

# Recurrence of planar graph limits

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## Abstract

We prove that any distributional limit of finite planar graphs in which the degree of the root has an exponential tail is almost surely recurrent. As a corollary, we obtain that the uniform infinite planar triangulation and quadrangulation (UIPT and UIPQ) are almost surely recurrent, resolving a conjecture of Angel, Benjamini and Schramm.

We also settle another related problem of Benjamini and Schramm. We show that in any bounded degree, finite planar graph the probability that the simple random walk started at a uniform random vertex avoids its initial location for  $T$  steps is at most  $\frac{C}{\log T}$ .

## 1. Introduction

A *distributional limit* of finite graphs  $G_n$  is a random rooted infinite graph  $(U, \rho)$  with the property that neighborhoods of  $G_n$  around a random vertex converge in distribution to neighborhoods of  $U$  around  $\rho$ ; see precise definitions below. This limit was defined by Benjamini and Schramm [9]. Their motivation was the study of infinite random planar maps, a widely studied model in the probability, combinatorics and statistical physics communities for generic two-dimensional geometries and quantum gravity. (See [21], [3], [15], [5] and the references within.) The canonical example of such a limit is Angel and Schramm's [5] *uniform infinite planar triangulation* (UIPT) and is obtained by taking the distributional limit of a uniform random triangulation on  $n$  vertices. Here a triangulation is a simple planar graph in which every face has three edges.

These authors conjectured that the UIPT is almost surely recurrent (see [5, Conj. 1.12] and [9, p. 3]). It is shown in [9] that a distributional limit of uniformly bounded degree finite planar graphs is almost surely recurrent. However, the degrees of random planar maps and the UIPT are unbounded so one cannot appeal to this result. In this paper we prove this conjecture.

**THEOREM 1.1.** *Let  $(U, \rho)$  be a distributional limit of planar graphs such that the degree of  $\rho$  has an exponential tail. Then  $U$  is almost surely recurrent.*

COROLLARY 1.2. *The UIPT is almost surely recurrent.*

The UIPT's recurrence conjecture has had strong circumstantial evidence supporting it. One such evidence is the recent result of Gill and Rohde [17] asserting that the natural Riemann surface associated with the UIPT (obtained by gluing equilateral triangles together according to the combinatorics of the graph) is almost surely parabolic; that is, Brownian motion on this surface is recurrent. Another such evidence, found by Benjamini and Curien [6], is that the UIPT is Liouville; that is, every bounded harmonic function on it is constant. Every recurrent graph is Liouville, and when  $G$  is a bounded degree planar graph the Liouville property implies recurrence [8]. (The bounded degree condition in the last statement is necessary.)

A popular variation of the UIPT is the *uniform infinite planar quadrangulation* (UIPQ) and is defined similarly with the role of triangulations replaced by quadrangulations. This model was constructed by Krikun [18] (see also [14]) and has received special attention since it is appealing to study it using bijections with random labeled trees.

COROLLARY 1.3. *The UIPQ is almost surely recurrent.*

Our approach allows us to answer another problem posed by Benjamini and Schramm (Problem 1.3 in [9]). Let  $G$  be a finite graph, and consider the simple random walk on it,  $(X_t)_{t \geq 0}$ , where  $X_0$  is a uniform random vertex of  $G$ . Let  $\phi(T, G)$  be the probability that  $X_t \neq X_0$  for all  $t = 1, \dots, T$ . For any  $D \geq 1$ , define

$$\phi_D(T) = \sup \{ \phi(T, G) : G \text{ is planar with degrees bounded by } D \}.$$

The almost sure recurrence of a distributional limit of planar graphs of bounded degree (the main result of [9]) is equivalent to  $\phi_D(T) \rightarrow 0$  as  $T \rightarrow \infty$  for any fixed  $D$ . In [9] it is asked "what is the rate of decay of this function?" The probability of avoiding the starting point for  $T$  steps on  $\mathbb{Z}^2$  is of order  $\frac{1}{\log T}$ , so this lower bounds  $\phi_D(T)$  since  $\mathbb{Z}^2$  is a distributional limit of finite planar graphs. Here we provide a matching upper bound.

THEOREM 1.4. *For any  $D \geq 1$ , there exists  $C < \infty$  such that for any  $T \geq 2$ ,*

$$\phi_D(T) \leq \frac{C}{\log T}.$$

Our results are related to two active research areas: graph limits and random planar maps. Let us briefly expand on each in order to introduce some definitions and background.

1.1. *Distributional graph limits.* The notion of the distributional limit of a sequence of graphs was introduced by Benjamini and Schramm [9]. With

slightly different generality this was studied by Aldous and Steele [2] under the name “local weak limit” and by Aldous and Lyons [1] under the name “random weak limit.” This limiting procedure is best suited for graphs with bounded average degree and is hence natural in the setting of finite planar graphs. Convergence of sequences of dense graphs requires quite a different treatment (see [22], [11]) though some interesting connections between the two are emerging (see [10]).

A rooted graph is a pair  $(G, \rho)$ , where  $G$  is a graph and  $\rho$  is a vertex of  $G$ . For any integer  $r \geq 0$ , we write  $B_G(\rho, r)$  for the ball around  $\rho$  of radius  $r$  in the graph distance. The space of rooted graphs is endowed with a natural metric: the distance between  $(G, \rho)$  and  $(G', \rho')$  is  $\frac{1}{\alpha+1}$ , where  $\alpha$  is the supremum over all  $r$  such that  $B_G(\rho, r)$  and  $B_{G'}(\rho', r)$  are isomorphic as rooted graphs. Let  $G_n$  be a sequence of finite graphs, and let  $\rho_n$  be a random vertex of  $G_n$  drawn according to some probability measure on the vertices of  $G_n$ . We say that  $(G_n, \rho_n)$  has distributional limit  $(U, \rho)$ , where  $(U, \rho)$  is a random rooted graph, if for every fixed  $r > 0$ , the random variable  $B_{G_n}(\rho_n, r)$  converges in distribution to  $B_U(\rho, r)$ .

It makes sense to choose the random root according to the stationary distribution in  $G_n$  (in which the probability of choosing a vertex is proportional to its degree) because then the resulting limit  $(U, \rho)$  is invariant under the random walk measure; that is,  $(U, \rho)$  has the same distribution as  $(U, X_1)$ , where  $X_1$  is a uniform random neighbor of  $\rho$ . It is also common to choose the root according to the uniform distribution on the vertices of  $G_n$ . This may lead to a different distributional limit. However, in our setting this does not matter, as we now explain. Let  $\rho_n^\pi$  and  $\rho_n^u$  be random roots of  $G_n$  drawn according to the stationary and uniform distributions, respectively. If the average degree of  $G_n$  is bounded by some number  $D$  and  $G_n$  has no isolated vertices, then it is immediate that for any event  $A$  on rooted graphs, we have  $\mathbf{P}((G_n, \rho_n^u) \in A) \leq D\mathbf{P}((G_n, \rho_n^\pi) \in A)$ . Hence, the distributional limit of  $(G_n, \rho_n^u)$  is absolutely continuous with respect to the limit of  $(G_n, \rho_n^\pi)$ . In fact, an appeal to Hölder’s inequality shows that if the degree distribution of  $G_n$  has a bounded  $(1 + \varepsilon)$ -moment, then the two limits are mutually absolutely continuous with respect to each other. (However, we do not use this fact in this paper.)

1.2. *Random planar maps.* Random planar maps is a widely studied topic at the intersection of probability, combinatorics and statistical physics. We give here a very brief account of this topic and refer the interested reader to [5], [21] and the many references within. The enumerative study of planar maps was initiated by Tutte [30] who counted the number of planar graphs of a given size of various classes, including triangulations. Cori and Vauquelin [13],

Schaeffer [28] and Chassaing and Schaeffer [12] provided robust bijections between planar maps and labeled trees — the specifics of these bijections change depending on the class of the planar maps considered, and many variations and extensions are known. The common theme to all of these is that random planar maps can be constructed from random labeled trees. This approach has shed a new light on the asymptotic geometry of random maps and spurred a new line of research: limits of large planar random maps. There are two natural notions of limits of random planar maps: the scaling limit and the aforementioned distributional limit.

In the study of scaling limits of random planar maps, one considers the random finite map  $T_n$  on  $n$  vertices as a random metric space induced by the graph distance, scales the distances properly (it turns out that  $n^{-1/4}$  is the correct scaling) and studies its limit in the Gromov-Hausdorff sense. The existence of such a limit was first suggested by Chassaing and Schaefer [12], Le Gall [19], and Marckert and Mokkadem [24], who named it the *Brownian map*. The challenges involved in this line of research are substantial. Existence and uniqueness of the limit are the first step, but even more challenging is the issue of universality; that is, that random planar maps of different classes exhibit the same limit, up to parametrization. For the case of random  $p$ -angulations, this research has recently culminated in the work of Le Gall [20], who established this for  $p = 3$  and all even  $p$ , and independently by Miermont [25] for the case  $p = 4$ . It remains open to prove this for all  $p$ .

The study of distributional limits, while bearing some similarities, is independent of the scaling limit questions. Let  $G_n$  be a random planar triangulation and  $\rho_n$  a random vertex chosen uniformly (or according to the stationary measure, as mentioned above). Angel and Schramm [5] showed that a distributional limit exists and that it is a one-ended infinite triangulation of the plane almost surely. They termed this limit as the *uniform infinite planar triangulation* (UIPT). The uniform infinite planar quadrangulation (UIPQ) was later constructed by Krikun [18].

The research in this area is focused on almost sure geometric properties of this limiting geometry. It is an interesting geometry and the comparison of it with the usual Euclidean geometry is especially striking. It is invariant, planar and polynomially growing, yet very fractal. Angel [4] showed that a ball of radius  $r$  has volume  $r^{4+o(1)}$  and the boundary component, separating this ball from infinity, has size  $r^{2+o(1)}$  (see also [12]). This suggests that the random walk on the UIPQ/UIPQ should be subdiffusive; that is, that the typical distance of the random walk from the origin after  $t$  steps is  $t^{\beta+o(1)}$  for some  $\beta \in [0, 1/2)$ . Benjamini and Curien [7] show that  $\beta \leq 1/3$  in the UIPQ; however, it is believed that the correct value is  $\beta = 1/4$ .

1.3. *Sharpness.* [Theorem 1.1](#) is sharp in the following sense. For any  $\alpha \in (0, 1)$ , there exists a distributional limit of planar graphs  $(U, \rho)$  such that  $\mathbf{P}(\deg(\rho) \geq k) \leq Ce^{-ck^\alpha}$  for some  $C, c$  that is transient almost surely. Indeed, let  $T_h$  be a binary tree of height  $h$  and replace each edge at height  $k = 1, \dots, h$  from the leaves by a disjoint union of  $k^{1/\alpha}$  paths of length 2 (or parallel edges). In the distributional limit of  $T_h$  as  $h \rightarrow \infty$ , almost surely, the effective resistance from the root to infinity is at most  $2 \sum_{k=1}^\infty k^{-1/\alpha} < \infty$ ; hence, it is transient. Furthermore, the probability that the degree of a uniformly chosen vertex of  $T_h$  is at least  $k$  can easily be computed to be of order  $e^{-ck^\alpha}$ .

## 2. Preliminaries on circle packing and electric networks

2.1. *Circle packing.* Our proof relies, as in [9], on the theory of circle packing, which we briefly describe below. We refer the reader to [29] and [27] for further information on this fascinating topic. A circle packing is a collection of circles in the plane with disjoint interiors. The tangency graph of a circle packing is a planar graph  $G = (V, E)$  in which the vertex set  $V$  is the set of circles and two circles are neighbors if they are tangent in the packing. The degree of a circle in the packing is its degree in the tangency graph. See [Figure 1](#). The Koebe-Andreev-Thurston Circle Packing Theorem (see [29]) asserts that for any finite planar graph  $G = (V, E)$ , there exists a circle packing in the plane that has a tangency graph isomorphic to  $G$ . Furthermore, if  $G$  is a triangulation, then this packing is unique up to Möbius transformations of the plane and reflections along lines. We will frequently use a simple but important fact known as the Ring Lemma [26]. If a circle  $C$  is completely surrounded by  $D$  other circles  $C_0, \dots, C_{D-1}$  (that is,  $C_i$  is tangent to  $C_{i+1 \bmod D}$  and to  $C$ ), then the ratio  $r/r_i$  between the radius of  $C$  and  $C_i$  is bounded above by a constant depending only on  $D$ . Thus, in a circle packing of a bounded degree triangulation (every inner circle is completely surrounded), the ratio of radii of

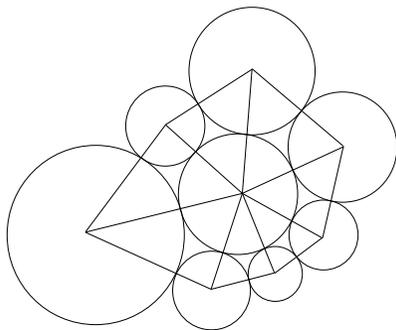


Figure 1. A circle packing and its tangency graph.

every two tangent circles is bounded above and below by a constant depending only on  $D$ , with the possible exception of the three boundary circles. The Circle Packing Theorem and the Ring Lemma are the only facts about circle packing that we will use in this paper.

*2.2. Electric networks.* We use some classical facts about electric networks and their connections to random walks; we refer the reader to [23] for further information. Let  $G = (V, E)$  be a finite graph with nonnegative edge weights  $\{c_e\}_{e \in E}$ . We call these weights conductances, and their inverses,  $R_e = c_e^{-1}$ , are called resistances. (By convention,  $0^{-1} = \infty$ .) For any two vertices  $a \neq z$ , define the effective resistance  $R_{\text{eff}}(a \leftrightarrow z; \{R_e\})$  between  $a$  and  $z$  as the minimum energy  $\mathcal{E}(\theta) = \sum_{e \in E} R_e [\theta(e)]^2$  of any unit flow  $\theta$  from  $a$  to  $z$ . The unit flow attaining this minimum is called the unit current flow. We often write  $R_{\text{eff}}(a \leftrightarrow z)$  when all the conductances are 1.

Given two disjoint sets of vertices  $A$  and  $Z$ , the effective resistance

$$R_{\text{eff}}(A \leftrightarrow Z; \{R_e\})$$

between  $A$  and  $Z$  is the effective resistance between the two corresponding vertices in the graph obtained from  $G$  by contracting the sets  $A$  and  $Z$  into single vertices and retaining the same resistances on the remaining edges. For convenience, if either  $A$  or  $Z$  are empty sets, then  $R_{\text{eff}}(A \leftrightarrow Z; \{R_e\}) = \infty$ . Now we may define effective resistances on infinite graphs — in this case we will only compute effective resistances between disjoint sets  $A$  and  $Z$  such that  $V \setminus (A \cup Z)$  is finite. A typical example is the effective resistance between a chosen vertex  $\rho$  and the complement of a finite set containing  $\rho$ . When  $G$  is infinite, we define the effective resistance from  $a$  to  $\infty$  as

$$R_{\text{eff}}(a \leftrightarrow \infty; \{R_e\}) = \lim_{n \rightarrow \infty} R_{\text{eff}}(a \leftrightarrow G \setminus B_n; \{R_e\}),$$

where  $\{B_n\}$  is any sequence of finite vertex sets that exhaust  $G$ . (The limit does not depend on the choice of exhausting sequence.)

For a function  $g : V \rightarrow \mathbb{R}$ , the Dirichlet energy is defined as

$$\mathcal{E}(g) = \sum_{e=(x,y) \in E} c_e [g(x) - g(y)]^2.$$

We will use the dual definition of effective resistance; that is, the discrete Dirichlet principle (see Exercise 2.13 of [23]) stating that

$$(2.1) \quad \frac{1}{R_{\text{eff}}(A \leftrightarrow Z; \{R_e\})} = \min \left\{ \mathcal{E}(g) : g : V \rightarrow \mathbb{R}, g|_A = 0, g|_Z = 1 \right\}.$$

Consider the network random walk  $(X_n)_{n \geq 0}$  on  $G$  with transition probabilities  $p(x, y) = c_{(x,y)} [\sum_{y': (x,y') \in E} c_{(x,y')}]^{-1}$ , and write  $\mathbf{P}_x$  for the probability measure of a network random walk started at  $X_0 = x$ . Write  $\tau$  for the stopping time  $\tau = \min\{n \geq 1 : X_n \in \{a, z\}\}$ . It is classical (stemming from the fact that

the minimizer of (2.1) is the unique harmonic function with the corresponding boundary values; see [23]) that

$$(2.2) \quad R_{\text{eff}}(a \leftrightarrow z; \{R_e\}) = \frac{1}{\mathbf{P}_a(X_\tau = z) \sum_{y:(a,y) \in E} c_{(a,y)}}.$$

This gives a useful electrical interpretation of recurrence. An infinite network  $(G; \{R_e\})$  is recurrent if and only if  $R_{\text{eff}}(a \leftrightarrow \infty; \{R_e\}) = \infty$ . It is not too hard to see that this implies the following two useful criteria for recurrence/transience. First, an infinite graph is  $G$  is recurrent if and only if for some vertex  $a$  there exists  $c > 0$  such that for any integer  $m \geq 0$ , there exists a finite vertex set  $B$  such that

$$(2.3) \quad R_{\text{eff}}(B_G(a, m) \leftrightarrow G \setminus B; \{R_e\}) \geq c;$$

see [23, Lemma 9.22]. Secondly, a network is transient if and only if there exists a unit flow from some vertex  $a$  to  $\infty$  with finite energy.

Another classical connection between random walks and effective resistances is known as the commute time identity [23], stating that

$$(2.4) \quad \mathbb{E}_a \tau_z + \mathbb{E}_z \tau_a = 2R_{\text{eff}}(a \leftrightarrow z) \sum_{e \in E} c_e,$$

where  $\tau_v$  is the hitting time of  $v$  and  $\mathbb{E}_x$  is the corresponding expectation operator of  $\mathbf{P}_x$ . We will also use the following bound, which is an immediate consequence of (2.2). Given a finite network and three vertices  $x, y, z$ , we have

$$(2.5) \quad \frac{1}{R_{\text{eff}}(x \leftrightarrow \{y, z\})} \leq \frac{1}{R_{\text{eff}}(x \leftrightarrow y)} + \frac{1}{R_{\text{eff}}(x \leftrightarrow z)}.$$

Finally, we will use the following easy bound.

LEMMA 2.1. *Let  $G = (V, E)$  be a finite network with resistances  $\{R_e\}$  and two vertices  $a$  and  $z$ . Let  $A \subset V$  such that  $a \in A$  and  $z \notin A$ , and define  $R_e^A = R_e$  for each edge  $e$  that has both endpoints in  $A$  and  $R_e^A = \infty$  otherwise. Then*

$$R_{\text{eff}}(a \leftrightarrow z; \{R_e\}) \leq R_{\text{eff}}(A \leftrightarrow z; \{R_e\}) + \max_{v \in A} R_{\text{eff}}(a \leftrightarrow v; \{R_e^A\}).$$

*Proof.* Assume without loss of generality that  $\max_{v \in A} R_{\text{eff}}(a \leftrightarrow v; \{R_e^A\}) < \infty$ . Consider the unit current flow  $\theta^A$  from  $A$  to  $z$  in  $(G, R_e)$ , and for each  $v \in A \setminus \{a\}$ , let  $\theta^{v,A}$  be the unit current flow in  $(G, \{R_e^A\})$  from  $a$  to  $v$ . For each  $v \in A$ , write

$$\alpha_v = \sum_{u:u \notin A, u \sim v} \theta^A(v, u).$$

Since  $\theta^A$  is the unit current flow, we have that  $\sum_{v \in A} \alpha_v = 1$  and  $\alpha_v \geq 0$  for all  $v \in A$ . We define a new flow  $\theta$  from  $a$  to  $z$  in  $(G, R_e)$  by setting  $\theta(e) = \theta^A(e)$

for any edge  $e$  that has at least one endpoint not in  $A$ . If  $e$  has both endpoints in  $A$ , we set  $\theta(e) = \sum_{v \in A \setminus \{a\}} \alpha_v \theta^{v,A}(e)$ . This defines a unit flow from  $a$  to  $z$ . The contribution to the energy of  $\theta$  coming from edges having at least one endpoint not in  $A$  is the energy of  $\theta^A$  that equals  $R_{\text{eff}}(A \leftrightarrow z; \{R_e\})$ , and the contribution coming from edges with two endpoints in  $A$  is at most

$$\begin{aligned} \sum_e R_e \left[ \sum_{v \in A \setminus \{a\}} \alpha_v \theta^{v,A}(e) \right]^2 &\leq \sum_{v \in A \setminus \{a\}} \alpha_v \sum_e R_e [\theta^{v,A}(e)]^2 \\ &= \sum_{v \in A \setminus \{a\}} \alpha_v R_{\text{eff}}(a \leftrightarrow v; \{R_e^A\}) \\ &\leq \max_{v \in A} R_{\text{eff}}(a \leftrightarrow v; \{R_e^A\}), \end{aligned}$$

where the first inequality is by Jensen’s inequality. □

### 3. Distributional limits of bounded degree graphs and circle packing

Let  $G_n$  be a sequence of finite planar graphs of bounded degree, and assume that it has distributional limit  $(U, \rho)$ . The main result of [9] is that  $U$  is almost surely recurrent. Our goal in this section is to prove the following theorem, providing a quantitative bound on the growth of the resistance.

**THEOREM 3.1.** *Let  $(U, \rho)$  be the distributional limit of finite planar graphs of bounded degree. Then  $(U, \rho)$  almost surely satisfies the following. There exists  $c > 0$  such that for any  $k \geq 0$ , there exist a finite set  $B_k \subset U$  with  $|B_k| \leq c^{-1}k$  and*

$$R_{\text{eff}}(\rho \leftrightarrow U \setminus B_k) \geq c \log k.$$

We begin with some basic estimates relating circle packing and resistances.

**3.1. Circle packing and resistance.** Given a circle packing  $P = \{C_v : v \in G\}$  of a planar graph  $G = (V, E)$  and a domain  $D \subset \mathbb{R}^2$ , we write  $V_D \subset V$  for the set of vertices such that their corresponding circles have centers in  $D$ . We also write  $B_{\text{euc}}(p, r)$  for the Euclidean ball of radius  $r$  around  $p$ .

**LEMMA 3.2.** *Let  $P = \{C_v : v \in G\}$  be a circle packing of a finite planar graph  $G = (V, E)$  such that the ratio of radii of two tangent circles is bounded by  $K$ . Then for any  $\alpha > 1$ , there exists  $c = c(K, \alpha) > 0$  such that for all  $r > 0$  and all  $p \in \mathbb{R}^2$ ,*

$$R_{\text{eff}}\left(V_{B_{\text{euc}}(p,r)} \leftrightarrow V_{\mathbb{R}^2 \setminus B_{\text{euc}}(p,\alpha r)}\right) \geq c,$$

*provided that both sets  $V_{B_{\text{euc}}(p,r)}$  and  $V_{\mathbb{R}^2 \setminus B_{\text{euc}}(p,\alpha r)}$  are nonempty.*

*Proof.* In the case where  $|V_{B_{\text{euc}}(p,r)}| = 1$ , the resistance is at least the inverse of the degree of the vertex. Since the ratio between the radii of tangent circles is bounded, the degrees of  $G$  are bounded by some  $D = D(K) < \infty$ ,

and so the resistance is at least  $D^{-1}$  in this case. Thus, let us assume that  $|V_{B_{\text{euc}}(p,r)}| > 1$ . Define a function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  by

$$f(x) = \begin{cases} 0 & \text{if } x \in B_{\text{euc}}(p, r), \\ \frac{\|x-p\|-r}{(\alpha-1)r} & \text{if } x \in B_{\text{euc}}(p, \alpha r) \setminus B_{\text{euc}}(p, r), \\ 1 & \text{if } x \in \mathbb{R}^2 \setminus B_{\text{euc}}(p, \alpha r). \end{cases}$$

Note that  $f$  is  $((\alpha - 1)r)^{-1}$ -Lipschitz. We define  $g : V \rightarrow \mathbb{R}$  by setting  $g(v) = f(\rho_v)$ , where  $\rho_v$  is the center of  $C_v$ , and we bound its Dirichlet energy. For every edge  $(u, v)$ ,

$$\|g(u) - g(v)\| \leq \frac{\|\rho_u - \rho_v\|}{(\alpha - 1)r} = \frac{(r_u + r_v)}{(\alpha - 1)r} \leq \frac{(K + 1)r_u}{(\alpha - 1)r}.$$

Edges for which both  $\rho_u$  and  $\rho_v$  are not in  $B_{\text{euc}}(p, \alpha r)$  contribute 0 to the energy. Since  $|V_{B_{\text{euc}}(p,r)}| > 1$ , for every edge  $(u, v)$  that has one of  $\rho_u$  or  $\rho_v$  in  $B_{\text{euc}}(p, \alpha r)$  the circles  $C_u$  and  $C_v$  are both contained in  $B_{\text{euc}}(p, K_2 r)$  for some  $K_2(K, \alpha) < \infty$ . Since the interiors of the circles  $\{C_v\}_{v \in V}$  are disjoint, the contribution to the energy is at most

$$\sum_{(u,v) \in E} \|g(u) - g(v)\|^2 \leq \frac{D \cdot (K + 1)^2 \cdot \text{Area}[B_{\text{euc}}(p, K_2 r)]}{((\alpha - 1)r)^2} \leq K_3,$$

where  $K_3 = K_3(K, \alpha) < \infty$ , concluding our proof by (2.1). □

**COROLLARY 3.3.** *Let  $P$  be a finite circle packing in  $\mathbb{R}^2$  such that the ratio of radii of two tangent circles is bounded by  $K$  and such that there exists a circle in  $P$  entirely contained in  $B_{\text{euc}}(0, 1)$ . Then there exists a constant  $c = c(K) > 0$  such that for all radii  $r \geq 2$ , we have*

$$R_{\text{eff}}(V_{B_{\text{euc}}(0,1)} \leftrightarrow V_{\mathbb{R}^2 \setminus B_{\text{euc}}(0,r)}) \geq c \log r,$$

provided that  $V_{\mathbb{R}^2 \setminus B_{\text{euc}}(0,r)}$  is nonempty.

*Proof.* Since there is a circle entirely contained in  $B_{\text{euc}}(0, 1)$ , using the bounded ratio assumption, we get that there exists  $C = C(K) \in (1, \infty)$  such that for all  $r' \geq 1$ , there is no edge between  $V_{B_{\text{euc}}(0,r')}$  and  $V_{\mathbb{R}^2 \setminus B_{\text{euc}}(0,Cr')}$ . Assume that  $r \geq C$ , and consider the  $k$  disjoint annuli  $A_1, \dots, A_k$  where  $k = \lfloor \log_C r \rfloor$  and  $A_j = B_{\text{euc}}(0, C^j) \setminus B_{\text{euc}}(0, C^{j-1})$  so that  $A_j$  is contained in  $B_{\text{euc}}(0, r) \setminus B_{\text{euc}}(0, 1)$  for all  $j$ . There are no edges in  $G$  between  $V_{A_j}$  and  $V_{A_\ell}$  for  $1 \leq j \leq \ell - 2 \leq k - 2$ ; hence, each  $V_{A_j}$  is a cut-set separating  $V_{A_{j-1}}$  from  $V_{A_{j+1}}$ . By Lemma 3.2, we have  $R_{\text{eff}}(V_{A_j} \leftrightarrow V_{A_{j+2}}) \geq c$  for some  $c = c(K) > 0$ . Summing these resistances using the series law (see [23]) yields that  $R_{\text{eff}}(V_{A_0} \leftrightarrow V_{A_k}) \geq ck/2$ . Finally, if  $2 \leq r \leq C$ , then the resistance is bounded below by another constant using Lemma 3.2. □

**3.2. Finite planar graphs.** We recall the key lemma of [9]. Let  $\mathcal{C} \subset \mathbb{R}^2$  be a finite set of points. For each  $w \in \mathcal{C}$ , we write  $\rho_w$  for its *isolation radius*; that is,  $\rho_w = \inf\{|v - w| : v \in \mathcal{C} \setminus \{w\}\}$ . Given  $\delta > 0, s > 0$  and  $w \in \mathcal{C}$ , we say that  $w$  is  $(\delta, s)$ -supported if in the disk of radius  $\delta^{-1}\rho_w$  around  $w$  there are at least  $s$  points of  $\mathcal{C}$  outside of any disk of radius  $\delta\rho_w$ ; in other words, if

$$\inf_{p \in \mathbb{R}^2} \left| \mathcal{C} \cap B(w, \delta^{-1}\rho_w) \setminus B(p, \delta\rho_w) \right| \geq s.$$

Benjamini and Schramm [9, Lemma 2.3] prove that for all  $\delta \in (0, 1)$ , there is a constant  $c = c(\delta)$  such that for every finite  $\mathcal{C} \subset \mathbb{R}^2$  and every  $s \geq 2$ , the set of  $(\delta, s)$ -supported points in  $\mathcal{C}$  has cardinality at most  $c|\mathcal{C}|/s$ . In the following we bound  $c(\delta)$ .

**LEMMA 3.4.** *There exists a universal constant  $A > 0$  such that for all  $\delta \in (0, 1/2)$  and  $s \geq 2$  and any finite set  $\mathcal{C} \subset \mathbb{R}^2$ , the number of  $(\delta, s)$ -supported points in  $\mathcal{C}$  is at most  $\frac{A|\mathcal{C}|\delta^{-2} \log(\delta^{-1})}{s}$ .*

*Proof.* To understand the proof of this one must first read [9, Lemma 2.3]. Our lemma is a straightforward calculation of the constants appearing in the last paragraph of the proof of [9, Lemma 2.3]. Indeed, in the notation of [9],  $c = c(\delta)$  is  $c = 2c_0^{-1}c_1$ . We estimate these constants below. Given  $\delta$ ,  $c_1(\delta)$  is the number of cities in any square  $S$  and is at most  $A_0\delta^{-2}$  for some universal  $A_0 < \infty$ . The probability  $c_0(\delta)$  that there exists a square that has edge length in the range  $[4\delta^{-1}\rho_w, 5\delta^{-1}\rho_w]$  is readily seen to be at least  $A_1 \log^{-1}(\delta)$  for some universal  $A_1 > 0$ .  $\square$

**COROLLARY 3.5.** *Let  $G$  be a finite planar triangulation, and let  $P = \{C_v : v \in G\}$  be an arbitrary circle packing of  $G$ . Let  $\rho$  be a random uniform vertex of  $G$ , and let  $\widehat{P} = \{\widehat{C}_v : v \in G\}$  be the circle packing obtained from  $P$  by translating and dilating so that  $\widehat{C}_\rho$  has radius 1 and is centered around the origin. Then there exists a universal constant  $A > 0$  such that for any  $r \geq 2$  and any  $s \geq 2$ ,*

$$\mathbf{P}(\forall p \in \mathbb{R}^2 \quad |V_{B_{\text{Euc}}(0,r)} \setminus B_{\text{Euc}}(p,r^{-1})| \geq s) \leq \frac{Ar^2 \log r}{s}.$$

*Proof.* Apply Lemma 3.4 with  $\mathcal{C}$  being the set of centers of  $P$  and  $\delta = r^{-1}$  and  $s$ . We deduce that the number of centers of  $P$  that are  $(r^{-1}, s)$ -supported is at most  $As^{-1}|G|r^2 \log r$ . Since  $\widehat{P}$  is a triangulation, any circle  $C_w$  with radius  $r_w$  not in the boundary (the boundary has three circles, which contributes a negligible  $3|G|^{-1}$  to the probability) has  $r_w \leq \rho_w \leq Cr_w$  for some universal constant  $C > 0$ , concluding the proof.  $\square$

**LEMMA 3.6.** *Let  $G = (V, E)$  be a finite planar graph with degrees at most  $D$ , and let  $\rho$  be a random uniform vertex. Then there exists  $c = c(D) > 0$  such*

that for all  $k \geq 1$ ,

$$\mathbf{P}\left(\exists B \subset V \text{ with } |B| \leq c^{-1}k \text{ and } R_{\text{eff}}(\rho \leftrightarrow V \setminus B) \geq c \log k\right) \geq 1 - c^{-1}k^{-1/3} \log k,$$

where we interpret  $R_{\text{eff}}(\rho \leftrightarrow V \setminus B) = \infty$  when  $B = V$ .

*Proof.* Without loss of generality it is enough to prove this for  $k$  large enough. Assume first that  $G$  is a triangulation, and consider the circle packing  $\widehat{P}$  from Corollary 3.5. Apply this corollary with  $r = k^{1/3}$  and  $s = k$ . We get that with probability at least  $1 - Ak^{-1/3} \log k$  there exists  $p \in \mathbb{R}^2$  such that  $|V_{B_{\text{euc}}(0,r) \setminus B_{\text{euc}}(p,r^{-1})}| \leq k$ . We proceed by analyzing two cases. If  $|V_{B_{\text{euc}}(p,r^{-1})}| \leq 1$ , then we set  $B = V_{B_{\text{euc}}(0,r)}$  so that  $|B| \leq k + 1$ . In this scenario, if  $V_{\mathbb{R}^2 \setminus B_{\text{euc}}(0,r)} = \emptyset$ , then  $B = V$  and the assertion holds trivially. If  $V_{\mathbb{R}^2 \setminus B_{\text{euc}}(0,r)} \neq \emptyset$ , then by Corollary 3.3, we have

$$R_{\text{eff}}(\rho \leftrightarrow U \setminus B) \geq c \log k,$$

where  $c = c(D) > 0$ .

In the case where  $|V_{B_{\text{euc}}(p,r^{-1})}| \geq 2$ , we take  $B = V_{B_{\text{euc}}(0,r) \setminus B_{\text{euc}}(p,r^{-1})}$ . Since  $G$  is a triangulation and  $C_\rho$  is a circle at the origin of radius 1, by the Ring Lemma we deduce that there exists some  $c' = c'(D) > 0$  such that the center of any circle other than  $C_\rho$  is of distance at least  $1 + c'$  from the origin. Hence, when  $r$  is large enough we must have that  $\|p\| \geq 1 + c'/2$ . Clearly, one of the circles with centers in  $B_{\text{euc}}(p, r^{-1})$  must have radius at most  $r^{-1}$ . Hence,  $B_{\text{euc}}(p, 2r^{-1})$  entirely contains a circle, and so we may scale and dilate so that Corollary 3.3 gives

$$R_{\text{eff}}(V_{B_{\text{euc}}(p,2r^{-1})} \leftrightarrow V_{\mathbb{R}^2 \setminus B_{\text{euc}}(p,c'/2)}) \geq c \log r;$$

therefore,

$$(3.1) \quad R_{\text{eff}}(\rho \leftrightarrow V_{B_{\text{euc}}(p,2r^{-1})}) \geq c \log r.$$

Also, by Corollary 3.3, we have

$$(3.2) \quad R_{\text{eff}}(\rho \leftrightarrow V_{\mathbb{R}^2 \setminus B_{\text{euc}}(0,r)}) \geq c \log r.$$

Inequalities (3.1) and (3.2) together with (2.5) conclude the proof when  $G$  is a triangulation.

If  $G$  is not a triangulation, then we add edges and vertices to extend  $G$  into a finite planar triangulation  $T$  in the zigzag fashion as in [9, Proof of Theorem 1.1]. After this procedure the maximal degree and the number of vertices have multiplied by at most a universal constant  $K$ . Let  $\rho_T$  be a uniform random vertex of  $T$ , by the proof in the case of triangulations, with probability at least  $1 - Ak^{-1/3} \log k$  there exists  $B' \subset T$  with  $|B'| \leq c^{-1}k$  and  $R_{\text{eff}}(\rho_T \leftrightarrow U \setminus B') \geq c \log k$ . We take  $B = B' \cap V(G)$ . Obviously  $|B| \leq c^{-1}k$ ,

and by Rayleigh's monotonicity the effective resistance only grew. Lastly,  $\mathbf{P}(\rho_T \in V(G)) \geq 1/K$ , so by incorporating  $K$  into the constant  $A$ , we conclude the proof.  $\square$

*Proof of Theorem 1.4.* Let  $G = (V, E)$  be a finite planar graph with degree bounded by  $D$ , and let  $T \geq 2$  be an arbitrary integer. Assume without loss of generality that  $G$  is connected. Let  $X_0$  be a uniform random vertex. Apply Lemma 3.6 with  $k = T^{1/3}$  so that with probability at least  $1 - c^{-1}T^{-1/9} \log T$  there exists  $B \subset V$  with  $|B| \leq c^{-1}T^{1/3}$  and

$$R_{\text{eff}}(X_0 \leftrightarrow V \setminus B) \geq c \log T.$$

If this event occurred, by the commute time identity (2.4) and the fact that  $R_{\text{eff}}(a \leftrightarrow z)$  is at most the graph distance between  $a$  and  $z$ , we have that

$$\mathbb{E}_{X_0} \tau_{V \setminus B} \leq 2D|B|R_{\text{eff}}(X_0 \leftrightarrow V \setminus B) \leq 2Dc^{-2}T^{2/3},$$

where  $\tau_{V \setminus B}$  is the hitting time of the random walk at  $V \setminus B$ . If  $\tau_{X_0} \geq T$ , then either  $\tau_{V \setminus B} \geq T$  or  $\tau_{X_0} > \tau_{V \setminus B}$ . Hence, by Markov's inequality and (2.2),

$$\mathbf{P}_{X_0}(\tau_{X_0} \geq T) \leq \frac{2Dc^{-2}T^{2/3}}{T} + \frac{1}{cD \log T}.$$

Putting all these together gives that

$$\varphi_D(T) \leq c^{-1}T^{-1/9} \log T + \frac{c^{-2}DT^{2/3}}{T} + \frac{1}{cD \log T} \leq \frac{C}{\log T}$$

for some  $C = C(D) > 0$ .  $\square$

*Proof of Theorem 3.1.* For any  $k = 1, 2, \dots$ , write  $\mathcal{A}_k$  for the complement of the event

$$\{\exists B \subset U \text{ with } |B| \leq c^{-1}k \text{ and } R_{\text{eff}}(\rho \leftrightarrow U \setminus B) \geq c \log k\}.$$

Lemma 3.6 gives that  $\mathbf{P}(\mathcal{A}_k) \leq c^{-1}k^{-1/3} \log k$  since  $(U, \rho)$  is a distributional limit of finite planar graphs of bounded degree. Borel-Cantelli implies that  $\mathcal{A}_{2^j}$  occurs for only finitely many values of  $j$ . If  $\mathcal{A}_{2^j}$  does not occur, then for all  $2^{j-1} \leq k \leq 2^j$  there exists  $B \subset U$  with  $|B| \leq 2c^{-1}k$  and  $R_{\text{eff}}(\rho \leftrightarrow U \setminus B) \geq c \log k$ .  $\square$

## 4. Reducing to bounded degrees

### 4.1. Bounded degree distributional limits with markings.

LEMMA 4.1. *Let  $G = (V, E)$  be a finite network with two distinguished vertices  $a \neq z$ . Let  $\{R_e\}$  and  $\{R'_e\}$  be two sets of resistances on  $E(G)$ , and let  $S \subset V \setminus \{a, z\}$  be such that  $R_e = R'_e$  for any  $e \notin S \times S$ . Then*

$$\left| \mathbf{P}_a(\tau_z < \tau_a) - \mathbf{P}'_a(\tau_z < \tau_a) \right| \leq \mathbf{P}_a(\tau_S < \tau_{\{a, z\}}),$$

where  $\mathbf{P}$  and  $\mathbf{P}'$  are the network random walks with resistances  $R$  and  $R'$ , respectively.

*Proof.* The proof is immediate by coupling the two random walks until they hit  $S \cup \{a, z\}$ . □

Next, we consider a triplet  $(U, \rho, M)$ , where  $(U, \rho)$  is a random rooted graph as before and  $M$  is a marking function  $M : E(U) \rightarrow \mathbb{R}^+$ . Conditioned on  $(U, \rho, M)$ , consider the simple random walk  $(X_n)_{n \geq 0}$  where  $X_0 = \rho$ . We say that  $(U, \rho, M)$  is *stationary* if  $(U, \rho, M)$  has the same distribution as  $(U, X_1, M)$  in the space of isomorphism classes of rooted graphs with markings. (This concept is described with more details in [1].) Given a marking  $M$ , we extend it to  $M : E(U) \cup V(U) \rightarrow \mathbb{R}$  by putting  $M(v) = \max_{e:v \in e} M(e)$  for  $v \in V(U)$ . We say that  $(U, \rho, M)$  has an exponential tail with exponent  $\beta > 0$  if  $\mathbf{P}(M(\rho) \geq s) \leq 2e^{-\beta s}$  for all  $s \geq 0$ .

LEMMA 4.2. *Let  $(U, \rho, M)$  be a stationary, bounded degree random rooted graph with markings that has an exponential tail with exponent  $\beta > 0$ . Then almost surely there exists  $K < \infty$  such that for any finite subset  $B \subset V(U)$  containing  $\rho$ , of size at least  $K$ , we have*

$$\left| \mathbf{P}_\rho(\tau_{U \setminus B} < \tau_\rho) - \mathbf{P}'_\rho(\tau_{U \setminus B} < \tau_\rho) \right| \leq \frac{1}{|B|},$$

where  $\mathbf{P}$  and  $\mathbf{P}'$  are the network random walk with resistances  $R$  and  $R'$ , respectively, where  $R_e = 1$  for all  $e \in E(U)$  and  $R'$  are any resistances satisfying  $R'_e = 1$  whenever  $M(e) \leq 30\beta^{-1} \log |B|$ .

*Proof.* For any two integers  $T, s \geq 1$ , let  $\mathcal{A}_{T,s}$  denote the event

$$\mathcal{A}_{T,s} = \left\{ \mathbf{P}_\rho(\exists t \leq T : M(X_t) \geq s) \leq T^3 e^{-\beta s/2} \right\},$$

and note that this event is measurable with respect to  $(U, \rho, M)$ . Stationarity together with exponential tail implies that for any integer  $t \geq 0$ ,

$$\mathbb{E}_{(U,\rho,M)} \left[ \mathbf{P}_\rho(M(X_t) \geq s) \right] \leq 2e^{-\beta s};$$

hence, the union bound gives

$$\mathbb{E}_{(U,\rho,M)} \left[ \mathbf{P}_\rho(\exists t \leq T : M(X_t) \geq s) \right] \leq 2T e^{-\beta s}.$$

By Markov's inequality,

$$\mathbf{P}(\mathcal{A}_{T,s}^c) \leq \frac{2e^{-\beta s/2}}{T^2}.$$

Borel-Cantelli implies that almost surely  $\mathcal{A}_{T,s}$  occurs for all but finitely many values of  $T \in \mathbb{N}$  and  $s \in \mathbb{N}$ . For any finite set  $B \subset U$  that contains  $\rho$ , by

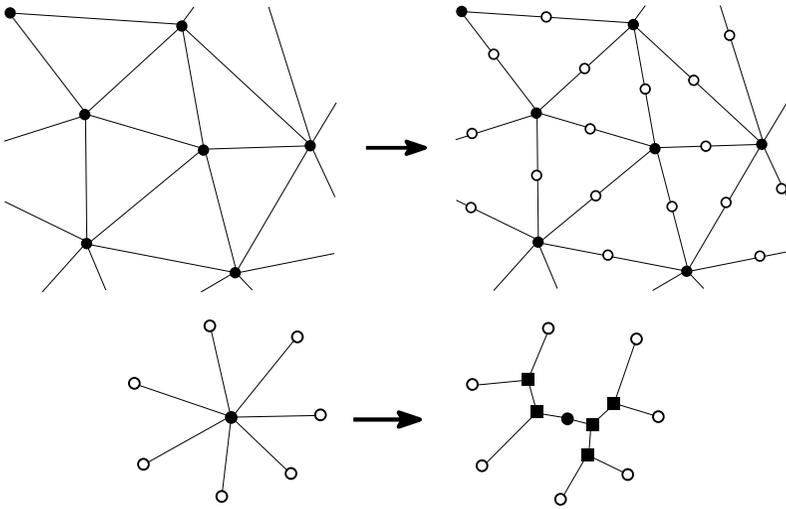


Figure 2. The star-tree transform. Stage 1 (top): subdividing edges. Stage 2 (bottom): replacing stars with binary trees.

the commute time identity (2.4), the fact that  $R_{\text{eff}}(\rho \leftrightarrow U \setminus B)$  is at most the graph distance between  $\rho$  and  $U \setminus B$  and Markov's inequality

$$\mathbf{P}_\rho(\tau_{U \setminus B} \geq T) \leq \frac{2D|B|^2}{T},$$

where  $D$  is the degree bound. Write  $S = \{v : v \in V(U) \text{ and } M(v) \geq s\}$ . Then for any  $T, s$  such that  $\mathcal{A}_{T,s}$  occurs, we have

$$\mathbf{P}_\rho(\tau_S < \tau_{\{\rho\} \cup (U \setminus B)}) \leq \frac{2D|B|^2}{T} + T^3 e^{-\beta s/2}.$$

Now, take  $T = 4D|B|^3$  and  $s = 30\beta^{-1} \log |B|$  so that the right-hand side is at most  $|B|^{-1}$  when  $|B|$  is large enough, and apply Lemma 4.1.  $\square$

4.2. *The star-tree transform.* Let  $G$  be a graph. We define the *star-tree* transform  $G^*$  of  $G$  as the graph of maximal degree at most 3 obtained by the following operations (see Figure 2).

- (1) We subdivide each edge  $e$  of  $G$  by adding a new vertex  $w_e$  of degree 2. Denote the resulting intermediate graph by  $G'$ .
- (2) Replace each vertex  $v$  of  $G$  and its incident edges in  $G'$  by a balanced binary tree  $T_v$  with  $\deg(v)$  leaves, which we identify with  $v$ 's neighbors in  $G'$ . When  $G$  is planar we choose this identification so as to preserve planarity; otherwise, this is an arbitrary identification. We denote by  $w_v$  the root of  $T_v$ . Denote the resulting graph by  $G^*$ .

LEMMA 4.3. *Let  $G$  be an infinite connected graph, and let  $G^*$  be its star-tree transform and equip  $G^*$  with edge resistances  $R$  as follows. On each edge  $e$  of the binary tree  $T_v$  we put  $R_e = 1/\deg(v)$ , where  $\deg(v)$  is the degree of  $v$  in  $G$ . Then if  $(G^*, R)$  is recurrent,  $G$  is recurrent.*

*Proof.* Assume that  $G$  is transient. Equivalently, assume that there is a unit flow  $\theta$  on  $G$ , from some  $a \in V(G)$  to infinity, with  $\mathcal{E}(\theta) < \infty$ . Given this flow we will construct a unit flow  $\theta^*$  on  $G^*$ , from  $w_a$  to infinity, such that  $\mathcal{E}(\theta^*) \leq 4\mathcal{E}(\theta)$ , thus showing that  $G^*$  is also transient.

First we define a flow  $\theta'$  from  $a$  to infinity in  $G'$  in the natural manner: for each edge  $e = (x, y)$  of  $G$ , we set  $\theta'(x, w_e) = \theta'(w_e, y) = \theta(x, y)$ . Obviously  $\mathcal{E}(\theta') = 2\mathcal{E}(\theta)$ . Next we introduce some notation. Let  $v$  be some vertex of  $G$ . Put  $k = \deg(v)$ , and note that the height of  $T_v$  is  $h = \lceil \log_2 k \rceil$  and that some leaves may occur at height  $h - 1$ . Recall that at stage 2 of the transform we obtain a correspondence between the  $k$  leaves of the tree and the neighbors of  $v$  in  $G'$ . Denote the latter vertices by  $v_1, \dots, v_k$ , and let  $e_1, \dots, e_k$  be the unique incident edges in  $T_v$ , respectively. Associate with each edge  $e \in T_v$  of the tree a string  $M(e) \in \{0, 1\}^{\leq h}$  of 0's and 1's of length at most  $h$ . The string  $M(e)$  "codes" the location of the edge in  $T_v$  by recording left turns with 0 and right turns with 1 so that the height of  $e$  is  $|M(e)|$ . (Edges touching the root  $w_v$  have height 1.)

We now construct the flow  $\theta^*$ . For each edge  $e = (x, y)$  of  $T_v$ , assume that  $(x, y)$  points towards the root  $w_v$  (recall that  $\theta^*$  should be antisymmetric), and set

$$(4.1) \quad \theta^*(e) = \sum_{j: M(e) \preceq M(e_j)} \theta'(v_j, v),$$

where two strings  $S_1$  and  $S_2$  satisfy  $S_1 \preceq S_2$  if  $S_1$  is a prefix of  $S_2$ . Let us first verify that this is a unit flow from  $a$  to  $\infty$ . Indeed, let  $u$  be a vertex in the tree that is not a leaf or the root, and denote its two children by  $u_1, u_2$  and its father by  $u^+$ . By our construction, we have  $\theta^*(u_1, u) + \theta^*(u_2, u) = \theta^*(u, u^+)$ . If  $u = w_v$  and  $v \neq a$ , then  $\theta^*(u_1, u) + \theta^*(u_2, u) = 0$  since  $\theta'$  was a flow. If  $u = w_a$ , then  $\theta^*(u_1, u) + \theta^*(u_2, u) = -1$ . Lastly, when  $u$  is a leaf of  $T_v$ , the corresponding vertex  $v_j$  has degree 2 and the flow passing through it is precisely the same as in  $\theta'$ .

Next we bound the energy  $\mathcal{E}(\theta^*)$  in terms of  $\mathcal{E}(\theta')$ . By (4.1) and Cauchy-Schwarz inequality, for any edge  $e$  of  $T_v$  at height  $\ell$ , the contribution to  $\mathcal{E}(\theta^*)$  from  $e$  is

$$R_e [\theta^*(e)]^2 = \frac{1}{k} \left[ \sum_{j: M(e) \preceq M(e_j)} \theta'(v_j, v) \right]^2 \leq \frac{2^{h-\ell}}{k} \sum_{j: M(e) \preceq M(e_j)} [\theta'(v_j, v)]^2.$$

When summing the right-hand side over all edges of  $T_v$ , the term  $[\theta'(v_j, v)]^2$  appears once for each level  $\ell = 1, \dots, h$  with coefficient  $k^{-1}2^{h-\ell}$ . Hence, the total contribution of the edges of  $T_v$  to  $\mathcal{E}(\theta^*)$  is at most

$$\sum_{\ell=1}^h \sum_{j=1}^k \frac{2^{h-\ell}}{k} [\theta'(v_j, v)]^2 \leq 2 \sum_{j=1}^k [\theta'(v_j, v)]^2.$$

Thus, when summing over all  $v$ , we obtain that

$$\mathcal{E}(\theta^*) \leq 2\mathcal{E}(\theta') = 4\mathcal{E}(\theta),$$

concluding the proof. □

### 5. Proof of main results

We begin by fixing some notation. Recall that we are given finite graphs  $G_n$ , that  $\rho_n$  is a randomly chosen vertex drawn from the stationary measure on  $G_n$  and that  $(U, \rho)$  is the distributional limit of this sequence. We write  $(G_n^*, \rho_n^*)$  and  $(U^*, \rho^*)$  for the result of the star-tree transform on  $G_n$  and  $U$  with the roots  $\rho_n^*$  and  $\rho^*$  chosen to be uniform vertices of the trees  $T_{\rho_n}$  (in  $G_n^*$ ) and  $T_\rho$  (in  $U^*$ ). We also set markings on  $G_n^*$  and  $U^*$  by putting  $M(e) = \deg(v)$  for any edge  $e$  in the tree  $T_v$ , where  $\deg(v)$  is the degree of  $v$  in  $G_n$  or  $U$ , respectively.

LEMMA 5.1. *The triplet  $(U^*, \rho^*, M)$  has an exponential tail.*

*Proof.* Note that  $M(\rho^*) = \deg(v)$ , where  $v$  is either  $\rho$  or one of its neighbors. Hence it suffices to show that if  $(U, \rho)$  is a distributional limit in which  $\deg(\rho)$  has an exponential tail, then  $D(\rho) = \max_{u:(u,\rho) \in E} \deg(u)$  also has an exponential tail. Indeed,

$$\mathbf{P}(D(\rho) \geq k) \leq \mathbf{P}(\deg(\rho) \geq k) + \mathbf{P}(\deg(\rho) \leq k \text{ and } D(\rho) \geq k).$$

The probability of the first term on the right-hand side decays exponentially. For the second term, we have

$$\mathbf{P}(\deg(X_1) \geq k \mid \deg(\rho) \leq k \text{ and } D(\rho) \geq k) \geq k^{-1},$$

where  $X_1$  is a random uniform neighbor of  $\rho$ . By stationarity,  $\mathbf{P}(\deg(X_1) \geq k)$  decays exponentially, concluding the proof. □

We now provide a proof for the intuitive fact that  $(U^*, \rho^*)$  is the distributional limit of  $(G_n^*, \rho_n^*)$ ; see [Figure 3](#).

LEMMA 5.2. *The star-tree transform is continuous on the space of distributions on rooted graphs.*

*Proof.* Let  $(H^*, h^*)$  be the star-tree transform of  $(H, h)$  as defined above. Then for any fixed  $m > 0$ , the distribution of  $B_{H^*}(h^*, m)$  is determined by the distribution of  $B_H(h, m)$  since the star-tree transform only increases distances. □

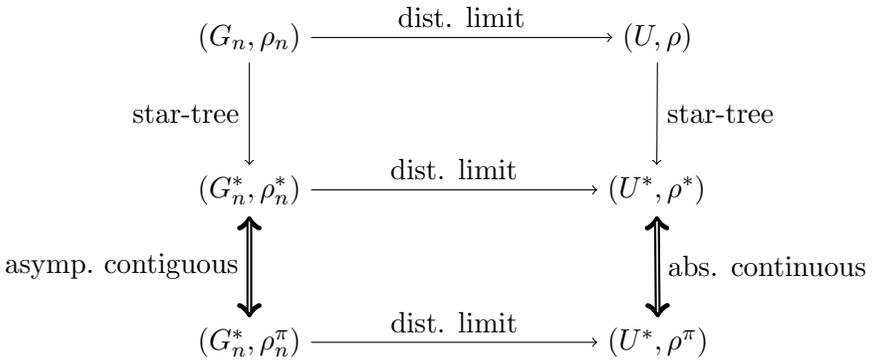


Figure 3. Commutative diagram.

COROLLARY 5.3.  $(U^*, \rho^*)$  is the distributional limit of  $(G_n^*, \rho_n^*)$ .

Note that  $(G_n^*, \rho_n^*)$  and  $(U^*, \rho^*)$  are *not* stationary with respect to the simple random walk. To overcome this small technicality, let  $\rho_n^\pi$  be a random root chosen from the stationary distribution on  $G_n^*$  and write  $(U^*, \rho^\pi)$  for an arbitrary subsequential distributional limit. Note that both  $(G_n^*, \rho_n^\pi, M)$  and  $(U^*, \rho^\pi, M)$  are stationary.

LEMMA 5.4. *There exists a universal constant  $C > 0$  such that for any  $n$ ,*

$$C^{-1} \leq \frac{\mathbf{P}_{(G_n^*, \rho_n^*)}(A)}{\mathbf{P}_{(G_n^*, \rho_n^\pi)}(A)} \leq C$$

for any event  $A$  on random rooted graphs.

*Proof.* Since  $G_n^*$  has bounded degree, the probability that  $\rho_n^\pi = v$  for any vertex  $v$  is, up to a multiplicative constant,  $\frac{1}{|G_n^*|}$ . The same holds for  $(G_n^*, \rho_n^*)$  because  $\rho_n^*$  was chosen uniformly from  $T_{\rho_n}$ , which has size proportional to  $\text{deg}(\rho_n)$ , and  $\rho_n$  was chosen with probability proportional to  $\text{deg}(\rho_n)$ . Hence, for any fixed  $n$  and an event  $A$  of random rooted graphs, we have that  $P_{(G_n^*, \rho_n^*)}(A)$  and  $P_{(G_n^*, \rho_n^\pi)}(A)$  are the same up to a multiplicative constant.  $\square$

COROLLARY 5.5. *There exists a universal constant  $C > 0$  such that for any event  $A$ ,*

$$C^{-1} \leq \frac{\mathbf{P}_{(U^*, \rho^*)}(A)}{\mathbf{P}_{(U^*, \rho^\pi)}(A)} \leq C.$$

In particular,  $(U^*, \rho^*)$  and  $(U^*, \rho^\pi)$  are absolutely continuous with respect to each other.

*Proof.* Immediate from Lemma 5.4 and Corollary 5.3.  $\square$

*Proof of Theorem 1.1.* For convenience, we use the standard  $O$ -notation. Given two sequences of nonnegative numbers  $f(k), g(k)$ , we write  $f = O(g)$  or  $g = \Omega(f)$  if there exists  $C < \infty$  such that  $f(k) \leq Cg(k)$  for sufficiently large  $k$ . We write  $f = \Theta(g)$  if  $f = O(g)$  and  $g = O(f)$ .

By definition,  $(U^*, \rho^\pi, M)$  is a distributional limit of bounded degree finite planar graph. By Lemma 5.1 and Corollary 5.5, it has an exponential tail. Apply Theorem 3.1 to get that almost surely there exist subsets  $B_k \subset U^*$  with  $|B_k| = O(k)$  such that

$$(5.1) \quad R_{\text{eff}}(\rho^\pi \leftrightarrow U^* \setminus B_k) \geq \Omega(\log k).$$

Assume without loss of generality that  $|B_k| = \Theta(k)$ . (Otherwise we can always add vertices and the resistance above only grows.) We define three different networks with underlying graph  $U^*$  by specifying the edge resistances for each edge  $e$  as follows,

$$R_e^{\text{unit}} = 1 \quad \text{and} \quad R_e^{\text{mark}} = M(e)^{-1}.$$

Given some  $s > 0$ , we define

$$R_e^s = \begin{cases} 1 & \text{if } M(e) \leq s, \\ M(e)^{-1} & \text{otherwise.} \end{cases}$$

Now, since  $|B_k| = \Theta(k)$  and (5.1), Lemma 4.2 together with (2.2) gives

$$R_{\text{eff}}(\rho^\pi \leftrightarrow U^* \setminus B_k; \{R_e^{C \log k}\}) \geq \Omega(\log k)$$

for some  $C < \infty$  depending only on the exponential decay rate. For any  $m \geq 0$ , by Lemma 2.1 with  $A = B_{U^*}(\rho^\pi, m)$ , we have

$$R_{\text{eff}}(B_{U^*}(\rho^\pi, m) \leftrightarrow U^* \setminus B_k; \{R_e^{C \log k}\}) \geq \Omega(\log k) - m$$

since  $R_e^{C \log k} \leq 1$  for all  $e$ ; hence,  $R_{\text{eff}}(\rho^\pi \leftrightarrow v) \leq m$  for all  $v \in B_{U^*}(\rho^\pi, m)$ . We have  $R_e^{\text{mark}} \geq [C \log k]^{-1} R_e^{C \log k}$  for all  $e$ . Hence,

$$R_{\text{eff}}(\partial B_{U^*}(\rho^\pi, m) \leftrightarrow U^* \setminus B_k; \{R_e^{\text{mark}}\}) \geq \Omega(1) - O(m/\log k).$$

All this occurs almost surely in  $(U^*, \rho^\pi)$ . Corollary 5.5 shows that almost surely

$$R_{\text{eff}}(\partial B_{U^*}(\rho^*, m) \leftrightarrow U^* \setminus B_k; \{R_e^{\text{mark}}\}) \geq \Omega(1) - O(m/\log k)$$

for any fixed  $m \geq 0$ . By (2.3) we deduce that the network  $(U^*, \rho^*)$  with edge resistances  $\{R_e^{\text{mark}}\}$  is almost surely recurrent. Lemma 4.3 implies that  $U$  is almost surely recurrent, concluding our proof.  $\square$

*Proof of Corollaries 1.2 and 1.3.* Follows immediately since the UIPT and UIPQ are distributional limits of finite planar graphs and it is known that the degree of the root has an exponential tail; see [5, Lemma 4.1, 4.2] and [16] for the UIPT and [7, Prop. 9] for the UIPQ.  $\square$

*Acknowledgements.* We are indebted to Omer Angel for many fruitful discussions and for comments on a previous version of this manuscript. We are grateful to Itai Benjamini, Tom Meyerovitch, Grégory Miermont, Gourab Ray and Juan Souto for many useful conversations and to Nicolas Curien for bringing [7, Prop. 9] to our attention. We also thank the participants of the circle packing reading seminar held in UBC at the fall of 2011.

## References

- [1] D. ALDOUS and R. LYONS, Processes on unimodular random networks, *Electron. J. Probab.* **12** (2007), 1454–1508. MR 2354165. Zbl 1131.60003. <http://dx.doi.org/10.1214/EJP.v12-463>.
- [2] D. ALDOUS and J. M. STEELE, The objective method: probabilistic combinatorial optimization and local weak convergence, in *Probability on Discrete Structures*, *Encyclopaedia Math. Sci.* **110**, Springer-Verlag, New York, 2004, pp. 1–72. MR 2023650. Zbl 1037.60008.
- [3] J. AMBJØRN, B. DURHUUS, and T. JONSSON, *Quantum Geometry. A Statistical Field Theory Approach*, *Cambridge Monog. Math. Phys.*, Cambridge Univ. Press, Cambridge, 1997. MR 1465433. Zbl 0993.82500. <http://dx.doi.org/10.1017/CBO9780511524417>.
- [4] O. ANGEL, Growth and percolation on the uniform infinite planar triangulation, *Geom. Funct. Anal.* **13** (2003), 935–974. MR 2024412. Zbl 1039.60085. <http://dx.doi.org/10.1007/s00039-003-0436-5>.
- [5] O. ANGEL and O. SCHRAMM, Uniform infinite planar triangulations, *Comm. Math. Phys.* **241** (2003), 191–213. MR 2013797. Zbl 1098.60010. [http://dx.doi.org/10.1007/978-1-4419-9675-6\\_16](http://dx.doi.org/10.1007/978-1-4419-9675-6_16).
- [6] I. BENJAMINI and N. CURIEN, Ergodic theory on stationary random graphs, *Electron. J. Probab.* **17** (2012), 1–20. <http://dx.doi.org/10.1214/EJP.v17-2401>.
- [7] I. BENJAMINI and N. CURIEN, Simple random walk on the uniform infinite planar quadrangulation: Subdiffusivity via pioneer points, *Geom. Funct. Anal.*, to appear. [arXiv 1202.5454](https://arxiv.org/abs/1202.5454).
- [8] I. BENJAMINI and O. SCHRAMM, Harmonic functions on planar and almost planar graphs and manifolds, via circle packings, *Invent. Math.* **126** (1996), 565–587. MR 1419007. Zbl 0868.31008. <http://dx.doi.org/10.1007/s002220050109>.
- [9] I. BENJAMINI and O. SCHRAMM, Recurrence of distributional limits of finite planar graphs, *Electron. J. Probab.* **6** (2001), 13 pp. MR 1873300. Zbl 1010.82021. <http://dx.doi.org/10.1214/EJP.v6-96>.
- [10] C. BORGS, J. T. CHAYES, J. KAHN, and L. LOVÁSZ, Left and right convergence of graphs with bounded degree, *Random Structures Algorithms* **42** (2013), 1–28, published online: 4 Apr 2012. <http://dx.doi.org/10.1002/rsa.20414>.
- [11] C. BORGS, J. T. CHAYES, L. LOVÁSZ, V. T. SÓS, and K. VESZTERGOMBI, Convergent sequences of dense graphs. I. Subgraph frequencies, metric properties and testing, *Adv. Math.* **219** (2008), 1801–1851. MR 2455626. Zbl 1213.05161. <http://dx.doi.org/10.1016/j.aim.2008.07.008>.

- [12] P. CHASSAING and G. SCHAEFFER, Random planar lattices and integrated superBrownian excursion, *Probab. Theory Related Fields* **128** (2004), 161–212. [MR 2031225](#). [Zbl 1041.60008](#). <http://dx.doi.org/10.1007/s00440-003-0297-8>.
- [13] R. CORI and B. VAUQUELIN, Planar maps are well labeled trees, *Canad. J. Math.* **33** (1981), 1023–1042. [MR 0638363](#). [Zbl 0415.05020](#). <http://dx.doi.org/10.4153/CJM-1981-078-2>.
- [14] N. CURIEN, L. MÉNARD, and G. MIERMONT, A view from infinity of the uniform infinite planar quadrangulation, preprint. [arXiv 1201.1052](#).
- [15] B. DUPLANTIER and S. SHEFFIELD, Liouville quantum gravity and KPZ, *Invent. Math.* **185** (2011), 333–393. [MR 2819163](#). [Zbl 1226.81241](#). <http://dx.doi.org/10.1007/s00222-010-0308-1>.
- [16] Z. GAO and L. B. RICHMOND, Root vertex valency distributions of rooted maps and rooted triangulations, *European J. Combin.* **15** (1994), 483–490. [MR 1292958](#). [Zbl 0807.05026](#). <http://dx.doi.org/10.1006/eujc.1994.1050>.
- [17] J. T. GILL and S. ROHDE, On the Riemann surface type of random planar maps, *Rev. Mat. Iberoamericana*, to appear.
- [18] M. KRIKUN, Local structure of random quadrangulations, preprint.
- [19] J.-F. LE GALL, The topological structure of scaling limits of large planar maps, *Invent. Math.* **169** (2007), 621–670. [MR 2336042](#). [Zbl 1132.60013](#). <http://dx.doi.org/10.1007/s00222-007-0059-9>.
- [20] J.-F. LE GALL, Uniqueness and universality of the Brownian map, *Ann. Probab.*, to appear. [arXiv 1105.4842](#).
- [21] J.-F. LE GALL and G. MIERMONT, Scaling limits of random trees and planar maps, Lecture notes for the Clay Mathematical Institute Summer School in Buzios, July 11 - August 7, 2010.
- [22] L. LOVÁSZ and B. SZEGEDY, Limits of dense graph sequences, *J. Combin. Theory Ser. B* **96** (2006), 933–957. [MR 2274085](#). [Zbl 1113.05092](#). <http://dx.doi.org/10.1016/j.jctb.2006.05.002>.
- [23] R. LYONS and Y. PERES, *Probability on Trees and Networks*, Cambridge University Press, in preparation. Available at <http://mypage.iu.edu/~rdlyons/prbtree/book.pdf>.
- [24] J.-F. MARCKERT and A. MOKKADEM, Limit of normalized quadrangulations: the Brownian map, *Ann. Probab.* **34** (2006), 2144–2202. [MR 2294979](#). [Zbl 1117.60038](#). <http://dx.doi.org/10.1214/009117906000000557>.
- [25] G. MIERMONT, The Brownian map is the scaling limit of uniform random plane quadrangulations, *Acta Math*, to appear. [arXiv 1104.1606](#).
- [26] B. RODIN and D. SULLIVAN, The convergence of circle packings to the Riemann mapping, *J. Differential Geom.* **26** (1987), 349–360. [MR 0906396](#). [Zbl 0694.30006](#). Available at <http://projecteuclid.org/euclid.jdg/1214441375>.
- [27] S. ROHDE, Oded Schramm: from circle packing to SLE, *Ann. Probab.* **39** (2011), 1621–1667. [MR 2884870](#). [Zbl 1236.60004](#). [http://dx.doi.org/10.1007/978-1-4419-9675-6\\_1](http://dx.doi.org/10.1007/978-1-4419-9675-6_1).
- [28] G. SCHAEFFER, Conjugaison d’arbres et cartes combinatoires aléatoires, Ph.D. thesis, Université Bordeaux I, 1998.

- [29] K. STEPHENSON, *Introduction to Circle Packing. The Theory of Discrete Analytic Functions*, Cambridge Univ. Press, Cambridge, 2005. [MR 2131318](#). [Zbl 1074.52008](#).
- [30] W. T. TUTTE, A census of planar triangulations, *Canad. J. Math.* **14** (1962), 21–38. [MR 0130841](#). [Zbl 0103.39603](#).

(Received: June 4, 2012)

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