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The GIT stability of polarized varieties via discrepancy

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Dedicated to Professor Shigefumi Mori on his Kanreki (60th birthday)

Abstract

We prove that various GIT semistabilities of polarized varieties imply semi-log-canonicity.

1. Introduction

For the study of the moduli of polarized varieties, Geometric Invariant Theory [Mum65] (GIT, for short) is an important basis because it constructs the moduli spaces as quotient schemes of the Hilbert schemes. In that theory, we must put restrictions on the objects to classify, which we call *stability*, the GIT stability. It is a quite difficult and interesting problem to explicitly understand the stability notion.

The projective moduli variety of stable curves $\overline{M_g}$ is constructed in GIT by permitting ordinary double points (nodes) to curves ([DM69], [KM76], [Mum77], [Gie82]), which is sometimes called the Deligne-Mumford compactification. We note that semistable polarized curves have only nodal singularities.

In this paper, we give its higher dimensional generalization and show that the general effect of singularities on stability is determined by the *discrepancy*, an invariant of singularity which was developed along the minimal model program. This is our new point of view. Recall that the discrepancy is defined under the following conditions, which ensure that the canonical divisor K_X or the canonical sheaf ω_X is in a tractable class (cf., e.g., [Ale96]).

Definition 1.1. An algebraic scheme X is said to *satisfy* $(*)$ when the following conditions hold:

(i) X is equidimensional and reduced.

(ii) X satisfies Serre's S_2 condition.

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(iii) Codimension-1 points of X are Gorenstein.
(iv) K_X is $\mathbb{Q}$-Cartier, in the sense that $\mathcal{O}_X(nK_X) := (\omega_X^{\otimes n})^{\vee\vee}$ is an invertible sheaf for some $n \in \mathbb{Z}_{>0}$, where $\mathcal{F}^\vee := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$.

Then, our main result is the following.

Theorem 1.2. *Let X be a projective scheme satisfying $(*)$ and L be an ample line bundle on X. Then, if (X, L) is* K*-semistable, X has only semi-log-canonical singularities.*

We will also explain that Theorem 1.2 should be the best possible , as we will see in the statements of Theorem 1.5.

As already mentioned, the definition of *semi-log-canonicity* is based on the *discrepancy.* The theory of discrepancy originally stemmed out of necessity in the way of extending minimal models for surfaces by the Italian school to higher dimensions after Mori [Mor82], as they should be allowed to have some mild singularities. Indeed, it forms a core notion in the minimal model program (the MMP, for short).

Along the development of the MMP, the semi-log-canonicity was first introduced by Kollár and Shepherd-Barron [KSB88] for surfaces and extended by Alexeev [Ale96] to higher dimensions. Their original purpose was to construct the compactified moduli spaces for varieties of general type *not* by GIT theory, but by MMP techniques. For the case of curves, semi-log-canonical singularities are simply smooth points or nodes. Semi-log-canonical surface singularities are classified by Kollár-Shepherd-Barron [KSB88, Th. (4.24)].

Now, let us explain the other side, i.e., the stability notion. While the GIT stability was originally intended to construct moduli spaces as mentioned at the beginning, the K-(*semi*)*stability* is a version of GIT-stability notion that was firstly introduced by Tian [Tia97] to describe when a Fano manifold has a Kähler-Einstein metric. Subsequently, Donaldson [Don02] extended the notion to general polarized varieties with an expectation of correspondence with the existence of Kähler metrics whose scalar curvature are constant (cscK, for short). We follow Donaldson's formulation [Don02] in this paper. We note that it is defined algebro-geometrically, although the introduction is motivated by differential geometry.

Thus, roughly speaking, our Main Theorem 1.2 bridges in a fresh way these two theories in algebraic geometry, i.e., birational geometry and GIT stability (in a broader sense). In addition, due to the conjectural correspondence with metrics side, one could hope that stability or moduli problems have further connections with differential geometry.

We should make some comments on Theorem 1.2. Firstly, we remark on the Fano case. In this paper, X is said to be a $(*)$-*Fano scheme* if X is a projective scheme satisfying $(*)$ and $-K_X$ is ample (we do not *a priori* assume normality of X). In this case, we can prove the following stronger result by slightly different arguments.

THEOREM 1.3. *If X is a $(*)$-Fano scheme as above and $(X, \mathcal{O}_X(-mK_X))$ is* K*-semistable with $m \in \mathbb{Z}_{>0}$, then X is log terminal.* (*In particular, X should be normal.*)

Secondly, let us comment on other stability notions. Recall that Mumford and Gieseker studied *asymptotic* (*Chow* and *Hilbert*) *stabilities*, which were the original stability notions for polarized varieties ([Gie82], [Mum77] etc). It is well known that these asymptotic (Chow or Hilbert) semistabilities imply K-semistability (cf. [RT07, §3]). Furthermore, there are more stability notions introduced recently by Donaldson ([Don10]), called $\bar{\mathrm{K}}$-stability and b-stability. It seems that these two notions are expected to be equivalent at least for smooth case, and we can see that $\bar{\mathrm{K}}$-semistability is also stronger than K-semistability. Therefore, we have

COROLLARY 1.4. (i) *Let X be a projective scheme satisfying $(*)$ and L be an ample line bundle on X. Then, if (X, L) is asymptotically* (*Chow or Hilbert*) *semistable, X has only semi-log-canonical singularities.*

(ii) *Let X be a projective scheme satisfying $(*)$ and L be an ample line bundle on X. Then, if (X, L) is $\bar{\mathrm{K}}$-semistable, X has only semi-log-canonical singularities.*

(iii) *If X is a $(*)$-Fano scheme and $(X, \mathcal{O}_X(-mK_X))$ with $m \in \mathbb{Z}_{>0}$ is asymptotically* (*Chow or Hilbert*) *semistable or $\bar{\mathrm{K}}$-semistable, then X is log terminal.* (*In particular, X should be normal.*)

A final but an important remark about Theorem 1.2 is that the following converse has already been proved for the Calabi-Yau case ([Oda]) and the canonically polarized case ([Oda12a]). In this sense, Theorem 1.2 is the best possible as mentioned earlier.

THEOREM 1.5. (i) ([Oda]). *A semi-log-canonical polarized variety (X, L) with numerically trivial canonical divisor K_X is* K*-semistable.*

(ii) ([Oda12a]). *A semi-log-canonical* (*pluri*)*canonically polarized variety*

$$(X, \mathcal{O}_X(mK_X))$$

with $m \in \mathbb{Z}_{>0}$ is K*-stable.*

We caution that, on the other hand, the singularities do not determine stabilities *in general*, as it is well known that there are smooth but not semistable polarized manifolds. We also remark that it has been known for a few decades that the asymptotic (semi)stability version of Theorem 1.5 does *not* hold (cf.

[SB83], [Oda12a]). The author supposes that this phenomenon ought to be a major reason why the relation between discrepancy and stability of polarized varieties has been unexpected so far.

The bare structure of the proof of Theorem 1.2 is that, assuming nonsemi-log-canonicity of X (i.e., X has "bad" singularities), we construct a "destabilizing" one-parameter subgroup by making use of a certain birational model of X and $X \times \mathbb{A}^1$. For the proof, we define the *S-coefficient*, which is an invariant of certain ideals of $X \times \mathbb{A}^1$. Very roughly speaking, to those ideals we associate the one-parameter subgroups.

The birational model of X that we shall use is the (relative) *semi-log canonical model* whose existence has been conjectured in the theory of the log minimal model program (LMMP, for short), at least for the normal case. The existence is recently verified in [OX].

In our standpoint, Shah [Sha81] introduced our key invariant S-coefficient for isolated singularities by an argument based on Eisenbud-Mumford's local stability theory [Mum77] and applied it to give certain list of semistable surface singularities, which gave us one of the major inspirations for Theorem 1.2.

Our paper is organized as follows. In the next section, we will review the basic stability notions for polarized varieties and some preparatory materials related to the log minimal model program. In Section 3, we will formulate an invariant of polarized varieties (with an ideal of certain type attached), which we call the *S-coefficient*, as a generalization of "a_I" in [Sha81]. Actually, the S-coefficient can be regarded as the leading coefficient of some series of the Donaldson-Futaki invariants, which can be calculated by the formula in Theorem (3.2) proven in [Oda], [Wan12]. Then we give technical details to the (birational geometric part of) proof of Theorems 1.2 and 1.3.

Conventions. Throughout, we work over an algebraically closed field k with characteristic 0. A polarization means an *ample* invertible sheaf, and a polarized scheme means an algebraic scheme X equipped with an ample invertible sheaf L. (X, L) always denotes a polarized scheme, and except in Section 2.1 and a small part of Section 3.1, it is assumed to satisfy $(*)$ as in the statement of Theorem 1.2. (For example, an arbitrary reduced projective hypersurface, or more generally, a (global) reduced complete intersection satisfies the conditions.)

$\mathrm{NN}(X)$, $\mathrm{NLC}(X)$, $\mathrm{NSLC}(X)$, and $\mathrm{NKLT}(X)$ denote nonnormal locus, nonlog-canonical locus, nonsemi-log-canonical locus, and non-Kawamata-log-terminal locus of X, respectively. X^{ν} denotes the normalization of a given variety X. $a(E;X)$ denotes the discrepancy of a divisor E over a normal variety X, and $a(E;X,D)$ denotes the discrepancy of E over a normal pair (X,D) (i.e., a pair of a normal variety X and its Weil divisor D with $\mathbb{Q}$-Cartier $K_X + D$).

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2. **Preliminaries**

In this section, we review the basics for K-stability, discrepancy, and log canonical model.

2.1. *K-stability*. K-stability was introduced first, under differential geometric background, by Tian in [Tia97]. It was reformulated and extended later by Donaldson [Don02]. Donaldson's version of K-stability and K-polystability were slightly amended recently by [LX11], while the semistability notion remains the same (see also [Oda12b], [Sto11]). Recall that it is the motivation for introducing K-(semi, poly)stability to seek the GIT-counterpart of the existence of special Kähler metric. Indeed, according to Professors Gang Tian and Toshiki Mabuchi, the "K" in K-stability stands for the K-energy (Mabuchi energy), a functional on the space of Kähler metrics whose critical points are canonical Kähler metrics and at last the "K" in the K-energy came from "K"ähler.

For the definition of the stability, we need the concept of "test configuration" following Donaldson [Don02]. Our notation (and even expression) mostly follows [RT07].

Definition 2.1. A *test configuration* (resp. *semi-test configuration*) for a polarized complete scheme (X, L) is a quasi-projective scheme $\mathcal{X}$ with an invertible sheaf $\mathcal{M}$ on it with

(i) a $\mathbb{G}_m$ action on $(\mathcal{X}, \mathcal{M})$,
(ii) a proper flat morphism $\alpha\colon \mathcal{X} \to \mathbb{A}^1$,

such that α is $\mathbb{G}_m$-equivariant for the usual action on $\mathbb{A}^1$:

$$\begin{aligned}\mathbb{G}_m \times \mathbb{A}^1 &\longrightarrow \mathbb{A}^1,\\ (t,x) &\longmapsto tx.\end{aligned}$$

$\mathcal{M}$ is relatively ample (resp. relatively semi-ample), and $(\mathcal{X},\mathcal{M})|_{\alpha^{-1}(\mathbb{A}^1\setminus\{0\})}$ is $\mathbb{G}_m$-equivariantly isomorphic to $(X, L^{\otimes r})\times(\mathbb{A}^1\setminus\{0\})$ for some positive integer r, called *exponent*, with the natural action of $\mathbb{G}_m$ on the latter and the trivial action on the former.

PROPOSITION 2.2 ([RT07, Prop. 3.7]). *In the above situation, a one-parameter subgroup of* $\mathrm{GL}(H^0(X, L^{\otimes r}))$ *is equivalent to the data of a test configuration* $(\mathcal{X},\mathcal{M})$ *of* (X, L) *with the polarization* $\mathcal{M}$ *very ample* (*over* $\mathbb{A}^1$) *and of exponent* r *for* $r \gg 0$.

In fact, let $\lambda\colon \mathbb{G}_m \to \mathrm{GL}(H^0(X, L^{\otimes r}))$ be a one-parameter subgroup. Then consider the natural action $\lambda\times\rho$ of $\mathbb{G}_m$ on $(\mathbb{P}(H^0(X, L^{\otimes r}))\times\mathbb{A}^1, \mathcal{O}(1))$ as a polarized variety, where ρ is the multiplication action on $\mathbb{A}^1$. Then the closure of the orbit $\mathcal{X} := \overline{((\lambda\times\rho)(\mathbb{G}_m))(X\times\{1\})}$ is a test configuration with the natural polarization $\mathcal{O}(1)|_{\mathcal{X}}$ and the restriction of the natural action on $(\mathbb{P}(H^0(X, L^{\otimes r}))\times\mathbb{A}^1, \mathcal{O}(1))$. This is called the *DeConcini-Procesi family* of λ by Mabuchi. The fact that any (very ample) test configuration can be obtained in this way follows from the fact that an arbitrary $\mathbb{G}_m$-equivariant vector bundle over $\mathbb{A}^1$ should be equivariantly trivial (cf. [Don05, Lemma 2]). Therefore, the test configuration can be regarded as *geometrization* of a one-parameter subgroup.

Now, let us define the Donaldson-Futaki invariants for test configurations whose positivity define K-stability. As a preparation, let us note that the *total weight* of an action of $\mathbb{G}_m$ on some finite-dimensional vector space will mean the sum of all weights in this paper. Here the *weights* mean the exponents of eigenvalues which should be powers of t. Take a test configuration $(\mathcal{X},\mathcal{M})$, and suppose that the exponent r is 1. Otherwise, we can similarly proceed by considering $(X, L^{\otimes r})$ instead of (X, L). We denote the total weight of the induced action on $(\alpha_*\mathcal{M}^{\otimes U})|_0$ by $w(U)$ and $\dim X$ as n. It is a polynomial of U of degree $n+1$. On the other hand, we write $P(u) := \dim H^0(X, L^{\otimes u})$. Let us take the $rP(r)$-th power of the action of $\mathbb{G}_m$ on $\mathcal{M}|_0$ and multiply suitable power of t so that the action on the vector space $(\alpha_*\mathcal{M}^{\otimes r})|_{\{0\}}$ would be in the special linear group $\mathrm{SL}((\alpha_*\mathcal{M}^{\otimes r})|_{\{0\}})$. Then, the corresponding normalized weight on $(\alpha_*\mathcal{M}^{\otimes U})|_0$ is $\tilde{w}_{r,Ur} := w(u)rP(r) - w(r)uP(u)$, where $u := Ur$. It is a polynomial of the form $\sum_{i=0}^{n+1} e_i(r)u^i$ of degree $n+1$ in u for $u \gg 0$. Further, the coefficients $e_i(r)$ are again polynomials of degree $n+1$ in r for $r \gg 0$: $e_i(r) = \sum_{j=0}^{n+1} e_{i,j}r^j$ for $r \gg 0$. Since the weight is normalized, $e_{n+1,n+1} = 0$. $e_{n+1,n}$ is called the *Donaldson-Futaki invariant* of the test configuration, which we will denote by $\mathrm{DF}(\mathcal{X},\mathcal{M})$. Note that $(n+1)!e_{n+1}(r)r^{n+1}$ has meaning as the Chow weight of $X \subset \mathbb{P}(H^0(X,L^{\otimes r}))$ with respect to the SL-normalization of the one-parameter subgroups associated to $(\mathcal{X},\mathcal{M}^{\otimes r})$ via Proposition 2.2 for $r \gg 0$ (cf. [Mum77, Lemma 2.11]).

For an arbitrary *semi*-test configuration $(\mathcal{X},\mathcal{M})$, we can define the (normalized) Chow weight or the Donaldson-Futaki invariant in completely similar way from the total weights of the induced $\mathbb{G}_m$-action on $(\alpha_*\mathcal{M}^{\otimes U})|_{\{0\}}$ for $U \gg 0$. Also note that the homogeneity $\mathrm{DF}(\mathcal{X},\mathcal{M}^{\otimes c}) = c^{2n}\,\mathrm{DF}(\mathcal{X},\mathcal{M})$ easily follows from the definition.

Now, we can recall the definition of K-stability as follows.

Definition 2.3 (cf. [Sto11], [Oda12b]). A test configuration $(\mathcal{X},\mathcal{L})$ is said to be *almost trivial* if $\mathcal{X}$ is $\mathbb{G}_m$-equivariantly isomorphic to the product test configuration away from a closed subset of codimension at least 2.

Definition 2.4. (i) A polarized complete scheme (X,L) is K-*stable* (resp. K-*semistable*) if for any test configurations of (X,L) that are not almost trivial, with exponent r, the leading coefficient $e_{n+1,n}$ of $e_{n+1}(r)$ (the Donaldson-Futaki invariant) is positive (resp. nonnegative).

(ii) A polarized complete scheme (X,L) is K-*polystable* if it is K-semistable, and the Donaldson-Futaki invariant of a test configuration $(\mathcal{X},\mathcal{M})$ is 0 if and only if $\mathcal{X}$ is isomorphic to $X \times \mathbb{A}^1$ away from a closed subset of codimension at least 2.

Although we only use K-semistability in this paper, we also remark on other notions. We should note that the original "K-stability" of [Don02] is what is called "K-*poly*stability" in [RT07]. We follow the convention of [RT07] at this point. We further note that it is possible to re-define asymptotic stability by the quantities introduced above, associated to test configurations, due to Proposition 2.2.

About other stability notions, we only note that stability notions are related as follows, without giving their definitions and proofs. For the details, we refer to [RT07] and [Don10].

Claim 2.5. (i) *Asymptotically Chow stable* $\Rightarrow$ *Asymptotically Hilbert stable* $\Rightarrow$ *Asymptotically Hilbert semistable* $\Rightarrow$ *Asymptotically Chow semistable* $\Rightarrow$ K-*semistable.*

(ii) $\bar{\mathrm{K}}$-*stable* $\Rightarrow$ $\bar{\mathrm{K}}$-*semistable* $\Rightarrow$ K-*semistable.*

Hence, among these notions, K-semistability is the weakest notion. It is the reason why Corollary 1.4 should follow from Theorems 1.2 and 1.3.

2.2. *Singularities via discrepancy.* We now explain the discrepancy and some classes of mild singularities. Consult [KM98, §2.3] and [Kol92, Ch. 12] for the details. Let us first treat the normal case. Let (X, D) be a normal pair, i.e., a pair of a normal variety X and an effective $\mathbb{Q}$-divisor D such that $K_X + D$ is $\mathbb{Q}$-Cartier. Let $\pi : X' \to X$ be a log resolution of D; i.e., π is a proper birational morphism such that X' is smooth and $\pi^{-1}\operatorname{Supp}(D) \cup E$ has a simple normal crossing divisor support, where E is the exceptional divisor of π. Then, we denote

$$K_{X'} - \pi^*(K_X + D) = \sum_i a(E_i; (X, D))E_i,$$

where $a(E_i; (X, D)) \in \mathbb{Q}$ and E_i run over the set of divisors of X' supported on the exceptional locus or the support $\operatorname{Supp}(\pi_*^{-1}D)$ of $\pi_*^{-1}D$, the strict transform of D. We sometimes simply write $a(E_i; (X, D))$ as $a(E_i; X)$ if $D = 0$, and we write $a(E_i)$ if the pair in concern is obvious from the context.

The pair (X, D) is called *log canonical* (resp. *Kawamata log terminal*) if and only if $a(E_i; (X, D)) \geq -1$ (resp. $a(E_i; (X, D)) > -1$) for any E_i. These notions are independent of the choice of the log resolution. We simply call X *log canonical* (resp. *log terminal*) when $(X, 0)$ is log canonical (resp. Kawamata log terminal).

The semi-log-canonicity is an extension to the nonnormal case of the notion of log-canonicity. We introduce those notions without divisors, i.e., in the nonlog setting, at this stage of argument. (We will need some log versions as well in Section 5, where we introduce those definitions.)

Let X be a projective variety, which is reduced, equidimensional, $\mathbb{Q}$-Gorenstein, Gorenstein in codimension 1, and satisfies the Serre condition S_2 (as we assumed). Let $\nu\colon X^\nu \to X$ be the normalization morphism and attach a conductor divisor $\operatorname{cond}(\nu)$ on X^ν, which is defined by $K_{X^\nu} = \nu^*K_X + \operatorname{cond}(\nu)$. From the assumption, $(X^\nu, \operatorname{cond}(\nu))$ is a log pair (i.e., $K_{X^\nu} + \operatorname{cond}(\nu)$ is $\mathbb{Q}$-Cartier). Then, the *semi-log-canonicity* of X are defined simply as the log canonicity of the normalized pair, $(X^\nu, \operatorname{cond}(\nu))$.

For the curve case, the semi-log-canonicity is equivalent to the fact that the curve is nodal or smooth. For the surface case, that class of singularities is also classified by Kollár-Shepherd-Barron [KSB88]. For the higher dimensional case, it is well known that a semi-log-canonical variety has only normal crossing singularity in codimension 1, so that repeatedly taking a general hyperplane section leads to a nodal curve.

2.3. *Log canonical model.* To construct "de-stabilizing" test configurations for nonsemi-log-canonical polarized varieties, we need a birational model called (*relative*) *log canonical model.* The definition is as follows.

Definition 2.6. Let (X, D) be a normal pair, i.e., X is a normal variety attached with a $\mathbb{Q}$-divisor such that $K_X + D$ is $\mathbb{Q}$-Cartier. We say that a birational projective morphism $\pi\colon B \to (X, D)$ gives *a* (*relative*) *log canonical model* of (X, D) (or of X if $D = 0$) if with the divisor E_{red}, which denotes the sum of π-exceptional prime divisors with coefficients 1, the pair (B, E_{red}) satisfies

(1) (B, E_{red}) is a log canonical pair,
(2) $K_B + E_{\mathrm{red}}$ is ample over X.

The existence is established in [OX]. We use the letter B, and we regard it as a certain blow up of X. Indeed, this model is a log canonical model of a log resolution with a boundary supported on the exceptional set in the sense of log minimal model program.

3. The S-coefficients

In this section, we introduce the concept of *S-coefficients* that control asymptotic behaviors for Donaldson-Futaki invariants of certain series of test configurations, and we establish some basic properties.

3.1. *Review of the formula for Donaldson-Futaki invariants.* In this subsection, let us recall the formula for the Donaldson-Futaki invariants we shall use from [Oda]. Note that a slightly different version of the formula had been also proved independently by Xiaowei Wang in [Wan12].

Firstly we define a class of ideals, which we shall use for our study of stability. Let (X, L) be an n-dimensional polarized complete variety (which is *not* necessarily normal).

Definition 3.1. A coherent ideal sheaf $\mathcal{J}$ of $X \times \mathbb{A}^1$ is called a *flag ideal* if $\mathcal{J} = I_0 + I_1 t + \cdots + I_{N-1}t^{N-1} + (t^N)$, where $I_0 \subseteq I_1 \subseteq \cdots \subseteq I_{N-1} \subseteq \mathcal{O}_X$ is the sequence of coherent ideal sheaves. (It is equivalent to the fact that the corresponding subscheme is supported on the central fiber $X \times \{0\}$ and is $\mathbb{G}_m$-invariant under the natural action of $\mathbb{G}_m$ on $X \times \mathbb{A}^1$.)

Let us introduce some notation. We set $\bar{\mathcal{L}} := p_1^* L$ on $X \times \mathbb{P}^1$ and its restriction $\mathcal{L} := p_1^* L|_{(X \times \mathbb{A}^1)}$, where p_i is the i-th projection morphism from $X \times \mathbb{A}^1$ or $X \times \mathbb{P}^1$. Let us write the blow up $\bar{\mathcal{B}}(:= Bl_{\mathcal{J}}(X \times \mathbb{P}^1)) \to X \times \mathbb{P}^1$ or its restriction to $\mathcal{B}(:= Bl_{\mathcal{J}}(X \times \mathbb{A}^1)) \to X \times \mathbb{A}^1$ by Π. Its natural exceptional divisor will be written as E, i.e., $\mathcal{O}_{\mathcal{B}}(-E') = \Pi^{-1}\mathcal{J}$. (We shall use the symbol (prime) $'$ for denoting exceptional divisors to indicate that they are exceptional divisors of the $(n+1)$-dimensional variety $X \times \mathbb{A}^1$, not of X.)

Let us assume r is sufficiently large so that $(\Pi^* \mathcal{L}^{\otimes r})(-E')$ is (relatively) semi-ample (over $\mathbb{A}^1$). Consider the Donaldson-Futaki invariant of the (semi) test configuration $(\mathcal{B}, (\Pi^* \mathcal{L})^{\otimes r}(-E'))$. Let us recall our formula for that.

THEOREM 3.2 ([Oda, Th. 3.2]). *Let (X, L) and $\mathcal{B}$, $\mathcal{J}$ be as above. We assume that exponent $r = 1$.* (*This is just to make the formula easier. For general r, put $L^{\otimes r}$ and $\bar{\mathcal{L}}^{\otimes r}$ in the place of L and $\bar{\mathcal{L}}$.*) *Furthermore, we assume* *that X is equidimentional, reduced, satisfies the S_2 condition, whose codimension-1 points are Gorenstein and have the Q-Cartier canonical divisor K_X, and $\mathcal{B}$ is Gorenstein in codimension 1. Then the corresponding Donaldson-Futaki invariant* $\mathrm{DF}((Bl_{\mathcal{J}}(X\times\mathbb{A}^1),\mathcal{L}(-E')))$ *is*

$$\begin{aligned}\frac{1}{2(n!)((n+1)!)}\Big\{&-n(L^{n-1}.K_X)((\Pi^*\bar{\mathcal{L}})(-E'))^{n+1}\\&+(n+1)(L^n)(((\Pi^*\bar{\mathcal{L}})(-E'))^n.\Pi^*(p_1^*K_X))\\&+(n+1)(L^n)(((\Pi^*\bar{\mathcal{L}})(-E'))^n.K_{\bar{\mathcal{B}}/X\times\mathbb{P}^1})\Big\}.\end{aligned}$$

In the above, the intersection numbers $(L^{n-1}.K_X)$ and (L^n) are taken on X. On the other hand, $K_{\bar{\mathcal{B}}/X\times\mathbb{P}^1}:=K_{\bar{\mathcal{B}}}-\Pi^*K_{X\times\mathbb{P}^1}$ is an exceptional divisor on $\bar{\mathcal{B}}$, and thus $(((\Pi^*\bar{\mathcal{L}})(-E'))^n.\Pi^*(p_1^*K_X))$ and $(((\Pi^*\bar{\mathcal{L}})(-E'))^n.K_{\bar{\mathcal{B}}/X\times\mathbb{P}^1})$ are intersection numbers taken on $\bar{\mathcal{B}}$.

We call the sum of first two terms the *canonical divisor part* since they involve intersection numbers with K_X or its pullback, and the last term will be called the *discrepancy term* since it reflects discrepancies over X. We remark that although not all semi-test configurations are of the form $(\mathcal{B},(\Pi^*\mathcal{L})^{\otimes r}(-E'))$, it is sufficient for K-(semi)stability to check the Donaldson-Futaki invariants of the special semi-test configurations ([Oda]).

3.2. *S-coefficient as a leading coefficient of Donaldson-Futaki invariants.* We define the S-coefficient, the key invariant, as follows.

Definition 3.3. Let us fix (X,L) in Theorem 3.2 above and fix a flag ideal $\mathcal{J}$. Suppose that $\bar{\mathcal{B}}$ is Gorenstein in codimension 1 so that the canonical divisor class $K_{\bar{\mathcal{B}}}$ is well defined. Then, the *S-coefficient* for that flag ideal $\mathcal{J}$ is defined as an intersection number $(\mathcal{L}^s.(-E')^{n-s}.K_{\bar{\mathcal{B}}/(X\times\mathbb{P}^1)})$ taken on $\bar{\mathcal{B}}$. We denote it by $S_{(X,L)}(\mathcal{J})$, where s denotes the dimension of $\mathrm{Supp}(\mathcal{O}_{X\times\mathbb{A}^1}/\mathcal{J})$. We note that homogeneity $S_{(X,L^{\lambda_1})}(\mathcal{J}^{\lambda_2})=\lambda_1^s\lambda_2^{n-s}S_{(X,L)}(\mathcal{J})$ follows from the definition.

The main motivation for above definition is the following meaning of S-coefficient, as leading coefficient of Donaldson-Futaki invariants.

PROPOSITION 3.4. *Let (X,L) and $\mathcal{J}$ be as above. Then, the following hold*:

(i) *The sequence of Donaldson-Futaki invariants* $\mathrm{DF}(Bl_{\mathcal{J}}(X\times\mathbb{A}^1),\mathcal{L}^{\otimes r}(-E'))$ *for $r\gg 0$ forms a polynomial.*

(ii) *Its coefficient of r^d is 0 for $d>n+s$ and equals*

$$\frac{\binom{n}{s}(L^n)}{2(n!)^2}S_{(X,L)}(\mathcal{J})$$

for $d=n+s$.

Hence, if $S_{(X,L)}(\mathcal{J})<0$ for some flag ideal $\mathcal{J}$, then (X,L) is not K*-semistable.*

To prove Proposition 3.4 and to analyze the positivity of the S-coefficients later, we shall use the following general properties of intersection numbers. As it follows from a standard arguments, we omit the proof. However, we give statements here for the readers' convenience as it shall be a key for our estimation.

LEMMA 3.5. *Let $\mathcal{X}$ be an arbitrary $n+1$-dimensional equidimensional complete scheme, and let $\pi\colon \bar{\mathcal{B}} \to \mathcal{X}$ a surjective, generically finite morphism. Then*

(i) *We have*

$$(\pi^* D_1. \cdots .\pi^* D_s.E'_1. \cdots .E'_{n+1-s}) = 0$$

for arbitrary Cartier divisors $D_1, \ldots, D_s$ on $\mathcal{X}$, and arbitrary Cartier divisors $E'_1, \ldots, E'_{n+1-s}$ with $\dim(\pi(\cap \mathrm{Supp}(E'_l))) < s$.

(ii) *We have*

$$(\pi^* D_1. \cdots .\pi^* D_s.E'_1. \cdots .E'_{n+1-s}) > 0$$

for arbitrary ample Cartier divisors $D_1, \ldots, D_s$ on $\mathcal{X}$, arbitrary ample Cartier divisors $E'_1, \ldots, E'_{n-s}$ on $\bar{\mathcal{B}}$, and an arbitrary effective Weil divisor E'_{n+1-s} on $\bar{\mathcal{B}}$ with $\dim(\pi(E'_{n+1-s})) = s$.

Proof of Proposition 3.4. Replacing L by $L^{\otimes r}$ and $\mathcal{L}$ by $\bar{\mathcal{L}}^{\otimes r}$ for the formula in Theorem 3.2, we have the formula of $\mathrm{DF}(Bl_{\mathcal{J}}(X \times \mathbb{A}^1), \mathcal{L}^{\otimes r}(-E'))$. From that, Proposition 3.4(i) easily follows.

Further, Lemma 3.5(i) applied to $\pi = \Pi\colon Bl_{\mathcal{J}}(X \times \mathbb{P}^1) \to X \times \mathbb{P}^1$ by taking $D_i := H \times \mathbb{P}^1$, where $H \in |L^{\otimes m}|$ $(m \in \mathbb{Z}_{>0})$, $E'_i = E'$ for $i \leq n - s$ and $E'_{n+1-s} = K_{\bar{\mathcal{B}}/(X \times \mathbb{P}^1)}$, implies that Proposition 3.4(ii) is straightforward. $\square$

3.3. *S-coefficients and discrepancy.* In this subsection, we shall show a criterion on positivity of S-coefficients, which gives a relation with discrepancy.

Let us assume, from now on, that X is an equidimensional reduced projective variety that satisfies the S_2 condition and whose codimension-1 points are Gorenstein. Thus, we can define the Weil divisor class K_X which we assume to be $\mathbb{Q}$-Cartier. If all codimension-1 points of $\mathcal{B}$ are Gorenstein, we set $K_{\bar{\mathcal{B}}/(X \times \mathbb{P}^1)} := K_{\bar{\mathcal{B}}} - \Pi^*(K_X \times \mathbb{P}^1) = \sum a(E'_i)E'_i$.

PROPOSITION 3.6. *Let X be as above and L be an ample line bundle on X. Moreover, assume that there is a flag ideal $\mathcal{J}$ whose blow up $\mathcal{B}$ is Gorenstein in codimension 1 as noted above. Furthermore, assume that the discrepancies $a(E'_i)$ satisfy the following:*

> *$a(E'_i) \leq 0$ for all the exceptional prime divisors E'_i on $\mathcal{B}$ that dominate s (maximal)-dimensional components of $\mathrm{Supp}(\mathcal{O}/\mathcal{J})$ and, moreover, there exists at least one such i with $a(E'_i) < 0$.*

Then, we have $S_{(X,L)}(\mathcal{J}) < 0$.

Hence, by combining Proposition 3.6 with Proposition 3.4, we have the following criteria for when a polarized variety can be not K-semistable.

Corollary 3.7. *Let X be as above and assume that there is a flag ideal $\mathcal{J}$ whose blow up $\mathcal{B}$ is Gorenstein in codimension* 1 *and that the discrepancies $a(E_i)$ satisfy the following*:

> $a(E'_i) \leq 0$ *for all the exceptional prime divisors E'_i on $\mathcal{B}$ that dominate s (maximal)-dimensional components of* $\operatorname{Supp}(\mathcal{O}/\mathcal{J})$ *and, moreover, there exists at least one such i with $a(E'_i) < 0$.*

Then (X, L) is not K*-semistable for an arbitrary polarization L.*

Proof of Proposition 3.6. We have

$$S_{(X,L)}(\mathcal{J}) := (\mathcal{L}^s.(-E')^{n-s}.K_{\bar{\mathcal{B}}/(X\times\mathbb{P}^1)}) = (\mathcal{L}^s.(\bar{\mathcal{L}}^{\otimes r} - E')^{n-s}.K_{\bar{\mathcal{B}}/(X\times\mathbb{P}^1)}). \tag{1}$$

Indeed, equality (1) follows from Lemma 3.5(i) applied to $\pi = \Pi\colon Bl_{\mathcal{J}}(X\times\mathbb{P}^1) \to X\times\mathbb{P}^1$ by taking $D_i := H\times\mathbb{P}^1$ for $i \leq s+1$ where $H \in |L^{\otimes m}|$ $(m \in \mathbb{Z}_{>0})$ and for $i > s+1$, $E'_i = \Pi^*(H\times\mathbb{P}^1)$ or $E'_i = E'$ or $E'_i = K_{\bar{\mathcal{B}}/(X\times\mathbb{P}^1)}$.

Moreover, the last term $(\mathcal{L}^s.(\bar{\mathcal{L}}^{\otimes r} - E')^{n-s}.K_{\bar{\mathcal{B}}/(X\times\mathbb{P}^1)})$ is positive due to Lemma 3.5(ii) applied to $\Pi\colon Bl_{\mathcal{J}}(X\times\mathbb{P}^1) \to X\times\mathbb{P}^1$ again by taking, this time, $D_i := H\times\mathbb{P}^1$ for $i \leq s$, $E'_i (i \leq n-s)$ to be an ample compactification of an ample divisor that belongs to $|(\Pi^*\mathcal{L})^{\otimes r}(-E')|$ on $\mathcal{B}$ to $\bar{\mathcal{B}}$ with $r > 1$, and $E'_{n+1-s} := K_{\bar{\mathcal{B}}/(X\times\mathbb{P}^1)}$. □

4. The normal case

As an application of the theory of S-coefficients prepared in the previous section, we partially prove Theorem 1.2 for the normal case in this section. More precisely, let X be a normal variety of pure dimension n, having the $\mathbb{Q}$-Cartier canonical divisor.

Proof of Theorem 1.2 *for normal X*. Thanks to Corollary 3.7, it is sufficient to construct a flag ideal $\mathcal{J}$ satisfying the following property.

Property 4.1. *The blow up $\mathcal{B}$ of $\mathcal{J}$ is normal. Furthermore, if we let $K_{\mathcal{B}/X\times\mathbb{A}^1} = \sum a(E'_i)E'_i$, then we have $a(E'_i) < 0$ for the discrepancy for an arbitrary Π-exceptional divisor E'_i.*

We will construct such $\mathcal{J}$ in the following two steps. Without loss of generality, we can assume that X is irreducible.

Step 1. Firstly, we construct a coherent ideal sheaf I of X, satisfying the following property. We denote the blow up of X along I by $\pi\colon B = Bl_I(X) \to X$.

Property 4.2. *The blow up B is normal. Furthermore, if s is the dimension of* $(\mathrm{Supp}(\mathcal{O}_X/I))$, *then we have $a(E_i;X) < -1$ for the discrepancy for an arbitrary π-exceptional divisor E_i.*

We construct such an I, using the (relative) log canonical model (cf. Section 2.3) as follows. Suppose $\pi\colon B \to X$ is the (relative) log canonical model of X, which exists due to [OX, Th. 1.1]. Then, we take the coherent ideal sheaf $I := (\pi)_*\mathcal{O}_B\big(m(K_{B/X} + E_{\mathrm{red}})\big)$ for sufficiently divisible $m \in \mathbb{Z}_{>0}$ and the total exceptional divisor E_{red}, and then $Bl_I(X) \simeq B$. Therefore, this I satisfies Property 4.2.

Step 2. The next step starts with taking I as constructed in the previous step. Using this, we will construct the flag ideal $\mathcal{J}$ satisfying Property 4.1 as follows. From the construction, we have $\dim(\mathrm{Supp}(\mathcal{O}_X/I)) \leq \dim(X) - 2$. Let us take sufficiently divisible positive integers m, N, and let us define $\mathcal{J} := \overline{(I + (t^m))^N}$, where the overline denotes the integral closure of the coherent ideal. Since it is an invariant ideal with respect to the natural $\mathbb{G}_m$ action on $X \times \mathbb{A}^1$, $\mathcal{J}$ is a flag ideal as well. We note that $\mathcal{C} := Bl_{I+(t)}(X \times \mathbb{A}^1)$ is *the deformation to the normal cone* (cf. [Ful84], [RT07]) but simply taking it is *not* sufficient for our purpose in general. Geometrically speaking, taking $I + (t^m)$ as above, instead of the simplest $I + (t)$, corresponds to taking the base change of $\mathcal{C}$ by m-th roots of t (i.e., $s \mapsto t := s^m$) and $\mathcal{B} := Bl_{\mathcal{J}}(X \times \mathbb{A}^1)$ is the normalization of the base change (cf. [Vas05]).

Let us think of the more detailed geometric structure of the deformation to the normal cone $\mathcal{C}$ and its modification $\mathcal{B}$. We know that its central fiber consists of two parts: the strict transform of $X \times \{0\}$ canonically isomorphic to $B = Bl_I(X)$ (we will identify them from now on) and the exceptional divisors F'_i that intersect as $F'_i \cap B = E_i$ whose generic points η_i are regular. Indeed, étale locally we can write $t = xy^{c_i}$ with étale local coordinates (i.e., regular parameters) x, y such that $(x = 0)$ corresponds to B and $(y = 0)$ corresponds to F'_i.

Based on the above facts, we obtain an étale local description of $\mathcal{B} \to \mathcal{C}$ explicitly around the generic point η_i of $F'_i \cap B$ as follows. We can take an étale local coordinate system $(u, y, z_1, \ldots, z_{n-2}, s)$ of $\mathcal{B}$ around η_i, and that of $\mathcal{C}$: $(x, y, z_1, \ldots, z_{n-2}, t)$ around the fiber of η_i that are related by the following equations. Here, t denotes the original coordinate of $\mathcal{C}$ corresponding to the $\mathbb{A}^1$ direction:

$$x = u^{c_i}, t = s^m.$$

We denote the preimage of F'_i by E'_i, which is irreducible. Then, from the above local description, it directly follows that

$$a(E'_i; X \times \mathbb{A}^1) = b_i\big(a(E_i;X) + 1\big), \tag{2}$$

where each $b_i := \frac{m}{c_i}$ is a positive integer as m is sufficiently divisible. Therefore, $a(E_i'; X \times \mathbb{A}^1) < 0$ follows from Property 4.2 in the previous step of construction. This completes the proof of Theorem 1.2 for the normal varieties case. □

5. Nonnormal case

To give a proof of the Main Theorem 1.2 in full generality, we introduce a nonnormal generalization of the (relative) log-canonical model, which we used in the previous section for normal case. A reduced equidimensional variety X is called *demi-normal* if X is S_2, whose codimension-1 points are regular or ordinary nodes.

Definition 5.1. Let X be a demi-normal projective variety. We call a biratonal projective morphism $\pi\colon B \to X$ a (*relative*) *semi-log-canonical model* if π is isomorphic over open locus of X with complement's codimension greater than 1, which satisfies the following two conditions. Here, E_{red} denotes the sum of π-exceptional prime divisors with coefficients 1.

(1) (B, E_{red}) is a semi-log-canonical pair.
(2) $K_B + E_{\mathrm{red}}$ is ample over X.

The existence of such models for any X is again proven in [OX, Cor. 1.3]. Given this birational model, the proof of Theorem 1.2 below is similar to the case where X is normal.

Proof of Theorem 1.2. Take the (relative) semi-log-canonical model $\pi\colon B \to X$ of X, which exists due to [OX, Cor. 1.3]. Here, we note that all the generic points of π-exceptional divisors are regular, by the definition of the model. Then, if we apply the negativity lemma [KM98, Lemma (3.39)] to these normalizations, we have $a_i < -1$ for any i, where $K_{B/X} = \sum a_i E_i$. Therefore, if we take $I := \pi_*(\omega_{B/X}^{[l]}(lE))^{**}$ with sufficiently divisible positive integer l, where $E_{\mathrm{red}} := \sum E_i$ denotes the total exceptional divisor of π, it would be a coherent ideal sheaf by Serre's S_2-condition of X. Further, it satisfies $Bl_I(X) \cong B$ by the relative ampleness of $K_{B/X} + E_{\mathrm{red}}$.

Let us consider a flag ideal $\mathcal{J}' = I + (t^m)$ on $X \times \mathbb{A}^1$ for sufficiently divisible positive integer m, its blow up $\mathcal{C} = Bl_{\mathcal{J}'}(X \times \mathbb{A}^1)$, and its normalization $\mathcal{C}^\nu \to \mathcal{C}$. We denote by $\Pi\colon \mathcal{C}^\nu \to X^\nu \times \mathbb{A}^1$ the associated morphism. We can prove $K_{\mathcal{C}^\nu} - \Pi^*(K_{X^\nu} \times \mathbb{A}^1 + \mathrm{cond}(\nu) \times \mathbb{A}^1) = \sum a_i' H_i'$ with $a_i' = b_i(a_i + 1) < 0$, where b_i are some positive integers for each exceptional divisor H_i' and $\mathrm{cond}(\nu)$ is the conductor divisor of the normalization. The proof is in a completly similar manner as in the previous section, by taking $(X^\nu, \mathrm{cond}(\nu))$ instead of X with the normality assumption.

We use the partial normalization $\mathcal{B}$ of $\mathcal{C}$, which was introduced in the proof of [Oda, Prop. 3.8]. The definition is $\mathcal{B} := \mathrm{Spec}_{\mathcal{O}_{\mathcal{C}}}(i_*\mathcal{O}_{X \times (\mathbb{A} \setminus \{0\})} \cap \mathcal{O}_{\mathcal{C}^\nu})$, where

$i\colon X \times (\mathbb{A}^1 \setminus \{0\}) \hookrightarrow X \times \mathbb{A}^1$ is the open immersion. Let f be the associated morphism from $\mathcal{C}^\nu$ to $\mathcal{B}$. Similar to the argument in the first half of Step 2 in Section 5, we can take a flag ideal $\mathcal{J}$ whose blow up is $\Pi\colon \mathcal{B} = Bl_{\mathcal{J}}(X \times \mathbb{A}^1) \to X \times \mathbb{A}^1$. Let us recall the following lemma.

LEMMA 5.2 ([Oda, Lemma 3.9]). *The morphism $f\colon \mathcal{C}^\nu \to \mathcal{B}$ is an isomorphism over an open neighborhood of the generic points of the central fiber.*

Thus, similarly as in the comparison of discrepancies (2), we have $K_{\mathcal{B}/X\times\mathbb{A}^1} = \sum a'_i E'_i$ with $a'_i = b_i(a_i+1) < 0$ where b_i are the positive integers introduced above, and E'_i is the strict transform of H'_i.

Therefore, we complete the proof Theorem 1.2 thanks to Corollary 3.7. $\square$

6. **Fano case**

Proof of Theorem 1.3. Here we do not use the notion of S-coefficients. Instead, the proof is done by more directly analyzing the formula for the Donaldson-Futaki invariants in Theorem 3.2.

Let us take a flag ideal $\mathcal{J} := \overline{(I+(t))^N}$, where $I \subset \mathcal{O}_X$ corresponds to the reduced subscheme supported on $\mathrm{NN}(X)$, the nonnormal locus of X, and N is a sufficiently divisible positive integer. We note that $\mathrm{NN}(X)$ is purely codimension 1 in X and its components are all generically normal crossing divisors. Consider the (semi) test configuration of the blow up type $(\mathcal{B}, (\Pi^*\mathcal{L})^{\otimes r}(-E'))$ for the flag ideal $\mathcal{J}$. Then, the S-coefficient becomes 0 and the leading coefficient of $\mathrm{DF}(\mathcal{B}, (\Pi^*\mathcal{L})^{\otimes r}(-E'))$ with respect to the variable r has the same signature as $((\Pi^*\bar{\mathcal{L}})^{n-1}.E'^2)$, which follows from Theorem 3.2. This can be shown to be negative by cutting X for $s := \dim(\mathrm{Supp}(\mathcal{O}_{X\times\mathbb{A}^1}/\mathcal{J})) = n-1$ by general hypersurface sections in $|L^{\otimes m}|$ for $m \gg 0$, which reduces to the $n=1$ case.

Thus, we can assume that X is normal. Let us assume that X is log canonical but not log terminal (i.e., strictly log canonical) and derives a contradiction. In the sense of the log minimal model program, a log resolution with Kawamata-log-terminal boundary $(\tilde{X}, (1-\varepsilon)E_{\mathrm{red}})$ with $0 < \varepsilon \ll 1$ should have a log canonical model B over X, by [BCHM10, Th. 1.2]. Note that B should be log terminal and so the morphism $B \to X$ is not an isomorphism, which is again a blow up of a certain coherent ideal sheaf I. We further remark that the model of Section 2.3 corresponds to the $\varepsilon = 0$ case. As in Section 4, we construct a flag ideal $\mathcal{J} := \overline{(I+(t^m))^N}$, where m, N are sufficiently divisible positive integers, and its blow up $\mathcal{B} := Bl_{\mathcal{J}}(X \times \mathbb{A}^1)$. Then, $K_{\mathcal{B}/X\times\mathbb{A}^1} = 0$ so that the discrepancy term vanishes.

On the other hand, as $s < n$, the canonical divisor part of the formula in Theorem 3.2 is negative by [Oda12a, proof of Theorem 2.13]. Hence, (X, L) should be not K-semistable. This completes the proof of Theorem 1.3. $\square$

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