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The universal relation between scaling exponents in first-passage percolation

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Abstract

It has been conjectured in numerous physics papers that in ordinary first-passage percolation on integer lattices, the fluctuation exponent χ and the wandering exponent ξ are related through the universal relation $\chi = 2\xi - 1$, irrespective of the dimension. This is sometimes called the KPZ relation between the two exponents. This article gives a rigorous proof of this conjecture assuming that the exponents exist in a certain sense.

1. Introduction

Consider the space $\mathbb{R}^d$ with Euclidean norm $|\cdot|$, where $d \geq 2$. Consider $\mathbb{Z}^d$ as a subset of this space, and say that two points x and y in $\mathbb{Z}^d$ are nearest neighbors if $|x-y| = 1$. Let $E(\mathbb{Z}^d)$ be the set of nearest neighbor bonds in $\mathbb{Z}^d$. Let $t = (t_e)_{e\in E(\mathbb{Z}^d)}$ be a collection of independent and identically distributed nonnegative random variables. In first-passage percolation, the variable t_e is usually called the 'passage time' through the edge e, alternately called the 'edge-weight' of e. We will sometimes refer to the collection t of edge-weights as the 'environment.' The total passage time, or total weight, of a path P in the environment t is simply the sum of the weights of the edges in P and will be denoted by $t(P)$ in this article. The first-passage time $T(x, y)$ from a point x to a point y is the minimum total passage time among all lattice paths from x to y. For all our purposes, it will suffice to consider self-avoiding paths; henceforth, 'lattice path' will refer to only self-avoiding paths.

Note that if the edge-weights are continuous random variables, then with probability one there is a unique 'geodesic' between any two points x and y. This is denoted by $G(x, y)$ in this paper. Let $D(x, y)$ be the maximum deviation (in Euclidean distance) of this path from the straight line segment joining x and y (see Figure 1).

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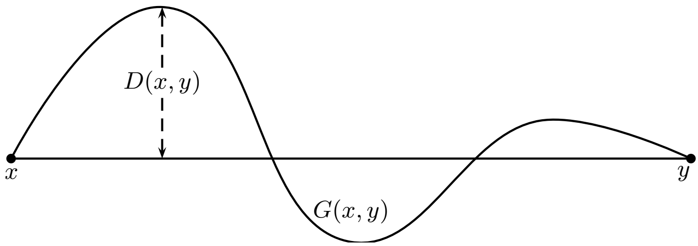

Figure 1. The geodesic $G(x,y)$ and the deviation $D(x,y)$.

Although invented by mathematicians [11], the first-passage percolation and related models have attracted considerable attention in the theoretical physics literature. (See [21] for a survey.) Among other things, the physicists are particularly interested in two 'scaling exponents,' sometimes denoted by χ and ξ in the mathematical physics literature. The *fluctuation exponent* χ is a number that quantifies the order of fluctuations of the first-passage time $T(x,y)$. Roughly speaking, for any x,y,

$$\text{the typical value of } T(x,y) - \mathbb{E}T(x,y) \text{ is of the order } |x-y|^{\chi}.$$

The *wandering exponent* ξ quantifies the magnitude of $D(x,y)$. Again, roughly speaking, for any x,y,

$$\text{the typical value of } D(x,y) \text{ is of the order } |x-y|^{\xi}.$$

There have been several attempts to give precise mathematical definitions for these exponents (see [23] for some examples), but I could not find a consensus in the literature. The main hurdle is that no one knows whether the exponents actually exist, and if they do, in what sense.

There are many conjectures related to χ and ξ. The main among these, to be found in numerous physics papers [14], [15], [16], [19], [20], [21], [24], [25], [30], including the famous paper of Kardar, Parisi and Zhang [15], is that although χ and ξ may depend on the dimension, they always satisfy the relation

$$\chi = 2\xi - 1.$$

A well-known conjecture from [15] is that when $d = 2$, $\chi = 1/3$ and $\xi = 2/3$. Yet another belief is that $\chi = 0$ if d is sufficiently large. Incidentally, due to its connection with [15], I have heard in private conversations the relation $\chi = 2\xi - 1$ being referred to as the 'KPZ relation' between χ and ξ.

There are a number of rigorous results for χ and ξ, mainly from the late eighties and early nineties. One of the first nontrivial results is due to Kesten [18, Th. 1], who proved that $\chi \le 1/2$ in any dimension. To date, the only improvement on Kesten's result is due to Benjamini, Kalai and Schramm [6], who proved that for first-passage percolation in $d \geq 2$ with binary edge-weights,

$$\sup_{v \in \mathbb{Z}^d,\ |v|>1} \frac{\mathrm{Var} T(0,v)}{|v|/\log|v|} < \infty. \tag{1}$$

Benaïm and Rossignol [5] extended this result to a large class of edge-weight distributions that they call 'nearly gamma' distributions. The definition of a nearly gamma distribution is as follows. A positive random variable X is said to have a nearly gamma distribution if it has a continuous probability density function h supported on an interval I (which may be unbounded), and its distribution function H satisfies, for all $y \in I$,

$$\Phi' \circ \Phi^{-1}(H(y)) \leq A\sqrt{y}h(y),$$

for some constant A, where Φ is the distribution function of the standard normal distribution. Although the definition may seem a bit strange, Benaïm and Rossignol [5] proved that this class is actually quite large, including e.g., exponential, gamma, beta and uniform distributions on intervals.

The only nontrivial lower bound on the fluctuations of passage times is due to Newman and Piza [26] and Pemantle and Peres [27], who showed that in $d = 2$, $\mathrm{Var}T(0,v)$ must grow at least as fast as $\log|v|$. Better lower bounds can be proved if one can show that with high probability, the geodesics lie in 'thin cylinders' [7].

For the wandering exponent ξ, the main rigorous results are due to Licea, Newman and Piza [23] who showed that $\xi^{(2)} \geq 1/2$ in any dimension, and $\xi^{(3)} \geq 3/5$ when $d = 2$, where $\xi^{(2)}$ and $\xi^{(3)}$ are exponents defined in their paper that may be equal to ξ.

Besides the bounds on χ and ξ mentioned above, there are some rigorous results relating χ and ξ through inequalities. Wehr and Aizenman [29] proved the inequality $\chi \geq (1-(d-1)\xi)/2$ in a related model, and the version of this inequality for first-passage percolation was proved by Licea, Newman and Piza [23]. The closest that anyone came to proving $\chi = 2\xi - 1$ is a result of Newman and Piza [26], who proved that $\chi' \geq 2\xi - 1$, where χ' is a related exponent that may be equal to χ. This has also been observed by Howard [13] under different assumptions.

Incidentally, in the model of Brownian motion in a Poissonian potential, Wüthrich [31] proved the equivalent of the KPZ relation assuming that the exponents exist.

The following theorem establishes the relation $\chi = 2\xi - 1$ assuming that the exponents χ and ξ exist in a certain sense (to be defined in the statement of the theorem) and that the distribution of edge-weights is nearly gamma.

THEOREM 1.1. *Consider the first-passage percolation model on $\mathbb{Z}^d$, $d \geq 2$, with independent and identically distributed edge-weights. Assume that the* *distribution of edge-weights is 'nearly gamma' in the sense of Benaïm and Rossignol* [5] *(which includes exponential, gamma, beta and uniform distributions, among others) and has a finite moment generating function in a neighborhood of zero. Let χ_a and ξ_a be the smallest real numbers such that for all $\chi' > \chi_a$ and $\xi' > \xi_a$, there exists $\alpha > 0$ such that*

$$\sup_{v\in\mathbb{Z}^d\setminus\{0\}} \mathbb{E}\exp\biggl(\alpha\frac{|T(0,v)-\mathbb{E}T(0,v)|}{|v|^{\chi'}}\biggr) < \infty, \tag{A1}$$

$$\sup_{v\in\mathbb{Z}^d\setminus\{0\}} \mathbb{E}\exp\biggl(\alpha\frac{D(0,v)}{|v|^{\xi'}}\biggr) < \infty. \tag{A2}$$

Let χ_b and ξ_b be the largest real numbers such that for all $\chi' < \chi_b$ and $\xi' < \xi_b$, there exists $C > 0$ such that

$$\inf_{v\in\mathbb{Z}^d,\ |v|>C} \frac{\mathrm{Var}(T(0,v))}{|v|^{2\chi'}} > 0, \tag{A3}$$

$$\inf_{v\in\mathbb{Z}^d,\ |v|>C} \frac{\mathbb{E}D(0,v)}{|v|^{\xi'}} > 0. \tag{A4}$$

Then $0 \le \chi_b \le \chi_a \le 1/2$, $0 \le \xi_b \le \xi_a \le 1$ and $\chi_a \ge 2\xi_b - 1$. Moreover, if it so happens that $\chi_a = \chi_b$ and $\xi_a = \xi_b$, and these two numbers are denoted by χ and ξ, then they must necessarily satisfy the relation $\chi = 2\xi - 1$.

Note that if $\chi_a = \chi_b$ and $\xi_a = \xi_b$ and these two numbers are denoted by χ and ξ, then χ and ξ are characterized by the properties that for every $\chi' > \chi$ and $\xi' > \xi$, there are some positive α and C such that for all $v \ne 0$,

$$\mathbb{E}\exp\biggl(\alpha\frac{|T(0,v)-\mathbb{E}T(0,v)|}{|v|^{\chi'}}\biggr) < C \quad\text{and}\quad \mathbb{E}\exp\biggl(\alpha\frac{D(0,v)}{|v|^{\xi'}}\biggr) < C,$$

and for every $\chi' < \chi$ and $\xi' < \xi$, there are some positive B and C such that for all v with $|v| > C$,

$$\mathrm{Var}(T(0,v)) > B|v|^{2\chi'} \quad\text{and}\quad \mathbb{E}D(0,v) > B|v|^{\xi'}.$$

It seems reasonable to expect that if the two exponents χ and ξ indeed exist, then they should satisfy the above properties.

Incidentally, a few months after the first draft of this paper was put up on arXiv, Auffinger and Damron [4] were able to replace a crucial part of the proof of Theorem 1.1 with a simpler argument that allowed them to remove the assumption that the edge-weights are nearly-gamma.

Section 2 has a sketch of the proof of Theorem 1.1. The rest of the paper is devoted to the actual proof. Proving that $0 \le \chi_b \le \chi_a \le 1/2$ and $0 \le \xi_b \le \xi_a \le 1$ is a routine exercise; this is done in Section 3. Proving that $\chi_a \ge 2\xi_b - 1$ is also relatively easy and similar to the existing proofs of analogous inequalities, e.g., in [26], [13]. This is done in Section 6. The 'hard part' is proving the opposite inequality; that is, $\chi \le 2\xi - 1$ when $\chi = \chi_a = \chi_b$ and $\xi = \xi_a = \xi_b$. This is done in Sections 7, 8 and 9.

2. Proof sketch

I will try to give a sketch of the proof in this section. I have found it very hard to aptly summarize the main ideas in the proof without going into the details. This proof-sketch represents the end-result of my best efforts in this direction. If the interested reader finds the proof sketch too obscure, I would like to request him to return to this section after going through the complete proof, whereupon this high-level sketch may shed some illuminating insights.

Throughout this proof sketch, C will denote any positive constant that depends only on the edge-weight distribution and the dimension. Let $h(x) := \mathbb{E}(T(0,x))$. The function h is subadditive. Therefore the limit

$$g(x) := \lim_{n\to\infty} \frac{h(nx)}{n}$$

exists for all $x \in \mathbb{Z}^d$. The definition can be extended to all $x \in \mathbb{Q}^d$ by taking $n \to \infty$ through a subsequence, and it can be further extended to all $x \in \mathbb{R}^d$ by uniform continuity. The function g is a norm on $\mathbb{R}^d$.

The function g is a norm, and hence much more well behaved than h. If $|x|$ is large, $g(x)$ is supposed to be a good approximation of $h(x)$. A method developed by Ken Alexander [1], [2] uses the order of fluctuations of passage times to infer bounds on $|h(x) - g(x)|$. In the setting of Theorem 1.1, Alexander's method yields that for any $\varepsilon > 0$, there exists C such that for all $x \neq 0$,

$$g(x) \le h(x) \le g(x) + C|x|^{\chi_a+\varepsilon}. \tag{2}$$

This is formally recorded in Theorem 4.1. In the proof of the main result, the above approximation will allow us to replace the expected passage time $h(x)$ by the norm $g(x)$.

In Lemma 5.1, we prove that there is a unit vector x_0 and a hyperplane H_0 perpendicular to x_0 such that for some $C > 0$, for all $z \in H_0$,

$$|g(x_0 + z) - g(x_0)| \le C|z|^2.$$

Similarly, there is a unit vector x_1 and a hyperplane H_1 perpendicular to x_1 such that for some $C > 0$, for all $z \in H_1$, $|z| \le 1$,

$$g(x_1 + z) \ge g(x_1) + C|z|^2.$$

The interpretations of these two inequalities is as follows. In the direction x_0, the unit sphere of the norm g is 'at most as curved as an Euclidean sphere' and in the direction x_1, it is 'at least as curved as an Euclidean sphere.'

Now take a look at Figure 2. Think of m as a fraction of n. By the definition of the direction of curvature x_1 and Alexander's approximation (2), for any $\varepsilon > 0$,

$$\begin{aligned}
&\text{Expected passage time of the path } P \\
&\quad \geq g(mx_1 + z) + g(nx_1 - (mx_1 + z)) + O(n^{\chi+\varepsilon}) \\
&\quad = mg(x_1 + z/m) + (n-m)g(x_1 + z/(n-m)) + O(n^{\chi+\varepsilon}) \\
&\quad \geq ng(x_1) + C|z|^2/n + O(n^{\chi+\varepsilon}) \\
&\quad \geq \mathbb{E}(T(0, nx_1)) + C|z|^2/n + O(n^{\chi+\varepsilon}).
\end{aligned}$$

Suppose $|z| = n^{\xi}$. Then $|z|^2/n = n^{2\xi-1}$. Fluctuations of $T(0, nx_1)$ are of order n^{χ}. Thus, if $2\xi - 1 > \chi$, then P cannot be a geodesic from 0 to nx_1. This sketch is formalized into a rigorous argument in Section 6 to prove that $\chi_a \geq 2\xi_b - 1$.

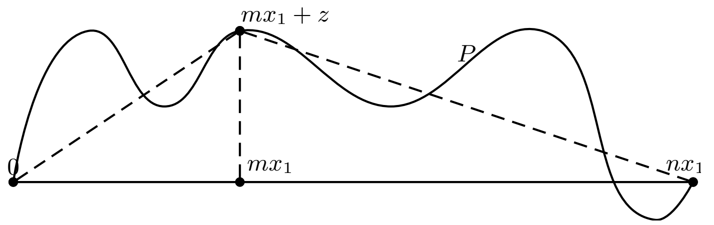

Figure 2. Proving $\chi \geq 2\xi - 1$.

Next, let me sketch the proof of $\chi \leq 2\xi - 1$ when $\chi > 0$. The methods developed in [7] for first-passage percolation in thin cylinders have some bearing on this part of the proof. Recall the direction of curvature x_0. Let $a = n^{\beta}$, $\beta < 1$. Let $m = n/a = n^{1-\beta}$. Under the conditions $\chi > 2\xi - 1$ and $\chi > 0$, we will show that there is a $\beta < 1$ such that

$$(\star) \qquad T(0, nx_0) = \sum_{i=0}^{m-1} T(iax_0, (i+1)ax_0) + o(n^{\chi}).$$

This will lead to a contradiction, as follows. Let $f(n) := \mathrm{Var} T(0, nx_0)$. Then by Benaïm and Rossignol [5], $f(n) \leq Cn/\log n$. Under $(\star)$, by the Harris-FKG inequality,

$$\begin{aligned}
f(n) = \mathrm{Var} T(0, nx_0) &\geq m \mathrm{Var} T(0, ax_0) + o(n^{2\chi}) \\
&= n^{1-\beta} f(n^{\beta}) + o(n^{2\chi}).
\end{aligned}$$

If β is chosen sufficiently small, the first term on the right will dominate the second. Consequently,

$$(\dagger) \qquad \liminf_{n\to\infty} \frac{f(n)}{n^{1-\beta} f(n^{\beta})} \geq 1.$$

Choose $n_0 > 1$, and define $n_{i+1} = n_i^{1/\beta}$ for each i. Let $v(n) := f(n)/n$. Then $v(n_i) \le C/\log n_i \le C\beta^i$. But by (†), $\liminf v(n_{i+1})/v(n_i) \ge 1$, and so for all i large enough, $v(n_{i+1}) \ge \beta^{1/2} v(n_i)$. In particular, there is a positive constant c such that for all i, $v(n_i) \ge c\beta^{i/2}$. Since $\beta < 1$, this gives a contradiction for i large, therefore proving that $\chi \le 2\xi - 1$.

Let me now sketch a proof of ($\star$) under the conditions $\chi > 2\xi - 1$ and $\chi > 0$. Let $a = n^\beta$ and $b = n^{\beta'}$, where $\beta' < \beta < 1$. Consider a cylinder of width n^ξ around the line joining 0 and nx_0. Partition the cylinder into alternating big and small cylinders of widths a and b respectively. Call the boundary walls of these cylinders $U_0, V_0, U_1, V_1, \ldots, V_{m-1}, U_m$, where m is roughly $n^{1-\beta}$ (see Figure 3).

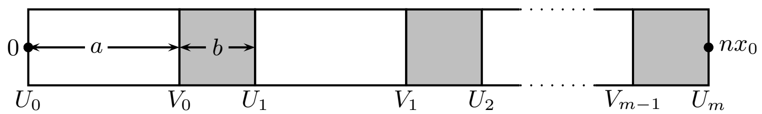

Figure 3. Cylinder of width n^ξ around the line joining 0 and nx_0.

Let $G_i := G(U_i, V_i)$; that is, the path with minimum passage time between any vertex in U_i and any vertex in V_i. Let u_i and v_i be the endpoints of G_i. Let $G'_i := G(v_i, u_{i+1})$. The concatenation of the paths G'_0, G_1, G'_1, $G_2, \ldots,$ G'_{m-1}, G_m is a path from U_0 to U_m. Therefore,

$$T(U_0, U_m) \le \sum_{i=1}^{m-1} T(U_i, V_i) + \sum_{i=0}^{m-1} T(v_i, u_{i+1}).$$

Next, let $G := G(U_0, U_m)$. Let u'_i be the first vertex in U_i visited by G, and let v'_i be the first vertex in V_i visited by G. If G stays within the cylinder throughout, then $T(u'_i, v'_i) \ge T(U_i, V_i)$ and $T(v'_i, u'_{i+1}) \ge T(V_i, U_{i+1})$. Thus,

$$T(U_0, U_m) \ge \sum_{i=0}^{m-1} T(U_i, V_i) + \sum_{i=0}^{m-1} T(V_i, U_{i+1}).$$

Thus, if $G(U_0, U_m)$ stays in a cylinder of width n^ξ, then

$$\begin{aligned} 0 &\le T(U_0, U_m) - \sum_{i=0}^{m-1} (T(U_i, V_i) + T(V_i, U_{i+1})) \\ &\le \sum_{i=0}^{m-1} (T(v_i, u_{i+1}) - T(V_i, U_{i+1})). \end{aligned}$$

Therefore,

$$\left| T(U_0, U_m) - \sum_{i=0}^{m-1} (T(U_i, V_i) + T(V_i, U_{i+1})) \right| \le \sum_{i=0}^{m-1} M_i,$$

where $M_i := \max_{v,v' \in V_i,\ u,u' \in U_{i+1}} |T(v,u) - T(v',u')|$. Note that the errors M_i come only from the small blocks. By curvature estimate in direction x_0, for any $v, v' \in V_i$ and $u, u' \in U_{i+1}$,

$$|\mathbb{E}T(v,u) - \mathbb{E}T(v',u')| \le C(n^\xi)^2/n^{\beta'} = Cn^{2\xi-\beta'}.$$

Fluctuations of $T(v,u)$ are of order $n^{\beta'\chi}$. If $2\xi - 1 < \chi$, then we can choose β' so close to 1 that $2\xi - \beta' < \beta'\chi$. That is, fluctuations dominate while estimating M_i. Consequently, M_i is of order $n^{\beta'\chi}$. Thus, total error $= n^{1-\beta+\beta'\chi}$. Since $\beta' < \beta$ and $\chi > 0$, this gives us the opportunity of choosing β', β such that the exponent is $< \chi$. This proves ($\star$) for passage times from 'boundary to boundary.' Proving ($\star$) for 'point to point' passage times is only slightly more complicated. The program is carried out in Sections 7 and 8.

Finally, for the case $\chi = 0$, we have to prove that $\xi \ge 1/2$. This was proved by Licea, Newman and Piza [23] for a different definition of the wandering exponent. The argument does not seem to work with our definition. A proof is given in Section 9; I will omit this part from the proof sketch.

3. *A priori* bounds

In this section we prove the *a priori* bounds $0 \le \chi_b \le \chi_a \le 1/2$ and $0 \le \xi_b \le \xi_a \le 1$. First, note that the inequalities $\chi_b \le \chi_a$ and $\xi_b \le \xi_a$ are easy. For example, if $\chi_b > \chi_a$, then for any $\chi_a < \chi' < \chi'' < \chi_b$, (A1) implies that

$$\sup_{v \in \mathbb{Z}^d \setminus \{0\}} \frac{\mathrm{Var}(T(0,v))}{|v|^{2\chi'}} < \infty,$$

and hence for any sequence v_n such that $|v_n| \to \infty$,

$$\lim_{n\to\infty} \frac{\mathrm{Var}(T(0,v_n))}{|v_n|^{2\chi''}} = 0,$$

which contradicts (A3). A similar argument shows that $\xi_b \le \xi_a$.

To show that $\chi_b \ge 0$, let E_0 denote the set of all edges incident to the origin. Let $\mathcal{F}_0$ denote the sigma-algebra generated by $(t_e)_{e \notin E_0}$. Since the edge-weight distribution is nondegenerate, there exists $c_1 < c_2$ such that for an edge e, $\mathbb{P}(t_e < c_1) > 0$ and $\mathbb{P}(t_e > c_2) > 0$. Therefore,

$$\mathbb{P}(\max_{e \in E_0} t_e < c_1) > 0, \quad \mathbb{P}(\min_{e \in E_0} t_e > c_2) > 0. \tag{3}$$

Let $(t'_e)_{e \in E_0}$ be an independent configuration of edge weights. Define $t'_e = t_e$ if $e \notin E_0$. Let $T'(0,v)$ be the first-passage time from 0 to a vertex v in the new environment t'. If $t_e < c_1$ and $t'_e > c_2$ for all $e \in E_0$, then $T'(0,v) > T(0,v) + c_2 - c_1$. Thus, by (3), there exists $\delta > 0$ such that for any v with $|v| \ge 2$,

$$\mathbb{E}\mathrm{Var}(T(0,v)|\mathcal{F}_0) = \frac{1}{2}\mathbb{E}(T(0,v) - T'(0,v))^2 > \delta.$$

Therefore $\mathrm{Var}(T(0,v)) > \delta$, and so $\chi_b \ge 0$.

To show that $\xi_b \geq 0$, note that there is an $\epsilon > 0$ small enough such that for any $v \in \mathbb{Z}^d$ with $|v| \geq 2$, there can be at most one lattice path from 0 to v that stays within distance ϵ from the straight line segment joining 0 to v. Fix such a vertex v and such a path P. If the number of edges in P is sufficiently large, one can use the nondegeneracy of the edge-weight distribution to show by an explicit assignment of edge weights that

$$\mathbb{P}(P \text{ is a geodesic}) < \delta,$$

where $\delta < 1$ is a constant that depends only on the edge-weight distribution (and not on v or P). This shows that for $|v|$ sufficiently large, $\mathbb{E}D(0, v)$ is bounded below by a positive constant that does not depend on v, thereby proving that $\xi_b \geq 0$.

Let us next show that $\chi_a \leq 1/2$. Essentially, this follows from [18, Th. 1] or [28, Prop. 8.3], with a little bit of extra work. Below, we give a proof using [5, Th. 5.4]. First, note that there is a constant C_0 such that for all v,

$$\mathbb{E}T(0, v) \leq C_0|v|_1, \tag{4}$$

where $|v|_1$ is the ℓ_1 norm of v. From the assumptions about the distribution of edge-weights, [5, Th. 5.4] implies that there are positive constants C_1 and C_2 such that for any $v \in \mathbb{Z}^d$ with $|v|_1 \geq 2$, and any $0 \leq t \leq |v|_1$,

$$\mathbb{P}\left(|T(0, v) - \mathbb{E}T(0, v)| \geq t\sqrt{\frac{|v|_1}{\log |v|_1}}\right) \leq C_1 e^{-C_2 t}. \tag{5}$$

Fix a path P from 0 to v with $|v|_1$ edges. Recall that $t(P)$ denotes the sum of the weights of the edges in P. Since the edge-weight distribution has finite moment generating function in a neighborhood of zero and (4) holds, it is easy to see that there are positive constants C_3, C_4 and C_4' such that if $|v|_1 > C_3$, then for any $t > |v|_1$,

$$\begin{aligned}
&\mathbb{P}\left(|T(0, v) - \mathbb{E}T(0, v)| \geq t\sqrt{\frac{|v|_1}{\log |v|_1}}\right) \\
&\quad \leq \mathbb{P}\left(T(0, v) \geq C_0|v|_1 + t\sqrt{\frac{|v|_1}{\log |v|_1}}\right) \\
&\quad \leq \mathbb{P}\left(t(P) \geq C_0|v|_1 + t\sqrt{\frac{|v|_1}{\log |v|_1}}\right) \leq e^{C_4|v|_1 - C_4' t\sqrt{|v|_1/\log|v|_1}}.
\end{aligned} \tag{6}$$

Combining (5) and (6), it follows that there are constants C_5, C_6 and C_7 such that for any v with $|v|_1 > C_5$,

$$\mathbb{E}\exp\left(C_6 \frac{|T(0, v) - \mathbb{E}T(0, v)|}{\sqrt{|v|_1/\log|v|_1}}\right) \leq C_7.$$

Appropriately increasing C_7, one sees that the above inequality holds for all v with $|v|_1 \geq 2$. In particular, $\chi_a \leq 1/2$.

Finally, let us prove that $\xi_a \leq 1$. Consider a self-avoiding path P starting at the origin, containing m edges. By the strict positivity of the edge-weight distributions, for any edge e,

$$\lim_{\theta \to \infty} \mathbb{E}(e^{-\theta t_e}) = 0.$$

Now, for any $\theta, c > 0$,

$$\mathbb{P}(t(P) \leq cm) = \mathbb{P}(e^{-t(P)/c} \geq e^{-m}) \leq (e\mathbb{E}(e^{-t_e/c}))^m.$$

Thus, given any $\delta > 0$, there exists c small enough such that for any m and any self-avoiding path P with m edges,

$$\mathbb{P}(t(P) \leq cm) \leq \delta^m.$$

Since there are at most $(2d)^m$ paths with m edges, therefore there exists c small enough such that

$$\mathbb{P}(t(P) \leq cm \text{ for some } P \text{ with } m \text{ edges}) \leq 2^{-m-1},$$

and therefore

$$\mathbb{P}(t(P) \leq cm \text{ for some } P \text{ with} \geq m \text{ edges}) \leq 2^{-m}. \tag{7}$$

There is a constant $B > 0$ such that for any $t \geq 1$ and any vertex $v \neq 0$, if $D(0, v) \geq t|v|$, then $G(0, v)$ has at least $Bt|v|$ edges. Therefore from (7),

$$\mathbb{P}(D(0, v) \geq t|v|) \leq \mathbb{P}(T(0, v) \geq Bt|v|/c) + 2^{-Bt|v|}.$$

As in (6), there is a constant C such that if P is a path from 0 to v with $|v|_1$ edges,

$$\mathbb{P}(T(0, v) \geq Bt|v|/c) \leq \mathbb{P}(t(P) \geq Bt|v|/c) \leq e^{C|v| - Bt|v|/c}.$$

Combining the last two displays shows that for some α small enough,

$$\sup_{v \neq 0} \mathbb{E} \exp\left(\alpha \frac{D(0, v)}{|v|}\right) < \infty,$$

and thus, $\xi_a \leq 1$.

4. **Alexander's subadditive approximation theory**

The first step in the proof of Theorem 1.1 is to find a suitable approximation of $\mathbb{E}T(0, x)$ by a convex function $g(x)$. For $x \in \mathbb{Z}^d$, define

$$h(x) := \mathbb{E}T(0, x). \tag{8}$$

It is easy to see that h satisfies the subadditive inequality

$$h(x + y) \leq h(x) + h(y).$$

By the standard subadditive argument, it follows that

$$g(x) := \lim_{n\to\infty} \frac{h(nx)}{n} \tag{9}$$

exists for each $x \in \mathbb{Z}^d$. In fact, $g(x)$ may be defined similarly for $x \in \mathbb{Q}^d$ by taking $n \to \infty$ through a sequence of n such that $nx \in \mathbb{Z}^d$. The function g extends continuously to the whole of $\mathbb{R}^d$, and the extension is a norm on $\mathbb{R}^d$ (see e.g., [2, Lemma 1.5]). Note that by subadditivity,

$$g(x) \le h(x) \text{ for all } x \in \mathbb{Z}^d. \tag{10}$$

Since the edge-weight distribution is continuous in the setting of Theorem 1.1, it follows by a well-known result (see [17]) that $g(x) > 0$ for each $x \neq 0$. Let e_i denote the ith coordinate vector in $\mathbb{R}^d$. Since g is symmetric with respect to interchange of coordinates and reflections across all coordinate hyperplanes, it is easy to show, using subadditivity, that

$$|x|_\infty \le g(x)/g(e_1) \le |x|_1 \text{ for all } x \neq 0, \tag{11}$$

where $|x|_p$ denotes the ℓ_p norm of the vector x.

How well does $g(x)$ approximate $h(x)$? Following the work of Kesten [17], [18], Alexander [1], [2] developed a general theory for tackling such questions. One of the main results of Alexander [2] is that under appropriate hypotheses on the edge-weights, there exists some $C > 0$ such that for all $x \in \mathbb{Z}^d \backslash \{0\}$,

$$g(x) \le h(x) \le g(x) + C|x|^{1/2} \log |x|.$$

Incidentally, Alexander has recently been able to obtain slightly improved results for nearly gamma edge-weights [3]. It turns out that under the hypotheses of Theorem 1.1, Alexander's argument goes through almost verbatim to yield the following result.

THEOREM 4.1. *Consider the setup of Theorem 1.1. Let g and h be defined as in (9) and (8) above. Then for any $\chi' > \chi_a$, there exists $C > 0$ such that for all $x \in \mathbb{Z}^d$ with $|x| > 1$,*

$$g(x) \le h(x) \le g(x) + C|x|^{\chi'} \log |x|.$$

Sacrificing brevity for the sake of completeness, I will now prove Theorem 4.1 by copying Alexander's argument with only minor changes at the appropriate points.

Fix $\chi' > \chi_a$. Since $0 \le \chi_a \le 1/2$, so χ' can be chosen to satisfy $0 < \chi' < 1$.

Let $B_0 := \{x : g(x) \le 1\}$. Given $x \in \mathbb{R}^d$, let H_x denote a hyperplane tangent to the boundary of $g(x)B_0$ at x. Note that if the boundary is not smooth, the choice of H_x may not be unique. Let H_x^0 be the hyperplane through the origin that is parallel to H_x. There is a unique linear functional g_x on $\mathbb{R}^d$ satisfying

$$g_x(y) = 0 \text{ for all } y \in H_x^0, \quad g_x(x) = g(x).$$

For each $x \in \mathbb{R}^d$, $C > 0$ and $K > 0$, let

$$Q_x(C, K) := \{y \in \mathbb{Z}^d : |y| \le K|x|,\ g_x(y) \le g(x),\ h(y) \le g_x(y) + C|x|^{\chi'} \log |x|\}.$$

The following key result is taken from [2].

LEMMA 4.2 (Alexander [2, Th. 1.8]). *Consider the setting of Theorem 4.1. Suppose that for some $M > 1$, $C > 0$, $K > 0$ and $a > 1$, the following holds. For each $x \in \mathbb{Q}^d$ with $|x| \ge M$, there exists an integer $n \ge 1$, a lattice path γ from 0 to nx and a sequence of sites $0 = v_0, v_1, \ldots, v_m = nx$ in γ such that $m \le an$ and $v_i - v_{i-1} \in Q_x(C, K)$ for all $1 \le i \le m$. Then the conclusion of Theorem 4.1 holds.*

Before proving that the conditions of Lemma 4.2 hold, we need some preliminary definitions and results. Define

$$s_x(y) := h(y) - g_x(y), \quad y \in \mathbb{Z}^d.$$

By the definition of g_x and the fact that g is a norm, it is easy to see that

$$|g_x(y)| \le g(y), \tag{12}$$

and by subadditivity, $g(y) \le h(y)$. Therefore $s_x(y) \ge 0$. Again from subadditivity of h and linearity of g_x,

$$s_x(y + z) \le s_x(y) + s_x(z) \quad \text{for all } y, z \in \mathbb{Z}^d. \tag{13}$$

Let $C_1 := 320d^2/\alpha$, where α is from the statement of Theorem 1.1. As in [2], define

$$\begin{aligned}
Q_x &:= Q_x(C_1, 2d + 1),\\
G_x &:= \{y \in \mathbb{Z}^d : g_x(y) > g(x)\},\\
\Delta_x &:= \{y \in Q_x : y \text{ adjacent to } \mathbb{Z}^d \backslash Q_x,\ y \text{ not adjacent to } G_x\},\\
D_x &:= \{y \in Q_x : y \text{ adjacent to } G_x\}.
\end{aligned}$$

The following lemma is simply a slightly altered copy of Lemma 3.3 in [2].

LEMMA 4.3. *Assume the conditions of Theorem 1.1. Then there exists a constant C_2 such that if $|x| \ge C_2$, the following hold:*

(i) *If $y \in Q_x$, then $g(y) \le 2g(x)$ and $|y| \le 2d|x|$.*
(ii) *If $y \in \Delta_x$, then $s_x(y) \ge C_1|x|^{\chi'}(\log |x|)/2$.*
(iii) *If $y \in D_x$, then $g_x(y) \ge 5g(x)/6$.*

Proof. (i) Suppose $g(y) > 2g(x)$ and $g_x(y) \leq g(x)$. Then using (10) and (12),

$$2g(x) < g(y) \leq h(y) = g_x(y) + s_x(y) \leq g(x) + s_x(y),$$

so from (11), $s_x(y) > g(x) > C_1|x|^{\chi'} \log|x|$, provided $|x| \geq C_2$. Thus $y \notin Q_x$ and the first conclusion in (i) follows. The second conclusion then follows from (11).

(ii) Note that $z = y \pm e_i$ for some $z \in \mathbb{Z}^d \cap Q_x^c \cap G_x^c$ and $i \leq d$. From (i), we have $|y| \leq 2d|x|$, so $|z| \leq (2d+1)|x|$, provided $|x| > 1$. Since $z \notin Q_x$, we must then have $s_x(z) > C_1|x|^{\chi'} \log|x|$, while using (12),

$$h(\pm e_i) = s_x(\pm e_i) + g_x(\pm e_i) \geq s_x(\pm e_i) - g(\pm e_i).$$

Consequently, by (13), if $|x| \geq C_2$,

$$\begin{aligned} s_x(y) &\geq s_x(z) - s_x(\pm e_i) \\ &\geq C_1|x|^{\chi'} \log|x| - h(\pm e_i) - g(\pm e_i) \\ &\geq C_1|x|^{\chi'}(\log|x|)/2. \end{aligned}$$

(iii) As in (ii), we have $z = y \pm e_i$ for some $z \in \mathbb{Z}^d \cap G_x$ and $i \leq d$. Therefore using (11) and (12),

$$g_x(y) = g_x(z) - g_x(\pm e_i) \geq g_x(z) - g(\pm e_i) \geq 5g(x)/6$$

for all $|x| \geq C_2$. □

Let us call the $m+1$ sites in Lemma 4.2 marked sites. If m is unrestricted, it is easy to find inductively a sequence of marked sites for any path γ from 0 to nx, as follows. One can start at $v_0 = 0$, and given v_i, let v'_{i+1} be the first site (if any) in γ, coming after v_i, such that $v'_{i+1} - v_i \notin Q_x$; then let v_{i+1} be the last site in γ before v'_{i+1} if v'_{i+1} exists; otherwise let $v_{i+1} = nx$ and end the construction. If $|x|$ is large enough, then it is easy to deduce from (11) and (12) that all neighbors of the origin must belong to Q_x and therefore $v_{i+1} \neq v_i$ for each i, and hence the construction must end after a finite number of steps. We call the sequence of marked sites obtained from a self-avoiding path γ in this way 'the Q_x-skeleton of γ.'

Given such a skeleton $(v_0, \ldots, v_m)$, abbreviated (v_i), of some lattice path, we divide the corresponding indices into two classes, corresponding to 'long' and 'short' increments:

$$\begin{aligned} S((v_i)) &:= \{i : 0 \leq i < m-1,\ v_{i+1} - v_i \in \Delta_x\}, \\ L((v_i)) &:= \{i : 0 \leq i < m-1,\ v_{i+1} - v_i \in D_x\}. \end{aligned}$$

Note that the final index m is in neither class, and by Lemma 4.3(ii),

$$j \in S((v_i)) \ \text{ implies } \ s_x(v_{j+1} - v_j) > C_1|x|^{\chi'}(\log|x|)/2. \tag{14}$$

The next result is analogous to Proposition 3.4 in [2].

PROPOSITION 4.4. *Assume the conditions of Theorem 1.1. There exists a constant C_3 such that if $|x| \geq C_3$, then for sufficiently large n, there exists a lattice path from 0 to nx with Q_x-skeleton of $2n+1$ or fewer vertices.*

Proof. Let $(v_0, \ldots, v_m)$ be a Q_x-skeleton of some lattice path, and let

$$Y_i := \mathbb{E}T(v_i, v_{i+1}) - T(v_i, v_{i+1}).$$

Then by (A1) of Theorem 1.1 and Lemma 4.3(i), there are constants $C_4 := \alpha/(2d)^{\chi'} \geq \alpha/2d$ and C_5 such that for $0 \leq i \leq m-1$,

$$\mathbb{E}\exp(C_4|Y_i|/|x|^{\chi'}) \leq C_5. \tag{15}$$

Let $Y_0', Y_1', \ldots, Y_{m-1}'$ be independent random variables with Y_i' having the same distribution as Y_i. Let $T(0, w; (v_j))$ be the minimum passage time among all lattice paths from 0 to a site w with Q_x-skeleton (v_j). By [17, eq. (4.13)] or [1, Th. 2.3], for all $t \geq 0$,

$$\mathbb{P}\left(\sum_{i=0}^{m-1} Y_i' \geq t\right) \geq \mathbb{P}\left(\sum_{i=0}^{m-1} \mathbb{E}T(v_i, v_{i+1}) - T(0, v_m; (v_j)) \geq t\right).$$

Now by (15),

$$\mathbb{P}\left(\sum_{i=0}^{m-1} Y_i' \geq t\right) \leq e^{-C_4 t/|x|^{\chi'}} C_5^m.$$

Let $C_6 := 20d^2/\alpha$. Taking $t = C_6 m|x|^{\chi'} \log|x|$, the above display shows that there is a constant C_7 such that for all $|x| \geq C_7$,

$$\mathbb{P}\left(\sum_{i=0}^{m-1} \mathbb{E}T(v_i, v_{i+1}) - T(0, v_m; (v_j)) \geq C_6 m|x|^{\chi'} \log|x|\right) \leq (C_5 e^{-10d\log|x|})^m.$$

From the definition of a Q_x-skeleton, it is easy to see that there is a constant C_8 such that there are at most $(C_8|x|^d)^m$ Q_x-skeletons with $m+1$ vertices. Therefore, the above display shows that there are constants C_9 and C_{10} such that when $|x| \geq C_9$,

$$\mathbb{P}\Bigg(\sum_{i=0}^{m-1} \mathbb{E}T(v_i, v_{i+1}) - T(0, v_m; (v_j)) \geq C_6 m|x|^{\chi'} \log|x| \\ \text{for some } Q_x\text{-skeleton with } m+1 \text{ vertices}\Bigg) \leq e^{-C_{10} m \log|x|}.$$

This in turn yields that for some constant C_{11}, for all $|x| \geq C_{11}$,

$$\mathbb{P}\Bigg(\sum_{i=0}^{m-1} \mathbb{E}T(v_i, v_{i+1}) - T(0, v_m; (v_j)) \geq C_6 m|x|^{\chi'} \log |x| \\ \text{for some } m \geq 1 \text{ and some } Q_x\text{-skeleton with } m+1 \text{ vertices}\Bigg) \\ \leq 2e^{-C_{10} \log |x|}. \tag{16}$$

Now let $\omega := \{t_e : e \text{ is an edge in } \mathbb{Z}^d\}$ be a fixed configuration of passage times (to be further specified later), and let $(v_0, \dots, v_m)$ be the Q_x-skeleton of a route from 0 to nx. Then since $v_{i+1} - v_i \in Q_x$,

$$mg(x) \geq \sum_{i=0}^{m-1} g_x(v_{i+1} - v_i) = g_x(nx) = ng(x).$$

Therefore,

$$n \leq m. \tag{17}$$

From the concentration of first-passage times,

$$\mathbb{P}(T(0, nx) \leq ng(x) + n) \to 1 \quad \text{as } n \to \infty,$$

so by (16), if n is large, there exists a configuration ω and a Q_x-skeleton $(v_0, \dots, v_m)$ of a path from 0 to nx such that

$$T(0, nx; (v_j)) = T(0, nx) \leq ng(x) + n \tag{18}$$

and

$$\sum_{i=0}^{m-1} \mathbb{E}T(v_i, v_{i+1}) - T(0, nx; (v_j)) < C_6 m|x|^{\chi'} \log |x|. \tag{19}$$

Thus for some constant C_{12}, if $|x| \geq C_{12}$, then by (17), (18) and (19),

$$\sum_{i=0}^{m-1} \mathbb{E}T(v_i, v_{i+1}) < ng(x) + n + C_6 m|x|^{\chi'} \log |x| \\ \leq ng(x) + 2C_6 m|x|^{\chi'} \log |x|. \tag{20}$$

But by (14),

$$\sum_{i=0}^{m-1} \mathbb{E}T(v_i, v_{i+1}) = \sum_{i=0}^{m-1} (g_x(v_{i+1} - v_i) + s_x(v_{i+1} - v_i)) \\ \geq g_x(nx) + C_1 |S((v_i))| |x|^{\chi'} (\log |x|)/2$$

which, together with (20), yields

$$|S((v_i))| \leq 4C_6 m/C_1 = m/4. \tag{21}$$

At the same time, using Lemma 4.3(iii),

$$\sum_{i=0}^{m-1} \mathbb{E}T(v_i, v_{i+1}) = \sum_{i=0}^{m-1} (g_x(v_{i+1} - v_i) + s_x(v_{i+1} - v_i)) \\ \geq 5|L((v_i))|g(x)/6.$$

With (20), (11) and the assumption that $\chi' < 1$, this implies that there is a constant C_{13} such that, provided $|x| \geq C_{13}$,

$$|L((v_i))| \leq 6n/5 + \frac{12C_6 m|x|^{\chi'} \log |x|}{6g(e_1)|x|/\sqrt{d}} \leq 6n/5 + m/8.$$

This and (21) give

$$m = |L((v_i))| + |S((v_i))| + 1 \leq 6n/5 + 3m/8 + 1$$

which, for n large, implies $m \leq 2n$, proving the proposition. □

Proof of Theorem 4.1. Lemma 4.2 and Proposition 4.4 prove the conclusion of Theorem 4.1 for x with sufficiently large Euclidean norm. To prove this for all x with $|x| > 1$, one simply has to increase the value of C. □

5. Curvature bounds

The unit ball of the g-norm, usually called the 'limit shape' of first-passage percolation, is an object of great interest and intrigue in this literature. Very little is known rigorously about the limit shape, except for a fundamental result about convergence to the limit shape due to Cox and Durrett [8], some qualitative results of Kesten [17] who proved, in particular, that the limit shape may not be an Euclidean ball, an important result of Durrett and Liggett [9] who showed that the boundary of the limit shape may contain straight lines, and some bounds on the rate of convergence to the limit shape [18], [2]. In particular, it is not even known whether the limit shape may be strictly convex in every direction (except for the related continuum model of 'Riemannian first-passage percolation' [22] and first-passage percolation with stationary ergodic edge-weights [10]).

The following proposition lists two properties of the limit shape that are crucial for our purposes.

Proposition 5.1. *Let g be defined as in (9), and assume that the distribution of edge-weights is continuous. Then there exists $x_0 \in \mathbb{R}^d$ with $|x_0| = 1$, a constant $C \geq 0$ and a hyperplane H_0 through the origin perpendicular to x_0 such that for all $z \in H_0$,*

$$|g(x_0 + z) - g(x_0)| \leq C|z|^2.$$

There also exist $x_1 \in \mathbb{R}^d$ *with* $|x_1| = 1$ *and a hyperplane* H_1 *through the origin perpendicular to* x_1 *such that for all* $z \in H_1$,

$$g(x_1 + z) \geq \sqrt{1 + |z|^2}\, g(x_1).$$

Proof. The proof is similar to that of [26, Lemma 5]. Let $B(0, r)$ denote the Euclidean ball of radius r centered at the origin, and let

$$B_g(0, r) := \{x : g(x) \leq r\}$$

denote the ball of radius r centered at the origin for the norm g. Let r be the smallest number such that $B_g(0, r) \supseteq B(0, 1)$. Let x_0 be a point of intersection of $\partial B_g(0, r)$ and $\partial B(0, 1)$. Let H_0 be a hyperplane tangent to $\partial B_g(0, r)$ at x_0, translated to contain the origin. Note that $x_0 + H_0$ is also a tangent hyperplane for $B(0, 1)$ at x_0, since it touches $B(0, 1)$ only at x_0. Therefore H_0 is perpendicular to x_0. Now for any $z \in H_0$, the point $y := (x_0 + z)/|x_0 + z|$ is a point on $\partial B(0, 1)$ and hence contained in $B_g(0, r)$. Therefore,

$$g(x_0) = r \geq g(y) = \frac{1}{|x_0 + z|} g(x_0 + z) = \frac{1}{\sqrt{1 + |z|^2}} g(x_0 + z).$$

Since $g(x_0 + z)$ grows like $|z|$ as $|z| \to \infty$, this shows that there is a constant C such that

$$g(x_0 + z) \leq g(x_0) + C|z|^2$$

for all $z \in H_0$. Also, since $x_0 + z \notin B_g(0, r)$ for $z \in H_0 \backslash \{0\}$, therefore $g(x_0) \leq g(x_0 + z)$ for all $z \in H_0$. This proves the first assertion of the proposition.

For the second, we proceed similarly. Let r be the largest number such that $B_g(0, r) \subseteq B(0, 1)$. Let x_1 be a point in the intersection of $\partial B_g(0, r)$ and $\partial B(0, 1)$. Let H_1 be the hyperplane tangent to $\partial B(0, 1)$ at x_1, translated to contain the origin. Note that this is simply the hyperplane through the origin that is perpendicular to x_1. Since $B(0, 1)$ contains $B_g(0, r)$, and $y := (x_1 + z)/|x_1 + z|$ is a point in $\partial B(0, 1)$, therefore

$$g(x_1) = r \leq g(y) = \frac{1}{|x_1 + z|} g(x_1 + z) = \frac{1}{\sqrt{1 + |z|^2}} g(x_1 + z).$$

This completes the argument. □

6. Proof of $\chi_a \geq 2\xi_b - 1$

We will prove by contradiction. Suppose that $2\xi_b - 1 > \chi_a$. Choose ξ' such that

$$\frac{1 + \chi_a}{2} < \xi' < \xi_b.$$

Note that $\xi' < 1$. Let x_1 and H_1 be as in Proposition 5.1. Let n be a positive integer, to be chosen later. Throughout this proof, C will denote any positive constant that does not depend on n. The value of C may change from line to line. Also, we will assume without mention that 'n is large enough' wherever required.

Let y be the closest point in $\mathbb{Z}^d$ to nx_1. Note that

$$|y - nx_1| \le \sqrt{d}. \tag{22}$$

Let L denote the line passing through 0 and nx_1, and let L' denote the line segment joining 0 to nx_1 (but not including the endpoints). Let V be the set of all points in $\mathbb{Z}^d$ whose distance from L' lies in the interval $[n^{\xi'}, 2n^{\xi'}]$. Take any $v \in V$. We claim that there is a constant C (not depending on n) such that for any $v \in V$,

$$g(v) + g(nx_1 - v) \ge g(nx_1) + Cn^{2\xi' - 1}. \tag{23}$$

Let us now prove this claim. Let w be the projection of v onto L along H_1 (i.e. the perpendicular projection). To prove (23), there are three cases to consider. First suppose that w lies in L'. Note that $w/|w| = x_1$. Let $v' := v/|w|$ and $z := v' - x_1 = (v - w)/|w|$.

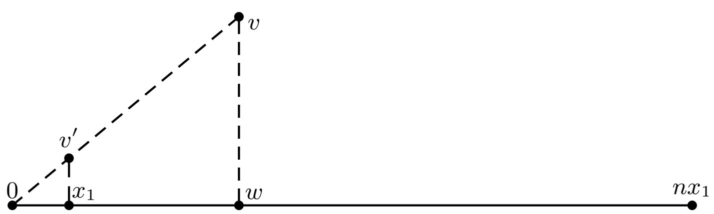

Figure 4. The relative positions of x_1, v', v, w, nx_1.

Note that $z \in H_1$. Thus by Proposition 5.1,

$$g(v') = g(x_1 + z) \ge \sqrt{1 + |z|^2}\, g(x_1).$$

Consequently,

$$g(v) \ge |w|\sqrt{1 + |z|^2}\, g(x_1). \tag{24}$$

Next, let $w' := nx_1 - w$. Note that $w'/|w'| = x_1$. Let $v'' := (nx_1 - v)/|w'|$ and

$$z' := v'' - x_1 = (w - v)/|w'|.$$

Then $z' \in H_1$, and hence by Proposition 5.1,

$$g(v'') = g(x_1 + z') \ge \sqrt{1 + |z'|^2}\, g(x_1).$$

Consequently,

$$g(nx_1 - v) \ge |w'|\sqrt{1 + |z'|^2}\, g(x_1). \tag{25}$$

Since $v \in V$, therefore $|v - w| \ge n^{\xi'}$. Again, $|w| + |w'| = n$. Thus,

$$\min\{|z|, |z'|\} \ge n^{\xi' - 1}.$$

Combining this with (24), (25), (11) and the fact that $\xi' < 1$, we have

$$\begin{aligned} g(v) + g(nx_1 - v) &\geq (|w| + |w'|)\sqrt{1 + n^{2\xi'-2}}g(x_1) \\ &= \sqrt{1 + n^{2\xi'-2}}g(nx_1) \\ &\geq g(nx_1) + Cn^{2\xi'-1}. \end{aligned}$$

Next, suppose that w lies in $L\backslash L'$, on the side closer to nx_1. As above, let $z := (v - w)/|w|$. As in (24), we conclude that

$$g(v) \geq |w|\sqrt{1 + |z|^2}g(x_1). \tag{26}$$

By the definition of V, the distance between v and nx_1 must be greater than $n^{\xi'}$. But in this case,

$$|v - nx_1|^2 = (|w| - n)^2 + |v - w|^2 = (|w| - n)^2 + |w|^2|z|^2,$$

and we also have $n \leq |w| \leq 3n$. Thus, either $|w|^2|z|^2 > n^{2\xi'}/2$ (which implies $|z|^2 \geq Cn^{2\xi'-2}$), or $|w| \geq n + n^{\xi'}/\sqrt{2}$. Since $\xi' > 2\xi' - 1$, therefore by (26), in either situation, we have

$$g(v) \geq g(nx_1) + Cn^{2\xi'-1}.$$

Similarly, if w lies in $L\backslash L'$, on the side closer to 0, then

$$g(nx_1 - v) \geq g(nx_1) + Cn^{2\xi'-1}.$$

This completes the proof of (23). Now (23) combined with Theorem 4.1, (22) and the fact that $2\xi' - 1 > \chi_a$ implies that if n is large enough, then for any $v \in V$,

$$h(v) + h(y - v) \geq h(y) + Cn^{2\xi'-1}. \tag{27}$$

Choose χ_1, χ_2 such that $\chi_a < \chi_1 < \chi_2 < 2\xi' - 1$. Then by (A1) of Theorem 1.1, there is a constant C such that for n large enough,

$$\mathbb{P}(T(0, y) > h(y) + n^{\chi_2}) \leq e^{-Cn^{\chi_2 - \chi_1}}.$$

Now, for any $v \in V$, both $|v|$ and $|y - v|$ are bounded above by Cn. Therefore, again by (A1),

$$\mathbb{P}(T(0, v) < h(v) - n^{\chi_2}) \leq e^{-Cn^{\chi_2 - \chi_1}},$$

$$\mathbb{P}(T(v, y) < h(y - v) - n^{\chi_2}) \leq e^{-Cn^{\chi_2 - \chi_1}}.$$

This, together with (27), shows that if n is large enough, then for any $v \in V$,

$$\mathbb{P}(T(0, y) = T(0, v) + T(v, y)) \leq e^{-Cn^{\chi_2 - \chi_1}}.$$

Since the size of V grows polynomially with n, this shows that

$$\mathbb{P}(T(0, y) = T(0, v) + T(v, y) \text{ for some } v \in V) \leq e^{-Cn^{\chi_2 - \chi_1}}.$$

Note that if the geodesic from 0 to y passes through V, then $T(0, y) = T(0, v) + T(v, y)$ for some $v \in V$. If $D(0, y) > n^{\xi'}$, then the geodesic must pass through V. Thus, the above inequality implies that

$$\mathbb{P}(D(0, y) > n^{\xi'}) \le e^{-Cn^{\chi_2 - \chi_1}}.$$

By (A2) of Theorem 1.1, this gives

$$\begin{aligned}\mathbb{E}D(0, y) &\le n^{\xi'} + \mathbb{E}(D(0, y)1_{\{D(0,y)>n^{\xi'}\}}) \\ &\le n^{\xi'} + \sqrt{\mathbb{E}(D(0, y)^2)\mathbb{P}(D(0, y) > n^{\xi'})} \\ &\le n^{\xi'} + C_1 n^{C_1} e^{-C_2 n^{\chi_2 - \chi_1}}.\end{aligned}$$

Taking $n \to \infty$, this shows that (A4) of Theorem 1.1 is violated (since $\xi' < \xi_b$), leading to a contradiction to our original assumption that $\chi_a < 2\xi_b - 1$. Thus, $\chi_a \ge 2\xi_b - 1$.

7. Proof of $\chi \le 2\xi - 1$ when $0 < \chi < 1/2$

In this section and the rest of the manuscript, we assume that $\chi_a = \chi_b$ and $\xi_a = \xi_b$ and denote these two numbers by χ and ξ.

Again we prove by contradiction. Suppose that $0 < \chi < 1/2$ and $\chi > 2\xi - 1$. Fix $\chi_1 < \chi < \chi_2$, to be chosen later. Choose ξ' such that

$$\xi < \xi' < \frac{1 + \chi}{2}.$$

Define

$$\begin{aligned}\beta' &:= \frac{1}{2} + \frac{\xi'}{1 + \chi}, \\ \beta &:= 1 - \frac{\chi}{2} + \frac{\chi}{2}\beta', \\ \varepsilon &:= (1 - \beta)\left(1 - \frac{\chi}{2}\right).\end{aligned}$$

We need several inequalities involving the numbers β', β and ε. Since

$$0 < \frac{\xi'}{1 + \chi} < \frac{1}{2},$$

therefore

$$\frac{1}{2} < \beta' < 1. \tag{28}$$

Since $\chi < 1$ and $\xi' < (1 + \chi)/2 < 1$,

$$\beta' > \frac{1}{2} + \frac{\xi'}{2} > \xi'. \tag{29}$$

Since β is a convex combination of 1 and β' and $\chi > 0$,

$$\beta' < \beta < 1. \tag{30}$$

Since $0 < \chi < 1$ and $0 < \beta < 1$,

$$0 < \varepsilon < 1 - \beta. \tag{31}$$

Since β' is the average of 1 and $2\xi'/(1+\chi) \in (0,1)$, therefore β' is strictly bigger than $2\xi'/(1+\chi)$ and hence

$$\begin{aligned} 2\xi' - \beta' &< 2\xi' - \frac{2\xi'}{1+\chi} \\ &= \frac{2\xi'}{1+\chi}\chi < \beta'\chi. \end{aligned} \tag{32}$$

By (30), this implies that

$$2\xi' - \beta < 2\xi' - \beta' < \beta'\chi < \beta\chi. \tag{33}$$

Next, by (28),

$$1 - \beta + \beta'\chi = \frac{\chi}{2}(1+\beta') < \chi. \tag{34}$$

And finally by (28),

$$\beta\chi + 1 - \beta - \varepsilon = \beta\chi + (1-\beta)\frac{\chi}{2} < \chi. \tag{35}$$

Let q be a large positive integer, to be chosen later. Throughout this proof, we will assume without mention that q is 'large enough' wherever required. Also, C will denote any constant that does not depend on our choice of q but may depend on all other parameters.

Let r be an integer between $\frac{1}{2}q^{(1-\beta-\varepsilon)/\varepsilon}$ and $2q^{(1-\beta-\varepsilon)/\varepsilon}$, recalling that by (31), $1-\beta-\varepsilon > 0$. Let $k = rq$. Let a be a real number between $q^{\beta/\varepsilon}$ and $2q^{\beta/\varepsilon}$. Let $n = ak$. Note that $n = arq$, which gives $\frac{1}{2}q^{1/\varepsilon} \le n \le 4q^{1/\varepsilon}$. From this it is easy to see that there are positive constants C_1 and C_2, depending only on β and ε, such that

$$C_1 n^{\varepsilon} \le q \le C_2 n^{\varepsilon}, \tag{36}$$

$$C_1 n^{1-\beta} \le k \le C_2 n^{1-\beta}, \tag{37}$$

$$C_1 n^{\beta} \le a \le C_2 n^{\beta}, \tag{38}$$

$$C_1 n^{1-\beta-\varepsilon} < r < C_2 n^{1-\beta-\varepsilon}. \tag{39}$$

Let $b := n^{\beta'}$. Note that by (30), b is negligible compared to a if q is large. Note also that, although r, k and q are integers, a, n and b need not be.

Let x_0 and H_0 be as in Proposition 5.1. For $0 \le i \le k$, define

$$\begin{aligned} U_i' &:= H_0 + iax_0, \\ V_i' &:= H_0 + (ia + a - b)x_0. \end{aligned}$$

Let U_i be the set of points in $\mathbb{Z}^d$ that are within distance $\sqrt{d}$ from U_i'. Let V_i be the set of points in $\mathbb{Z}^d$ that are within distance $\sqrt{d}$ from V_i'.

For $0 \le i \le k$, let y_i be the closest point in $\mathbb{Z}^d$ to iax_0, and let z_i be the closest point in $\mathbb{Z}^d$ to $(ia+a-b)x_0$, applying some arbitrary rule to break ties. Note that if $x \in \mathbb{R}^d$, and $y \in \mathbb{Z}^d$ is closest to x, then $|x-y| \le \sqrt{d}$. Therefore $y_i \in U_i$ and $z_i \in V_i$. Figure 5 gives a pictorial representation of the above definitions, assuming for simplicity that $U_i = U_i'$ and $V_i = V_i'$.

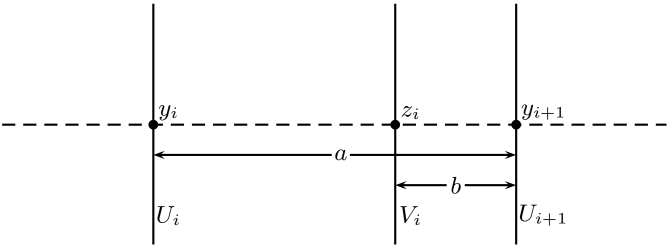

Figure 5. Diagrammatic representation of y_i, z_i, U_i and V_i.

Let U_i^o be the subset of U_i that is within distance $n^{\xi'}$ from y_i. Similarly let V_i^o be the subset of V_i that is within distance $n^{\xi'}$ from z_i.

For any $A, B \subseteq \mathbb{Z}^d$, let $T(A,B)$ denote the minimum passage time from A to B. Let $G(A,B)$ denote the (unique) geodesic from A to B, so that $T(A,B)$ is the sum of edge-weights of $G(A,B)$.

Fix any two integers $0 \le l < m \le k$ such that $m - l > 3$. Consider the geodesic $G := G(y_l, y_m)$. Since $x_0 \notin H_0$, it is easy to see that G must 'hit' each U_i and V_i, $l \le i \le m-1$. Arranging the vertices of G in a sequence starting at y_l and ending at y_m, for each $l \le i < m$, let u_i' be the first vertex in U_i visited by G and let v_i' be the first vertex in V_i visited by G. Let $u_m' := y_m$. Note that G visits these vertices in the order $u_l', v_l', u_{l+1}', v_{l+1}', \ldots, v_{m-1}', u_m'$. Figure 6 gives a pictorial representation of the points u_i' and v_i' on the geodesic G. Let T_i' be the sum of edge-weights of the portion of G from u_i' to v_i'. Let E be the event that $u_i' \in U_i^o$ and $v_i' \in V_i^o$ for each i. If E happens, then clearly

$$T_i' \ge T(U_i^o, V_i^o).$$

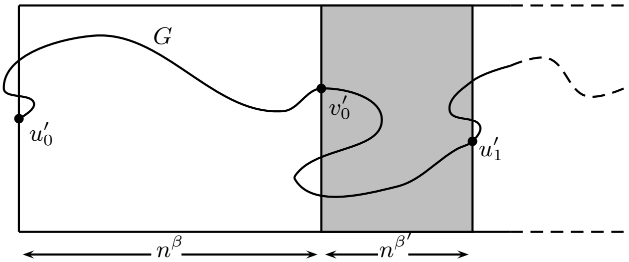

Figure 6. Location of $u_0', v_0', u_1', v_1', \ldots$ on the geodesic G.

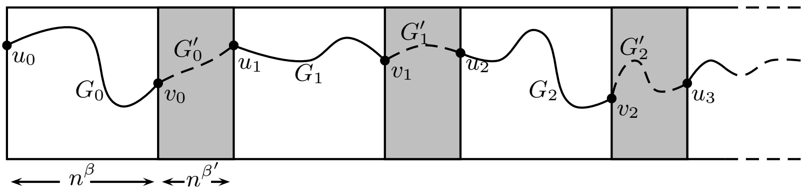

Figure 7. The paths $G_0, G'_0, G_1, G'_1, \ldots$.

Similarly, note that weight of the part of G from v'_i to u'_{i+1} must exceed or equal $T(v'_i, u'_{i+1})$. Therefore, if E happens, then

$$
(40) \qquad T(y_l, y_m) \geq \sum_{i=l}^{m-1} T'_i + \sum_{i=l}^{m-1} T(v'_i, u'_{i+1}) \\
\geq \sum_{i=l}^{m-1} T(U_i^o, V_i^o) + \sum_{i=l}^{m-1} T(v'_i, u'_{i+1}).
$$

Next, for each $0 \leq i < k$, let $G_i := G(U_i^o, V_i^o)$. Let u_i and v_i be the end-points of G_i. Let $G'_i := G(v_i, u_{i+1})$. Figure 7 gives a picture illustrating the paths G_i and G'_i. The concatenation of the paths $G(y_l, v_l)$, G'_l, G_{l+1}, G'_{l+1}, $G_{l+2}, \ldots, G'_{m-2}$, G_{m-1}, $G(v_{m-1}, y_m)$ is a path from y_l to y_m (need not be self-avoiding). Therefore,

$$
(41) \qquad T(y_l, y_m) \leq T(y_l, v_l) + \sum_{i=l+1}^{m-1} T(U_i^o, V_i^o) + \sum_{i=l}^{m-2} T(v_i, u_{i+1}) \\
+ T(v_{m-1}, y_m).
$$

Define

$$
\Delta_{l,m} := T(y_l, y_m) - \sum_{i=l}^{m-1} (T(U_i^o, V_i^o) + T(V_i^o, U_{i+1}^o)).
$$

Combining (40) and (41) implies that if E happens, then

$$
|\Delta_{l,m}| \leq \sum_{i=l}^{m-1} |T(V_i^o, U_{i+1}^o) - T(v'_i, u'_{i+1})| + \sum_{i=l}^{m-2} |T(V_i^o, U_{i+1}^o) - T(v_i, u_{i+1})| \\
+ |T(U_l^o, V_l^o) - T(y_l, v_l)| + |T(V_{m-1}^o, U_m^o) - T(v_{m-1}, y_m)|.
$$

Thus, if

$$
M_i := \max_{v, v' \in V_i^o, \ u, u' \in U_{i+1}^o} |T(v, u) - T(v', u')|,
$$

$$
N_i := \max_{u, u' \in U_i^o, \ v, v' \in V_i^o} |T(u, v) - T(u', v')|
$$

and the event E happens, then

$$|\Delta_{l,m}| \le 2\sum_{i=l}^{m-1} M_i + N_l. \tag{42}$$

For a random variable X, let $\|X\|_p := (\mathbb{E}|X|^p)^{1/p}$ denote its L^p norm. It is easy to see that $\|\Delta_{l,m}\|_4 \le n^C$, where we recall that C stands for any constant that does not depend on our choice of the integer q but may depend on χ, ξ, ξ' and the distribution of edge weights. Take any $\xi_1 \in (\xi, \xi')$. By (A2) of Theorem 1.1, $\mathbb{P}(E^c) \le e^{-Cn^{\xi'-\xi_1}}$. Together with (42), this shows that for some constants C_3 and C_4,

$$\begin{aligned}\|\Delta_{l,m}\|_2 &\le \|\Delta_{l,m}1_{E^c}\|_2 + \|\Delta_{l,m}1_E\|_2 \\ &\le \|\Delta_{l,m}\|_4(\mathbb{P}(E^c))^{1/4} + \|\Delta_{l,m}1_E\|_2 \\ &\le n^{C_3}e^{-C_4 n^{\xi'-\xi_1}} + 2\sum_{i=l}^{m-1}\|M_i\|_2 + \|N_l\|_2.\end{aligned} \tag{43}$$

Fix $0 \le i \le k-1$ and $v \in V_i^o$, $u \in U_{i+1}^o$. Let x be the nearest point to v in V_i' and y be the nearest point to u in U_{i+1}'. Then by definition of V_i' and U_{i+1}', there are vectors $z, z' \in H_0$ such that $|z|$ and $|z'|$ are bounded by $Cn^{\xi'}$, and $x = (ia + a - b)x_0 + z$ and $y = (ia + a)x_0 + z'$. Thus by Proposition 5.1,

$$\begin{aligned}|g(y-x) - g(bx_0)| &= |g(bx_0 + z' - z) - g(bx_0)| \\ &= b|g(x_0 + (z'-z)/b) - g(x_0)| \\ &\le \frac{C|z'-z|^2}{b} \le Cn^{2\xi'-\beta'}.\end{aligned}$$

Thus, for any $v, v' \in V_i^o$ and $u, u' \in U_{i+1}^o$,

$$|g(u-v) - g(u'-v')| \le Cn^{2\xi'-\beta'}.$$

Note also that $|y - x| \le C(n^{\beta'} + n^{\xi'}) \le Cn^{\beta'}$ by (29). This, together with Theorem 4.1, shows that for any $v, v' \in V_i^o$ and $u, u' \in U_{i+1}^o$,

$$|\mathbb{E}T(v,u) - \mathbb{E}T(v',u')| \le Cn^{2\xi'-\beta'} + Cn^{\beta'\chi_2}\log n.$$

By (32), this implies

$$|\mathbb{E}T(v,u) - \mathbb{E}T(v',u')| \le Cn^{\beta'\chi_2}\log n. \tag{44}$$

Let

$$M := \max_{v\in V_i^o,\ u\in U_{i+1}^o} \frac{|T(v,u) - \mathbb{E}T(v,u)|}{|u-v|^{\chi_2}}.$$

By (A1) of Theorem 1.1,

$$\begin{aligned}\mathbb{E}(e^{\alpha M}) &\le \sum_{v\in V_i^o,\ u\in U_{i+1}^o} \mathbb{E}\exp\biggl(\alpha\frac{|T(v,u)-\mathbb{E}T(v,u)|}{|u-v|^{\chi_2}}\biggr)\\ &\le C|V_i^o||U_{i+1}^o| \le Cn^C.\end{aligned}$$

This implies that $\mathbb{P}(M>t)\le Cn^C e^{-\alpha t}$, which in turn gives $\|M\|_2 \le C\log n$. Let

$$M' := \max_{v\in V_i^o,\ u\in U_{i+1}^o} |T(v,u)-\mathbb{E}T(v,u)|.$$

Since by (29), $|u-v|\le C(n^{\beta'}+n^{\xi'})\le Cn^{\beta'}$ for all $v\in V_i^o$, $u\in U_{i+1}^o$, therefore $M'\le Cn^{\beta'\chi_2}M$. Thus,

$$\|M'\|_2 \le Cn^{\beta'\chi_2}\log n.$$

From this and (44) it follows that

$$\|M_i\|_2 \le Cn^{\beta'\chi_2}\log n.$$

By an exactly similar sequence of steps, replacing β' by β everywhere and using (33) instead of (32), one can deduce that

$$\|N_i\|_2 \le Cn^{\beta\chi_2}\log n.$$

Combining with (43) this gives

$$\|\Delta_{l,m}\|_2 \le Cn^{\beta\chi_2}\log n + C(m-l)n^{\beta'\chi_2}\log n, \tag{45}$$

since the exponential term in (43) is negligible compared to the rest.

Now, from the definition of $\Delta_{l,m}$, the fact that $k=rq$ and the triangle inequality, it is easy to see that

$$\biggl|T(y_0,y_k)-\sum_{j=0}^{r-1}T(y_{jq},y_{(j+1)q})\biggr| \le |\Delta_{0,k}| + \sum_{j=0}^{r-1}|\Delta_{jq,(j+1)q}|.$$

Thus by (45), (39) and (37),

$$\begin{aligned}\biggl\|T(y_0,y_k)-\sum_{j=0}^{r-1}T(y_{jq},y_{(j+1)q})\biggr\|_2 &\le \|\Delta_{0,k}\|_2 + \sum_{j=0}^{r-1}\|\Delta_{jq,(j+1)q}\|_2\\ &\le C(r+1)n^{\beta\chi_2}\log n + Ckn^{\beta'\chi_2}\log n\\ &\le Cn^{1-\beta-\varepsilon+\beta\chi_2}\log n + Cn^{1-\beta+\beta'\chi_2}\log n.\end{aligned} \tag{46}$$

For any two random variables X and Y,

$$\begin{aligned}\Bigl|\sqrt{\mathrm{Var}(X)}-\sqrt{\mathrm{Var}(Y)}\Bigr| &= \bigl|\|X-\mathbb{E}X\|_2 - \|Y-\mathbb{E}Y\|_2\bigr|\\ &\le \|(X-\mathbb{E}X)-(Y-\mathbb{E}Y)\|_2\\ &\le \|X-Y\|_2 + |\mathbb{E}X-\mathbb{E}Y| \le 2\|X-Y\|_2.\end{aligned} \tag{47}$$

Therefore it follows from (46) that

$$(48)\qquad \left|(\mathrm{Var}T(y_0,y_k))^{1/2} - \left(\mathrm{Var}\sum_{j=0}^{r-1} T(y_{jq}, y_{(j+1)q})\right)^{1/2}\right| \\ \le Cn^{1-\beta-\varepsilon+\beta\chi_2}\log n + Cn^{1-\beta+\beta'\chi_2}\log n.$$

For any $x, y \in \mathbb{Z}^d$, $T(x,y)$ is an increasing function of the edge weights. So by the Harris-FKG inequality [12], $\mathrm{Cov}(T(x,y), T(x',y')) \ge 0$ for any x, y, x', $y' \in \mathbb{Z}^d$. Therefore by (A3) of Theorem 1.1 and (38), (39) and (36),

$$(49)\qquad \begin{aligned}\mathrm{Var}\sum_{j=0}^{r-1} T(y_{jq}, y_{(j+1)q}) &\ge \sum_{j=0}^{r-1}\mathrm{Var}T(y_{jq}, y_{(j+1)q})\\ &\ge C\sum_{j=0}^{r-1}|y_{jq} - y_{(j+1)q}|^{2\chi_1}\\ &\ge Cr(aq)^{2\chi_1} \ge Cn^{(1-\beta-\varepsilon)+(\beta+\varepsilon)2\chi_1}.\end{aligned}$$

By the inequalities (34) and (35), we see that if χ_1 and χ_2 are chosen sufficiently close to χ, then χ_1 is strictly bigger than both $1-\beta-\varepsilon+\beta\chi_2$ and $1-\beta+\beta'\chi_2$. Therefore by (48) and (49), and since $1-\beta-\varepsilon+(\beta+\varepsilon)2\chi_1 > 2\chi_1$,

$$\mathrm{Var}T(y_0,y_k) \ge Cn^{(1-\beta-\varepsilon)+(\beta+\varepsilon)2\chi_1}.$$

By (31) and the assumption that $\chi < 1/2$, we again have that if χ_1 is chosen sufficiently close to χ,

$$(1-\beta-\varepsilon)+(\beta+\varepsilon)2\chi_1 > 2\chi.$$

Since $|y_0 - y_k| \le Cak \le Cn$ by (38) and (37), therefore taking $q \to \infty$ (and hence $n \to \infty$) gives a contradiction to (A1) of Theorem 1.1, thereby proving that $\chi \le 2\xi - 1$ when $0 < \chi < 1/2$.

8. **Proof of** $\chi \le 2\xi - 1$ **when** $\chi = 1/2$

Suppose that $\chi = 1/2$ and $\chi > 2\xi - 1$. Define χ_1, χ_2, x_0, H_0, ξ', β, β', ε, q, a, r, k, n, y_i and z_i exactly as in Section 7, considering a, r, k and n as functions of q. Then all steps go through, except the very last, where we used $\chi < 1/2$ to get a contradiction. Therefore all we need to do is modify this last step to get a contradiction in a different way. This is where we need the sublinear variance inequality (1). As before, throughout the proof C denotes any constant that does not depend on q.

For each real number $m \ge 1$, let w_m be the nearest lattice point to mx_0. Note that $y_i = w_{ia}$. Let

$$f(m) := \mathrm{Var}T(0, w_m).$$

Note that there is a constant C_0 such that $f(m) \le C_0 m$ for all m. Again by (A3), there is a $C_1 > 0$ such that for all m,

$$f(m) \ge C_1 m^{2\chi_1}. \tag{50}$$

Now, $|(w_{(j+1)aq} - w_{jaq}) - w_{aq}| \le C$. Again, as a consequence of (47), we have that for any two random variables X and Y,

$$\begin{aligned}\bigl|\mathrm{Var}(X) - \mathrm{Var}(Y)\bigr| &= \Bigl|\sqrt{\mathrm{Var}(X)} - \sqrt{\mathrm{Var}(Y)}\Bigr|\Bigl(\sqrt{\mathrm{Var}(X)} + \sqrt{\mathrm{Var}(Y)}\Bigr) \\ &\le 2\|X-Y\|_2\bigl(2\sqrt{\mathrm{Var}(X)} + 2\|X-Y\|_2\bigr).\end{aligned} \tag{51}$$

By (51) and the subadditivity of first-passage times,

$$\begin{aligned}\mathrm{Var}(T(w_{jaq}, w_{(j+1)aq})) &\ge f(aq) - C\sqrt{f(aq)} - C \\ &\ge f(n/r) - C\sqrt{n/r}.\end{aligned}$$

Therefore by the Harris-FKG inequality,

$$\mathrm{Var}\Biggl(\sum_{j=0}^{r-1} T(w_{jaq}, w_{(j+1)aq})\Biggr) \ge rf(n/r) - C\sqrt{nr}. \tag{52}$$

Now, by (34) and (35), if χ_2 is sufficiently close to χ, then both $1-\beta-\varepsilon+\beta\chi_2$ and $1-\beta+\beta'\chi_2$ are strictly smaller than $1/2$. Therefore by (46), (51) and the fact that $f(n) \le Cn$,

$$\begin{aligned}&\Biggl|f(n) - \mathrm{Var}\Biggl(\sum_{j=0}^{r-1} T(w_{jaq}, w_{(j+1)aq})\Biggr)\Biggr| \\ &\qquad \le C\sqrt{n}(n^{1-\beta-\varepsilon+\beta\chi_2}\log n + n^{1-\beta+\beta'\chi_2}\log n).\end{aligned}$$

Combining this with (52) gives

$$f(n) \ge rf(n/r) - C\sqrt{nr} - C\sqrt{n}(n^{1-\beta-\varepsilon+\beta\chi_2}\log n + n^{1-\beta+\beta'\chi_2}\log n).$$

Again by (39) and (50),

$$rf(n/r) \ge Cn^{(1-\beta-\varepsilon)+(\beta+\varepsilon)2\chi_1}.$$

Combining (39) with the last two displays, it follows that we can choose χ_1 and χ_2 so close to $1/2$ that as $q \to \infty$,

$$\liminf \frac{f(n)}{rf(n/r)} \ge 1.$$

In particular, for any $\delta > 0$, there exists an integer $q(\delta)$ such that if $q \ge q(\delta)$, then

$$f(n) \ge (1-\delta)rf(n/r). \tag{53}$$

Fix $\delta = (1-\beta-\varepsilon)/2$ and choose $q(\delta)$ satisfying the above criterion. Note that $q(\delta)$ can be chosen as large as we like. Let $m_0 := aq = n/r$ and $m_1 = n$. The above inequality implies that

$$\frac{f(m_1)}{m_1} \geq (1-\delta)\frac{f(m_0)}{m_0}.$$

Note that by (36), if $q(\delta)$ is chosen sufficiently large to begin with, then

$$m_1^{\varepsilon/(\beta+\varepsilon)} > Cq^{1/(\beta+\varepsilon)} > q(\delta).$$

We now inductively define an increasing sequence $m_2, m_3, \ldots$ as follows. Suppose that m_{i-1} has been defined such that

$$m_{i-1}^{\varepsilon/(\beta+\varepsilon)} > q(\delta). \tag{54}$$

Let

$$q_i := [m_{i-1}^{\varepsilon/(\beta+\varepsilon)}] + 1,$$

where $[x]$ denotes the integer part of a real number x. By (54), $q_i \geq q(\delta)$. Let $a_i := m_{i-1}/q_i$. Then if $q(\delta)$ is chosen large enough,

$$a_i \geq \frac{2}{3} m_{i-1}^{\beta/(\beta+\varepsilon)} \geq \frac{1}{2} q_i^{\beta/\varepsilon}$$

and

$$a_i \leq m_{i-1}^{\beta/(\beta+\varepsilon)} \leq q_i^{\beta/\varepsilon}.$$

Let r_i be an integer between $q_i^{(1-\beta-\varepsilon)/\varepsilon}$ and $2q_i^{(1-\beta-\varepsilon)/\varepsilon}$. Let $k_i = r_i q_i$ and $n_i = a_i k_i = a_i r_i q_i = r_i m_{i-1}$. If we carry out the argument of Section 7 with q_i, r_i, k_i, a_i, n_i in place of q, r, k, a, n, then, since $q_i \geq q(\delta)$, as before we arrive at the inequality

$$f(n_i) \geq (1-\delta) r_i f(n_i/r_i) = (1-\delta) r_i f(m_{i-1}).$$

Define $m_i := n_i$. Then the above inequality shows that

$$\frac{f(m_i)}{m_i} \geq (1-\delta)\frac{f(m_{i-1})}{m_{i-1}}. \tag{55}$$

Note that since r_i is a positive integer and $m_i = r_i m_{i-1}$, therefore $m_i \geq m_{i-1}$. In particular, (54) is satisfied with m_i in place of m_{i-1}. This allows us to carry on the inductive construction such that (55) is satisfied for each i.

Now, the above construction shows that if the initial q was chosen large enough, then for each i,

$$m_i = r_i m_{i-1} \geq q_i^{(1-\beta-\varepsilon)/\varepsilon} m_{i-1} \geq m_{i-1}^{1/(\beta+\varepsilon)}.$$

Therefore, for all $i \geq 2$,

$$m_i \geq m_1^{(\beta+\varepsilon)^{-(i-1)}}.$$

So, by (1), there exists a constant C_3 such that

$$\frac{f(m_i)}{m_i} \le \frac{C}{\log m_i} \le C_3(\beta + \varepsilon)^{i-1}.$$

However, (55) shows that there is $C_4 > 0$ such that

$$\frac{f(m_i)}{m_i} \ge C_4(1-\delta)^{i-1}.$$

Since $1 - \delta > \beta + \varepsilon$, we get a contradiction for sufficiently large i.

9. Proof of $\chi \le 2\xi - 1$ when $\chi = 0$

As usual, we prove by contradiction. Assume that $\chi = 0$ and $2\xi - 1 < \chi$. Then $\xi < 1/2$. Choose ξ_1, ξ' and ξ'' such that $\xi < \xi_1 < \xi'' < \xi' < 1/2$. From this point on, however, the proof is quite different from the case $\chi > 0$. Recall that $t(P)$ is the sum of edge-weights of a path P in the environment $t = (t_e)_{e \in E(\mathbb{Z}^d)}$. This notation is used several times in this section. First, we need a simple lemma about the norm g.

LEMMA 9.1. *Assume that the edge-weight distribution is continuous, and let L denote the infimum of its support. Then there exists $M > L$ such that for all $x \in \mathbb{R}^d \setminus \{0\}$, $g(x) \ge M|x|_1$, where $|x|_1$ is the ℓ_1 norm of x.*

Proof. Since g is a norm on $\mathbb{R}^d$,

$$M := \inf_{x \neq 0} \frac{g(x)}{|x|_1} > 0,$$

and the infimum is attained. Choose $x \neq 0$ such that $g(x) = M|x|_1$. Define a new set of edge-weights s_e as $s_e := t_e - L$. Then s_e are nonnegative and independent and identically distributed. Let the function g^s be defined for these new edge-weights the same way g was defined for the old weights. Similarly, define h^s and T^s. Since any path P from a point y to a point z must have at least $|z - y|_1$ many edges, therefore $s(P) \le t(P) - L|z - y|_1$. Thus,

$$T^s(y, z) \le T(y, z) - L|z - y|_1.$$

In particular, $h^s(y) \le h(y) - L|y|_1$ for any y. Considering a sequence y_n in $\mathbb{Z}^d$ such that $y_n/n \to x$, we see that

$$\begin{aligned} g^s(x) &= \lim_{n\to\infty} \frac{h^s(y_n)}{n} \le \lim_{n\to\infty} \frac{h(y_n) - L|y_n|_1}{n} \\ &= g(x) - L|x|_1 = (M - L)|x|_1. \end{aligned}$$

Since t_e has a continuous distribution, s_e has no mass at 0. Therefore, by a well-known result (see [17]), $g^s(x) > 0$. This shows that $M > L$. □

Choose β, ε' and ε so small that $0 < \varepsilon' < \varepsilon < \beta < (\xi'' - \xi_1)/d$. Choose x_0 and H_0 as in Proposition 5.1. Let n be a positive integer, to be chosen arbitrarily large at the end of the proof. Again, as usual, C denotes any positive constant that does not depend on our choice of n.

Choose a point $z \in H_0$ such that $|z| \in [n^{\xi'}, 2n^{\xi'}]$. Let $v := nx_0/2 + z$. Then by Proposition 5.1 and the fact that $\xi' < 1/2$,

$$\begin{aligned} |g(v) - g(nx_0/2)| &= (n/2)|g(x_0 + 2z/n) - g(x_0)| \\ &\leq C|z|^2/n \leq Cn^{2\xi'-1} \leq C. \end{aligned} \tag{56}$$

Similarly,

$$|g(nx_0 - v) - g(nx_0/2)| \leq Cn^{2\xi'-1} \leq C. \tag{57}$$

Let w be the closest lattice point to v, and let y be the closest lattice point to nx_0. Then $|w - v|$ and $|y - nx_0|$ are bounded by $\sqrt{d}$. Therefore, inequalities (56) and (57) imply that

$$|g(y) - (g(w) + g(y - w))| \leq C. \tag{58}$$

Figure 8 has an illustration of the relative locations of y and w, together with some other objects that will be defined below.

By Theorem 4.1 and the assumption that $\chi = 0$, $|h(y)-g(y)|$, $|h(w)-g(w)|$ and $|h(y-w)-g(y-w)|$ are all bounded by Cn^{ε}. Again by (A1) of Theorem 1.1 and the assumption that $\chi = 0$, the probabilities $\mathbb{P}(|T(0,w) - h(w)| > n^{\varepsilon})$, $\mathbb{P}(|T(w,y) - h(y - w)| > n^{\varepsilon})$ and $\mathbb{P}(|T(0,y) - h(y)| > n^{\varepsilon})$ are all bounded by $e^{-Cn^{\varepsilon-\varepsilon'}}$. These observations, together with (58), imply that there are constants C_1 and C_2, independent of our choice of n, such that

$$\mathbb{P}(|T(0,y) - (T(0,w) + T(w,y))| > C_1 n^{\varepsilon}) \leq e^{-C_2 n^{\varepsilon-\varepsilon'}}. \tag{59}$$

Let $T_o(0,y)$ be the minimum passage time from 0 to y among all paths that do not deviate by more than $n^{\xi''}$ from the straight line segment joining 0 and y. By assumption (A2) of Theorem 1.1,

$$\mathbb{P}(T_o(0,y) = T(0,y)) \geq 1 - e^{-Cn^{\xi''-\xi_1}}.$$

Combining this with (59), we see that if E_1 is the event

$$E_1 := \{|T_o(0,y) - (T(0,w) + T(w,y))| \leq C_1 n^{\varepsilon}\}, \tag{60}$$

where C_1 is the constant from (59), then there is a constant C_3 such that

$$\mathbb{P}(E_1) \geq 1 - e^{-C_3 n^{\xi''-\xi_1}} - e^{-C_3 n^{\varepsilon-\varepsilon'}}. \tag{61}$$

Let V be the set of all lattice points within ℓ_1 distance n^{β} from w. Let ∂V denote the boundary of V in $\mathbb{Z}^d$; that is, all points in V that have at least one neighbor outside of V. Let w_1 be the first point in $G(0,w)$ that belongs to ∂V, when the points are arranged in a sequence from 0 to w. Let w_2 be the last point in $G(w,y)$ that belongs to ∂V, when the points are arranged in a sequence from w to y. Let G_1 denote the portion of $G(0,w)$ connecting w_1 and w, and let G_2 denote the portion of $G(w,y)$ connecting w and w_2. Let G_0 be the portion of $G(0,w)$ from 0 to w_1 and let G_3 be the portion of $G(w,y)$ from w_2 to y. Note that G_0 and G_3 lie entirely outside of V. Figure 8 provides a schematic diagram to illustrate the above definitions.

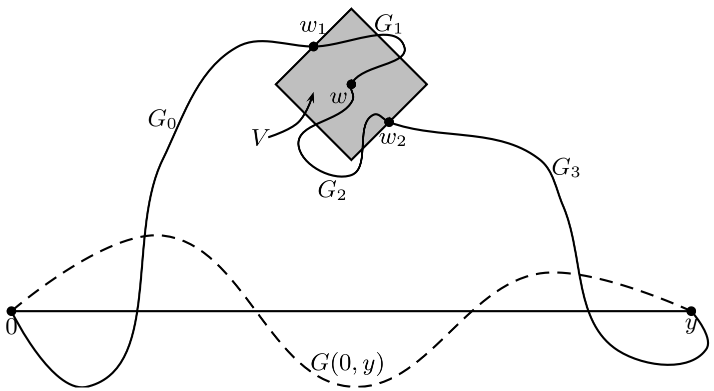

Figure 8. Schematic diagram for V, w, w_1, w_2 and G_0, G_1, G_2, G_3.

Let L and M be as in Lemma 9.1. Choose L', M' such that $L < L' < M' < M$. Take any $u \in \partial V$. By Lemma 9.1, $g(u-w) \geq M|u-w|_1$. Therefore by Theorem 4.1,

$$h(u-w) \geq M|u-w|_1 - C|u-w|^{\varepsilon} \geq M|u-w|_1 - Cn^{\beta\varepsilon}.$$

Now, $|u-w|_1 \geq Cn^{\beta}$. Therefore by assumption (A1) of Theorem 1.1 and the above inequality,

$$\begin{aligned}&\mathbb{P}(T(u,w) < M'|u-w|_1)\\ &\quad\leq \mathbb{P}(|T(u,w)-h(u-w)| > (M-M')|u-w|_1 - Cn^{\beta\varepsilon})\\ &\quad\leq \mathbb{P}(|T(u,w)-h(u-w)| > Cn^{\beta}) \leq e^{-n^{\beta-\varepsilon'}/C}.\end{aligned}$$

Since there are at most n^C points in ∂V, the above bound shows that

$$\mathbb{P}(T(u,w) < M'|u-w|_1 \text{ for some } u \in \partial V) \leq n^C e^{-n^{\beta-\varepsilon'}/C}.$$

In particular, if E_2 and E_3 are the events

$$\begin{aligned}E_2 &:= \{t(G_1) \geq M'|w-w_1|_1\},\\ E_3 &:= \{t(G_2) \geq M'|w-w_2|_1\},\end{aligned}$$

then there is a constant C_4 such that

$$\mathbb{P}(E_2 \cap E_3) \geq 1 - n^{C_4} e^{-n^{\beta-\varepsilon'}/C_4}. \tag{62}$$

Let $E(V)$ denote the set of edges between members of V. Let $(t'_e)_{e\in E(V)}$ be a collection of independent and identically distributed random variables, independent of the original edge-weights, but having the same distribution. For $e \notin E(V)$, let $t'_e := t_e$. Let E_4 be the event

$$E_4 := \{t'_e \le L' \text{ for each } e \in E(V)\}.$$

If E_4 happens, then there is a path P_1 from w_1 to w and a path P_2 from w to w_2 such that $t'(P_1) \le L'|w - w_1|_1$ and $t'(P_2) \le L'|w - w_2|_1$. Let P be the concatenation of the paths G_0, P_1, P_2 and G_3. Since $t'(G_0) = t(G_0)$ and $t'(G_3) = t(G_3)$, therefore under E_4,

$$t'(P) \le t(G_0) + t(G_3) + L'|w - w_1|_1 + L'|w - w_2|_1.$$

On the other hand, under $E_2 \cap E_3$,

$$\begin{aligned} T(0, w) + T(w, y) &= t(G_0) + t(G_1) + t(G_2) + t(G_3) \\ &\ge t(G_0) + t(G_3) + M'|w - w_1|_1 + M'|w - w_2|_1. \end{aligned}$$

Consequently, if E_1, E_2, E_3, E_4 all happen simultaneously, then there is a (deterministic) positive constant C_5 such that

$$T_o(0, y) \ge t'(P) + C_5 n^\beta - C_1 n^\varepsilon,$$

where C_1 is the constant in the definition (60) of E_1. Since $\beta < \xi'' < \xi'$ and $x_0 \notin H_0$, the edges within distance $n^{\xi''}$ of the line segment joining 0 and y have the same weights in the environment t' as in t. Since $\beta > \varepsilon$, this observation and the above display proves that $E_1 \cap E_2 \cap E_3 \cap E_4$ implies $D'(0, y) \ge n^{\xi''}$, where $D'(0, y)$ is the value of $D(0, y)$ in the new environment t'. (To put it differently, if $E_1 \cap E_2 \cap E_3 \cap E_4$ happens, then there is a path P that has less t'-weight than the least t'-weight path within distance $n^{\xi''}$ of the straight line connecting 0 to y, and therefore $D'(0, y)$ must be greater than or equal to $n^{\xi''}$.)

Now note that the event E_4 is independent of E_1, E_2 and E_3. Moreover, since $L' > L$, there is a constant C_6 such that $\mathbb{P}(E_4) \ge e^{-C_6 n^{\beta d}}$. Combining this with (61), (62) and the last observation from the previous paragraph, we get

$$\begin{aligned} \mathbb{P}(D'(0, y) \ge n^{\xi''}) &\ge \mathbb{P}(E_1 \cap E_2 \cap E_3 \cap E_4) \\ &= \mathbb{P}(E_1 \cap E_2 \cap E_3)\mathbb{P}(E_4) \\ &\ge (1 - e^{-C_3 n^{\xi'' - \xi_1}} - e^{-C_3 n^{\varepsilon - \varepsilon'}} - n^{C_4} e^{-n^{\beta - \varepsilon'}/C_4}) e^{-C_6 n^{\beta d}} \\ &\ge e^{-C_7 n^{\beta d}}. \end{aligned}$$

Now $D'(0, y)$ has the same distribution as $D(0, y)$. But by (A2) of Theorem 1.1, $\mathbb{P}(D(0, y) \ge n^{\xi''}) \le e^{-C_8 n^{\xi'' - \xi_1}}$, and $\beta d < \xi'' - \xi_1$ by our choice of β. Together with the above display, this gives a contradiction, thereby proving that $\chi \le 2\xi - 1$ when $\chi = 0$.

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