyle q_\varepsilon} & \downarrow{\scriptstyle \mathrm{pr}_1} & \downarrow{\scriptstyle \mathrm{st}_{\Gamma_\varepsilon}} \\
\overline{\mathscr{W}}(\widehat{\Gamma}_\varepsilon) & & & \\
& \searrow{\scriptstyle \mathrm{st}_{\widehat{\Gamma}_\varepsilon}} & & \\
& & \overline{\mathscr{M}}(\widehat{\Gamma}) \xrightarrow{\ \hat\rho\ } \overline{\mathscr{M}}(\Gamma).
\end{array}
$$

And summing over all $\varepsilon \in G$, we have the following:

$$(70)$$

$$
\begin{array}{ccc}
\displaystyle\bigcup_{\varepsilon \in G} \overline{\mathscr{W}}(\Gamma_\varepsilon) & \xrightarrow{\ \tilde{i}\ } & \overline{\mathscr{W}}_{g,k}(\boldsymbol{\gamma}) \\
\Big\downarrow{\scriptstyle \sum_{\varepsilon \in G}\mathrm{st}_{\Gamma_\varepsilon}} & & \Big\downarrow{\scriptstyle \mathrm{st}} \\
\overline{\mathscr{M}}(\Gamma) & \xrightarrow{\ i\ } & \overline{\mathscr{M}}_{g,k}.
\end{array}
$$

We have $\rho = i \circ \hat\rho$. In the second diagram, note that the square is not Cartesian. In fact, by Propositions 2.2.17 and 2.2.18, it fails to be Cartesian by a factor of $|\langle \varepsilon \rangle|$ on each term.

LEMMA 4.2.4. *For any $\alpha \in H_*(\overline{\mathscr{W}}_{g,k}(\boldsymbol{\gamma}))$, we have the relation*

$$(71) \qquad i^* \mathrm{st}_* \alpha = \sum_{\varepsilon \in G} |\langle \varepsilon \rangle| (\mathrm{st}_{\Gamma_\varepsilon})_* \tilde{i}^* \alpha.$$

COROLLARY 4.2.5. *The virtual fundamental classes pushed down to $\overline{\mathscr{M}}(\Gamma)$ are related by the equality*

$$(72) \qquad i^* \mathrm{st}_* \left[\overline{\mathscr{W}}_{g,k}(\boldsymbol{\gamma})\right]^{\mathrm{vir}} = \sum_{\varepsilon \in G} |\langle \varepsilon \rangle| (\mathrm{st}_{\Gamma_\varepsilon})_* \tilde{i}^* \left[\overline{\mathscr{W}}_{g,k}(\boldsymbol{\gamma})\right]^{\mathrm{vir}}.$$

Proof of Lemma 4.2.4. The orbifold $\bigcup_{\varepsilon \in G} \overline{\mathscr{W}}(\Gamma_\varepsilon)$ is the inverse image $\mathrm{st}^{-1}(\overline{\mathscr{M}}(\Gamma))$. We would like to be able to apply a push-pull/pull-push relation, but st is not transverse to i, so this will not work.

Instead, we deform the map st in a small neighborhood of $\bigcup_{\varepsilon \in G} \overline{\mathscr{W}}(\Gamma_\varepsilon)$, and we deform in a normal direction to get a new map $\widehat{\mathrm{st}}$ that is transverse to i and so that the inverse image $\widehat{\mathrm{st}}^{-1}(\overline{\mathscr{M}}(\Gamma))$ lies in the normal bundle of $\tilde{i}$. So

we have the following diagram:

$$
\begin{array}{ccc}
\widehat{\mathrm{st}}^{-1}(\overline{\mathscr{M}}(\Gamma)) & \xrightarrow[\hat{\tilde{\imath}}]{} & \overline{\mathscr{W}}_{g,k}(\boldsymbol{\gamma}) \\
\downarrow{\widehat{\mathrm{st}}_\Gamma} & & \downarrow{\widehat{\mathrm{st}}} \\
\overline{\mathscr{M}}(\Gamma) & \xrightarrow[i]{} & \overline{\mathscr{M}}_{g,k}.
\end{array}
$$

(73)

For any $\alpha \in H_*(\overline{\mathscr{W}}_{g,k}(\boldsymbol{\gamma}))$, we have $\mathrm{st}_*\alpha = \widehat{\mathrm{st}}_*\alpha$, since $\widehat{\mathrm{st}}$ is a deformation of st. Now the standard push-pull/pull-push relation, which is a special case of the clean intersection formula [Qui71, Prop. 3.3], says that we have

$$
\widehat{\mathrm{st}}_{\Gamma*}\hat{\tilde{\imath}}^*\alpha = i^*\widehat{\mathrm{st}}_*\alpha = i^*\mathrm{st}_*\alpha.
$$

But since $\mathrm{st}^{-1}(\overline{\mathscr{M}}(\Gamma))$ lies in the normal bundle of $\tilde{\imath}$, we can factor the map $\widehat{\mathrm{st}}_\Gamma$ as

$$
\widehat{\mathrm{st}}_\Gamma = \mathrm{st}_\Gamma \circ \mathrm{pr},
$$

where pr is the projection of the normal bundle down to $\bigcup_{\varepsilon \in G}\overline{\mathscr{W}}(\Gamma_\varepsilon)$. Moreover, since $\widehat{\mathrm{st}}$ is a deformation of st, we have that the pullbacks $\hat{\imath}^*a$ and $\mathrm{pr}^*\tilde{\imath}^*a$ are equal. The map pr is finite when restricted to $\mathrm{st}^{-1}(\overline{\mathscr{M}}(\Gamma))$, and for each ε, we denote by $\deg(\mathrm{pr}_\varepsilon)$ its degree over the component $\overline{\mathscr{W}}(\Gamma_\varepsilon)$. Therefore,

$$
\begin{aligned}
i^*\mathrm{st}_*\alpha &= \widehat{\mathrm{st}}_{\Gamma*}\hat{\imath}^*\alpha \\
&= \mathrm{st}_{\Gamma*}\mathrm{pr}_*\mathrm{pr}^*\tilde{\imath}^*\alpha \\
&= \sum_{\varepsilon \in G}\deg(\mathrm{pr}_\varepsilon)\mathrm{st}_{\Gamma_\varepsilon*}\tilde{\imath}^*\alpha.
\end{aligned}
$$

Now, it is easy to see that $\deg(\mathrm{pr}_\varepsilon)$ is equal to the number of nonisomorphic W-curves over a generic smooth curve that degenerate to a given generic nodal W-curve in $\overline{\mathscr{W}}(\Gamma_\varepsilon)$. As described in Propositions 2.2.17 and 2.2.18, after accounting for automorphisms, this number is $|\langle\varepsilon\rangle|$. $\square$

Now, we prove Axioms C2 and C3. To simplify computations, we choose a basis $B := \{\mu_{\gamma,i}\}$ of $\mathscr{H}_W$ with each $\mu_{\gamma,i} \in \mathscr{H}_\gamma$, and we write all the Casimir elements in terms of this basis.

In the case of Axiom C3 (the case that Γ is a loop), we have

$$
\begin{aligned}
&\Lambda_{g-1,k+2}(\alpha_1,\ldots,\alpha_k,\mu,\nu)\eta^{\mu\nu} \\
&= \sum_{\gamma \in G}\frac{|G|^{g-1}}{\deg(\mathrm{st}_{\widehat{\Gamma}_\gamma})}\mathrm{PD}\,(\mathrm{st}_{\widehat{\Gamma}_\gamma})_*\left(\left[\overline{\mathscr{W}}(\widehat{\Gamma}_\gamma)\right]^{\mathrm{vir}} \cap \prod_{i=1}^k \alpha_i \cup \mu_{\gamma,i} \cup \nu_{\gamma,i}\right)\eta^{\mu_{\gamma,i}\nu_{\gamma,i}} \\
&= \sum_{\gamma \in G}\frac{|G|^{g-1}}{\deg(\mathrm{st}_{\widehat{\Gamma}_\gamma})}\mathrm{PD}\,(\mathrm{st}_{\widehat{\Gamma}_\gamma})_*\left(\left\langle\left[\overline{\mathscr{W}}(\widehat{\Gamma}_\gamma)\right]^{\mathrm{vir}}\right\rangle_\pm \cap \prod_{i=1}^k \alpha_i\right)
\end{aligned}
$$

$$= \sum_{\gamma \in G} \frac{|G|^{g-1}}{\deg(\mathrm{st}_{\widehat{\Gamma}_\gamma}) \deg(q_\gamma)} \mathrm{PD}\,(\mathrm{st}_{\widehat{\Gamma}_\gamma})_* \left((q_\gamma)_* \mathrm{pr}_2^* \left[\overline{\mathscr{W}}(\Gamma_\gamma) \right]^{\mathrm{vir}} \cap \prod_{i=1}^k \alpha_i \right)$$

$$= \sum_{\gamma \in G} \frac{|G|^{g-1}}{\deg(\mathrm{pr}_1)} \mathrm{PD}\,(\mathrm{pr}_1)_* \left(\mathrm{pr}_2^* \left[\overline{\mathscr{W}}(\Gamma_\gamma) \right]^{\mathrm{vir}} \cap \prod_{i=1}^k \alpha_i \right)$$

$$= \sum_{\gamma \in G} \frac{|G|^{g-1}}{\deg(\mathrm{st}_{\Gamma_\gamma})} \mathrm{PD}\,\hat{\rho}^*(\mathrm{st}_{\Gamma_\gamma})_* \left(\left[\overline{\mathscr{W}}(\Gamma_\gamma) \right]^{\mathrm{vir}} \cap \prod_{i=1}^k \alpha_i \right)$$

$$= \sum_{\gamma \in G} \frac{|G|^{g-1}}{|G|^{2g-3}|G/\langle\gamma\rangle|} \mathrm{PD}\,\hat{\rho}^*(\mathrm{st}_{\Gamma_\gamma})_* \left(\left[\overline{\mathscr{W}}(\Gamma_\gamma) \right]^{\mathrm{vir}} \cap \prod_{i=1}^k \alpha_i \right)$$

$$= \sum_{\gamma \in G} \frac{|G|^g |\langle\gamma\rangle|}{|G|^{2g-1}} \mathrm{PD}\,\hat{\rho}^*(\mathrm{st}_{\Gamma_\gamma})_* \tilde{i}^* \left(\left[\overline{\mathscr{W}}_{g,k}(\boldsymbol{\gamma}) \right]^{\mathrm{vir}} \cap \prod_{i=1}^k \alpha_i \right)$$

$$= \frac{|G|^g}{\deg(\mathrm{st})} \mathrm{PD}\,\rho^* \mathrm{st}_* \left(\left[\overline{\mathscr{W}}_{g,k}(\boldsymbol{\gamma}) \right]^{\mathrm{vir}} \cap \prod_{i=1}^k \alpha_i \right)$$

$$= \rho^* \Lambda_{g,k}(\alpha_1, \ldots, \alpha_k)$$

The second equality follows from the fact that the Casimir element in cohomology is dual to the pairing in homology. The sixth follows from the explicit computation of $\deg(\mathrm{st}_\Gamma)$ in Proposition 2.2.18 and the seventh from the connected graphs axiom (Axiom 3). The eighth equality follows from equation (72).

The case of Axiom C2 is similar, but simpler, because there is only one choice of decoration γ for the edge of Γ. In this case, we have

$$\Lambda_{g_1,k_1+1}(\alpha_{i_1}, \ldots, \alpha_{i_{k_1}}, \mu) \eta^{\mu\nu} \Lambda_{g_2,k_2+1}(\nu, \alpha_{i_{k_1+1}}, \ldots, \alpha_{i_{k_1+k_2}})$$

$$= \frac{|G|^g}{\deg(\mathrm{st}_{\widehat{\Gamma}})} \mathrm{PD}\,(\mathrm{st}_{\widehat{\Gamma}})_* \left(\left[\overline{\mathscr{W}}(\widehat{\Gamma}) \right]^{\mathrm{vir}} \cap \prod_{i=1}^k \alpha_i \cup \mu \cup \nu \right) \eta^{\mu\nu}$$

$$= \frac{|G|^g}{\deg(\mathrm{st}_{\widehat{\Gamma}})} \mathrm{PD}\,(\mathrm{st}_{\widehat{\Gamma}})_* \left(\left\langle \left[\overline{\mathscr{W}}(\widehat{\Gamma}) \right]^{\mathrm{vir}} \right\rangle_\pm \cap \prod_{i=1}^k \alpha_i \right)$$

$$= \frac{|G|^g}{\deg(\mathrm{st}_{\widehat{\Gamma}}) \deg(q)} \mathrm{PD}\,(\mathrm{st}_{\widehat{\Gamma}})_* \left((q)_* \mathrm{pr}_2^* \left[\overline{\mathscr{W}}(\Gamma) \right]^{\mathrm{vir}} \cap \prod_{i=1}^k \alpha_i \right)$$

$$= \frac{|G|^g}{\deg(\mathrm{pr}_1)} \mathrm{PD}\,(\mathrm{pr}_1)_* \left(\mathrm{pr}_2^* \left[\overline{\mathscr{W}}(\Gamma) \right]^{\mathrm{vir}} \cap \prod_{i=1}^k \alpha_i \right)$$

$$= \frac{|G|^g}{\deg(\mathrm{st}_\Gamma)} \mathrm{PD}\,\rho^*(\mathrm{st}_\Gamma)_* \left(\left[\overline{\mathscr{W}}(\Gamma) \right]^{\mathrm{vir}} \cap \prod_{i=1}^k \alpha_i \right)$$

$$= \frac{|G|^g}{\deg(\mathrm{st})/|\langle\gamma\rangle|} \mathrm{PD}\,\rho^*(\mathrm{st}_\Gamma)_* \left(\left[\overline{\mathscr{W}}(\Gamma) \right]^{\mathrm{vir}} \cap \prod_{i=1}^k \alpha_i \right)$$

$$= \frac{|G|^g |\langle \gamma \rangle|}{\deg(\mathrm{st})} \mathrm{PD}\, \rho^*(\mathrm{st}_\Gamma)_* \left(\left[\overline{\mathscr{W}}(\Gamma) \right]^{\mathrm{vir}} \cap \prod_{i=1}^{k} \alpha_i \right)$$

$$= \rho^* \Lambda_{g,k}(\alpha_1, \ldots, \alpha_k).$$

Axiom C4a and Axiom C4b follow immediately from the forgetting tails axiom. $\qquad \square$

Definition 4.2.6. Define correlators

$$\langle \tau_{l_1}(\alpha_1), \ldots, \tau_{l_k}(\alpha_k) \rangle_g^{W,G} := \int_{\left[\overline{\mathscr{M}}_{g,k} \right]} \Lambda_{g,k}^{W,G}(\alpha_1, \ldots, \alpha_k) \prod_{i=1}^{k} \psi_i^{l_i}.$$

Definition 4.2.7. Let $\{\alpha_0, \ldots, \alpha_s\}$ be a basis of the state space $\mathscr{H}_W$ such that $\alpha_0 = \mathbf{1}$, and let $\mathbf{t} = (\mathbf{t}_0, \mathbf{t}_1, \ldots)$ with $\mathbf{t}_l = (t_l^{\alpha_0}, t_l^{\alpha_1}, \ldots, t_l^{\alpha_s})$ be formal variables. Denote by $\Phi^{W,G}(\mathbf{t}) \in \lambda^{-2}\mathbb{C}[[\mathbf{t}, \lambda]]$ the (large phase space) potential of the theory:

$$\Phi^{W,G}(\mathbf{t}) := \sum_{g \geq 0} \Phi_g^{W,G}(\mathbf{t})$$

$$:= \sum_{g \geq 0} \lambda^{2g-2} \sum_k \frac{1}{k!} \sum_{l_1, \ldots, l_k} \sum_{\alpha_1, \ldots, \alpha_k} \langle \tau_{l_1}(\alpha_1) \cdots \tau_{l_k}(\alpha_k) \rangle_g^{W,G} t_{l_1}^{\alpha_1} \cdots t_{l_k}^{\alpha_k}.$$

In [Man99, Thm. III.4.3], Manin shows that a cohomological field theory in genus zero is equivalent to a formal Frobenius manifold.

COROLLARY 4.2.8. *The genus-zero theory defines a formal Frobenius manifold structure on* $\mathbb{Q}[[\mathscr{H}_{W,G}^*]]$ *with pairing* $\langle \, , \rangle^{W,G}$ *and (large phase space) potential* $\Phi_0^{W,G}(\mathbf{t})$.

Three very important constraints are the *string* and *dilaton* equations and the *topological recursion relations* (see [Man99, §VI.5.2] and [JKV01, §5.2]).

THEOREM 4.2.9. *The potential* $\Phi^{W,G}(\mathbf{t})$ *satisfies analogues of the string and dilaton equations and the topological recursion relations.*

Proof. Let $\vartheta : \overline{\mathscr{M}}_{g,k+1} \longrightarrow \overline{\mathscr{M}}_{g,k}$ denote the universal curve, and let $D_{i,k+1}$ denote the class of the image of the i-th section in $\overline{\mathscr{M}}_{g,k}$.

The dilaton and string equations for $\Phi^{W,G}$ follow directly from the forgetting tails axiom and from fact that the gravitational descendants ψ_i satisfy $\vartheta^*(\psi_i) = \psi_i + D_{i,k+1}$.

The topological recursion relations hold because of the relation

$$\psi_i = \sum_{\substack{T_+ \sqcup T_- = [k] \\ k, k-1 \in T^+ \\ i \in T_-}} \delta_{0;T_+}$$

on $\overline{\mathscr{M}}_{0,k}$, where $\delta_{0;T_+}$ is the boundary divisor in $\overline{\mathscr{M}}_{0,k}$ corresponding to a graph with a single edge and one vertex labeled by tails in T_+. $\square$

For more details about these equations in the A_n case, see [JKV01, §5.2].

5. ADE-singularities and mirror symmetry

The construction of this paper corresponds to the A-model of the Landau-Ginzburg model. A particular invariant from our theory is the ring $\mathscr{H}_{W,G}$. The Milnor ring, or local algebra, $\mathscr{Q}_W$ of a singularity can be considered as the B-model. One outstanding conjecture of Witten is the self-mirror phenomenon for ADE-singularities. This conjecture states that for any simple (i.e., ADE) singularity W, the ring $\mathscr{H}_{W,\langle J \rangle}$ is isomorphic, as a Frobenius algebra, to the Milnor ring $\mathscr{Q}_W$ of the same singularity.

This is the main topic of this section. More precisely, we prove the following theorem, which resolves the conjecture and serves as the first step toward the proof of the integrable hierarchy theorems in the next section.

THEOREM 5.0.10 (Theorem 1.0.7). (1) *Except for D_n with n odd, the ring $\mathscr{H}_{W,\langle J \rangle}$ of any simple (ADE) singularity W with symmetry group $\langle J \rangle$ is isomorphic, as a Frobenius algebra, to the Milnor ring $\mathscr{Q}_W$ of the same singularity.*

(2) *The ring $\mathscr{H}_{D_n,G_{D_n}}$ of D_n with the maximal diagonal symmetry group G_{D_n} is isomorphic, as a Frobenius algebra, to the Milnor ring $\mathscr{Q}_{x^{n-1}y+y^2} \cong \mathscr{Q}_{A_{2n-3}}$.*

(3) *The ring $\mathscr{H}_{W,G_W}$ of $W = x^{n-1}y + y^2$ ($n \geq 4$) with the maximal diagonal symmetry group is isomorphic, as a Frobenius algebra, to the Milnor ring $\mathscr{Q}_{D_n}$ of D_n.*

Note that the self-mirror conjecture is not quite correct. In particular, in the case of D_n for n odd, the maximal symmetry group is generated by J, but the ring $\mathscr{H}_{W,G_W} = \mathscr{H}_{W,\langle J \rangle}$ is not isomorphic to $\mathscr{Q}_{D_n}$. Instead it is isomorphic to the Milnor ring of the singularity $\widetilde{W} := x^{n-1}y + y^2$, and conversely, the ring $\mathscr{H}_{W',G_{W'}}$ is isomorphic to the Milnor ring $\mathscr{Q}_{D_n}$, so, in fact, the mirror of D_n is $W' = x^{n-1}y + y^2$.

This is a special case of the construction of Berglund and Hübsch [BH93] for invertible singularities. Specifically, consider a singularity W of the form

$$W = \sum_{i=1}^{N} W_j \quad \text{with} \quad W_j = \prod_{l=1}^{N} x_\ell^{b_{\ell,j}}$$

and with $b_{\ell,j} \in \mathbb{Z}^{\geq 0}$. As we did in the proof of Lemma 2.1.8, we form the $N \times N$ matrix $B := (b_{\ell j})$. Berglund and Hübsch conjectured that the mirror

partner to W should be the singularity corresponding to B^T; that is,

$$W^T := \sum_{\ell=1}^{N} W_\ell^T, \quad \text{where} \quad W_\ell^T = \prod_{j=1}^{N} x_j^{b_{\ell,j}}.$$

Using this construction, we find that the mirror partner to $D_n = x^{n-1} + xy^2$ should be the singularity $D_n^T = x^{n-1}y + y^2$. This singularity is isomorphic to A_{2n-3}, so the Milnor ring of W is isomorphic to the Milnor ring of A_{2n-3}. But this isomorphism of singularities does not give an isomorphism of A-model theories. Indeed, Theorem 5.0.10 shows that the ring $\mathscr{H}_{D_n^T, G_{D_n^T}}$ of D_n^T is not isomorphic to the ring $\mathscr{H}_{A_{2n-3}, G_{A_{2n-3}}}$, but rather it it isomorphic to $\mathscr{Q}_{D_n}$.

The Berglund and Hübsch construction also explains the self-duality of A_n and $E_{6,7,8}$. In addition, their elegant construction opens a door to the further development of the subject of Landau-Ginzburg mirror symmetry. Since the initial post of this article in 2007, much progress on Landau-Ginzburg mirror symmetry has been made by Krawitz and his collaborators [Kra10], [KPA$^+$10].

We note that Kaufmann [Kau06], [Kau03], [Kau02] has made a computation for a different, algebraic construction of an "orbifolded Landau-Ginzburg model" which gives mirror symmetry results that match the results of Theorem 5.0.10. In particular, in his theory, just as in ours, the D_n case for n odd is also not self-dual, but rather is mirror dual to D_n^T.

5.1. **Relation between $\mathscr{Q}_W$ and $H^N(\mathbb{C}^N, W^\infty, \mathbb{C})$.** As we mentioned earlier, the Milnor ring $\mathscr{Q}_W$ represents a B-model structure. In order to obtain the correct action, we consider $\mathscr{Q}_W \omega$, where $\omega = dx_1 \wedge \cdots \wedge dx_N$. Here an element of $\mathscr{Q}_W \omega$ is of the form $\phi \omega$, where $\phi \in \mathscr{Q}_W$ and $\gamma \in G_W$ acts on both ϕ and ω. The A-model analogy is the relative cohomology groups $H^N(\mathbb{C}^N, W^\infty, \mathbb{C})$. It was an old theorem of Wall [Wal80a], [Wal80b] that they are isomorphic as G_W-spaces. Wall's theorem could almost be viewed as a sort of mirror symmetry theorem itself.

An "honest" mirror symmetry theorem should exchange the A-model for one singularity with the B-model for a *different* singularity. However, it is technically convenient for us to use Wall's isomorphism to label the class of $H^N(\mathbb{C}^N, W^\infty, \mathbb{C})$. For the A-model state space, we need to consider $H^N(\mathbb{C}^N, W^\infty, \mathbb{C})^{\langle J \rangle}$ with the intersection pairing. It is well known that Wall's isomorphism can be improved to show that

$$\left(H^N(\mathbb{C}^N, W^\infty, \mathbb{C})^{\langle J \rangle}, \langle \,,\rangle \right) \cong \left((\mathscr{Q}_W \omega)^{\langle J \rangle}, \mathrm{Res} \right).$$

(See a nice treatment in [Cec91].) It is clear that the above isomorphism also holds for the invariants of any admissible group G. With the above isomorphism, we have the identifications

$$(74) \qquad \mathscr{H}_{W,G} = \bigoplus_{\gamma \in G} \left(H^{\mathrm{mid}}(\mathbb{C}^{N_\gamma}, W_\gamma^\infty, \mathbb{Q}) \right)^G \cong \bigoplus_{\gamma \in G} \left(\mathscr{Q}_{W_\gamma} \omega_\gamma \right)^G,$$

where ω_γ is the restriction of the volume form ω to the fixed locus $\mathrm{Fix}\,\gamma$. The space $\bigoplus_\gamma(\mathscr{Q}_{W_\gamma}\omega_\gamma)^G$ arises in the orbifolded Landau-Ginzburg models studied by Intriligator-Vafa and Kaufmann in [IV90], [Kau06], [Kau03], [Kau02].

For computational purposes, it is usually easier to work with the sums of Milnor rings, so we will use the identification (74) for the remainder of the paper. However, we would like to emphasize that while $\mathscr{Q}_W$ has a natural ring structure, $H^N(\mathbb{C}^N, W^\infty, \mathbb{C})$ does not have any natural ring structure. Moreover, the ring structure induced on the state space is *not* the same as the one induced by the Milnor rings via the isomorphism (74). Furthermore, $\mathscr{Q}_W$ has an internal grading, while the degree of $H^N(\mathbb{C}^N, W^\infty, \mathbb{C})$ is just N. Hence, they are very different objects, and readers should not be confused by their similarity.

Before we start an explicit computation, we make several additional remarks.

Remark 5.1.1. One point of confusion is the notation of degree in singularity theory versus that of Gromov-Witten theory. Throughout the rest of paper, we will use $\deg_\mathbb{C}$ to denote the degree in singularity theory (i.e., the degree of the monomial) and $\deg_W$ to denote its degree as a cohomology class in Gromov-Witten or quantum singularity theory. We have

$$\deg_W = 2\deg_\mathbb{C}.$$

Remark 5.1.2. The local algebra, or Milnor ring, $\mathscr{Q}_W$ carries a natural nondegenerate pairing defined by

$$\langle f, g \rangle = \mathrm{Res}_{x=0}\frac{fg\,dx_1 \wedge \cdots \wedge dx_N}{\frac{\partial W}{\partial x_1} \cdots \frac{\partial W}{\partial x_N}}.$$

The pairing can be also understood as follows. The residue

$$\mathrm{Res}(f) := \mathrm{Res}_{x=0}\frac{f\,dx_1 \wedge \cdots \wedge dx_N}{\frac{\partial W}{\partial x_1} \cdots \frac{\partial W}{\partial x_N}}$$

has the following properties:

 (1) $\mathrm{Res}(f) = 0$ if $\deg_\mathbb{C}(f) < \hat{c}_W$.
 (2) $\mathrm{Res}\left(\frac{\partial^2 W}{\partial x_i \partial x_j}\right) = \mu$, where $\mu := \dim_\mathbb{C}(\mathscr{Q}_W)$ is the Milnor number.

Modulo the Jacobian ideal, any polynomial f can be uniquely expressed as $f = C\left(\frac{\partial^2 W}{\partial x_i \partial x_j}\right) + f'$, with $\deg_\mathbb{C}(f') < \hat{c}_W$. This implies that

$$\mathrm{Res}(f) = C\mu.$$

Remark 5.1.3. For any $G \leq \mathrm{Aut}(W)$, the action of the group G on the line bundles of the W-structure and on relative homology is *inverse* to the action on sheaves of sections, on relative cohomology, on the local ring, and on germs of differential forms. For instance, the element we have called J acts on homology

and on the line bundles of the W-structure as $(\exp(2\pi i q_1), \ldots, \exp(2\pi i q_N))$, but it acts on $\mathscr{Q}_W$ and on $\mathscr{Q}_W \omega$ as

$$J \cdot x_1^{m_1} \cdots x_N^{m_N} = e^{-2\pi i \sum_i m_i q_i} x_1^{m_1} \cdots x_N^{m_N}$$

and

$$J \cdot x_1^{m_1} \cdots x_N^{m_N} dx_1 \wedge \cdots \wedge dx_N = e^{-2\pi i \sum_i (m_i+1) q_i} x_1^{m_1} \cdots x_N^{m_N} dx_1 \wedge \cdots \wedge dx_N.$$

5.2. Self-mirror cases.

5.2.1. *The singularity A_n.* The maximal diagonal symmetry group of $A_n = x^{n+1}$ is precisely the group $\langle J \rangle$. The $\langle J \rangle$-invariants of the theory in the case of A_n agree with the theory of $(n+1)$-spin curves in [JKV01]. In that paper it is proved that the associated Frobenius algebra is isomorphic to the A_n Milnor ring (local algebra) and the Frobenius manifold is isomorphic to the Saito Frobenius manifold for A_n.

5.2.2. *The exceptional singularity E_7.* Consider now the case of $E_7 = x^3 + xy^3$. We have

$$q_x = 1/3, \text{ and } q_y = 2/9.$$

By equation (43), we get

$$\hat{c}_{E_7} = 8/9.$$

Furthermore, if $\xi = \exp(2\pi i/9)$, then J acts by (ξ^3, ξ^2), and

$$\Theta_x^J = 1/3, \qquad \Theta_y^J = 2/9.$$

It is easy to check that the maximal symmetry group is generated by J:

$$G_{E_7} = \langle J \rangle \cong \mathbb{Z}/9\mathbb{Z}.$$

Denote

$$\mathbf{e}_0 := dx \wedge dy \in H^{\mathrm{mid}}(\mathbb{C}^{N_{J^0}}, W_{J^0}^\infty, \mathbb{Q}),$$

$$\mathbf{e}_k := dx \in H^{\mathrm{mid}}(\mathbb{C}^{N_{J^k}}, W_{J^k}^\infty, \mathbb{Q}) \text{ for } k = 3, 6,$$

and

$$\mathbf{e}_k := 1 \in H^{\mathrm{mid}}(\mathbb{C}^{N_{J^k}}, W_{J^k}^\infty, \mathbb{Q}) \text{ for } 3 \nmid k.$$

We also denote the element $\mathbf{1} := \mathbf{e}_1$.

Using this notation, the G_{E_7}-space $\bigoplus_{k \in \mathbb{Z}/9\mathbb{Z}} H^{\mathrm{mid}}(\mathbb{C}^{N_{J^k}}, W_{J^k}^\infty, \mathbb{Q})$ can be described as follows:

$$(75) \quad H^{\mathrm{mid}}(\mathbb{C}^{N_{J^k}}, W_{J^k}^\infty, \mathbb{Q})$$

$$= \begin{cases} E_7 = \mathrm{span}(\mathbf{e}_0, x^1 \mathbf{e}_0, x^2 \mathbf{e}_0, y \mathbf{e}_0, y^2 \mathbf{e}_0, xy \mathbf{e}_0, x^2 y \mathbf{e}_0) & \text{if } k = 0, \\ A_2 = \mathrm{span}(\mathbf{e}_k, x \mathbf{e}_k) & \text{if } k \equiv 3, 6 \pmod 9, \\ A_1 = \mathrm{span}(\mathbf{e}_k) & \text{if } 3 \nmid k. \end{cases}$$

The G_{E_7}-invariant elements of this space form the state space of the E_7 theory:

$$\mathscr{H}_{E_7} = \mathrm{span}(y^2\mathbf{e}_0, \mathbf{1}, \mathbf{e}_2, \mathbf{e}_4, \mathbf{e}_5, \mathbf{e}_7, \mathbf{e}_8).$$

We now compute the genus-zero, three-point correlators for the G_{E_7}-invariant terms of the theory. First, the degree shift

$$\iota_{J^k} = \sum_{i=1}^{N}(\Theta_i^{J^k} - q_i)$$

and the W-degree

$$\deg_W(x^i y^j \mathbf{e}_k) = \deg(\mathbf{x}^i y^j \mathbf{e}_k) + 2\iota_{J^k} = N_{J^k} + 2\iota_{J^k}$$

depend only on k. For example, we have

$$\iota_{J^2} = (\Theta_x^{J^2} - q_x) + (\Theta_y^{J^2} - q_y) = (2/3 - 1/3) + (4/9 - 2/9) = 5/9$$

and

$$\deg_W(\mathbf{e}_2) = \deg(\mathbf{e}_2) + 2\iota_{J^2} = 0 + 10/9.$$

The complete set of numbers ι and $\deg_W$ are given by the following table:

k	0	1	2	3	4	5	6	7	8
ι_{J^k}	$-5/9$	0	$5/9$	$1/9$	$6/9$	$2/9$	$-2/9$	$3/9$	$8/9$
$\deg_W(x^i y^j e_k)$	$8/9$	0	$10/9$	$11/9$	$12/9$	$4/9$	$5/9$	$6/9$	$16/9$

For each genus-zero, three-point correlator $\langle a\mathbf{e}_{k_1}, b\mathbf{e}_{k_2}, c\mathbf{e}_{k_3}\rangle_0^{E_7}$, we have $g = 0$, $k = 3$, and we compute from equation (52) that

$$D = -\mathrm{index}(\mathscr{L}_x) - \mathrm{index}(\mathscr{L}_y) = \hat{c}_W(0-1) + \sum_{j=1}^{3}\iota_{J^{k_j}} = -8/9 + \sum_{j=1}^{3}\iota_{J^{k_j}}.$$

The dimension axiom (equation (53)) states that the the correlator will vanish unless

$$\dim_{\mathbb{R}}(\overline{\mathscr{M}}_{0,3}) = -2D - \sum_{j=1}^{3}N_{J^{k_j}}.$$

That means the correlator will vanish unless

$$0 = -2\hat{c}_{E_7} + 2\sum_{j=1}^{3}\iota_{J^{k_j}} + \sum_{j=1}^{3}N_{J^{k_j}} = -2\hat{c}_W + \sum_{j=1}^{3}\deg_W(\mathbf{e}_{k_j}).$$

A straightforward computation shows this only occurs for the following correlators:

$$\langle y^2\mathbf{e}_0, y^2\mathbf{e}_0, \ \mathbf{1}\rangle_0^{E_7}, \ \ \langle y^2\mathbf{e}_0, \mathbf{e}_5, \mathbf{e}_5\rangle_0^{E_7}, \langle \mathbf{1}, \mathbf{1}, \mathbf{e}_8\rangle_0^{E_7}, \ \ \langle \mathbf{1}, \mathbf{e}_2, \mathbf{e}_7\rangle_0^{E_7},$$

$$\langle \mathbf{1}, \mathbf{e}_4, \mathbf{e}_5\rangle_0^{E_7}, \ \ \langle \mathbf{e}_5, \mathbf{e}_7, \mathbf{e}_7\rangle_0^{E_7}.$$

Now we compute when the line bundles $\mathscr{L}_x$ and $\mathscr{L}_y$, defining the E_7-structure, are concave. Since we are in genus zero, this occurs precisely when

the degree of the desingularization of each line bundle (see equation (77)) is negative:

$$(76) \qquad 0 > \deg(|\mathscr{L}_x|) = \left(q_x(2g - 2 + k) - \sum_{l=1}^{k} \Theta_x^{\gamma_l} \right) = 1/3 - \sum_{l=1}^{3} \Theta_x^{J^{k_l}}$$

and

$$(77) \qquad 0 > \deg(|\mathscr{L}_y|) = \left(q_y(2g - 2 + k) - \sum_{l=1}^{k} \Theta_y^{\gamma_l} \right) = 2/9 - \sum_{l=1}^{3} \Theta_y^{J^{k_l}}.$$

This occurs precisely for the correlators

$$\langle \mathbf{1}, \mathbf{1}, \mathbf{e}_8 \rangle_0^{E_7}, \langle \mathbf{1}, \mathbf{e}_2, \mathbf{e}_7 \rangle_0^{E_7}, \langle \mathbf{1}, \mathbf{e}_4, \mathbf{e}_5 \rangle_0^{E_7}, \langle \mathbf{e}_5, \mathbf{e}_7, \mathbf{e}_7 \rangle_0^{E_7}.$$

For these concave cases, the virtual cycle must be Poincaré dual to the top (zeroth) Chern class of the bundle $R^1\pi_*(\mathscr{L}_1 \oplus \mathscr{L}_2) = 0$, which is 1. Thus these correlators are all 1.

The correlator $\langle y^2\mathbf{e}_0, y^2\mathbf{e}_0\mathbf{1} \rangle_0^{E_7}$, is just the residue pairing of the element y^2 with itself in the J^0-sector $H^{\mathrm{mid}}(\mathbb{C}^{N_{J^0}}, W_{j^0}^{\infty}, \mathbb{Q}) = \mathscr{Q}_{E_7}$. The Hessian $h := \frac{\partial^2 W}{\partial x_i \partial x_j}$ of W is $36x^2y - 9y^4 = -21y^4$ in $\mathscr{Q}_W$, and by Remark 5.1.2, we have $\langle \mathbf{1e}_0, h\mathbf{e}_0 \rangle^{E_7} = \mu_{E_7} = 7$, so

$$\langle y^2\mathbf{e}_0, y^2\mathbf{e}_0, \mathbf{1} \rangle^{E_7} = \langle \mathbf{1e}_0, y^4\mathbf{e}_0 \rangle^{E_7} = \langle \mathbf{1e}_0, -h/21\mathbf{e}_0 \rangle^{E_7} = -1/3.$$

Finally, we will compute the correlator $\langle y^2\mathbf{e}_0, \mathbf{e}_5, \mathbf{e}_5 \rangle_0^{E_7}$ by using the Composition Law (Axiom 6). The cycle $\left[\overline{\mathscr{W}}_{0,4}(E_7; J^5, J^5, J^5, J^5) \right]^{\mathrm{vir}}$ corresponds to a cycle on $\overline{\mathscr{W}}_{0,4}(E_7; J^5, J^5, J^5, J^5)$ of (real) dimension $6g - 6 + 2k - 2D = 2 = \dim_{\mathbb{R}} \overline{\mathscr{W}}_{0,4}(E_7; J^5, J^5, J^5, J^5)$, and thus it is just a constant times the fundamental cycle.

In this case we can compute that the line bundles $|\mathscr{L}_x|$ and $|\mathscr{L}_y|$ have degrees -2 and 0, respectively, and thus for each fiber (isomorphic to $\mathbb{CP}^1$) of the universal curve $\mathscr{C}$ over $\overline{\mathscr{W}}_{0,4}(E_7; J^5, J^5, J^5, J^5)$, we have $H^0(\mathbb{CP}^1, |\mathscr{L}_x| \oplus |\mathscr{L}_y|) = 0 \oplus \mathbb{C}$ and $H^1(\mathbb{CP}^1, |\mathscr{L}_x| \oplus |\mathscr{L}_y|) = \mathbb{C} \oplus 0$. The Witten map from H^0 to H^1 is $(3\bar{x}^2 + \bar{y}^3, 2\bar{x}\bar{y})$. This map has degree -3, so by the Index-Zero Axiom (Axiom 5b), the cycle $\left[\overline{\mathscr{W}}_{0,4}(E_7; J^5, J^5, J^5, J^5) \right]^{\mathrm{vir}}$ is -3 times the fundamental cycle. Pushing down to the moduli of pointed curves (see equation (63)) gives $\Lambda_{0,4}^{E_7}(\mathbf{e}_5, \mathbf{e}_5, \mathbf{e}_5, \mathbf{e}_5) = -3$, and the pullback along the gluing map ρ gives $\rho^*\Lambda_{0,4}^{E_7}(\mathbf{e}_5, \mathbf{e}_5, \mathbf{e}_5, \mathbf{e}_5) = -3$.

By the Composition Axiom, we have

$$-3 = \sum_{i,j} \Lambda_{0,3}^{E_7}(\mathbf{e}_5, \mathbf{e}_5, \alpha_i) \eta^{\alpha_i \beta_j} \Lambda_{0,3}^{E_7}(\beta_j, \mathbf{e}_5, \mathbf{e}_5).$$

However, the only nonzero three-point class of the form $\Lambda_{0,3}^{E_7}(\mathbf{e}_5, \mathbf{e}_5, \alpha_i)$ is $\Lambda_{0,3}^{E_7}(\mathbf{e}_5, \mathbf{e}_5, y^2\mathbf{e}_0)$. Thus, we have

$$-3 = -3\left(\Lambda_{0,3}^{E_7}(\mathbf{e}_5, \mathbf{e}_5, y^2\mathbf{e}_0)\right)^2,$$

and so

(78) $$\langle \mathbf{e}_5, \mathbf{e}_5, y^2\mathbf{e}_0 \rangle_0^{E_7} = \int_{\mathscr{M}_{0,3}} \Lambda_{0,3}^{E_7}(\mathbf{e}_5, \mathbf{e}_5, y^2\mathbf{e}_0) = \pm 1.$$

Now the fact that our pairing matches $\mathscr{H}_\gamma$ with $\mathscr{H}_{\gamma^{-1}}$ means that it pairs $\mathscr{H}_{J^k}$ with $\mathscr{H}_{J^{9-k}}$ and if $k \neq 0$, then the sectors $(\mathscr{H}_{J^k})^J \cong (\mathscr{H}_{J^{9-k}})^J$ are one-dimensional, spanned by $\mathbf{e}_k$ and the pairing gives $\langle \mathbf{e}_k, \mathbf{e}_{9-k} \rangle^{E_7} = 1$. We can use these correlators as the structure constants for an algebra on the invariant state space. If we define a map $\phi_\alpha : \mathbb{C}[X, Y] \longrightarrow \mathscr{H}_{E_7}$ by

$$X \mapsto \alpha^3 \mathbf{e}_7 \quad \text{and} \quad Y \mapsto \alpha^2 \mathbf{e}_5$$

for any $\alpha \in \mathbb{C}^*$, then we can make ϕ into a surjective homomorphism as follows:

$$1 \mapsto \mathbf{1} = \mathbf{e}_1, \quad X^2 \mapsto \alpha^6 \mathbf{e}_4, \quad XY \mapsto \alpha^5 \mathbf{e}_2$$
$$X^2Y \mapsto \alpha^8 \mathbf{e}_8, \quad Y^2 \mapsto \mp 3\alpha^4 y^2 \mathbf{e}_0.$$

Moreover, we have the relations

$$\phi(X) \star \phi(Y)^2 = 0$$

and

(79) $$\phi(Y)^3 = \phi(Y) \star (\mp 3\alpha^4 y^2 \mathbf{e}_0) = \mp 3\alpha^6 \sum_{\alpha,\beta} \langle \mathbf{e}_5, y^2\mathbf{e}_0, \alpha \rangle_0^{E_7} \eta^{\alpha\beta} \beta$$

$$= -3\alpha^6 \mathbf{e}_4 = -3\phi(X)^2.$$

So the kernel of ϕ contains XY^2 and $Y^3 + 3X^2$, but

$$\mathscr{Q}_{E_7} = \mathbb{C}[X, Y]/(XY^2, Y^3 + 3X^2)$$

has the same dimension as $\mathscr{H}_{E_7}$; therefore,

$$\mathscr{Q}_{E_7} = \mathbb{C}[X, Y]/(XY^2, 3X^2 + Y^3) \xrightarrow{\phi_\alpha} (\mathscr{H}_{E_7}^G, \star)$$

is an isomorphism of graded algebras for any choice of $\alpha \in \mathbb{C}^*$.

We wish to choose α so that the isomorphism ϕ_α also preserves the pairing. The pairing for $\mathscr{Q}_{E_7}$ has

$$\langle 1, X^2Y \rangle_{\mathscr{Q}_{E_7}} = \frac{1}{9} \quad \text{and} \quad \langle Y^2, Y^2 \rangle_{\mathscr{Q}_{E_7}} = -\frac{1}{3},$$

whereas for $\mathscr{H}_{E_7}$, the pairing is given by

$$\langle 1, \mathbf{e}_8 \rangle_{\mathscr{H}_{E_7}} = 1 \quad \text{and} \quad \langle \mp y^2\mathbf{e}_0, \mp y^2\mathbf{e}_0 \rangle_{\mathscr{H}_{E_7}} = -3.$$

This shows that the pairings differ by a constant factor of 9, and since $\phi(X^2Y)$ and $\phi(Y^4)$ both have degree 8 in α, choosing $\alpha^8 = 1/9$ makes ϕ into an isomorphism of graded Frobenius algebras

$$\mathcal{Q}_{E_7} \cong (\mathcal{H}^G_{E_7}, \star).$$

5.2.3. *The exceptional singularities E_6 and E_8.* Our ring $\mathcal{H}_{W,G_W}$ for both of the exceptional singularities $E_6 = x^3 + y^4$ and $E_8 = x^3 + y^5$ with maximal symmetry group G_W can be computed easily using the Sums of Singularities Axiom (Axiom 8). In this case, we have

$$(80) \qquad \mathcal{H}_{E_6, G_{E_6}} \cong \mathcal{H}_{A_2, G_{A_2}} \otimes \mathcal{H}_{A_3, G_{A_3}} \cong \mathcal{Q}_{A_2} \otimes \mathcal{Q}_{A_3} \cong \mathcal{Q}_{E_6},$$

$$(81) \qquad \mathcal{H}_{E_8, G_{E_8}} \cong \mathcal{H}_{A_2, G_{A_2}} \otimes \mathcal{H}_{A_4, G_{A_4}} \cong \mathcal{Q}_{A_2} \otimes \mathcal{Q}_{A_4} \cong \mathcal{Q}_{E_8},$$

where the second isomorphism of each row follows from the A_n case. Note that in both cases we have $\langle J \rangle = G_W$.

Later, when we compute the four-point correlators, it will be useful to have these isomorphisms described explicitly.

Explicit Isomorphism for E_6. Define $E_6 := x^3 + y^4$. The invariants are generated by the elements $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_5, \mathbf{e}_7, \mathbf{e}_{10}, \mathbf{e}_{11}$, where $\mathbf{e}_i := 1 \in H^{\mathrm{mid}}(\mathbb{C}^{N_{J^i}}, W^\infty_{J^i}, \mathbb{Q})$.

Computations similar to those done above show that the isomorphism of graded Frobenius algebras $\mathcal{Q}_{E_6} \longrightarrow \mathcal{H}_{E_6, G_{E_6}}$ is given by

$$Y \mapsto \alpha^3 \mathbf{e}_5 \quad \text{and} \quad X \mapsto \alpha^4 \mathbf{e}_{10},$$

with $\alpha^{10} = 1/12$.

Explicit Isomorphism for E_8. Define $E_8 := x^3 + y^5$. The invariants are generated by the elements $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_4, \mathbf{e}_7, \mathbf{e}_8, \mathbf{e}_{11}, \mathbf{e}_{13}, \mathbf{e}_{14}$, where

$$\mathbf{e}_i := 1 \in H^{\mathrm{mid}}(\mathbb{C}^{N_{J^i}}, W^\infty_{J^i}, \mathbb{Q}).$$

Again, computations similar to those done above show that the isomorphism of graded Frobenius algebras $\mathcal{Q}_{E_8} \longrightarrow \mathcal{H}_{E_8, G_{E_8}}$ is given by

$$Y \mapsto \alpha^3 \mathbf{e}_7 \quad \text{and} \quad X \mapsto \alpha^5 \mathbf{e}_{11},$$

with $\alpha^{14} = 1/15$.

5.2.4. *The singularity D_{n+1} with n odd and symmetry group $\langle J \rangle$.* Consider the case of D_{n+1} with $W = x^n + xy^2$ and with n odd. The weights are $q_x = 1/n$ and $q_y = (n-1)/2n$, and the central charge is $\hat{c}_{D_{n+1}} = (n-1)/n$. The exponential grading operator J is

$$J = (\xi^2, \xi^{n-1}), \quad \text{where } \xi = \exp(2\pi i/2n).$$

And J has order n in the group $G_{D_{n+1}} = \langle (\xi^2, \xi) \rangle \cong \mathbb{Z}/2n\mathbb{Z}$.

As described in Section 2.3, we may restrict to the sectors that come from the subgroup $\langle J \rangle$ by restricting the virtual cycle for D_{n+1} to the locus

corresponding to the moduli space for W'-curves, with $W' := x^n + xy^2 + x^{(n+1)/2}y$. For the rest of this example, we assume this restriction has been made. To simplify computations later, we find it easier to take $a \in (0, n]$ instead of the more traditional range of $[0, n)$.

Denote

$$\mathbf{e}_n := dx \wedge dy \in H^{\mathrm{mid}}(\mathbb{C}^{N_{J^n}}, W_{J^n}^\infty, \mathbb{Q}) = H^{\mathrm{mid}}(\mathbb{C}^{N_{J^0}}, W_{J^0}^\infty, \mathbb{Q}),$$

$$\mathbf{e}_a := 1 \in H^{\mathrm{mid}}(\mathbb{C}^{N_{J^a}}, W_{J^a}^\infty, \mathbb{Q}) \text{ for } a \neq n$$

so that the $G_{D_{n+1}}$-space $\bigoplus_{k \in \mathbb{Z}/n\mathbb{Z}} H^{\mathrm{mid}}(\mathbb{C}^{N_{J^k}}, W_{J^k}^\infty, \mathbb{Q})$ can be described as

(82) $H^{\mathrm{mid}}(\mathbb{C}^{N_{J^k}}, W_{J^k}^\infty, \mathbb{Q})$

$$= \begin{cases} D_{n+1} = \langle \mathbf{e}_n, x^1 \mathbf{e}_n, x^2 \mathbf{e}_n, \ldots, x^{n-1}\mathbf{e}_n, y\mathbf{e}_n \rangle & \text{if } k = n, \\ A_1 = \langle \mathbf{e}_k \rangle & \text{if } k \not\equiv 0 \pmod{n}. \end{cases}$$

The $\langle J \rangle$-invariant elements form our state space

$$\mathscr{H}_{D_{n+1}} = \langle x^{(n-1)/2}\mathbf{e}_n, y\mathbf{e}_n, \mathbf{e}_1, \ldots, \mathbf{e}_{(n-1)} \rangle.$$

To prove that $(\mathscr{H}_{D_{n+1}}, \star) \cong \mathscr{Q}_{D_{n+1}}$, we will choose constants $\alpha, \beta \in \mathbb{C}$ so that the ring homomorphism

$$\phi : \mathbb{C}[X, Y] \longrightarrow \mathscr{H}_{D_{n+1}},$$

defined by $X \mapsto \mathbf{e}_3$ and $Y \mapsto \alpha(x^{\frac{n-1}{2}}\mathbf{e}_n) + \beta(y\mathbf{e}_n)$, induces an isomorphism from $\mathscr{Q}_{D_{n+1}}$ to $\mathscr{H}_{D_{n+1}, \langle J \rangle}$.

To determine properties of the homomorphism, we must better understand the genus-zero, three-point correlators for the $\langle J \rangle$-invariant terms of the theory.

The degree $\deg_W(x^i y^j \mathbf{e}_a)$ is determined only by a and is given as follows:

$$\deg_W(x^i y^j \mathbf{e}_a) = \begin{cases} \frac{a-1}{n} & \text{if } a \text{ is odd and } a \in (0, n], \\ \frac{n+a-1}{n} & \text{if } a \text{ is even and } a \in (0, n). \end{cases}$$

For the genus-zero, three-point correlators, denote the relevant sectors by J^{a_i} for $i \in \{1, 2, 3\}$. Using the dimension axiom, we see that the virtual cycle vanishes unless

$$2\hat{c}_{D_{n+1}} = \sum_l \deg_W(\mathbf{e}_{a_l}).$$

This occurs precisely when

(83) $$\sum_i a_i = 2n + 1 - nE,$$

where E denotes the number of a_i that are even. Since $0 < a_i \leq n$ for all i, we have $\sum a_i \geq 3$, and so $0 \leq E \leq 1$.

Using equation (77) for the degree of the bundles $|\mathscr{L}_x|$ and $|\mathscr{L}_y|$, we have the following two cases:

(1) If $E = 1$, then $\deg(|\mathscr{L}_x|) = \deg(|\mathscr{L}_y|) = -1$. In this case the concavity axiom shows that the correlator is 1.
(2) If $E = 0$, then at most two of the a_i can be n. There are three cases:
  (a) If $E = 0$ and none of the a_i is n, then $\deg(|\mathscr{L}_x|) = -2$ and $\deg(|\mathscr{L}_y|) = 0$.
  (b) If $E = 0$ and exactly one a_i is n, then $\deg(|\mathscr{L}_x|) = -1$ and $\deg(|\mathscr{L}_y|) = 0$.
  (c) If $E = 0$ and exactly two of the a_i are n, then $\deg(|\mathscr{L}_x|) = \deg(|\mathscr{L}_y|) = 0$.

For Case 2, first note that all correlators of the form

$$\langle v, v', \mathbf{1} \rangle_{0,3}^{D_{n+1}}$$

for $v, v' \in \mathscr{H}$ are simply the pairing of v with v' in $\mathscr{H}$. In particular,

$$\langle x^{(n-1)/2}\mathbf{e}_n, x^{(n-1)/2}\mathbf{e}_n, \mathbf{1} \rangle_{0,3}^{D_{n+1}} = \langle x^{(n-1)/2}\mathbf{e}_n, x^{(n-1)/2}\mathbf{e}_n \rangle = \frac{1}{2n},$$

$$\langle y\mathbf{e}_n, y\mathbf{e}_n, \mathbf{1} \rangle_{0,3}^{D_{n+1}} = \langle y\mathbf{e}_n, y\mathbf{e}_n \rangle = \frac{-1}{2},$$

and

$$\langle x^{(n-1)/2}\mathbf{e}_n, y\mathbf{e}_n, \mathbf{1} \rangle_{0,3}^{D_{n+1}} = \langle x^{(n-1)/2}\mathbf{e}_n, y\mathbf{e}_n \rangle = 0.$$

For Case 2a, the line bundles $|\mathscr{L}_x|$ and $|\mathscr{L}_y|$ have degrees -2 and 0, respectively, and thus $H^0(\mathbb{CP}^1, |\mathscr{L}_x| \oplus |\mathscr{L}_y|) = 0 \oplus \mathbb{C}$, $H^1(\mathbb{CP}^1, |\mathscr{L}_x| \oplus |\mathscr{L}_y|) = \mathbb{C} \oplus 0$, and the Witten map from H^0 to H^1 is $(n\bar{x}^{n-1} + \bar{y}^2, 2\bar{x}\bar{y})$. This map has degree -2, so, as in previous arguments, the Index-Zero Axiom (Axiom 5b) shows that the correlator, is -2.

Case of $n > 3$. If we assume that $n > 3$, and letting μ and ν range through the basis $\{x^{(n-1)/2}\mathbf{e}_n, y\mathbf{e}_n, \mathbf{e}_1, \ldots, \mathbf{e}_{(n-1)}\}$, we have

$$\mathbf{e}_3 \star \mathbf{e}_3 = \sum_{\mu,\nu} \langle \mathbf{e}_3, \mathbf{e}_3, \mu \rangle \eta^{\mu\nu} \nu$$

$$= \sum_{\nu} \langle \mathbf{e}_3, \mathbf{e}_3, \mathbf{e}_{n-5} \rangle \eta^{\mathbf{e}_{n-5}\nu} \nu$$

$$= \langle \mathbf{e}_3, \mathbf{e}_3, \mathbf{e}_{n-5} \rangle \mathbf{e}_5 = \mathbf{e}_5.$$

Similar computations show that for $l < (n-1)/2$, we have

$$\mathbf{e}_3^l = \mathbf{e}_{2l+1}.$$

In the case of $\mathbf{e}_3^{(n-1)/2}$, we have

$$\mathbf{e}_3^{(n-1)/2} = \mathbf{e}_3 \star \mathbf{e}_3^{(n-3)/2}$$

$$= \langle \mathbf{e}_3, \mathbf{e}_{(n-2)}, x^{(n-1)/2}\mathbf{e}_n \rangle 2nx^{(n-1)/2}\mathbf{e}_n + \langle \mathbf{e}_3, \mathbf{e}_{(n-2)}, y\mathbf{e}_n \rangle (-2)y\mathbf{e}_n.$$

To simplify notation, we denote

$$r := \langle \mathbf{e}_3, \mathbf{e}_{(n-2)}, x^{(n-1)/2}\mathbf{e}_n \rangle \quad \text{and} \quad s := \langle \mathbf{e}_3, \mathbf{e}_{(n-2)}, y\mathbf{e}_n \rangle$$

so that

$$\mathbf{e}_3^{(n-1)/2} = (2nrx^{(n-1)/2}\mathbf{e}_n - 2sy\mathbf{e}_n).$$

Note that a computation like the one done above for Case 2a shows that the restriction of the virtual cycle $\left[\overline{\mathscr{W}}_{0,4,D_{n+1}}(J^3, J^{(n-2)}, J^3, J^{(n-2)})\right]^{\mathrm{vir}}$ to the boundary is zero-dimensional and equals -2. The Composition Axiom applied to this class shows that

$$-2 = 2nr^2 - 2s^2.$$

This shows that

$$\begin{aligned}
\mathbf{e}_3^{(n+1)/2} &= \mathbf{e}_3 \star \left((2nrx^{(n-1)/2}\mathbf{e}_n - 2sy\mathbf{e}_n)\right) \\
&= 2nr^2\mathbf{e}_2 - 2s^2\mathbf{e}_2 \\
&= -2\mathbf{e}_2.
\end{aligned}$$

Proceeding in this manner, we find that

$$\mathbf{e}_3^l = -2\mathbf{e}_{(2l-n+1)} \quad \text{if} \quad (n+1)/2 \le l \le n-1.$$

We wish to choose constants α and β so that the homomorphism

$$\phi : \mathbb{C}[X,Y] \longrightarrow \mathscr{H}_{D_{n+1},\langle J\rangle}, \quad 1 \mapsto \mathbf{e}_1, \ X \mapsto \mathbf{e}_3, \ Y \mapsto \alpha x^{(n-1)/2}\mathbf{e}_n + \beta y\mathbf{e}_n$$

has both XY and $nX^{n-1} + Y^2$ in its kernel, but so that $\phi(Y)$ is not in the span $\langle \phi(1), \phi(X), \dots, \phi(X^{(n-1)})\rangle$.

A straightforward calculation shows that

$$\phi(Y^2) = \left(\frac{\alpha^2}{2n} - \frac{\beta^2}{2}\right)\mathbf{e}_{n-1}.$$

Combining this with our previous calculations, we require

$$\frac{\alpha^2}{2n} - \frac{\beta^2}{2} = 2n.$$

Moreover, one easily computes that $\phi(XY) = (\alpha r + \beta s)\mathbf{e}_2$, and so $\alpha = -\beta s/r$. This gives

$$\beta = \pm 2nr, \quad \text{and thus} \quad \alpha = \mp 2ns.$$

With these choices of α and β it is easy to check that $\phi(Y)$ is not in the span $\langle \phi(1), \phi(X), \dots, \phi(X^{(n-1)})\rangle$. This means that ϕ is surjective and the ideal $(XY, nX^{n-1}+Y^2)$ lies in its kernel, and thus it induces the desired isomorphism of graded rings $\bar{\phi} : \mathscr{Q}_{D_{n+1}} \longrightarrow (\mathscr{H}_{D_{n+1},\langle J\rangle}, \star)$.

As in the case of E_7, we wish to rescale $\bar{\phi}$ to make it also an isomorphism of Frobenius algebras. The pairing for $\mathscr{Q}_{D_{n+1}}$ is

$$\langle X^{n-1}, 1\rangle = 1/2n \quad \text{and} \quad \langle Y^2, 1\rangle = -1/2,$$

whereas the paring for $\mathscr{H}_{D_{n+1},\langle J\rangle}$ has

$$\langle \mathbf{e}_3^{n-1}, \mathbf{1}\rangle = \langle -2\mathbf{e}_{n-1}, \mathbf{1}\rangle = -2 \quad \text{and} \quad \langle \bar{\phi}(Y^2), \mathbf{1}\rangle = -n\langle \bar{\phi}(X^{n-1}), \mathbf{1}\rangle = 2n.$$

Thus the pairing of $\mathscr{H}_{D_{n+1},\langle J\rangle}$ is a constant $-4n$ times the pairing of $\mathscr{Q}_{D_{n+1}}$. Since both rings are graded and the pairing respects the grading, rescaling the homomorphism $\bar{\phi}$ by an appropriate factor (namely, $\bar{\phi}(X) = \sigma^2 \mathbf{e}_3$, and $\bar{\phi}(Y) = \sigma^{n-1}(\alpha x^{(n-1)/2}\mathbf{e}_n + \beta y \mathbf{e}_n)$, with $\sigma^{2n-2} = 1/(-4n)$), shows that we can construct an isomorphism of graded Frobenius algebras

$$\mathscr{Q}_{D_{n+1}} \cong (\mathscr{H}_{D_{n+1},\langle J\rangle}, \star).$$

Case of $n = 3$. In the case that $n = 3$ we can determine all the correlators just by the selection rule (equation (83)) and the pairing. Specifically, we have the correlators

$$\langle \mathbf{e}_1, \mathbf{e}_1, \mathbf{e}_2 \rangle_0^{D_4} = 1, \qquad \langle x\mathbf{e}_3, x\mathbf{e}_3, \mathbf{e}_1 \rangle_0^{D_4} = 1/6,$$
$$\langle y\mathbf{e}_3, y\mathbf{e}_3, \mathbf{e}_1 \rangle_0^{D_4} = -1/2, \quad \langle x\mathbf{e}_3, y\mathbf{e}_3, \mathbf{e}_1 \rangle_0^{D_4} = 0,$$

and all other three-point correlators vanish.

It is easy to verify that the map $\phi : \mathbb{C} \longrightarrow \mathscr{H}_{D_4,\langle J\rangle}$ taking $X \mapsto x\mathbf{e}_3$ and $Y \mapsto y\mathbf{e}_3$ induces an isomorphism of graded Frobenius algebras

$$\mathscr{Q}_{D_4} \cong (\mathscr{H}_{D_4,\langle J\rangle}, \star).$$

5.3. Simple singularities that are not self-mirror.

5.3.1. *The singularity D_{n+1} with its maximal Abelian symmetry group.* In this subsection we will show that, regardless of whether n is even or odd, the ring $\mathscr{H}_{D_{n+1},G_{D_{n+1}}}$ with its maximal symmetry group $G_{D_{n+1}}$ is isomorphic, as a Frobenius algebra, to the Milnor ring $\mathscr{Q}_{x^n y + y^2}$ of $D_{n+1}^T = x^n y + y^2$.

Regardless of whether n is even or odd, the maximal Abelian symmetry group $G := G_{D_{n+1}}$ of $D_{n+1} = x^n + xy^2$ is isomorphic to $\mathbb{Z}/2n\mathbb{Z}$. It is generated by $\lambda := (\zeta^{-2}, \zeta^1)$, with $\zeta = \exp(2\pi i/2n)$. We have $J = \lambda^{n-1}$. If n is even, then J generates the entire group G, but if n is odd, it generates a subgroup of index 2 in G. The case of D_{n+1} with n odd and with symmetry group $\langle J\rangle$ has already been treated in Section 5.2.4.

Define

$$\mathbf{e}_0 := dx \wedge dy \in H^{\mathrm{mid}}(\mathbb{C}^{N_{\lambda^0}}, W_{\lambda^0}^\infty, \mathbb{Q}),$$

$$\mathbf{e}_a := 1 \in H^{\mathrm{mid}}(\mathbb{C}^{N_{\lambda^a}}, W_{\lambda^a}^\infty, \mathbb{Q}) \text{ for } 0 < a < n \text{ or } n < a < 2n,.$$

After computing G-invariants, we find that the state space $\mathscr{H}_{D_{n+1},G}$ is spanned by the elements

$$y\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{n-1}, \mathbf{e}_{n+1}, \mathbf{e}_{n+2}, \ldots, \mathbf{e}_{2n-1}.$$

We have

(84)
$$\Theta_x^{\lambda^a} = \begin{cases} a/n & \text{if } 0 \le a < n, \\ a/n - 1 & \text{if } n \le a < 2n, \end{cases}$$

$$(85) \qquad \Theta_y^{\lambda^a} = \begin{cases} 0 & \text{if } a = 0, \\ 1 - a/2n & \text{if } 0 < a < 2n, \end{cases}$$

and

$$(86) \qquad \deg_W(\mathbf{e}_a) = \begin{cases} \frac{a-1}{n} + 1 & \text{if } 0 \le a < n, \\ \frac{a-1}{n} - 1 & \text{if } n < a < 2n. \end{cases}$$

For three-point correlators of the form $\langle \varkappa_1, \varkappa_2, \varkappa_3 \rangle_{0,3}^{D_{n+1}}$, with each $\varkappa_i$ in the λ^{a_i}-sector, the dimension axiom gives the selection rule

$$2\hat{c}_{D_{n+1}} = \sum_{i=1}^{3} \deg_W(\varkappa_i)$$

which, using equation (86), gives

$$\sum_{i=1}^{3} a_i = 2nB - n + 1,$$

where B is the number of a_i greater than n.

Similarly, we compute the degree of each of the line bundles in the D_{n+1}-structure to be

$$(87) \qquad \begin{aligned} \deg(|\mathscr{L}_x|) &= 1 - B \\ \deg(|\mathscr{L}_y|) &= R + B - 3, \end{aligned}$$

where R is the number of broad sectors $\varkappa_i \in H^{\mathrm{mid}}(\mathbb{C}^{N_{\lambda^0}}, W_{\lambda^0}^{\infty}, \mathbb{Q})$. A straightforward case-by-case analysis of the possible choices for B and R shows that (up to reindexing) the only correlators that do not vanish for dimensional reasons are the following:

$$\langle \mathbf{e}_{n+a}, \mathbf{e}_{n+b}, \mathbf{e}_{n+1-a-b} \rangle_{0,3}^{D_{n+1}} \text{ with } 0 < a, b \text{ and } a + b \le n,$$

$$\langle y\mathbf{e}_0, \mathbf{e}_{n+1+a}, \mathbf{e}_{2n-a} \rangle_{0,3}^{D_{n+1}} \text{ with } 0 < a < n - 1,$$

$$\langle y\mathbf{e}_0, y\mathbf{e}_0, \mathbf{e}_{n+1} \rangle_{0,3}^{D_{n+1}} = \eta_{y\mathbf{e}_0, y\mathbf{e}_0} = -\frac{1}{2}.$$

Using equation (87), we see that correlators of the first type are all concave and so are equal to 1. Those of the second type can be computed using the Composition Axiom; specifically, the Index-Zero Axiom shows that the restriction of the virtual cycle $\left[\overline{\mathscr{W}}_{0,4}(D_{n+1}; \lambda^{n+1+a}, \lambda^{n+1+a}, \lambda^{2n-a}, \lambda^{2n-a}) \right]^{\mathrm{vir}}$ to the boundary is -2 times the fundamental cycle. The Composition Axiom now shows that

$$\rho^* \left(\langle y\mathbf{e}_0, \mathbf{e}_{n+1+a}, \mathbf{e}_{2n-a} \rangle_{0,3}^{D_{n+1}} \right)^2 \eta^{y\mathbf{e}_0, y\mathbf{e}_0} = -2,$$

which gives

$$(88) \qquad \langle y\mathbf{e}_0, \mathbf{e}_{n+1+a}, \mathbf{e}_{2n-a} \rangle_{0,3}^{D_{n+1}} = \pm 1.$$

Using these computations, it is now straightforward to check that, regardless of the choice of sign in equation (88), the map

$$\phi : \mathscr{Q}_{x^n y + y^2} = \mathbb{C}[X, Y]/(X^{n-1}Y, X^n + 2Y) \longrightarrow (\mathscr{H}_{D_{n+1}, G}, \star),$$

defined by

$$X^i \mapsto \begin{cases} \mathbf{e}_{n+1+i} & \text{for } 0 \leq i < n-1, \\ \mp 2y\mathbf{e}_0 & \text{for } i = n-1, \\ \mathbf{e}_{i-n+1} & \text{for } n \leq i < 2n-1, \end{cases} \quad \text{and} \quad Y \mapsto -\frac{X^n}{2} = -\frac{\mathbf{e}_1}{2},$$

is an isomorphism of graded algebras. The pairing on

$$\mathscr{Q}_{x^n y + y^2} = \mathbb{C}[X, Y]/(X^{n-1}Y, X^n + 2Y)$$

is given by

$$\langle X^{2n-2}, 1 \rangle^{\mathscr{Q}_{D_{n+1}}} = -1/n,$$

whereas the pairing on $\mathscr{H}_{D_{n+1}, G_{D_{n+1}}}$ is easily seen to be given by

$$\langle \phi(X^{2n-2}), \mathbf{1} \rangle^{\mathscr{H}_{D_{n+1}}} = \langle \mathbf{e}_{n-1}, \mathbf{e}_{n+1} \rangle^{\mathscr{H}_{D_{n+1}}} = 1.$$

Since ϕ and the pairing both preserve the grading, we can rescale ϕ to be $\phi(X) = \alpha \mathbf{e}_{n+2}$ and $\phi(Y) = -\alpha^n \mathbf{e}_1/2$ with $\alpha^{2n-2} = -1/n$ to obtain an isomorphism of graded Frobenius algebras:

$$\mathscr{Q}_{x^n y + y^2} \cong (\mathscr{H}_{D_{n+1}, G_{D_{n+1}}}, \star).$$

5.3.2. *The mirror partner D_{n+1}^T of D_{n+1}.* The mirror partner of D_{n+1} is the singularity $D_{n+1}^T := x^n y + y^2$. In this subsection we show that the ring $\mathscr{H}_{D_{n+1}^T}$ of D_{n+1}^T with its maximal Abelian symmetry group is isomorphic, as a Frobenius algebra, to the Milnor ring $\mathscr{Q}_{D_{n+1}}$. Since we have already shown that the ring $\mathscr{H}_{D_{n+1}}$ with its maximal Abelian symmetry group is isomorphic to the Milnor ring of D_{n+1}^T, this will complete the proof that, at least at the level of Frobenius algebras, D_{n+1}^T is indeed the mirror partner of D_{n+1}.

For this singularity, the weights are $q_x = 1/2n$ and $q_y = 1/2$, and the central charge is $\hat{c}_{D_{n+1}^T} = (n-1)/n$. If $\xi := \exp(2\pi i/2n)$, then the exponential grading operator is $J = (\xi, \xi^n)$. The element J generates the maximal Abelian symmetry group $\langle J \rangle = G_W \cong \mathbb{Z}/2n\mathbb{Z}$.

Denote

$$\mathbf{e}_0 := dx \wedge dy \in H^{\mathrm{mid}}(\mathbb{C}^{N_{J^0}}, W_{J^0}^\infty, \mathbb{Q}),$$

$$\mathbf{e}_a := 1 \in H^{\mathrm{mid}}(\mathbb{C}^{N_{J^a}}, W_{J^a}^\infty, \mathbb{Q}) \text{ for } 0 < a < 2n.$$

The G_W-invariant state space is

$$\mathscr{H}_W = \mathscr{H}_{W, G_W} = \langle x^{(n-1)} \mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_5, \ldots, \mathbf{e}_{2n-1} \rangle.$$

As always, we have

$$\deg_W(x^{n-1}\mathbf{e}_0) = \hat{c}_W = 2\frac{n-1}{2n}.$$

Also, we have $\Theta_x^{J^a} = a/2n$ for $a \in \{0, \ldots, 2n-1\}$ and $\Theta_y^{J^a} = a/2 \pmod 1$, so the degree of any element $\varkappa$ in the J^a-sector is given as follows:

$$\deg_W(\varkappa) = 2\frac{a-1}{2n} \quad \text{if } a \text{ is odd and } a \in (0, 2n).$$

For the genus-zero, three-point correlators $\langle \varkappa_1, \varkappa_2, \varkappa_3 \rangle_{0,3}^W$ with homogeneous elements $\varkappa_i \in \mathscr{H}_{J^{a_i}}$, the dimension axiom gives that the virtual cycle vanishes unless

$$2\hat{c}_{D_{n+1}^T} = \sum_l \deg_W(\varkappa_i).$$

This occurs precisely when

(89) $$\sum_i a_i = 2n + 1 - nR,$$

where R denotes the number of a_i that are equal to 0; that is, the number of broad sectors.

Equation (89) shows that $R \in \{0, 1, 2\}$. And a simple computation shows that the degree of the W-structure line bundle $\mathscr{L}_x$ is not integral if $R = 1$, so we have only the two cases of $R = 0$ and $R = 2$. In the case of $R = 2$, equation (89) shows that the only nonvanishing correlator is

$$\langle \mathbf{e}_1, x^{n-1}\mathbf{e}_0, x^{n-1}\mathbf{e}_0 \rangle_{0,3}^W = \eta_{x^{n-1}\mathbf{e}_0, x^{n-1}\mathbf{e}_0} = -\frac{1}{n}.$$

In the case that $R = 0$, we have $\deg(\mathscr{L}_x) = \deg(\mathscr{L}_y) = -1$, so by concavity, these correlators are all 1.

Now define a map $\phi : \mathbb{C}[X, Y] \longrightarrow \mathscr{H}_{D_{n+1}^T}$ by $X^i \mapsto \mathbf{e}_{2i+1}$ and $Y \mapsto nx^{n-1}\mathbf{e}_0$. It is straightforward to check that ϕ is a graded surjective homomorphism with kernel $(nX^{n-1} + Y^2, XY)$. So ϕ defines an isomorphism of graded algebras

$$\mathscr{Q}_{D_{n+1}} \cong (\mathscr{H}_{D_{n+1}^T}, \star).$$

The pairing on each of these algebras also respects the grading, and the two pairings differ by a constant multiple of $2n$. So rescaling the homomorphism ϕ by $X^i \mapsto \sigma^{2i}\mathbf{e}_{2i+1}$ and $Y \mapsto n\sigma^{n-1}x^{n-1}\mathbf{e}_0$ with $\sigma^{2n-2} = 1/2n$ makes an isomorphism of graded Frobenius algebras.

This shows that D_{n+1}^T is indeed a mirror partner to D_{n+1}, and it completes the proof of Theorem 1.0.7.

6. **ADE-hierarchies and the generalized Witten conjecture**

The main motivation for Witten to introduce his equation was the following conjecture.

CONJECTURE 6.0.1 (ADE-integrable hierarchy conjecture). *The total potential functions of the* A, D, *and* E *singularities with group* $\langle J \rangle$ *are* τ-functions *of the corresponding* A, D, *and* E *integrable hierarchies.*

The A_n-case was established recently by Faber-Shadrin-Zvonkin [FSZ10]. One of our main results is the resolution of Witten's integrable hierarchies conjecture for the D and E series. It turns out that Witten's conjecture needs a modification in the D_n case for n odd. This modification is extremely interesting because it reveals a surprising role that mirror symmetry plays in integrable hierarchies.

6.1. **Overview of the results on integrable hierarchies.** Let us start from the ADE-hierarchies. As we mentioned in the introduction, there are two equivalent versions of ADE-integrable hierarchies—that of Drinfeld-Sokolov [DS84] and that of Kac-Wakimoto [KW89]. The version directly relevant to us is the Kac-Wakimoto ADE-hierarchies because the following beautiful work of Frenkel-Givental-Milanov reduces the problem to an explicit problem in Gromov-Witten theory. Let us describe their work.

Let W be a nondegenerate quasi-homogeneous singularity, and let ϕ_i ($i \leq \mu$) be the monomial basis of the Milnor ring with $\phi_1 = 1$. Consider the miniversal deformation space $\mathbb{C}^\mu$ where a point $\lambda = (t_1, \ldots, t_\mu)$ parametrizes the polynomial $W + t_1\phi_1 + t_2\phi_2 \cdots + t_\mu\phi_\mu$. We can assign a degree to t_i such that the above perturbed polynomial has the degree one; i.e., $\deg(t_i) = 1 - \deg(\phi_i)$. The tangent space T_λ carries an associative multiplication $\circ$ and a Euler vector field $E = \sum_i \deg(t_i)\partial_{t_i}$ with the unit $e = \frac{\partial W_\lambda}{\partial t_1}$. It is more subtle to construct a metric. We can consider residue pairing

$$\langle f, g \rangle_\lambda = \mathrm{Res}_{x=0} \frac{fg\omega}{\frac{\partial W_\lambda}{\partial x_1} \cdots \frac{\partial W_\lambda}{\partial x_N}}$$

using a holomorphic n-form ω. A deep theorem of Saito [Sai81] states that one can choose a *primitive form* ω such that the induced metric is flat. Together, it defines a Frobenius manifold structure on a neighborhood of zero of $\mathbb{C}^\mu$. We should mention that there is no explicit formula for the primitive form in general. However, it is known that for ADE-singularities, the primitive form can be chosen to be a constant multiple of standard volume form; i.e., $c\,dx$ for A_n and $c\,dx \wedge dy$ for DE series.

Furthermore, one can define a potential function $\mathscr{F}$, playing the role of genus-zero Gromov-Witten theory with only primary fields. It is constructed

as follows. We want to work in flat coordinates s_i with the property that $\deg_{\mathbb{C}}(s_i) = \deg_{\mathbb{C}}(t_i)$ and $\langle s_i, s_j \rangle$ are constant. The flat coordinates depend on the flat connection of the metric and hence the primitive form. Its calculation is important and yet a difficult problem. Nevertheless, we know that the flat coordinates exist thanks to the work of Saito [Sai81]. Then, consider the three-point correlator $C_{ijk} = \langle \partial_{s_i}, \partial_{s_j}, \partial_{s_k} \rangle$ as a function near zero in $\mathbb{C}^\mu$. We can integrate C_{ijk} to obtain $\mathscr{F}$. Here, we normalize $\mathscr{F}$ such that $\mathscr{F}$ has leading term of degree three. We can differentiate $\mathscr{F}$ by the Euler vector field. It has the property $L_E \mathscr{F} = (\hat{c}_W - 3)\mathscr{F}$. Namely, $\mathscr{F}$ is homogeneous of degree $\hat{c}_W - 3$. The last condition means that, in the Taylor expansion

$$\mathscr{F} = \sum a(n_1, \ldots, n_\mu) \frac{s_1^{n_1} \cdots s_\mu^{n_\mu}}{n_1! \cdots n_\mu!},$$

we have $a(n_1, \ldots, n_\mu) \neq 0$ only when $\sum n_i - \sum n_i(1 - \deg_{\mathbb{C}}(s_i)) = \sum \deg_{\mathbb{C}}(s_i) = \hat{c}_W - 3$. Note that the degree in the Frobenius manifold is different from that of the A-model. For example, the unit e has degree 1 instead of zero. The A-model degree is 1 minus the B-model degree. With this relation in mind, we will treat the insertion s_i with degree $1 - \deg_{\mathbb{C}}(s_i)$. Then, the above formula is precisely the selection rule of quantum singularity theory.

It is known that the Frobenius manifold of a singularity is semisimple in the sense that the Frobenius algebra on T_λ at a generic point λ is semisimple. On any semisimple Frobenius manifold, Givental constructed a *formal* Gromov-Witten potential function. We will only be interested in the case that the Frobenius manifold is the one corresponding to the miniversal deformation space of a quasi-homogeneous singularity W. We denote it by

$$\mathscr{D}_{W,\text{formal}} = \exp\left(\sum_{g \geq 0} h^{2g-2} \mathscr{F}^g_{\text{formal}} \right).$$

The construction of $\mathscr{D}_{W,\text{formal}}$ is complicated, but we only need its following formal properties:

(1) $\mathscr{F}^0_{\text{formal}}$ agrees with $\mathscr{F}$ for primitive fields, i.e., with no descendants.
(2) $\mathscr{F}^g_{\text{formal}}$ satisfies the same selection rule as a Gromov-Witten theory with $C_1 = 0$ and dimension $\hat{c}_W$.
(3) $\mathscr{D}_{W,\text{formal}}$ satisfies all the formal axioms of Gromov-Witten theory.

The first property is obvious from the construction. The second property is a consequence of the fact that $\mathscr{D}_{W,\text{formal}}$ satisfies the dilaton equation and Virasoro constraints. A fundamental theorem of Frenkel-Givental-Milanov [GM05], [FGM10] is

THEOREM 6.1.1. *For ADE-singularities, $\mathscr{D}_{W,\text{formal}}$ is a τ-function of the corresponding Kac-Wakimoto ADE-hierarchies.*

Remark 6.1.2. Givental-Milanov first constructed a Hirota-type equation for Givental's formal total potential function. Later, Frenkel-Givental-Milanov proved that Givental-Milanov's Hirota equation is indeed the same as that of Kac-Wakimoto.

Our main theorem is

THEOREM 6.1.3. (1) *Except for D_n with n odd and D_4, the total potential functions of* DE-*singularities with the group $\langle J \rangle$ are equal to the corresponding Givental formal Gromov-Witten potential functions up to a linear change of variables.*

(2) $\mathscr{D}_{D_n, G_{\max}} = \mathscr{D}_{A_{2n-3}, \text{formal}}$, *up to a linear change of variables.*

(3) *For $D_n^T = x^{n-1}y + y^2$ $(n > 4)$, $\mathscr{D}_{D_n^T, G_{\max}} = \mathscr{D}_{D_n, \text{formal}}$, up to a linear change of variables.*

Using the theorem of Frenkel-Givental-Milanov, we obtain

COROLLARY 6.1.4. (1) *Except for D_n with n odd and D_4, the total potential function of* DE-*singularities with the group $\langle J \rangle$ is a τ-function of the corresponding Kac-Wakimoto hierarchies (and hence Drinfeld-Sokolov hierarchies).*

(2) *The total potential function of all D_n-singularities for $n > 4$ with the maximal diagonal symmetry group is a τ-function of the A_{2n-3} Kac-Wakimoto hierarchies (and hence Drinfeld-Sokolov hierarchies).*

(3) *The total potential function of $D_n^T = x^{n-1}y + y^2$ $(n > 4)$ with the maximal diagonal symmetry group is a τ-function of the D_n Kac-Wakimoto hierarchies (and hence Drinfeld-Sokolov hierarchies).*

Remark 6.1.5. There is a technical issue in Givental's formal theory, as follows. For any semisimple point t of Saito's Frobenius manifold, he defined an *ancestor potential* $\mathscr{A}_t$. From this he obtains a descendant potential function $\mathscr{D} = \hat{S}_t \mathscr{A}_t$, where $\hat{S}_t$ is certain quantization of a symplectic transformation S_t determined by the Frobenius manifold. Then, he showed $\mathscr{D}$ is independent of t. However, to compare with our A-model calculation, we need to expand $\mathscr{D}$ as formal power series at $t = 0$. Although $\mathscr{D}$ is expected to have a power series expansion at $t = 0$, we have been informed that a proof is not yet in the literature. Our strategy to avoid this problem is to show that (i) the A- and B-models have isomorphic Frobenius manifolds, and (ii) in the ADE cases the ancestor functions of both models are completely determined by their respective Frobenius manifolds. Therefore, the A- and B-model have the same ancestor potentials and hence the same descendant potentials.

Definition 6.1.6. An ancestor correlator is defined as

$$\langle \tau_{l_1}(\alpha_1), \ldots, \tau_{l_n}(\alpha_n) \rangle_g^{W,G}(t) = \sum_k \langle \tau_{l_1}(\alpha_1), \ldots, \tau_{l_n}(\alpha_n), \overbrace{t, \ldots, t}^{k \text{ copies}} \rangle_g^{W,G}.$$

Then, we define ancestor generating function $\mathscr{F}_g^{W,G}(t)$ of our theory with these correlators similarly.

Givental ancestor potential is defined for semisimple points $t \neq 0$. In the above definition, t is only a formal variable. To be able to choose an actual value $t \neq 0$, we need to show that the ancestor correlator is convergent for that choice of t. This is done in the following lemma.

LEMMA 6.1.7. *Choose a basis T^i of $\mathscr{H}_{W,G}$, and write $t = \sum_i t_i T^i$. For the simple* (ADE) *singularities, the ancestor correlator $\langle \tau_{l_1}(\alpha_1), \ldots, \tau_{l_n}(\alpha_n) \rangle_g^{W,G}(t)$ is a polynomial in the variables t_i. Furthermore, if $l_1 = \cdots = l_n = 0$, (i.e., if there are no ψ-classes), the ancestor correlator is also a polynomial in the variables $\alpha_1, \ldots, \alpha_n$.*

Proof. Consider correlator $\langle \tau_1(\alpha_1), \ldots, \tau_n(\alpha_n), T_{i_1}, \ldots, T_{i_k} \rangle_g^{W,G}$. The dimension condition is

$$2((\hat{c}_w - 3)(1 - g) + n + k) = \sum_i (2l_i + \deg_W \alpha_i) + \sum_j \deg_W T_{i_j}.$$

This implies that

$$\sum_j (2 - \deg_W T_{i_j}) = \sum_i (2l_i + \deg_W \alpha_i) - 2((\hat{c}_w - 3)(1 - g) + n).$$

Therefore, if we redefine $\deg_W' T_{i_j} := 2 - \deg_W T_{i_j}$, the ancestor correlator is homogeneous of a fixed degree. When W is an ADE-singularity, it is straightforward to check that $2 - \deg_W T_{i_j} > 0$. Hence, it must be a polynomial. The same argument implies the second case. $\qquad\square$

This lemma shows that we can consider $\mathscr{F}_t^{W,G}$ and $\mathscr{A}_t^{W,G}$ for a semisimple point $t \neq 0$.

The proof of the main theorem depends on four key ingredients. The first ingredient is the reconstruction theorem for the ADE-theory, which shows that the two ancestor potentials are both determined by their corresponding Frobenius manifolds. The second step is to show that the Frobenius manifolds are completely determined by genus-zero, three-point correlators and certain explicit four-point correlators. The third ingredient is the *Topological Euler class axiom for narrow sectors*, which enables us to compute all the three-point and required four-point correlators. The last ingredient is the mirror symmetry of ADE-singularities we proved in last section. The required modification in the D_n case when n is odd is transparent from mirror symmetry.

6.2. **Reconstruction theorem.** In this subsection, we will establish the reconstruction theorem simultaneously for ADE-quantum singularity theory and Givental's formal Gromov-Witten theory in the ADE-case. We use the facts that

(i) both theories satisfy the formal axioms of Gromov-Witten theories,
(ii) they both have the same selection rules,
(iii) they both have isomorphic quantum rings up to a mirror transformation.

The last fact has been established in the previous section. To simplify the notation, we state the theorem for the quantum singularity theory of the A-model.

We start with the higher genus reconstruction using an idea of Faber-Shadrin-Zvonkine [FSZ10].

THEOREM 6.2.1. *If $\hat{c} < 1$, then the ancestor potential function is uniquely determined by the genus-zero primary potential (i.e., without gravitational descendants). If $\hat{c} = 1$, then the ancestor potential function is uniquely determined by its genus-zero and genus-one primary potentials.*

The proof of Theorem 6.2.1 is a direct consequence of the following two lemmas, using the Faber-Shadrin-Zvonkine reduction technique. For this argument, we always assume that $\hat{c} \leq 1$.

LEMMA 6.2.2. *Let $\alpha_i \in \mathscr{H}_{\gamma_i,G}$ for all $i \in \{1, \ldots, n\}$, and let β be any product of ψ classes. If $\hat{c} < 1$, then the integral*

$$\int_{\overline{\mathscr{M}}_{g,n+k}} \beta \cdot \Lambda^W_{g,n+k}(\alpha_1, \ldots, \alpha_n, T_{i_1}, \ldots, T_{i_k})$$

vanishes if $\deg \beta < g$ for $g \geq 1$. If $\hat{c} = 1$, then the above integral vanishes when $\deg \beta < g$ for $g \geq 2$.

Proof. The integral $\int_{\overline{\mathscr{M}}_{g,n+k}} \beta \cdot \Lambda^W_{g,n+k}(\alpha_1, \ldots, \alpha_n, T_{i_1}, \ldots, T_{i_k})$ does not vanish only if

$$\deg \beta = 3g - 3 + n + k - D - \sum_{\tau=1}^{n+k} N_{\gamma_\tau}/2,$$

where $D = \hat{c}(g-1) + \sum_\tau \iota_{\gamma_\tau}$. Recall that $\iota_\gamma = \sum_{i=1}^N (\Theta_i^\gamma - q_i)$.
Now we have the inequality

$$(90) \qquad \deg \beta = (3 - \hat{c})(g-1) + \sum_{\tau=1}^{n+k}(1 - \iota_{\gamma_\tau})$$

$$= (3 - \hat{c})(g-1) + \sum_{\tau=1}^{n+k}(1 - \hat{c} + \hat{c} - \iota_{\gamma_\tau} - N_{\gamma_\tau}/2)$$

$$\geq (3 - \hat{c})(g-1) + (n+k)(1 - \hat{c}),$$

where we used the fact (easily verified for the simple singularities AD and E) that the complex degree $\deg_{\mathbb{C}} \alpha_\gamma = \iota_\gamma + N_\gamma/2$ of a class $\alpha_\gamma \in \mathscr{H}_\gamma$ always satisfies

$$\deg_{\mathbb{C}} \alpha_\gamma = \iota_\gamma + N_\gamma/2 \leq \hat{c}.$$

Hence if $g \geq 2$, we have $\deg \beta \geq g$. If $g = 1$, then $\deg \beta > 0$ for $\hat{c} < 1$ and $\deg \beta \geq 0$ for $\hat{c} = 1$, where the equality holds if and only if $\deg_{\mathbb{C}} \alpha_{\gamma_\tau} = \hat{c}$ for all τ. $\square$

The following lemma treats the integral for higher-degree ψ classes. It was proved in [FSZ10], where it was called *g-reduction*.

LEMMA 6.2.3. *Let P be a monomial in the ψ and κ-classes in $\overline{\mathscr{M}}_{g,k}$ of degree at least g for $g \geq 1$ or at least 1 for $g = 0$. Then the class P can be represented by a linear combination of dual graphs, each of which has at least one edge.*

Proof of Theorem 6.2.1. Take any correlators

$$\langle \tau_{d_1}(\alpha_1) \cdots \tau_{d_k}(\alpha_n), T_{i_1}, \ldots, T_{i_k} \rangle_{g,n+k}$$
$$= \int_{\overline{\mathscr{M}}_{g,n+k}} \psi_1^{d_1} \cdots \psi_k^{d_n} \Lambda_{g,n+k}^W(\alpha_1, \ldots, \alpha_n, T_{i_1}, \ldots, T_{i_k}).$$

The total degree of the ψ-classes must either match the hypothesis of Lemma 6.2.2 or match the hypothesis of Lemma 6.2.3. If the total degree is small, then it vanishes by Lemma 6.2.2. If it is large, then the integral is changed to the integral over the boundary classes while decreasing the degree of the total integrated ψ or κ classes. Applying the degeneration and composition laws, the genus of the moduli spaces involved will also decrease. It is easy to see that one can continue this process until the original integral is represented by a linear combination of integrals over moduli spaces of genus zero and genus one, without gravitational descendants. $\square$

Remark 6.2.4. There is an alternative higher-genus reconstruction, using Teleman's recent announcement [Tel12] of a proof of Givental's conjecture [Giv01]. However, in the ADE-case the above argument is much simpler and achieves the same goal.

The above theorem implies that all the ancestor correlators are determined by genus-zero ancestor correlators without ψ classes. On the B-model side, Givental's genus-zero generating function is equal to Saito's genus-zero generating function. Hence, it is well defined at $t = 0$. Furthermore, Lemma 6.1.7 shows that both the A- and B-model genus-zero functions without descendants are polynomials and are defined over the entire Frobenius manifold. Finally, we observe that the genus-zero ancestor generating function is determined by the ordinary genus-zero generating function (i.e., at $t = 0$). Therefore, it is enough to compare the ordinary genus zero generating functions.

Next, we consider the reconstruction of genus-zero correlators using the Witten-Dijkgraaf-Verlinde-Verlinde Equation (WDVV).

Definition 6.2.5. We call a class γ *primitive* if it cannot be written as $\gamma = \gamma_1 \star \gamma_2$ for $0 < \deg_{\mathbb{C}}(\gamma_i) < \deg_{\mathbb{C}}(\gamma)$ (or, in the case of our A-model singularity theory, $0 < \deg_W(\gamma_i) < \deg_W(\gamma)$).

We have the following lemma.

LEMMA 6.2.6 (Reconstruction Lemma). *Any genus-zero k-point correlator of the form $\langle \gamma_1, \ldots, \gamma_{k-3}, \alpha, \beta, \varepsilon \star \phi \rangle_0$ can be rewritten as*

$$
(91) \qquad \langle \gamma_1, \ldots, \gamma_{k-3}, \alpha, \beta, \varepsilon \star \phi \rangle = S + \langle \gamma_1, \ldots, \gamma_{k-3}, \alpha, \varepsilon, \beta \star \phi \rangle_0
$$
$$
+ \langle \gamma_1, \ldots, \gamma_{k-3}, \alpha \star \varepsilon, \beta, \phi \rangle_0
$$
$$
- \langle \gamma_1, \ldots, \gamma_{k-3}, \alpha \star \beta, \varepsilon, \phi \rangle_0,
$$

where S is a linear combination of genus-zero correlators with fewer than k insertions.

Moreover, all the genus-zero k-point correlators $\langle \gamma_1, \ldots, \gamma_k \rangle_0$ are uniquely determined by the pairing, by the three-point correlators, and by correlators of the form $\langle \alpha_1, \ldots, \alpha_{k'-2}, \alpha_{k'-1}, \alpha_{k'} \rangle_0$ for $k' \leq k$ and such that α_i primitive for all $i \leq k' - 2$.

Proof. Choose a basis $\{\delta_i\}$ such that $\delta_0 = \varepsilon \star \phi$, and let δ_i' be the dual basis with respect to the pairing (i.e., $\langle \delta_i, \delta_j' \rangle = \delta_{ij}$). Using WDVV and the definition of the multiplication $\star$, we have the formula

$$
(92)
$$
$$
\langle \gamma_1, \ldots, \gamma_{k-3}, \alpha, \beta, \varepsilon \star \phi \rangle_0 = \langle \gamma_1, \ldots, \gamma_{k-3}, \alpha, \beta, \varepsilon \star \phi \rangle_0 \langle \delta_0', \varepsilon, \phi \rangle_0
$$
$$
= \sum_{k-3=I \cup J} \sum_{\ell} \langle \gamma_{i \in I}, \alpha, \varepsilon, \delta_\ell \rangle_0 \langle \delta_\ell', \phi, \beta, \gamma_{j \in J} \rangle_0
$$
$$
- \sum_{\substack{k-3=I \cup J \\ J \neq \emptyset}} \sum_{\ell} \langle \gamma_{i \in I}, \alpha, \beta, \delta_\ell \rangle_0 \langle \delta_\ell', \phi, \varepsilon, \gamma_{j \in J} \rangle_0.
$$

All of the terms on the right-hand side are k'-point correlators with $k' < k$ except

$$
\sum_{\ell} \langle \gamma_{i \leq k-3}, \alpha, \varepsilon, \delta_\ell \rangle_0 \langle \delta_\ell', \phi, \beta \rangle_0
$$
$$
+ \sum_{\ell} \langle \alpha, \varepsilon, \delta_\ell \rangle_0 \langle \delta_\ell', \phi, \beta, \gamma_{j \leq k-3} \rangle_0 - \sum_{\ell} \langle \alpha, \beta, \delta_\ell \rangle_0 \langle \delta_\ell', \phi, \varepsilon, \gamma_{j \leq k-3} \rangle_0
$$
$$
= \langle \gamma_{j \leq k-3}, \alpha, \varepsilon, \phi \star \beta \rangle_0 + \langle \alpha \star \varepsilon, \phi, \beta, \gamma_{j \leq k-3} \rangle_0 - \langle \alpha \star \beta, \varepsilon, \phi, \gamma_{j \leq k-3} \rangle_0,
$$

as desired. This proves equation (91).

Now, suppose that $\langle \gamma_1, \ldots, \gamma_k \rangle_0$ is such that γ_k is not primitive, so $\gamma_k = \varepsilon \star \phi$ with ε primitive. Applying equation (91) shows that $\langle \gamma_1, \ldots, \gamma_k \rangle_0$ can be rewritten as a linear combination of correlators S with fewer insertions plus

three more terms:

$$\langle \gamma_1, \ldots, \gamma_k \rangle_0 = S + \langle \gamma_{j \leq k-3}, \gamma_{k-2}, \varepsilon, \gamma_{k-1} \star \phi \rangle_0$$
$$+ \langle \gamma_{j \leq k-3}, \gamma_{k-2} \star \varepsilon, \gamma_{k-1}, \phi \rangle_0 - \langle \gamma_{j \leq k-3}, \gamma_{k-2} \star \gamma_{k-1}, \varepsilon, \phi \rangle_0.$$

Note that we have replaced $\gamma_{k-2}, \gamma_{k-1}, \gamma_k$ in the original correlator by $\gamma_{k-2}, \varepsilon, \phi$ in the first and third terms and by $\gamma_{k-1}, \phi, \gamma_{k-2} \star \varepsilon$ in the second term. So the first and third terms now have a primitive class ε where there was originally γ_{k-1}. The second term has replaced γ_k by ϕ, which has lower degree. We repeat the above argument on the second term $\langle \gamma_{j \leq k-3}, \gamma_{k-2} \star \varepsilon, \gamma_{k-1}, \phi \rangle_0$ to show that the original correlator $\langle \gamma_1, \ldots, \gamma_{k-3}, \gamma_{k-2}, \gamma_{k-1}, \gamma_k \rangle_0$ can be rewritten in terms of correlators that are either shorter $(k' < k)$ or that have replaced one of the three classes $\gamma_{k-2}, \gamma_{k-1}$, or γ_k by a primitive class.

Now move the primitive class into the set $\gamma_{i \leq k-3}$. Pick another nonprimitive class and continue the induction. In this way, we can replace all the insertions by primitive classes except the last two. $\square$

Definition 6.2.7. We call a correlator a *basic correlator* if it is of the form described in the previous lemma; that is, if all insertions are primitive but the last two.

For a basic correlator, we still have the dimension formula

(93)
$$\sum_i \deg_{\mathbb{C}}(a_i) = \hat{c} + k - 3.$$

This gives the following lemmas.

LEMMA 6.2.8. *If $\deg_{\mathbb{C}}(a) \leq \hat{c}$ for all classes a and if P is the maximum complex degree of any primitive class, then all the genus-zero correlators are uniquely determined by the pairing and k-point correlators with*

(94)
$$k \leq 2 + \frac{1 + \hat{c}}{1 - P}.$$

Proof. Let $\langle a_1, \ldots, a_{k-2}, a_{k-1}, a_k \rangle_0$ be a basic correlator, so $a_{i \leq k-2}$'s are primitive. Then, $\deg_{\mathbb{C}}(a_i) \leq P$ for $i \leq k - 2$ and $deg_{\mathbb{C}}(a_{k-1}), deg_{\mathbb{C}}(a_k) \leq \hat{c}$. By the dimension formula, we have

$$\hat{c} + k - 3 \leq (k - 2)P + 2\hat{c}.   \square$$

LEMMA 6.2.9. *All the genus-zero correlators for the $A_n, D_{n+1}, E_6, E_7, E_8$, and D_{n+1}^T singularities, in either the* A-*model or the* B-*model, are uniquely determined by the pairing, the three-point correlators, and the four-point correlators.*

Proof. Since the pairing, the three-point correlators and the selection rules in the A-model and the B-model have been shown to be mirror, it suffices to prove the conclusion in the A-model side.

Let P be the maximum complex degree of any primitive class. It is easy to obtain the data for these singularities:

$$A_n : P = \tfrac{1}{n+1}, \ \hat{c} = \tfrac{n-1}{n+1}, \qquad\qquad E_6 : P = \tfrac{1}{3}, \ \hat{c} = \tfrac{5}{6},$$
$$E_7 : P = \tfrac{1}{3}, \ \hat{c} = \tfrac{8}{9}, \qquad\qquad E_8 : P = \tfrac{1}{3}, \ \hat{c} = \tfrac{14}{15},$$
$$D_{n+1}(n \text{ even}) : P = \tfrac{1}{n}, \ \hat{c} = \tfrac{n-1}{n}, \qquad D_{n+1}(n \text{ odd}) : P = \tfrac{n-1}{2n}, \ \hat{c} = \tfrac{n-1}{n},$$
$$D_{n+1}^T : P = \tfrac{n-1}{2n}, \ \hat{c} = \tfrac{n-1}{n}.$$

By formula (94), we know that

(1) $k \le 4$ for A_n, E_6, E_7, E_8, and $D_{n+1}(n \text{ even})$ singularities,
(2) $k \le 5$ for $D_{n+1}(n \text{ odd})$ and D_{n+1}^T singularities.

For the singularities $D_{n+1}(n \text{ odd})$ and D_{n+1}^T, we need a more refined estimate.

For the singularity D_{n+1} (n odd), we have the isomorphism

$$(\mathscr{H}_{D_{n+1},\langle J\rangle}, \star) \cong \mathscr{Q}_{D_{n+1}}.$$

Here $\mathscr{Q}_{D_{n+1}}$ is generated by $\{1, X, \dots, X^{n-1}, Y\}$ and satisfies the relations $nX^{n-1} + Y^2 \equiv 0$ and $XY \equiv 0$. X and Y are the only primitive forms, and they have complex degrees as follows:

$$\deg_{\mathbb{C}} X = \frac{1}{n}, \deg_{\mathbb{C}} Y = \frac{n-1}{2n}.$$

The basic genus-zero, five-point correlators may have the form $\langle X, Y, Y, \alpha, \beta \rangle_0$. By the dimension formula (93) for $k = 5$, we have

$$\deg_{\mathbb{C}} \alpha + \deg_{\mathbb{C}} \beta = \hat{c} + 2 - \frac{n-1}{n} - \frac{1}{n} = \frac{2n-1}{n} > \frac{2n-2}{n} = 2\hat{c}.$$

This is impossible, since for any element a, we have $\deg_{\mathbb{C}}(a) \le \hat{c}$. Similarly we can rule out the existence of the basic five-point functions of the form $\langle X, X, X, \alpha, \beta \rangle_0$ and $\langle X, X, Y, \alpha, \beta \rangle_0$. Therefore the only possible basic five-point functions have the form $\langle Y, Y, Y, \alpha, \beta \rangle_0$. In this case, we have the degree formula

$$\deg_{\mathbb{C}} \alpha + \deg_{\mathbb{C}} \beta = \frac{3n+1}{2n}.$$

Because of the fact that $X \star Y \equiv 0$, and for dimension reasons, α or β cannot contain Y. So the only possible form of the basic five-point correlators are

$$\langle Y, Y, Y, X^i, X^{\frac{3n+1}{2}-i} \rangle_0, \ i > 0.$$

Using formula (91) with $\alpha = Y, \beta = X^i, \varepsilon = X$, and $\phi = X^{\frac{3n-1}{2}-i}$, we have

$$
\begin{aligned}
\langle Y, Y, Y, X^i, X^{\frac{3n+1}{2}-i}\rangle_0 &= S + \langle Y, Y, Y, X, X^{\frac{3n-1}{2}}\rangle_0 \\
&\quad + \langle Y, Y, X \star Y, X^i, X^{\frac{3n-1}{2}-i}\rangle_0 \\
&\quad - \langle Y, Y, Y \star X^i, X^{\frac{3n-1}{2}-i}, X\rangle_0 \\
&= S.
\end{aligned}
$$

This shows that any basic, genus-zero, five-point correlators can be uniquely determined by two-, three-, and four-point correlators.

For the D_{n+1}^T singularity, we have the isomorphism

$$
(\mathscr{H}_{D_{n+1}^T}, \star) \cong \mathscr{Q}_{D_{n+1}} = \mathbb{C}[X,Y]/\langle nX^{n-1} + Y^2, XY\rangle
$$

and the degrees for the primitive classes X and Y

$$
\deg_\mathbb{C} X = \frac{1}{n}, \quad \deg_\mathbb{C} Y = \frac{n-1}{2n}.
$$

Hence the reduction from basic five-point correlators to the fewer-point correlators is exactly the same as for the singularity D_{n+1} with n odd. $\qquad\square$

The Reconstruction Lemma yields more detailed information for the basic correlators as well.

THEOREM 6.2.10. (1) *All genus-zero correlators in the A_{n-1} case for both our* (A-*model*) *and the Saito* (B-*model*) *theory are uniquely determined by the pairing, the three-point correlators, and a single four-point correlator of the form* $\langle X, X, X^{n-2}, X^{n-2}\rangle_0$, *where X denotes the primitive class that is the image of x via the Frobenius algebra isomorphism from* $\mathscr{Q}_{A_n} = \mathbb{C}[x]/(x^{n-1})$.

(2) *All genus-zero correlators in the D_{n+1} case of our* (A-*model*) *theory with maximal symmetry group and in the D_{n+1}^T case of the Saito* (B-*model*) *are uniquely determined by the pairing, the three-point correlators, and a single four-point correlator of the form* $\langle X, X, X^{2n-2}, X^{2n-2}\rangle_0$. *Again, X denotes the primitive class that is the image of x via the Frobenius algebra isomorphism from* $\mathscr{Q}_{D_{n+1}} = \mathbb{C}[x,y]/(nx^{n-1} + y^2, xy)$.

(3) *All genus-zero correlators in the D_{n+1}^T case of our* (A-*model*) *theory, in the D_{n+1} case of our theory with n odd and symmetry group $\langle J\rangle$, and in the D_{n+1} case of the Saito* (B-*model*) *theory, are uniquely determined by the pairing, the three-point correlators, and four-point correlators of the form* $\langle X, X, X^{n-1}, X^{n-2}\rangle_0$ *and* $\langle X, X, Y, X^2\rangle_0$. *The second of these occurs only in the case that $n = 3$. Here X and Y denote the primitive classes that are the*

images of x and y, respectively, via the Frobenius algebra isomorphism from $\mathscr{Q}_{D_{n+1}^T} = \mathbb{C}[x,y]/(x^{n-1}y, x^n + 2y)$.

(4) *In the E_6 case of our theory (A-model) with maximal symmetry group and in the E_6 case of the Saito (B-model) theory, all genus-zero correlators are uniquely determined by the pairing, the three-point correlators, and the correlators* $\langle Y, Y, Y^2, XY^2 \rangle_0$ *and* $\langle X, X, XY, XY \rangle_0$. *Here X and Y denote the primitive classes that are the images of x and y, respectively, via the Frobenius algebra isomorphism from* $\mathscr{Q}_{E_6} = \mathbb{C}[x,y]/(x^2, y^3)$.

(5) *In the E_7-case of our theory (A-model) with maximal symmetry group and in the E_7 case of the Saito (B-model) theory, all genus-zero correlators are uniquely determined by the pairing, the three-point correlators, and the correlators* $\langle X, X, X^2, XY \rangle_0$, $\langle X, Y, X^2, X^2 \rangle_0$, *and* $\langle Y, Y, XY, X^2Y \rangle_0$. *Here X and Y denote the primitive classes that are the images of x and y, respectively, via the Frobenius algebra isomorphism from* $\mathscr{Q}_{E_7} = \mathbb{C}[x,y]/(3x^2 + y^3, xy^2)$.

(6) *In the E_8-case of our theory (A-model) with maximal symmetry group and in the E_8 Saito (B-model) theory, all genus-zero correlators are uniquely determined by the pairing, the three-point correlators, and by the correlators* $\langle Y, Y, Y^3, XY^3 \rangle_0$, *and* $\langle X, X, X, XY^3 \rangle_0$. *Here X and Y denote the primitive classes that are the images of x and y, respectively, via the Frobenius algebra isomorphism from* $\mathscr{Q}_{E_8} = \mathbb{C}[x,y]/(x^2, y^4)$.

Proof. Applying Lemma 6.2.9, all genus zero correlators are uniquely determined by the pairing, three- or four-point correlators. Let us study the genus zero four-point correlators in more detail.

In the A_{n-1} case, X is the only ring generator and hence the only primitive class. It has $\deg_{\mathbb{C}} X = 1/(n+1)$. A dimension count shows that the only four-point correlator of the form $\langle X, X, \alpha, \beta \rangle_0$ is $\langle X, X, X^{n-2}, X^{n-2} \rangle_0$.

A similar argument shows that in the D_{n+1} A-model with the maximal symmetry group and D_{n+1}^T B-model cases the only basic four-point correlator is $\langle X, X, X^{2n-2}, X^{2n-2} \rangle_0$.

In the case of the D_{n+1}^T A-model, and for the D_{n+1} A-model for n odd with symmetry group J, and for the D_{n+1} B-model, the central charges are the same, $\hat{c} = \frac{n-1}{n}$, and all have only two primitive classes X and Y with the same degrees

$$\deg_{\mathbb{C}} X = \frac{1}{n}, \ \deg_{\mathbb{C}} Y = \frac{n-1}{2n}.$$

Hence the basic four-point correlators are the same for the three cases. Let us consider the case D_{n+1} A-model for n odd with symmetry group J. There are several cases for the form of the basic four-point correlators.

Case A: form $\langle X, X, \alpha, \beta \rangle_0$. The dimension formula shows that $\deg_{\mathbb{C}} \alpha + \deg_C \beta = \frac{2n-3}{n}$. So the only possibility is $\langle X, X, X^{n-1}, X^{n-2} \rangle_0$.

Case B: form $\langle X, Y, \alpha, \beta \rangle_0$. By the dimension formula, we have $\deg_{\mathbb{C}} \alpha + \deg_C \beta = \frac{3n-3}{2n}$. There are two cases:

Case B1: α, β do not contain Y. Then for $j > 1$, the correlator has the form $\langle X, Y, X^i, X^j \rangle_0$. Setting $\alpha = Y, \beta = X^i, \varepsilon = X^{j-1}, \phi = X$ in formula (91), we have

$$
\begin{aligned}
\langle X, Y, X^i, X^j \rangle_0 &= S + \langle X, Y, X^{j-1}, X^{i+1} \rangle_0 \\
&\quad + \langle X, Y \star X^{j-1}, X^i, X \rangle_0 - \langle X, Y \star X^i, X^{j-1}, X \rangle_0 \\
&= S + \langle X, Y, X^{j-1}, X^{i+1} \rangle_0 = \cdots = S + \langle X, Y, X, X^{i_0} \rangle_0.
\end{aligned}
$$

The dimension formula shows that the only four-point correlator $\langle X, Y, X, X^i \rangle_0$ does not vanish only if $n = 3$ and in this case $i = 2$.

Case B2: α, β contain Y. In this case $\langle X, Y, \alpha, \beta \rangle_0$ has the form $\langle Y, Y, X, \beta \rangle_0$ which can be included in the following Case C.

Case C: form $\langle Y, Y, \alpha, \beta \rangle_0$. We have the degree formula $\deg_{\mathbb{C}} \alpha + \deg_C \beta = 1$. There are two cases:

Case C1: α, β do not contain Y. We have the form $\langle Y, Y, X^i, X^j \rangle_0$ with $j > 1$. Let $\alpha = Y, \beta = X^i, \varepsilon = X^{j-1}, \phi = X$ in formula (91); we obtain

$$
\begin{aligned}
\langle Y, Y, X^i, X^j \rangle_0 &= S + \langle Y, Y, X^{j-1}, X^{i+1} \rangle_0 \\
&\quad + \langle Y, Y \star X^{j-1}, X^i, X \rangle_0 - \langle Y, Y \star X^i, X^{j-1}, X \rangle_0 \\
&= S + \langle Y, Y, X^{j-1}, X^{i+1} \rangle_0 = \cdots = S + \langle Y, Y, X, X^{n-1} \rangle_0.
\end{aligned}
$$

Now

$$
\begin{aligned}
\langle Y, Y, X, X^{n-1} \rangle_0 &= \langle X, Y, Y, X^{n-1} \rangle_0 \\
&= S + \langle X, Y, X, Y \star X^{n-2} \rangle_0 \\
&\quad + \langle X, X \star Y, Y, X^{n-2} \rangle_0 - \langle X, Y^2, X^{n-2}, X \rangle_0 \\
&= S.
\end{aligned}
$$

Case C2: α, β contain Y. The basic correlator has the form $\langle Y, Y, Y, X^{\frac{n+1}{2}} \rangle_0$. Similarly, we have

$$
\begin{aligned}
\langle Y, Y, Y, X^{\frac{n+1}{2}} \rangle_0 &= S + \langle Y, Y, X, Y \star X^{\frac{n-1}{2}} \rangle_0 \\
&\quad + \langle Y, Y \star X, Y, X^{\frac{n-1}{2}} \rangle_0 - \langle Y, Y^2, X^{\frac{n-1}{2}}, X \rangle_0 \\
&= S.
\end{aligned}
$$

In summary, if $n > 3$, then the basic four-point correlator is only

$$
\langle X, X, X^{n-1}, X^{n-2} \rangle_0;
$$

if $n = 3$, then the basic four-point correlators are

$$
\langle X, X, X^2, X \rangle_0, \quad \langle X, Y, X, X^2 \rangle_0.
$$

In the E_6 case, the primitive classes are X, Y. The dimension condition shows that the only four-point correlators with two primitive insertions are

$$\langle Y, Y, Y^2, XY^2 \rangle_0, \ \ \langle X, X, X, XY^2 \rangle_0, \ \ \langle X, X, XY, XY \rangle_0.$$

Applying equation (91) and the fact that $X^2 = 0$, we can reduce $\langle X, X, X, XY^2 \rangle_0$ to $\langle X, X, XY, XY \rangle_0$.

In the E_7 case, the primitive classes are X and Y with $\deg_{\mathbb{C}} X = 1/3$, $\deg_{\mathbb{C}} Y = 2/9$, and $\hat{c} = 8/9$. The dimension condition shows that the only basic four-point correlators are

$$\langle X, X, X^2, XY \rangle_0, \ \ \langle X, X, X, X^2Y \rangle_0, \ \ \langle X, Y, X^2, X^2 \rangle_0,$$
$$\langle X, Y, Y^2, X^2Y \rangle_0, \ \ \langle Y, Y, XY, X^2Y \rangle_0.$$

We can use equation (91) to further reduce $\langle X, X, X, X^2Y \rangle_0$ to the remaining four and to reduce $\langle X, Y, Y^2, X^2Y \rangle_0 = \langle Y, X, Y^2, X^2Y \rangle_0$ to $\langle Y, Y, XY, X^2Y \rangle_0$.

Finally, in the E_8 case, a dimension count shows that the only basic four-point correlators are

$$\langle X, X, X, XY^3 \rangle_0, \ \ \langle X, X, XY, XY^2 \rangle_0, \ \ \langle Y, Y, Y^3, XY^3 \rangle_0.$$

Again equation (91) shows that $\langle X, X, XY, XY^2 \rangle_0$ can be expressed in terms of $\langle X, X, X, XY^3 \rangle_0$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

6.3. **Computation of the basic four-point correlators in the A-model.**

6.3.1. *Computing classes in complex codimension one.*

Definition 6.3.1. Let $\mathbf{\Gamma}_{g,k,W}$ denote the set of all connected single-edged W-graphs of genus g with k tails decorated by elements of $\mathscr{H}_W$. Further, denote by $\mathbf{\Gamma}_{g,k,W,\mathrm{cut}}$ the set of all W-graphs with no edges (possibly disconnected), but with one pair of tails labeled $+$ and $-$, respectively, such that gluing the tail $+$ to the tail $-$ gives an element of $\mathbf{\Gamma}_{g,k,W}$. Furthermore, we require that the decorations γ_+ and γ_- satisfy $\gamma_+ \gamma_- = 1$.

Similarly, let $\mathbf{\Gamma}_{g,k,W}(\gamma_1, \ldots, \gamma_k)$ and $\mathbf{\Gamma}_{g,k,W,\mathrm{cut}}(\gamma_1, \ldots, \gamma_k)$ denote the subset of $\mathbf{\Gamma}_{g,k,W}$ and of $\mathbf{\Gamma}_{g,k,W}$, respectively, consisting of decorated W-graphs with the i-th tail decorated by γ_i for each $i \in \{1, \ldots, k\}$.

For any graph $\Gamma_{\mathrm{cut}} \in \mathbf{\Gamma}_{g,k,W,\mathrm{cut}}$, we denote by $\Gamma \in \mathbf{\Gamma}_{g,k,W}$ the uniquely determined graph in $\mathbf{\Gamma}_{g,k,W}$ obtained by gluing the two tails $+$ and $-$. We further denote the underlying undecorated graph by $|\Gamma|$, and we denote the closure in $\overline{\mathscr{M}}_{g,k}$ of the locus of stable curves with dual graph $|\Gamma|$ by $\overline{\mathscr{M}}(|\Gamma|)$. Finally, denote the Poincaré dual of this locus by $\left[\overline{\mathscr{M}}(|\Gamma|)\right] \in H^*(\overline{\mathscr{M}}_{g,k}, \mathbb{C})$.

Remark 6.3.2. In genus zero, the local group at an edge (or at the tails labelled $+$ and $-$) is completely determined by the local group at each of the tails.

THEOREM 6.3.3. *Assume the W-structure is concave (i.e., $\pi_* \left(\bigoplus_{i=1}^{t} \mathscr{L}_i \right) = 0$) with all marks narrow. If the i-th mark is labeled with group element γ_i and if the complex codimension D is 1, then the class $\Lambda_{g,k}^W(\mathbf{e}_{\gamma_1}, \ldots, \mathbf{e}_{\gamma_k}) \in H^*(\overline{\mathscr{M}}_{g,k}, \mathbb{C})$ is given by the following:*

$$(95) \quad \Lambda_{g,k}^W(\mathbf{e}_{\gamma_1}, \ldots, \mathbf{e}_{\gamma_k})$$

$$= \sum_{\ell=1}^{N} \left[\left(\frac{q_\ell^2}{2} - \frac{q_\ell}{2} + \frac{1}{12} \right) \kappa_1 - \sum_{i=1}^{k} \left(\frac{1}{12} - \frac{1}{2}\Theta_\ell^{\gamma_i}(1 - \Theta_\ell^{\gamma_i}) \right) \psi_i \right.$$

$$\left. + \frac{1}{2} \sum_{\Gamma_{\mathrm{cut}} \in \mathbf{\Gamma}_{g,k,W,\mathrm{cut}}(\gamma_1, \ldots, \gamma_k)} \left(\frac{1}{12} - \frac{1}{2}\Theta_\ell^{\gamma_+}(1 - \Theta_\ell^{\gamma_+}) \right) \left[\overline{\mathscr{M}}(|\Gamma|) \right] \right].$$

Proof. The proof follows from the orbifold Grothendieck-Riemann-Roch (oGRR) theorem [Toe99]. After we finished this paper, we became aware of an elegant alternative treatment of this type of problem by Chiodo [Chi08]. We now review the oGRR theorem in the case we are interested in, namely, orbicurves. For more details on oGRR, see [Tse10, App. A].

For any k-pointed family of stable orbicurves $(\mathscr{C} \xrightarrow{\pi} T, \sigma_1, \ldots, \sigma_k)$ over a scheme T, with W-structure $(\mathscr{L}_1, \ldots, \mathscr{L}_N, \phi_1, \ldots, \phi_s)$, if the W-structure has type $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_k)$ then the inertia stack $\bigwedge \mathscr{C} = \coprod_{g \in G} \mathscr{C}_{(g)}$ consists of the following sectors:

$$\bigwedge \mathscr{C} = \mathscr{C} \sqcup \coprod_{i=1}^{k} \coprod_{j=1}^{r_i-1} \mathscr{S}_i(\gamma_i^j) \sqcup \coprod_{\Gamma \in \mathbf{\Gamma}_{g,k,W}(\boldsymbol{\gamma})} \coprod_{j=1}^{r_\Gamma-1} \mathscr{Z}_\Gamma(\gamma_\Gamma^j).$$

Here r_i is the order of the element γ_i and r_Γ is the order of the element γ_Γ. Also, $\mathscr{S}_i(\gamma_i^j) := \mathscr{S}_i$ is the i-th gerbe-section of π; that is, the image of σ_i with the orbifold structure inherited from $\mathscr{C}$. The notation $\mathscr{S}_i(\gamma_i^j)$ just indicates that this is part of the γ_i^j-sector of $\bigwedge \mathscr{C}$. Similarly, $\mathscr{Z}_\Gamma(\gamma_\Gamma^j) := \mathscr{Z}_\Gamma$ is the locus of nodes in $\mathscr{C}$ with dual graph Γ lying in the γ_Γ^j-sector. As in the case of marks, the nodal sector $\mathscr{Z}_\Gamma$ should be given the orbifold structure it inherits from $\mathscr{C}$.

Let $\upsilon : \bigwedge \mathscr{C} \longrightarrow \mathscr{C}$ denote the obvious union of inclusions. Furthermore, let $I\pi : \bigwedge \mathscr{C} \longrightarrow T$ denote the composition $I\pi = \pi \circ \upsilon$. And let $\rho : K(\bigwedge \mathscr{C}) \longrightarrow K(\bigwedge \mathscr{C}) \otimes \mathbb{C}$ denote the Atiyah-Segal decomposition

$$\rho(E) = \sum_\zeta \zeta E_{\gamma,\zeta},$$

where for each sector $\mathscr{C}_{(\gamma)}$, the sum runs over eigenvalues ζ of the action of γ on E, and $E_{\gamma,\zeta}$ denotes the eigenbundle of E where γ acts as ζ.

Define

$$\widetilde{\mathrm{Ch}} = \mathrm{Ch} \circ \rho \circ \upsilon^* : K(\mathscr{C}) \longrightarrow H^*\left(\bigwedge \mathscr{C}, \mathbb{C}\right)$$

and

$$\widetilde{\mathrm{Td}}(E) := \frac{\mathrm{Td}((\upsilon^* E)_1)}{\mathrm{Ch}(\rho \circ \lambda_{-1}(\sum_{\zeta \neq 1} (\upsilon^* E)_\zeta)^\vee)}.$$

The oGRR theorem states that for any bundle E on $\mathscr{C}$, we have

(96) $$\widetilde{\mathrm{Ch}}(R\pi_* E) = I\pi_*(\widetilde{\mathrm{Ch}}(E)\widetilde{\mathrm{Td}}(T_\pi))$$

Writing this out explicitly for one of the W-structure bundles $\mathscr{L}_\ell$ on our W-curve $\mathscr{C} \xrightarrow{\pi} T$, we have

(97) $\mathrm{Ch}(\pi_* \mathscr{L}_\ell \ominus R^1 \pi_* \mathscr{L}_\ell) = \pi_*(\mathrm{Ch}(\mathscr{L}_\ell)\mathrm{Td}(T_\pi))$

$$+ \sum_{i=1}^{k} \sum_{j=1}^{r_i-1} \pi_* \left(\frac{\exp\left(2\pi i \Theta_\ell^{\gamma_i^j} c_1(\upsilon^* \mathscr{L}_\ell)\right)}{(1 - \exp(2\pi i j q_\ell c_1(\upsilon^* K)))} \right) + \frac{1}{2} \sum_{\Gamma_{\mathrm{cut}} \in \mathbf{\Gamma}_{g,k,W,\mathrm{cut}}(\gamma)}$$

$$\times \sum_{j=1}^{r_{\Gamma_{\mathrm{cut}}}-1} \pi_* \left(\frac{\exp\left(2\pi i \Theta_\ell^{\gamma_+^j} c_1(\upsilon^* \mathscr{L}_\ell)\right)}{(1 - \exp(2\pi i j q_\ell c_1(\upsilon^* K)))(1 - \exp(-2\pi i j q_\ell c_1(\upsilon^* K)))} \right).$$

For our present purposes, we need only compute the codimension-one part of this sum. Denote the first Chern class of $\mathscr{L}_\ell$ on $\mathscr{C}$ by L_ℓ. Note that because $\mathscr{L}_\ell$ is part of the W-structure and because the singularity is nondegenerate (so the matrix B has maximal rank), we have

$$L_\ell = c_1(\mathscr{L}_\ell) = q_\ell K_{\log}.$$

Copying Mumford's argument given in [Mum83, §5], one computes that the codimension-one part of the untwisted sector contribution to this sum is

$$\pi_* \left(L_\ell^2/2 - L_\ell K/2 + K^2/12 + \frac{1}{24} \sum_{\Gamma_{\mathrm{cut}} \in \mathbf{\Gamma}_{g,k,W,\mathrm{cut}}(\gamma)} i_{\Gamma *}(1) \right),$$

where i_Γ is the inclusion into $\mathscr{C}$ of the nodes corresponding to the edge of Γ.

For each sector $\mathscr{S}_i(\gamma_i^j)$, the induced map π_* is just $\frac{1}{r_i}\sigma_i^*$; therefore, on these sectors we have

$$\pi_* L_\ell = \frac{1}{r_i}\sigma_i^* L_\ell = \frac{q_\ell}{r_i}\sigma_i^* K_{\log} = 0.$$

Now γ_i acts on the canonical bundle K at the mark $\mathscr{S}_i$ by multiplication by $\xi_i := \exp(2\pi i/r_i)$, and it acts on $\mathscr{L}_\ell$ at $\mathscr{S}_i$ by $\exp(2\pi i \Theta_\ell^{\gamma_i}) = \xi_i^{a_i}$ for $a_i :=$

$r\Theta_\ell^{\gamma_i} \in [0, r_i) \cap \mathbb{Z}$. Expanding the denominator in equation (97), one sees that the codimension-one part of the contribution from the marks is

$$(98) \qquad \sum_{i=1}^{k} \sum_{j=1}^{r_i-1} \frac{\xi_i^{(a_i+1)j}}{r_i(1-\xi_i^j)^2} \widetilde{\psi}_i.$$

Similarly, letting $\xi_\Gamma := \exp(2\pi i/r_\Gamma)$ and choosing $a_\Gamma := r_\Gamma \Theta_\ell^{\gamma_\Gamma} \in [0, r_\Gamma) \cap \mathbb{Z}$ so that $\xi_\Gamma^{a_\Gamma} = \exp(2\pi i \Theta_\ell^{\gamma_\Gamma})$, one sees that the contribution to equation (97) from the nodes is

$$(99) \qquad \frac{1}{2} \sum_{\Gamma_{\mathrm{cut}} \in \mathbf{\Gamma}_{g,k,W,\mathrm{cut}}(\boldsymbol{\gamma})} \sum_{j=1}^{r_{\Gamma_{\mathrm{cut}}}-1} \frac{-\xi_i^{(a_{\Gamma_{\mathrm{cut}}}+1)j}}{(1-\xi_{\Gamma_{\mathrm{cut}}}^j)^2} \pi_*(i_{\Gamma*}(1)).$$

A long but elementary computation shows that for any primitive r-th root ζ of unity and any $a \in [0, r) \cap \mathbb{Z}$, we have[‡]

$$(100) \qquad \sum_{j=1}^{r-1} \frac{\zeta^{(a+1)j}}{(1-\zeta^j)^2} = \frac{1-r^2}{12} + \frac{1}{2}a(r-a).$$

Using the definition $\kappa_1 = \pi_*(c_1(K_{\log}))^2 = \pi_*(c_1(K))^2 + \sum_{i=1}^{k} \widetilde{\psi}_i$, together with equation (100) and the fact that $a_i/r_i = \Theta_\ell^{\gamma_i}$ and $a_\Gamma/r_\Gamma = \Theta_\ell^{\gamma_\Gamma}$, we now have

$$\mathrm{Ch}(\pi_*\mathscr{L}_\ell \ominus R^1\pi_*\mathscr{L}_\ell)$$

$$= \left(\frac{q_\ell^2}{2} - \frac{q_\ell}{2} + \frac{1}{12}\right)\kappa_1 - \sum_{i=1}^{k} \frac{\widetilde{\psi}_i}{12} + \sum_{\Gamma \in \mathbf{\Gamma}_{g,k,W}(\boldsymbol{\gamma})} \frac{\pi_* i_{\Gamma*}(1)}{12}$$

$$- \sum_{i=1}^{k} \frac{r_i}{12}\left(\frac{1}{r_i^2} - 1 + 6\,\Theta_\ell^{\gamma_i}(1-\Theta_\ell^{\gamma_i})\right)\widetilde{\psi}_i$$

$$- \frac{1}{2} \sum_{\Gamma_{\mathrm{cut}} \in \mathbf{\Gamma}_{g,k,W,\mathrm{cut}}(\boldsymbol{\gamma})} \frac{r_{\Gamma_{\mathrm{cut}}}^2}{12}\left(\frac{1}{r_{\Gamma_{\mathrm{cut}}}^2} - 1 + 6\Theta_\ell^{\gamma+}(1-\Theta_\ell^{\gamma+})\right)\pi_* i_{\Gamma*}(1)$$

$$= \left(\frac{q_\ell^2}{2} - \frac{q_\ell}{2} + \frac{1}{12}\right)\kappa_1 - \sum_{i=1}^{k}\left(\frac{1}{12} - \frac{1}{2}\Theta_\ell^{\gamma_i}(1-\Theta_\ell^{\gamma_i})\right)\psi_i$$

$$+ \frac{1}{2} \sum_{\Gamma_{\mathrm{cut}} \in \mathbf{\Gamma}_{g,k,W,\mathrm{cut}}(\boldsymbol{\gamma})} r_{\Gamma_{\mathrm{cut}}}\left(\frac{1}{12} - \frac{1}{2}\Theta_\ell^{\gamma+}(1-\Theta_\ell^{\gamma+})\right)\left[\overline{\mathscr{W}}(\Gamma)\right],$$

where the last equality follows from the fact that $\widetilde{\psi}_i = \psi_i/r_i$ and $\pi_* i_{\Gamma*}(1) = \left[\overline{\mathscr{W}}(\Gamma)\right]/r_{\Gamma_{\mathrm{cut}}}$.

[‡]We would like to thank H. Tracy Hall for showing us this relation.

Finally, in the concave case, we have $\pi_*(\mathscr{L}_\ell) = 0$, so pushing down to $\overline{\mathscr{M}}_{g,k}$ gives

$$
\begin{aligned}
\Lambda_{g,k}^W(\mathbf{e}_{\gamma_1}, \ldots, \mathbf{e}_{\gamma_k}) &= \frac{1}{\deg(\mathrm{st})} \sum_{\ell=1}^N \mathrm{st}_* c_1(-R^1\pi_*\mathscr{L}_\ell) \\
&= \sum_{\ell=1}^N \left[\left(\frac{q_\ell^2}{2} - \frac{q_\ell}{2} + \frac{1}{12} \right) \kappa_1 - \sum_{i=1}^k \left(\frac{1}{12} - \frac{1}{2}\Theta_\ell^{\gamma_i}(1 - \Theta_\ell^{\gamma_i}) \right) \psi_i \right. \\
&\quad \left. + \frac{1}{2} \sum_{\Gamma_{\mathrm{cut}} \in \mathbf{\Gamma}_{g,k,W,\mathrm{cut}}(\boldsymbol{\gamma})} \left(\frac{1}{12} - \frac{1}{2}\Theta_\ell^{\gamma_+}(1 - \Theta_\ell^{\gamma_+}) \right) \left[\overline{\mathscr{M}}(|\Gamma|) \right] \right]
\end{aligned}
$$

since κ_1 and ψ_i on $\overline{\mathscr{W}}_{g,k}(W)$ are equal to the pullbacks $\mathrm{st}^*\kappa_1$ and $\mathrm{st}^*\psi_i$, respectively, and $\left[\overline{\mathscr{W}}(\Gamma) \right] = \mathrm{st}^* \left[\overline{\mathscr{M}}(|\Gamma|) \right] / r_\Gamma$. $\qquad \square$

6.3.2. *Four-point correlators for E_7.* Now we compute the genus-zero four-point correlators for E_7 with symmetry group $G_{E_7} = \langle J \rangle$. We will continue to use the notation of Section 5.2.2. By Theorem 6.2.10(5) we need only compute the following correlators to completely determine the Frobenius manifold, and thereby the entire cohomological field theory:

$$
\langle Y, Y, XY, X^2Y \rangle_0^{E_7}, \quad \langle X, Y, X^2, X^2 \rangle_0^{E_7}, \quad \langle X, X, X^2, XY \rangle_0^{E_7}.
$$

We use the identification of X, Y with the A-model classes from last section. To simplify the notation, we choose $\alpha = 1$ instead of $\alpha^8 = \frac{1}{9}$. Later, we will re-scale the primitive form to take care of discrepancy between the pairing. These correspond to the correlators

$$
\langle \mathbf{e}_2, \mathbf{e}_5, \mathbf{e}_5, \mathbf{e}_8 \rangle_0^{E_7}, \quad \langle \mathbf{e}_4, \mathbf{e}_4, \mathbf{e}_5, \mathbf{e}_7 \rangle_0^{E_7}, \quad \langle \mathbf{e}_2, \mathbf{e}_4, \mathbf{e}_7, \mathbf{e}_7 \rangle_0^{E_7}.
$$

These are all concave and have only narrow markings, so we may use Theorem 6.3.3 to compute them. To apply that theorem, we need to use the fact that

$$
\int_{\overline{\mathscr{M}}_{0,4}} \kappa_1 = \int_{\overline{\mathscr{M}}_{0,4}} \psi_i = \int_{\overline{\mathscr{M}}_{0,4}} \left[\overline{\mathscr{M}}(|\Gamma|) \right] = 1
$$

for every $i \in \{1, \ldots, 4\}$ and every graph $\Gamma \in \mathbf{\Gamma}_{0,4}$. We also need to compute the group element γ_Γ for each of the four-pointed, genus-zero, decorated W-graphs. This is uniquely determined by the fact that the sum of the powers of J on each three-point correlator must be congruent to $1 \mod 9$. We will work out the details in the case of $\langle \mathbf{e}_2, \mathbf{e}_4, \mathbf{e}_7, \mathbf{e}_7 \rangle_0^{E_7}$—the others are computed in a similar manner.

There are three graphs in $\mathbf{\Gamma}_{0,4,E_7}(J^2, J^4, J^7, J^7)$. The first we will denote by Γ_1. It is depicted in Figure 1. There are two cut graphs $\Gamma_{1,\mathrm{cut}}, \Gamma'_{1,\mathrm{cut}} \in \mathbf{\Gamma}_{0,4,E_7,\mathrm{cut}}(J^2, J^4, J^7, J^7)$ that glue to give the graph Γ_1. These have the tail $+$ labeled with $\gamma_+ = J^4$ and the tail $-$ labeled with $\gamma_- = J^5$, or in the second
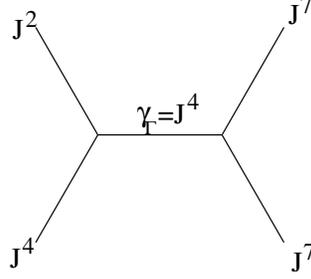
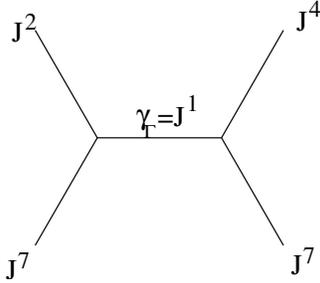Figure 1. The graph $\Gamma_1 \in \mathbf{\Gamma}_{0,4,E_7}(J^2, J^4, J^7, J^7)$.

Figure 2. The graph Γ_2. Two of the three graphs in $\mathbf{\Gamma}_{0,4,E_7}(J^2, J^4, J^7, J^7)$ are decorated as in this figure.

case, $\gamma_+ = J^5$ and $\gamma_- = J^4$. The formula gives the same result for each of these two cases, canceling the factor of $\frac{1}{2}$ outside the sum for this term.

The other two graphs are both decorated as in Figure 2. We will abuse notation and denote both of them by Γ_2 and simply count the contribution of Γ_2 twice. The edge of Γ_2 is labeled with $\gamma_+ = J$ or $\gamma_+ = J^8$, and again, the contribution to the formula from these two choices is identical and cancels the factor of $\frac{1}{2}$ outside the sum.

Now, it is easy to check that the degree of $\mathscr{L}_x$ is -1, so $R^1\pi_*\mathscr{L}_x = 0$ and this will not contribute to the correlator. We have

$$\langle X, X, X^2, XY \rangle_0^{E_7} = \int_{\overline{\mathcal{M}}_{0,4}} \Lambda_{0,4}^{E_7}(\mathbf{e}_2, \mathbf{e}_4, \mathbf{e}_7, \mathbf{e}_7)$$

$$= \left(\frac{q_y^2}{2} - \frac{q_y}{2} + \frac{1}{12} \right) - \sum_{i=1}^{4} \left(\frac{1}{12} - \frac{1}{2}\Theta_y^{\gamma_i}(1 - \Theta_y^{\gamma_i}) \right) + \left(\frac{1}{12} - \frac{1}{2}\Theta_y^{\gamma_{\Gamma_1}}(1 - \Theta_y^{\gamma_{\Gamma_1}}) \right)$$

$$+ 2\left(\frac{1}{12} - \frac{1}{2}\Theta_y^{\gamma_{\Gamma_2}}(1 - \Theta_y^{\gamma_{\Gamma_2}}) \right)$$

$$= \left(\frac{q_y^2}{2} - \frac{q_y}{2} \right) + \left(\frac{1}{2}\Theta_y^{J^2}(1 - \Theta_y^{J^2}) \right) + \left(\frac{1}{2}\Theta_y^{J^4}(1 - \Theta_y^{J^4}) + 2\left(\frac{1}{2}\Theta_y^{J^7}(1 - \Theta_y^{J^7}) \right) \right)$$

$$- \left(\frac{1}{2} \Theta_y^{J^4} (1 - \Theta_y^{J^4}) \right) - 2 \left(\frac{1}{2} \Theta_y^{J^1} (1 - \Theta_y^{J^1}) \right)$$

$$= \left(\frac{4}{2 \cdot 81} - \frac{2}{2 \cdot 9} \right) + \left(\frac{1}{2} \cdot \frac{4}{9} \cdot \frac{5}{9} \right) + \left(\frac{1}{2} \cdot \frac{8}{9} \cdot \frac{1}{9} \right) + 2 \left(\frac{1}{2} \cdot \frac{5}{9} \cdot \frac{4}{9} \right)$$

$$- \left(\frac{1}{2} \cdot \frac{8}{9} \cdot \frac{1}{9} \right) - 2 \left(\frac{1}{2} \cdot \frac{2}{9} \cdot \frac{7}{9} \right)$$

$$= \frac{1}{9}.$$

Similar computations show that

$$\langle X, Y, X^2, X^2 \rangle_0^{E_7} = -\frac{1}{9}$$

and

$$\langle Y, Y, XY, X^2 Y \rangle_0^{E_7} = \frac{1}{3}.$$

6.3.3. *Four-point correlators for E_6.* By Theorem 6.2.10 we need only compute the correlators $\langle Y, Y, Y^2, XY^2 \rangle_0^{E_6}$ and $\langle X, X, XY, XY \rangle_0^{E_6}$. Here, again, we choose $\alpha = 1$. These correspond to the correlators $\langle \mathbf{e}_{10}, \mathbf{e}_{10}, \mathbf{e}_7, \mathbf{e}_{11} \rangle_0^{E_6}$ and $\langle \mathbf{e}_5, \mathbf{e}_5, \mathbf{e}_2, \mathbf{e}_2 \rangle_0^{E_6}$.

The correlators in question are easily seen to be concave. Applying Theorem 6.3.3 in a manner similar to the previous computations, we find that

$$\langle Y, Y, Y^2, XY^2 \rangle_0^{E_6} = \langle \mathbf{e}_{10}, \mathbf{e}_{10}, \mathbf{e}_7, \mathbf{e}_{11} \rangle_0^{E_6} = \frac{1}{4}$$

and

$$\langle X, X, XY, XY \rangle_0^{E_6} = \langle \mathbf{e}_5, \mathbf{e}_5, \mathbf{e}_2, \mathbf{e}_2 \rangle_0^{E_6} = \frac{1}{3}.$$

6.3.4. *Four-point correlators for E_8.* By Theorem 6.2.10 we need only compute the correlators $\langle Y, Y, Y^3, XY^3 \rangle_0$ and $\langle X, X, X, XY^3 \rangle_0$ with $\alpha = 1$. These correspond to the correlators $\langle \mathbf{e}_7, \mathbf{e}_7, \mathbf{e}_4, \mathbf{e}_{14} \rangle_0$ and $\langle \mathbf{e}_{11}, \mathbf{e}_{11}, \mathbf{e}_{11}, \mathbf{e}_{14} \rangle_0$.

The correlators in question are easily seen to be concave. Applying Theorem 6.3.3 in a manner similar to the previous computations, we find that

$$\langle Y, Y, Y^3, XY^3 \rangle_0^{E_8} = \langle \mathbf{e}_7, \mathbf{e}_7, \mathbf{e}_4, \mathbf{e}_{14} \rangle_0^{E_8} = \frac{1}{5}$$

and

$$\langle X, X, X, XY^3 \rangle_0^{E_8} = \langle \mathbf{e}_{11}, \mathbf{e}_{11}, \mathbf{e}_{11}, \mathbf{e}_{14} \rangle_0^{E_8} = \frac{1}{3}.$$

6.3.5. *Four-point correlators for D_{n+1} with n odd and symmetry group $\langle J \rangle$.* Next consider the case of D_{n+1} for n odd with symmetry group $\langle J \rangle$. We will use the notation of Section 5.2.4 but with $\sigma = 1$ instead. By Theorem 6.2.10 we need only compute the correlator

$$\langle X, X, X^{n-1}, X^{n-2} \rangle_0^{D_{n+1}} = \langle \mathbf{e}_3, \mathbf{e}_3, -2\mathbf{e}_3^{n-1}, -2\mathbf{e}_3^{n-2} \rangle_0^{D_{n+1}}.$$
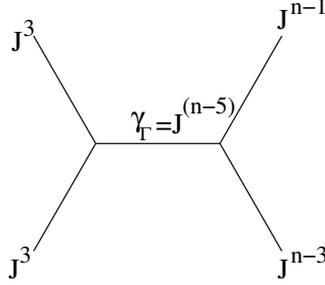
Figure 3. The graph $\Gamma_1 \in \mathbf{\Gamma}_{0,4,D_{n+1}}(J^3, J^3, J^{n-1}, J^{n-3})$.
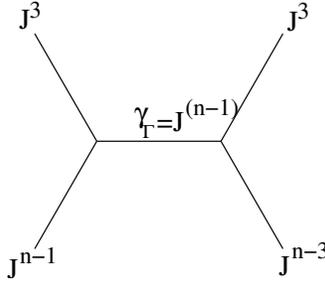

Figure 4. The graph Γ_2. Two of the three graphs in $\mathbf{\Gamma}_{0,4,D_{n+1}}(J^3, J^3, J^{n-1}, J^{n-3})$ are decorated as in this figure.

To apply Theorem 6.3.3 we need only compute the group element acting at the node over the three boundary graphs.

There are three uncut graphs in $\mathbf{\Gamma}_{0,4,D_{n+1}}(\mathbf{e}_3, \mathbf{e}_3, \mathbf{e}_{n-1}, \mathbf{e}_{n-3})$. The first we will denote by Γ_1. It is depicted in Figure 3. As before, the choice of labeling the internal edge with $+$ and $-$ gives each term in the sum twice and will exactly cancel the factor of $\frac{1}{2}$ in each case. The edge of Γ_1 is labeled with $\gamma_{\Gamma_1} = J^{-a}$ for $a = n - 5$, assuming $n > 3$. This gives

$$\Theta_x^{\gamma_{\Gamma_1}}(1 - \Theta_x^{\gamma_{\Gamma_1}}) = \frac{5(n-5)}{n^2} \quad \text{and} \quad \Theta_y^{\gamma_{\Gamma_1}}(1 - \Theta_y^{\gamma_{\Gamma_1}}) = \frac{n^2 - 25}{4n^2}.$$

The other two graphs are both decorated as in Figure 4. We will abuse notation and denote both of them by Γ_2 and simply count the contribution of Γ_2 twice. The edge of Γ_2 is labeled with $\gamma_{\Gamma_2} = J^{(n-1)}$. This gives

$$\Theta_x^{\gamma_{\Gamma_2}}(1 - \Theta_x^{\gamma_{\Gamma_2}}) = \frac{n-1}{n^2} \quad \text{and} \quad \Theta_y^{\gamma_{\Gamma_2}}(1 - \Theta_y^{\gamma_{\Gamma_2}}) = \frac{n^2 - 1}{4n^2}.$$

Putting these into equation (95) gives

$$\langle \mathbf{e}_3, \mathbf{e}_3, \mathbf{e}_3^{n-1}, \mathbf{e}_3^{n-2} \rangle_0^{D_{n+1}} = 1/n \quad \text{and} \quad \langle X, X, X^{n-1}, X^{n-2} \rangle_0^{D_{n+1}} = 1/n.$$

In the case of $n = 3$ we have to compute the correlators $\langle X, X, X^2, X \rangle_0^{D_4} = \langle x\mathbf{e}_3, x\mathbf{e}_3, \mathbf{e}_2/6, x\mathbf{e}_3 \rangle_0^{D_4}$ and $\langle X, X, Y, X^2 \rangle_0^{D_4} = \langle x\mathbf{e}_3, x\mathbf{e}_3, y\mathbf{e}_0, \mathbf{e}_2/6 \rangle_0^{D_4}$. Unfortunately, because of the broad sectors, we cannot use the standard tools for computing these correlators.

6.3.6. *Four-point correlators for D_{n+1} with maximal symmetry group.* By Theorem 6.2.10 we need only compute the correlator $\langle X, X, X^{2n-2}, X^{2n-2} \rangle_0^{D_{n+1}}$. Here, we use the corresponding notation from Section 5 with $\alpha = 1$. This corresponds to the correlator $\langle \mathbf{e}_{n+2}, \mathbf{e}_{n+2}, \mathbf{e}_{n-1}, \mathbf{e}_{n-1} \rangle_0^{D_{n+1}}$.

By equation (77) we compute that the degrees of the structure bundles are $\deg(|\mathscr{L}_x|) = -2$ and $\deg(|\mathscr{L}_y|) = -1$. This shows that the correlator is concave and that $R^1\pi_*\mathscr{L}_y = 0$, so the y terms makes no contribution to that correlator.

To apply Theorem 6.3.3 we need to know (using equation (84)) that

$$\Theta_x^{\lambda^{n+2}} = 2/n \quad \text{and} \quad \Theta_x^{\lambda^{n-1}} = (n-1)/n.$$

We also need to compute the contribution of the different boundary (nodal) terms. It is easy to check, in the same manner as we did in the case of $\langle \mathbf{e}_2, \mathbf{e}_4, \mathbf{e}_7, \mathbf{e}_7 \rangle_0^{E_7}$, that there is one graph Γ_1 with $\Theta_x^{\gamma_{\Gamma_1}} = (n-3)/n$ and two copies of a graph Γ_2 with $\Theta_x^{\gamma_{\Gamma_2}} = 0$.

By Theorem 6.3.3, we have

$$\langle \mathbf{e}_{n+2}, \mathbf{e}_{n+2}, \mathbf{e}_{n-1}, \mathbf{e}_{n-1} \rangle_0^{D_{n+1}} = \fint_{\overline{\mathscr{M}}_{0,4}} \Lambda_{0,4}^{D_{n+1}}(\mathbf{e}_{n+2}, \mathbf{e}_{n+2}, \mathbf{e}_{n-1}, \mathbf{e}_{n-1})$$

$$= \frac{1}{2}\left(\left(\frac{1}{n^2} - \frac{1}{n} \right) \fint_{\overline{\mathscr{M}}_{0,4}} \kappa_1 + \sum_{i=1}^{4} \Theta_x^{\gamma_i}(1 - \Theta_x^{\gamma_i}) \fint_{\overline{\mathscr{M}}_{0,4}} \psi_i \right.$$

$$\left. - \sum_{\Gamma \in \mathbf{\Gamma}_{0,4,E_7}(\mathbf{e}_{n+2}, \mathbf{e}_{n+2}, \mathbf{e}_{n-1}, \mathbf{e}_{n-1})} \Theta_x^{\gamma_\Gamma}(1 - \Theta_x^{\gamma_\Gamma}) \fint_{\overline{\mathscr{M}}_{0,4}} \left[\overline{\mathscr{M}}(|\Gamma|) \right] \right)$$

$$= \frac{1}{2}\left(\frac{1}{n^2} - \frac{1}{n} + 2\frac{2}{n}\frac{n-2}{n} + 2\frac{1}{n}\frac{n-1}{n} - \frac{3}{n}\frac{n-3}{n} \right) = \frac{1}{n}.$$

This gives

$$\langle X, X, X^{2n-2}, X^{2n-2} \rangle_0^{D_{n+1}} = \langle \mathbf{e}_{n+2}, \mathbf{e}_{n+2}, \mathbf{e}_{n-1}, \mathbf{e}_{n-1} \rangle_0^{D_{n+1}} = \frac{1}{n}.$$

6.3.7. *Four-point correlators for D_{n+1}^T.* By Theorem 6.2.10 we need only compute the correlator $\langle X, X, X^{n-1}, X^{n-2} \rangle_0^{D_{n+1}^T}$. Here, we choose $\sigma = 1$. This corresponds to the correlator $\langle \mathbf{e}_3, \mathbf{e}_3, \mathbf{e}_{2n-1}, \mathbf{e}_{2n-3} \rangle_0^{D_{n+1}^T}$.

A now-familiar computation shows that the correlator in question is concave (and all markings are narrow), so we may apply Theorem 6.3.3. Applying

that theorem in a manner similar to the previous computations, we find that

$$\langle X, X, X^{n-1}, X^{n-2}\rangle_0^{D_{n+1}^T} = \langle \mathbf{e}_3, \mathbf{e}_3, \mathbf{e}_{2n-1}, \mathbf{e}_{2n-3}\rangle_0^{D_{n+1}^T} = \frac{1}{2n}.$$

6.4. **Computation of the basic four-point correlators in the** B-**model.** The primary potentials on Saito's Frobenius manifolds of the A, D, and E singularities have been computed by a variety of computational methods (see [DVV91], [NY98], [Wit93a], [KTS92], etc.). However, these results are scattered in different papers and are difficult to follow. For the reader's convenience, we present explicit computations of the basic four-point correlators using the Noumi-Yamada formula for the flat coordinates of the A, D, and E singularities [Nou84], [NY98]. Recall that the primitive forms for the ADE-singularities are $C\,dx$ for A_n-case and $C\,dx \wedge dy$ for the DE-cases. The calculation of the flat coordinates does not depend on the leading constant C, but the pairing and potential function will be re-scaled by C.

6.4.1. *The Noumi-Yamada formula for flat coordinates.* We must make several definitions before writing the Noumi-Yamada formula for the flat coordinates.

Definition 6.4.1. Let $\mathscr{N}$ be the following set of exponents for a monomial basis of the Milnor ring $\mathscr{Q}_W$:

$$\mathscr{N} := \begin{cases} \{\nu \in \mathbb{N} : 0 \le \nu \le n-1\} & \text{if } W = A_n \text{ ,} \\ \{(\nu_1, 0) \in \mathbb{N}^2 : 0 \le \nu_1 \le n-2\} \cup \{(0,1)\} & \text{if } W = D_n, \\ \{(\nu_1, \nu_2) \in \mathbb{N}^2 : 0 \le \nu_1 \le 2,\, 0 \le \nu_2 \le 1\} & \text{if } W = E_6, \\ \{(\nu_1, \nu_2) \in \mathbb{N}^2 : 0 \le \nu_1 \le 2,\, 0 \le \nu_2 \le 1\} \cup \{0, 2\} & \text{if } W = E_7, \\ \{(\nu_1, \nu_2) \in \mathbb{N}^2 : 0 \le \nu_1 \le 3,\, 0 \le \nu_2 \le 1\} & \text{if } W = E_8. \end{cases}$$

For each $\nu \in \mathscr{N}$, we let $\phi_\nu = \mathbf{x}^\nu$ be the corresponding monomial in $\mathscr{Q}_W$. Recall that a miniversal deformation of W is a family of polynomials $W_\lambda = W + \sum_{\nu \in \mathscr{N}} t_\nu \phi_\nu$. We want to find flat coordinates $\{s_\nu\}$ with the property $\langle s_\nu, s_v \rangle = \delta_{\nu v}$. One can formally write s_ν in terms of power series in t_v. One special property of the simple singularities is that the s_ν are always a polynomial, but this is not true for general singularities.

Definition 6.4.2. For $W \in \mathbb{C}[x_1, \ldots, x_N]$ quasi-homogeneous, with the weight of each variable x_i equal to q_i, and for any $\nu \in \mathscr{N}$ we define the *weight of s_ν* to be

$$\sigma_\nu := \text{wt}(s_\nu) := 1 - \sum_{i=1}^{N} \nu_i q_i.$$

For any $\alpha \in \mathbb{N}^{\mathcal{N}}$, we define the *weight of* α to be

$$\mathrm{wt}(\alpha) := \langle \alpha, \sigma \rangle := \sum_{\nu \in \mathcal{N}} \alpha_\nu \sigma_\nu.$$

We also define a mapping $\ell : \mathbb{N}^{\mathcal{N}} \to \mathbb{N}^N$ by

$$\ell(\alpha) := \sum_{\nu \in \mathcal{N}} \nu \alpha_\nu \in \mathbb{N}^N.$$

THEOREM 6.4.3 (See [NY98, Thm. 1.1]). *The formula for the flat coordinates for the simple singularities with primitive form $\bigwedge_{i=1}^N dx_i$ is as follows:*

(101)
$$s_\nu = t_0 \delta_{\nu,0} + \sum_{\substack{\alpha \in \mathbb{N}^{\mathcal{N}} \\ \langle \sigma, \alpha \rangle = \sigma_\nu}} c_\nu(\ell(\alpha)) \frac{t^\alpha}{\alpha!},$$

where the function $c_\nu : \mathbb{N}^N \longrightarrow \mathbb{C}$ is given below.

Case (A_n). For any $\nu \in \mathcal{N} = \{0, 1, \ldots, n-1\}$, let $L(\nu) := \{\alpha \in \mathbb{N} : \alpha \equiv \nu \mod (n+1)\} = \{\nu + k(n+1) : k \geq 0\}$. Define

$$c_\nu(\alpha) = \begin{cases} (-1)^k \left(\frac{\nu+1}{n+1}; k\right) & \text{if } \alpha \in L(\nu), \\ 0 & \text{otherwise,} \end{cases}$$

where $(z; k) := \Gamma(z+k)/\Gamma(z)$ denotes the shifted factorial function.

Case (D_n). For any $\nu \in \mathcal{N}$, let $L(\nu) := \mathbb{N}^2 \cap (\nu + \mathrm{span}\{(n-1, 0), (1, 2)\}) = \{(\nu_1 + k_1(n-1) + k_2, \nu_2 + 2k_2 : k_2 \geq 0, k_1 \geq -(\nu_1 + k_2)/(n-1)\}$. Now define

$$c_\nu(\alpha) := \begin{cases} (-1)^{k_1+k_2} \left(\frac{\nu_1+1}{n-1} - \frac{\nu_2+1}{2(n-1)}; k_1\right) \left(\frac{\nu_2+1}{2}; k_2\right) & \text{if } \alpha \in L(\nu), \\ 0 & \text{otherwise,} \end{cases}$$

where $(z; k) := \Gamma(z+k)/\Gamma(z)$ denotes the shifted factorial function.

Case (E_6). For any $\nu \in \mathcal{N} = \{(\nu_1, \nu_2) : \nu_1 = 0, 1, 2, \nu_2 = 0, 1\}$, let $L(\nu) := \{(\alpha_1, \alpha_2) \in \mathbb{N}^2 : \alpha_1 \equiv \nu_1 \mod 4, \alpha_2 \equiv \nu_2 \mod 3\} = \{(\nu_1 + 4k_1, \nu_2 + 3k_2) : k_1, k_2 \geq 0\}$. Now define

$$c_\nu(\alpha) := \begin{cases} (-1)^{k_1+k_2} \left(\frac{\nu_1+1}{4}; k_1\right) \left(\frac{\nu_2+1}{3}; k_2\right) & \text{if } \alpha \in L(\nu), \\ 0 & \text{otherwise,} \end{cases}$$

where $(z; k) := \Gamma(z+k)/\Gamma(z)$ denotes the shifted factorial function.

Case (E_7). For any $\nu \in \mathcal{N}$, let $L(\nu) := \mathbb{N}^2 \cap \mathrm{span}\{(3, 0), (1, 3)\} = \{(\nu_1 + 3k_1 + k_2, \nu_2 + 3k_2); k_2 \geq 0, k_1 \geq -(\nu_2 + k_1)/3\}$. Now define

$$c_\nu(\alpha) := \begin{cases} (-1)^{k_1+k_2} \left(\frac{\nu_1+1}{3}; k_1\right) \left(\frac{\nu_2+1}{3} - \frac{\nu_1+1}{9}; k_2\right) & \text{for } \alpha \in L(\nu), \\ 0 & \text{otherwise,} \end{cases}$$

where $(z; k) := \Gamma(z+k)/\Gamma(z)$ denotes the shifted factorial function.

Case (E_8). For any $\nu \in \mathcal{N} = \{(\nu_1, \nu_2); \nu_1 \equiv 0, 1, 2, 3; \nu_2 = 0, 1\}$, let $L(\nu_1, \nu_2) = \{(\alpha_1, \alpha_2) \in \mathbb{N}^2 : \alpha_1 \equiv \nu_1 \mod 5, \alpha_2 \equiv \nu_2 \mod 3\} = \{(\nu_1 + 5k_1, \nu_2 + 3k_2); k_1, k_2 \geq 0\}$. Now define

$$c_\nu(\alpha) := \begin{cases} (-1)^{k_1+k_2}(\frac{\nu+1}{5}; k_1)(\frac{\nu_2+1}{3}; k_2) & \text{if } \alpha \in L(\nu), \\ 0 & \text{otherwise}, \end{cases}$$

where $(z; k) := \Gamma(z + k)/\Gamma(z)$ denotes the shifted factorial function.

6.4.2. *Four-point correlators for* E_7. We start with the primitive form $dx \wedge dy = dx_1 \wedge dx_2$. Assume that the deformation of E_7 is given by

$$W = x_1^3 + x_1 x_2^3 + t_1 x_1^2 x_2 + t_3 x_1^2 + t_4 x_1 x_2 + t_5 x_2^2 + t_6 x_1 + t_7 x_2 + t_9.$$

Then the flat coordinates s and t have the asymptotic expansion formula (to second order):

$$\begin{aligned} &t_1 \doteq s_1, && t_3 \doteq s_3, \\ &t_4 \doteq s_4 + \tfrac{4}{9} s_3 s_1, && t_5 \doteq s_5, \\ &t_6 \doteq s_6 + \tfrac{1}{3} s_5 s_1 + \tfrac{5}{18} s_3^2, && t_7 \doteq s_7 + \tfrac{1}{9} s_6 s_1 + \tfrac{1}{9} s_4 s_3, \\ &t_9 \doteq s_9 + \tfrac{2}{9} s_6 s_3 + \tfrac{1}{3} s_5 s_4. \end{aligned}$$

To compute the four-point correlators, we first use the residue formula computing the three-point correlators of the deformed chiral ring and then take the possible first order derivatives with respect to the flat coordinates. We have

$$\partial_{x_1} W = 3x_1^2 + x_2^3 + 2s_1 x_1 x_2 + 2s_3 x_1 + \left(s_4 + \frac{4}{9} s_1 s_3\right) x_2 + s_6 + \frac{1}{3} s_5 s_1,$$

$$\partial_{x_2} W = 3x_1 x_2^2 + s_1 x_1^2 + \left(s_4 + \frac{4}{9} s_1 s_3\right) x_1 + 2s_5 x_2 + s_7 + \frac{1}{9} s_6 s_1 + \frac{1}{9} s_4 s_3,$$

$$\begin{aligned} \text{Hess}(W) &= 36 x_1^2 x_2 - 9 x_2^4 + \text{lower order terms} \\ &= 63 x_1^2 x_2 + \text{lower order terms or } -21 x_2^4 + \text{lower order terms}, \end{aligned}$$

$$\begin{aligned} \hat{H}_W &:= \text{Hess}(W)/7 \\ &= 9 x_1^2 x_2 + \text{lower order terms or } -3 x_2^4 + \text{lower order terms}. \end{aligned}$$

Let $C_{ijk}(s) := \text{Res}_s(\frac{\partial W}{\partial s_i}, \frac{\partial W}{\partial s_j}, \frac{\partial W}{\partial s_k})$. Then

$$C_{ijk}(s) = \left(\frac{\partial W}{\partial s_i} \frac{\partial W}{\partial s_j} \frac{\partial W}{\partial s_k}\right) / \hat{H}_W \mod \text{Jac}_W.$$

For example, $C_{991}(0) = (x_1^2 x_2 \cdot 1 \cdot 1)/(9 x_1^2 x_2) = 1/9$. All the possible three-point correlators can be obtained below:

$$\begin{aligned} &C_{991}(0) = 1/9, && C_{946}(0) = 1/9, && C_{577}(0) = -1/3, \\ &C_{559}(0) = -1/3, && C_{667}(0) = 1/9, && C_{937}(0) = 1/9. \end{aligned}$$

Now, we change primitive form from $dx_1 \wedge dx_2$ to $9dx_1 \wedge dx_2$. This rescales the pairing and entire potential function by 9. The cubic term of the primary potential function is

$$F_3 = \frac{1}{12} s_1 s_9^2 + s_4 s_6 s_9 - \frac{3}{2} s_5 s_7^2 - \frac{3}{2} s_5^2 s_9 + \frac{1}{12} s_6^2 s_7 + s_3 s_7 s_9.$$

Recall that the ring structure with current rescaled pairing has already been proved to be isomorphic to the quantum ring in the A-model; and moreover, the three-point correlators in the B-model and the A-model are identical.

Using the isomorphism of the ring structure, we make the following identification between the basic four-point correlators in the A-model and those in the B-model:

$$\langle X, X, X^2, XY \rangle_0 \longleftrightarrow s_6^2 s_3 s_4,$$
$$\langle X, Y, X^2, X^2 \rangle_0 \longleftrightarrow s_6 s_7 s_3^2,$$
$$\langle Y, Y, XY, X^2Y \rangle_0 \longleftrightarrow s_7^2 s_4 s_1.$$

We have the formula for the four-point correlators

$$C_{ijkl} = \left. \frac{d}{ds_l} C_{ijk} \right|_{s=0}.$$

Now it is easy to obtain

$$C_{6634} = -1/9, \quad C_{6733} = 1/9, \quad \text{and} \quad C_{7741} = -1/3.$$

The part of the fourth-order term of the primary potential we need is

$$F_4 = -\frac{1}{18} s_3 s_4 s_6^2 - \frac{1}{6} s_1 s_4 s_7^2 + \frac{1}{18} s_3^2 s_6 s_7.$$

The computation here coincides with the result in [Nou] and in [KTS92] (under a quasi-homogeneous coordinate transformation).

6.4.3. *Four-point correlators for E_6.* Assume that the deformation of E_6 is given by

$$W = x_1^3 + x_2^4 + t_2 x_2^2 x_1 + t_5 x_1 x_2 + t_6 x_2^2 + t_8 x_1 + t_9 x_2 + t_{12}.$$

We choose the primitive form $12 dx_1 \wedge dx_2$. The metric and the third- and fourth-order terms of the potential are given below:

$$\eta_{ij} = \delta_{i,14-j} \text{ for } i, j \in \{2, 5, 6, 8, 9, 12\},$$
$$F_3 = s_6 s_8 s_{12} + s_5 s_9 s_{12} + \frac{1}{2} s_2 s_{12}^2 + \frac{1}{2} s_8 s_9^2,$$
$$F_4 = -\frac{1}{8} s_5 s_9 s_6^2 - \frac{1}{12} s_8^2 s_5^2 - \frac{1}{18} s_2 s_8^3 - \frac{1}{8} s_2 s_6 s_9^2.$$

We make the following identification between the A- and the B-models:

$$\langle Y, Y, Y^2, XY^2 \rangle \longleftrightarrow s_9^2 s_6 s_2 \quad \text{and} \quad \langle X, X, XY, XY \rangle \longleftrightarrow s_8^2 s_5^2,$$

and we get the basic four-point correlators in the B-model:

(102) $$C_{9962} = -\frac{1}{4} \quad \text{and} \quad C_{8855} = -\frac{1}{3}.$$

6.4.4. *Four-point correlators for E_8.* Assume that the deformation of E_8 is given by

$$W = x_1^3 + x_2^5 + t_1 x_2^3 x_1 + t_4 x_2^2 x_1 + t_6 x_2^3 + t_7 x_2 x_1 + t_9 x_2^2 + t_{10} x_1 + t_{12} x_2 + t_{15}.$$

We choose the primitive form $dx_1 \wedge dx_2$. In the same manner as before, we obtain

$$\eta_{ij} = \delta_{i,16-j}, \text{ for } i, j \in \{1, 4, 6, 7, 9, 10, 12, 15\},$$

$$F_3 = s_4 s_{12} s_{15} + s_7 s_9 s_{15} + s_6 s_{10} s_{15} + \frac{1}{2} s_1 s_{15}^2 + s_9 s_{10} s_{12} + \frac{1}{2} s_7 s_{12}^2,$$

$$F_4 = -\frac{1}{18} s_7^3 s_{10} - \frac{1}{10} s_6 s_7 s_9^2 - \frac{1}{10} s_7 s_6^2 s_{12} - \frac{1}{15} s_4 s_9^3 - \frac{1}{6} s_4 s_7 s_{10}^2$$
$$- \frac{1}{5} s_4 s_6 s_9 s_{12} - \frac{1}{18} s_1 s_{10}^3 - \frac{1}{10} s_1 s_9^2 s_{12} - \frac{1}{10} s_1 s_6 s_{12}^2.$$

By the following correspondence between the A- and the B-models,

$$\langle Y, Y, Y^3, XY^3 \rangle \longleftrightarrow s_{12}^2 s_6 s_1 s_2 \quad \text{and} \quad \langle X, X, X, XY^3 \rangle \longleftrightarrow s_{10}^3 s_1,$$

we get the basic four-point correlators in the B-model:

(103) $$C_{(12)(12)61} = -\frac{1}{5} \quad \text{and} \quad C_{(10)(10)(10)1} = -\frac{1}{3}.$$

6.4.5. *Four point correlators for D_{n+1}.* Assume that the deformation is given by

$$W = x_1^n + x_1 x_2^2 + \sum_{i=0}^{n-1} t_i x_1^i + t_{01} x_2.$$

Then we have the flat coordinates by Noumi's formula

$$\begin{cases} s_r \doteq t_r + \tilde{c}_r \sum_{k \geq 1} t_{r+k} t_{n-k}, \\ s_{01} \doteq t_{01}. \end{cases}$$

Here $\tilde{c}_r$ is just the Noumi-Yamada function $c_{r,0}$ defined before, but if $r + k = n - k$, then $\tilde{c}_r := c_{r,0}/2$.

Now the inverse function is given by

$$\begin{cases} t_r \doteq s_r - \tilde{c}_r \sum_{k \geq 1} s_{r+k} s_{n-k}, \\ t_{01} \doteq s_{01}. \end{cases}$$

We have the derivative formula

$$\frac{\partial t_r}{\partial s_j} = \begin{cases} 0 & \text{if } j < r, \\ (1 - \delta_{rj})(-c_r s_{n+r-j}) + \delta_{rj} & \text{if } j \geq r. \end{cases}$$

Here the indices should satisfy the restriction

$$n + r - j \geq 1, \; j \geq r + 1.$$

We have the basic computation

$$\partial_{x_1} W = n x_1^{n-1} + x_2^2 + \sum_{i=1}^{n-1} i t_i x_1^{i-1}, \;\; \partial_{x_2} W = 2 x_1 x_2 + t_{01}$$

$$\mathrm{Hess}_W = (-2)(n+1) x_2^2.$$

The $n + 1$ primary fields $\phi_i(s), 0 \leq i \leq n - 1$ and $\phi_{(01)}(s)$ are given as below, which are functions of the flat coordinates s:

$$\phi_i(s) = \frac{\partial W}{\partial s_i} = \sum_{j=0}^{n-1} \frac{\partial t_j}{\partial s_i} x_1^j,$$

$$\phi_{(01)}(s) = \frac{\partial W}{\partial s_{(01)}} = x_2.$$

Choose primitive form $2n \, dx_1 \wedge dx_2$. Then, we re-scale pairing and potential function by $2n$. Let $\langle \phi \rangle := 2n \mathrm{Res}_W \left(\frac{\phi}{\partial_{x_1} W \cdot \partial_{x_2} W} \right)$. Then in flat coordinates, we can normalize the metric η and the three-point functions such that

$$\eta_{pq} = \langle \phi_p \phi_q \rangle,$$

$$C_{pqr}(s) = \langle \phi_p \phi_q \phi_r \rangle.$$

Actually, $C_{pqr}(s)$ is the coefficient of the equality

$$\phi_p \phi_q \phi_r = C_{pqr} \cdot (\mathrm{Hess}_W / (n+1)) \mod \partial_{x_i} W.$$

After a straightforward calculation, we obtain

PROPOSITION 6.4.4. *The three-point correlators of D_{n+1} are given as follows*:

(104)
$$\begin{cases} C_{ijk} = \delta_{i+j+k,n-1} & \text{for } 0 \leq i,j,k \leq n-1, \\ C_{i(01)(01)} = -n\delta_{0i} & \text{for } 0 \leq i \leq n-1, \\ C_{(01)(01)(01)} = 0. \end{cases}$$

The four-point correlators are
(105)
$$\begin{cases} C_{ijkl} = \left(-\frac{1}{n} \right) \left(l - \left(n - i - j - \frac{1}{2} \right) \delta_{i+j \leq n-1} - \left(n - k - j - \frac{1}{2} \right) \delta_{k+j \leq n-1} \right. \\ \qquad\qquad \left. - \left(n - i - k - \frac{1}{2} \right) \delta_{i+k \leq n-1} \right) & \text{for } 0 \leq i,j,k \leq n-1, \\ C_{ij(01)(01)} = -\frac{1}{2} \delta_{i+j,n} & \text{for } 0 \leq i \leq n-1, \\ C_{i(01)(01)n-i} = -\frac{1}{2} & \text{for } 0 \leq i \leq n-1. \end{cases}$$

The function $\delta_{x \leq y}$ is defined as

$$\delta_{x \leq y} = \begin{cases} 1 & \text{if } x \leq y, \\ 0 & \text{if } x > y. \end{cases}$$

COROLLARY 6.4.5. *The basic four-point correlator for $n > 3$ is*

$$C_{11(n-1)(n-2)} = 1/2n.$$

6.4.6. *Four-point correlators for D_{n+1}^T.* The singularity D_{n+1}^T is isomorphic to $A_{2n-1} = x'^{2n} + y'^2$ by the quasi-homogeneous isomorphism

$$x = (2i)^{\frac{1}{n}} x', \ y = y' - ix'^n.$$

This induces an isomorphism of Saito's Frobenius manifolds with primitive forms $cdx \wedge dy \to c(2i)^{\frac{1}{n}} dx' \wedge dy'$.

6.4.7. *Four-point correlators for A_n.* The three- and four-point correlators have already been calculated in [Wit93a]. Suppose that the deformation is given by

$$W = x^{n+1} + \sum_{i=0}^{n-1} t_i x^i.$$

We choose primitive form dx. We list the metric, three- and four-point correlators below:

$$\eta_{ij} = (n+1)\delta_{i+j,n-1},$$
$$C_{ijk} = \delta_{i+j+k,n-1},$$
$$C_{ijkl} = -\frac{1}{n+1}\Big(l + (n-j-k)\delta_{j+k \leq n-1} + (n-i-k)\delta_{i+k \leq n-1}$$
$$+ (n-i-j)\delta_{i+j \leq n-1}\Big)$$

for $0 \leq i, j, k \leq n-1$.

6.5. **Proof of Theorem 6.1.3.** Because of our reconstruction theorem, to prove Theorem 6.1.3, it suffices to compare the two-point, three-point, and the basic four-point functions in our theory (A-model) with their analogues in the B-model.

We have established the isomorphism of Frobenius algebras in Section 5. This means that we have matched the unit, the pairing, and the multiplication and hence all three-point functions, by the explicit identification of state spaces. The remaining task is to match the four-point basic correlators. We shall keep the identification of the unit and multiplication fixed. The main idea is to explore the flexibility of re-scaling the primitive form by a constant. Re-scaling the primitive form by c corresponds to re-scaling h by $1/c$. Hence, it still satisfies the corresponding hierarchies. However, the corresponding Frobenious

manifold structure is different in general. This approach seems to give a better conceptual picture. For the reader's convenience, we shall list the explicit value of constant we used in the proof.

Our main technical tool is the following observation. Let F_3^A, F_4^A be the three- and basic four-point functions of the A-model and F_3^B, F_4^B be the three- and basic four-point functions of the B-model. Suppose that $F_3^A = F_3^B$. Now, we re-scale the primitive form by c and make an additional change of variable $s_i \to \lambda^{1-\deg_{\mathbb{C}}(s_i)} s_i$. Notice that the above change of variable preserves the unit e of the Frobenius algebras. This change of variables gives $F_3^B \to c\lambda^{\hat{c}_W} F_3^B$ and $F_4^B \to c\lambda^{\hat{c}_W+1} F_4^B$. If we choose $c = \lambda^{-\hat{c}_W}$, then F_3^B remains the same and $F_4^B \to \lambda F_4^B$. Since the linear map $s_i \to \lambda^{1-\deg_{\mathbb{C}}(s_i)} s_i$ preserves the unit, it preserves the metric as well.

6.5.1. E_7 *A-model versus* E_7 *B-model of primitive form* $(-1)^{-\frac{8}{9}} 9 dx_1 \wedge dx_2$. The Frobenius manifold in the A-model is given by the small phase space quantum cohomology. Take the flat coordinates $\{T_1, T_3, T_4, T_5, T_6, T_7, T_9\}$ corresponding to the primary fields $\{\mathbf{e}_8 = X^2Y, \mathbf{e}_4 = X^2, \mathbf{e}_2 = XY, \pm y^2 \mathbf{e}_0 = Y^2, \mathbf{e}_7 = X, \mathbf{e}_5 = Y, \mathbf{e}_1 = 1\}$. The three-point correlators give the cubic term of the primary potential F_3^A:

$$F_3^A = \frac{1}{2} T_9^2 T_1 + T_9 T_7 T_3 + T_9 T_6 T_4 - \frac{3}{2} T_9 T_5^2 - \frac{3}{2} T_7^2 T_5 + \frac{1}{2} T_7 T_6^2.$$

The basic four-point function is

(106) $$\frac{1}{18} T_6^2 T_3 T_4 + \frac{1}{6} T_7^2 T_4 T_1 - \frac{1}{18} T_6 T_7 T_3^2.$$

Choose primitive form $9 \, dx_1 \wedge dx_2$ on the B-model side. We obtain the same metric and the same cubic terms. However, $F_4^B = -F_4^A$. Then, we choose $\lambda = -1$ and $c = (-1)^{-\frac{8}{9}}$. It means that we choose primitive form $(-1)^{-\frac{8}{9}} 9 \, dx_1 \wedge dx_2$. The corresponding linear map between state spaces is

$$T_i \to (-1)^{1-\deg_{\mathbb{C}}(s_i)} s_i.$$

6.5.2. E_6 *A-model versus* E_6 *B-model of primitive form* $(-1)^{-\frac{5}{6}} 12 \, dx_1 \wedge dx_2$. Consider the A-model. Let the flat coordinates $\{T_2, T_5, T_6, T_8, T_9, T_{12}\}$ correspond to the primary fields $\{XY^2, XY, Y^2, X, Y, 1\}$. Then we obtain the three-point potential functions:

(107) $$F_3^A = \frac{1}{2} T_2 T_{12}^2 + T_5 T_9 T_{12} + T_6 T_8 T_{12} + \frac{1}{2} T_8 T_9^2.$$

The polynomial corresponding to the basic four-point correlators is

$$\frac{1}{8} T_2 T_6 T_9^2 + \frac{1}{12} T_5^2 T_8^2.$$

On the B-model side, we start from primitive form $12 \, dx_1 \wedge dx_2$ and a linear map between state spaces $T_i \to s_i$. It matches the unit, pairing, and

multiplications and hence $F_3^A = F_3^B$. But we have

(108)
$$F_4^B = -F_4^A.$$

Similar to the E_7, a choice of $\lambda = -1$ and $c = (-1)^{-\frac{5}{6}}$ will match the A-model to the B-model of the primitive form $(-1)^{-\frac{5}{6}} 12\, dx_1 \wedge dx_2$.

6.5.3. E_8 *A-model versus* E_8 *B-model of primitive form* $(-1)^{-\frac{14}{15}} dx_1 \wedge dx_2$. Let $\{T_1, T_4, T_6, T_7, T_9, T_{10}, T_{12}, T_{15}\}$ be the flat coordinates in the A-model corresponds to the primary fields $\{XY^3, XY^2, Y^3, XY, Y^2, X, Y, 1\}$. We can obtain the three-point potential, the basic polynomials, and the basic four-point potential:

(109) $\quad F_3^A = \dfrac{1}{2}T_1 T_{15}^2 + T_7 T_9 T_{15} + T_6 T_{10} T_{15} + T_4 T_{12} T_{15} + \dfrac{1}{2}T_7 T_{12}^2 + T_9 T_{10} T_{12},$

$$F_4^A = \frac{1}{10}T_1 T_6 T_{12}^2 + \frac{1}{18}T_1 T_{10}^3.$$

Choose primitive form $dx_1 \wedge dx_2$ and the linear map $T_i \to s_i$. Then, we match the unit, pairing, and multiplication. Hence, we have $F_3^A = F_3^B$. But $F_4^A = -F_4^B$. Then, a choice of $\lambda = -1$ and $c = (-1)^{-\frac{14}{15}}$ will match the A-model with the B-model.

6.5.4. $(D_{n+1}, \langle J \rangle, (n \text{ odd }))$ *A-model versus* D_{n+1} *B-model of primitive form* $(-1)^{1-\frac{n-1}{n}} 4n\, dx_1 \wedge dx_2$. Consider the A-model. Let $\{T_0, T_1, \ldots, T_{n-1}, T_{01}\}$ be the flat coordinates corresponding to the primary field $1, X, \ldots, X^{n-1}, Y$. Here $\{X, Y\}$ has already been identified with $\{\mathbf{e}_3, \pm 2nr x^{\frac{n-1}{2}} \mathbf{e}_n + \mp 2nsy\mathbf{e}_n\}$ in the state space $\mathscr{H}_{D_{n+1}, \langle J \rangle}$. We have the computation of the 2-point correlators (metric)

$$\langle X^{n-1}, 1 \rangle = -2 \quad \text{and} \quad \langle Y^2, 1 \rangle = 2n.$$

The three-point potential is

$$F_3^A = -2 \sum_{i+j+k=(n-1)} a_{ijk} T_i T_j T_k + nT_0 T_{01}^2,$$

where

$$a_{ijk} = \begin{cases} 1 & \text{if } i, j, k \text{ are mutually not equal,} \\ 1/2 & \text{if only two of } i, j, k \text{ are equal,} \\ 1/6 & \text{if } i = j = k, \end{cases}$$

and the basic four-point polynomial for $n > 3$ is

$$F_4^A = \frac{1}{2n}T_1^2 T_{n-2} T_{n-1}.$$

In the B-model, by choosing primitive form $-4n\, dx_1 \wedge dx_2$ and linear map $T_i \to s_i$, we have the same pairing as the A-model and cubic term $F_3^B = F_3^A$

and the basic four-point polynomial for $n > 3$:

$$-\frac{1}{2n}s_1^2 s_{n-2}s_{n-1}.$$

Then, a choice of $\lambda = -1$ and $c = (-1)^{-\frac{n-1}{n}}$ will match the A-model with the B-model.

6.5.5. $D_{n+1}(G_{D_{n+1}})$ A-*model versus* A_{2n-1} B-*model of primitive form* $2n(\frac{n}{5-4n})^{\frac{1-n}{n}} dx$. Recall that

$$X^i \mapsto \begin{cases} \mathbf{e}_{n+1+i} & \text{for } 0 \le i < n-1, \\ \mp 2y\mathbf{e}_0 & \text{for } i = n-1, \\ \mathbf{e}_{i-n+1} & \text{for } n \le i < 2n-1 \end{cases}$$

is an isomorphism of graded algebras $\mathscr{H}_{D_{n+1},G_{D_{n+1}}} \to \mathscr{Q}_{A_{2n-1}}$. The pairing on $\mathscr{Q}_{A_{2n-1}}$ is given by $\langle X^{2n-2}, 1 \rangle^{\mathscr{Q}_{A_{2n-1}}} = 1/2n$, whereas the pairing on $\mathscr{H}_{D_{n+1},G_{D_{n+1}}}$ is easily seen to be given by

$$\langle X^{2n-2}, \mathbf{1} \rangle^{\mathscr{H}_{D_{n+1}}} = \langle \mathbf{e}_{n-1}, \mathbf{e}_{n+1} \rangle^{\mathscr{H}_{D_{n+1}}} = 1.$$

The basic four-point correlator is

$$\langle X, X, X^{2n-2}, X^{2n-2} \rangle_0^{D_{n+1}} = \langle \mathbf{e}_{n+2}, \mathbf{e}_{n+2}, \mathbf{e}_{n-1}, \mathbf{e}_{n-1} \rangle_0^{D_{n+1}} = \frac{1}{n}.$$

We start with the primitive form $2n\,dx$ on the B-model side. Then, we have an isomorphism between the A-model and the B-model ring with pairing, and hence the potential functions have the same cubic terms, i.e., $F_3^A = F_3^B$. The basic four-point correlator of the B-model is $C_{11(2n-2)(2n-2)} = -(4n-5)$. Hence, $F_4^B = -\frac{n}{4n-5}F_4^A$. Now, a choice of $\lambda = -\frac{n}{4n-5}$ and $c = \lambda^{-\frac{n-1}{n}}$ will match the A-model with the B-model.

6.5.6. D_{n+1}^T A-*model versus* D_{n+1} B-*model of primitive form* $2n\,dx_1 \wedge dx_2$. In the D_{n+1}^T A-model, the state space $\mathscr{H}_{W,G_W}$ is generated by $n+1$ elements $\{nx^{n-1}\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_3, \ldots, \mathbf{e}_{2i+1}, \mathbf{e}_{2n-1}\}$. Identify $\mathbf{e}_{2i+1}$ with X^i and $nx^{n-1}\mathbf{e}_0$ with Y. We have computed the metric and the three-point correlators:

$$\langle X^i, X^j, X^k \rangle = 1 \quad \text{if} \quad i+j+k = n-1, \quad \langle 1, Y, Y \rangle = -n,$$

and the other three-point correlators are zero.

The basic four-point correlator is

$$\langle X, X, X^{n-1}, X^{n-2} \rangle = \frac{1}{2n}.$$

Let $\{T_0, T_1, \ldots, T_{n-1}, T_{01}\}$ be the flat coordinates with respect to the primary fields $\{1, X, \ldots, X^{n-1}, Y\}$. On the B-model side, we choose primitive form

$2n\,dx_1 \wedge dx_2$ and the linear map $T_i \to s_i$. Comparing the A-model and the B-model, we have the identity

(110) $$F^A_{3+4}(T) = F^B_{3+4}(T).$$

This shows that $F^A = F^B$ with primitive form $2n\,dx_1 \wedge dx_2$ and completes the proof of Theorem 6.1.3.

References

[AJ03]   D. Abramovich and T. J. Jarvis, Moduli of twisted spin curves, *Proc. Amer. Math. Soc.* **131** (2003), 685–699. MR 1937405. Zbl 1037.14008. http://dx.doi.org/10.1090/S0002-9939-02-06562-0.

[AV02]   D. Abramovich and A. Vistoli, Compactifying the space of stable maps, *J. Amer. Math. Soc.* **15** (2002), 27–75. MR 1862797. Zbl 0991.14007. http://dx.doi.org/10.1090/S0894-0347-01-00380-0.

[AGZV85] V. I. Arnol'd, S. M. Guseĭn-Zade, and A. N. Varchenko, *Singularities of Differentiable Maps. Vol. I*, *Monogr. Math.* **82**, Birkhäuser, Boston, MA, 1985. MR 0777682. Zbl 0554.58001. http://dx.doi.org/10.1007/978-1-4612-5154-5.

[Art91]   M. Artin, *Algebra*, Prentice Hall, Englewood Cliffs, NJ, 1991. MR 1129886. Zbl 0788.00001.

[BH93]   P. Berglund and T. Hübsch, A generalized construction of mirror manifolds, *Nuclear Phys. B* **393** (1993), 377–391. MR 1214325. Zbl 1245.14039. http://dx.doi.org/10.1016/0550-3213(93)90250-S.

[Cec91]   S. Cecotti, $N = 2$ Landau-Ginzburg vs. Calabi-Yau σ-models: nonperturbative aspects, *Internat. J. Modern Phys. A* **6** (1991), 1749–1813. MR 1098365. Zbl 0743.57022. http://dx.doi.org/10.1142/S0217751X91000939.

[CR04]   W. Chen and Y. Ruan, A new cohomology theory of orbifold, *Comm. Math. Phys.* **248** (2004), 1–31. MR 2104605. Zbl 1063.53091. http://dx.doi.org/10.1007/s00220-004-1089-4.

[Chi06a]   A. Chiodo, A construction of Witten's top Chern class in K-theory, in *Gromov-Witten Theory of Spin Curves and Orbifolds*, *Contemp. Math.* **403**, Amer. Math. Soc., Providence, RI, 2006, pp. 21–29. MR 2238814. Zbl 1105.14032. http://dx.doi.org/10.1090/conm/403/07593.

[Chi06b]   A. Chiodo, The Witten top Chern class via K-theory, *J. Algebraic Geom.* **15** (2006), 681–707. MR 2237266. Zbl 1117.14008. http://dx.doi.org/10.1090/S1056-3911-06-00444-9.

[Chi08]   A. Chiodo, Towards an enumerative geometry of the moduli space of twisted curves and rth roots, *Compos. Math.* **144** (2008), 1461–1496. MR 2474317. Zbl 1166.14018. http://dx.doi.org/10.1112/S0010437X08003709.

[CR10]   A. Chiodo and Y. Ruan, Landau-Ginzburg/Calabi-Yau correspondence for quintic three-folds via symplectic transformations, *Invent. Math.* **182**

(2010), 117–165. MR 2672282. Zbl 1197.14043. http://dx.doi.org/10. 1007/s00222-010-0260-0.

[DVV91] R. DIJKGRAAF, H. VERLINDE, and E. VERLINDE, Topological strings in $d < 1$, *Nuclear Phys. B* **352** (1991), 59–86. MR 1103047. http://dx.doi. org/10.1016/0550-3213(91)90129-L.

[DS84] V. G. DRINFEL′D and V. V. SOKOLOV, Lie algebras and equations of Korteweg-de Vries type, in *Current Problems in Mathematics*, *Itogi Nauki i Tekhniki* **24**, Akad. Nauk SSSR Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1984, pp. 81–180. MR 0760998.

[FSZ10] C. FABER, S. SHADRIN, and D. ZVONKINE, Tautological relations and the r-spin Witten conjecture, *Ann. Sci. Éc. Norm. Supér.* **43** (2010), 621– 658. MR 2722511. Zbl 1203.53090. Available at http://smf4.emath.fr/ Publications/AnnalesENS/4_43/html/ens_ann-sc_43_621-658.php.

[FJR11] H. FAN, T. J. JARVIS, and Y. RUAN, Quantum singularity theory for A_{r-1} and r-spin theory, 2011, To appear in *Annales de l'Institut Fourier*, *Special Volume on Geometry and Physics of the Landau-Ginzburg model*. arXiv 1012.0066.

[FJR] H. FAN, T. J. JARVIS, and Y. RUAN, The Witten equation and its virtual fundamental cycle. arXiv 0712.4025.

[FJJS12] A. FRANCIS, T. JARVIS, D. JOHNSON, and R. SUGGS, Landau-Ginzburg mirror symmetry for orbifolded Frobenius algebras, in *String-Math* 2011, *Proc. Sympos. Pure Math.* **85**, Amer. Math. Soc., Providence, RI, 2012, pp. 333–353. MR 2985340. http://dx.doi.org/10.1090/pspum/085/1389.

[FGM10] E. FRENKEL, A. GIVENTAL, and T. MILANOV, Soliton equations, vertex operators, and simple singularities, *Funct. Anal. Other Math.* **3** (2010), 47–63. MR 2734560. Zbl 1203.37108. http://dx.doi.org/10.1007/ s11853-010-0035-6.

[Giv98] A. GIVENTAL, A mirror theorem for toric complete intersections, in *Topological Field Theory, Primitive Forms and Related Topics* (Kyoto, 1996), *Progr. Math.* **160**, Birkhäuser, Boston, MA, 1998, pp. 141–175. MR 1653024. Zbl 0936.14031.

[Giv01] A. B. GIVENTAL, Gromov-Witten invariants and quantization of quadratic Hamiltonians, *Mosc. Math. J.* **1** (2001), 551–568, 645, dedicated to the memory of I. G. Petrovskii on the occasion of his 100th anniversary. MR 1901075. Zbl 1008.53072.

[GM05] A. B. GIVENTAL and T. E. MILANOV, Simple singularities and integrable hierarchies, in *The Breadth of Symplectic and Poisson Geometry*, *Progr. Math.* **232**, Birkhäuser, Boston, MA, 2005, pp. 173–201. MR 2103007. Zbl 1075.37025. http://dx.doi.org/10.1007/0-8176-4419-9_7.

[GS09a] J. GUFFIN and E. SHARPE, A-twisted Landau-Ginzburg models, *J. Geom. Phys.* **59** (2009), 1547–1580. MR 2583793. Zbl 1187.81188. http://dx.doi. org/10.1016/j.geomphys.2009.07.014.

[GS09b]   J. GUFFIN and E. SHARPE, A-twisted heterotic Landau-Ginzburg models, *J. Geom. Phys.* **59** (2009), 1581–1596. MR 2583794. Zbl 1187.81189. http://dx.doi.org/10.1016/j.geomphys.2009.07.013.

[HK]   C. HERTLING and R. KURBEL, On the classification of quasi-homogeneous singularities. arXiv 1009.0763.

[HM93]   T. HOLLOWOOD and J. L. MIRAMONTES, Tau-functions and generalized integrable hierarchies, *Comm. Math. Phys.* **157** (1993), 99–117. MR 1244861. Zbl 0796.35144. http://dx.doi.org/10.1007/BF02098021.

[HKQ09]   M.-x. HUANG, A. KLEMM, and S. QUACKENBUSH, Topological string theory on compact Calabi-Yau: modularity and boundary conditions, in *Homological Mirror Symmetry*, *Lecture Notes in Phys.* **757**, Springer-Verlag, New York, 2009, pp. 45–102. MR 2596635. Zbl 1166.81358.

[IV90]   K. INTRILIGATOR and C. VAFA, Landau-Ginzburg orbifolds, *Nuclear Phys. B* **339** (1990), 95–120. MR 1061738. http://dx.doi.org/10.1016/0550-3213(90)90535-L.

[Jar98]   T. J. JARVIS, Torsion-free sheaves and moduli of generalized spin curves, *Compositio Math.* **110** (1998), 291–333. MR 1602060. Zbl 0912.14010. http://dx.doi.org/10.1023/A:1000209527158.

[Jar00]   T. J. JARVIS, Geometry of the moduli of higher spin curves, *Internat. J. Math.* **11** (2000), 637–663. MR 1780734. Zbl 1094.14504. http://dx.doi.org/10.1142/S0129167X00000325.

[JKV01]   T. J. JARVIS, T. KIMURA, and A. VAINTROB, Moduli spaces of higher spin curves and integrable hierarchies, *Compositio Math.* **126** (2001), 157–212. MR 1827643. Zbl 1015.14028. http://dx.doi.org/10.1023/A:1017528003622.

[KW89]   V. G. KAC and M. WAKIMOTO, Exceptional hierarchies of soliton equations, in *Theta Tunctions—Bowdoin* 1987, *Part* 1 (Brunswick, ME, 1987), *Proc. Sympos. Pure Math.* **49**, Amer. Math. Soc., Providence, RI, 1989, pp. 191–237. MR 1013133. Zbl 0691.17014.

[Kau02]   R. M. KAUFMANN, Orbifold Frobenius algebras, cobordisms and monodromies, in *Orbifolds in Mathematics and Physics* (Madison, WI, 2001), *Contemp. Math.* **310**, Amer. Math. Soc., Providence, RI, 2002, pp. 135–161. MR 1950945. Zbl 1084.57027. http://dx.doi.org/10.1090/conm/310/05402.

[Kau03]   R. M. KAUFMANN, Orbifolding Frobenius algebras, *Internat. J. Math.* **14** (2003), 573–617. MR 1997832. Zbl 1083.57037. http://dx.doi.org/10.1142/S0129167X03001831.

[Kau06]   R. M. KAUFMANN, Singularities with symmetries, orbifold Frobenius algebras and mirror symmetry, in *Gromov-Witten Theory of Spin Curves and Orbifolds*, *Contemp. Math.* **403**, Amer. Math. Soc., Providence, RI, 2006, pp. 67–116. MR 2234885. Zbl 1116.14037. http://dx.doi.org/10.1090/conm/403/07596.

[KTS92]   A. KLEMM, S. THEISEN, and M. G. SCHMIDT, Correlation functions for topological Landau-Ginzburg models with $c \leq 3$, *Internat. J. Modern Phys.*

A **7** (1992), 6215–6244. MR 1187400. Zbl 0954.81543. http://dx.doi.org/
10.1142/S0217751X92002817.

[Kon92]   M. KONTSEVICH, Intersection theory on the moduli space of curves and the
          matrix Airy function, *Comm. Math. Phys.* **147** (1992), 1–23. MR 1171758.
          Zbl 0756.35081. http://dx.doi.org/10.1007/BF02099526.

[Kra10]   M. KRAWITZ, FJRW *rings and Landau-Ginzburg mirror symmetry*, Pro-
          Quest LLC, Ann Arbor, MI, 2010, Ph.D. thesis, University of Michigan.
          MR 2801653. Available at http://gateway.proquest.com/openurl?url_ver=
          Z39.88-2004&rft_val_fmt=info:ofi/fmt:kev:mtx:dissertation&res_dat=xri:
          pqdiss&rft_dat=xri:pqdiss:3429350.

[KPA+10] M. KRAWITZ, N. PRIDDIS, P. ACOSTA, N. BERGIN, and H. RATH-
          NAKUMARA, FJRW-rings and mirror symmetry, *Comm. Math. Phys.* **296**
          (2010), 145–174. MR 2606631. Zbl 1250.81087. http://dx.doi.org/10.
          1007/s00220-009-0929-7.

[Lee06]   Y.-P. LEE, Witten's conjecture and the Virasoro conjecture for genus up
          to two, in *Gromov-Witten Theory of Spin Curves and Orbifolds*, *Con-
          temp. Math.* **403**, Amer. Math. Soc., Providence, RI, 2006, pp. 31–
          42. MR 2234883. Zbl 1114.14034. http://dx.doi.org/10.1090/conm/403/
          07594.

[LLY97]   B. H. LIAN, K. LIU, and S.-T. YAU, Mirror principle. I, *Asian J. Math.*
          1 (1997), 729–763. MR 1621573. Zbl 0953.14026.

[Man99]   Y. I. MANIN, *Frobenius Manifolds*, *Quantum Cohomology*, *and Moduli
          Spaces*, *Amer. Math. Soc. Colloq. Publ.* **47**, Amer. Math. Soc., Providence,
          RI, 1999. MR 1702284. Zbl 0952.14032.

[Mar90]   E. MARTINEC, Criticality, catastrophes, and compactifications, in *Physics
          and Mathematics of Strings* (L. BRINK, D. FRIEDAN, and A. M.
          POLYAKOV, eds.), World Sci., Publ., Teaneck, NJ, 1990, pp. 389–433.
          MR 1104265. Zbl 0737.58060.

[MP06]   D. MAULIK and R. PANDHARIPANDE, A topological view of Gromov–
          Witten theory, *Topology* **45** (2006), 887–918. MR 2248516. Zbl 1112.
          14065. http://dx.doi.org/10.1016/j.top.2006.06.002.

[Mil]   J. S. MILNE, Fields and Galois theory. Available at http://www.jmilne.
          org/math/CourseNotes/FT421.pdf.

[Moc06]   T. MOCHIZUKI, The virtual class of the moduli stack of stable *r*-spin
          curves, *Comm. Math. Phys.* **264** (2006), 1–40. MR 2211733. Zbl 1136.
          14015. http://dx.doi.org/10.1007/s00220-006-1538-3.

[Mum83]   D. MUMFORD, Towards an enumerative geometry of the moduli space of
          curves, in *Arithmetic and Geometry*, *Vol. II*, *Progr. Math.* **36**, Birkhäuser,
          Boston, MA, 1983, pp. 271–328. MR 0717614. Zbl 0554.14008.

[Nou84]   M. NOUMI, Expansion of the solutions of a Gauss-Manin system at a point
          of infinity, *Tokyo J. Math.* **7** (1984), 1–60. MR 0752110. Zbl 0554.32009.
          http://dx.doi.org/10.3836/tjm/1270152991.

[Nou]   M. NOUMI, private communication.

[NY98]   M. NOUMI and Y. YAMADA, Notes on the flat structures associated with simple and simply elliptic singularities, in *Integrable Systems and Algebraic Geometry* (Kobe/Kyoto, 1997), World Sci. Publ., River Edge, NJ, 1998, pp. 373–383. MR 1672069. Zbl 0964.32025.

[OP06a]   A. OKOUNKOV and R. PANDHARIPANDE, The equivariant Gromov-Witten theory of $\mathbf{P}^1$, *Ann. of Math.* **163** (2006), 561–605. MR 2199226. Zbl 1105.14077. http://dx.doi.org/10.4007/annals.2006.163.561.

[OP06b]   A. OKOUNKOV and R. PANDHARIPANDE, Virasoro constraints for target curves, *Invent. Math.* **163** (2006), 47–108. MR 2208418. Zbl 1140.14047. http://dx.doi.org/10.1007/s00222-005-0455-y.

[Pol04]   A. POLISHCHUK, Witten's top Chern class on the moduli space of higher spin curves, in *Frobenius Manifolds*, *Aspects Math.* **E36**, Vieweg, Wiesbaden, 2004, pp. 253–264. MR 2115773. Zbl 1105.14010. http://dx.doi.org/10.1007/978-3-322-80236-1_10.

[PV01]   A. POLISHCHUK and A. VAINTROB, Algebraic construction of Witten's top Chern class, in *Advances in Algebraic Geometry Motivated by Physics* (Lowell, MA, 2000), *Contemp. Math.* **276**, Amer. Math. Soc., Providence, RI, 2001, pp. 229–249. MR 1837120. Zbl 1051.14007. http://dx.doi.org/10.1090/conm/276/04523.

[Qui71]   D. QUILLEN, Elementary proofs of some results of cobordism theory using Steenrod operations, *Advances in Math.* **7** (1971), 29–56 (1971). MR 0290382. Zbl 0214.50502. http://dx.doi.org/10.1016/0001-8708(71)90041-7.

[Sai81]   K. SAITO, Primitive forms for a universal unfolding of a function with an isolated critical point, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **28** (1981), 775–792 (1982). MR 0656053. Zbl 0523.32015.

[Sei08]   P. SEIDEL, *Fukaya Categories and Picard-Lefschetz Theory*, *Zurich Lect. Adv. Math.*, European Mathematical Society (EMS), Zürich, 2008. MR 2441780. Zbl 1159.53001. http://dx.doi.org/10.4171/063.

[Tay11]   P. TAYLOR, Commutative diagrams package, 2011. Available at http://www.paultaylor.eu/diagrams/.

[Tel12]   C. TELEMAN, The structure of 2D semi-simple field theories, *Invent. Math.* **188** (2012), 525–588. MR 2917177. Zbl 1248.53074. http://dx.doi.org/10.1007/s00222-011-0352-5.

[Toe99]   B. TOEN, Théorèmes de Riemann-Roch pour les champs de Deligne-Mumford, *K-Theory* **18** (1999), 33–76. MR 1710187. Zbl 0946.14004. http://dx.doi.org/10.1023/A:1007791200714.

[Tse10]   H.-H. TSENG, Orbifold quantum Riemann-Roch, Lefschetz and Serre, *Geom. Topol.* **14** (2010), 1–81. MR 2578300. Zbl 1178.14058. http://dx.doi.org/10.2140/gt.2010.14.1.

[VW89]   C. VAFA and N. WARNER, Catastrophes and the classification of conformal theories, *Phys. Lett. B* **218** (1989), 51–58. MR 0983349. http://dx.doi.org/10.1016/0370-2693(89)90473-5.

[Wal80a]   C. T. C. WALL, A note on symmetry of singularities, *Bull. London Math. Soc.* **12** (1980), 169–175. MR 0572095. Zbl 0427.32010. http://dx.doi.org/10.1112/blms/12.3.169.

[Wal80b]   C. T. C. WALL, A second note on symmetry of singularities, *Bull. London Math. Soc.* **12** (1980), 347–354. MR 0587705. Zbl 0424.58006. http://dx.doi.org/10.1112/blms/12.5.347.

[Wit91]   E. WITTEN, Two-dimensional gravity and intersection theory on moduli space, in *Surveys in Differential Geometry* (Cambridge, MA, 1990), Lehigh Univ., Bethlehem, PA, 1991, pp. 243–310. MR 1144529. Zbl 0757.53049.

[Wit93a]   E. WITTEN, Algebraic geometry associated with matrix models of two-dimensional gravity, in *Topological Methods in Modern Mathematics* (Stony Brook, NY, 1991), Publish or Perish, Houston, TX, 1993, pp. 235–269. MR 1215968. Zbl 0812.14017.

[Wit93b]   E. WITTEN, Phases of $N = 2$ theories in two dimensions, *Nuclear Phys. B* **403** (1993), 159–222. MR 1232617. Zbl 0910.14020. http://dx.doi.org/10.1016/0550-3213(93)90033-L.

[Wit]   E. WITTEN, private communication.

[Zin08]   A. ZINGER, Standard versus reduced genus-one Gromov-Witten invariants, *Geom. Topol.* **12** (2008), 1203–1241. MR 2403808. Zbl 1167.14009. http://dx.doi.org/10.2140/gt.2008.12.1203.

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