Finite time singularities for Lagrangian mean curvature flow

By André Neves

Abstract

Given any embedded Lagrangian on a four-dimensional compact Calabi-Yau, we find another Lagrangian in the same Hamiltonian isotopy class that develops a finite time singularity under mean curvature flow. This contradicts a weaker version of the Thomas-Yau conjecture regarding long time existence and convergence of Lagrangian mean curvature flow.

1. Introduction

One of the hardest open problems regarding the geometry of Calabi-Yau manifolds consists in determining when a given Lagrangian admits a minimal Lagrangian (SLag) in its homology class or Hamiltonian isotopy class. If such SLag exists, then it is area-minimizing, and so one could approach this problem by trying to minimize area among all Lagrangians in a given class. Schoen and Wolfson [12] studied the minimization problem and showed that, when the real dimension is four, a Lagrangian minimizing area among all Lagrangians in a given class exists, is smooth everywhere except finitely many points, but is not necessarily a minimal surface. Later Wolfson [19] found a Lagrangian sphere with nontrivial homology on a given K3 surface such that the Lagrangian that minimizes area among all Lagrangians in this class is not an SLag and the surface that minimizes area among all surfaces in this class is not Lagrangian. This shows the subtle nature of the problem.

In another direction, Smoczyk [14] observed that when the ambient manifold is Kähler-Einstein, the Lagrangian condition is preserved by the gradient flow of the area functional (mean curvature flow), and so a natural question is whether one can produce SLag’s using Lagrangian mean curvature flow. To that end, R. P. Thomas and S.-T. Yau [15, §7] considered this question and proposed a notion of “stability” for Lagrangians in a given Calabi-Yau, which we now describe.

© 2013 Department of Mathematics, Princeton University.
Let \((M^{2n}, \omega, J, \Omega)\) be a compact Calabi-Yau with metric \(g\), where \(\Omega\) stands for the unit parallel section of the canonical bundle. Given \(L \subseteq M\) Lagrangian, it is a simple exercise ([15, §2] for instance) to see that
\[
\Omega_L = e^{i\theta} \text{vol}_L,
\]
where \(\text{vol}_L\) denotes the volume form of \(L\) and \(\theta\) is a multivalued function defined on \(L\) called the \textit{Lagrangian angle}. All the Lagrangians considered will be \textit{zero-Maslov class}, meaning that \(\theta\) can be lifted to a well-defined function on \(L\). Moreover if \(L\) is zero-Maslov class with oscillation of Lagrangian angle less than \(\pi\) (called \textit{almost-calibrated}), there is a natural choice for the phase of \(\int_L \Omega\), which we denote by \(\phi(L)\). Finally, given any two Lagrangians \(L_1, L_2\), a connected sum operation \(L_1 \# L_2\) is defined in [15, §3] (more involved than a simply topological connected sum). We refer the reader to [15, §3] for the details.

\textit{Definition 1.1 (Thomas-Yau Flow-Stability).} Without loss of generality, suppose that the almost-calibrated Lagrangian \(L\) has \(\phi(L) = 0\). Then \(L\) is flow-stable if either of the following two happen:

- \(L\) Hamiltonian isotopic to \(L_1 \# L_2\), where \(L_1, L_2\) are two almost-calibrated Lagrangians, implies that
  \[
  [\phi(L_1), \phi(L_2)] \not\subseteq (\inf_L \theta, \sup_L \theta).
  \]
- \(L\) Hamiltonian isotopic to \(L_1 \# L_2\), where \(L_1, L_2\) are almost-calibrated Lagrangians, implies that
  \[
  \text{area}(L) \leq \int_{L_1} e^{-i\phi(L_1)} \Omega + \int_{L_2} e^{-i\phi(L_2)} \Omega.
  \]

\textit{Remark 1.2.} The notion of flow-stability defined in [15, §7] applies to a larger class than almost-calibrated Lagrangians. For simplicity, but also because the author (unfortunately) does not fully understand that larger class, we chose to restrict the definition to almost calibrated.

In [15, §7] it is then conjectured

\textit{Conjecture (Thomas-Yau Conjecture).} Let \(L\) be a flow-stable Lagrangian in a Calabi-Yau manifold. Then the Lagrangian mean curvature flow will exist for all time and converge to the unique SLag in its Hamiltonian isotopy class.

The intuitive idea is that if a singularity occurs, it is because the flow is trying to decompose the Lagrangian into “simpler” pieces and so, if we rule out this possibility, no finite time singularities should occur. Unfortunately, their stability condition is, in general, hard to check. For instance, the definition does not seem to be preserved by Hamiltonian isotopies, and so the existence of Lagrangians that are flow-stable and not SLag is a highly nontrivial problem. As a result, it becomes quite hard to disprove the conjecture because not many
examples of flow-stable Lagrangians are known. For this reason there has been
considerable interest in the following simplified version of the above conjecture
(see [17, §1.4]).

**Conjecture.** Let $M$ be Calabi-Yau and $\Sigma$ be a compact embedded Lagrangian submanifold with zero Maslov class. Then the mean curvature flow of $\Sigma$ exists for all time and converges smoothly to a special Lagrangian submanifold in the Hamiltonian isotopy class of $\Sigma$.

We remark that in [17] this conjecture is attributed to Thomas and Yau, but this is not correct because there is no mention of stability. For this reason, this conjecture, due to Mu-Tao Wang, is a weaker version of Thomas-Yau conjecture.

Schoen and Wolfson [13] constructed solutions to Lagrangian mean curvature flow that become singular in finite time and where the initial condition is homologous to a SLag $\Sigma$. On the other hand, we remark that the flow does distinguish between isotopy class and homology class. For instance, on a two-dimensional torus, a curve $\gamma$ with a single self-intersection that is homologous to a simple closed geodesic will develop a finite time singularity under curve shortening flow, while if we make the more restrictive assumption that $\gamma$ is isotopic to a simple closed geodesic, Grayson’s Theorem [5] implies that the curve shortening flow will exist for all time and sequentially converge to a simple closed geodesic.

The purpose of his paper is to prove

**Theorem 6.1.** Let $M$ be a four real-dimensional Calabi-Yau and $\Sigma$ an embedded Lagrangian. There is $L$ Hamiltonian isotopic to $\Sigma$ so that the Lagrangian mean curvature flow starting at $L$ develops a finite time singularity.

**Remark 1.3.** (1) If we take $\Sigma$ to be a SLag, the theorem implies the second conjecture is false because $L$ is then a zero-Maslov class Lagrangian.

(2) Theorem A provides the first examples of compact embedded Lagrangians that are not homologically trivial and for which mean curvature flow develops a finite time singularity. The main difficulty comes from the fact, due to the high codimension, that barrier arguments or maximum principle arguments do not seem to be as effective as in the codimension-one case and thus new ideas are needed.

(3) One way to picture $L$ is to imagine a very small Whitney sphere $N$ (a Lagrangian sphere with a single transverse self-intersection at $p$ in $\Sigma$) and consider $L = \Sigma \# N$. (See the local picture in Figure 1.)

(4) If $\Sigma$ is SLag, then for every $\varepsilon$ we can make the oscillation for the Lagrangian angle of $L$ lying in $[-\varepsilon, \pi + \varepsilon]$. Thus $L$ is not almost-calibrated and so does not qualified to be flow-stable in the sense of Thomas-Yau.
(5) It is a challenging open question whether or not one can find a Hamiltonian isotopic to a SLag with arbitrarily small oscillation of the Lagrangian angle such that mean curvature flow develops finite time singularities. More generally, it is a fascinating problem to state a Thomas-Yau type conjecture that has an easier to check hypothesis on the initial condition and allows (or not) for the formation of a restricted type of singularities.

Acknowledgements. The author would like to thank Sigurd Angenent for his remarks regarding Section 7.2 and Richard Thomas and Dominic Joyce for their kindness in explaining the notion of stability for the flow. He would also like to express his gratitude to Felix Schulze for his comments on Lemma 5.13 and to one of the referees for the extensive comments and explanations that greatly improved the exposition of this paper.

2. Preliminaries and sketch of proof

In this section we describe the main ideas that go into the proof of Theorem A, but first we have to introduce some notation.

2.1. Preliminaries. Fix \((M, J, \omega, \Omega)\) a four-dimensional Calabi-Yau manifold with Ricci flat metric \(g\), complex structure \(J\), Kähler form \(\omega\), and calibration form \(\Omega\). For every \(R\), set \(g_R = R^2g\), and consider \(G\) to be an isometric embedding of \((M, g_R)\) into some Euclidean plane \(\mathbb{R}^n\). \(L\) denotes a smooth Lagrangian surface contained in \(M\) and \((L_t)_{t \geq 0}\) a smooth solution to Lagrangian mean curvature flow with respect to one of the metrics \(g_R\). (Different \(R\) simply change the time scale of the flow.) It is simple to recognize the existence of \(F_t : L \rightarrow \mathbb{R}^n\) so that the surfaces \(L_t = F_t(L)\) solve the equation

\[
\frac{dF_t}{dt}(x) = H(F_t(x)) = \bar{H}(F_t(x)) + E(F_t(x), T_{F_t(x)}L_t),
\]

where \(H(F_t(x))\) stands for the mean curvature with respect to \(g_R\), \(\bar{H}(F_t(x))\) stands for the mean curvature with respect to the Euclidean metric and \(E\) is some vector valued function defined on \(\mathbb{R}^n \times G(2, n)\), with \(G(2, n)\) being the set of 2-planes in \(\mathbb{R}^n\). The term \(E\) can be made arbitrarily small by choosing \(R\) sufficiently large. In order to avoid introducing unnecessary notation, we will not be explicit whether we are regarding \(L_t\) being a submanifold of \(M\) or \(\mathbb{R}^n\).

Given any \((x_0, T)\) in \(\mathbb{R}^n \times \mathbb{R}\), we consider the backwards heat kernel

\[
\Phi(x_0, T)(x, t) = \exp\left(-\frac{|x-x_0|^2}{4(T-t)}\right),
\]

We need the following extension of Huisken’s monotonicity [6] formula which follows trivially from [16, formula (5.3)].
LEMMA 2.1 (Huisken’s monotonicity formula). Let \( f_t \) be a smooth family of functions with compact support on \( L_t \). Then
\[
\frac{d}{dt} \int_{L_t} f_t \Phi(x_0, T) d\mathcal{H}^2 = \int_{L_t} (\partial_t f_t - \Delta f_t) \Phi(x_0, T) d\mathcal{H}^2
\]
\[\quad - \int_{L_t} \left( H + \frac{E}{2} + \frac{(x - x_0)^2}{2(T - t_0)} \right) \Phi(x_0, T) d\mathcal{H}^2 + \int_{L_t} f_t \Phi(x_0, T) \frac{|E|^2}{4} d\mathcal{H}^2.\]

We denote
\[
A(r_1, r_2) = \{ x \in \mathbb{R}^n \mid r_1 < |x| < r_2 \}, \quad B_r = A(-1, r),
\]
and define the \( C^{2,\alpha} \) norm of a surface \( N \) at a point \( x_0 \) in \( \mathbb{R}^n \) as in [18, §2.5]. This norm is scale invariant and, given an open set \( U \), the \( C^{2,\alpha}(U) \) norm of \( N \) denotes the supremum in \( U \) of the pointwise \( C^{2,\alpha} \) norms. We say \( N \) is \( \nu \)-close in \( C^{2,\alpha} \) to \( N \) if there is an open set \( U \) and a function \( u : N \cap U \rightarrow \mathbb{R}^n \) so that \( \bar{N} = u(N \cap U) \) and the \( C^{2,\alpha} \) norm of \( u \) (with respect to the induced metric on \( N \)) is smaller than \( \nu \).

2.1.1. Definition of \( N(\varepsilon, R) \). Let \( c_1, c_2, \) and \( c_3 \) be three half-lines in \( \mathbb{C} \) so that \( c_1 \) is the positive real axis and \( c_2, c_3 \) are, respectively, the positive line segments spanned by \( e^{i\theta_2} \) and \( e^{i\theta_3} \), where \( \pi/2 < \theta_2 < \theta_3 < \pi \). These curves generate three Lagrangian planes in \( \mathbb{R}^4 \), which we denote by \( P_1, P_2, \) and \( P_3 \) respectively. Consider a curve \( \gamma(\varepsilon) : [0, +\infty) \rightarrow \mathbb{C} \) such that (see Figure 1)
\begin{itemize}
  \item \( \gamma(\varepsilon) \) lies in the first and second quadrant and \( \gamma(\varepsilon)^{-1}(0) = 0 \);
  \item \( \gamma(\varepsilon) \cap A(3, \infty) = c_1^+ \cap A(3, \infty) \) and \( \gamma(\varepsilon) \cap A(\varepsilon, 1) = (c_1^+ \cup c_2 \cup c_3) \cap A(\varepsilon, 1) \);
  \item \( \gamma(\varepsilon) \cap B_1 \) has two connected components \( \gamma_1 \) and \( \gamma_2 \), where \( \gamma_1 \) connects \( c_2 \) to \( c_1^+ \) and \( \gamma_2 \) coincides with \( c_3 \);
  \item the Lagrangian angle of \( \gamma_1 \), \( \arg \left( \gamma_1 \frac{d\gamma_1}{d\varepsilon} \right) \), has oscillation strictly smaller than \( \pi/2 \).
\end{itemize}

Set \( \gamma(\varepsilon, R) = R\gamma(\varepsilon/R) \). We define
\[
N(\varepsilon, R) = \{ (\gamma(\varepsilon, R)(s) \cos \alpha, \gamma(\varepsilon, R)(s) \sin \alpha) \mid s \geq 0, \alpha \in S^1 \}.
\]

We remark that one can make the oscillation for the Lagrangian angle of \( N(\varepsilon, R) \) as close to \( \pi \) as desired by choosing \( \theta_2 \) and \( \theta_3 \) very close to \( \pi/2 \).

2.1.2. Definition of a self-expander. A surface \( \Sigma \subseteq \mathbb{R}^4 \) is called a self-expander if \( H = \frac{\varepsilon^2}{2} \), which is equivalent to saying that \( \Sigma_t = \sqrt{t}\Sigma \) is a solution to mean curvature flow. We say that \( \Sigma \) is asymptotic to a varifold \( V \) if, when \( t \) tends to zero, \( \Sigma_t \) converges in the Radon measure sense to \( V \). For instance, Anciaux [1, §5] showed there is a unique curve \( \chi \) in \( \mathbb{C} \) so that
\[
S = \{ (\chi(s) \cos \alpha, \chi(s) \sin \alpha) \mid s \in \mathbb{R}, \alpha \in S^1 \}
\]
is a self-expander for Lagrangian mean curvature flow asymptotic to \( P_1 + P_2 \).
2.2. Sketch of Proof.

Theorem 6.1. Let $M$ be a four real-dimensional Calabi-Yau and $\Sigma$ an embedded Lagrangian. There is $L$ Hamiltonian isotopic to $\Sigma$ so that the Lagrangian mean curvature flow starting at $L$ develops a finite time singularity.

Remark 2.2. The argument to prove Theorem 6.1 has two main ideas. The first is to construct $L$ so that if the flow $(L_t)_{t \geq 0}$ exists smoothly, then $L_1$ and $L$ will be in different Hamiltonian isotopy classes. Unfortunately this does not mean the flow must become singular because Lagrangian mean curvature flow is not an ambient Hamiltonian isotopy. This is explained below in the First Step and the Second Step.

The second main idea is to note that $L_1$ is very close to an $SO(2)$-invariant Lagrangian $M_1$ that has the following property. The flow $(M_t)_{t \geq 1}$ develops a singularity at some time $T$ with the Lagrangian angle jumping by $2\pi$ at instant $T$. Because the solution $(L_t)_{t \geq 1}$ will be “nearby” $(M_t)_{t \geq 1}$, this jump
will also occur on \((L_t)_{t \geq 1}\) around time \(T\), which means that it must have a singularity as well.

**Sketch of proof.** It suffices to find a singular solution to Lagrangian mean curvature flow with respect to the metric \(g_R = R^2 g\) for \(R\) sufficiently large. Pick Darboux coordinates defined on \(B_4R\) that send the origin into \(p \in \Sigma\) so that \(T_p \Sigma\) coincides with the real plane oriented positively and the pullback metric at the origin is Euclidean. (We can increase \(R\) by making \(R\) larger.) The basic approach is to remove \(\Sigma \cap B_2R\) and replace it with \(N(\varepsilon, R) \cap B_2R\). Denote the resulting Lagrangian by \(L\) which, due to [4, Th. 1.1.A], we know to be Hamiltonian isotopic to \(\Sigma\).

Assume that the Lagrangian mean curvature flow \((L_t)_{t \geq 0}\) exists for all time. The goal is to get a contradiction when \(R, R\) are large enough and \(\varepsilon\) is small enough.

**First Step.** Because \(L \cap A(1, 2R)\) consists of three planes that intersect transversely at the origin, we will use standard arguments based on White’s Regularity Theorem [18] and obtain estimates for the flow in a smaller annular region. Hence, we will conclude the existence of \(R_1\) uniform so that \(L_t \cap A(R_1, R)\) for all \(1 \leq t \leq 2\) and the decomposition of \(L_t \cap B_R\) into two connected components \(Q_{1,t}, Q_{2,t}\) for all \(0 \leq t \leq 2\), where \(Q_{2,0} = P_3 \cap B_R\). Moreover, we will also show that \(Q_{2,t}\) is a small \(C^{2,\alpha}\) perturbation of \(P_3\) for all \(1 \leq t \leq 2\). This is done in Section 3, and the arguments are well-known among the experts.

**Second Step.** In Section 4 we show that \(Q_{1,1}\) must be close to \(S\), the smooth self-expander asymptotic to \(P_1\) and \(P_2\). (See (2) and Figure 2.) The geometric argument is that self-expanders act as attractors for the flow; i.e., because \(Q_{1,0}\) is very close to \(P_1 \cup P_2\) and \(\sqrt{t} S\) tends to \(P_1 \cup P_2\) when \(t\) tends to zero, then \(Q_{1,t}\) must be very close to \(\sqrt{t} S\) for all \(1 \leq t \leq 2\). It is crucial for this part of the argument that \((Q_{1,t})_{0 \leq t \leq 2}\) exists smoothly and that \(P_1 \cup P_2\) is not area-minimizing. (See Theorem 4.2 and Remark 4.3 for more details.) This step is the first main idea of this paper.

From the first two steps it follows that \(L_1\) is very close to a Lagrangian \(M_1\) generated by a curve \(\sigma\) like the one in Figure 3. Because \(Q_{1,0}\) is isotopic to \(P_1 \# P_2\) but \(Q_{1,1}\) is isotopic to \(P_2 \# P_1\) (in the notation of [15]), we have that \(M_1\) is not Hamiltonian isotopic to \(L\). Thus it is not possible to connect the two by an ambient Hamiltonian isotopy. Nonetheless, as it was explained to the author by Paul Seidel, it is possible to connect them by smooth Lagrangian immersions that are neither rotationally symmetric nor embedded. Unfortunately it is not known whether Lagrangian mean curvature flow is a Hamiltonian isotopy (only infinitesimal Hamiltonian deformation is known), and so there is no topological obstruction to go from \(L\) to \(L_1\) without singularities.
Naturally we conjecture that does not occur and that \((L_t)_{0 \leq t \leq 1}\) has a finite time singularity that corresponds to the flow developing a “neck-pinch” in order to get rid of the noncompact “Whitney Sphere” \(N(\varepsilon, R)\) we glued to \(\Sigma\). If the initial condition is simply \(N(\varepsilon, R)\) instead of \(L\), we showed in [9, §4] that this conjecture is true, but the arguments relied on the rotationally symmetric properties of \(N(\varepsilon, R)\) and thus cannot be extended to arbitrarily small perturbations like \(L\). If this conjecture were true, then the proof of Theorem 6.1 would finish here.

After several attempts, the author was unable to prove this conjecture, and this lead us to the second main idea of this paper described below. Again we stress that, conjecturally, this case will never occur without going through “earlier” singularities.

Third step. Denote by \((M_t)_{t \geq 1}\) the evolution by mean curvature flow of \(M_1\), the Lagrangian that corresponds to the curve \(\sigma\). In Theorem 5.3 we will show that \(M_t\) is SO(2)-invariant and can be described by curves \(\sigma_t\) that evolve the following way (see Figure 4). There is a singular time \(T\) so that for all \(t < T\), the curves \(\sigma_t\) look like \(\sigma\) but with a smaller enclosed loop. When \(t = T\), this enclosed loop collapses and we have a singularity for the flow. For \(t > T\), the curves \(\sigma_t\) will become smooth and embedded.

We can now describe the second main idea of this paper. (See Remark 5.2 and Corollary 5.5 for more details.) Because \(\sigma_t\) “loses” a loop when \(t\) passes through the singular time, winding number considerations will show that the Lagrangian angle of \(M_t\) must suffer a discontinuity of \(2\pi\). Standard arguments will show that, because \(L_1\) is very close to \(M_1\), then \(L_t\) will be very close to \(M_t\) as well and so the Lagrangian angle of \(L_t\) should also suffer a discontinuity of approximately \(2\pi\) when \(t\) passes through \(T\). But this contradicts the fact that \((L_t)_{t \geq 0}\) exists smoothly. \(\Box\)
2.3. Organization. The first step in the proof is done in Section 3. It consists mostly of standard but slightly technical results, all of which are well known. The second step is done in Section 4, and the third step is done in Section 5. Finally, in Section 6 the proof of Theorem 6.1 is made rigorous and in the appendix some basic results are collected.

Some parts of this paper are long and technical but can be skipped on a first reading. Section 3 can be skipped and consulted only when necessary. In Section 4 the reader can skip the proofs of Propositions 4.4 and 4.6 and read instead the outlines in Remarks 4.5 and 4.7. In Section 5 the reader can skip the proof of Theorem 5.3.

3. First Step: General Results

3.1. Setup of Section 3.

3.1.1. Hypothesis on ambient space. We assume the setting of Section 2.1 and the existence of a Darboux chart

$$\phi : B_{4R} \rightarrow M,$$

meaning $\phi^* \omega$ coincides with the standard symplectic form in $\mathbb{R}^4$ and $\phi^* J$ and $\phi^* \Omega$ coincide, respectively, with the standard complex structure and $dz_1 \wedge dz_2$ at the origin. Moreover, we assume that

- $\phi^* g_R$ is $1/R$-close in $C^3$ to the Euclidean metric,
- $G \circ \phi$ is $1/R$-close in $C^3$ to the map that sends $x$ in $\mathbb{R}^4$ to $(x, 0)$ in $\mathbb{R}^n$,
- the $C^{0, \alpha}$ norm of $E$ (defined in Section 2.1) is smaller than $1/R$, and
- $G(M) \cap B_{4R-1} \subseteq G \circ \phi (B_{4R}).$

For the sake of simplicity, given any subset $B$ of $M$, we freely identify $B$ with $\phi^{-1}(B)$ in $B_{4R}$ or $G(B)$ in $\mathbb{R}^n$. 

Figure 4. Evolution of $\sigma_t$. 

3.1.2. Hypothesis on Lagrangian $L$. We assume that $L \subseteq M$ Lagrangian is such that

\[(3) \quad L \cap B_{2R} = N(\varepsilon, R) \cap B_{2R} \quad \text{for some } R \geq 4R,\]

where $N(\varepsilon, R)$ was defined in (1). Thus $L \cap B_{2R}$ consists of two connected components $Q^1$ and $Q^2$, where

\[(4) \quad Q^1 \setminus B_\varepsilon = (P_1 + P_2) \cap A(\varepsilon, 2R) \quad \text{and } \quad Q^2 = P_3 \cap B_{2R}.\]

To be rigorous, one should use the notation $L_{\varepsilon,R}$ for $L$. Nonetheless, for the sake of simplicity, we prefer the latter. Finally, we assume the existence of $K_0$ so that

- area$(L \cap B_r(x)) \leq K_0 r^2$ for every $x \in M$ and $r \geq 0$, 
- the norm of second fundamental form of $M$ in $\mathbb{R}^n$ is bounded by $K_0$, 
- $\sup_{Q^1} |\theta| \leq \pi/2 - K_0^{-1}$ and we can find $\beta \in C^\infty(Q^1)$ such that $d\beta = \lambda$, and

\[(5) \quad |\beta(x)| \leq K_0(|x|^2 + 1) \quad \text{for all } x \in Q^1.\]

3.2. Main results. We start with two basic lemmas and then state the two main theorems.

**Lemma 3.1.** For all $\varepsilon$ small, $R$ large, and $T_1 > 0$, there is $D = D(T_1, K_0)$ so that

$\mathcal{H}^2(B_r(x) \cap L_t) \leq Dr^2$ for all $x \in \mathbb{R}^n$, $r > 0$, and $0 \leq t \leq T_1$.

**Proof.** Assuming a uniform bound on the second fundamental form of $M$ in $\mathbb{R}^n$, it is a standard fact that uniform area bounds for $L_t$ hold for all $0 \leq t \leq T_1$. (See, for instance, [9, Lemma A.3] if $g$ is the Euclidean metric. A general proof could be given along the same lines provided we use the modification of monotonicity formula given in Lemma 2.1.)

**Lemma 3.2.** For every $\delta$ small, $T_1 > 1$, and $R > 0$, there is $R = R(T_1, \delta, R)$ so that, for every $1 \leq t \leq T_1$, $L_t$ is $\delta$-close in $C^{2,\alpha}$ to the plane $P_1$ in the annular region $A(R, \overline{R})$.

**Proof.** Apply Lemma 3.9 with $\nu$ being $\delta$ given in this lemma, $S = 1$, and $\kappa = 1/T_1$. Because $L_0 \cap A(3R, 2R) = P_1 \cap A(3R, 2R)$, it is simple to see that conditions (a), (b), and (c) of Lemma 3.9 are satisfied for all $x_0 \in L_0 \cap A(R, \overline{R})$ provided we choose $R$ suitably large. Thus, the desired result follows from Lemma 3.9(ii).

The next theorem is one of the main results of this section. The proof will be given at the end of Section 3 and can be skipped on a first reading.

**Theorem 3.3.** Fix $\nu$. The constant $\Lambda_0$ mentioned below is universal.
There are \( \varepsilon_1 \) and \( R_1 \), depending on the planes \( P_1, P_2, P_3, K_0, \) and \( \nu \), such that if \( \varepsilon \leq \varepsilon_1 \) and \( R \geq 2R_1 \) in (3), then

(i) for every \( 0 \leq t \leq 2 \), the \( C^{2,\alpha}(A(R_1, R)) \) norm of \( L_t \) is bounded by \( \Lambda_0 t^{-1/2} \) and

\[
F_t(x) \in A(R_1, R) \implies |F_s(x) - x| < \Lambda_0 \sqrt{s} \text{ for all } 0 \leq s \leq 2;
\]

(ii) for every \( 1 \leq t \leq 2 \), \( L_t \cap A(R_1, R) \) is \( \nu \)-close in \( C^{2,\alpha} \) to \( L \).

Moreover, setting

\[
Q_{1,t} = F_t(Q^1 \cap B_R) \quad \text{and} \quad Q_{2,t} = F_t(Q^2 \cap B_R),
\]

we have that

(iii) for every \( 0 \leq t \leq 2 \),

\[
L_t \cap B_{R-\Lambda_0} \subseteq Q_{1,t} \cup Q_{2,t} \subseteq B_{R+\Lambda_0};
\]

(iv) for every \( 1 \leq t \leq 2 \), \( Q_{2,t} \) is \( \nu \)-close in \( C^{2,\alpha}(B_{R_1}) \) to \( P_3 \).

Remark 3.4. (1) We remark that Theorem 3.3(i) and (iii) have no \( \nu \) dependence in their statements and so could have been stated independently of Theorem 3.3(ii) and (iv).

(2) The content of Theorem 3.3(i) and (ii) is that for all \( \varepsilon \) small and \( R \) large we have good control of \( L_t \) on an annular region \( A(R_1, R) \) for all \( t \leq 2 \). This is expected because, as we explain next, for all \( \varepsilon \) small and \( R \) sufficiently large, \( L \cap A(1, 2R) \) has small \( C^{2,\alpha} \) norm and area ratios close to one. In the region \( A(1, R) \) this follows because, as defined in (3),

\[
L \cap A(1, R) = (P_1 \cup P_2 \cup P_3) \cap A(1, R).
\]

In the region \( A(R, 2R) \) this follows because the \( C^{2,\alpha} \) norm and the area ratios of \( L \cap A(R, 2R) \) tend to zero as \( R \) tends to infinity.

(3) The content of Theorem 3.3(iii) is that \( L_t \cap B_R \) has two distinct connected components for all \( 0 \leq t \leq 2 \), which we call \( Q_{1,t} \) and \( Q_{2,t} \). The idea is that initially \( L \cap B_R \) has two connected components and because we have control of the flow on the annulus \( A(R_1, R) \) due to Theorem 3.3(i), then no connected component in \( L_t \cap B_R \) can be “lost” or “gained.” Without the control on the annular region it is simple to construct examples where a solution to mean curvature flow in \( B_1(0) \) consists initially of disjoint straight lines and at a later time is a single connected component.

(4) Theorem 3.3(iv) is also expected because \( Q_{2,0} \) initially is just a disc and we have good control on \( \partial Q_{2,t} \) for all \( 0 \leq t \leq 2 \).

The next theorem collects some important properties of \( Q_{1,t} \). The proof will be given at the end of Section 3 and, because it is largely standard, can be skipped on a first reading.
Theorem 3.5. There are $D_1$, $R_2$, and $\varepsilon_2$ depending only on $K_0$ so that if $R \geq R_2$ and $\varepsilon \leq \varepsilon_2$ in (3), then for every $0 \leq t \leq 2$, the following properties hold:

(i) $\sup_{0 \leq t \leq 2} \sup_{Q_{1,t}} |\theta_t| \leq \pi/2 - 1/(2K_0)$.

(ii) $\mathcal{H}^2(\hat{B}_r(y)) \geq D_1 r^2$,
where $\hat{B}_r(y)$ denotes the intrinsic ball of radius $r$ in $Q_{1,t}$ centered at $y \in Q_{1,t}$ and $r < \text{dist}(y, \partial Q_{1,t})$.

(iii) All $Q_{1,t}$ are exact, and one can choose $\beta_t \in C^\infty(Q_{1,t})$ with
$$d\beta_t = \lambda = \sum_{i=1}^2 x_i dy_i - y_i dx_i$$
and
$$\frac{d}{dt} (\beta_t + 2t\theta_t) = \Delta(\beta_t + 2t\theta_t) + E_1,$$
where $E_1 = \sum_{i=1}^2 \nabla e_i \lambda(e_i)$ and $\{e_1, e_2\}$ is an orthonormal basis for $Q_{1,t}$.

(iv) $|\beta_t|(x) \leq D_1(|x|^2 + 1)$ for every $x \in Q_{1,t}$.

(v) If $\mu = x_1y_2 - x_2y_1$, then
$$\frac{d|\mu|^2}{dt} \leq \Delta|\mu|^2 - 2|\nabla \mu|^2 + E_2,$$
where $E_2 = (|x|^3 + 1)O(1/R)$.

Remark 3.6. (1) We comment on Theorem 3.5(i). Recall that we are assuming $\sup_{Q_1} |\theta| \leq \pi/2 - K_0^{-1}$, where $Q_1$ is defined in (4). Because $\theta_t$ evolves by the heat equation, we have
$$\sup_{0 \leq t \leq 2} \sup_{Q_{1,t}} |\theta_t| \leq \max \left\{ \sup_{Q_{1,0}} |\theta|, \sup_{0 \leq t \leq 2} \sup_{\partial Q_{1,t}} |\theta_t| \right\}.$$
Hence we need to control the Lagrangian angle along $\partial Q_{1,t}$ in order to obtain Theorem 3.5(i). The idea is to use the fact that $Q_1$ is very “flat” near $\partial Q_{1,0}$ to show that $F_l(Q_1)$ is a small $C^1$ perturbation of $Q_1$ near $\partial Q_{1,0}$, which means the Lagrangian angle along $\partial Q_{1,t}$ will not change much.

(2) Theorem 3.5(ii) is a consequence of the fact that $Q_{1,t}$ is almost-calibrated.

(3) Theorem 3.5(iii) and (v) are just derivations of evolution equations taking into account the error term one obtains from the metric $g_R$ (defined in Section 2.1) not being Euclidean.

(4) Theorem 3.5(iv) gives the expected growth for $\beta_t$ on $Q_{1,t}$, and its proof is a simple technical matter.
3.3. Abstract results. We derive some simple results that will be used to prove Theorems 3.3 and 3.5 as well as throughout the rest of the paper. They are presented in a fairly general setting in order to be used in various circumstances. The proofs are based on White’s Regularity Theorem and Huisken’s monotonicity formula.

Let $E$ be a vector valued function defined on $\mathbb{R}^n \times G(2, n)$, $\Sigma$ a smooth surface possibly with boundary, and $F_t : \Sigma \rightarrow \mathbb{R}^n$ a smooth solution to

\[
\frac{dF_t}{dt}(x) = H(F_t(x)) + E(F_t(x), T_{F_t(x)} M_t),
\]

where $M_t = F_t(\Sigma)$ and $F_0$ is the identity map.

In what follows, $\Omega$ denotes a closed set of $\mathbb{R}^n$ and we use the notation $\Omega(s) = \{ x \in \mathbb{R}^n | \text{dist}(x, \Omega) < s \}$.

We derive two lemmas that are well known among the experts. Denote $\bar{E} = \sup |E|_{0, \alpha}$, and let $\varepsilon_0$ be the constant given by White’s Regularity Theorem [18, Th. 4.1].

**Lemma 3.7.** Assume $T \leq 4$. There is $\Lambda = \Lambda(\bar{E}, n)$ so that for every $s \geq 0$, if

- (a) for all $0 \leq t \leq 2T$, $y \in \Omega(s + 2\Lambda \sqrt{T})$, and $l \leq 2T$,
  \[
  \int_{M_t} \Phi(y, l)d\mathcal{H}^2 \leq 1 + \varepsilon_0;
  \]

- (b) for all $0 \leq t \leq 2T$, $\partial M_t \cap \Omega(s + 2\Lambda \sqrt{T}) = \emptyset$;

then for every $0 \leq t \leq T$, we have

- (i) the $C^{2, \alpha}$ norm of $M_t$ on $\Omega(s + \Lambda \sqrt{T})$ is bounded by $\Lambda/\sqrt{t}$;
- (ii) $F_t(x) \in \Omega(s) \implies |F_t(x) - x| < \Lambda \sqrt{t}$ for all $0 \leq t \leq T$.

**Remark 3.8.** The content of Lemma 3.9 is that if we know the Gaussian density ratios at a scale smaller than $2T$ in a region $U$ are all close to one and $\partial M_t$ lies outside $U$ for all $t \leq 2T$, then we have good control of $M_t$ for all $0 \leq t \leq T$ on a slightly smaller region. The proof is a simple consequence of White’s Regularity Theorem.

**Proof.** Assume for all $0 \leq t \leq 2T$, $y \in \Omega(s + (\Lambda + 1) \sqrt{T})$, and $l \leq 2T$ that

\[
\int_{M_t} \Phi(y, l)d\mathcal{H}^2 \leq 1 + \varepsilon_0,
\]

where $\Lambda \geq 1$ is a constant to be chosen later. From White’s Regularity Theorem [18, Th. 4.1] there is $K_1 = K_1(\bar{E}, n)$ so that the $C^{2, \alpha}$ norm of $M_t$ on $\Omega(s + \Lambda \sqrt{T})$ is bounded by $K_1/\sqrt{t}$ and

\[
\sup_{M_t \cap \Omega(s + \Lambda \sqrt{T})} |A|^2 \leq \frac{K_1}{t}
\]
for every \( t \leq T \). Thus from (6), we obtain
\[
\left| \frac{dF_t(x)}{dt} \right| \leq \frac{K_1}{\sqrt{t}} + E
\]
whenever \( F_t(x) \in \Omega(s + \Lambda \sqrt{T}) \). Integrating the above inequality and using \( T \leq 2 \), we have the existence of \( K_2 = K_2(E, K_1) \) so that if
\[
F_t(x) \in \Omega(s + (\Lambda - K_2) \sqrt{T}) \text{ for some } 0 \leq t' \leq T,
\]
then
\[
F_t(x) \in \Omega(s + \Lambda \sqrt{T}) \quad \text{for every } 0 \leq t \leq T
\]
and
\[
|F_t(x) - x| < K_2 \sqrt{t} \quad \text{for every } 0 \leq t \leq T.
\]
Choose \( \Lambda = \max\{K_1, K_2\} \). Then (i) and (ii) follow at once. \( \square \)

**Lemma 3.9.** For every \( \nu, S \), and \( 0 < \kappa < 1 \), there is \( \delta, R \) so that if \( x_0 \in M_0 \) and

(a) the \( C^{2,\alpha} \) norm of \( M_0 \) in \( B_{R \sqrt{T}}(x_0) \) and the \( C^{0,\alpha}(\mathbb{R}^n \times G(2, M)) \) norm of \( E \) are smaller than \( \delta/\sqrt{T} \);

(b) \( \mathcal{H}^2(M_0 \cap B_r(x_0)) \leq 7\pi r^2 \) for all \( 0 \leq r \leq R \sqrt{T} \);

(c) \( \partial M_t \cap B_{R \sqrt{T}}(x_0) = \emptyset \) for all \( 0 \leq t \leq T \);

then the following hold:

(i) \( \int_{M_t} \Phi(y, l) d\mathcal{H}^2 \leq 1 + \varepsilon_0 \) for all \( y \in B_{(S+1) \sqrt{T}}(x_0) \), \( t \leq T \), and \( l \leq 2T \);

(ii) For every \( \kappa T \leq t \leq T \), there is a function
\[
 u_t : T_{x_0} M_0 \cap B_{(S+1) \sqrt{T}}(x_0) \rightarrow (T_{x_0} M_0)^{\perp}
\]
with
\[
\sup_{T_{x_0} M_0 \cap B_{(S+1) \sqrt{T}}} \left( |u_t|/\sqrt{T} + ||\nabla u_t|| + ||\nabla^2 u_t||_{0,\alpha} \sqrt{T} \right) \leq \nu
\]
and
\[
 M_t \cap B_{S \sqrt{T}}(x_0) \subseteq \{ u_t(x) + x, |x| \in T_{x_0} M_0 \cap B_{(S+1) \sqrt{T}}(x_0) \}.
\]

**Remark 3.10.** This lemma, roughly speaking, says that for every \( S \), there is \( R \) so that if the initial condition is very close to a disc in \( B_{R \sqrt{T}}(x_0) \) (condition (a) and (b)) and \( \partial M_t \) lies outside \( B_{R \sqrt{T}}(x_0) \) for all \( 0 \leq t \leq T \) (condition (c)), then we get good control of \( M_t \) inside \( B_{S \sqrt{T}}(x_0) \).

**Proof.** It suffices to prove this for \( T = 1 \) and \( x_0 = 0 \). Consider a sequence of flows \( (M^i_t)_{0 \leq t \leq 1} \) satisfying all the hypotheses with \( \delta_i \) converging to zero and \( R_i \) tending to infinity. The sequence of flows \( (M^i_t)_{t \geq 0} \) will converge weakly to \( (M_t)_{t \geq 0} \), a weak solution to mean curvature flow (see [7, §7.1]). The fact that the \( C^{2,\alpha}_{\text{loc}} \) norm of \( M^i_0 \) converges to zero implies that \( M^i_0 \) converges in \( C^{2,\alpha}_{\text{loc}} \) to
a union of planes. From (b) we conclude that \( M_0^i \) converges to a multiplicity one plane \( P \). Because \( \partial M_t^i \) lies outside \( B_{R_t} \) for all \( 0 \leq t \leq 1 \) with \( R_t \) tending to infinity and

\[
\lim_{i \to \infty} \int_{M_0^i} \Phi(y,l) d\mathcal{H}^2 = \int_P \Phi(y,l) d\mathcal{H}^2 = 1 \quad \text{for every } y \text{ and } l,
\]

we can still conclude from Huisken’s monotonicity formula that for all \( i \) sufficiently large,

\[
\int_{M_t^i} \Phi(y,l) d\mathcal{H}^2 \leq 1 + \varepsilon_0 \quad \text{for all } y \in B_{S+1}, \ t \leq 1, \text{ and } l \leq 2.
\]

This proves (i). Moreover, the above inequality also implies, via White’s Regularity Theorem, that \( M_t^i \) converges in \( C^{2,\alpha}_{\text{loc}} \) to \( P \) for all \( \kappa \leq t \leq 1 \) and so (ii) will also hold for all \( i \) sufficiently large. This implies the desired result. \( \square \)

3.4. Proof of Theorems 3.3 and 3.5.

Proof of Theorem 3.3. We first prove part (ii). Consider \( \delta \) and \( R \) given by Lemma 3.9 when \( \kappa = 1/2, \nu \) is the constant fixed in Theorem 3.3 and \( S \) is large to be chosen later. The same reasoning used in Remark 3.4(2) shows the existence of \( K_1 \) (depending on \( R \) and \( \delta \)) so that for all \( \varepsilon \) small and \( R \) sufficiently large, the \( C^{2,\alpha} \) norm of \( L \cap A(K_1, 2R - K_1) \) is smaller than \( \delta/2 \) and the area ratios with scale smaller than \( 2R \) are close to one. Thus, after relabelling \( K_1 \) to be \( K_1 - \sqrt{2}R \), we can apply Lemma 3.9(ii) (with \( T = 4 \)) to \( M_0^i = L \) for all \( x_0 \) in \( \Omega = L \cap A(K_1, 2R - K_1) \) and conclude Theorem 3.3(ii). Moreover, we also conclude from Lemma 3.9(i) that

\[
\int_{L_t} \Phi(y,l) d\mathcal{H}^2 \leq 1 + \varepsilon_0 \quad \text{for all } y \in \Omega(S), \ t \leq 4, \text{ and } l \leq 4,
\]

where \( \Omega(S) \) denotes the tubular neighbourhood of \( \Omega \) in \( \mathbb{R}^n \) with radius \( S \).

We now prove part (i). From (7), we see that hypotheses (a) and (b) of Lemma 3.7 are satisfied with \( T = 2, \ s = 0, \) and \( r = S - 2^{3/2} \Lambda \) (which we assume to be positive). Hence Lemma 3.7(i) gives that the \( C^{2,\alpha} \) norm of \( L_t \) in \( \Omega(S - 2^{1/2} \Lambda) \) is bounded by \( \Lambda/\sqrt{t} \). Theorem 3.3(i) follows from this provided

\[
L_t \cap A(K_1, 2R - K_1) \subset \Omega(S - 2^{3/2} \Lambda).
\]

This inclusion follows because, according to Brakke’s Clearing Out Lemma [7, §12.2] (which can be easily extended to our setting assuming small \( C^{0,\alpha} \) norm of \( |E| \)), there is a universal constant \( S_0 \) such that

\[
L_t \cap A(K_1, 2R - K_1) \subset \Omega(S_0) \text{ for all } 0 \leq t \leq 2.
\]

Thus we simply need to require \( S - 2^{3/2} \Lambda > S_0 \) in order to obtain (8). Furthermore, Lemma 3.7(ii) implies

\[
F_t(x) \in \Omega(S - 2^{3/2} \Lambda) \implies |F_s(x) - x| < \Lambda s^{1/2} \text{ for all } 0 \leq s \leq 2,
\]
which combined with (8) gives
\[ F_t(x) \in A(K_1, 2R - K_1) \implies |F_t(x) - x| < \Lambda s^{1/2} \text{ for all } 0 \leq s \leq 2, \]
and this proves the second statement of Theorem 3.3(i).

We now prove the first statement of Theorem 3.3(iii). Suppose
\[ L' \cap B_{R-\sqrt{2}\Lambda} \not\subset F'(L \cap B_R) = Q_{1,t'} \cup Q_{2,t'}, \]
meaning \( F_t(x) \in B_{R-\sqrt{2}\Lambda} \) but \( x \notin B_R \). By continuity there is \( 0 \leq t \leq t' \) so that
\[ F_t(x) \in A(K_1, R), \]
and this implies from (8) that \( F_t(x) \in \Omega(S - 2^{3/2}\Lambda) \), in which case we conclude from (9) that \( |F_t(x) - x| < \sqrt{2}\Lambda \), a contradiction. Similar reasoning shows the other inclusion in Theorem 3.3(iii).

Finally we show (iv). Apply Lemma 3.9, with \( S = K_1/\sqrt{2}, \kappa = 1/2, \) and \( \nu \) the constant fixed in this theorem, to \( M_0 = Q_{2,0} = P_3 \cap B_R \) where \( x_0 = 0 \). Note that hypotheses (a) and (b) of Lemma 3.9 are satisfied with \( T = 2 \) if one assumes \( R \) sufficiently large. Moreover, hypothesis (c) is also satisfied because due to Theorem 3.3(i) we have \( \partial Q_{2,t} \subset A(R - 2\Lambda_0, R + 2\Lambda_0) \). Thus \( Q_{2,t} \) is \( \nu \)-close in \( C^{2,\alpha}(B_{K_1}) \) to \( P_3 \) for every \( 1 \leq t \leq 2 \). \( \square 

**Proof of Theorem 3.5.** During this proof we will use Theorem 3.3(i) and (iii) with \( \nu = 1, \Lambda_0 \) is the constant given by that theorem.

From the maximum principle applied to \( \theta_t \) we know that
\[ \sup_{0 \leq t \leq 2} \sup_{Q_{1,t}} |\theta_t| \leq \max \left\{ \sup_{Q_{1,0}} |\theta|, \sup_{0 \leq t \leq 2} \sup_{\partial Q_{1,t}} |\theta_t| \right\}. \]
The goal now is to control the \( C^1 \) norm of \( Q_{1,t} \) along \( \partial Q_{1,t} \) so that we control \( \sup_{\partial Q_{1,t}} |\theta_t| \).

Given \( \eta \) small, consider \( R \) and \( \delta \) given by Lemma 3.9 when \( \nu = \eta, S = 2\Lambda_0, \) and \( \kappa = 1/2 \). We have
\[ \partial Q_{1,0} = Q^1 \cap \{|x| = R\}, \]
where \( Q^1 \) is defined in (4). Thus, for all \( R \) sufficiently large and \( \varepsilon \) small, we have that \( M_0 = Q^1 \) satisfies hypotheses (a) and (b) of Lemma 3.9 for every \( x_0 \in \partial Q_{1,0} \). Moreover \( \partial(F_t(Q^1)) \cap B_{R} = \emptyset \) by Theorem 3.3(i), and so hypothesis (c) is also satisfied because we are assuming \( R \geq 4R \) (see (3)).

This means \( F_t(Q^1) \cap B_{2\Lambda_0\sqrt{t}}(x_0) \) is graphical over \( T_{x_0}Q^1 \) with the \( C^1 \) norm being smaller than \( \eta \) for all \( 0 \leq t \leq 2 \). Thus we can choose \( \eta \) small so that
\[ \sup \{|\theta_t(y) - \theta_0(x_0)| : y \in F_t(Q^1) \cap B_{2\Lambda_0\sqrt{t}}(x_0)\} \leq 1/(2K_0). \]
Using Theorem 3.3(i) we see that for every $y \in \partial Q_{1,t}$, there is $x_0 \in \partial Q_{1,0}$ so that $y \in B_{2\Lambda_0 \sqrt{t}}(x_0)$. Thus from (10), we obtain
\[
\sup_{\partial Q_{1,t}} |\theta_t| \leq \sup_{\partial Q_{1,0}} |\theta| + 1/(2K_0)
\]
and this implies (i) because we are assuming $\sup_{\partial Q_1} |\theta| \leq \pi/2 - K_0^{-1}$.

We now prove (ii). Assume for a moment that the metric $g$ (defined in Section 2.1) is Euclidean in $B_{2R}$. Because $Q_{1,t}$ is almost-calibrated, we have from [9, Lemma 7.1] the existence of a constant $C$ depending only $K_0$ so that, for every open set $B$ in $Q_{1,t}$ with rectifiable boundary,
\[
\left(\mathcal{H}^2(B)\right)^{1/2} \leq C \text{length} (\partial B).
\]
It is easy to recognize the same is true (for some slightly larger $C$) if $g_R$ is very close to the Euclidean metric. Set
\[
\psi(r) = \mathcal{H}^2(\hat{B}_r(x))
\]
which has, for almost all $r < \text{dist}(y, \partial Q_{1,t})$, the derivative given by
\[
\psi'(r) = \text{length} (\partial \hat{B}_r(x)) \geq C^{-1}(\psi(r))^{1/2}.
\]
Hence, integration implies that for some other constant $C$, $\psi(r) \geq Cr^2$, and so (ii) is proven.

We now prove (iii). The Lie derivative of $\lambda_t = F^\ast_t(\lambda)$ is given by
\[
\mathcal{L}_H \lambda_t = dF^\ast_t (H \lambda) + dF^\ast_t (H \cdot 2\omega) = d(F^\ast_t (H \lambda) - 2\theta_t),
\]
and so we can find $\beta_t \in C^\infty(Q_{1,t})$ with $d\beta_t = \lambda$ and
\[
(11) \quad \frac{d\beta_t}{dt} = H \lambda - 2\theta_t.
\]
A simple computation shows that $\Delta \beta_t = H \lambda + \sum_{i=1}^2 \nabla_{e_i} \lambda (e_i)$, which proves (iii).

We now prove (iv). Combining Theorem 3.3(i) and (11), we have that
\[
\left| \frac{d\beta_t}{dt} (F_t(x)) \right| \leq \frac{\Lambda_0}{\sqrt{t}} |F_t(x)| + \pi - K_0^{-1}
\]
for every $x \in L \cap A(R_1 + 2\Lambda_0, R)$. Thus after integration in the $t$ variable, assuming $t \leq 2$, and recalling (5), we obtain a constant $C = C(K_0, \Lambda_0)$ such that
\[
|\beta_t(F_t(x))| \leq C(|F_t(x)| + |\beta(x)|) + C \leq C(|F_t(x)|^2 + 1).
\]
We are left to estimate $\beta_t$ on $A_t = F_t(Q_{1,0} \cap B_{R_1 + 2\Lambda_0})$. From Theorem 3.3(i) we know that $A_t \subseteq B_{C_1}(0)$ for some $C_1 = C_1(K_0, \Lambda_0, R_1)$ and thus, provided we assume $g_R$ to be sufficiently close to the Euclidean metric,
\[
|\nabla \beta_t(x)| = |\lambda| \leq 2C_1 \quad \text{for every } x \in A_t.
\]
Hence, if we fix \( x_1 \) in \( \partial A_t \), we can find \( C = C(K_0, \Lambda_0, R_1) \) so that for every \( y \) in \( A_t \),
\[
|\beta_t(y)| \leq |\beta_t(x_1)| + C \operatorname{dist}_{A_t}(x_1, y) \leq C(1 + \operatorname{dist}_{A_t}(x_1, y)),
\]
where \( \operatorname{dist}_{A_t} \) denotes the intrinsic distance in \( A_t \). Property (ii) of this theorem, \( A_t \subseteq B_{C_1}(0) \), and Lemma 3.1 are enough to bound uniformly the intrinsic diameter of \( A_t \) and thus bound \( \beta_t \) uniformly on \( A_t \). Hence (iv) is proven.

We now prove (v). In what follows, \( E_j^2 \) denotes any term with decay \( (|x|^j + 1)O(1/R) \). Given a coordinate function \( v = x_i \) or \( y_i \), \( i = 1, 2 \), we have
\[
\frac{dv}{dt} = \Delta v - \sum_{i=1}^{2} g(\nabla_v, e_i) = \Delta v + E_0^0,
\]
where \( V \) denotes the gradient of \( v \) with respect to \( g_R \). Thus,
\[
\frac{d\mu}{dt} = \Delta \mu + E_2^0 - 2g_R(X_1^T, Y_2^T) + 2g_R(Y_1^T, X_2^T),
\]
where \( X_i, Y_i, i = 1, 2 \) denote the gradient of the coordinate functions with respect to \( g_R \). If the ambient Calabi-Yau structure were Euclidean, then
\[
\langle X_1^T, Y_2^T \rangle - \langle Y_1^T, X_2^T \rangle = -\langle (JY_1^T), Y_2 \rangle - \langle Y_1^T, X_2 \rangle = \langle Y_1^T + Y_2^T, X_2 \rangle = -\langle Y_1, X_2 \rangle = 0.
\]
In general, it is easy to see that \( g_R(X_1^T, Y_2^T) - g_R(Y_1^T, X_2^T) = E_0^0 \) and so
\[
\frac{d\mu^2}{dt} \leq \Delta \mu^2 - 2|\nabla \mu|^2 + E_2^3. \quad \square
\]

4. Second Step: Self-expanders

The goal of this section is to prove the theorem below. For the reader’s convenience, we recall that the planes \( P_1, P_2 \) are defined in Section 2.1.1, \( K_0 \) is defined at the beginning of Section 3, \( Q^1 \) is defined in (4), and \( Q_{1,t} \) is defined in Theorem 3.3(iii). The self-expander equation is defined in Section 2.1.2, and the self-expander \( S \) is defined in (2) (see Figure 2).

**Theorem 4.1.** Fix \( S_0 \) and \( \nu \). There are \( \varepsilon_3 \) and \( R_3 \), depending on \( S_0, \nu, \) and \( K_0 \), such that if \( R \geq R_3, \varepsilon \leq \varepsilon_3 \) in (3), and

the flow \( Q_{1,t} \) exists smoothly for all \( 0 \leq t \leq 2 \),

then \( t^{-1/3}Q_{1,t} \) is \( \nu \)-close in \( C^{2,\alpha}(B_{S_0}) \) to \( S \) for every \( 1 \leq t \leq 2 \).

As we will see shortly, this theorem follows from Theorem 4.2. Recall that, as seen in Theorem 3.5(iii), we can find \( \beta_t \) on \( Q_{1,t} \) so that
\[
\frac{d\beta_t}{dt} = \lambda = \sum_{i=1}^{2} x_i dy_i - y_i dx_i.
\]
Theorem 4.2. Fix $S_0$ and $\nu$. There are $\varepsilon_4$, $R_4$, and $\delta$ depending on $S_0$, $\nu$, and $K_0$, such that if $R \geq R_4$, $\varepsilon \leq \varepsilon_4$ in (3), and

- the flow $Q_{1,t}$ exists smoothly for all $0 \leq t \leq 2$;

\[
\int_{Q_{1} \cap B_R} \beta^2 \exp(-|x|^2/8)d\mathcal{H}^2 \leq \delta;
\]

then $t^{-1/2}Q_{1,t}$ is $\nu$-close in $C^{2,\alpha}(B_{S_0})$ to a smooth embedded self-expander asymptotic to $P_1$ and $P_2$ for every $1 \leq t \leq 2$.

Remark 4.3. (1) If the ambient metric $g_R$ (defined in Section 2.1) were Euclidean, then $|\nabla \beta(x)| = |x^\perp|$, and thus $\beta$ would be constant exactly on cones. Hence, roughly speaking, the left-hand side of (12) measures how close $Q_{1,0}$ is to a cone.

(2) The content of the theorem is that given $\nu$ and $S_0$, there is $\delta$ so that if the initial condition is $\delta$-close, in the sense of (12), to a non area-minimizing configuration of two planes $P_1 + P_2$ and the flow exists smoothly for all $0 \leq t \leq 2$, then the flow will be $\nu$-close to a smooth self-expander in $B_{S_0}$ for all $1 \leq t \leq 2$.

(3) The result is false if one removes the hypothesis that the flow exists smoothly for all $0 \leq t \leq 2$. For instance, there are known examples [9, Th. 4.1] where $Q_{1,0}$ is very close to $P_1 + P_2$ (see [9, Fig. 1]) and a finite-time singularity happens for a very short time $T$. In this case, $Q_{1,T}$ can be seen as a transverse intersection of small perturbations of $P_1$ and $P_2$ (see [9, Fig. 2]) and we could continue the flow past the singularity by flowing each component of $Q_{1,T}$ separately, in which case $Q_{1,1}$ would be very close to $P_1 + P_2$ and this is not a smooth self-expander. The fact the flow exists smoothly will be crucial to prove Lemma 4.10.

(4) The result is also false if $P_1 + P_2$ is area-minimizing. The reason is that in this case the self-expander asymptotic to $P_1 + P_2$ is simply $P_1 + P_2$, which is singular at the origin and thus not smooth as it is guaranteed by Theorem 4.2. The fact that $P_1 + P_2$ is not area-minimizing will be crucial to prove Lemma 4.10.

(5) The strategy to prove Theorem 4.2 is the following. The first step (Proposition 4.4) is to show that if the left-hand side of (12) is very small, then

\[
\int_{Q_{1,1} \cap B_{R/2}} (\beta_1 + 2\theta_1)^2 \Phi(0,4-t)d\mathcal{H}^2 + \int_0^2 \int_{Q_{1,t}} |x^\perp - 2tH|^2 \Phi(0,4-t)d\mathcal{H}^2dt
\]
is also very small. The second step (Proposition 4.6) in the proof will be to show that if
\[
\int_{Q_{1,t} \cap B_{R/2}} (\beta_1 + 2\theta_1)^2 \Phi(0, 4 - t) d\mathcal{H}^2 + \int_0^2 \int_{Q_{1,t}} |x^1 - 2tH|^2 \Phi(0, 4 - t) d\mathcal{H}^2 dt
\]
is very small, then \(t^{-1/2}Q_{1,t}\) will be \(\nu\)-close in \(C^{2,\alpha}(B_{S_0})\) to a smooth self-expander. It is in this step that we use the fact that the flow exists smoothly and \(P_1 + P_2\) is not an area-minimizing configuration.

**Proof of Theorem 4.1.** The first step is to show that Theorem 4.2 can be applied, which amounts to showing that (12) holds if we choose \(\varepsilon\) sufficiently small and \(R\) sufficiently large. Thus, we obtain that \(t^{-1/2}Q_{1,t}\) is \(\nu\)-close in \(C^{2,\alpha}(B_{S_0})\) to a smooth embedded self-expander asymptotic to \(P_1\) and \(P_2\) for every \(1 \leq t \leq 2\). The second step is to show that self-expander must be \(S\).

**First Step.** We note \(Q^1 \cap A(1, R)\) (defined in (4)) coincides with \((P_1 \cup P_2) \cap A(1, R)\), and so the uniform control we have on \(\beta\) given by (5) implies that for all \(\delta\), there is \(r_1\) large depending on \(K_0\) and \(\delta\) so that
\[
(13) \quad \int_{Q^1 \cap A(r_1, R)} \beta^2 \exp(-|x|^2/8) d\mathcal{H}^2 \leq \delta/2
\]
for all \(\varepsilon\) small and \(R\) large. Also, if we make \(\varepsilon\) tend to zero and \(R\) tend to infinity in (3), it is straightforward to see that \(Q^1\) tends to \(P_1 \cup P_2\) smoothly on any compact set that does not contain the origin. Because \(\beta\) is constant on cones, we can choose \(\beta\) on \(Q^1\) so that
\[
\lim_{\varepsilon \to 0, R \to \infty} \int_{Q^1 \cap B_r} \beta^2 \exp(-|x|^2/8) d\mathcal{H}^2 = 0.
\]
Combining this with (13) we obtain that for all \(\varepsilon\) small and \(R\) large,
\[
\int_{Q^1 \cap B_R} \beta^2 \exp(-|x|^2/8) d\mathcal{H}^2 \leq \delta.
\]
Hence all the hypotheses of Theorem 4.2 hold.

**Second Step.** Let \(Q\) denote a smooth embedded Lagrangian self-expander asymptotic to \(P_1 + P_2\). Then \(Q_t = \sqrt{t}Q\) and \(\lim_{t \to 0^+} Q_t = P_1 + P_2\) as Radon measures. Thus, if we recall the function \(\mu = x_1y_2 - y_1x_2\) defined in Theorem 3.5(v), we have
\[
(14) \quad \lim_{t \to 0^+} \int_{Q_t} \mu^2 \Phi(0, T - t) d\mathcal{H}^2 = \int_{P_1 + P_2} \mu^2 \Phi(0, T) d\mathcal{H}^2 = 0.
\]
Using the evolution equation for \(\mu\) given in Theorem 3.5(v) \((E_2\) is identically zero) into Huisken's monotonicity formula (see Lemma 2.1), we have
\[
\frac{d}{dt} \int_{Q_t} \mu^2 \Phi(0, T - t) d\mathcal{H}^2 \leq 0.
\]
This inequality and (14) imply at once that
\[ \int_{Q_t} \mu^2 \Phi(0, 1) dH^2 = 0 \text{ for all } t \geq 0 \]
and so \( Q \subset \mu^{-1}(0) \). A trivial modification of Lemma 7.1 implies the existence of \( \gamma \) asymptotic to \( \chi \) (the curve defined in (2)) so that
\[ Q = \{ (\gamma(s) \cos \alpha, \gamma(s) \sin \alpha) \mid s \in \mathbb{R}, \alpha \in S^1 \}. \]

From [1, §5] we know that \( \chi = \gamma \), and so the result follows. \( \square \)

4.1. Proof of Theorem 4.2. Throughout this proof we assume that \( R \) is sufficiently large and \( \varepsilon \) is sufficiently small so that Theorem 3.3 (with \( \nu = 1 \)) and Theorem 3.5 apply. We also assume the flow \((Q_{1,t})_{0 \leq t \leq 2}\) exists smoothly.

For simplicity, denote \( Q_{1,t} \) simply by \( Q_t \). We also recall that the constant \( K_0 \), which will appear multiple times during this proof, was defined at the beginning of Section 3.

**Proposition 4.4.** Fix \( \eta \). There are \( \varepsilon_5 \) and \( R_5 \) depending on \( \eta \) and \( K_0 \) so that if \( \varepsilon \leq \varepsilon_5 \) and \( R \geq R_5 \) in (3), then
\begin{align*}
\sup_{0 \leq t \leq 2} \int_{Q_t \cap B_{R/2}} (\beta_t + 2t \theta_t)^2 \Phi(0, 4 - t) dH^2 &+ \int_0^2 \int_{Q_t \cap B_{R/2}} |x^\perp - 2tH|^2 \Phi(0, 4 - t) dH^2 dt \\
\leq \frac{\eta}{2} + \int_{Q \cap B_R} \beta^2 \Phi(0, 4) dH^2.
\end{align*}

**Remark 4.5.** The idea is to apply Huisken monotonicity formula for \((\beta_t + 2t \theta_t)^2\). Some extra (technical) work has to be done because \( Q_t \) has boundary and the ambient metric \( g_R \) (defined in Section 3) is not Euclidean.

**Proof.** Let \( \phi \in C^\infty(\mathbb{R}^4) \) such that \( 0 \leq \phi \leq 1 \),
\[ \phi = 1 \text{ on } B_{R/2}, \quad \phi = 0 \text{ on } B_{2R/3}, \quad |D\phi| + |D^2\phi| \leq \frac{\Lambda}{R}, \]
where \( \Lambda \) is some universal constant. By Theorem 3.3(i) we have that, provided we chose \( R \) large and \( \varepsilon \) small, \( \partial Q_t \cap B_{R/2} = \) and thus \( \phi \) has compact support in \( Q_t \).

Set \( \gamma_t = \beta_t + 2t \theta_t \). Then on \( Q_t \), we have from Theorem 3.5(iii) that
\[ \frac{d(\gamma_t\phi)}{dt} = \Delta(\gamma_t\phi)^2 - 2|\nabla \gamma_t|^2 \phi^2 + \langle H, D\phi \rangle \gamma_t^2 - 2\langle \nabla \gamma_t^2, D\phi \rangle - \gamma_t^2 \Delta \phi^2 + \phi^2 E_1. \]
Thus, using Theorem 3.3(i) to estimate $H$ and Theorem 3.5(iv), we have that for all $R$ large and $\varepsilon$ small,
\[
\frac{d(\gamma_t \phi)}{dt} \leq \Delta(\gamma_t \phi)^2 - 2|\nabla \gamma_t|^2 \phi^2 + \frac{C_1}{\sqrt{t} R}(|x|^4 + 1)(1 - \chi_{B/2}) + |E_1|,
\]
where $C_1 = C_1(K_0, \Lambda_0, D_1)$ and $\chi_{R/2}$ denotes the characteristic function of $B_{R/2}$. From Lemma 2.1, we conclude
\[
\frac{d}{dt} \int_{Q_t} (\gamma_t \phi)^2 \Phi(0, 4 - t) dH^2 + 2 \int_{Q_t} |\nabla \gamma_t|^2 \phi^2 \Phi(0, 4 - t) dH^2 \\
\leq \int_{Q_t} \left( \gamma_t^2 \frac{|E|^2}{4} + |E_1| \right) \Phi(0, 4 - t) dH^2 + \frac{C_1}{\sqrt{t} R} \int_{Q_t \setminus B_{R/2}} (|x|^4 + 1) \Phi(0, 4 - t) dH^2.
\]

We now estimate the two terms on the right-hand side. If $g_R$ (defined at the beginning of Section 3) were Euclidean, both terms $|E|^2$ and $E_1$ mentioned above would vanish. Otherwise it is easy to see that making $R$ sufficiently large so that $g_R$ becomes close to Euclidean, both terms can be made arbitrarily small. The growth of $\gamma_t$ is quadratic (Theorem 3.5(i) and (iv)), and so choosing $R$ sufficiently large and $\varepsilon$ sufficiently small, we have
\[
\int_{Q_t} \left( \gamma_t^2 \frac{|E|^2}{4} + |E_1| \right) \Phi(0, 4 - t) dH^2 \leq \frac{\eta}{8} \quad \text{for all } t \leq 2.
\]
Using that $|x| \geq |x|^2/2 + R^2/8$ outside $B_{R/2}$, it is easy to see that
\[
\Phi(0, 4 - t) \leq 2^{1/2} \Phi(0, 2(4 - t)) \exp(-R^2/(32(4 - t))) \quad \text{on } \mathbb{R}^4 \setminus B_{R/2}.
\]
Thus, for all $0 \leq t \leq 2$, the uniform area bounds given in Lemma 3.1 imply
\[
\int_{Q_t \setminus B_{R/2}} (|x|^4 + 1) \Phi(0, 4 - t) dH^2 \\
\leq C_2 \exp(-R^2/C_2) \int_{Q_t \setminus B_{R/2}} (|x|^4 + 1) \Phi(0, 2(4 - t)) dH^2 \leq C_3 \exp(-R^2/C_3),
\]
where $C_2$ and $C_3$ depend only on $K_0$. Therefore, we have
\[
\frac{d}{dt} \int_{Q_t} (\gamma_t \phi)^2 \Phi(0, 4 - t) dH^2 + 2 \int_{Q_t} |\nabla \gamma_t|^2 \phi^2 \Phi(0, 4 - t) dH^2 \\
\leq \frac{C_4}{\sqrt{t} R} \exp(-R^2/C_4) + \frac{\eta}{8},
\]
where $C_4 = C_4(C_1, C_3)$. Integrating this inequality, for all $t \leq 2$, we obtain
\[
(15) \quad \int_{Q_t} \gamma_t^2 \phi^2 \Phi(0, 4 - t) dH^2 + 2 \int_0^t \int_{Q_s} |\nabla \gamma_t|^2 \phi^2 \Phi(0, 4 - s) dH^2 ds \\
\leq \int_{Q^1 \cap B_R} \beta^2 \Phi(0, 4) dH^2 + 2^{3/2} C_4 R^{-1} \exp(-R^2/C_4) + \frac{\eta}{4}.
\]
If the metric \( g_R \) were Euclidean, then \(|\nabla \gamma_t|^2 = |x^\perp - 2tH|^2\). Hence the result follows from (15) if we assume \( R \) is large enough so that \( 2|\nabla \gamma_t|^2 \leq |x^\perp - 2tH|^2 \)

\[
2^{3/2}C_4 R^{-1} \exp(-R^2/C_4) \leq \frac{\eta}{4}.
\]

The next proposition is crucial to prove Theorem 4.2.

**Proposition 4.6.** Fix \( \nu \) and \( S_0 \). There are \( \varepsilon_6, R_6, \) and \( \eta \) depending on \( \nu, K_0, \) and \( S_0 \), such that if \( R \geq R_5 \), \( \varepsilon \leq \varepsilon_5 \) in (3), and

\[
(16) \quad \sup_{0 \leq t \leq 2} \int_{{Q}_i \cap B_{R_i/2}} (\beta_1 + 2t\theta_1)^2 \Phi(0, 4 - t) dH^2
+ \int_0^2 \int_{{Q}_i \cap B_{R_i/2}} \frac{|x^\perp - 2tH|^2 \Phi(0, 4 - t)}{dH^2} dt \leq \eta,
\]

then \( t^{-1/2}Q_t \) is \( \nu \)-close in \( C^{2,\alpha}(B_{S_0}) \) to a smooth embedded self-expander asymptotic to \( P_1 + P_2 \) for every \( 1 \leq t \leq 2 \).

**Remark 4.7.** The strategy to prove this proposition is the following. We argue by contradiction, and standard arguments will give us a sequence of flows \((Q_i^t)_{0 \leq t \leq 2}\) converging weakly to a Brakke flow \((\bar{Q}_t)_{0 \leq t \leq 2}\), where in (3) we have \( R_i \) tending to infinity, \( \varepsilon_i \) tending to zero, and

\[
(17) \quad \lim_{i \to \infty} \int_{{Q}_i \cap B_{r_1/2}} (\beta_1 + 2\theta_1)^2 \Phi(0, 4 - t) dH^2
+ \int_0^2 \int_{{Q}_i \cap B_{r_1/2}} \frac{|x^\perp - 2tH|^2 \Phi(0, 4 - t)}{dH^2} dt = 0.
\]

Standard arguments (Lemma 4.8) imply \( \bar{Q}_t \) is a self-expander with

\[
\lim_{t \to 0^+} \bar{Q}_t = P_1 + P_2.
\]

The goal is to show that \( \bar{Q}_1 \) is smooth because we could have, for instance, \( \bar{Q}_1 = P_1 + P_2 \).

The first step (Lemma 4.10) is to show that \( \bar{Q}_1 \) is not stationary. The idea is the following. If \( \bar{Q}_t \) were stationary, then \( \bar{Q}_t = \bar{Q}_1 \) for all \( t \) and so \( \bar{Q}_1 = \lim_{t \to 0^+} \bar{Q}_t = P_1 + P_2 \). On the other hand, from the control given in Theorem 3.3, we will be able to find \( r_1 > 0 \) large so that \( Q_i^1 \cap B_{r_1} \) is connected. (If the flow had a singularity, this would not necessarily be true.) Furthermore, we will deduce from (17) that

\[
\int_{{Q}_i \cap B_{r_1}} |\nabla \beta_1|^2 dH^2 = \lim_{i \to \infty} \int_{{Q}_i \cap B_{r_1}} |x^\perp|^2 dH^2 = 0.
\]
Hence we can invoke [9, Prop. A.1] and conclude that $\beta_i$ must tend to constant $\bar{\beta}$ in $L^2$. Combining this with (17), we have
\[
\lim_{i \to \infty} \int_{Q_i^t \cap B_{\bar{\beta} i}} (\beta_i + 2\theta_i^t)^2 d\mathcal{H}^2 = 0,
\]
and thus $\bar{Q}_1$ must be Special Lagrangian with Lagrangian angle $-\bar{\beta}/2$. This contradicts the choice of $P_1$ and $P_2$.

The second step (Lemma 4.11) is to show the existence of $l_1$ so that, for every $y \in \mathbb{C}^2$ and $l < l_1$, the Gaussian density ratios of $\bar{Q}_1$ centered at $y$ with scale $l$ defined by
\[
\Theta(y, l) = \int_{Q_1} \Phi(y, l) d\mathcal{H}^2
\]
are very close to one. If true, then standard theory implies $\bar{Q}_1$ is smooth and embedded. The (rough) idea for the second step is the following. If this step fails for some $y \in \mathbb{C}^2$, then $y$ should be in the singular set of $\bar{Q}_1$. Now $T_y\bar{Q}_1$ should be a union of (at least two) planes. Hence the Gaussian density ratios of $\bar{Q}_1$ at $y$ for all small scales should not only be away from one but actually bigger than or equal to two. We know from Huisken’s monotonicity formula that the Gaussian density ratios of $\bar{Q}_1$ at $y$ and scale $l$ are bounded from above by the Gaussian density ratios of $\bar{Q}_0 = P_1 + P_2$ at $y$ and scale $l + 1$. But these latter Gaussian ratios are never bigger than two (see Remark 4.12), which means equality must hold in Huisken’s monotonicity formula and so $\bar{Q}_t$ must be a self-shrinker. Now $\bar{Q}_t$ is also a self-expander, and thus it must be stationary. This contradicts the first step.

Proof. Consider a sequence $(R_i)$ converging to infinity and a sequence $(\varepsilon_i)$ converging to zero in (3) that give rise to a sequence of smooth flows $(Q_i^t)_{0 \leq t \leq 2}$ satisfying
\[
\sup_{0 \leq t \leq 2} \int_{Q_i^t \cap B_{\varepsilon_i/2}} (\beta_i + 2t\theta_i^t)^2 \Phi(0, 4 - t) d\mathcal{H}^2 + \int_0^2 \int_{Q_t \cap B_{\varepsilon_i/2}} |x^\perp - 2tH|^2 \Phi(0, 4 - t) d\mathcal{H}^2 dt \leq \frac{1}{i}.
\]
We will show the existence of a smooth self-expander $\bar{Q}_1$ asymptotic to $P_1$ and $P_2$ so that, after passing to a subsequence, $t^{-1/2}Q_i^t$ converges in $C^{2,\alpha}(B_{S_0})$ to $\bar{Q}_1$ for every $1 \leq t \leq 2$.

From compactness for integral Brakke motions [7, §7.1] we know that, after passing to a subsequence, $(Q_i^t)_{0 \leq t \leq 2}$ converges to an integral Brakke motion $(\bar{Q}_t)_{0 \leq t \leq 2}$, where $\bar{Q}_0$ converges in the varifold sense to the varifold $P_1 + P_2$. Furthermore,
\[
\lim_{i \to \infty} \int_0^2 \int_{Q_i^t \cap B_{\varepsilon_i/2}} |x^\perp - 2tH|^2 \Phi(0, 4 - t) d\mathcal{H}^2 dt = 0,
\]
which means
\begin{equation}
H = \frac{x^+}{2t} \text{ on } \bar{Q}_t \text{ for all } t > 0,
\end{equation}
and so \( \bar{Q}_t = \sqrt{t} \bar{Q}_1 \) as varifolds for every \( t > 0 \). (See proof of [10, Th. 3.1] for this last fact.)

**Lemma 4.8.** As \( t \) tends to zero, \( \bar{Q}_t \) converges, as Radon measures, to \( P_1 + P_2 \).

**Remark 4.9.** This lemma is needed because the Brakke flow theory only assures that the support of the Radon measure obtained from \( \lim_{t \to 0} \bar{Q}_t \) is contained in the support of \( \lim_{i \to \infty} Q^0_i = P_1 + P_2 \).

**Proof.** Set
\[ \mu_t(\phi) = \int_{\bar{Q}_t} \phi \, dH^2. \]
The Radon measure \( \nu = \lim_{t \to 0^+} \mu_t \) is well defined by [7, Th. 7.2] and satisfies, for every \( \phi \geq 0 \) with compact support,
\begin{equation}
\nu(\phi) \leq \lim_{i \to \infty} \int_{Q^0_i} \phi \, dH^2 = \int_{P_1 + P_2} \phi \, dH^2.
\end{equation}
It is simple to recognize that \( \nu \) must be either zero, \( P_1 \), \( P_2 \), or \( P_1 + P_2 \).

The measure \( \nu \) is invariant under scaling, meaning that if we set \( \phi_c(x) = \phi(cx) \), then
\[ \nu(\phi_c) = \lim_{t \to 0^+} \int_{\bar{Q}_t} \phi_c \, dH^2 = c^{-2} \lim_{t \to 0^+} \int_{c\bar{Q}_t} \phi \, dH^2 \]
\[ = c^{-2} \lim_{t \to 0^+} \int_{Q^0_{ct}} \phi \, dH^2 = c^{-2} \lim_{t \to 0^+} \int_{\bar{Q}_t} \phi \, dH^2 = c^{-2} \nu(\phi). \]
From Theorem 3.3(i) and Theorem 3.5(ii) we have that the support of \( \nu \) contains \( (P_1 + P_2) \cap A(K_1, \infty) \) which, combined with the invariance of the measure we just mentioned, implies the support of \( \nu \) coincides with \( P_1 \cup P_2 \). Thus \( \nu = P_1 + P_2 \), as we wanted to show. \( \square \)

**Lemma 4.10.** \( \bar{Q}_1 \) is not stationary.

**Proof.** If true, then \( \bar{Q}_1 \) needs to be a cone because \( x^+ = 2H = 0 \) and so, because \( \bar{Q}_t = \sqrt{t} \bar{Q}_1 \), they are also cones for all \( t > 0 \). Hence we must have (from varifold convergence) that for every \( r > 0 \),
\[ \lim_{i \to \infty} \int_0^2 \int_{Q_i^0 \cap B_r} |x^+|^2 \, dH^2 \, dt = 0, \]
which implies from (18) that
\[ \lim_{i \to \infty} \int_0^2 \int_{Q_i^0 \cap B_r} (t^2 |H|^2 + |x^+|^2) \, dH^2 \, dt = 0. \]
Therefore, we can assume without loss of generality that for every $r > 0$,

$$\lim_{i \to \infty} \int_{Q^1_i \cap B_r} (|H|^2 + |x^\perp|^2) d\mathcal{H}^2 = 0$$

and thus, by [9, Prop. 5.1], $\bar{Q}_1$ is a union of Lagrangian planes with possible multiplicities. We will argue that $\bar{Q}_1$ must be a Special Lagrangian, i.e., all the planes in $\bar{Q}_1$ must have the same Lagrangian angle. This gives us a contradiction for the following reason. On one hand, $\bar{Q}_t = \bar{Q}_1$ for all $t > 0$, which means $\lim_{t \to 0} \bar{Q}_t = \bar{Q}_1$. On the other hand, from Lemma 4.8, we have

$$\lim_{t \to 0} \bar{Q}_t = P_1 + P_2,$$

which means $\bar{Q}_1 = P_1 + P_2$ and therefore the Lagrangian angle of $P_1$ and $P_2$ must be identical (or differ by a multiple of $\pi$). This contradicts how $P_1$ and $P_2$ were chosen.

From Theorem 3.3(ii) (which we apply with $\nu = 1$) we have that for all $i$ sufficiently large, $Q_i^1 \cap A(R_1, R_i/2)$ is graphical over $(P_1 \cup P_2) \cap A(R_1, R_i/2)$ with the $C^{2,\alpha}$ norm uniformly bounded. Hence we can find $r_1 \geq R_1$ so that if we set $N_i = Q_i^1 \cap B_{3r_1}$, we have for all $i$ sufficiently large that $N_i \cap B_{2r_1}$ connected. We note that if $Q_i^1$ had a singularity for some $t < 1$ then $N_i$ could be two discs intersecting transversally near the origin and thus $N_i \cap B_{2r_1}$ would not be connected.

Furthermore, we obtain from (21) that

$$\lim_{i \to \infty} \int_{N_i} |\nabla \beta^i|^2 d\mathcal{H}^2 = \lim_{i \to \infty} \int_{N_i} |x^\perp|^2 d\mathcal{H}^2 = 0$$

and so, because of Theorem 3.5(ii), we can apply [9, Prop. A.1] and conclude the existence of a constant $\beta$ so that, after passing to a subsequence,

$$\lim_{i \to \infty} \int_{N_i \cap B_{r_1}} (\beta^i_1 - \bar{\beta})^2 d\mathcal{H}^2 = 0.$$ 

Recall that from (18), we have

$$\lim_{i \to \infty} \int_{Q_i^1 \cap B_{r_1}} (\bar{\beta}^i_1 + 2\theta^i_1)^2 d\mathcal{H}^2 = 0,$$

which combined with (22) implies

$$\lim_{i \to \infty} \int_{Q_i^1 \cap B_{r_1}} (\bar{\beta}^i + 2\theta^i_1)^2 d\mathcal{H}^2 = 0.$$ 

Therefore $\bar{Q}_1$ must be a Special Lagrangian cone with Lagrangian angle $-\bar{\beta}/2$. 

□

In the next lemma, $\varepsilon_0$ denotes the constant given by White’s Regularity Theorem [18].
Lemma 4.11. There is $l(t)$, a positive continuous function of $0 < t \leq 2$, so that

$$\int_{Q_t} \Phi(y, l)dH^2 \leq 1 + \varepsilon_0/2 \quad \text{for every } l \leq l(t), \ y \in \mathbb{R}^4, \ \text{and } t > 0.$$ 

Remark 4.12. During the proof the following simple formula will be used constantly. Given $y \in C^2$, let $d_1, d_2$ denote, respectively, the distance from $y$ to $P_1$ and $P_2$. Then

$$\int_{P_1 + P_2} \Phi(y, l)dH^2 = \exp(-d_1^2/(4l)) + \exp(-d_2^2/(4l)) \leq 2.$$ 

Proof. It suffices to prove the lemma for $t = 1$ because, as we have seen, $\bar{Q}_t = \sqrt{t}Q_1$ for all $t > 0$.

Claim. There is $C_1$ such that for every $l \leq 2$ and $y \in \mathbb{R}^4$,

$$\int_{\bar{Q}_1} \Phi(y, l)dH^2 \leq 2 - C_1^{-1}.$$ 

From the monotonicity formula for Brakke flows [8, Lemma 7],

$$\int_{\bar{Q}_1} \Phi(y, l)dH^2 + \int_0^1 \int_{\bar{Q}_1} \left| H + \frac{(x - y)\perp}{2(l + 1 - t)} \right|^2 \Phi(y, l + 1 - t)dH^2 dt = \int_{P_1 + P_2} \Phi(y, l + 1)dH^2 \leq 2.$$ 

Suppose there is a sequence $(y_i)$ and $(l_i)$ with $0 \leq l_i \leq 2$ such that

$$\int_{\bar{Q}_1} \Phi(y_i, l_i)dH^2 \geq 2 - \frac{1}{i}.$$ 

Then, from (25) we obtain

$$\lim_{i \to \infty} \int_{P_1 + P_2} \Phi(y_i, l_i + 1)dH^2 = 2$$

and so, from (23), $(y_i)$ must converge to zero. Assuming $(l_i)$ converges to $\bar{l}$, we have again from (25) that

$$\int_0^{1/2} \int_{\bar{Q}_t} \left| H + \frac{x\perp}{2(\bar{l} + 1 - t)} \right|^2 \Phi(0, \bar{l} + 1 - t)dH^2 dt$$

$$\leq \lim_{i \to \infty} \int_0^1 \int_{\bar{Q}_t} \left| H + \frac{(x - y_i)\perp}{2(l_i + 1 - t)} \right|^2 \Phi(y_i, l_i + 1 - t)dH^2 dt$$

$$\leq 2 - \lim_{i \to \infty} \int_{\bar{Q}_1} \Phi(y_i, l_i)dH^2 = 0.$$ 

As a result,

$$H + \frac{x\perp}{2(l + 1 - t)} = 0 \quad \text{on } \bar{Q}_t \text{ for all } 0 \leq t \leq 1/2,$$
and combining this with the fact that \(H = \frac{x^t}{2t}\) on \(\tilde{Q}_t\), we obtain that \(H = 0\) on \(\tilde{Q}_1 = t^{-1/2}\tilde{Q}_t\), which contradicts Lemma 4.10. Thus, (24) must hold.

To finish the proof we argue again by contradiction and assume the lemma does not hold. Hence, there is a sequence \((y_j)_{j\in\mathbb{N}}\) of points in \(\mathbb{R}^4\) and a sequence \((l_j)_{j\in\mathbb{N}}\) converging to zero for which
\[
\int_{\tilde{Q}_1} \Phi(y_j, l_j) d\mathcal{H}^2 \geq 1 + \frac{\varepsilon_0}{2}.
\]

The first thing we do is to show (26) implies the existence of \(m\) so that \(|y_j| \leq m\) for all \(j\). The reason is that from (25) we obtain
\[
\int_{P_1 + P_2} \Phi(y_j, l_j + 1) d\mathcal{H}^2 \geq 1 + \frac{\varepsilon_0}{2}
\]
and so, because \((l_j)\) tends to zero, we obtain from (23) that the sequence \((y_j)\) must be bounded.

The motivation for the rest of the argument is the following. The sequence \((y_j)\) has a subsequence that converges to \(\bar{y} \in \mathbb{C}^2\). From (26) we have that \(\bar{y}\) must belong to the singular set of \(\tilde{Q}_1\). The tangent cone to \(\tilde{Q}_1\) at \(\bar{y}\) is a union of (at least two) Lagrangian planes, and thus for all \(l\) very small, we must have
\[
\int_{\tilde{Q}_1} \Phi(\bar{y}, l) d\mathcal{H}^2 \geq 2 - \frac{1}{2C_1}.
\]
This contradicts (24).

Recalling that the flow \((Q^1_t)_{0 \leq t \leq 2}\) tends to \((\tilde{Q}_t)_{0 \leq t \leq 2}\), a standard diagonalization argument allows us to find a sequence of integers \((k_j)_{j\in\mathbb{N}}\) so that the blow-up sequence
\[
\tilde{Q}^j_s = l_j^{-1/2} \left( Q^j_{k_j + s l_j} - y_j \right), \quad 0 \leq s \leq 1
\]
has
\[
- \frac{1}{j} \leq \int_{\tilde{Q}_0} \Phi(0, u) d\mathcal{H}^2 - \int_{l_j^{-1/2}(\tilde{Q}_1 - y_j)} \Phi(0, u) d\mathcal{H}^2 \leq \frac{1}{j}
\]
for every \(1 \leq u \leq j\) and
\[
\int_1^{1+l_j} \int_{Q^j_{k_j \cap B_t(y_j)}} \left| H - \frac{x^t}{2t} \right|^2 d\mathcal{H}^2 dt \leq l_j^2.
\]
Thus, for every \(r > 0\), we have from (28) and \(|y_j| \leq m\) that
\[
\int_0^1 \int_{\tilde{Q}^j_s \cap B_r(0)} |H|^2 d\mathcal{H}^2 ds = l_j^{-1} \int_1^{1+l_j} \int_{Q^j_{k_j \cap B_t(y_j)}} |H|^2 d\mathcal{H}^2 dt
\]
\[
\leq l_j^{-1} \int_1^{1+l_j} \int_{Q^j_{k_j \cap B_t(y_j)}} \left| H - \frac{x^t}{2t} \right|^2 + \left| \frac{x^t}{2t} \right|^2 d\mathcal{H}^2 dt \leq l_j + C_2 l_j,
\]
where $C_2 = C_2(r, m, K_0)$. Therefore,

$$
\lim_{j \to \infty} \int_0^1 \int_{\tilde{Q}_j \cap B_r(0)} |H|^2 d\mathcal{H}^2 ds = 0
$$

and so $(\tilde{Q}_s)_{0 \leq s \leq 1}$ converges to an integral Brakke flow $(\tilde{Q}_s)_{0 \leq s \leq 1}$ with $\tilde{Q}_s = \tilde{Q}$ for all $s$. From Proposition 5.1 in [9] we conclude that $\tilde{Q}$ is a union of Special Lagrangian currents. Note that

$$
\int_{\tilde{Q}} \Phi(0,1) d\mathcal{H}^2 \geq 1 + \varepsilon_0
$$

and so $\tilde{Q}$ cannot be a plane with multiplicity one. The blow-down $C$ of $\tilde{Q}$ is a union of Lagrangian planes (those are the only Special Lagrangian cones in $\mathbb{R}^4$), and so

$$
(29) \quad \lim_{u \to \infty} \lim_{j \to \infty} \int_{\tilde{Q}_0} \Phi(0, u) d\mathcal{H}^2 = \lim_{u \to \infty} \int_{\tilde{Q}} \Phi(0, u) d\mathcal{H}^2 = \int_C \Phi(0,1) d\mathcal{H}^2 \geq 2.
$$

From (29) and (27) one can find $u_0$ such that for every $j$ sufficiently large, we have

$$
2 - \frac{1}{2C_1} \leq \int_{\tilde{Q}_0} \Phi(0, u_0) d\mathcal{H}^2 \leq \int_{j^{-1/2}(\bar{Q}_1 - y_j)} \Phi(0, u_0) d\mathcal{H}^2 + \frac{1}{j}
$$

$$
= \int_{\bar{Q}_1} \Phi(y_j, u_0l_j) d\mathcal{H}^2 + \frac{1}{j}.
$$

This contradicts (24) for all $j$ large.

The lemma we have just proven allows us to find $l_0$ so that for all $\hat{R}$ and all $i$ sufficiently large,

$$
\int_{\bar{Q}_1} \Phi(y, l) d\mathcal{H}^2 \leq 1 + \varepsilon_0 \quad \text{for all } y \in B_{\hat{R}}, \ l \leq l_0, \ \text{and } \frac{1}{2} \leq t \leq 2.
$$

Thus, from White’s Regularity Theorem [18] we have uniform bounds on the second fundamental form and all its derivatives on compact sets of $\bar{Q}_1$ for all $1 \leq t \leq 2$. This implies $\bar{Q}_t$ is smooth and $t^{-1/2}Q^i_t$ converges in $C^{2,\alpha}_{\text{loc}}$ to $\bar{Q}_1$, a smooth self-expander asymptotic to $P_1 + P_2$ by Lemma 4.8, which must be embedded due to Lemma 4.11. This finishes the proof of Proposition 4.6.

Apply Proposition 4.6 with $\nu$ and $S_0$ given by Theorem 4.2, and then apply Proposition 4.4 with $\eta$ being the one given by Theorem 4.2. Theorem 4.2 follows at once if we choose $\delta = \eta/2$, $\varepsilon_3 = \min\{\varepsilon_5, \varepsilon_6\}$, and $R_3 = \max\{R_5, R_6\}$. 

5. Third Step: Equivariant flow

5.1. Setup of Section 5. Consider a smooth curve $\sigma : [0, +\infty) \to \mathbb{C}$ so that

- $\sigma^{-1}(0) = 0$ and $\sigma \cup -\sigma$ is smooth at the origin.
- $\sigma$ has a unique self-intersection.
- Outside a large ball the curve $\sigma$ can be written as the graph of a function $u$ defined over part of the negative real axis with 
  \[ \lim_{r \to -\infty} |u|_{C^{2,\alpha}((-\infty, r])} = 0. \]
- For some $a$ small enough, we have 
  \[ \sigma \subseteq C_a = \{r \exp(i\theta) \mid r \geq 0, \pi/2 + 2a < \theta < \pi + a\}. \]

The curve $\sigma$ shown in Figure 3 has all these properties. Condition (30) is there for technical reasons that will be used during Lemma 5.6.

Denote by $A_1$ the area enclosed by the self-intersection of $\sigma$.

We assume that $L \subset M$ is a Lagrangian surface as defined in (3) and that $\varepsilon, R$ are such that Theorem 3.3 (with $\nu = 1$) and Theorem 3.5 hold. We also assume that the solution to Lagrangian mean curvature flow $(L_t)_{t \geq 0}$ satisfies the following condition.

($\star$) There is a constant $K_1$, a disc $D$, and $F_t : D \to \mathbb{C}^2$ a normal deformation defined for all $1 \leq t \leq 2$ so that

\[
L_t \cap B_{R/2} \subset F_t(D) \subset L_t \cap B_{R}
\]

and the $C^{2,\alpha}$ norm of $F_t$ is bounded by $K_1$.

5.2. Main result.

**Theorem 5.1.** Assume condition ($\star$) holds. There are $\eta_0$ and $R_5$, depending on $K_1$ and $\sigma$, so that if $R \geq R_5$ in (3) and $L_1$ is $\eta_0$-close in $C^{2,\alpha}(B_{R_5})$ to 

\[
M_1 = \{(\sigma(s) \cos \alpha, \sigma(s) \sin \alpha) \mid s \in [0, +\infty), \alpha \in S^1\},
\]

then $(L_t)_{t \geq 0}$ must have a singularity before $T_1 = 2A_1/\pi + 1$ (with $A_1$ defined in Section 5.1).

**Remark 5.2.** The content of the theorem is that if $L_1$ is very close to $M_1$ and $R$ sufficiently large, then the flow $(L_t)_{1 \leq t \leq T_1}$ must have a finite time singularity. The proof proceeds by contradiction, and we assume the existence of smooth flows $(L^i_t)_{0 \leq t \leq T_1}$ with $\bar{R}^i$ tending to infinity and $L^i_1$ converging to $M_1$ in $C^{2,\alpha}_{loc}$. Standard arguments show that $(L^i_t)_{1 \leq t \leq T_1}$ converges to $(M_t)_{1 \leq t \leq T_1}$ a (weak) solution to mean curvature flow starting at $M_1$. The rest of the argument will have two steps.
The first step, see Theorem 5.3(ii)–(iv), is to show the existence of a family of curves $\sigma_t$ so that

$$M_t = \{ (\sigma_t(s) \cos \alpha, \sigma_t(s) \sin \alpha) \mid s \in [0, +\infty), \alpha \in S^1 \}$$

and show that $(\sigma_t)_{t \geq 1}$ behaves as depicted in Figures 3 and 4. More precisely, there is a singular time $T_0$ so that $\sigma_t$ has a single self-intersection for all $1 \leq t < T_0$, $\sigma_{T_0}$ is embedded with a singular point, and $\sigma_t$ is an embedded smooth curve for $t > T_0$. Finally, and this will be important for the second step, we show in Theorem 5.3(i) that $L_t^i$ converges in $C^{2,\alpha}$ to $M_t$ in a small ball around the origin and outside a large ball for all $t \leq T_0 + 1$.

The second step (see details in Corollary 5.5) consists in considering the function

$$f(t) = \theta_t(\infty) - \theta_t(0),$$

where $\theta_t(0)$ is the Lagrangian angle of $M_t$ at $0 \in M_t$ and $\theta_t(\infty)$ is the “asymptotic” Lagrangian angle of $M_t$, which makes sense because, due to Lemma 3.2, $M_t$ is asymptotic to the plane $P_1$. On one hand, because the curve $\sigma_t$ changes from a curve with a single self-intersection to a curve that is embedded as $t$ crosses $T_0$, we will see that

$$\lim_{t \to T_0^-} f(t) = \lim_{t \to T_0^+} f(t) - 2\pi.$$

On the other hand, because $L_t^i$ is smooth and converges to $M_t$ in a small ball around the origin and outside a large ball for all $t \leq T_0 + 1$, we will see that the function $f(t)$ is continuous. This gives us a contradiction.

Proof of Theorem 5.1. We argue by contradiction and assume the theorem does not hold. In this case we can find $(L_t^i)_{0 \leq t \leq T_1}$ a sequence of smooth flows that satisfies condition $(\ast)$ with $R_i$ tending to infinity and $L_t^i$ converging to $M_1$ in $C^{2,\alpha}_{loc}$.

Compactness for integral Brakke motions [7, §7.1] implies that, after passing to a subsequence, $(L_t^i)_{0 \leq t \leq T_1}$ converges to an integral Brakke motion $(M_t)_{0 \leq t \leq T_1}$. The next theorem characterizes $(M_t)_{0 \leq t \leq T_1}$.

**Theorem 5.3.** There is $\delta_0$ small, $r$ small, $R$ large, $T_0 \in (1, T_1)$, and a continuous family of curves $\sigma_t : [0, +\infty) \to \mathbb{C}$ with

$$\sigma_1 = \sigma, \quad \sigma_t^{-1}(0) = 0 \text{ for all } 1 \leq t \leq T_0 + \delta_0,$$

and such that

(i) For all $1 \leq t \leq T_0 + \delta_0$,

- $M_t$ is smooth in $B_r \cup \mathbb{C} \setminus B_R$ and
- $L_t^i$ converges in $C^{2,\alpha}_{loc}$ to $M_t$ in $B_r \cup \mathbb{C} \setminus B_R$. 
(ii) For all \( 1 \leq t < T_0 \), \( \sigma_t \) is a smooth curve with a single self-intersection. Moreover,

\[
M_t = \{ (\sigma_t(s) \cos \alpha, \sigma_t(s) \sin \alpha) \mid s \in [0, +\infty), \alpha \in S^1 \}
\]

and

\[
\frac{dx}{dt} = \vec{k} - \frac{x^\perp}{|x|^2}.
\]

Finally, for each \( t < T_0 \), \( L^i_t \) converge in \( C^{2,\alpha}_{\text{loc}} \) to \( M_t \).

(iii) The curve \( \sigma_{T_0} \) has a singular point \( Q \) so that \( \sigma_{T_0} \setminus \{Q\} \) consists of two disjoint smooth embedded arcs and, away from \( Q \), \( \sigma_t \) converges to \( \sigma_{T_0} \) as \( t \) tends to \( T_0 \).

(iv) For all \( T_0 < t \leq T_0 + \delta_0 \), \( \sigma_t \) is a smooth embedded curve that satisfies (31) and (32). Moreover, for each \( T_0 < t \leq T_0 + \delta_0 \), \( L^i_t \) converge in \( C^{2,\alpha}_{\text{loc}} \) to \( M_t \).

**Remark 5.4.** (1) The content of this theorem is to justify the behavior shown in Figures 3 and 4. More precisely, Theorem 5.3(ii) and (iii) say that the solution \( (\sigma_t)_{t \geq 1} \) to (32) with \( \sigma_1 = \sigma \) will have a singularity at time \( T_0 \) that corresponds to the loop enclosed by the self-intersection of \( \sigma_t \) collapsing. Theorem 5.3(iv) says that after \( T_0 \), the curves \( \sigma_t \) become smooth and embedded.

(2) The behavior described above follows essentially from Angenent’s work [2], [3] on general one-dimensional curvature flows.

(3) We also remark that the fact \( M_t \) has the symmetries described in (31) up to the singular time \( T_0 \) is no surprise because that is equivalent to uniqueness of solutions with smooth controlled data. After the singular time \( T_0 \), there is no general principle justifying why \( M_t \) has the symmetries described in (31). The reason this occurs is because the function \( \mu \) defined in Theorem 3.5(v) evolves by the linear heat equation and is zero if and only if \( M_t \) can be expressed as in (31). (See Claim 1 in the proof of Theorem 5.3 for details.)

(4) Theorem 5.3(i) is necessary so that we can control the flow in neighborhood of the origin because the right-hand side of (32) is singular at the origin. It is important for Corollary 5.5 that the convergence mentioned in Theorem 5.3(i) holds for all \( t \leq T_0 + \delta_0 \) including the singular time.

(5) The proof is mainly technical and will be given at the end of this section.

**Corollary 5.5.** Assuming Theorem 5.3 we have that, for all \( i \) sufficiently large, \((L^i_t)_{1 \leq t \leq T_1}\) must have a finite time singularity.

In Remark 5.2 we sketched the idea behind the proof of this corollary.
Proof. From Theorem 5.3(i) we can find a small interval \( I \) containing \( T_0 \) (the singular time of \( \sigma_t \)) and pick \( a_t \in \sigma_t \cap A(r/3, r/2) \), \( b_t \in \sigma_t \cap A(2R, 3R) \) so that \( a_t, b_t \) are the endpoints of a segment \( \bar{\sigma}_t \subseteq \sigma_t \cap A(r/3, 3R) \) and the paths \( (a_t)_{t \in I}, (b_t)_{t \in I} \) are smooth. Consider the function

\[
 f(t) = \theta_t(b_t) - \theta_t(a_t) .
\]

We claim that

\[(33) \quad \lim_{t \to T_0^-} f(t) = \lim_{t \to T_0^+} f(t) = -2\pi .\]

Recall that the Lagrangian angle \( \theta_t \) equals, up to a constant, the argument of the complex number \( \sigma_t \sigma_t' \). Hence, for all \( t \in I \setminus \{ T_0 \} \), we have

\[
 \theta_t(b_t) - \theta_t(a_t) = \int_{\bar{\sigma}_t} d\theta_t = \int_{\bar{\sigma}_t} \langle \vec{k}, \nu \rangle dH^1 - \int_{\bar{\sigma}_t} \left\langle \frac{x}{|x|^2}, \nu \right\rangle dH^1 ,
\]

where \( \nu \) is the normal obtained by rotating the tangent vector to \( \bar{\sigma}_t \) counterclockwise and we are assuming that this segment is oriented from \( a_t \) to \( b_t \). The curves \( \bar{\sigma}_t \) are smooth near the endpoints by Theorem 5.3(i), have a single self-intersection for \( t < T_0 \) by Theorem 5.3(ii), and are embedded for \( t > T_0 \) by Theorem 5.3(ii) (see Figure 4). Thus, the rotation index of \( \sigma_t \) changes across \( T_0 \) and so

\[(34) \quad \lim_{t \to T_0^-} \int_{\bar{\sigma}_t} \langle \vec{k}, \nu \rangle dH^1 = \lim_{t \to T_0^+} \int_{\bar{\sigma}_t} \langle \vec{k}, \nu \rangle dH^1 + 2\pi .\]

The vector field \( X = x|\mathbf{x}|^{-2} \) is divergence free and so, because none of the segments \( \bar{\sigma}_t \) wind around the origin, the Divergence Theorem implies

\[(35) \quad \lim_{t \to T_0^-} \int_{\bar{\sigma}_t} \left\langle \frac{x}{|x|^2}, \nu \right\rangle dH^1 = \lim_{t \to T_0^+} \int_{\bar{\sigma}_t} \left\langle \frac{x}{|x|^2}, \nu \right\rangle dH^1 .\]

Claim (33) follows at once from (34) and (35).

From Theorem 5.3(i) we can choose a sequence of smooth paths \( (a_t^1)_{t \in I}, (b_t^1)_{t \in I} \) converging to \( (a_t)_{t \in I}, (b_t)_{t \in I} \) respectively, and such that \( a_t^1, b_t^1 \in L_t^1 \). Consider the function

\[
 f^1(t) = \theta_t^1(b_t^1) - \theta_t^1(a_t^1) .
\]

For every \( t \in I \setminus \{ T_0 \} \), we have from Theorem 5.3(ii) and (iv) that \( L_t^1 \) converges in \( C^{2,\alpha}_{\text{loc}} \) to \( M_t \). As a result,

\[(36) \quad f_t(t) \text{ converges to } f(t) \text{ for all } t \in I \setminus \{ T_0 \} .\]

Because the flow \( (L_t^1)_{t \in I} \) exists smoothly, the function \( f^1(t) \) is smooth and

\[
 \frac{df^1(t)}{dt} = \Delta \theta_t^1(b_t^1) + \langle \nabla \theta_t^1, db_t^1/\text{d}t \rangle - \Delta \theta_t^1(a_t^1) - \langle \nabla \theta_t^1, da_t^1/\text{d}t \rangle .
\]
Hence, Theorem 5.3(i) shows that \( df_i(t)/dt \) is uniformly bounded (independently of \( i \)) for all \( t \in I \). From (36) we obtain that the function \( f \) must be Lipschitz continuous, and this contradicts (33). \( \Box \)

This corollary gives us the desired contradiction and finishes the proof of the theorem. \( \Box \)

5.3. Proof of Theorem 5.3. Recall the function \( \mu = x_1 y_2 - y_1 x_2 \) defined in Theorem 3.5(v). We start by proving two claims.

**Claim 1.** \( M_t \subseteq \mu^{-1}(0) \) and \( |\nabla \mu| = 0 \) for almost all \( 1 \le t \le T_1 \).

From Lemma 2.1 and Theorem 3.5(v), we have

\[
\int_{L^i_1} \mu^2 \Phi(0,1)d\mathcal{H}^2 + \int_1^t \int_{L^i_s} |\nabla \mu|^2 \Phi(0,1+t-s)d\mathcal{H}^2 ds \le \int_{L^i_1} \mu^2 \Phi(0,1+t)d\mathcal{H}^2 + \int_1^t \int_{L^i_s} \frac{|E|^2}{4} \mu^2 + E_2 \Phi(0,1+t-s)d\mathcal{H}^2 ds.
\]

Because \( M_1 \subseteq \mu^{-1}(0) \) and \( E, E_2 \) converge uniformly to zero when \( i \) goes to infinity, we obtain

\[
\lim_{i \to \infty} \int_{L^i_1} \mu^2 \Phi(0,1+t)d\mathcal{H}^2 + \int_1^t \int_{L^i_s} \frac{|E|^2}{4} \mu^2 + E_2 \Phi(0,1+t-s)d\mathcal{H}^2 ds = \int_{M_1} \mu^2 \Phi(0,1+t)d\mathcal{H}^2 = 0,
\]

which combined with (37) implies

\[
\int_{M_1} \mu^2 \Phi(0,1)d\mathcal{H}^2 + \int_1^t \int_{M_s} |\nabla \mu|^2 \Phi(0,1+t-s)d\mathcal{H}^2 ds = 0.
\]

This proves the claim.

**Claim 2.** For every \( \delta \), there is \( R = R(\delta, T_1) \) so that, in the annular region \( A(R, \overline{R}_i) \), \( L^i_t \) is \( \delta \)-close in \( C^{2,\alpha} \) to the plane \( P_1 \) for all \( 1 \le t \le T_1 \) and \( i \) sufficiently large.

According to Lemma 3.2 there is a constant \( R = R(\delta, T_1, \overline{R}_i) \) so that, in the annular region \( A(R, \overline{R}_i) \), \( L^i_t \) is \( \delta \)-close in \( C^{2,\alpha} \) to \( P_1 \) for all \( 1 \le t \le T_1 \). Because \( L^i_1 \) converges to \( M_1 \), we can deduce from Theorem 3.3(i) that \( \overline{R}_i \) is bounded and thus the constant \( R \) depends only on \( \delta \) and \( T_1 \) and not on the index \( i \). This prove the claim.

**Definition of “singular time” \( T_0 \):** First we need to introduce some notation. Because condition (*) holds for the flow \( (L^i_t) \), there are a sequence of discs \( D_i \) of increasingly larger radius and normal deformations \( F^i_t : D_i \to C^2 \) so
that, for all $1 \leq t \leq 2$, $F_t^i(D_t) \subseteq L_t^i$, and $F_t^i$ converges in $C^{2,\alpha}_{\text{loc}}$ to $F_t : \mathbb{R}^2 \rightarrow \mathbb{C}^2$, where $M_t = F_t(\mathbb{R}^2)$.

Consider the following condition:

\begin{equation}
F_t^i \text{ converges in } C^{2,\alpha}_{\text{loc}} \text{ to } F_t : \mathbb{R}^2 \rightarrow \mathbb{C}^2, \text{ where } M_t = F_t(\mathbb{R}^2),
\end{equation}

and set

\begin{equation}
T_0 = \sup\{l \mid F_t^i \text{ is defined and condition (38) holds for all } t \leq l\} \cap [1, T_1].
\end{equation}

**Proof of Theorem 5.3(ii).** By the way $T_0$ was chosen and Claim 1, we have that $\mu^{-1}(0)$ is a smooth surface diffeomorphic to $\mathbb{R}^2$. Thus Lemma 7.1 implies the existence of $(\sigma_t)^{1 \leq t < T_0}$ so that (31) holds. From the definition of $T_0$ it is also straightforward to conclude that $L_t^i$ converges in $C^{2,\alpha}_{\text{loc}}$ to $M_t$ if $t < T_0$. We are left to argue that $\sigma_t$ has a single self-intersection for all $1 \leq t < T_0$. From Lemma 5.6 below we conclude that if $\sigma_t$ develops a tangential self-intersection, it must be away from the origin. It is easy to see from the flow (32) that this cannot happen.

**Lemma 5.6.** There exists $r$ so that $\sigma_t \cap B_r$ is embedded for all $1 \leq t < T_0$.

**Proof.** Recall the definition of $C_a$ in (30). We start by arguing that

\begin{equation}
\sigma_t \subseteq C_a \text{ for all } 1 \leq t < T_0.
\end{equation}

The boundary of the cone $C_a$ consists of two half-lines that are fixed points for the flow (32). From Claim 2 we see that $M_t$ is asymptotic to $P_1$ and so $\sigma_t$ does not intersect $\partial C_a$ outside a large ball. Thus, because $\sigma_1 \subseteq C_a$, we conclude from Lemma 7.3 that $\sigma_t \subseteq C_a$ for all $1 \leq t < T_0$.

Denote by $\Gamma$ a curve in $\mathbb{C}$ that is asymptotic at infinity to

\begin{equation}
\{r \exp(i(\pi/2 + 3a/2)) \mid r \geq 0\} \cup \{r \exp(i(\pi/2 + 3a/2)) \mid r \geq 0\}
\end{equation}

and generates, under the $S^1$ action described in (31), a Special Lagrangian asymptotic to two planes (Lawlor Neck). In particular, the curves $\Gamma_\delta = \delta \Gamma$ are fixed points for the flow (32) for all $\delta$ and, because of (40) and (41), $\sigma_t$ does not intersect $\Gamma_\delta$ outside a large ball for all $1 \leq t < T_0$.

From the description of $\sigma$ given at the beginning of Section 5, we find $\delta_0$ so that for every $\delta < \delta_0$, the curve $\Gamma_\delta$ intersects $\sigma$ only once. Hence, we can apply [3, Variation on Th. 1.3] and conclude that $\Gamma_\delta$ and $\sigma_t$ intersect only once for all $1 \leq t < T_0$ and all $\delta < \delta_0$. It is simple to see that this implies the result we want to show provided we choose $r$ small enough. □

**Proof of Theorem 5.3(i).** This follows from Claim 2 and the next lemma.

**Lemma 5.7.** There are $r$ small and $\delta$ small so that $M_t \cap B_r$ is smooth, embedded, and $L_t^i$ converges in $C^{2,\alpha}(B_r)$ to $M_t \cap B_r$ for all $1 \leq t \leq T_0 + \delta$. 

In particular, the curve $\sigma_t \cup -\sigma_t$ is smooth and embedded near the origin with bounds on its $C^{2,\alpha}$ norm for all $1 \leq t \leq T_0 + \delta$.

Remark 5.8. The key step to show Lemma 5.7 is to argue that $(M_t)_{t \geq 1}$ develops no singularity at the origin at time $T_0$. The idea is the following. First principles will show that a sequence of blow-ups at the origin $(\sigma_j^t)_{s < 0}$ of $(\sigma_t)_{t < T_0}$ converge in $C^{1,1/2}_{\text{loc}}(\mathbb{R}^2 - \{0\})$ to a union of half-lines. But Lemma (5.6) implies $\sigma_j^t$ is embedded in $B_1$ for all $j$ sufficiently large and so it must converge to a single half-line. White’s Regularity Theorem implies no singularity occurs.

Proof. From the way $T_0$ was chosen (39) and Lemma 5.6 we know the existence of $r$ so that $M_t \cap B_r$ is smooth, embedded, and $L_t$ converges in $C^{2,\alpha}(B_r)$ to $M_t \cap B_r$ for all $1 \leq t < T_0$. To extend this to hold up to $t = T_0$ (with possible smaller $r$) it suffices to show that $(M_t)_{1 \leq t < T_0}$ develops no singularity at the origin at time $T_0$.

Choose a sequence $(\lambda_j)_{j \in \mathbb{N}}$ tending to infinity, and set $M_j^t = \lambda_j M_{T_0 + t/\lambda_j^2}$, for all $t < 0$.

From [9, Lemma 5.4] we have the existence of a union of planes $Q$ with support contained in $\mu^{-1}(0)$ such that, after passing to a subsequence and for almost all $t < 0$, $M_j^t$ converges in the varifold sense to $Q$ and

$$\lim_{j \to \infty} \int_{M_j^t} (|H|^2 + |x|^2) \exp(-|x|^2) dH^2 = 0.$$ 

From (5.3) we can find curves $\sigma_j^t$ so that

$$M_j^t = \{(\sigma_j^t(s) \cos \alpha, \sigma_j^t(s) \sin \alpha) | s \in [0, +\infty), \alpha \in S^1\}.$$ 

We obtain from (42) that for almost all $t$ and every $0 < \eta < 1$,

$$\lim_{j \to \infty} \int_{\sigma_j^t \cap A(\eta, \eta^{-1})} |k|^2 + |x|^2 dH^1 = 0,$$

which implies that $\sigma_j^t$ converges in $C^{1,1/2}_{\text{loc}}(\mathbb{R}^2 - \{0\})$ to a union of half-lines with endpoints at the origin. Lemma 5.6 implies that for all $j$ sufficiently large $\sigma_j^t$ is embedded inside the unit ball. Thus $\sigma_j^t$ must converge to a single half-line and so $Q$ is a multiplicity one plane. Thus there can be no singularity at time $T_0$ at the origin.

We now finish the proof of the lemma. So far we have proven that $M_t$ is smooth and embedded near the origin for all $1 \leq t \leq T_0$. Thus we can find $l_0$ small so that

$$\int_{M_t} \Phi(x, l) dH^2 \leq 1 + \varepsilon_0$$

for every $x \in B_{2l_0}, l \leq 4l_0^2$, and $1 \leq t \leq T_0$. 


The monotonicity formula implies that
\[ \int_{M_t} \Phi(x,l) dH^2 \leq 1 + \varepsilon_0 \text{ for every } x \in B_{l_0}, l \leq l_0^2, \text{ and } 1 \leq t \leq T_0 + l_0^2. \]
Because \( L_i \) converges to \( M_t \) as Radon measures, White’s Regularity Theorem implies uniform \( C^{2,\alpha} \) bounds in \( B_{l_0/2} \) for \( L_i \) whenever \( i \) is sufficiently large and \( t \leq T_0 + l_0^2 \). The lemma then follows straightforwardly. \( \square \)

**Proof of Theorem 5.3(iii).** We need two lemmas first.

**Lemma 5.9.** \( T_0 < T_1 \).

**Remark 5.10.** The idea is to show that if \( T_0 \geq T_1 \), then the loop of \( \sigma_t \) created by its self-intersection would have negative area.

**Proof.** Suppose \( T_0 = T_1 \). Denote by \( q_t \) the single self-intersection of \( \sigma_t \), by \( c_t \) the closed loop with endpoint \( q_t \), by \( \alpha_t \in [-\pi, \pi] \) the exterior angle that \( c_t \) has at the vertex \( q_t \), by \( \nu \) the interior unit normal, and by \( A_t \) the area enclosed by the loop. From the Gauss-Bonnet Theorem, we have
\[ \int_{c_t} \langle \vec{k}, \nu \rangle dH^1 + \alpha_t = 2\pi \implies \int_{c_t} \langle \vec{k}, \nu \rangle dH^1 \geq \pi. \]
A standard formula shows that
\[ \frac{d}{dt} A_t = -\int_{c_t} \left( \frac{x \times \nu}{|x|^2} \right) dH^1 \leq -\pi + \int_{c_t} \left( \frac{x}{|x|^2}, \nu \right) dH^1 = -\pi, \]
where the last identity follows from the Divergence Theorem combined with the fact that \( c_t \) does not contain the origin in its interior. Hence \( 0 \leq A_t \leq A_1 - (t-1)\pi \). Making \( t \) tending to \( T_1 = 2A_1/\pi + 1 \), we obtain a contradiction. \( \square \)

**Lemma 5.11.** The curve \( \sigma_t \) must become singular when \( t \) tends to \( T_0 \).

**Remark 5.12.** The flow \( (M_t)_{t \geq 0} \) is only a weak solution to mean curvature flow which means that, in principle, \( \sigma_{T_0} \) could be a smooth curve with a self-intersection and, right after, \( \sigma_t \) could split off the self-intersection and become instantaneously a disjoint union of a circle with a half-line. This lemma shows that, because \( M_t \) is a limit of smooth flows \( L_i \), this phenomenon cannot happen. The proof is merely technical.

**Proof.** We are assuming \( F^i_t \) converges in \( C^{2,\alpha}_{\text{loc}} \) to \( F_t \) for all \( t < T_0 \). Assuming \( \sigma_{T_0} \) is smooth, we have from parabolic regularity that \( (\sigma_t)_{t \leq T_0} \) is a smooth flow. Thus, \( M_{T_0} \) is also smooth and the maps \( F_t \) converge smoothly to a map \( F_{T_0} : \mathbb{R}^2 \rightarrow \mathbb{C}^2 \). Therefore, there is a constant \( C \) that bounds the \( C^2 \) norm of \( F_t \) for all \( T_0 - 1 \leq t \leq T_0 \). Hence, using Claim 2 to control the \( C^{2,\alpha} \) norm of \( F^i_t \) outside a large ball, we obtain that for \( \hat{t} < T_0 \) and \( i \) sufficiently large, the \( C^2 \) norm of \( F^i_t \) is bounded by \( 2C \). Looking at the evolution equation of \( |A|^2 \)
it is then a standard application of the maximum principle to find \( \delta = \delta(C) \) such that the second fundamental form of the immersion \( F_t^\delta \) is bounded by \( 4C \) for all \( \bar{t} \leq t \leq \bar{t} + \delta \). Therefore, choosing \( \bar{t} \) such that \( T_0 < \bar{t} + \delta \), parabolic regularity implies condition (38) holds for all \( t \) slightly larger than \( T_0 \) which, due to Lemma 5.9, contradicts the maximality of \( T_0 \). \( \square \)

Claim 2 and Lemma 5.7 give us control of the flow (32) outside an annulus. Hence we apply Theorem 7.2 and conclude the singular curve \( \sigma_{T_0} \) contains a point \( Q \) distinct from the origin such that \( \sigma_{T_0} \setminus \{Q\} \) consists of two smooth disjoint arcs and, away from the singular point, the curves \( \sigma \) converge smoothly to \( \sigma_{T_0} \) (see Figure 4).

**Proof of Theorem 5.3(iv).** From Theorem 5.3(i) and Claim 1, we can apply Lemma 7.1 and conclude that \( M_t \) can be described by a one-dimensional varifold \( \sigma_t \subset C \) for almost all \( T_0 < t < T_1 \).

In [2, §8] Angenent constructed an embedded smooth solution \( (\gamma_t)_{t > 0} \) that tends to \( \sigma_{T_0} \) when \( t \) tends to zero and that looks like the solution described on Figure 4. The next lemma is the key to showing Theorem 5.3(iv).

**Lemma 5.13.** There is \( \delta \) small so that \( \gamma_t = \sigma_{T_0 + t} \) for all \( 0 < t < \delta \).

**Remark 5.14.** This lemma amounts to showing that there is a unique (weak) solution to the flow (32) that starts at \( \sigma_{T_0} \).

The idea to prove this lemma, which we now sketch, is well known among the specialists. Consider \( \gamma^+_{i, t}, \gamma^-_{i, t} \) two sequences of smooth embedded curves with an endpoint at the origin and converging to \( \sigma_{T_0} \), with \( \gamma^+_{i, t}, \gamma^-_{i, t} \) lying above and below \( \sigma_{T_0} \), respectively. There is a region \( A_i \) that has \( \sigma_{T_0} \subseteq A_i \) and \( \partial A_i = \gamma^+_{i, t} \cup \gamma^-_{i, t} \). Denote the flows starting at \( \gamma^+_{i, t} \) and \( \gamma^-_{i, t} \) by \( \gamma^+_{i, t, +} \) and \( \gamma^-_{i, t, -} \), respectively, and use \( A_i(t) \) to denote the region below \( \gamma^+_{i, t, +} \) and above \( \gamma^-_{i, t, -} \).

For the sake of the argument, we can assume that \( A_i \) is finite and tends to zero when \( i \) tends to infinity. A simple computation will show that \( \text{area}(A_i(t)) \leq \text{area}(A_i) \) and so, like \( A_i \), the area of \( A_i(t) \) tends to zero when \( i \) tends to infinity. The avoidance principle for the flow implies that \( \sigma_{T_0 + t}, \gamma_t \subseteq A_i(t) \) for all \( i \) and \( t \), and thus, making \( i \) tend to infinity, we obtain that \( \sigma_{T_0 + t} = \gamma_t \).

The proof requires some technical work to go around the fact that the curves \( \gamma^+_{i, t}, \gamma^-_{i, t} \) are noncompact and thus \( A_i \) could be infinity.

**Proof.** Let \( \gamma^+_{i, t}, \gamma^-_{i, t} : [0, +\infty] \to \mathbb{C} \) be two sequences of smooth embedded curves converging to \( \sigma_{T_0} \) with \( \gamma^+_{i, t}, \gamma^-_{i, t} \) lying above (below) \( \sigma_{T_0} \) and such that

\[
(\gamma^\pm_{i, t})^{-1}(0) = 0, \quad \gamma^\pm_{i, t} \cup -\gamma^\pm_{i, t} \text{ is smooth}, \quad \theta^+_{i, t}(0) < \theta_{T_0}(0) < \theta^-_{i, t}(0).
\]

The convergence is assumed to be strong on compact sets not containing the cusp point of \( \sigma_{T_0} \). Denote by \( \gamma^\pm_{i, t} \) the solution to the equivariant flow (32) with initial condition \( \gamma^\pm_{i, t} \). Short time existence was proven in [9, §4] provided we
assume controlled behavior at infinity. The same arguments used to study $\sigma_t$ (namely Lemma 5.7) show that embeddedness is preserved and no singularity of $\gamma_{\pm,t}^i$ can occur at the origin. Hence an immediate consequence of Theorem 7.2 is that the flow exists smoothly for all time.

From the last condition in (43) we know that $\gamma_{+,t}^i \cap \gamma_{-,t}^i$ intersects transversely at the origin. Furthermore we can choose $\gamma_{+,t}^i, \gamma_{-,t}^i$ to be not asymptotic to each other at infinity. Thus we can apply Lemma 7.3 and conclude that $\gamma_{+,t}^i$ and $\gamma_{-,t}^i$ intersect each other only at the origin. Hence there is an open region $A_i(t) \subset \mathbb{C}$ so that $\gamma_{+,t}^i \cup \gamma_{-,t}^i = \partial A_i(t)$.

From Claim 2 we know that $\gamma_{T_0}$ is asymptotic to a straight line. Thus we can reason as in the proof of Theorem 3.5(i) and conclude the existence of $R_i$ tending to infinity so that $\gamma_{\pm,t}^i \cap A(R_i/2, 2R_i)$ is graphical over the real axis with $C^1$ norm smaller than $1/i$ for all $0 \leq t \leq 1$.

Consider $B_i(t) = A_i(t) \cap \{(x, y) \mid x \geq -R_i\}$. This region has the origin as one of its “vertices” and is bounded by three smooth curves. The top curve is part of $\gamma_{+,t}^i$, the bottom curve is part of $\gamma_{-,t}^i$, and left-side curve is part of $\{x = -R_i\}$. Using the fact that

$$\text{area}(B_i(t)) = \int_{\partial B_i(t)} \lambda,$$

differentiation shows that

$$\frac{d}{dt}\text{area}(B_i(t)) = -\left(\theta_{+,t}^i(-R_i) - \theta_{-,t}^i(-R_i)\right) + \left(\theta_{+,t}^i(0) - \theta_{-,t}^i(0)\right),$$

where $\theta_{\pm,t}^i(-R_i)$ denote the Lagrangian angle of $\gamma_{\pm,t}^i$ at the intersection with $\{x = -R_i\}$ and $\theta_{\pm,t}^i(0)$ denotes the Lagrangian angle of $\gamma_{\pm,t}^i$ at the origin. Because $\gamma_{+,t}^i$ lies above $\gamma_{-,t}^i$ and they intersect at the origin, we have $\theta_{+,t}^i(0) \leq \theta_{-,t}^i(0)$. Thus,

$$\frac{d}{dt}\text{area}(B_i(t)) \leq -\left(\theta_{+,t}^i(-R_i) - \theta_{-,t}^i(-R_i)\right).$$

Recalling that $\gamma_{\pm,t}^i \cap A(R_i/2, 2R_i)$ is graphical over the real axis with $C^1$ norm smaller than $1/i$ for all $t \leq 1$, we have that the term on the right side of the above inequality tends to zero when $i$ tends to infinity. Finally the curves can be chosen so that $\text{area}(B_i(0)) \leq 1/i$ and thus

$$\lim_{i \to \infty} \text{area}(B_i(t)) \leq \lim_{i \to \infty} \text{area}(B_i(0)) = 0.$$

We now argue the existence of $\delta$ so that

$$\gamma_t, \sigma_{T_0+t} \subseteq A_i(t) \text{ for all } t \leq \delta \text{ and all } i.$$

The inclusion for $\gamma_t$ follows from Lemma 7.3. Next we want to deduce the inclusion for the varifolds $\sigma_{T_0+t}$ (recall Lemma 7.1), which does not follow directly from Lemma 7.3 because $\sigma_{T_0+t}$ might not be smooth. We remark that
the right-hand side of (32) is the geodesic curvature with respect to the metric 
\[ h = (x_1^2 + y_1^2)(dx_1^2 + dy_1^2). \]
Because \((M_t)_{t \geq 1}\) is a Brakke flow, it is not hard to deduce from Lemma 7.1 that \((\sigma_t)_{t \geq 1}\) is also a Brakke flow with respect to the metric \(h\). This metric is singular at the origin and has unbounded curvature but fortunately, due to Claim 2 and Lemma 5.7, we already know that \(\sigma_t\) is smooth in a neighborhood of the origin and outside a compact set for all \(t \leq T_0 + \delta\). Thus, the Inclusion Theorem proven in [7, 10.7 Inclusion Theorem] adapts straightforwardly to our setting, and this implies \(\sigma_{T_0+t} \subseteq A_i(t)\) for all \(t \leq \delta\).

Combining (44) with (45) we obtain that 
\[ \gamma_t = \sigma_{T_0+t} \quad \text{all } 0 < t < \delta. \]
\[ \square \]

From Lemma (5.13) we obtain that \(M_t\) is smooth, embedded, and satisfies (31), (32) for all \(T_0 < t < T_0 + \delta\). Finally, from the fact that \(M_t\) is embedded, it follows in a straightforward manner from White’s Regularity Theorem that \(L_t\) converges in \(C^{2,\alpha}_{\text{loc}}\) to \(M_t\). This completes the proof of Theorem 5.3.

6. Main Theorem

**Theorem 6.1.** For any embedded closed Lagrangian surface \(\Sigma\) in \(M\), there is \(L\) Lagrangian in the same Hamiltonian isotopy class so that the Lagrangian mean curvature flow with initial condition \(L\) develops a finite time singularity.

**Proof.** Setup. Given \(\overline{R}\) large we can find a metric \(g_R = R^2 g\) (see Section 2.1) so that the hypothesis on ambient space described in Section 3.1.1 are satisfied. Pick \(p \in \Sigma\), and assume the Darboux chart \(\phi\) sends the origin into \(p \in \Sigma\) and \(T_p\Sigma\) coincides with the real plane \(\mathbb{R} \oplus i\mathbb{R} \subseteq \mathbb{C}^2\) oriented positively.

We can assume \(\Sigma \cap B_{4\overline{R}}\) is given by the graph of the gradient of some function defined over the real plane, where the \(C^2\) norm can be made arbitrarily small. It is simple to find \(\Sigma\) Hamiltonian isotopic to \(\Sigma\) that coincides with the real plane in \(B_{3\overline{R}}\). Denote by \(L\) the Lagrangian that is obtained by replacing \(\Sigma \cap B_{3\overline{R}}\) with \(N(\varepsilon, \overline{R})\) defined in (1). Using [4, Th. 1.1.A] we obtain at once that \(L\) is Hamiltonian isotopic to \(\overline{\Sigma}\) and hence to \(\Sigma\) as well. Moreover, there is \(K_0\) depending only on \(\Sigma\) so that the hypothesis on \(L\) described in Section 3.1.2 are satisfied for all \(\overline{R}\) large.

We recall once more that \(L\) depends on \(\varepsilon, R, \overline{R}\) and that \(\overline{R} \geq 4R\). Assume the Lagrangian mean curvature flow \((L_t)_{t \geq 0}\) with initial condition \(L\) exists smoothly for all time.

**First Step.** Pick \(\nu_0\) small (to be fixed later), and choose \(\varepsilon, R, \overline{R}\) so that Theorem 3.3 (with \(\nu = \nu_0\)) and Theorem 3.5 hold. Thus, there is \(R_1 = R_1(\nu_0, K_0)\) so that (see Theorem 3.3(ii))

(A) for every \(1 \leq t \leq 2\), \(L_t \cap A(R_1, \overline{R})\) is \(\nu_0\)-close in \(C^{2,\alpha}\) to \(L\).
Moreover, from Theorem 3.3(iii) and (iv), \( L_t \cap B_{R_1} \) is contained in two connected components \( Q_{1,t} \cup Q_{2,t} \) where

(B) for every \( 1 \leq t \leq 2 \), \( Q_{2,t} \) is \( \nu_0 \)-close in \( C^{2,\alpha}(B_{R_1}) \) to \( P_3 \).

Second Step. We need to control \( Q_{1,t} \). Apply Theorem 4.1 with \( S_0 = R_1 \) and \( \nu = \nu_0 \). Thus for all \( \varepsilon \) small and \( R \) large, we have that

(C) for every \( 1 \leq t \leq 2 \), \( t^{-1/2}Q_{1,t} \) is \( \nu_0 \)-close in \( C^{2,\alpha}(B_{R_1}) \) to \( S \), where \( S \) is the self-expander defined in (2) (see Figure 2). One immediate consequence of (A), (B), and (C) is the existence of \( K_1 \) so that for all \( \varepsilon \) small and \( R \) large, we have

(\( \star \star \)) the existence of a disc \( D \), and \( F_t : D \to \mathbb{C}^2 \) a normal deformation defined for all \( 1 \leq t \leq 2 \), so that

\[ L_t \cap B_{R/2} \subset F_t(D) \subset L_t \cap B_R \]

and the \( C^{2,\alpha} \) norm of \( F_t \) is bounded by \( K_1 \).

Third Step. Fix \( \varepsilon \) and \( R \) in the definition of \( L \) so that (A), (B), (C), and (\( \star \star \)) hold, but let \( R \) tend to infinity. We then obtain a sequence of smooth flows \( (L^i_t)_{t \geq 0} \), where \( L^i_0 \) converges strongly to \( N(\varepsilon, R) \) defined in (1) (see Figure 1).

Lemma 6.2. If \( \nu_0 \) is chosen small enough, there is a curve \( \sigma \subset \mathbb{C} \) with all the properties described in Section 5.1 (see Figure 3) and such that \( L^i_1 \) tends in \( C^{2,\alpha}_{\text{loc}} \) to

\[
(46) \quad M_1 = \{(\sigma(s) \cos \alpha, \sigma(s) \sin \alpha) \mid \alpha \in S^1, s \in [0, +\infty)\}.
\]

Assuming this lemma, we will show that \( (L^i_t)_{t \geq 0} \) must have a singularity for all \( i \) sufficiently large, which finishes the proof of Theorem 6.1. Indeed, because the flow \( (L^i_t)_{t \geq 0} \) has property (\( \star \star \)), we have at once that condition (\( \star \)) of Section 5.1 is satisfied. Hence Lemma 6.2 implies that \( L^i_1 \) satisfies the hypothesis of Theorem 5.1 for all \( i \) sufficiently large, and thus Theorem 5.1 implies that \( (L^i_t)_{t \geq 0} \) must have a finite time singularity.

Proof of Lemma 6.2. From condition (\( \star \star \)) we have that \( L^i_1 \) converges in \( C^{2,\alpha}_{\text{loc}} \) to a smooth Lagrangian \( M_1 \) diffeomorphic to \( \mathbb{R}^2 \). Moreover, from (B) and (C) we see that we can choose \( \nu_0 \) small so that \( M_1 \) is embedded in a small neighborhood the origin. We argue that \( M_1 \subset \mu^{-1}(0) \), where the function \( \mu = x_1y_2 - y_1x_2 \) was defined in Theorem 3.5(v). From Lemma 2.1 and Theorem 3.5(v), we have

\[
(47) \quad \int_{L^i_1} \mu^2 \Phi(0, 1) d\mathcal{H}^2 + \int_0^1 \int_{L^i_t} |\nabla \mu|^2 \Phi(0, 2 - t) d\mathcal{H}^2 dt \leq \int_{L^i_0} \mu^2 \Phi(0, 2) d\mathcal{H}^2 + \int_0^1 \int_{L^i_t} \left( \frac{|E|^2}{4} \mu^2 + E_2 \right) \Phi(0, 2 - t) d\mathcal{H}^2 dt.
\]
The terms $E, E_2$ converge uniformly to zero when $i$ goes to infinity because the ambient metric converges to the Euclidean one. Moreover $N(\varepsilon, R) \subset \mu^{-1}(0)$, and so we obtain from (47) that

$$
\int_{M_1^1} \mu^2 \Phi(0, 1) d\mathcal{H}^2 = \lim_{i \to \infty} \int_{L_i^1} \mu^2 \Phi(0, 1) d\mathcal{H}^2 \\
\leq \lim_{i \to \infty} \int_{L_i^0} \mu^2 \Phi(0, 2) d\mathcal{H}^2 + \int_0^1 \int_{L_i^1} \left( \frac{|E|^2}{4} \mu^2 + E_2 \right) \Phi(0, 2 - t) d\mathcal{H}^2 dt \\
= \int_{N(\varepsilon, R)} \mu^2 \Phi(0, 2) d\mathcal{H}^2 = 0.
$$

Hence, $M_1 \subset \mu^{-1}(0)$ and we can apply Lemma 7.1 to conclude the existence of a curve $\sigma$ so that (46) holds.

In order to check that $\sigma$ has the properties described in Section 5.1, it suffices to see that $\sigma$ has a single self-intersection and is contained in the cone $C_\alpha$ (defined in (30)) because the remaining properties follow from $M_1$ being diffeomorphic to $\mathbb{R}^2$, embedded near the origin, and asymptotic to the plane $P_1$ (Lemma 3.2).

Recall that $\gamma(\varepsilon, R)$, $\chi$, and $c_3$ are the curves in $\mathbb{C}$ that define, respectively, the Lagrangian $N(\varepsilon, R)$, the self-expander $S$, and the plane $P_3$. Now $M_1$, being the limit of $L_i^1$, also satisfies (A), (B), and (C). Hence we know that $\sigma$ is $\nu_0$-close in $C^{2, \alpha}$ to $\gamma(\varepsilon, R)$ in $\mathbb{C} \setminus B_{R_1}$ and that $\sigma \cap B_{R_1}$ has two connected components, one $\nu_0$-close in $C^{2, \alpha}(B_{R_1})$ to $c_3$ and the other $\nu_0$-close in $C^{2, \alpha}(B_{R_1})$ to $\chi$. It is simple to see that if $\nu_0$ is small, then indeed all the desired properties for $\sigma$ follow. \hfill \Box

7. Appendix

7.1. Lagrangians with symmetries. Recall that $\mu(x_1, y_1, x_2, y_2) = x_1y_2 - x_2y_1$, and consider two distinct conditions on $M$.

(C1) $M$ is an integral Lagrangian varifold which is a smooth embedded surface in a neighborhood of the origin;

(C2) There is a smooth Lagrangian immersion $F : \mathbb{R}^2 \to \mathbb{C}^2$ so that $M = F(\mathbb{R}^2)$ and $M$ is a smooth embedded surface in a neighborhood of the origin.

Lemma 7.1. Assume $M \subseteq \mu^{-1}(0)$. If (C1) holds, then there is a one-dimensional integral varifold $\gamma \subset \mathbb{C}$ so that for every function $\phi$ with compact support,

$$
\int_M \phi d\mathcal{H}^2 = \int_{\gamma} |z| \int_0^{2\pi} \phi(z \cos \alpha, z \sin \alpha) d\alpha d\mathcal{H}^1.
$$
If \((C2)\) holds, then there is a smooth immersed curve \(\gamma : [0, \infty) \to \mathbb{C}\) with \(\gamma^{-1}(0) = 0\), and

\[
M = \{(\gamma(s) \cos \alpha, \gamma(s) \sin \alpha) \mid s \in [0, +\infty), \theta \in S^1\}.
\]  

(49)

In both cases the curve (or varifold) \(\gamma \cup -\gamma\) is smooth near the origin.

Proof. Consider the vector field

\[
X = -JD\mu = (-x_2, -y_2, x_1, y_1).
\]

A simple computation shows that for any Lagrangian plane \(P\) with orthonormal basis \(\{e_1, e_2\}\), we have

\[
\text{div}_P X = \sum_{i=1}^{2} \langle D(e_i)X, e_i \rangle = 0.
\]

(50)

Finally, consider \((F_\alpha)_{\alpha \in S^1}\) to be the one-parameter family of diffeomorphisms in \(SU(2)\) such that

\[
\frac{d}{d\alpha} F_\alpha(x) = X(F_\alpha(x)).
\]

Consider the functions \(f_1(x) = \arctan(x_2/x_1)\) and \(f_2(x) = \arctan(y_2/y_1)\), which are defined, respectively, in \(U_1 = \{x_1 \neq 0\}\), \(U_2 = \{y_1 \neq 0\}\). Assume that \((C1)\) holds. We now make several remarks that will be important when one applies the co-area formula.

First, \(F_\alpha(M) = M\); i.e.,

\[
\int_M \phi \circ F_\alpha d\mathcal{H}^2 = \int_M \phi d\mathcal{H}^2 \text{ for all } \phi \text{ with compact support}.
\]

Because \(M \subseteq \mu^{-1}(0)\) and \(M\) is Lagrangian, we have that \(X\) is a tangent vector to \(M\). Hence,

\[
\frac{d}{d\alpha} \int_M \phi \circ F_\alpha d\mathcal{H}^2 = \int_M \langle D(\phi \circ F_\alpha), X \rangle d\mathcal{H}^2 = -\int_M (\phi \circ F_\alpha) \text{div}_M X d\mathcal{H}^2 = 0,
\]

where the last identity follows from (50).

Second, on \(M\) we have \(|\nabla f_i|(x) = |x|^{-1}\). Indeed, for every \(x \in U_i\) with \(\mu(x) = 0\), it is a simple computation to see that \(Df_i \in \text{span}\{X(x), JX(x)\}\) and thus, because \(X\) is a tangent vector,

\[
|\nabla f_i|(x) = |\langle Df_i(x), X(x) \rangle| |x|^{-1} = |x|^{-1}.
\]

Third, for almost all \(\alpha\) and \(i = 1, 2\), \(f_i^{-1}(\alpha) \cap M\) is a one-dimensional varifold. Moreover, a simple computation shows \(f_i \circ F_\alpha(x) = \alpha + f_i(x)\) for all \(x \in U_i\) and all \(\alpha \in (-\pi/2, \pi/2)\), and thus

\[
F_\alpha(f_i^{-1}(0) \cap M) = f_i^{-1}(\alpha) \cap F_\alpha(M) = f_i^{-1}(\alpha) \cap M.
\]
Fourth, the fact that \( M \subseteq \mu^{-1}(0) \) implies that \( f_i^{-1}(0) \cap M \) has support contained in \( \{x_2 = y_2 = 0\} = \mathbb{C} \). Moreover, \( f_1 = f_2 \) on \( M \), and so we set
\[
\Gamma = f_1^{-1}(0) \cap M = f_2^{-1}(0) \cap M.
\]

Fifth, one can check that \( F_\pi \) coincides with the antipodal map \( A \). Thus
\[
A(\Gamma) = A(f_i^{-1}(0)) \cap A(M) = f_i^{-1}(0) \cap M = \Gamma.
\]
As a result, there is a one-dimensional varifold \( \gamma \) such that \( \Gamma = A(\gamma) + \gamma \). (The choice of \( \gamma \) is not unique.)

Finally, we can apply the co-area formula and obtain for every \( \phi \) with compact support in \( U_i \),
\[
\int_M \phi d\mathcal{H}^2 = \int_{-\pi/2}^{\pi/2} \int_{f_i^{-1}(\alpha) \cap M} \frac{\phi}{|\nabla f_i|} \, d\mathcal{H}^1 \, d\alpha = \int_{-\pi/2}^{\pi/2} \int_{f_i^{-1}(\alpha) \cap M} |x| \phi \, d\mathcal{H}^1 \, d\alpha
\]
\[
= \int_{-\pi/2}^{\pi/2} \int_{f_i^{-1}(0) \cap M} |F_\alpha(x)| (\phi \circ F_\alpha) \, d\mathcal{H}^1 \, d\alpha
\]
\[
= \int_{\Gamma} |x| \int_{-\pi/2}^{\pi/2} \phi(F_\alpha(x)) \, d\alpha \, d\mathcal{H}^1
\]
\[
= \int_{\Gamma} |z| \int_{-\pi/2}^{\pi/2} \phi(z \cos \alpha, z \sin \alpha) \, d\alpha \, d\mathcal{H}^1
\]
\[
= \int_{\Gamma} |z| \int_{0}^{2\pi} \phi(z \cos \alpha, z \sin \alpha) \, d\alpha \, d\mathcal{H}^1.
\]
This proves (48) for functions with support contained in \( U_i \). Because \( M \) is smooth and embedded near the origin, it is straightforward to extend that formula to all functions with compact support.

Assume that (C2) holds. From what we have done it is straightforward to obtain the existence of a curve \( \gamma : I \rightarrow \mathbb{C} \), where \( I \) is a union of intervals, so that (49) holds. The fact that \( M \) is diffeomorphic to \( \mathbb{R}^2 \) implies that \( \gamma \) is connected and that \( \gamma^{-1}(0) \) must be nonempty. The condition that \( M \) is embedded when restricted to a small neighborhood of the origin implies that \( \gamma^{-1}(0) \) must have only one element, which we set to be zero. Finally, the fact that the map \( F \) is an immersion is equivalent to the curve \( \gamma \cup -\gamma \) being smooth at the origin. \( \square \)

7.2. Regularity for equivariant flow. In [2] and [3] Angenent developed the regularity theory for a large class of parabolic flows of curves in surfaces. We collect the necessary results, along with an improvement done in [11], which will be used in our setting.

Let \( \gamma_t : [0, a] \rightarrow \mathbb{C}, 0 \leq t < T, \) be a one-parameter family of smooth curves so that
(A1) There is \( r > 0 \) and \( p \in \mathbb{C} \) so that for all \( 0 \leq t < T \), \( \gamma_t(0) = 0 \), \( \gamma_t(a) \in B_r(p) \), \( \gamma_t \) has no self-intersections in \( B_{2r}(0) \cup B_{2r}(p) \), and the curvature of \( \gamma_t \) along with \( \frac{x}{|x|^2} \) and all its derivatives are bounded (independently of \( t \)) in \( B_{2r}(0) \cup B_{2r}(p) \).

(A2) Away from the origin and for all \( 0 \leq t < T \), the curves \( \gamma_t \) solve the equation

\[
\frac{dx}{dt} = \vec{k} - \frac{x}{|x|^2}.
\]

A simple modification of [3, Th. 1.3] implies that, for \( t > 0 \), the self-intersections of \( \gamma_t \) are finite and nonincreasing with time.

**Theorem 7.2.** There is a continuous curve \( \gamma_T \) and a finite number of points \( \{Q_1, \ldots, Q_m\} \subseteq \mathbb{C} \setminus (B_{2r}(0) \cup B_{2r}(p)) \) such that \( \gamma_T \setminus \{Q_1, \ldots, Q_m\} \) consists of smooth arcs and away from the singular points the curves \( \gamma_t \) converge smoothly to \( \gamma_T \). Any two smooth arcs intersect only in finitely many points.

For each of the singular points \( Q_i \) and for each small \( \varepsilon \), the number of self-intersections of \( \gamma_T \) in \( B_{\varepsilon}(Q_i) \) is strictly less than the number of self-intersections of \( \gamma_{t_j} \) in \( B_{\varepsilon}(Q_i) \) for some sequence \( (t_j)_{j \in \mathbb{N}} \) converging to \( T \).

**Proof.** Condition (A1) implies that the curves \( \gamma_t \) converge smoothly in \( B_{2r}(0) \cup B_{2r}(p) \) as \( t \) tends to \( T \). A slight modification of [2, Th. 4.1] shows that the quantity

\[
\int_{\gamma_t} |\vec{k}| dH^1
\]

is uniformly bounded. Indeed the only change one has to make concerns the existence of boundary terms when integration by parts is performed. Fortunately, (A1) implies that the contribution from the boundary terms is uniformly bounded and so all the other arguments in [2, Th. 4.1] carry through.

The fact that the total curvature is uniformly bounded and that, on \( \mathbb{C} \setminus B_{2r}(0) \), the deformation vector \( \vec{k} - \frac{x}{|x|^2} \) satisfies conditions \((V_1^\ast), (V_2), (V_3), (V_4^\ast)\), and \((S)\) of [2], shows that we can apply [3, Th. 5.1] to conclude the existence of a continuous curve \( \gamma_T \) and a finite number of points \( \{Q_1, \ldots, Q_m\} \subseteq \mathbb{C} \setminus (B_{2r}(0) \cup B_{2r}(p)) \) such that \( \gamma_T \setminus \{Q_1, \ldots, Q_m\} \) consists of smooth arcs and away from the singular points the curves \( \gamma_t \) converge smoothly to \( \gamma_T \). We note that [3, Th. 5.1] is applied to closed curves, but an inspection of the proof shows that all the arguments are local and so they apply with no modifications to \( \gamma_t \) provided hypothesis (A1) holds.

Oaks [11, Th. 6.1] showed that for each of the singular points \( Q_i \) and for each small \( \varepsilon \), there is a sequence \( (t_j)_{j \in \mathbb{N}} \) converging to \( T \) so that \( \gamma_{t_j} \) has self-intersections in \( B_{\varepsilon}(Q_i) \) and either a closed loop of \( \gamma_{t_j} \) in \( B_{\varepsilon}(Q_i) \) contracts as \( t_j \) tends to \( T \) or else there are two distinct arcs in the smooth part of \( \gamma_T \) that coincide in a neighborhood of \( Q_i \) (see [3, Fig. 6.2]). Using the fact that the
deformation vector is analytic in its arguments on \( \mathbb{C} \setminus B_{2r}(0) \), we can argue as in [3, pp. 200–201] and conclude that the smooth part of \( \gamma_T \) must in fact be real analytic in \( \mathbb{C} \setminus B_{2r}(0) \). Therefore, any two smooth arcs intersect only in finitely many points and this excludes the second possibility.

\[ \square \]

7.3. Nonavoidance principle for equivariant flow.

**Lemma 7.3.** For each \( j = 1, 2 \), consider smooth curves \( \sigma_{j,t} : [-a, a] \to \mathbb{C} \) defined for all \( 0 \leq t \leq T \) so that

(i) \( \sigma_{j,t}(-s) = -\sigma_{j,t}(s) \) for all \( 0 \leq t \leq T \) and \( s \in [-a, a] \).

(ii) The curves \( \gamma_t \) solve the equation

\[
\frac{dx}{dt} = \vec{k} - \frac{x}{|x|^2}.
\]

(iii) \( \sigma_{1,0} \cap \sigma_{2,0} = \{0\} \) (nontangential intersection) and \( (\partial \sigma_{1,t}) \cap \sigma_{2,t} = \sigma_{1,t} \cap (\partial \sigma_{2,t}) = \emptyset \) for all \( 0 \leq t \leq T \).

For all \( 0 \leq t \leq T \), we have \( \sigma_{1,t} \cap \sigma_{2,t} = \{0\} \).

**Proof.** Away from the origin, it is simple to see the maximum principle holds and so two disjoint solutions cannot intersect for the first time away from the origin. Thus it suffices to focus on what happens around the origin. Without loss of generality we assume that \( \sigma_{j,t}(s) = (s, f_{j,t}(s)) \) for all \( s \in [-\delta, \delta] \) for all \( t \leq T_1 \). The functions \( \alpha_{j,t}(s) = s^{-1}f_{j,t}(s) \) are smooth by (i), and so we consider \( u_t = \alpha_{1,t} - \alpha_{2,t} \) which, form (iii), we can assume to be initially positive and \( u_t(\delta) = u_t(-\delta) > 0 \) for all \( t \leq T_1 \). It is enough to show that \( u_t \) is positive for all \( t \leq T_1 \). We have at once that

\[
\frac{df_{j,t}}{dt} = (\arctan(\alpha_{j,t}))' + \frac{f_{j,t}''}{1 + (f_{j,t}')^2}.
\]

\[ \Longrightarrow \frac{d\alpha_{j,t}}{dt} = \frac{\alpha_{j,t}''}{1 + (\alpha_{j,t}' + \alpha_{j,t})^2} + \frac{\alpha_{j,t}'}{s} \frac{1}{1 + \alpha_{j,t}^2} + \frac{\alpha_{j,t}'}{s} \frac{2}{1 + (s\alpha_{j,t}' + \alpha_{j,t})^2}. \]

The functions \( s^{-1}\alpha_{j,t}' \) are smooth for all \( s \), and so we obtain

\[
\frac{du_t}{dt} = \frac{u_t''}{1 + C_1^2} + C_2u_t' + C_3u_t + \frac{u_t'}{s}C_4^2,
\]

where \( C_k \) are smooth time dependent bounded functions for \( k = 1, \ldots, 4 \).

Suppose \( T_1 \) is the first time at which \( u_t \) becomes zero, and consider \( v_t = u_te^{-Ct} + \varepsilon(t - T_1) \) with \( \varepsilon \) small and \( C \) large. The function \( v_t \) becomes zero for a first time \( t \leq T_1 \) at some point \( s_0 \) for all small positive \( \varepsilon \). At that time we have \( u_t'(s_0) \geq 0 \), \( u_t'(s_0) = 0 \), and thus, with an obvious abuse of notation,

\[
0 \geq \frac{dv_t}{dt}(s_0) = \varepsilon + \frac{d}{dt}(u_te^{-Ct})(s_0) \geq \varepsilon + \frac{u_t'(s_0)}{s_0}C_4^2e^{-Ct}. \]
If $s_0$ is not zero, the last term on the right is zero. If $s_0$ is zero, then the last term on the right is $u''_t(0)C_4^2e^{-Ct}$, which is nonnegative. In any case we get a contradiction. □

References


(Received: September 8, 2010)
(Revised: March 30, 2012)

Imperial College London, London, UK
E-mail: aneves@imperial.ac.uk