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Quasisymmetric rigidity of square Sierpiński carpets

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Abstract

We prove that every quasisymmetric self-homeomorphism of the standard 1/3-Sierpiński carpet S_3 is a Euclidean isometry. For carpets in a more general family, the standard $1/p$-Sierpiński carpets S_p, $p \geq 3$ odd, we show that the groups of quasisymmetric self-maps are finite dihedral. We also establish that S_p and S_q are quasisymmetrically equivalent only if $p = q$. The main tool in the proof for these facts is a new invariant—a certain discrete modulus of a path family—that is preserved under quasisymmetric maps of carpets.

1. Introduction

In this paper we establish rigidity properties of Sierpiński carpets under quasisymmetric maps. In order to formulate our results, we first discuss some background and fix terminology.

The well-known standard Sierpiński carpet S_3 is a self-similar fractal in $\mathbb{R}^2$ defined as follows. Let

$$Q_0 = \{(x, y) \in \mathbb{R}^2 \colon 0 \leq x \leq 1,\, 0 \leq y \leq 1\}$$

denote the closed unit square in $\mathbb{R}^2$. We subdivide Q_0 into 3×3 subsquares of equal size in the obvious way and remove the interior of the middle square. The resulting set Q_1 consists of eight squares of sidelength 1/3. Inductively, Q_{n+1}, $n \geq 1$, is obtained from Q_n by subdividing each of the remaining squares in the subdivision of Q_n into 3×3 subsquares and removing the interiors of the middle squares. The *standard Sierpiński carpet* S_3 is the intersection of all the sets Q_n, $n \geq 0$ (see Figure 1). For arbitrary $p \geq 3$ odd, the *standard*

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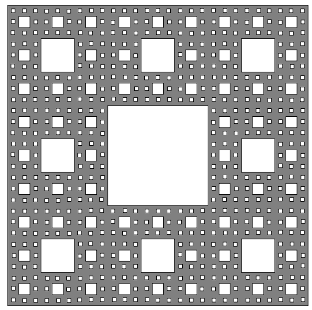

Figure 1. The standard Sierpiński carpet S_3.

$1/p$-Sierpiński carpet S_p is the subset of the plane obtained in a similar way by subdividing the square Q_0 into $p \times p$ subsquares of equal size, removing the interior of the middle square, and repeating these operations as above.

In general, a (*Sierpiński*) *carpet* is a metrizable topological space S homeomorphic to the standard Sierpiński carpet S_3. According to the topological characterization of Whyburn [Why58], S is a carpet if and only if it is a planar continuum of topological dimension 1 that is locally connected and has no local cut points.

Let

$$\mathbb{S}^2 = \{(x,y,z) \in \mathbb{R}^3 \colon x^2 + y^2 + z^2 = 1\}$$

denote the unit sphere in $\mathbb{R}^3$. In the following we often identify $\mathbb{S}^2$ with the extended complex plane $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ by stereographic projection. For subsets of $\mathbb{S}^2$, a more explicit characterization of carpets can be given as follows [Why58]. A set $S \subseteq \mathbb{S}^2$ is a carpet if and only if it can be written as

$$S = \mathbb{S}^2 \setminus \bigcup_{i \in \mathbb{N}} D_i, \tag{1.1}$$

where for each $i \in \mathbb{N}$, the set $D_i \subseteq \mathbb{S}^2$ is a Jordan region and the following conditions are satisfied: S has empty interior, $\operatorname{diam}(D_i) \to 0$ as $i \to \infty$, and $\overline{D}_i \cap \overline{D}_j = \emptyset$ for $i \neq j$. This characterization implies that all the sets S_p, $p \geq 3$ odd, are indeed carpets.

A Jordan curve in a carpet S is called a *peripheral circle* if its complement in S is a connected set. If $S \subseteq \mathbb{S}^2$ is a carpet, written as in (1.1), then the peripheral circles of S are precisely the boundaries ∂D_i of the Jordan regions D_i, $i \in \mathbb{N}$.

Let $f \colon X \to Y$ be a homeomorphism between two metric spaces (X, d_X) and (Y, d_Y). The map f is called *quasisymmetric* if there exists a homeomorphism $\eta \colon [0, \infty) \to [0, \infty)$ such that

$$\frac{d_Y(f(u), f(v))}{d_Y(f(u), f(w))} \leq \eta\left(\frac{d_X(u,v)}{d_X(u,w)}\right)$$

whenever $u, v, w \in X$, $u \neq w$. If we want to emphasize the distortion function η, we say that f is *η-quasisymmetric*. When we speak of a quasisymmetric map f from X to Y, then it is understood that f is a homeomorphism of X onto Y and that the underlying metrics on the spaces have been specified. Unless otherwise indicated, a carpet as in (1.1) is equipped with the spherical metric. The carpets S_p will carry the Euclidean metric. Note that for a compact subset K of $\mathbb{C} \subseteq \mathbb{C} \cup \{\infty\} \cong \mathbb{S}^2$, the Euclidean and the spherical metrics are comparable. So for the notion of a quasisymmetric map on K, it does not matter which of these two metrics we choose on K.

It is immediate that restrictions, inverses, and compositions of quasisymmetric maps are quasisymmetric. If there is a quasisymmetric map between two metric spaces X and Y, we say that X and Y are *quasisymmetrically equivalent*. The quasisymmetric self-maps on a metric space X, i.e., the quasisymmetric homeomorphisms of X onto itself, form a group that we denote by $\mathrm{QS}(X)$. If two metric spaces X and Y are quasisymmetrically equivalent, then $\mathrm{QS}(X)$ and $\mathrm{QS}(Y)$ are isomorphic groups.

From the topological point of view, all carpets are the same and so the topological universe of all carpets consists of a single point. A much richer structure emerges if we look at metric carpets from the point of view of quasiconformal geometry. In this case, we identify two metric carpets if and only if they are quasisymmetrically equivalent. Even if we restrict ourselves to carpets contained in $\mathbb{S}^2$, then the set of quasisymmetric equivalence classes of carpets is uncountable.

One way to see this is to invoke a rigidity result that has recently been established in [BKM09]. To formulate it, we call a carpet $S \subseteq \mathbb{S}^2$ *round* if its peripheral circles are geometric circles. So if S is written as in (1.1), then each Jordan region D_i is an open spherical disk. According to [BKM09], two round carpets S and S' of measure zero are quasisymmetrically equivalent only if they are *Möbius equivalent*, i.e., one is the image of the other under a Möbius transformation on $\widehat{\mathbb{C}} \cong \mathbb{S}^2$. Since the group of Möbius transformations depends on six real parameters and the set of round carpets is a family depending on essentially a countably infinite set of real parameters (specifying the radii and the locations of the centers of the complementary disks of the round carpet), it easily follows that the set of quasisymmetric equivalence classes of round carpets in $\mathbb{S}^2$ has the cardinality of the continuum.

Among the round carpets there is a particular class of carpets that are distinguished by their symmetry. Namely, suppose that K is a convex subset of hyperbolic 3-space $\mathbb{H}^3$ with nonempty interior and nonempty totally geodesic boundary, suppose that there exists a group G of isometries of $\mathbb{H}^3$ that leave K invariant, and suppose that G acts cocompactly and properly discontinuously on K. If we identify $\mathbb{S}^2$ with the boundary at infinity $\partial_\infty \mathbb{H}^3$ of $\mathbb{H}^3$, then the limit set $\Lambda_\infty(G) \subseteq \partial_\infty \mathbb{H}^3 = \mathbb{S}^2$ of G is a round carpet. The group G induces an action on $\mathbb{S}^2$ by Möbius transformations that leave $S = \Lambda_\infty(G)$ invariant. Moreover, this action is cocompact on triples of S. We can consider G as a subgroup of $\mathrm{QS}(S)$. An immediate consequence is that $\mathrm{QS}(S)$ is infinite and that there are only finitely many distinct orbits of peripheral circles under the action of G, and hence of $\mathrm{QS}(S)$, on S. In this sense, S is very symmetric.

According to an open conjecture by Kapovich and Kleiner [KK00], up to virtual isomorphism the groups G as above are precisely the Gromov hyperbolic groups whose boundaries at infinity are Sierpiński carpets. In order to get a better understanding of the relevant issues in this problem, it seems desirable to characterize the carpets S that arise from such groups G from the point of view of their quasiconformal geometry.

To formulate these questions more precisely, let $\mathcal{S}$, $\mathcal{R}$, $\mathcal{G}$, respectively, denote the set of all quasisymmetric equivalence classes of all carpets in $\mathbb{S}^2$, all round carpets, and all *round group carpets*, i.e., all carpets arising as limit sets $\Lambda_\infty(G)$ of groups G as above. Then $\mathcal{G} \subseteq \mathcal{R} \subseteq \mathcal{S}$. Let $[S]$ denote the quasisymmetric equivalence class of a carpet $S \subseteq \mathbb{S}^2$.

An obvious problem is where $[S_p]$ is placed in the universe $\mathcal{S}$. It follows from the main result in [Bon11] that each standard carpet S_p is quasisymmetrically equivalent to a round carpet. Hence $[S_p] \in \mathcal{R}$. The question whether actually $[S_p] \in \mathcal{G}$ arose in discussions with B. Kleiner and the first author about ten years ago. At the time this problem was considered as completely inaccessible, and one stood helpless in front of these and other problems of quasiconformal geometry. (Another well-known hard problem related to carpets is the question of the (Ahlfors regular) conformal dimension of S_3; see [MT10] for general background and [Kig10] for specific results on S_3.)

The main results of this paper give answers to some of these questions. We consider them as an important step in a better understanding of the quasiconformal geometry of Sierpiński carpets and hope that some of our results and techniques may be useful for progress on the Kapovich-Kleiner conjecture.

THEOREM 1.1. *Every quasisymmetric self-map of the standard Sierpiński carpet S_3 is a Euclidean isometry.*

The isometries of S_3 are given by the Euclidean symmetries that leave S_3, and also the unit square Q_0, invariant. They form a dihedral group with eight elements. We conjecture that also for $p > 3$, each quasisymmetric self-map of S_p is an isometry. (See Remark 8.3 for more discussion.)

We are able to prove that $\mathrm{QS}(S_p)$ is a finite dihedral group for each odd p.

THEOREM 1.2. *For every odd integer $p \geq 3$, the group of quasisymmetric self-maps $\mathrm{QS}(S_p)$ of the standard Sierpiński carpet S_p is finite dihedral.*

Theorems 1.1 and 1.2 are quite unexpected as the group of *all* homeomorphisms on S_p is large. For example, if u and v are two points in S_p that do not lie on a peripheral circle of S_p, then there exists a homeomorphism $f\colon S_p \to S_p$ with $f(u) = v$.

Every bi-Lipschitz homeomorphism between metric spaces (that is, every homeomorphism that distorts distances by an at most bounded multiplicative amount) is a quasisymmetry. So Theorems 1.1 and 1.2 remain true if one only considers bi-Lipschitz homeomorphisms instead of quasisymmetries. In general, these maps form a rather restricted subclass of all quasisymmetries. In view of this, one may wonder whether the bi-Lipschitz versions of Theorems 1.1 and 1.2 are easier to establish. Our methods do not offer any simplifications for this more restricted class, and it seems that there is no straightforward way to benefit from the stronger bi-Lipschitz hypothesis on the maps.

An immediate consequence of Theorem 1.2 is that $[S_p] \notin \mathcal{G}$. Indeed, $\mathrm{QS}(S_p)$ is a finite group, while $\mathrm{QS}(S)$ is infinite if $[S] \in \mathcal{G}$. So the points $[S_3]$, $[S_5]$, $[S_7], \dots$ lie in $\mathcal{R} \setminus \mathcal{G}$. As the following theorem shows, these points are actually all distinct.

THEOREM 1.3. *Two standard Sierpiński carpets S_p and S_q, $p, q \geq 3$ odd, are quasisymmetrically equivalent if and only if $p = q$.*

It was previously known that if $|p - q|$ is large, then S_p and S_q cannot be quasisymmetrically equivalent; more precisely, if $p > q$ say, and

$$1 + \frac{\log(p-1)}{\log p} > \frac{\log(q^2-1)}{\log q},$$

then $[S_p] \neq [S_q]$. Here the quantity on the right of the inequality is the Hausdorff dimension of S_q, while the quantity on the left is a lower bound for the (Ahlfors regular) conformal dimension of S_p, i.e., for the infimum of the Hausdorff dimensions of all Ahlfors regular metric spaces quasisymmetrically equivalent to S_p. So the inequality guarantees that $[S_p] \neq [S_q]$. The bi-Lipschitz version of Theorem 1.3 is easy to establish. Namely, if $p \neq q$, then S_p and S_q have different Hausdorff dimensions. So there cannot be any bi-Lipschitz homeomorphism between these spaces, because bi-Lipschitz maps preserve Hausdorff dimension.

One of the main difficulties in the proof of Theorem 1.1 is that we have no *a priori* normalization of a quasisymmetric self-map f of S_3. If we knew in advance, for example, that f sends each corner of the unit square to another corner, then this statement would immediately follow from the following theorem, which is relatively easy to establish.

THEOREM 1.4. *Let S and $\widetilde{S}$ be square carpets of measure zero in rectangles $K = [0,a] \times [0,1] \subseteq \mathbb{R}^2$ and $\widetilde{K} = [0,\tilde{a}] \times [0,1] \subseteq \mathbb{R}^2$, respectively, where $a, \tilde{a} > 0$.*

If f is an orientation-preserving quasisymmetric homeomorphism from S onto $\widetilde{S}$ that takes the corners of K to the corners of $\widetilde{K}$ such that $f(0) = 0$, then $a = \tilde{a}$, $S = \widetilde{S}$, and f is the identity on S.

Here the expression *square carpet in a rectangle* is used in the specific sense of the more general concept of a *square carpet in a closed Jordan domain* defined in Section 6. A quasisymmetric map between carpets in $\mathbb{S}^2$ is called *orientation-preserving* if it has an extension to a homeomorphism on $\mathbb{S}^2$ with this property.

Theorem 1.4 is analogous to the uniqueness part of [Sch93, Th. 1.3]. Our proof is similar in spirit, but we use the classical conformal modulus instead of a discrete version of it.

Another situation where a natural normalization implies a strong rigidity statement is for square carpets in $\mathbb{C}^*$-cylinders.

THEOREM 1.5. *Let S and $\widetilde{S}$ be square carpets of measure zero in $\mathbb{C}^*$-cylinders A and $\widetilde{A}$, respectively. Suppose that f is an orientation-preserving quasisymmetric homeomorphism of S onto $\widetilde{S}$ that maps the inner and outer boundary components of A onto the inner and outer boundary components of $\widetilde{A}$, respectively. Then f is (the restriction of) a map of the form $z \mapsto f(z) = az$, where $a \in \mathbb{C} \setminus \{0\}$.*

See Section 4 for the relevant definitions. Similar rigidity results for other types of carpets were established in [Mer10] (for slit carpets) and in [Mer12] (for round carpets in general Jordan domains).

We now discuss some of the ideas in the proof of our main results and give a general outline of the paper. Most of the results have been announced in [Bon06].

The main new tool used in proving Theorems 1.1–1.3 is *carpet modulus*, a version of Schramm's transboundary modulus [Sch95] for path families adapted to Sierpiński carpets. This is discussed in Section 2. We also need a notion of carpet modulus that takes a group action into account; see Section 3. The crucial feature of carpet modulus is that it is invariant under quasisymmetric maps in a suitable sense (see Lemma 2.1).

We denote by O the boundary of the unit square Q_0 and by M (for fixed $p \geq 3$ odd) the boundary of the first square removed from Q_0 in the construction of S_p ("the middle square"). Then the pair $\{O, M\}$ is distinguished by an extremality property for carpet modulus among all pairs of peripheral circles of S_p (Lemma 5.1). It follows that every quasiymmetric self-map f of S_p must preserve the pair O and M, i.e., $\{f(O), f(M)\} = \{O, M\}$ (Corollary 5.2). In principle, f may interchange O and M, but by a more refined analysis we will later establish that $f(O) = O$ and $f(M) = M$ (Lemma 8.1). This is quite in contrast to the behavior of general homeomorphisms on a carpet: if we have two finite families each consisting of the same number of distinct peripheral circles of a carpet S, then we can find a self-homeomorphism of S that sends one family to the other family.

The proof of Corollary 5.2 relies on some previous work. In Section 4 we collect certain uniformization and rigidity results that were established in [Bon11] and [BKM09], and we derive some consequences. Among these results is Proposition 4.9, which gives an explicit description of extremal mass distributions for carpet modulus of certain path families. This is an important ingredient in the proof of Corollary 5.2. Corollaries 4.4, 4.5, 4.6, and 4.7 in Section 4 give information on quasisymmetric maps on certain carpets under various normalizing conditions for points and peripheral circles.

In Section 6 we prove Theorems 1.4 and 1.5. This is essentially independent of the rest of the paper, but Theorem 1.4 will later be used in the proof of Theorem 1.3.

The fact that every quasisymmetric self-map of S_p preserves the pair $\{O, M\}$ already has some strong consequences. For example, combined with the results in Section 4, one can easily derive that the group $\mathrm{QS}(S_p)$ is finite (Corollary 5.3). To push the analysis further and to arrive at proofs of Theorems 1.1–1.3, we need one additional essential idea; namely, we will investigate weak tangent spaces of the carpets S_p and induced quasisymmetric maps on these weak tangents (see Section 7). In particular, we prove that the weak tangent of S_p at a corner of O cannot be mapped to the weak tangent of S_p at a corner of M by a (suitably normalized) quasisymmetric map (Proposition 7.3). Actually, we conjecture that such maps only exist if the weak tangents are isometric, but Proposition 7.3 is the only result in this direction that we are able to prove.

Using these statements on weak tangents, we will give proofs of Theorems 1.1–1.3 in the following Section 8. Overall, the ideas in these proofs are very similar. In order to establish Theorem 1.3, for example, one wants to apply Theorem 1.4. For this, one essentially has to show that a quasisymmetric map $f\colon S_p \to S_q$ preserves the set of corners of O. Let M_p and M_q denote the boundary of the middle square for S_p and S_q, respectively. Using the extremality property for the pair $\{O, M\}$, one can show that $\{f(O), f(M_p)\} = \{O, M_q\}$. This leads to various combinatorial possibilities, and in each case one analyzes what happens to the corners of O under the map f. The case $f(O) = O$ leads to a favorable situation, where the set of corners of O is preserved and where one can apply Theorem 1.4 to conclude $S_p = S_q$. One wants to rule out the existence of the map f in the other cases, for example when $f(O) = M_q$. In all these cases, one eventually ends up with a contradiction to Proposition 7.3.

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2. **Carpet modulus**

We first make some remarks about notation and terminology used in the rest of the paper. We denote the imaginary unit in $\mathbb{C}$ by $\boldsymbol{i}$. Let (X, d) be a metric space. If $x \in X$ and $r > 0$, we denote by $B(x, r)$ the open ball and by $\overline{B}(x, r)$ the closed ball in X that has radius $r > 0$ and is centered at x. If $\lambda > 0$ and $B = B(x, r)$, we let λB be the open ball of radius λr centered at x. If $A \subseteq X$, then $\operatorname{diam}(A)$ is the diameter, χ_A the characteristic function, and $\#A \in \mathbb{N}_0 \cup \{\infty\}$ the cardinality of A. If $B \subseteq X$ is another set, then we let

$$\operatorname{dist}(A, B) := \inf\{d(a, b) \colon a \in A,\, b \in B\}$$

be the distance between A and B.

If X is a set, then id_X is the identity map on X. If $f \colon X \to Y$ is a map between two sets X and Y, and $A \subseteq X$, then $f|A$ denotes the restriction of f to A.

Unless otherwise indicated, our ambient metric space is the sphere $\mathbb{S}^2$ equipped with the spherical metric induced by the standard Riemannian structure on $\mathbb{S}^2$. In this metric space the balls are spherical disks.

A *Jordan region* in $\mathbb{S}^2$ is an open connected set bounded by a Jordan curve, i.e., a set homeomorphic to a circle. A *closed Jordan region* in $\mathbb{S}^2$ is the closure of a Jordan region.

A *path* γ in a metric space X is a continuous map $\gamma \colon I \to X$ of a finite interval I, i.e., a set of the form $[a, b], [a, b), (a, b]$, or (a, b), where $a < b$ are real numbers, into the space X. If γ is a map from (a, b), we say that the path is *open.* As is standard, we often denote by γ also the image set $\gamma(I)$ in X. The limits $\lim_{t\to a} \gamma(t)$ and $\lim_{t\to b} \gamma(t)$, if they exist, are called *end points* of γ. If $A, B \subseteq X$, then we say that γ *connects* A and B if γ has end points and one of them lies in A and the other in B. A path is called a *subpath* of γ it is of the form $\gamma|J$ for some interval $J \subseteq I$. We denote the length of γ by $\operatorname{length}(\gamma)$. The path γ is called *rectifiable* if it has finite length and *locally rectifiable* if $\gamma|J$ is a rectifiable path for every compact subinterval $J \subseteq I$.

Let σ denote the spherical measure and ds the spherical line element on $\mathbb{S}^2$ induced by the standard Riemannian metric. A *density* ρ is a nonnegative

Borel function defined on $\mathbb{S}^2$. The density ρ provides a pseudo-metric with line element $\rho\, ds$. If Γ is a family of paths in $\mathbb{S}^2$, then the *conformal modulus* of Γ, denoted $\operatorname{mod}(\Gamma)$, is defined to be the infimum of the *mass*

$$\int \rho^2\, d\sigma$$

over all *admissible* densities ρ, i.e., all densities such that for *ρ-length* of each locally rectifiable path $\gamma \in \Gamma$, we have the inequality

$$\int_\gamma \rho\, ds \geq 1.$$

If ρ is admissible and has minimal mass among all densities admissible for Γ, then ρ is called *extremal*. Often it is convenient to change the spherical metric that was the underlying *base metric* in the definition of $\operatorname{mod}(\Gamma)$ to another conformally equivalent metric. This leads to the same quantity $\operatorname{mod}(\Gamma)$. (See [Bon11, Rem. 6.1] for more discussion.)

Conformal modulus is monotone [LV73, §4.2, p. 133]: If Γ and Γ' are two path families in $\mathbb{S}^2$ such that every path $\gamma \in \Gamma$ contains a subpath $\gamma' \in \Gamma'$, then

$$\operatorname{mod}(\Gamma) \leq \operatorname{mod}(\Gamma').$$

In particular, this inequality holds if $\Gamma \subseteq \Gamma'$.

Conformal modulus is also countably subadditive [LV73, §4.2, p. 133]: For any countable union $\Gamma = \bigcup_i \Gamma_i$ of path families Γ_i in $\mathbb{S}^2$, we have

$$\operatorname{mod}(\Gamma) \leq \sum_i \operatorname{mod}(\Gamma_i).$$

Here and in the following we adopt the convention that if the range of an index such as i above is not specified, then it is extended over $\mathbb{N}$; i.e., it runs through $1, 2, \dots$.

An important property of conformal modulus is its invariance under conformal and its quasi-invariance under quasiconformal maps. The latter means that if Γ is a family of paths contained in a region $D \subseteq \mathbb{S}^2$ and if $f\colon D \to \widetilde{D}$ is a quasiconformal map onto another region $\widetilde{D} \subseteq \mathbb{S}^2$, then

$$\frac{1}{K}\operatorname{mod}(\Gamma) \leq \operatorname{mod}(f(\Gamma)) \leq K\operatorname{mod}(\Gamma), \tag{2.1}$$

where $f(\Gamma) := \{f \circ \gamma : \gamma \in \Gamma\}$ and K depends only on the dilatation of f [LV73, Th. 3.2, p. 171]. For the basic definitions and general background on quasiconformal maps, see [Ahl66], [LV73], [Väi71]. We use the "metric definition" of quasiconformal maps and allow them to be orientation-reversing.

If a certain property for paths in a family Γ holds for all paths outside an exceptional family $\Gamma_0 \subseteq \Gamma$ with $\operatorname{mod}(\Gamma_0) = 0$, we say that it holds for *almost every path* in Γ.

Now let $S \subseteq \mathbb{S}^2$ be a carpet as in (1.1) and Γ be a family of paths in $\mathbb{S}^2$. Then we define the *carpet modulus* of Γ (*with respect to* S), denoted by $\operatorname{mod}_S(\Gamma)$, as follows. Let ρ be a *mass distribution* defined on the peripheral circles of S, i.e., a function ρ that assigns to each peripheral circle $C_i = \partial D_i$ of S a nonnegative number $\rho(C_i)$. If γ is a path in $\mathbb{S}^2$, the *ρ-length* of γ is

$$\sum_{\gamma \cap C_i \neq \emptyset} \rho(C_i).$$

We say that a mass distribution ρ is *admissible* for $\operatorname{mod}_S(\Gamma)$ if for almost every path $\gamma \in \Gamma$ the ρ-length of γ is ≥ 1; so we require that there exists a family $\Gamma_0 \subseteq \Gamma$ with $\operatorname{mod}(\Gamma_0) = 0$ such that

$$\sum_{\gamma \cap C_i \neq \emptyset} \rho(C_i) \geq 1$$

for every path $\gamma \in \Gamma \setminus \Gamma_0$. We call Γ_0 an *exceptional family* for ρ. Now we set

$$\operatorname{mod}_S(\Gamma) = \inf_\rho \left\{ \sum_i \rho(C_i)^2 \right\},$$

where the infimum is taken over all mass distributions ρ that are admissible for $\operatorname{mod}_S(\Gamma)$. The sum $\sum_i \rho(C_i)^2$ is called the (*total*) *mass* of ρ, denoted $\operatorname{mass}(\rho)$. Often we will consider a mass distribution ρ of finite mass as an element in the Banach space ℓ^2 of square summable sequences. By definition, ℓ^2 consists of all sequences $a = (a_i)$ with $a_i \in \mathbb{R}$ for $i \in \mathbb{N}$ and

$$\|a\|_{\ell^2} := \left(\sum_i a_i^2 \right)^{1/2} < \infty.$$

It is straightforward to check that the carpet modulus is monotone and countably subadditive. A crucial property of carpet modulus is its invariance under quasiconformal maps.

Lemma 2.1. *Let D be a region in $\mathbb{S}^2$, let S be a carpet contained in D, and let Γ be a path family such that $\gamma \subseteq D$ for each $\gamma \in \Gamma$. If $f\colon D \to \widetilde{D}$ is a quasiconformal map onto another region $\widetilde{D} \subseteq \mathbb{S}^2$, $\widetilde{S} := f(S)$, and $\widetilde{\Gamma} := f(\Gamma)$, then*

$$\operatorname{mod}_{\widetilde{S}}(\widetilde{\Gamma}) = \operatorname{mod}_S(\Gamma).$$

Proof. Note that $\widetilde{S}$ is also a carpet. Then the peripheral circles of S and of $\widetilde{S}$ correspond to each other under the map f. So if C_i, $i \in \mathbb{N}$, is the family of peripheral circles of S, then $f(C_i)$, $i \in \mathbb{N}$, is the family of peripheral circles of $\widetilde{S}$.

Let ρ be an admissible mass distribution for $\operatorname{mod}_S(\Gamma)$ with an exceptional path family Γ_0. Then the function $\tilde{\rho}$ that takes the value $\rho(C_i)$ at the peripheral circle $f(C_i)$ of $\widetilde{S} = f(S)$ is admissible for $\widetilde{\Gamma}$ with an exceptional path family

$\widetilde{\Gamma}_0 = f(\Gamma_0)$. Indeed, the $\tilde{\rho}$-length of every path $\tilde{\gamma} = f \circ \gamma$, $\gamma \in \Gamma$, is the same as the ρ-length of γ, and the vanishing of the conformal modulus of $\widetilde{\Gamma}_0$ is guaranteed by (2.1). The mass distributions $\tilde{\rho}$ and ρ have the same total mass. Therefore $\operatorname{mod}_{\widetilde{S}}(\widetilde{\Gamma}) \leq \operatorname{mod}_S(\Gamma)$. We also have the converse inequality, since f^{-1} is also quasiconformal [LV73, §3.2, p. 17]. $\square$

The role of the exceptional family Γ_0 in the definition of carpet modulus is somewhat subtle. One can define an alternative concept of carpet modulus without the exceptional family by requiring the admissibility condition for all paths in the family. Then one has invariance under arbitrary homeomorphisms (in the sense of the previous lemma), but it turns out that this modulus is useless, because it is trivial (i.e., equal to 0 or ∞) for many path families.

Allowing an exceptional path family Γ_0 in the definition of admissibility guarantees that for some relevant families Γ an admissible mass distribution exists and that we have $0 < \operatorname{mod}_S(\Gamma) < \infty$. Moreover, the vanishing of conformal modulus of a path family is invariant under quasiconformal maps. This is essentially the reason why for our notion of carpet modulus we have a quasiconformal invariance statement as given by the previous lemma.

An admissible mass distribution ρ is called *extremal* for $\operatorname{mod}_S(\Gamma)$ if

$$\operatorname{mass}(\rho) = \operatorname{mod}_S(\Gamma).$$

An elementary convexity argument shows that if $\operatorname{mod}_S(\Gamma) < \infty$ and an extremal mass distribution exists, then it is unique. Proposition 2.4 below guarantees existence of an extremal mass distribution. To prove this proposition, we need some auxiliary results. We first set up some notation.

We let L^2 be the space of all functions f on $\mathbb{S}^2$ that are square-integrable with respect to spherical measure σ, and set

$$\|f\|_{L^2} := \left(\int f^2 \, d\sigma \right)^{1/2}.$$

For two quantities A and B, we write $A \lesssim B$ if there exists a constant $C \geq 0$ (depending on some obvious ambient parameters) such that $A \leq CB$.

A version of the following lemma can be found in [Boj88]; see also [Hei01, Exercise 2.10].

Lemma 2.2. *Let $\lambda \geq 1$, and let I be a countable index set. Suppose that B_i, $i \in I$, is a collection of spherical disks in $\mathbb{S}^2$ and that a_i, $i \in I$, are nonnegative real numbers. Then there exists a constant $C \geq 0$ that depends only on λ such that*

$$\left\| \sum_{i \in I} a_i \chi_{\lambda B_i} \right\|_{L^2} \leq C \left\| \sum_{i \in I} a_i \chi_{B_i} \right\|_{L^2}. \tag{2.2}$$

Proof. We may assume that $\sum_{i\in I} a_i\chi_{B_i} \in L^2$. Let $\phi \in L^2$, and let $M(\phi)$ denote the uncentered maximal function of ϕ. (For the definition and the basic properties of the maximal function operator, see [Ste70, Ch. 1].) Then there is an absolute constant c such that

$$\begin{aligned}\left|\int\left(\sum_{i\in I} a_i\chi_{\lambda B_i}\right)\phi\, d\sigma\right| &= \left|\sum_{i\in I} a_i \int_{\lambda B_i}\phi\, d\sigma\right|\\ &\le c\lambda^2\sum_{i\in I} a_i\int_{B_i} M(\phi)\, d\sigma = c\lambda^2\int\left(\sum_{i\in I} a_i\chi_{B_i}\right)M(\phi)\, d\sigma\\ &\le c\lambda^2\left\|\sum_{i\in I} a_i\chi_{B_i}\right\|_{L^2}\|M(\phi)\|_{L^2}.\end{aligned}$$

It is known (see, e.g., [Ste70, Th. 1(c), p. 5]) that the maximal function satisfies the inequality

$$\|M(\phi)\|_{L^2} \le H\|\phi\|_{L^2},$$

where H is an absolute constant. The self-duality of L^2 now gives inequality (2.2) with $C = cH\lambda^2$. $\square$

A *quasicircle* in a metric space X is a Jordan curve that is quasisymmetrically equivalent to the unit circle in $\mathbb{R}^2$ equipped with the Euclidean metric. We say that a family $\{C_i : i \in I\}$ of Jordan curves in a metric space X consists of *uniform quasicircles* if there exists a homeomorphism $\eta\colon [0,\infty) \to [0,\infty)$ such that every curve C_i in the family is the image of an η-quasisymmetric map of the unit circle.

LEMMA 2.3. *Let S be a carpet in $\mathbb{S}^2$ whose peripheral circles are uniform quasicircles, and let Γ be a path family in $\mathbb{S}^2$. If $\operatorname{mod}_S(\Gamma)=0$, then $\operatorname{mod}(\Gamma)=0$.*

Proof. We write S as in (1.1) and set $C_i = \partial D_i$ for $i \in \mathbb{N}$. Under the given hypotheses suppose that $\operatorname{mod}_S(\Gamma) = 0$. It follows from the definitions that then Γ cannot contain any constant paths. So if for $k \in \mathbb{N}$ we define

$$\Gamma_k := \{\gamma \in \Gamma\colon \operatorname{diam}(\gamma) \ge 1/k\},$$

then $\Gamma = \bigcup_k \Gamma_k$, and it is enough to show that $\operatorname{mod}(\Gamma_k) = 0$ for every $k \in \mathbb{N}$. By monotonicity of carpet modulus, we have $\operatorname{mod}_S(\Gamma_k) = 0$. This means that we are actually reduced to proving the statement of the lemma under the additional assumption that $\operatorname{diam}(\gamma) \ge \delta$ for all $\gamma \in \Gamma$, where $\delta > 0$.

Since $\operatorname{mod}_S(\Gamma) = 0$, for each $n \in \mathbb{N}$ there exists $\rho_n \in \ell^2$ with $\|\rho_n\|_{\ell^2} < 1/2^n$ and an exceptional family $\widetilde{\Gamma}_n \subseteq \Gamma$ with $\operatorname{mod}(\widetilde{\Gamma}_n) = 0$ such that

$$\sum_{\gamma\cap C_i\neq\emptyset} \rho_n(C_i) \ge 1$$

for all $\gamma \in \Gamma \setminus \widetilde{\Gamma}_n$. Let $\rho = \sum_n \rho_n$. Then

$$\operatorname{mass}(\rho) = \|\rho\|_{\ell^2}^2 < \infty.$$

Moreover, if $\widetilde{\Gamma} := \bigcup_n \widetilde{\Gamma}_n$, then $\operatorname{mod}(\widetilde{\Gamma}) = 0$ and

$$\sum_{\gamma \cap C_i \neq \emptyset} \rho(C_i) = \infty \tag{2.3}$$

for all $\gamma \in \Gamma \setminus \widetilde{\Gamma}$.

Since we assume that the peripheral circles $C_i = \partial D_i$ of S are uniform quasicircles, there exists $\lambda \geq 1$ with the following property (see, e.g., [Bon11, Prop. 4.3]). For each $i \in \mathbb{N}$, there exists $x_i \in \mathbb{S}^2$, and $0 < r_i \leq R_i$ with $R_i/r_i \leq \lambda$ such that

$$B(x_i, r_i) \subseteq D_i \subseteq B(x_i, R_i).$$

Now we consider the density $\tilde{\rho}$ on the sphere defined by

$$\tilde{\rho} = \sum_i \frac{\rho(C_i)}{R_i} \chi_{B(x_i, 2R_i)}.$$

If $4R_i < \delta$, then every path $\gamma \in \Gamma$ that meets C_i must also meet the complement of $B(x_i, 2R_i)$, since $\operatorname{diam}(\gamma) \geq \delta$. So γ will meet both complementary components of $B(x_i, 2R_i) \setminus \overline{B}(x_i, R_i)$. If γ is locally rectifiable, this implies

$$\int_\gamma \chi_{B(x_i, 2R_i)}\, ds \geq R_i.$$

Since there are only finitely many $i \in \mathbb{N}$ with $4R_i \geq \delta$, it follows from (2.3) that

$$\int_\gamma \tilde{\rho}\, ds = \infty$$

for every locally rectifiable path $\gamma \in \Gamma \setminus \widetilde{\Gamma}$. On the other hand, by Lemma 2.2,

$$\int \tilde{\rho}^2\, d\sigma \lesssim \sum_i \frac{\rho(C_i)^2}{R_i^2} \sigma(B(x_i, r_i)) \lesssim \operatorname{mass}(\rho) < \infty.$$

This implies $\operatorname{mod}(\Gamma \setminus \widetilde{\Gamma}) = 0$, and so

$$\operatorname{mod}(\Gamma) \leq \operatorname{mod}(\Gamma \setminus \widetilde{\Gamma}) + \operatorname{mod}(\widetilde{\Gamma}) = 0.$$

Hence $\operatorname{mod}(\Gamma) = 0$ as desired. □

Proposition 2.4. *Let S be a carpet in $\mathbb{S}^2$ whose peripheral circles are uniform quasicircles, and let Γ be an arbitrary path family in $\mathbb{S}^2$ with $\operatorname{mod}_S(\Gamma) < \infty$. Then the extremal mass distribution for $\operatorname{mod}_S(\Gamma)$ exists; i.e., the infimum in the definition of $\operatorname{mod}_S(\Gamma)$ is attained as a minimum.*

Proof. Let (ρ_n) be a sequence of admissible mass distributions for $\operatorname{mod}_S(\Gamma)$ such that $\operatorname{mass}(\rho_n) \to \operatorname{mod}_S(\Gamma)$ as $n \to \infty$. If C_i, $i \in \mathbb{N}$, are the peripheral circles of S, then each ρ_n is given by the sequence $\rho_n = (\rho_n(C_i))$ of weights it assigns to the peripheral circles.

Since $\operatorname{mod}_S(\Gamma) < \infty$, we may assume that for a suitable constant, we have $\operatorname{mass}(\rho_n) \leq C$ for all n. By passing to a subsequence using a standard diagonalization argument, we may also assume that the limit

$$\rho(C_i) := \lim_{n \to \infty} \rho_n(C_i)$$

exists for each $i \in \mathbb{N}$. We claim that the mass distribution $\rho = (\rho(C_i))$ is extremal.

First, it is clear that $\operatorname{mass}(\rho) \leq \operatorname{mod}_S(\Gamma)$. Indeed, for every $\varepsilon > 0$ and $m \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that

$$\sum_{i=1}^{m} \rho_n(C_i)^2 \leq \operatorname{mod}_S(\Gamma) + \varepsilon$$

for all $n \geq N$. Taking the limit as $n \to \infty$, we get

$$\sum_{i=1}^{m} \rho(C_i)^2 \leq \operatorname{mod}_S(\Gamma) + \varepsilon$$

for all $m \in \mathbb{N}$. Since m and ε are arbitrary, this gives $\operatorname{mass}(\rho) \leq \operatorname{mod}_S(\Gamma)$.

To complete the proof we have to show that ρ is admissible, which is harder to establish. By Mazur's Lemma (see, e.g., [Yos80, Th. 2, p. 120]) there is a sequence of convex combinations $(\tilde{\rho}_N)$, where

$$\tilde{\rho}_N = \sum_{n=1}^{N} \lambda_n^N \rho_n, \quad \lambda_n^N \geq 0, \quad \sum_{n=1}^{N} \lambda_n^N = 1,$$

that converges to ρ in ℓ^2. Every element of the sequence $(\tilde{\rho}_N)$ is admissible for Γ, where the exceptional path family $\widetilde{\Gamma}_N$ for $\tilde{\rho}_N$ is the union of the exceptional path families for ρ_n, $n = 1, 2, \dots, N$. Since $(\tilde{\rho}_N)$ converges to ρ in ℓ^2, it is also a minimizing sequence for $\operatorname{mod}_S(\Gamma)$.

By passing to a subsequence, we may assume that

$$||\tilde{\rho}_N - \rho||_{\ell^2} \leq \frac{1}{2^N} \tag{2.4}$$

for all $N \in \mathbb{N}$.

Let

$$\Gamma_\infty = \left\{ \gamma \in \Gamma \colon \limsup_{N \to \infty} \sum_{\gamma \cap C_i \neq \emptyset} |\tilde{\rho}_N(C_i) - \rho(C_i)| \neq 0 \right\}$$

and

$$\Gamma_N = \left\{ \gamma \in \Gamma \colon \sum_{\gamma \cap C_i \neq \emptyset} |\tilde{\rho}_N(C_i) - \rho(C_i)| \geq \frac{1}{N} \right\}.$$

Then $\Gamma_\infty \subseteq \bigcap_n \bigcup_{N\geq n} \Gamma_N$. Indeed, let $\gamma \in \Gamma_\infty$ be arbitrary. Then there exists $\delta > 0$ and a sequence of natural numbers (N_k) with $N_k \to \infty$ as $k \to \infty$ such that

$$\sum_{\gamma\cap C_i \neq \emptyset} |\tilde{\rho}_{N_k}(C_i) - \rho(C_i)| \geq \delta$$

for all $k \in \mathbb{N}$. Now let n be arbitrary. We choose k so large that $N_k \geq n$ and $1/N_k \leq \delta$. Then $\gamma \in \Gamma_{N_k} \subseteq \bigcup_{N\geq n} \Gamma_N$. Hence $\gamma \in \bigcap_n \bigcup_{N\geq n} \Gamma_N$ as desired.

It follows that the mass distributions

$$\rho_{\infty,n} = \sum_{N=n}^{\infty} N|\tilde{\rho}_N - \rho|, \; n = 1, 2, \ldots,$$

are admissible for $\operatorname{mod}_S(\Gamma_\infty)$. Since $\operatorname{mass}(\rho_{\infty,n}) \to 0$ as $n \to \infty$ by (2.4), this implies that $\operatorname{mod}_S(\Gamma_\infty) = 0$. Invoking Lemma 2.3, we conclude that $\operatorname{mod}(\Gamma_\infty) = 0$.

If γ is in $\Gamma \setminus (\Gamma_\infty \cup \bigcup_N \widetilde{\Gamma}_N)$, then

$$\begin{aligned}\sum_{\gamma\cap C_i\neq\emptyset} \rho(C_i) &\geq \limsup_{N\to\infty}\Bigg(\sum_{\gamma\cap C_i\neq\emptyset} \tilde{\rho}_N(C_i) - \sum_{\gamma\cap C_i\neq\emptyset} |\tilde{\rho}_N(C_i) - \rho(C_i)|\Bigg)\\ &\geq 1 - \limsup_{N\to\infty}\Bigg(\sum_{\gamma\cap C_i\neq\emptyset} |\tilde{\rho}_N(C_i) - \rho(C_i)|\Bigg) = 1.\end{aligned}$$

Moreover,

$$\operatorname{mod}\Bigg(\Gamma_\infty \cup \bigcup_N \widetilde{\Gamma}_N\Bigg) \leq \operatorname{mod}(\Gamma_\infty) + \sum_N \operatorname{mod}(\widetilde{\Gamma}_N) = 0.$$

It follows that ρ is admissible for Γ as desired. □

3. Carpet modulus with respect to a group

Let S be a carpet in $\mathbb{S}^2$. In this section we assume that S is written as in (1.1) and that $C_i = \partial D_i$, $i \in \mathbb{N}$, denote the peripheral circles of S. Let G be a group of homeomorphisms of S. If $g \in G$ and $C \subseteq S$ is a peripheral circle of S, then $g(C)$ is also a peripheral circle of S. So the whole orbit $\mathcal{O} = \{g(C) : g \in G\}$ of C under the action of G consists of peripheral circles of S. If Γ is a family of paths in $\mathbb{S}^2$, we define the carpet modulus $\operatorname{mod}_{S/G}(\Gamma)$ of Γ with respect to the action of G as follows. A (*invariant*) *mass distribution* ρ is a nonnegative function defined on the peripheral circles of S that takes the same value on each peripheral circle in the same orbit; so $\rho(g(C)) = \rho(C)$ for all $g \in G$ and all peripheral circles C of S. Such a mass distribution is *admissible* for $\operatorname{mod}_{S/G}(\Gamma)$ if there exists an *exceptional family* $\Gamma_0 \subseteq \Gamma$ with $\operatorname{mod}(\Gamma_0) = 0$ and

$$\sum_{\gamma\cap C_i\neq\emptyset} \rho(C_i) \geq 1 \tag{3.1}$$

for all $\gamma \in \Gamma \setminus \Gamma_0$.

If ρ is a mass distribution and $\mathcal{O}$ an orbit of peripheral circles, we set $\rho(\mathcal{O}) := \rho(C)$, where $C \in \mathcal{O}$. We define the (*total*) *mass* of ρ as

$$\mathrm{mass}_{S/G}(\rho) = \sum_{\mathcal{O}} \rho(\mathcal{O})^2,$$

where the sum is taken over all orbits of peripheral circles under the action of G.

The *carpet modulus of* Γ *with respect to the group* G is defined as

$$\mathrm{mod}_{S/G}(\Gamma) = \inf_{\rho} \{\mathrm{mass}_{S/G}(\rho)\},$$

where the infimum is taken over all admissible mass distributions ρ. An admissible mass distribution ρ realizing this infimum is called *extremal* for $\mathrm{mod}_{S/G}(\Gamma)$.

Note that each orbit contributes with exactly one term to the total mass of a mass distribution. In contrast, the admissibility condition is similar to the one for carpet modulus: Each peripheral circle that intersects the curve γ contributes a term to the sum in (3.1), and we may get multiple contributions from each orbit; we restrict ourselves though to invariant mass distributions that are constant on each orbit of the action of G on peripheral circles.

Carpet modulus with respect to a group has similar monotonicity and subadditivity properties as carpet modulus and conformal modulus. We formulate its invariance property under quasiconformal maps explicitly.

LEMMA 3.1. *Let D be a region in $\mathbb{S}^2$, S a carpet contained in D, G a group of homeomorphisms on S, and Γ a path family such that $\gamma \subseteq D$ for each $\gamma \in \Gamma$. If $f\colon D \to \widetilde{D}$ is a quasiconformal map onto another region $\widetilde{D} \subseteq \mathbb{S}^2$, and we define $\widetilde{S} := f(S)$, $\widetilde{\Gamma} := f(\Gamma)$, and $\widetilde{G} := (f|S) \circ G \circ (f|S)^{-1}$, then*

$$\mathrm{mod}_{\widetilde{S}/\widetilde{G}}(\widetilde{\Gamma}) = \mathrm{mod}_{S/G}(\Gamma).$$

Proof. Note that $\widetilde{S}$ is a carpet and $f|S$ is homeomorphism from S onto $\widetilde{S}$. Hence $\widetilde{G}$ is a group of homeomorphisms on $\widetilde{S}$. The argument is now along the same lines as the proof of Lemma 2.1, and we omit the details. $\square$

The following proposition gives a criterion for the existence of an extremal mass distribution for carpet modulus with respect to the group.

PROPOSITION 3.2. *Let S be a carpet in $\mathbb{S}^2$ whose peripheral circles are uniform quasicircles, let G be a group of homeomorphisms of S, and let Γ be a path family in $\mathbb{S}^2$ with $\mathrm{mod}_{S/G}(\Gamma) < \infty$.*

Suppose that for each $k \in \mathbb{N}$, there exists a family of peripheral circles $\mathcal{C}_k$ of S and a constant $N_k \in \mathbb{N}$ with the following properties:

(i) *if $\mathcal{O}$ is any orbit of peripheral circles of S under the action of G, then $\#(\mathcal{O} \cap \mathcal{C}_k) \leq N_k$ for all $k \in \mathbb{N}$;*

(ii) *if* Γ_k *is the family of all paths in* Γ *that only meet peripheral circles in* $\mathcal{C}_k$, *then* $\Gamma = \bigcup_k \Gamma_k$.

Then an extremal mass distribution for $\mathrm{mod}_{S/G}(\Gamma)$ *exists*; *i.e.*, *the infimum in the definition of* $\mathrm{mod}_{S/G}(\Gamma)$ *is attained.*

Proof. We first observe that one has an analog of Lemma 2.3; namely, if in addition to the given hypotheses $\mathrm{mod}_{S/G}(\Gamma) = 0$, then $\mathrm{mod}(\Gamma) = 0$.

The proof of this implication is very similar to the proof of Lemma 2.3. As in the proof of this lemma, we can make the additional assumption that there exists $\delta > 0$ such that $\mathrm{diam}(\gamma) \geq \delta$ for all $\gamma \in \Gamma$. Using that $\mathrm{mod}_{S/G}(\Gamma) = 0$, one can find an invariant mass distribution ρ with $\mathrm{mass}_{S/G}(\rho) < \infty$ and a family $\widetilde{\Gamma} \subseteq \Gamma$ with $\mathrm{mod}(\widetilde{\Gamma}) = 0$ such that

$$\sum_{\gamma \cap C_i \neq \emptyset} \rho(C_i) = \infty \tag{3.2}$$

for all $\gamma \in \Gamma \setminus \widetilde{\Gamma}$.

Since peripheral circles of S, represented as in (1.1), are uniform quasicircles, there exists $\lambda \geq 1$ such that for each $i \in \mathbb{N}$, we can find $x_i \in \mathbb{S}^2$ and $0 < r_i \leq R_i$ with

$$B(x_i, r_i) \subseteq D_i \subseteq B(x_i, R_i),$$

and $R_i/r_i \leq \lambda$.

Now fix $k \in \mathbb{N}$, consider the family $\Gamma'_k := (\Gamma \setminus \widetilde{\Gamma}) \cap \Gamma_k$ of all paths in $\Gamma \setminus \widetilde{\Gamma}$ that only intersect peripheral circles in the family $\mathcal{C}_k$, and let

$$\tilde{\rho} = \sum_{C_i \in \mathcal{C}_k} \frac{\rho(C_i)}{R_i} \chi_{B(x_i, 2R_i)}.$$

Using our hypothesis (i) and Lemma 2.2, we see that

$$\int \tilde{\rho}^2 \, d\sigma \lesssim \sum_{C_i \in \mathcal{C}_k} \rho(C_i)^2 \leq N_k \mathrm{mass}_{S/G}(\rho) < \infty.$$

On the other hand, similarly as in the proof of Lemma 2.3, by (3.2) we have

$$\int_\gamma \tilde{\rho} \, ds = \infty$$

for every locally rectifiable path $\gamma \in \Gamma'_k$. It follows that $\mathrm{mod}(\Gamma'_k) = 0$.

Our hypothesis (ii) implies that $\Gamma = \widetilde{\Gamma} \cup \bigcup_k \Gamma'_k$. Since all families in the last union have vanishing modulus, we conclude $\mathrm{mod}(\Gamma) = 0$ as desired.

Now the proof of the statement is almost identical to the proof of Proposition 2.4. The only differences are that we use mass distributions that are constant on each orbit of peripheral circles under the action of G and that for control on the masses of the relevant distributions, we use an ℓ^2-space indexed by these orbits. Note that hypothesis (ii) passes to every subfamily of Γ, so we can apply the first part of the proof to the family that corresponds to Γ_∞ in the proof of Proposition 2.4. We omit the details. $\square$

If ψ is a homeomorphism of the carpet $S \subseteq \mathbb{S}^2$, we denote by $\langle\psi\rangle$ the cyclic group of homeomorphisms on S generated by ψ. If Γ is a path family in $\mathbb{S}^2$ and Ψ is a homeomorphism on $\mathbb{S}^2$, then Γ is called Ψ-*invariant* if $\Psi(\Gamma) = \Gamma$. The following lemma gives a precise relationship between the carpet modulus with respect to a cyclic group and its subgroups.

Lemma 3.3. *Let S be a carpet in $\mathbb{S}^2$, let $\Psi\colon \mathbb{S}^2 \to \mathbb{S}^2$ be a quasiconformal map with $\Psi(S) = S$, and define $\psi := \Psi|S$. Suppose that Γ is a Ψ-invariant path family in $\mathbb{S}^2$ such that for every peripheral circle C of S that meets some path in Γ, we have $\psi^n(C) \neq C$ for all $n \in \mathbb{Z} \setminus \{0\}$.*

Then

$$\operatorname{mod}_{S/\langle\psi^k\rangle}(\Gamma) = k \operatorname{mod}_{S/\langle\psi\rangle}(\Gamma)$$

for every $k \in \mathbb{N}$.

Proof. Fix $k \in \mathbb{N}$. Let $\varepsilon > 0$, and let ρ be an admissible mass distribution for $\operatorname{mod}_{S/\langle\psi\rangle}(\Gamma)$ such that

$$\operatorname{mass}_{S/\langle\psi\rangle}(\rho) \leq \operatorname{mod}_{S/\langle\psi\rangle}(\Gamma) + \varepsilon.$$

Here it is understood that ρ is invariant in the sense that it is constant on orbits of peripheral circles under the action of $\langle\psi\rangle$. Then ρ is also constant on orbits under the action of $\langle\psi^k\rangle$, and hence admissible for $\operatorname{mod}_{S/\langle\psi^k\rangle}(\Gamma)$.

Each orbit of a peripheral circle under the action of $\langle\psi\rangle$ consists of at most k orbits under the action of $\langle\psi^k\rangle$. Therefore, we obtain

$$\begin{aligned}\operatorname{mod}_{S/\langle\psi^k\rangle}(\Gamma) &\leq \operatorname{mass}_{S/\langle\psi^k\rangle}(\rho) \\ &\leq k \operatorname{mass}_{S/\langle\psi\rangle}(\rho) \leq k \operatorname{mod}_{S/\langle\psi\rangle}(\Gamma) + k\varepsilon.\end{aligned}$$

Since ε is arbitrary, we conclude

$$\operatorname{mod}_{S/\langle\psi^k\rangle}(\Gamma) \leq k \operatorname{mod}_{S/\langle\psi\rangle}(\Gamma).$$

Conversely, let $\varepsilon > 0$, and let ρ be an admissible invariant mass distribution for $\operatorname{mod}_{S/\langle\psi^k\rangle}(\Gamma)$ such that

$$\operatorname{mass}_{S/\langle\psi^k\rangle}(\rho) \leq \operatorname{mod}_{S/\langle\psi^k\rangle}(\Gamma) + \varepsilon.$$

Note that if C is a peripheral circle of S and no path in Γ meets C, then by Ψ-invariance of Γ no path in Γ meets any of the peripheral circles in the orbit of C under $\langle\psi\rangle$. This implies that we may assume that $\rho(C) = 0$ for all peripheral circles C that do not meet any path in Γ.

Consider $\tilde\rho$ given by

$$\tilde\rho = \frac{1}{k}(\rho + \rho\circ\psi + \cdots + \rho\circ\psi^{k-1}).$$

Here $\rho \circ \psi^j$ denotes the mass distribution that assigns the value $\rho(\psi^j(C))$ to a peripheral circle C of S.

We have that $\rho \circ \psi^k = \rho$, since ρ is constant on orbits of peripheral circles under the action of ψ^k. This implies that $\tilde{\rho} \circ \psi = \tilde{\rho}$ and so $\tilde{\rho}$ is constant on orbits of peripheral circles under the action of ψ.

Let Γ_0 be an exceptional family for ρ, and define

$$\widetilde{\Gamma}_0 = \bigcup_{n \in \{-(k-1), \dots, -1, 0\}} \Psi^n(\Gamma_0).$$

Since $\operatorname{mod}(\Gamma_0) = 0$, we have $\operatorname{mod}(\widetilde{\Gamma}_0) = 0$. Moreover, the Ψ-invariance of Γ implies that

$$\sum_{\gamma \cap C_i \neq \emptyset} \tilde{\rho}(C_i) \geq 1$$

for all $\gamma \in \Gamma \setminus \widetilde{\Gamma}_0$. Hence $\tilde{\rho}$ is admissible for $\operatorname{mod}_{S/\langle\psi\rangle}(\Gamma)$.

It follows that

$$\operatorname{mod}_{S/\langle\psi\rangle}(\Gamma) \leq \operatorname{mass}_{S/\langle\psi\rangle}(\tilde{\rho}).$$

Since ρ assigns 0 to all peripheral circles of S that do not meet any path in Γ, the same is true for $\tilde{\rho}$. So if C is a peripheral circle of S with $\tilde{\rho}(C) \neq 0$, then C meets some path in Γ and so our hypotheses imply that the peripheral circles $\psi^n(C)$, $n \in \mathbb{Z}$, are all distinct. Hence the $\langle\psi\rangle$-orbit of C consists of precisely k orbits of C under $\langle\psi^k\rangle$. It follows that

$$\operatorname{mass}_{S/\langle\psi^k\rangle}(\tilde{\rho}) = k \operatorname{mass}_{S/\langle\psi\rangle}(\tilde{\rho}).$$

Moreover, the convexity of the norm in ℓ^2 implies that

$$\operatorname{mass}_{S/\langle\psi^k\rangle}(\tilde{\rho}) \leq \operatorname{mass}_{S/\langle\psi^k\rangle}(\rho).$$

We conclude

$$\begin{aligned} k \operatorname{mod}_{S/\langle\psi\rangle}(\Gamma) &\leq k \operatorname{mass}_{S/\langle\psi\rangle}(\tilde{\rho}) \;=\; \operatorname{mass}_{S/\langle\psi^k\rangle}(\tilde{\rho}) \\ &\leq \operatorname{mass}_{S/\langle\psi^k\rangle}(\rho) \;\leq\; \operatorname{mod}_{S/\langle\psi^k\rangle}(\Gamma) + \varepsilon. \end{aligned}$$

Since ε was arbitrary, this gives the other desired inequality

$$k \operatorname{mod}_{S/\langle\psi\rangle}(\Gamma) \leq \operatorname{mod}_{S/\langle\psi^k\rangle}(\Gamma).$$

The statement follows. □

4. **Auxiliary results**

In this section we state results from [Bon11] and [BKM09] that are used in this paper and derive some consequences.

Proposition 4.1. *Let S be a carpet in $\mathbb{S}^2$ whose peripheral circles are uniform quasicircles, and let f be a quasisymmetric map of S onto another carpet $\widetilde{S} \subseteq \mathbb{S}^2$. Then there exists a quasiconformal map F on $\mathbb{S}^2$ whose restriction to S is f.*

This follows immediately from [Bon11, Prop. 5.1].

Suppose $\{C_i : i \in I\}$ is a family of continua in a metric space X; i.e., each set C_i is a compact connected set consisting of more than one point. These continua are said to be *uniformly relatively separated* if the pairwise relative distances are uniformly bounded away from zero; i.e., there exists $\delta > 0$ such that

$$\Delta(C_i, C_j) := \frac{\operatorname{dist}(C_i, C_j)}{\min\{\operatorname{diam}(C_i), \operatorname{diam}(C_j)\}} \geq \delta$$

for any two distinct i and j. The uniform relative separation property is preserved under quasisymmetric maps; see [Bon11, Cor. 4.6].

Recall that a carpet $S \subseteq \mathbb{S}^2$ is called *round* if its peripheral circles are geometric circles. So if S is written as in (1.1), then each Jordan region D_i is an open spherical disk.

Theorem 4.2 (Uniformization by round carpets). *If S is a carpet in $\mathbb{S}^2$ whose peripheral circles are uniformly relatively separated uniform quasicircles, then there exists a quasisymmetric map of S onto a round carpet.*

This is [Bon11, Cor. 1.2].

Theorem 4.3 (Quasisymmetric rigidity of round carpets). *Let S be a round carpet in $\mathbb{S}^2$ of measure zero. Then every quasisymmetric map of S onto any other round carpet is the restriction of a Möbius transformation.*

This is [BKM09, Th. 1.2]. Here, by definition, a *Möbius transformation* is a fractional linear transformation on $\mathbb{S}^2 \cong \widehat{\mathbb{C}}$, or the complex-conjugate of such a map. So we allow a Möbius transformation to be orientation-reversing.

Let $S \subseteq \mathbb{S}^2$ be a carpet, and let $f\colon S \to \mathbb{S}^2$ be a homeomorphic embedding. Then f has a homeomorphic extension to a homeomorphism $F\colon \mathbb{S}^2 \to \mathbb{S}^2$. (See the proof of Lemma 5.3 in [Bon11].) We call f *orientation-preserving* if F (and hence every homeomorphic extension of f) is orientation-preserving on $\mathbb{S}^2$. On a more intuitive level, the map f is orientation-preserving if the following is true. If we orient any peripheral circle C of S so that S lies "to the left" of C with this orientation, then the image carpet $f(S)$ lies "to the left" of its peripheral circle $f(C)$ equipped with the induced orientation.

Corollary 4.4 (Three-Circle Theorem). *Let S be a carpet in $\mathbb{S}^2$ of measure zero whose peripheral circles are uniformly relatively separated uniform quasicircles. Let C_1, C_2, C_3 be three distinct peripheral circles of S. If f*

and g are two orientation-preserving quasisymmetric self-maps of S such that $f(C_i) = g(C_i)$ for $i = 1, 2, 3$, then $f = g$.

This follows from [Bon11, Th. 1.5] applied to $f^{-1} \circ g$.

Corollary 4.5. *Let S be a carpet in $\mathbb{S}^2$ of measure zero whose peripheral circles are uniformly relatively separated uniform quasicircles. Let C be a peripheral circle of S, and p, q two distinct points on C, and G be the group of all orientation-preserving quasisymmetric self-maps of S that fix p and q. Then $G = \{\mathrm{id}_S\}$ or G is an infinite cyclic group.*

In other words, either G is trivial or isomorphic to $\mathbb{Z}$.

Proof. By Theorem 4.2 there exists a quasisymmetric map f of S onto a round carpet $\widetilde{S}$. Using Proposition 4.1 we can extend f to a quasiconformal map on $\mathbb{S}^2$. Since quasiconformal maps preserve the class of sets of measure zero [LV73, Th. 1.3, p. 165], $\widetilde{S}$ has measure zero as well. According to Theorem 4.3, the conjugate group $\widetilde{G} = f \circ G \circ f^{-1}$ consists of the restrictions of orientation-preserving Möbius transformations with two fixed points $\tilde{p}, \tilde{q}$ on a peripheral circle $\widetilde{C}$ of $\widetilde{S}$. By post-composing f with a Möbius transformation we may assume that $\tilde{p} = 0$, $\tilde{q} = \infty$, and that $\widetilde{C}$ is the extended real line. Moreover, we may assume that $\widetilde{S}$ is contained in the upper half-plane. Then the maps in $\widetilde{G}$ are of the form $z \mapsto \lambda z$ with $\lambda > 0$. The multiplicative group of the factors λ that arise in this way must be discrete (this follows from the fact that peripheral circles are mapped to peripheral circles) and hence forms a cyclic group. It follows that $\widetilde{G}$, and hence also G, is the trivial group consisting only of the identity map or an infinite cyclic group. □

Corollary 4.6. *Let S be a carpet in $\mathbb{S}^2$ of measure zero whose peripheral circles are uniformly relatively separated uniform quasicircles. Let C_1 and C_2 be two distinct peripheral circles of S, and let G be the group of all orientation-preserving quasisymmetric self-maps of S that fix C_1 and C_2 setwise. Then G is a finite cyclic group.*

Proof. As in the proof of Corollary 4.5, we can reduce to the case that S is a round carpet of measure zero. By applying an auxiliary Möbius-transformation if necessary, we may also assume that C_1 and C_2 are Euclidean circles both centered at 0. Then by Theorem 4.3, each element in G is (the restriction of) a rotation around 0. Moreover, G must be a discrete group as it maps peripheral circles of S to peripheral circles. Hence G is finite cyclic. □

Corollary 4.7. *Let S be a carpet in $\mathbb{S}^2$ of measure zero whose peripheral circles are uniformly relatively separated uniform quasicircles, C_1 and C_2 be two distinct peripheral circles of S, and $p \in S$. If f is an orientation-preserving* *quasisymmetric self-map of* S *such that* $f(C_1)=C_1$, $f(C_2)=C_2$, *and* $f(p)=p$, *then* f *is the identity on* S.

Proof. By the argument as in the proof of Corollary 4.6, we can reduce to the case where S is a round carpet, C_1 and C_2 are circles both centered at 0, and f is a rotation around 0. Since C_1 and C_2 are distinct peripheral circles of S, these sets are disjoint and bound two disjoint Jordan regions. This implies that S is contained in the Euclidean annulus with boundary components C_1 and C_2. Since $p \in S$, it follows that $p \neq 0, \infty$. Since f is a rotation around 0 and fixes p, the map f must be the identity on S. $\square$

The metric on $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ induced by the length element $|dz|/|z|$ is called the *flat metric* (*on* $\mathbb{C}^*$). Equipped with this metric, $\mathbb{C}^*$ is isometric to an infinite cylinder of circumference 2π. The following terminology is suggested by this geometric picture.

A $\mathbb{C}^*$-*cylinder* A is a set of the form

$$A = \{z \in \mathbb{C} \colon r \leq |z| \leq R\},$$

where $0 < r < R < \infty$. The boundary components $\{z \in \mathbb{C} : |z| = r\}$ and $\{z \in \mathbb{C} : |z| = R\}$ are called the *inner* and *outer* boundary components of A, respectively. A $\mathbb{C}^*$-*square* Q is a Jordan region of the form

$$Q = \{\rho e^{i\theta} \colon a < \rho < b,\ \alpha < \theta < \beta\},$$

where $0 < \log(b/a) = \beta - \alpha < 2\pi$. We call the quantity

$$\ell_{\mathbb{C}^*}(Q) := \log(b/a) = \beta - \alpha$$

its *side length*, the set $\{ae^{i\theta} \colon \alpha \leq \theta \leq \beta\}$ the *bottom side*, and the set $\{be^{i\theta} \colon \alpha \leq \theta \leq \beta\}$ the *top side* of Q. The sets $\{\rho e^{i\alpha} \colon a \leq \rho \leq b\}$ and $\{\rho e^{i\beta} \colon a \leq \rho \leq b\}$ are referred to as the *vertical sides* of Q. The *corners* of Q are the four points that are end points of one of the sides of Q.

A *square carpet* T *in a* $\mathbb{C}^*$-*cylinder* A is a carpet that can be written as

$$T = A \setminus \bigcup_i Q_i,$$

where the sets Q_i, $i \in I$, are $\mathbb{C}^*$-squares whose closures are pairwise disjoint and contained in the interior of A. Very similar terminology was employed in [Bon11]. Note that in contrast to [Bon11], our $\mathbb{C}^*$-cylinders A are closed and the $\mathbb{C}^*$-squares Q are open sets.

THEOREM 4.8 (Cylinder Uniformization Theorem). *Let* S *be a carpet of measure zero in* $\mathbb{S}^2$ *whose peripheral circles are uniformly relatively separated uniform quasicircles, and let* C_1 *and* C_2 *be distinct peripheral circles of* S. *Then there exists a quasisymmetric map* f *from* S *onto a square carpet* T *in a* $\mathbb{C}^*$-*cylinder* A *such that* $f(C_1)$ *is the inner boundary component of* A *and* $f(C_2)$ *is the outer one.*

This is [Bon11, Th. 1.6]. In this statement, S is equipped with the spherical metric as by our convention adopted in the introduction. For the metric on T, one can choose the spherical metric, the Euclidean metric, or the flat metric on $\mathbb{C}^*$; they are all comparable on A and hence on T.

Let S be a carpet in $\mathbb{S}^2$ and C_1 and C_2 be two distinct peripheral circles of S that bound the complementary components D_1 and D_2 of S, respectively. We denote by $\Gamma(C_1, C_2; S)$ the family of all open paths γ in $\mathbb{S}^2 \setminus (\overline{D}_1 \cup \overline{D}_2)$ that connect $\overline{D}_1$ and $\overline{D}_2$.

The following proposition gives an explicit description for the extremal mass distribution for the carpet modulus $\operatorname{mod}_S(\Gamma(C_1, C_2; S))$ under suitable conditions.

Proposition 4.9. *Let S be a carpet of measure zero in $\mathbb{S}^2$ whose peripheral circles are uniformly relatively separated uniform quasicircles, and let C_1 and C_2 be two distinct peripheral circles of S. Then an extremal mass distribution ρ for $\operatorname{mod}_S(\Gamma(C_1, C_2; S))$ exists, has finite and positive total mass, and is given as follows. Let f be a quasisymmetric map of S onto a square carpet T in a $\mathbb{C}^*$-cylinder $A = \{z \in \mathbb{C} \colon r \leq |z| \leq R\}$ such that C_1 corresponds to the inner and C_2 to the outer boundary component of A. Then $\rho(C_1) = \rho(C_2) = 0$, and for the peripheral circles $C \neq C_1, C_2$ of S, we have*

$$\rho(C) = \frac{\ell_{\mathbb{C}^*}(Q)}{\log(R/r)},$$

where Q is the $\mathbb{C}^*$-square bounded by $f(C)$.

This is [Bon11, Cor. 12.2]. Note that a map f as in this proposition exists by the previous Theorem 4.8. The map f is actually unique up to scaling and rotation around 0 as follows from Theorem 1.5, which we will prove in Section 6. It follows from Proposition 4.1 that f has a quasiconformal extension to $\mathbb{S}^2$, and so T is also a set of measure zero [LV73, Th. 1.3, p. 165]. From the explicit description of the extremal mass distribution, it follows that

$$0 < \operatorname{mod}_S(\Gamma(C_1, C_2; S)) = \frac{2\pi}{\log(R/r)} < \infty.$$

In [Bon11] the proof of Proposition 4.9 was fairly straightforward but had to rely on Theorem 4.8, whose proof was rather involved. The only consequence of Proposition 4.9 that we will use is that for the extremal density ρ, we have $\rho(C) > 0$ for all peripheral circles $C \neq C_1, C_2$. It is an interesting question whether a direct proof of this statement can be given without resorting to Theorem 4.8.

5. **Distinguished pairs of peripheral circles**

In the following it is convenient to use the term *square* also for the boundary of a solid Euclidean square in the usual sense. It will be clear from the context which meaning of square is intended. If Q is a square in either sense, then we denote by $\ell(Q)$ its Euclidean side length. A *corner* of Q is an end point of a side of Q.

With our terminology, we can refer to the peripheral circles of the standard Sierpiński carpets S_p, $p \geq 3$ odd, simply as squares. These squares arising as peripheral circles of S_p form a family of uniform quasicircles in the Euclidean metric, since each of them can be mapped to the boundary ∂Q_0 of the solid unit square Q_0 by a Euclidean similarity. This family is also uniformly relatively separated in the Euclidean metric, because if C and C' are two distinct squares in this family, then for their Euclidean distance, we have

$$\begin{aligned}\operatorname{dist}(C, C') &\geq \frac{p-1}{2} \min\{\ell(C), \ell(C')\} \\ &= \frac{p-1}{2\sqrt{2}} \min\{\operatorname{diam}(C), \operatorname{diam}(C')\}.\end{aligned}$$

Since Euclidean distance and spherical distance on $S_p \subseteq \mathbb{S}^2 \cong \widehat{\mathbb{C}}$ are comparable, it follows that the family of peripheral circles is uniformly relatively separated and consists of uniform quasicircles also with respect to the spherical metric. Since S_p has measure zero in addition, we can apply to S_p the results that were stated in Section 4. Moreover, by the comparability of Euclidean and spherical metric on S_p, the class of quasisymmetric self-maps on S_p is the same for both metrics.

Our goal in this section is to prove that any quasisymmetric self-map of S_p preserves the outer and the middle squares as a pair. By definition, the *outer square* O is the peripheral circle that corresponds to the boundary of the original unit square in the construction of S_p. The *middle square* M is the boundary of the open middle square removed from the unit square in the first step of the construction of S_p. It is the unique peripheral circle different from O that is invariant under all Euclidean isometries of S_p. Note that these isometries of S_p form a dihedral group with eight elements.

LEMMA 5.1. *Let $p \geq 3$ be odd, and let C, C' be any* (*unordered*) *pair of distinct peripheral circles of S_p other than M, O. Then*

$$\operatorname{mod}_{S_p}(\Gamma(C, C'; S_p)) < \operatorname{mod}_{S_p}(\Gamma(M, O; S_p)).$$

Proof. The self-similarity of the carpet S_p and the monotonicity property of the modulus give

$$\operatorname{mod}_{S_p}(\Gamma(C, C'; S_p)) \leq \operatorname{mod}_{S_p}(\Gamma(M, O; S_p)). \tag{5.1}$$

Indeed, if l and l' are the side lengths of the squares C and C', respectively, then we may assume that $l \leq l'$. Then $l \leq 1/p^2$. This implies that there exists a copy $S \subset S_p$, $S \neq S_p$, of S_p, rescaled by the factor pl, so that C corresponds to M, the middle square. Then the outer square o of S is the rescaled copy of O, and the interior region of o is disjoint from C'. Hence every path in $\Gamma(C, C'; S_p)$ meets o (possibly in one of its end points) and so contains a subpath in $\Gamma(C, o; S)$. (See Figure 2 for an illustration of this situation.) Therefore,

$$\operatorname{mod}_{S_p}(\Gamma(C, C'; S_p)) \leq \operatorname{mod}_{S_p}(\Gamma(C, o; S)).$$

On the other hand,

$$\operatorname{mod}_{S_p}(\Gamma(C, o; S)) = \operatorname{mod}_S(\Gamma(C, o; S)),$$

since every path in $\Gamma(C, o; S)$ meets exactly the same peripheral circles of S and S_p. Moreover,

$$\operatorname{mod}_S(\Gamma(C, o; S)) = \operatorname{mod}_{S_p}(\Gamma(M, O; S_p)),$$

by Lemma 2.1. Inequality (5.1) follows.

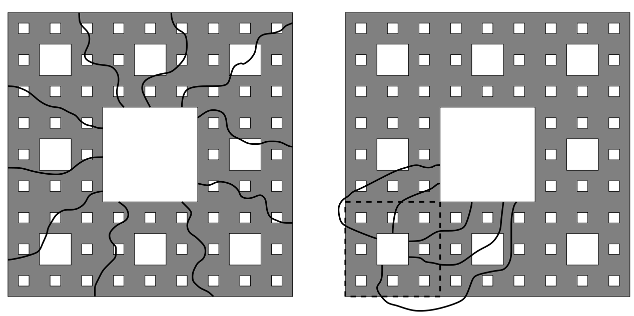

Figure 2. The part of the carpet on the right bounded by the dashed line is a rescaled copy S of S_p.

To reach a contradiction, assume now that

$$\operatorname{mod}_{S_p}(\Gamma(C, C'; S_p)) = \operatorname{mod}_{S_p}(\Gamma(M, O; S_p)). \tag{5.2}$$

Note that all carpet moduli considered above are finite by Proposition 4.9, and so an extremal mass distribution exists for each of them by Proposition 2.4. Then (5.2), the preceding discussion, and the uniqueness of the extremal mass distributions implies that the extremal mass distribution for $\operatorname{mod}_{S_p}(\Gamma(C, C'; S_p))$ is obtained from the extremal mass distribution for $\operatorname{mod}_{S_p}(\Gamma(M, O; S_p))$ by "transplanting" it to S using a suitable Euclidean similarity between S and S_p (similarly as in the proof of Lemma 2.1). Hence the extremal mass distribution for $\operatorname{mod}_{S_p}(\Gamma(C, C'; S_p))$ is supported only on the set of peripheral circles of S_p that are also peripheral circles of S. This is, however, not the case as follows from Proposition 4.9, and we arrive at a contradiction. $\square$

COROLLARY 5.2. *Let $p \geq 3$ be odd. Then every quasisymmetric self-map of S_p preserves the middle and the outer squares M and O as a pair.*

So if $f\colon S_p \to S_p$ is a quasisymmetric map, then $\{f(M), f(O)\} = \{M, O\}$. This allows the possibility that f interchanges M and O, i.e., that $f(M) = O$ and $f(O) = M$. We will later see that actually $f(M) = M$ and $f(O) = O$ (Lemma 8.1).

Proof. Assume that f maps the pair M, O to some pair of peripheral circles C, C'. By Proposition 4.1, the map f extends to a quasiconformal homeomorphism F on $\mathbb{S}^2$. In particular, $\Gamma(C, C'; S_p) = F(\Gamma(M, O; S_p))$. Lemma 2.1 then implies

$$\operatorname{mod}_{S_p}(\Gamma(C, C'; S_p)) = \operatorname{mod}_{S_p}(\Gamma(M, O; S_p)). \tag{5.3}$$

By Lemma 5.1, this is only possible if $\{C, C'\} = \{M, O\}$. $\square$

COROLLARY 5.3. *Let $p \geq 3$ be odd. Then the group* $\mathrm{QS}(S_p)$ *of quasisymmetric self-maps of S_p is finite.*

Proof. According to Corollary 5.2, the middle square M and the outer square O of S_p are preserved as a pair under every quasisymmetric self-map of S_p. Moreover, by Corollary 4.6, the group G of all orientation-preserving quasisymmetric self-maps f of S_p with $f(M) = M$ and $f(O) = O$ is finite cyclic. If $f_1, f_2 \in \mathrm{QS}(S_p)$ are orientation-reversing, then $f_1^{-1} \circ f_2 \in \mathrm{QS}(S_p)$ is orientation-preserving. Likewise, if $f_1, f_2 \in \mathrm{QS}(S_p)$ interchange M and O, then $f_1^{-1} \circ f_2$ preserves both M and O setwise. This implies that G is a subgroup of $\mathrm{QS}(S_p)$ with index at most 4. Since G is a finite group, $\mathrm{QS}(S_p)$ is finite as well. $\square$

6. Quasisymmetric rigidity of square carpets

In this section we prove quasisymmetric rigidity results for square carpets in rectangles and for square carpets in $\mathbb{C}^*$-cylinders.

By definition, a *square carpet S in a closed Jordan region $D \subseteq \mathbb{R}^2 \cong \mathbb{C}$* is a carpet $S \subseteq D$ so that ∂D is a peripheral circle of S and all other peripheral circles are squares with sides parallel to the coordinate axes (see Figure 3). We equip such a carpet with the Euclidean metric.

We will now prove Theorem 1.4. In the ensuing proofs all metric concepts refer to the Euclidean metric on $\mathbb{R}^2 \cong \mathbb{C}$. Moreover, we will use the Euclidean metric also as a base metric in the definition of conformal modulus of a path family.

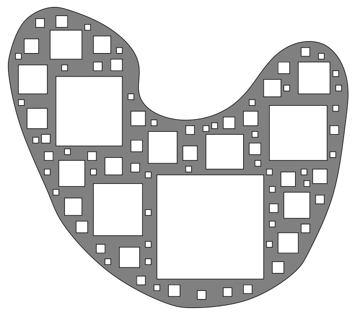

Figure 3. A square carpet in a closed Jordan region.

Proof of Theorem 1.4. Without loss of generality we may assume that $\tilde{a} \leq a$. Suppose f is a quasisymmetric map as in the statement. Note that the peripheral circles of S are uniform quasicircles; this is clear if we use the Euclidean metric, but on K the Euclidean and the spherical metrics are comparable, and so the peripheral circles of S are uniform quasicircles also with respect to the spherical metric. Hence by Proposition 4.1, the map f extends to a quasiconformal map F on $\mathbb{S}^2$.

We denote by C_i, $i \in \mathbb{N}$, the peripheral circles of S distinct from ∂K and by Q_i the closed solid square bounded by C_i. We set $\widetilde{C}_i := f(C_i)$. Then the sets $\widetilde{C}_i$, $i \in \mathbb{N}$, are the peripheral circles of $\widetilde{S}$ distinct from $\partial \widetilde{K}$. For $i \in \mathbb{N}$, let $\widetilde{Q}_i$ be the closed solid square bounded by $\widetilde{C}_i$. Note that then $\widetilde{Q}_i = F(Q_i)$.

For $t \in [0, a]$, we denote by γ_t the path $u \mapsto t + \boldsymbol{i}u$, $u \in [0, 1]$, and let $\Gamma = \{\gamma_t : t \in [0, a]\}$. So Γ consists precisely of the closed vertical line segments connecting the horizontal sides of K. Then $\operatorname{mod}(\Gamma) = a$ and the function $\rho_0 \equiv 1$ on K is an extremal density for $\operatorname{mod}(\Gamma)$. If $\tilde{\rho}$ is another extremal density for $\operatorname{mod}(\Gamma)$, then $\tilde{\rho}(z) = 1$ for almost every $z \in K$.

We define a Borel density ρ_1 on K as follows. For $z \in K$, we set $\rho_1(z) = \ell(\widetilde{Q}_i)/\ell(Q_i)$ if $z \in Q_i$ for some $i \in I$, and $\rho_1(z) = 0$ otherwise. We will show that $\int_\gamma \rho_1 \, ds \geq 1$ for almost every path $\gamma \in \Gamma$. This will make it possible to adjust ρ_1 on a set of measure zero so that the resulting density is admissible for Γ.

To see this, let E be the set of all $t \in [0, a]$ for which $\gamma_t \cap S$ has positive length, i.e., $E = \{t \in [0, a] : \int_{\gamma_t} \chi_S \, ds > 0\}$. Since S has measure zero, it follows from Fubini's theorem that E has 1-dimensional measure zero.

Moreover, since quasiconformal maps are absolutely continuous on almost every line [LV73, Th. 3.1, p. 170], there exists a set $E' \subseteq [0, a]$ of 1-dimensional measure zero such that the map $u \mapsto F(t + \boldsymbol{i}u)$ is absolutely continuous on $[0, 1]$ for each $t \in [0, a] \setminus E'$.

If we define $E_0 = E \cup E' \subseteq [0,a]$, then E_0 also has 1-dimensional measure zero. Moreover, if $t \in [0,a] \setminus E$, then the map $u \mapsto F(t + \boldsymbol{i}u)$ is absolutely continuous on $[0,1]$, and $\gamma_t \cap S$ has length zero. It follows that $F(\gamma_t \cap S) = F(\gamma_t) \cap \widetilde{S}$ also has length zero. By our normalization assumption, the map f, and hence also its extension F, sends each horizontal side of K to a horizontal side of $\widetilde{K}$. Thus $F(\gamma_t)$ joins the horizontal sides of $\widetilde{K}$ and we conclude that

$$\int_{\gamma_t} \rho_1 \, ds = \sum_{\gamma_t \cap Q_i \neq \emptyset} \ell(\widetilde{Q}_i) = \sum_{F(\gamma_t) \cap \widetilde{Q}_i \neq \emptyset} \ell(\widetilde{Q}_i) \geq 1.$$

For $z \in K$, define $\rho_2(z) = \infty$ if $z \in E_0 \times [0,1]$ and $\rho_2(z) = 0$ otherwise. Then $\int_{\gamma_t} \rho_2 \, ds = \infty$ for $t \in E_0$. It follows that $\rho = \rho_1 + \rho_2$ is admissible for Γ. Moreover, since $\rho_2(z) = 0$ for almost every $z \in K$, we have

$$\int_K \rho^2 \, dA = \int_K \rho_1^2 \, dA = \sum_i \ell(\widetilde{Q}_i)^2 = \widetilde{a} \leq a = \operatorname{mod}(\Gamma),$$

where dA indicates integration with respect to 2-dimensional Lebesgue measure. Therefore, $\widetilde{a} = a$, $\widetilde{K} = K$, and ρ is extremal for $\operatorname{mod}(\Gamma)$. Hence $\rho(z) = 1$ for almost every $z \in K$, which in turn implies that $\ell(Q_i) = \ell(\widetilde{Q}_i)$ for all $i \in \mathbb{N}$. So each square Q_i has the same side length as its image square $\widetilde{Q}_i = F(Q_i)$.

We will next show that actually $Q_i = \widetilde{Q}_i$ for each $i \in \mathbb{N}$. To see this, let $i \in \mathbb{N}$ be arbitrary, and consider the family Γ' of all open line segments that are parallel to the real axis and connect the left vertical side of K to the left vertical side of Q_i (so these sides face each other). Let $\gamma \in \Gamma'$, and assume in addition that $\gamma \cap S$ has length zero and that F is absolutely continuous on γ. Then the intersection $F(\gamma) \cap \widetilde{S}$ also has length zero, and we have

$$\operatorname{length}(\gamma) = \sum_{\gamma \cap Q_j \neq \emptyset} \ell(Q_j).$$

Since F sends each peripheral square to a square of the same side length, we conclude that for the length L of the projection of $F(\gamma)$ to the real axis, we have

$$L \leq \sum_{F(\gamma) \cap \widetilde{Q}_j \neq \emptyset} \ell(\widetilde{Q}_j) = \sum_{\gamma \cap Q_j \neq \emptyset} \ell(Q_j) = \operatorname{length}(\gamma).$$

In other words, the length of the projection of γ to the real axis (which is equal to the length of γ) does not increase under the map F.

Similarly as above, one can see that the family Γ'_0 of all line segments $\gamma \in \Gamma'$ for which $\gamma \cap S$ has positive length or for which F is not absolutely continuous on γ has modulus zero. In particular, for each $\gamma \in \Gamma'$, there are line segments $\gamma' \in \Gamma' \setminus \Gamma'_0$ that are arbitrarily close to γ. Since the length of the projection of each such line segment γ' to the real axis does not increase under F, it cannot increase for any $\gamma \in \Gamma'$ either.

We conclude that the distance of $\widetilde{Q}_i$ to the left vertical side of $\widetilde{K} = K$ is bounded from above by the distance of Q_i to the left vertical side of K. Using the same argument for the inverse map, we conclude that these distances are actually equal. We can apply the same reasoning for other pairs of respective sides of K and Q_i. Hence Q_i and $\widetilde{Q}_i$ are squares in $K = \widetilde{K}$ with the same distances to all sides of K. This implies $Q_i = \widetilde{Q}_i$.

Moreover, we can actually deduce that F maps each side of $C_i = \partial Q_i$ into itself. Indeed, let $p \in C_i$ be a point on the left vertical side of C_i, say. If Γ' is as above, then there is a line segment $\gamma \in \Gamma'$ that has p as one of its end points. Then $F(p) \in \widetilde{C}_i = C_i$ is one end point of $F \circ \gamma$, while the other end point lies on the left vertical side of K. As we have seen, the length of the projection of $F \circ \gamma$ to the real axis is bounded by the length of γ, which is equal to the distance of C_i to the left vertical side of K. This is only possible if $F(p)$ lies on the left vertical side of C_i. So F maps this side into itself and, similarly, each side of C_i into itself. This in turn implies F must fix each corner of C_i.

Since $i \in \mathbb{N}$ was arbitrary, we conclude that $S = \widetilde{S}$ and that F, and hence also f, fixes the corners of all squares Q_i, $i \in \mathbb{N}$.

Now every subset of a carpet that meets all but finitely many peripheral circles is dense. In particular, the set D consisting of all corners of the squares C_i, $i \in \mathbb{N}$, is a dense subset of S. Since f is the identity on D, it follows that the map f is the identity on S. $\square$

We now prove Theorem 1.5. The argument is very similar to the proof of Theorem 1.4. In the proof, metric notions refer to the flat metric on $\mathbb{C}^*$ given by the length element $|dz|/|z|$. We will also use it as a base metric for modulus. For terminology related to $\mathbb{C}^*$-cylinders and $\mathbb{C}^*$-squares used in the ensuing proof, see the discussion before Theorem 4.8.

Proof of Theorem 1.5. Let f be as in the statement. Each $\mathbb{C}^*$-square Q (equipped with the flat metric on $\mathbb{C}^*$) that satisfies $\ell(Q) \leq \pi$ is isometric to a Euclidean square of the same sidelength. So the family of all peripheral circles of S that bound such $\mathbb{C}^*$-squares consists of uniform quasicircles. There are only finitely many peripheral circles not in this family, namely the boundary components of A and boundaries of complementary components of S that are $\mathbb{C}^*$-squares Q with $\pi < \ell(Q) < 2\pi$. Since each of these finitely many peripheral circles (equipped with the flat metric on $\mathbb{C}^*$) is bi-Lipschitz equivalent to the unit circle, it follows that the family of *all* peripheral circles of S consists of uniform quasicircles. So again by Proposition 4.1, we can extend the map f to a quasiconformal map F on $\mathbb{S}^2$. Then F, as the map f, is orientation-preserving.

We may assume that the inner components of A and $\widetilde{A}$ are equal to the unit circle and that the outer boundary components of A and $\widetilde{A}$ are equal to $\{z \in \mathbb{C} : |z| = R\}$ and $\{z \in \mathbb{C} : |z| = \widetilde{R}\}$, respectively, where $1 < R \leq \widetilde{R}$.

We denote by Q_i, $i \in \mathbb{N}$, the (open) $\mathbb{C}^*$-squares whose boundaries give the peripheral circles of S distinct from the boundary components of A, and we set $\widetilde{Q}_i = F(Q_i)$ for $i \in \mathbb{N}$. Then $\partial\widetilde{Q}_i$, $i \in \mathbb{N}$, is the family of all peripheral circles of $\widetilde{S}$ distinct from the boundary components of $\widetilde{A}$.

We now consider the family Γ of closed radial line segments joining the boundary components of A. Then $\mathrm{mod}(\Gamma) = 2\pi/\log(R)$, and $\rho_0 = 1/\log(R)$ is the essentially unique extremal density (with the flat metric as the underlying base metric). On the other hand, we define a density ρ_1 on A such that

$$\rho_1(z) = \frac{\ell_{\mathbb{C}^*}(\widetilde{Q}_i)}{\log(\widetilde{R})\ell_{\mathbb{C}^*}(Q_i)}$$

if $z \in Q_i$ for some $i \in \mathbb{N}$, and $\rho(z) = 0$ elsewhere on A. As in the proof of Theorem 1.4, one shows that up to adjustment on a set of measure zero, ρ_1 is admissible for Γ. Moreover,

$$\int_A \rho_1^2 \, dA_{\mathbb{C}^*} = \frac{1}{\log^2(\widetilde{R})} \sum_i \ell_{\mathbb{C}^*}(\widetilde{Q}_i)^2 = \frac{2\pi}{\log(\widetilde{R})} \leq \frac{2\pi}{\log(R)} = \mathrm{mod}(\Gamma),$$

where $dA_{\mathbb{C}^*}$ means integration with respect to the area element induced by the flat metric. Hence $R = \widetilde{R}$, $A = \widetilde{A}$, and ρ_1 (up to a change on a set of measure zero) is extremal for $\mathrm{mod}(\Gamma)$. We conclude that $\rho_1 = 1/\log(R)$ almost everywhere on A, which implies that $\ell_{\mathbb{C}^*}(Q_i) = \ell_{\mathbb{C}^*}(\widetilde{Q}_i)$ for all $i \in \mathbb{N}$. So again, each $\mathbb{C}^*$-square Q_i has the same side length as its image $\widetilde{Q}_i$ under F.

Using this and arguments similar to the ones in the proof of Theorem 1.4, one can show that for each $i \in \mathbb{N}$, the squares Q_i and $\widetilde{Q}_i$ have the same distances to the inner and outer boundary components of A and that F maps the bottom and top sides of Q_i to the bottom and top sides of $\widetilde{Q}_i$, respectively. This implies that F sends the corners of Q_i to the corners of $\widetilde{Q}_i$. Since F is orientation-preserving, the cyclic order of the corners is preserved under the map F. It follows that for each $i \in \mathbb{N}$, there exits a rotation r_i around 0 such that $r_i(Q_i) = F(Q_i) = \widetilde{Q}_i$ and such that $r_i(c) = f(c) = F(c)$ if c is a corner of Q_i.

So far, we exclusively used the behavior of F on radial directions. We will now investigate the behavior of F on "circular directions." To do this, we consider the circular projection

$$P_i := \{t \in (0, \infty) : te^{i\alpha} \in Q_i \text{ for some } \alpha \in [0, 2\pi]\}$$

of Q_i to the positive real axis. Each set P_i, $i \in \mathbb{N}$, is an open subinterval of $(1, R)$.

If $P_i \cap P_j \neq \emptyset$ for $i, j \in \mathbb{N}$, $i \neq j$, then the circular projections of the squares Q_i and Q_i overlap, and so we can find a family Γ' of closed circular arcs γ, each contained in a circle of radius $t \in P_i \cap P_j$ centered at 0, that join two vertical sides of Q_i and Q_j facing each other. This family Γ' has positive modulus, and similarly as in the proof of Theorem 1.4, one can show that the length of the radial projection of each path $\gamma \in \Gamma'$ to the unit circle does not increase under the map F. Applying the same argument to the other pair of vertical sides of Q_i and Q_j facing each other, we conclude that the circular distance of Q_i and Q_j is the same as the circular distance of the image squares $\widetilde{Q}_i$ and $\widetilde{Q}_j$. This implies that $r_i = r_j$.

We can write

$$U := \bigcup_i P_i = \bigcup_{k \in J} M_k,$$

where J is a countable index set and the sets M_k, $k \in J$, are pairwise disjoint open subintervals of $(1, R)$ forming the connected components of U. Suppose that $M_k = (a_k, b_k)$, where $1 \leq a_k < b_k \leq R$, and set $A_k := \{z \in \mathbb{C} : a_k < |z| < b_k\}$. For every $i \in \mathbb{N}$, there exists precisely one $k \in J$ such that $Q_i \subseteq A_k$. Moreover, since any two points $u, v \in M_k$ can be connected by a chain of intervals $P_i \subseteq A_k$, it follows that $r_i = r_j$ whenever $P_i, P_j \subseteq M_k$ for some $k \in J$. So for each $k \in J$, there exists a rotation $\tilde{r}_k$ around 0 such that $r_i = \tilde{r}_k$ whenever $Q_i \subseteq A_k$.

We claim that

$$f|S \cap A_k = \tilde{r}_k|S \cap A_k \tag{6.1}$$

for each $k \in J$. To see this, let $k \in J$ and $z_0 \in S \cap A_k$ be arbitrary. Since the set of corners of the $\mathbb{C}^*$-squares Q_i is dense in S, there exists a sequence (c_n) of such corners with $c_n \to z_0$ as $n \to \infty$. If c_n is a corner of the $\mathbb{C}^*$-square Q_{i_n}, then $Q_{i_n} \subseteq A_k$ for sufficiently large n. For these n, we have $\tilde{r}_k(c_n) = r_{i_n}(c_n) = f(c_n)$. Passing to the limit $n \to \infty$, we conclude that indeed $\tilde{r}_k(z_0) = f(z_0)$ as desired.

We also have

$$f|\partial Q_i = r_i|\partial Q_i \tag{6.2}$$

for each $i \in \mathbb{N}$. Indeed, let $i \in \mathbb{N}$ be arbitrary. By (6.1), it is clear that f and r_i agree on the interior of each vertical side of Q_i, because these interiors are contained in a suitable set $S \cap A_k$. Let u be a point on one of the other sides of Q_i, say on the bottom side of Q_i. Pick a corner v of Q_i on the same side. We will construct a sequence (k_j) in J, and sequences (u_j) and (v_j) of points such that $u_j, v_j \in S \cap A_{k_j}$ for $j \in \mathbb{N}$, and $u_j \to u$ and $v_j \to v$ as $j \to \infty$. Then by (6.1), we have $\tilde{r}_{k_j}(u_j) = f(u_j)$ and $\tilde{r}_{k_j}(v_j) = f(v_j)$. By passing to a subsequence if necessary, we may assume that the rotations $\tilde{r}_{k_j}$ converge to a rotation r' uniformly on A as $j \to \infty$. Then $r'(u) = f(u)$ and $r'(v) = f(v)$. Since v is a corner of Q_i, we also have $r_i(v) = f(v) = r'(v)$. Hence $r' = r_i$, and so $r_i(u) = r'(u) = f(u)$ as desired.

To produce the sequences (u_j) and (v_j), we consider the set of $E = [1, R] \setminus U$. This is the set of all radii of circles centered at 0 that lie in S. Since S has measure zero, E has 1-dimensional measure zero.

Suppose that $u = se^{i\alpha}$, where $1 < s = |u| = |v| < R$ and $\alpha \in [0, 2\pi]$. Since E is a set of measure zero, we can find a sequence (s_j) of "good radii" such that $s_j \in (1, s) \setminus E$ and $s_j \to s$ as $j \to \infty$. Then there exists $k_j \in J$ such that $s_j \in A_{k_j}$ for $j \in \mathbb{N}$. Define $u'_j = s_j e^{i\alpha}$ for $j \in \mathbb{N}$. Then $u'_j \in A_{k_j} \subseteq A$ for $j \in \mathbb{N}$, and $u'_j \to u$ as $j \to \infty$, but the sequence (u'_j) is not necessarily contained in S. To achieve this, we shift each point u'_j on the circle $\{z : |z| = s_j\}$ if necessary. More precisely, if $u'_j \in S$, we let $u_j := u'_j$. If u'_j does not lie in S, then u'_j is contained in one of the $\mathbb{C}^*$-squares Q_l, $l \in \mathbb{N}$. We can then move u'_j on the circle $\{z : |z| = s_j\}$ to a point u_j on one of the vertical sides of Q_l. Note that $Q_l \neq Q_i$, and so the diameter of Q_l is small if j is large, since $\mathbb{C}^*$-squares $Q_l \neq Q_i$ exceeding a given size cannot be arbitrarily close to Q_i. So we have $u_j \in S \cap A_{k_j}$ and $u_j \to u$ as $j \to \infty$. A sequence (v_j) with $v_j \in S \cap A_{k_j}$ for $j \in \mathbb{N}$ and with $v_j \to v$ as $j \to \infty$ is constructed similarly. Note that our construction guarantees that u_j and v_j lie in S and in the same set A_{k_j}, which was crucial for the argument in the previous paragraph.

Now that we have established (6.2), we can finish the argument as follows. The proof of Proposition 5.1 in [Bon11] combined with (6.2) shows that one can find a quasiconformal extension $\widetilde{F}$ of f to $\mathbb{S}^2$ such that $\widetilde{F}|Q_i = r_i|Q_i$ for each $i \in \mathbb{N}$. Then $\widetilde{F}$ is conformal on each Q_i. Since $\widetilde{F}$ is quasiconformal and the squares Q_i fill A up to a set of measure zero, it follows that $\widetilde{F}$ is a 1-quasiconformal map on the interior of A. Hence $\widetilde{F}$ is a conformal map on the interior of A [LV73, Th. 5.1, p. 28]. Since $\widetilde{F} = r_1$ on Q_1, it follows that $\widetilde{F}|A$ is a rotation around 0. Then on S, the map $f = \widetilde{F}|A$ also agrees with such a rotation. The statement follows. □

7. **Weak tangent spaces**

In this section we discuss some facts about weak tangents of the carpets S_p. The most important result here is Proposition 7.3, which will be crucial in the proofs of our main theorems.

In general, weak tangent spaces can be defined as Gromov-Hausdorff limits of pointed metric spaces obtained by rescaling the underlying metric. (See [BBI01, Chs. 7, 8] and [DS97, Ch. 8] for the general definitions, and see [BK02] for applications very similar in spirit to the present paper.) As we will need this only for the carpets S_p, we will first present a suitable definition for arbitrary subsets of $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \cong \mathbb{S}^2$ and then further adjust the definition for the carpets S_p.

If $a, b \in \mathbb{C}$, $a \neq 0$, and $M \subseteq \widehat{\mathbb{C}}$, we denote by $aM + b$ the image of M under the Möbius transformation $z \mapsto az + b$ on $\widehat{\mathbb{C}}$. Let A be a subset of $\widehat{\mathbb{C}}$ with a distinguished point $z_0 \in A$, $z_0 \neq \infty$. We say that a closed set $A_\infty \subseteq \widehat{\mathbb{C}}$ is a *weak tangent* of A (*at* z_0) if there exists a sequence (λ_n) of positive real numbers with $\lambda_n \to \infty$ such that the sets $A_n := \lambda_n(A - z_0)$ tend to A_∞ as

$n \to \infty$ in the sense of Hausdorff convergence on $\widehat{\mathbb{C}}$ equipped with the spherical metric. (See [BBI01, Ch. 7, §7.3.1] for the definition of Hausdorff convergence of sets in a metric space.) In this case, we use the notation

$$A_\infty = \lim_{n\to\infty}(A, z_0, \lambda_n).$$

So a weak tangent of A at z_0 is obtained by extracting a limit from "blowing up" A at z_0 by suitable scaling factors $\lambda_n \to \infty$. Every weak tangent of A contains 0 and, if A is not a singelton set, also the point ∞.

A set $A \subseteq \widehat{\mathbb{C}}$ has weak tangents at each point $z_0 \in A \setminus \{\infty\}$, because for every sequence (λ_n) of positive numbers with $\lambda_n \to \infty$, there is a subsequence (λ_{n_k}) such that the sequence of the sets $A_{n_k} = \lambda_{n_k}(A - z_0)$ converges as $k \to \infty$ ([BBI01, Th. 7.3.8, p. 253]). In general, weak tangents at a point are not unique. In particular, if $\lambda > 0$ and A_∞ is a weak tangent of A at a point, then λA_∞ is also a weak tangent.

It is advantageous to avoid this scaling ambiguity of weak tangents for the standard carpets S_p, $p \geq 3$ odd, and restrict the scaling factors λ_n used in the definition of a weak tangent to powers of p. So in the following, *a weak tangent of S_p at a point $z_0 \in S_p$* is a closed set $A_\infty \subseteq \widehat{\mathbb{C}}$ such that

$$A_\infty = \lim_{n\to\infty}(S_p, z_0, p^{k_n}),$$

where $k_n \in \mathbb{N}_0$ and $k_n \to \infty$ as $n \to \infty$. If this limit exists along the full sequence (p^n), i.e., if

$$A_\infty = \lim_{n\to\infty}(S_p, z_0, p^n)$$

exists, then A_∞ is the unique weak tangent of S_p at z_0. We equip each weak tangent of S_p with the spherical metric unless otherwise indicated.

We now exhibit some points in S_p, where we have unique weak tangents, and set up some notation for the weak tangents thus obtained.

Fix an odd integer $p \geq 3$. At the point 0, the carpet S_p has the unique weak tangent

$$W_{\pi/2} := \lim_{n\to\infty}(S_p, 0, p^n) = \{\infty\} \cup \bigcup_{n\in\mathbb{N}_0} p^n S_p. \tag{7.1}$$

The existence of this and similar limits below can easily be justified by observing that by self-similarity of S_p, the relevant sets involved form an increasing sequence. Here (7.1) follows from the inclusions $p^n S_p \subseteq p^{n+1} S_p$ for $n \in \mathbb{N}_0$.

Similarly, at each corner of O, there exists a unique weak tangent of S_p obtained by a suitable rotation of the set $W_{\pi/2}$ around 0.

Let $m = 1/2$ be the midpoint of the bottom side of O. Then at the point m the carpet S_p has the unique weak tangent

$$W_\pi := \lim_{n\to\infty}(S_p, 1/2, p^n) = \{\infty\} \cup \bigcup_{n\in\mathbb{N}_0} p^n (S_p - m).$$

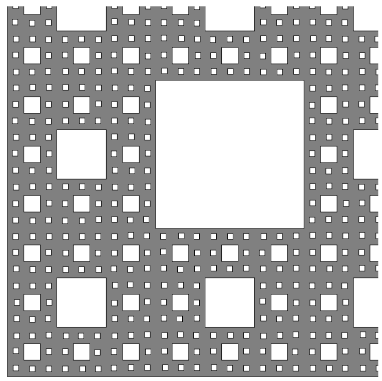

Figure 4. The weak tangent space $W_{\pi/2}$ for $p = 3$.

Moreover, if z_0 is any midpoint of a side of a square that is a peripheral circle of S_p, then S_p has a unique weak tangent at z_0 obtained by a suitable rotation of the set W_π around 0. This easily follows from the fact that for the existence and uniqueness of a weak tangent at a point z_0, only an arbitrary small (not necessarily open) relative neighborhood of z_0 in S_p is relevant, as the complement of such a neighborhood will "disappear to infinity" if the set is blown up at z_0. Moreover, by self-similarity of S_p, for each of these midpoints z_0, we can choose a relative neighborhood N of z_0 in S_p so that a suitable Euclidean similarity maps N to S_p and z_0 to m, and where the scaling factor of the similarity is an integer power of p.

Let

$$c := \frac{p-1}{2p} + \frac{p-1}{2p}\boldsymbol{i}$$

be the lower left corner of M. Then at c, the carpet S_p has a unique weak tangent

$$W_{3\pi/2} := \lim_{n\to\infty}(S_p, c, p^n) = \{\infty\} \cup \bigcup_{n\in\mathbb{N}_0} p^n\Big(\boldsymbol{i}S_p \cup (-\boldsymbol{i})S_p \cup (-1)S_p\Big).$$

Note that $W_{3\pi/2}$ can be obtained by pasting together three copies of $W_{\pi/2}$. If z_0 is any corner of a square $C \neq O$ that is a peripheral circle of S_p, then S_p has a unique weak tangent at z_0 obtained by a suitable rotation of the set $W_{3\pi/2}$ around 0.

The angles $\pi/2$, π, $3\pi/2$ used as indices of the above weak tangents indicate that the corresponding space is contained in the closure of the quarter, half-, and three-quarter plane, respectively. Mostly, it will be clear from the context what p is, so we will usually omit an additional label p from the notation; if we want to indicate p, then we write $W_{\pi/2}(p)$ for the weak tangent $W_{\pi/2}$ of S_p, etc.

For $p \geq 3$ odd, we denote by D_p the set of all midpoints of sides and all corners of squares that are peripheral circles of S_p. It follows from our previous discussion that at each point $z_0 \in D_p$, the carpet S_p has a unique weak tangent W isometric to one of the sets $W_{\pi/2}$, W_π, or $W_{3\pi/2}$. The weak tangent W of S_p at $z_0 \in D_p$ can always be written as

$$W = \{\infty\} \cup \bigcup_{n \in \mathbb{N}_0} p^n(N - z_0), \tag{7.2}$$

where N is a suitable relative neighborhood of z_0 in S_p such that

$$p^n(N - z_0) \subseteq p^{n+1}(N - z_0) \tag{7.3}$$

for all $n \in \mathbb{N}_0$. Actually, we can choose N to be a rescaled copy of S_p (if W is isometric to $W_{\pi/2}$ or W_π) or a union of three rescaled copies of S_p (if W is isometric to $W_{3\pi/2}$). Note that (7.2) and (7.3) imply that $p^n W = W$ for all $n \in \mathbb{Z}$.

Lemma 7.1. *Let $p \geq 3$ be odd, $z_0 \in D_p$, and W be the weak tangent of S_p at z_0. Then W is a carpet of measure zero. If W is equipped with the spherical metric, then the peripheral circles of W form a family of uniform quasicircles that are uniformly relatively separated.*

Proof. We know that up to rotation around 0, the set W is equal to one of the weak tangents $W_{\pi/2}$, W_π, $W_{3\pi/2}$. So it is enough to show the statement for these weak tangents. We will do this for $W_{\pi/2}$. The proofs for W_π and $W_{3\pi/2}$ are the same with minor modifications.

First note that $W_{\pi/2}$ is a carpet, since it can be represented as in (1.1). Moreover, this set has measure zero, because by (7.2) it can be written as a countable union of sets of measure zero.

Let

$$\Omega = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0 \text{ and } \mathrm{Im}(z) > 0\}$$

be the open quarter-plane whose closure (in $\widehat{\mathbb{C}}$) contains $W_{\pi/2}$. Then $\partial\Omega$ is a peripheral circle of $W_{\pi/2}$. Since $\partial\Omega$ can be mapped to the unit circle by a bi-Lipschitz map, this peripheral circle is a quasicircle.

All other peripheral circles of $W_{\pi/2}$ are squares; actually, they are precisely the squares of the form $C' = p^n C$, where $p \in \mathbb{N}_0$ and C is a peripheral circle of S_p different from the outer square O. As all of these peripheral circles are similar to O, and O is bi-Lipschitz equivalent to the unit circle, the peripheral circles $C' \neq \partial\Omega$ of $W_{\pi/2}$ are uniform quasicircles in the Euclidean metric. This is equivalent to a uniform lower bound for certain (metric) cross-ratios of points on these peripheral circles (see [Bon11, Prop. 4.4(iv)]). Since cross-ratios for points in $\mathbb{C}$ are the same in the Euclidean and in the chordal metric (the restriction of the Euclidean metric on $\mathbb{R}^3$ to $\mathbb{S}^2 \cong \widehat{\mathbb{C}}$), it follows that the peripheral circles $C \neq \partial\Omega$ of W form a family of uniform quasicircles in the chordal metric. Since chordal and spherical metric on $\widehat{\mathbb{C}}$ are comparable, we also get a family of uniform quasicircles in the spherical metric. If we add the quasi-circle $\partial\Omega$ to this collection, we still have a family of uniform quasicircles in the spherical metric.

The uniform relative separation property of the peripheral circles of $W_{\pi/2}$ can be established similarly by passing from the Euclidean to the chordal and the spherical metrics. Namely, first note that if C and C' are peripheral circles of $W_{\pi/2}$ and $C \neq C'$, then for their Euclidean distance, we have

$$\operatorname{dist}(C, C') \geq \frac{p-1}{2} \min\{\ell(C), \ell(C')\},$$

where $\ell(C)$ and $\ell(C')$ denote the Euclidean side lengths of C and C', respectively, with the convention $\ell(\partial\Omega) = \infty$. This follows from the fact that if $\ell(C) \leq \ell(C')$ say, then there exists a rescaled copy S of S_p in $W_{\pi/2}$ with $C \subseteq S$ such that C corresponds to the middle square of S_p, and C' meets S at most in points of the peripheral circle o of S that corresponds to O.

The relative uniform separation of the family of all peripheral circles of $W_{\pi/2}$ with respect to the Euclidean metric follows. Again this is equivalent to a uniform lower bound for certain metric cross-ratios. (This follows from [Bon11, Lemma 4.6]; to include $\partial\Omega$, we need a slightly extended form of this lemma where one of the sets is allowed to have infinite diameter, but the statement and the proof of the lemma can easily be adjusted.) Since cross-ratios are unchanged if we pass to the chordal metric, it follows that the family of peripheral circles of $W_{\pi/2}$ is uniformly relatively separated with respect to the chordal metric, and hence also with respect to the spherical metric. $\square$

We are interested in quasisymmetric maps $g\colon W \to W'$ between weak tangents W of S_p and weak tangents W' of S_q. Note that $0, \infty \in W, W'$. We call g *normalized* if $g(0) = 0$ and $g(\infty) = \infty$.

Lemma 7.2. *Let $p, q \geq 3$ be odd, $z_0 \in D_p$, $w_0 \in D_q$, and let $f\colon S_p \to S_q$ be a quasisymmetric map with $f(z_0) = w_0$. Then f induces a normalized quasisymmetric map g between the weak tangent W of S_p at z_0 and the weak tangent W' of S_q at w_0.*

Proof. By Proposition 4.1, we can extend f to a quasiconformal homeomorphism $F\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. By our discussion earlier in this section, there exists a relative neighborhood N of z_0 in S_p and a relative neighborhood N' of w_0 in S_q such that $W \setminus \{\infty\} = \bigcup_{n \in \mathbb{N}_0} p^n(N - z_0)$ and $W' \setminus \{\infty\} = \bigcup_{n \in \mathbb{N}_0} q^n(N' - w_0)$.

Moreover, by (7.3) we may assume $p^{-n}(N-z_0) \subseteq N-z_0$ and $q^{-n}(N'-w_0) \subseteq N'-w_0$ for all $n \in \mathbb{N}_0$.

Pick a point $u_0 \in N-z_0$, $u_0 \neq 0$. Then for each $n \in \mathbb{N}_0$, we have

$$z_0 + p^{-n}u_0 \in N \setminus \{z_0\} \subseteq S_p,$$

and so $F(z_0+p^{-n}u_0) \neq w_0, \infty$. Hence we can choose a unique number $k(n) \in \mathbb{Z}$ as follows. If we define the map $F_n \colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ by

$$F_n(u) = q^{k(n)}\big(F(z_0+p^{-n}u) - w_0\big)$$

for $u \in \widehat{\mathbb{C}}$, then

$$1 \leq |F_n(u_0)| < q.$$

Note that $k(n) \to \infty$ as $n \to \infty$. Since $F(\infty) \notin S_q$, and so $F(\infty) \neq w_0$, this implies that $F_n(\infty) \to \infty$ as $n \to \infty$. We also have $F_n(0) = 0$. So the images of 0, ∞, and $u_0 \neq 0, \infty$ under F_n have mutual spherical distance uniformly bounded from below independent of n. Moreover, each map F_n is obtained from F by pre- and post-composing by Möbius transformations. Hence the sequence (F_n) is uniformly quasiconformal, and it follows that we can find a subsequence of (F_n) that converges uniformly on $\widehat{\mathbb{C}}$ to a quasiconformal map F_∞ [LV73, Th. 5.1(3), p. 73]. By passing to yet another subsequence if necessary, we can also assume that we have uniform convergence of the inverse maps in the subsequence to F_∞^{-1}.

In this way, we can find sequences (k_n) and (l_n) of natural numbers with $k_n \to \infty$ and $l_n \to \infty$ as $n \to \infty$ such that if we define

$$\widetilde{F}_n(u) = q^{k_n}\big(F(z_0+p^{-l_n}u) - w_0\big)$$

for $u \in \widehat{\mathbb{C}}$, then $\widetilde{F}_n \to F_\infty$ and $\widetilde{F}_n^{-1} \to F_\infty^{-1}$ uniformly on $\widehat{\mathbb{C}}$ as $n \to \infty$. Then $F_\infty(0) = 0$ and $F_\infty(\infty) = \infty$. Moreover, since F_∞ is quasiconformal, this map is a quasisymmetry on $\widehat{\mathbb{C}}$ [HK98, Th. 4.9].

So to prove the statement of the lemma, it suffices to show that $F_\infty(W) = W'$, because then $g := F_\infty|W$ is an induced normalized quasisymmetric map between W and W' as desired.

Let $u \in W$ be arbitrary. If $u = \infty$, then $F_\infty(u) = \infty \in W'$. If $u \in W \setminus \{\infty\}$, then $u \in p^m(N-z_0)$ for some $m \in \mathbb{N}_0$, and so

$$z_0 + p^{-l_n}u \in z_0 + (N-z_0) = N \subseteq S_p$$

for large n. Since $z_0 + p^{-l_n}u \to z_0$ as $n \to \infty$, it follows that

$$F(z_0+p^{-l_n}u) \subseteq N'$$

for large n, and so $\widetilde{F}_n(u) \in W'$. Since W' is closed, we have

$$F_\infty(u) = \lim_{n\to\infty} \widetilde{F}_n(u) \in W'.$$

Hence $F_\infty(W) \subseteq W'$.

Note that

$$\widetilde{F}_n^{-1}(w) = p^{l_n}\left(F^{-1}(w_0 + q^{-k_n}w) - z_0\right)$$

for $w \in \widehat{\mathbb{C}}$. So we can apply the same argument to the inverse maps and conclude that $F_\infty^{-1}(W') \subseteq W$. It follows that $F_\infty(W) = W'$ as desired. $\square$

The previous lemma is an instance of the more general fact that a quasisymmetric map between two standard carpets induces a normalized quasisymmetric map between weak tangents. It is likely that such a map only exists if the weak tangents are isometric. If this were the case, then this would put strong restrictions on the original quasisymmetric map. Unfortunately, we are only able to prove one result in this direction.

PROPOSITION 7.3. *Let $p \geq 3$ be odd. Then there is no normalized quasisymmetric map from $W_{\pi/2}(p)$ onto $W_{3\pi/2}(p)$.*

The proof will occupy the rest of this section. It strongly relies on the fact that $W_{3\pi/2}(p)$ consists of three isometric copies of $W_{\pi/2}(p)$ (which is used in combination with some monotonicity properties of carpet modulus with respect to a group). Because of this, the argument does not generalize to other pairs of weak tangents of S_p and we cannot prove that there is no normalized quasisymmetric map between $W_{\pi/2}(p)$ and $W_\pi(p)$ or between $W_{3\pi/2}(p)$ and $W_\pi(p)$. If this were true, the proofs of Theorems 1.1 and 1.3 would admit some simplifications.

We fix an odd number $p \geq 3$ in the following. Weak tangents refer to S_p, and we write $W_{\pi/2} = W_{\pi/2}(p)$, etc.

Let G and $\widetilde{G}$ denote the groups of normalized orientation-preserving quasisymmetric self-maps of $W_{\pi/2}$ and $W_{3\pi/2}$, respectively. Then G and $\widetilde{G}$ both contain the map $z \mapsto \mu(z) := pz$ induced by multiplication by p, and so it follows from Lemma 7.1 and Corollary 4.5 that G and $\widetilde{G}$ are infinite cyclic. Let ϕ be a generator of G. It is actually very likely that G is generated by μ and that we can take $\phi = \mu$, but there seems to be no easy proof for this statement. So the subsequent argument cannot rely on this, which causes some complications. In any case, we have $\mu = \phi^s$ for some $s \in \mathbb{Z} \setminus \{0\}$. By replacing ϕ by ϕ^{-1} if necessary, we may assume that $s > 0$.

By Lemma 7.1 and Proposition 4.1 there exists a quasiconformal map $\Phi\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ whose restriction to $W_{\pi/2}$ is equal to ϕ. Then $\Phi(0) = 0$ and $\Phi(\infty) = \infty$. Let

$$\Omega := \{z \in \mathbb{C}\colon \operatorname{Re}(z) > 0 \text{ and } \operatorname{Im}(z) > 0\}.$$

Then $C_0 := \partial\Omega$ is a peripheral circle of $W_{\pi/2}$ and we have $\Phi(\partial\Omega) = \phi(\partial\Omega) = \partial\Omega$. Since ϕ, and hence also Φ, is orientation-preserving, these maps fix the positive real axis and the positive imaginary axis setwise. It follows that $\Phi(\Omega) = \Omega$.

Let Γ be the family of all open paths in the region Ω that connect the positive real and positive imaginary axes. By what we have just seen, the path family Γ is Φ-invariant. The peripheral circles of $W_{\pi/2}$ that meet some path in Γ are precisely the peripheral circles $C \neq C_0 = \partial\Omega$. (This is why we chose Γ to consist of open paths.) It follows from Corollary 4.7 that $\phi^n(C) \neq C$ for all $n \in \mathbb{Z} \setminus \{0\}$ and all peripheral circles C of $W_{\pi/2}$ that meet some path in Γ. So we can apply Lemma 3.3 and conclude that

$$\operatorname{mod}_{W_{\pi/2}/\langle\mu\rangle}(\Gamma) = \operatorname{mod}_{W_{\pi/2}/\langle\phi^s\rangle}(\Gamma) = s \operatorname{mod}_{W_{\pi/2}/G}(\Gamma). \tag{7.4}$$

LEMMA 7.4. *We have* $0 < \operatorname{mod}_{W_{\pi/2}/G}(\Gamma) < \infty$.

Proof. By (7.4) it is enough to show that

$$0 < \operatorname{mod}_{W_{\pi/2}/\langle\mu\rangle}(\Gamma) < \infty.$$

To establish the inequality $\operatorname{mod}_{W_{\pi/2}/\langle\mu\rangle}(\Gamma) < \infty$, it suffices to exhibit an admissible mass distribution of finite mass.

If $C \neq C_0 = \partial\Omega$ is a peripheral circle of $W_{\pi/2}$, we denote by $\theta(C)$ the angle under which C is seen from the origin; i.e., $\theta(C)$ is the length of the circular arc obtained as the image of C under the radial projection map $z \in \mathbb{C} \setminus \{0\} \mapsto \operatorname{pr}(z) := z/|z|$ to the unit circle. We set $\rho(C_0) := 0$, and $\rho(C) := \frac{2}{\pi}\theta(C)$ for all peripheral circles $C \neq C_0$. We claim that ρ is admissible for $\operatorname{mod}_{W_{\pi/2}/\langle\mu\rangle}(\Gamma)$.

To see this, first note that ρ is constant on orbits of peripheral circles under the action of the group $\langle\mu\rangle$. Let Γ_0 denote the family of all paths γ in Γ that are not locally rectifiable or for which $\gamma \cap W_{\pi/2}$ has positive length. Since $W_{\pi/2}$ is a set of measure zero, we have $\operatorname{mod}(\Gamma_0) = 0$.

If $\gamma \in \Gamma$, then $\operatorname{pr}(\gamma) = \alpha := \{e^{it} : 0 < t < \pi/2\}$ and the projection map pr is a Lipschitz map on γ. So if $\gamma \in \Gamma \setminus \Gamma_0$, then up to a set of measure zero, $\operatorname{pr}(\gamma) = \alpha$ is covered by the projections $\operatorname{pr}(C)$ of the peripheral circles that meet γ. It follows that

$$\sum_{\gamma\cap C\neq\emptyset} \rho(C) = \frac{2}{\pi} \sum_{\gamma\cap C\neq\emptyset} \theta(C) \geq 1$$

for all $\gamma \in \Gamma \setminus \Gamma_0$. So ρ is indeed admissible.

To find a mass bound for ρ, note that every $\langle\mu\rangle$-orbit of a peripheral circle $C \neq C_0$ has a unique element contained in the set $F := \overline{\mu(Q_0) \setminus Q_0}$. (Recall that $Q_0 = [0,1] \times [0,1] \subseteq \mathbb{R}^2 \cong \mathbb{C}$ denotes the unit square.) Moreover, there exists a constant $K > 0$ such that

$$\theta(C) \leq K\ell(C)$$

for all peripheral circles C of $W_{\pi/2}$ with $C \subseteq F$. It follows that

$$\operatorname{mass}_{W_{\pi/2}/\langle\mu\rangle}(\rho) = \frac{4}{\pi^2} \sum_{C\subseteq F} \theta(C)^2 \lesssim \sum_{C\subseteq F} \ell(C)^2 = \operatorname{Area}(F) = p^2 - 1,$$

where Area(F) denotes the Euclidean area of F. Hence ρ is an admissible density for $\operatorname{mod}_{W_{\pi/2}/\langle\mu\rangle}(\Gamma)$ with finite mass as desired.

To show that $\operatorname{mod}_{W_{\pi/2}/\langle\mu\rangle}(\Gamma) > 0$, we argue by contradiction and assume that $\operatorname{mod}_{W_{\pi/2}/\langle\mu\rangle}(\Gamma) = 0$. For $k \in \mathbb{N}$, let $\mathcal{C}_k$ denote the set of all peripheral circles C of $W_{\pi/2}$ with $C \subseteq F_k := \overline{\mu^k(Q_0) \setminus \mu^{-k}(Q_0)}$. Then every orbit $\mathcal{O}$ of a peripheral circle $C \neq C_0$ under the action of $\langle\mu\rangle$ has exactly $2k$ elements in common with $\mathcal{C}_k$. Hence $\#(\mathcal{O} \cap \mathcal{C}_k) \leq N_k := 2k$. Moreover, since every path $\gamma \in \Gamma$ lies in F_k for sufficiently large k, we have $\Gamma = \bigcup_k \Gamma_k$, where Γ_k denotes the family of all paths in Γ that only meet peripheral circles in $\mathcal{C}_k$. This and the previous considerations imply that the hypotheses of Proposition 3.2 are satisfied. Hence there exists an extremal mass distribution for $\operatorname{mod}_{W_{\pi/2}/\langle\mu\rangle}(\Gamma)$.

By our assumption, $\operatorname{mod}_{W_{\pi/2}/\langle\mu\rangle}(\Gamma) = 0$. This is only possible if every path in Γ belongs to the exceptional family for $\operatorname{mod}_{W_{\pi/2}/\langle\mu\rangle}(\Gamma)$. We conclude that $\operatorname{mod}(\Gamma) = 0$; but obviously $\operatorname{mod}(\Gamma) = \infty$, and we obtain a contradiction. $\square$

Let H be the group of homeomorphisms of $\widehat{\mathbb{C}}$ generated by the reflections in the real and in the imaginary axes. Then H consists of precisely four elements.

We may assume that the quasiconformal map Φ whose restriction to $W_{\pi/2}$ is equal to the generator ϕ of G has the property that it is equivariant under H in the sense that $\Phi \circ \alpha = \alpha \circ \Phi$ for all $\alpha \in H$.

Indeed, if the original map Φ does not have this property, then we restrict it to the first quadrant and extend this restriction by successive reflections in real and imaginary axes to the whole sphere. The new map Φ obtained in this way is clearly an orientation-preserving homeomorphism with the desired equivariance property. It is also quasiconformal away from the real and positive imaginary axes. Since sets of finite 1-dimensional Hausdorff measures form removable singularities for quasiconformal maps [LV73, Th. 3.2, p. 202], Φ will actually be a quasiconformal map on $\widehat{\mathbb{C}}$. As before, $\Phi | W_{\pi/2} = \phi$.

Let

$$\widetilde{\Omega} := \{z \in \mathbb{C} : \operatorname{Re}(z) < 0 \text{ or } \operatorname{Im}(z) < 0\}.$$

Then $\widetilde{\Omega}$ is a three-quarter plane whose closure contains $W_{3\pi/2}$, and $C_0 = \partial\Omega = \partial\widetilde{\Omega}$ is a peripheral circle of $W_{3\pi/2}$. The set $W_{3\pi/2}$ consists of three copies of $W_{\pi/2}$ that can be obtained by successive reflections in the real and positive imaginary axes. By its equivariance property, the map Φ restricts to a normalized orientation-preserving quasisymmetric self-map $\psi := \Phi | W_{3\pi/2}$ of $W_{3\pi/2}$.

Recall that $\widetilde{G}$ denotes the infinite cyclic group consisting of all normalized orientation-preserving quasisymmetric self-maps of $W_{3\pi/2}$. Then we have $\psi \in \widetilde{G}$, and $\langle\psi\rangle$ is an infinite cyclic subgroup of $\widetilde{G}$.

Let $\widetilde{\Gamma}$ be the family of all open paths in $\widetilde{\Omega}$ that join the positive real and the positive imaginary axes.

LEMMA 7.5. *We have* $\operatorname{mod}_{W_{3\pi/2}/\langle\psi\rangle}(\widetilde{\Gamma}) \le \frac{1}{3}\operatorname{mod}_{W_{\pi/2}/G}(\Gamma)$.

Proof. Essentially, this follows from an application of a suitable "serial law" to modulus with respect to a group.

More precisely, suppose that ρ is an arbitrary admissible invariant mass distribution for $\operatorname{mod}_{W_{\pi/2}/G}(\Gamma)$ with exceptional family Γ_0. We want to use ρ to define a suitable admissible mass distribution $\tilde{\rho}$ for $\operatorname{mod}_{W_{3\pi/2}/\langle\psi\rangle}(\widetilde{\Gamma})$. For the special peripheral circle C_0 of $W_{3\pi/2}$, we set $\tilde{\rho}(C_0)=0$. If $\widetilde{C}$ is any peripheral circle of $W_{3\pi/2}$ with $\widetilde{C}\neq C_0$, then there exists a unique element $\alpha\in H$ such that $\alpha(\widetilde{C})$ is a peripheral circle of $W_{\pi/2}$. We set $\tilde{\rho}(\widetilde{C}) := \frac{1}{3}\rho(\alpha(\widetilde{C}))$.

By the equivariance property of Φ and the fact that ρ is constant on orbits of $G=\langle\phi\rangle$, it follows that $\tilde{\rho}$ is constant on orbits of $\langle\psi\rangle$.

Let $\widetilde{\Gamma}_0$ be the family of all paths in $\widetilde{\Gamma}$ that have a subpath that can be mapped to a path in Γ_0 by an element $\alpha\in H$. Since $\operatorname{mod}(\Gamma_0)=0$, we have $\operatorname{mod}(\widetilde{\Gamma}_0)=0$.

Let $\gamma\in\widetilde{\Gamma}$ be arbitrary. Then γ has three disjoint open subpaths (one for each quarter-plane of $\widetilde{\Omega}$) that are mapped to a path in Γ by a suitable element in H. Let γ_i, $i=1,2,3$, denote these image paths in Γ. In addition, if $\gamma\notin\widetilde{\Gamma}_0$, then $\gamma_i\notin\Gamma_0$ for $i=1,2,3$; so

$$\sum_{\gamma\cap\widetilde{C}\neq\emptyset}\widetilde{\rho}(\widetilde{C}) \ge \frac{1}{3}\sum_{i=1}^{3}\sum_{\gamma_i\cap C\neq\emptyset}\rho(C)\ge 1$$

for all $\gamma\in\widetilde{\Gamma}\setminus\widetilde{\Gamma}_0$. Hence $\widetilde{\rho}$ is admissible for $\operatorname{mod}_{W_{3\pi/2}/\langle\psi\rangle}(\widetilde{\Gamma})$ and it follows that

$$\operatorname{mod}_{W_{3\pi/2}/\langle\psi\rangle}(\widetilde{\Gamma}) \le \operatorname{mass}_{W_{3\pi/2}/\langle\psi\rangle}(\tilde{\rho}) \le \tfrac{1}{3}\operatorname{mass}_{W_{\pi/2}/G}(\rho).$$

Since ρ was an arbitrary admissible mass distribution for $\operatorname{mod}_{W_{\pi/2}/G}(\Gamma)$, the statement follows. □

Proof of Proposition 7.3. We use the notation introduced above and denote by G and $\widetilde{G}$ infinite cyclic groups of all normalized orientation-preserving quasisymmetric self-maps of $W_{\pi/2}$ and $W_{3\pi/2}$, respectively. As before, let Γ and $\widetilde{\Gamma}$ be the family of all paths in Ω and $\widetilde{\Omega}$, respectively, that join the positive real and the positive imaginary axes.

To reach a contradiction, we assume that there is a normalized quasisymmetric map f from $W_{\pi/2}$ onto $W_{3\pi/2}$. Precomposing f by the reflection in the line $L=\{z\in\mathbb{C} : \operatorname{Re}(z)=\operatorname{Im}(z)\}$ if necessary, we may assume that f is orientation-preserving. Then $\widetilde{G}=f\circ G\circ f^{-1}$, and $\tilde{\phi} := f\circ\phi\circ f^{-1}$ is a generator for $\widetilde{G}$. By Lemma 7.1 and Proposition 4.1, the map f extends to a quasiconformal map F on $\widehat{\mathbb{C}}$. Then $\widetilde{\Gamma} = F(\Gamma)$, and so Lemma 3.1 gives

$$\operatorname{mod}_{W_{3\pi/2}/\widetilde{G}}(\widetilde{\Gamma}) = \operatorname{mod}_{W_{\pi/2}/G}(\Gamma).$$

Let $\psi = \Phi|W_{3\pi/2} \in \widetilde{G}$ be the map considered above. Then $\psi = \tilde{\phi}^m$ for some $m \in \mathbb{Z} \setminus \{0\}$, and it follows from Corollary 4.7 and Lemma 3.3 (see the argument that we used to establish (7.4)) that

$$\operatorname{mod}_{W_{3\pi/2}/\langle\psi\rangle}(\widetilde{\Gamma}) = |m| \operatorname{mod}_{W_{3\pi/2}/\widetilde{G}}(\widetilde{\Gamma}).$$

Hence by Lemma 7.5, we have

$$\begin{aligned}\operatorname{mod}_{W_{\pi/2}/G}(\Gamma) &= \operatorname{mod}_{W_{3\pi/2}/\widetilde{G}}(\widetilde{\Gamma})\\ &= \tfrac{1}{|m|}\operatorname{mod}_{W_{3\pi/2}/\langle\psi\rangle}(\widetilde{\Gamma})\\ &\leq \tfrac{1}{3|m|}\operatorname{mod}_{W_{\pi/2}/G}(\Gamma).\end{aligned}$$

This is only possible if $\operatorname{mod}_{W_{\pi/2}/G}(\Gamma) = 0$ or $\operatorname{mod}_{W_{\pi/2}/G}(\Gamma) = \infty$; this contradicts Lemma 7.4, and the statement follows. □

8. **Proof of Theorems 1.1–1.3**

We fix an odd integer $p \geq 3$. As before, we assume that the standard Sierpiński carpet S_p is obtained by subdividing the unit square $Q_0 = [0,1] \times [0,1]$ in the first quadrant of $\mathbb{C} \cong \mathbb{R}^2$. In this section it is convenient to mostly use real notation; so (x_0, y_0) is the point in $\mathbb{R}^2$ with x-coordinate x_0 and y-coordinate y_0. As before, we use 0 to denote the origin in $\mathbb{R}^2$.

We equip S_p with the restriction of the Euclidean metric. The carpet S_p has four lines of symmetries; one of them is the diagonal $D := \{(x,y) \in \mathbb{R}^2 : x = y\}$ and another the vertical line $V := \{(x,y) \in \mathbb{R}^2 : x = 1/2\}$. We denote the reflections in D and V by R_D and R_V, respectively. The maps R_D and R_V generate the group of Euclidean isometries of S_p, which consists of eight elements.

If f is a quasisymmetric self-map of S_p, then by Corollary 5.2 the outer square O and the middle square M of S_p are preserved as a pair; so $\{f(O), f(M)\} = \{O, M\}$. We will now show that $f(O) = M$ is actually impossible.

Lemma 8.1. *Let f be a quasisymmetric self-map of S_p, $p \geq 3$ odd. Then $f(O) = O$ and $f(M) = M$.*

Proof. Let $f \in \mathrm{QS}(S_p)$ be arbitrary. We know that $f(O) \in \{O, M\}$. It is enough to show $f(O) = O$, because then necessarily $f(M) = M$. We argue by contradiction and assume that $f(O) = M$. Then $f(M) = O$, and so f interchanges O and M.

By Corollary 5.3, the group $\mathrm{QS}(S_p)$ of all quasisymmetric self-maps of S_p is finite. Let G be the subgroup of $\mathrm{QS}(S_p)$ consisting of all quasisymmetric self-maps g of S_p with $g(O) = O$ and $g(M) = M$. Then G is also finite and contains the isometry group of S_p. Moreover, if G_0 is the set of all maps in G that are orientation-preserving, then G_0 is a subgroup in G of index 2; indeed, G can be written as the disjoint union

$$G = G_0 \cup G_0 R_D \tag{8.1}$$

of two right cosets of G_0.

If $z \in S_p$ is arbitrary, we denote by

$$\mathcal{O}(z) = \{g(z) : g \in G\}$$

the orbit of z under the action of G. Let

$$c = ((p-1)/(2p), (p-1)/(2p))$$

be the left lower corner of the square M, and let $w_0 = f(0) \in M$. In the following we will consider the orbits $\mathcal{O}(c)$ and $\mathcal{O}(w_0)$. Both are subsets of M. Since G contains the isometry group of S_p, the orbits $\mathcal{O}(w_0)$ and $\mathcal{O}(c)$ have the same symmetries as S_p.

It follows from Corollary 4.7 that if $g \in G_0$ has a fixed point in S_p, then g is the identity on S_p. So if $z \in S_p$ is arbitrary, then the map $g \in G_0 \mapsto g(z)$ is injective. Since $R_D(0) = 0$, it follows from (8.1) that $\#\mathcal{O}(0) = \#G_0$.

We also have $R_D(c) = c$, and so $\#\mathcal{O}(c) = \#G_0$. Moreover, the map $g \in G \mapsto f \circ g \circ f^{-1} \in G$ is an automorphism of G. This implies that

$$\mathcal{O}(w_0) = \{(f \circ g \circ f^{-1})(w_0) : g \in G\} = \{(f \circ g)(0) : g \in G\} = f(\mathcal{O}(0)).$$

Hence

$$\#\mathcal{O}(w_0) = \#\mathcal{O}(0) = \#G_0 = \#\mathcal{O}(c),$$

and so the orbits $\mathcal{O}(w_0)$ and $\mathcal{O}(c)$ have the same number of elements.

We will now show that this is impossible. First note that $c \notin \mathcal{O}(w_0)$, and so $\mathcal{O}(c) \cap \mathcal{O}(w_0) = \emptyset$. Indeed, suppose on the contrary that $c \in \mathcal{O}(w_0)$. Then there exists $g \in G$ with $c = g(w_0) = (g \circ f)(0)$. Then $h := g \circ f$ is a quasisymmetric self-map of S_p with $h(0) = c$. By Lemma 7.2, the map h induces a normalized quasisymmetric map from $W_{\pi/2}$, the weak tangent of S_p at 0, onto $W_{3\pi/2}$, the weak tangent of S_p at c. This is impossible by Proposition 7.3.

By symmetry, $\mathcal{O}(c)$ contains all corners of M, while $\mathcal{O}(w_0)$ contains none of the corners of M by what we have just seen.

Let

$$m' = (1/2, (p-1)/(2p))$$

be the midpoint of the bottom side of M. We want to show that m' belongs to neither $\mathcal{O}(w_0)$ nor $\mathcal{O}(c)$. Indeed, suppose that $m' \in \mathcal{O}(w_0)$. Similarly as above, we can then find a quasisymmetric self-map h of S_p with $h(0) = m'$.

By precomposing h by R_D if necessary, we may assume that h is orientation-preserving. Since the weak tangent of S_p at m' is isometric to W_π, we get an induced normalized quasisymmetric map $h_1\colon W_{\pi/2} \to W_\pi$.

We necessarily have $h(O) = M$ and $h(M) = O$. Consider the map $R_V \circ h \circ R_D$. Then h and $R_V \circ h \circ R_D$ are orientation-preserving quasisymmetric self-maps of S_p that act in the same way on the origin and on the peripheral circles O and M. By Corollary 4.7 it follows that $h = R_V \circ h \circ R_D$. This shows that h maps the set $S_p \cap D$ onto $S_p \cap V$. Since we know that $h(c) \in h(M) = O$, this only leaves two possibilities for the point $h(c)$, namely $(1/2, 0)$ and $(1/2, 1)$. Since at both points the weak tangents of S_p are isometric to W_π, we get an induced normalized quasisymmetric map $h_2\colon W_{3\pi/2} \to W_\pi$. Then $h_2^{-1} \circ h_1$ is a normalized quasisymmetric map from $W_{\pi/2}$ onto $W_{3\pi/2}$. We get a contradiction to Proposition 7.3, showing that $m' \notin \mathcal{O}(w_0)$.

The proof that $m' \notin \mathcal{O}(c)$ runs along similar lines. Again we argue by contradiction and assume $m' \in \mathcal{O}(c)$. Then we can find an orientation-preserving quasisymmetric self-map h of S_p with $h(c) = m'$. This gives a normalized quasisymmetric map $h_1\colon W_{3\pi/2} \to W_\pi$. We must have $h(M) = M$ and $h(O) = O$. Then h and $R_H \circ h \circ R_D$ are orientation-preserving quasisymmetric self-maps of S_p that act in the same way on c and on the peripheral circles O and M. Therefore, $h = R_V \circ h \circ R_D$, and so h maps the set $S_p \cap D$ onto $S_p \cap V$. This only leaves the possibilities $(1/2, 0)$ or $(1/2, 1)$ for the point $h(0)$. In any case, we get an induced normalized quasisymmetric map $h_2\colon W_{\pi/2} \to W_\pi$, and by considering $h_1^{-1} \circ h_2$, again a contradiction to Proposition 7.3. Hence $m' \notin \mathcal{O}(c)$.

To summarize, we know that the sets $\mathcal{O}(c)$ and $\mathcal{O}(w_0)$ are disjoint subsets of M with the same symmetries as S_p, and none of these sets contains m'. This implies that each side of M contains an even number of points in $\mathcal{O}(c)$ and $\mathcal{O}(w_0)$, since we have reflection symmetry about the midpoint of each side.

Since no corner of M is in $\mathcal{O}(w_0)$ and each side of M contains the same even number of points in $\mathcal{O}(w_0)$, it follows that

$$\#\mathcal{O}(w_0) = 8k$$

for some $k \in \mathbb{N}_0$. So $\#\mathcal{O}(w_0)$ is divisible by 8. On the other hand, $\mathcal{O}(c)$ contains the corners of M. Since each corner belongs to two sides, we have

$$\#\mathcal{O}(c) = 8l - 4$$

for some $l \in \mathbb{N}$, and so $\#\mathcal{O}(c)$ is not divisible by 8. Since we know that $\#\mathcal{O}(w_0) = \#\mathcal{O}(c)$, this is a contradiction. So $f(O) = M$ is impossible, and we must have $f(O) = O$. $\square$

One can make the logic of the previous proof a little more transparent if one follows a slightly different (albeit longer) route. Namely, if a map

$f \in \mathrm{QS}(S_p)$ with $f(O) = M$ exists, then, by using a counting argument as above, one can find such a map f that sends the origin to one of the natural candidates adapted to the symmetries of S_p, namely to a corner of M or to the midpoint of one of the sides of M. Arguing as in the previous proof based on Proposition 7.3, one can rule out these possibilities, and one again reaches a contradiction.

As before, let O be the outer and M the middle square of S_p. We denote the orbit of a point $z \in S_p$ by the group $\mathrm{QS}(S_p)$ of quasisymmetric self-maps of S_p by $\mathcal{O}(z)$. Now that we know that every map $f \in \mathrm{QS}(S_p)$ preserves O and M setwise, the group G introduced in the previous proof is actually equal to $\mathrm{QS}(S_p)$. Define $m = (1/2, 0)$. Then m is the midpoint of the bottom side of O.

Lemma 8.2. *Let $z \in O$ be arbitrary. If $\mathcal{O}(z) \neq \mathcal{O}(0), \mathcal{O}(m)$, then $\#\mathcal{O}(z)$ is divisible by* 8. *Moreover*, $\#\mathcal{O}(0)$ *and* $\#\mathcal{O}(m)$ *are divisible by* 4, *but not by* 8, *and* $\mathcal{O}(0) \cap \mathcal{O}(m) = \emptyset$.

Proof. If $z \in O$, then the orbit $\mathcal{O}(z) \subseteq O$ has the same symmetries as S_p; so each side of O contains the same number of points in $\mathcal{O}(z)$, and $\#\mathcal{O}(z)$ must be divisible by 4. If $\mathcal{O}(z) \neq \mathcal{O}(0), \mathcal{O}(m)$, then $\mathcal{O}(z)$ does not contain any corners of O, nor any midpoint of a side of O. Hence each side of O contains an even number $2k$, $k \in \mathbb{N}$, of points in $\mathcal{O}(z)$ and $\#\mathcal{O}(z) = 8k$. In this case, $\#\mathcal{O}(z)$ is divisible by 8.

We want to show that $\mathcal{O}(0) \cap \mathcal{O}(m) = \emptyset$. We argue by contradiction and assume that $\mathcal{O}(0) \cap \mathcal{O}(m) \neq \emptyset$. Then we can find a map $f \in \mathrm{QS}(S_p)$ with $f(0) = m$, and pre-composing f with R_D if necessary, we may assume that f is orientation-preserving. Similarly as in the proof of Lemma 8.1, this leads to a contradiction; namely, we first get an induced normalized quasisymmetry $f_1\colon W_{\pi/2} \to W_\pi$. Moreover, from Corollary 4.7 we conclude that $f = R_V \circ f \circ R_D$, and so f maps $S_p \cap D$ onto $S_p \cap V$; hence the lower left corner c of M must be mapped to the intersection $M \cap V$, which consists of two points where the weak tangents are isometric to W_π. This gives an induced normalized quasisymmetry $f_2\colon W_{3\pi/2} \to W_\pi$. Considering $f_2^{-1} \circ f_1$, we get a contradiction from Proposition 7.3.

It follows that $\mathcal{O}(0)$ contains the corners of O but not any midpoint of a side. So the number of points in $\mathcal{O}(0)$ on each side is an even number $2r$, $r \in \mathbb{N}$, and we conclude $\#\mathcal{O}(0) = 8r - 4$.

Finally, the number of points in $\mathcal{O}(m)$ on each side is an odd number $2l - 1$, $l \in \mathbb{N}$, since $\mathcal{O}(m)$ contains the midpoint of the side. Since none of the corners of O belongs to $\mathcal{O}(m)$, we have $\#\mathcal{O}(0) = 8l - 4$. Hence neither $\#\mathcal{O}(0)$ nor $\#\mathcal{O}(m)$ is divisible by 8. □

Exactly the same statement with essentially the same proof is true for orbits of points in M, if in Lemma 8.2 we replace 0 by a corner of M and m by a midpoint of a side of M.

Proof of Theorem 1.1. Let f be a quasisymmetric self-map of S_3. We want to show that f is a Euclidean isometry of S_3. To see this, we may assume f is orientation-preserving, for otherwise we can compose this map with the reflection R_V that lies in the isometry group of S_3. By Lemma 8.1 we know that if O is the outer and M the middle square of S_3, then $f(O) = O$ and $f(M) = M$.

There are eight peripheral circles of S_3 that are squares of sidelength 1/9. We call them *second generation squares* as they are the boundaries of the solid squares that were removed in the second step of the construction. (In the first step, the square bounded by M was removed from Q_0.) Four second generation squares, the *corner squares*, have distance 1/9 to precisely two sides of the unit square Q_0; the four other ones, the *side squares*, have distance 1/9 to exactly one side of Q_0.

Before continuing, we give a general outline of the ensuing argument. We will show that f must map some second generation square to another one. This will lead to various combinatorial possibilities. We will analyze them in detail. In some cases we can invoke the Three-Circle Theorem, Corollary 4.4, to identify f with an isometry as desired. In the other cases, a map with the given mapping behavior on a second generation square will not exist. The strategy for ruling out the existence of such "ghost maps" is this: Using symmetries and again the Three-Circle Theorem, we will be able to restrict the possibilities for the image of the origin under f. The map f will induce a quasisymmetry of the weak tangent $W_{\pi/2}$ of S_3 at 0 to a weak tangent of S_3 at $p = f(0)$. As we will see, this always leads to a normalized quasisymmetry from $W_{\pi/2}$ onto $W_{3\pi/2}$. Invoking Proposition 7.3, we will then get a contradiction ruling out the existence of the map.

We now proceed to presenting the details.

Claim. The map f sends some second generation square to another second generation square.

Among the eight second generation squares, let C_0 be one for which

$$\operatorname{mod}_{S_3}(\Gamma(C_0, O; S_3))$$

is largest, and define $C_1 = f(C_0)$. Then C_1 is a peripheral circle of S_3 and hence a square. As in the proof of Corollary 5.2, Lemma 2.1 implies that

$$\operatorname{mod}_{S_3}(\Gamma(C_0, O; S_3)) = \operatorname{mod}_{S_3}(\Gamma(C_1, O; S_3)).$$

For establishing the claim, it suffices to show that C_1 has sidelength $1/9$ and is hence a second generation square. Since $f(O) = O$ and $f(M) = M$, we have $C_1 = f(C_0) \notin \{O, M\}$, and so the sidelength of C_1 is at most $1/9$. The monotonicity of carpet modulus and the self-similarity of S_3 imply that this side length cannot be strictly smaller than $1/9$. Indeed, suppose that this is the case. Then there exists a unique carpet $S \subseteq S_3$ that can be mapped to S_3 by a Euclidean similarity so that C_1 corresponds to a second generation square of S_3. Let o denote the outer peripheral circle of S corresponding to O. By our assumption that C_1 is not a second generation square, we have $S \neq S_3$, and so S is a proper subset of S_3.

By definition of C_0 and scale invariance of carpet modulus, we have

$$\operatorname{mod}_S(\Gamma(C_1, o; S)) \leq \operatorname{mod}_{S_3}(\Gamma(C_0, O; S_3)).$$

On the other hand, an argument as in the proof of Lemma 5.1 gives

$$\operatorname{mod}_{S_3}(\Gamma(C_1, O; S_3)) < \operatorname{mod}_S(\Gamma(C_1, o; S)).$$

The previous three modulus relations combined lead to a contradiction, and the claim follows.

Having established that the image $C_1 = f(C_0)$ of some second generation square C_0 is also a second generation square, we now distinguish several cases depending on the type of the squares C_0 and C_1, i.e., whether they are corner or sides squares. These cases will exhaust all possibilities.

Case 1: C_0 and C_1 are corner squares.

Then there exists an isometry T of S_3 given by a rotation that maps C_0 to C_1. Then f and T act in the same way on three peripheral circles O, M, C_0 of S_3. Since f and T are orientation-preserving, it follows from Corollary 4.4 that $f = T$. Hence f is an isometry of S_3.

Case 2. C_0 is a corner square, and C_1 is a side square.

By pre- and post-composing f by suitable rotations, we may assume that C_0 is the corner square that has distance $1/9$ to both the x- and y-axes and that C_1 is the side square that has distance $1/9$ to the x-axis.

Then f and $R_V \circ f \circ R_D$ are orientation-preserving quasisymmetric self-maps of S_3 that act in the same way on O, M, C_0. Again Corollary 4.4 allows us to conclude that $f = R_V \circ f \circ R_D$, and so $f(D \cap S_3) = V \cap S_3$. This only leaves two possibilities for the image of the origin under f, namely the points $(\frac{1}{2}, 0)$ or $(\frac{1}{2}, 1)$; since the weak tangents of S_3 at these points are isometric to W_π, we get an induced normalized quasisymmetric map $f_1 \colon W_{\pi/2} \to W_\pi$.

Moreover, the lower left corner $c = (\frac{1}{3}, \frac{1}{3})$ of M is mapped by f to either $(\frac{1}{2}, \frac{1}{3})$ or $(\frac{1}{2}, \frac{2}{3})$. The weak tangent of S_3 at $(\frac{1}{3}, \frac{1}{3})$ is equal to $W_{3\pi/2}$, and the weak tangents at both points $(\frac{1}{2}, \frac{1}{3})$ or $(\frac{1}{2}, \frac{2}{3})$ are isometric to W_π. As above, the map f induces a normalized quasisymmetric map $f_2 \colon W_{3\pi/2} \to W_\pi$. Then the map $f_2^{-1} \circ f_1$ is a normalized quasisymmetric map from $W_{\pi/2}$ onto $W_{3\pi/2}$. This contradicts Proposition 7.3, and a map f as in this case does not exist.

Case 3. C_0 is a side square, and C_1 is a corner square.

Then we consider f^{-1} and reduce to Case 2. This shows that a map f as in this case does not exist.

Case 4. C_0 and C_1 are side squares.

Then there exists a rotation T of S_3 that maps O, M, C_0 in the same way as f, and as in Case 1, we conclude that $f = T$. Hence f is an isometry of S_3.

Cases 1–4 exhaust all possibilities, and we have shown that in each case the map f is an isometry or does not exist. Theorem 1.1 follows. □

Remark 8.3. For larger p, the number of second generation squares of S_p increases and the above case analysis seems to meet insurmountable obstacles. In a previous version of this paper it was incorrectly stated that one can still carry the argument through for S_5, but our intended case analysis contained a gap. This was pointed out to us by Guy David.

A more natural approach is to first prove rigidity statements for weak tangents of S_p. In view of Theorem 1.4 or Theorem 1.5, one may speculate whether a normalized quasisymmetry between two weak tangents of a carpet S_p only exists if the weak tangents are similar; i.e., one is the image of the other by a Euclidean similarity. If this is the case, then by considering $W_{\pi/2}$, one can conclude that under any quasisymmetry f of S_p, the origin must be mapped to a corner of the unit square, and it would easily follow from Corollary 4.7 that f is an isometry of S_p.

Unfortunately, we cannot even rule out the existence of a normalized quasisymmetric map between the weak tangents $W_{\pi/2}$ and W_π of S_p. This caused some complications in the previous proof that we were able to overcome by ad hoc arguments.

Proof of Theorem 1.2. Let $G = \mathrm{QS}(S_p)$ be the group of all quasisymmetric self-maps of S_p, and let G_0 be the subgroup of all orientation-preserving maps in G. Then G_0 is a subgroup in G of index 2 and G_0 is finite cyclic as follows from Lemma 8.1 and Corollary 4.6.

Consider the orbit $\mathcal{O}(0)$ of the origin under G. Since $R_D(0) = 0$, this set is equal to the orbit of 0 under G_0. Since each element in G preserves the outer square O, we know that $\mathcal{O}(0)$ consists of points on O. Moreover, $\mathcal{O}(0)$ is symmetric with respect to all symmetries of S_p. We equip O with positive orientation, so that S_p lies on the left if we run through O with this orientation. Let $z_0 = 0, z_1, \dots, z_{n-1}, z_n = z_0$, where $n \in \mathbb{N}$, be the points in $\mathcal{O}(0)$ in cyclic order on the oriented curve O, and let α_i for $i = 0, \dots, n-1$ be the corresponding subarcs of O with end points z_i and z_{i+1}.

There exists an element $r \in G_0$ with $r(z_0) = z_1$. Then $r(\alpha_0)$ is a subarc of O that has the initial point z_1, is positively oriented on O, since r is orientation-preserving, and has its end point in $\mathcal{O}(0)$. Moreover, $r(\alpha_0)$ does not contain any point from $\mathcal{O}(0)$ in its interior, because this is true for α_0. Hence the end point of $r(\alpha_0)$ must be z_2, and so $r(\alpha_0) = \alpha_1$. Repeating this argument successively for the arcs $\alpha_1, \dots, \alpha_{n-1}$, we conclude that $r(\alpha_i) = \alpha_{i+1}$ for all $i = 0, \dots, n-1$, where $\alpha_n = \alpha_0$. In particular, $r(z_i) = z_{i+1}$, and so $z_i = r^i(0)$ for $i = 0, \dots, n$.

This implies that r generates G_0; indeed, if $g \in G_0$ is arbitrary, then by what we have just seen, there exists $i \in \{0, \dots, n-1\}$ such that $g(0) = z_i = r^i(0)$. Then $g^{-1} \circ r^i$ is an orientation-preserving element in G that fixes the origin and the peripheral circles O and M. Hence $g^{-1} \circ r^i = e$, where $e = \mathrm{id}_{S_p}$, and so $g = r^i$. Since $r^n(0) = r^n(z_0) = z_n = z_0 = 0$, the same argument shows that $r^n = e$. Moreover, since the points $z_i = r^i(0)$ for $i = 0, \dots, n-1$ are all distinct, n is the order of g.

Let $s = R_D$ be the reflection in D. Then $s \in G$ is orientation-reserving and, since G_0 has index 2 in G, it follows that s and r generate G. Since the orbit $\mathcal{O}(0)$ is invariant under s, the arc $s(\alpha_0)$ has its end points in $\mathcal{O}(0)$. There are no points from the orbit in its interior, one of the end points is $z_0 = 0$, and $s(\alpha_0)$ is traversed in negative orientation if we traverse α_0 positively. Hence $s(\alpha_0) = \alpha_{n-1}$, and so $s(z_1) = z_{n-1}$. It follows that $(s \circ r)^2 \in G$ is orientation-preserving and

$$\begin{aligned}(s \circ r)^2(0) &= (s \circ r \circ s)(r(0)) \ = \ (s \circ r \circ s)(z_1) \\ &= (s \circ r)(z_{n-1}) \ = \ s(z_0) = s(0) = 0.\end{aligned}$$

Similarly as before, we conclude that $(s \circ r)^2 = e$. So we have the relations $s^2 = r^n = (s \circ r)^2 = e$ for the generators s and r of G. Moreover, the element r has order n. The elements s and $s \circ r$ are orientation-reversing, and so they have order 2. This implies that G is a finite dihedral group. ☐

Remark 8.4. Let G_0 be the group of orientation-preserving maps in $G = \mathrm{QS}(S_p)$. As we have seen in the preceding proof, G_0 is a cyclic subgroup of G with index 2. Moreover, the order n of G_0 is equal to the cardinality of the orbit of $\mathcal{O}(0)$ of 0 under G. So by Lemma 8.1, the order n of G_0 is divisible by 4, but not by 8. Of course, if our conjecture is true that every element in G is an isometry, then G_0 consists of four rotations and $n = 4$.

Proof of Theorem 1.3. Let $p, q \geq 3$ be odd integers, and suppose that there exists a quasisymmetric map $f \colon S_p \to S_q$. We want to show that $p = q$.

In the following we use the subscript p in our notation if we refer to objects related to S_p and q if we refer to S_q. So M_p denotes the middle square of S_p, etc.

Let G_p and G_q be the groups of quasisymmetric self-maps of S_p and S_q, respectively. Note that by Corollary 5.3 the groups G_p and G_q are finite, and, f being quasisymmetric, conjugates G_p and G_q. This implies that if $\mathcal{O}_p$ denotes the orbit of 0 under G_p, then $\mathcal{O}_q := f(\mathcal{O}_p)$ is the orbit of $f(0)$ under G_q.

It follows from Lemmas 2.1 and 5.1 that f maps the pair $\{O_p, M_p\}$ consisting of the outer square and middle square of S_q to the corresponding pair $\{O_q, M_q\}$. Hence $f(O_p) = O_q$ or $f(O_p) = M_q$ and, in particular, $f(0) \in O_q$ or $f(0) \in M_q$.

By Lemma 8.2, the number $\#\mathcal{O}_p$ is not divisible by 8. Since $\mathcal{O}_q$ has the same cardinality as $\mathcal{O}_p$, the number $\#\mathcal{O}_q$ is not divisible by 8 either. Applying Lemma 8.2 and the remark after this lemma, we conclude that the orbit $\mathcal{O}_q$ of $f(0) \in M_q \cup O_q$ under G_q must be equal to the orbit of a corner of O_q or M_q, or the orbit of a midpoint of a side of O_q or M_q.

Let

$$c_q = ((q-1)/(2q), (q-1)/(2q))$$

be the lower left corner of M_q, and let $m = (1/2, 0)$ and

$$m'_q = (1/2, (q-1)/(2q))$$

be the midpoint of the bottom side of O_q and M_q, respectively. By what we have seen, $f(0)$ must belong to an orbit of one of the four points $0, c_q, m, m'_q$ under G_q. By composing f with a suitable element in G_q, we may actually assume that $f(0) \in \{0, c_q, m, m'_q\}$. By pre-composing f with R_D if necessary, we may in addition assume that f is orientation-preserving.

We are led to four cases that we now analyze.

Case 1. $f(0) = 0$.

Then $f(O_p) = O_q$ and $f(M_p) = M_q$. The map $f^{-1} \circ R_D \circ f \circ R_D$ is an orientation-preserving quasisymmetry in G_p, fixes the point 0, and the peripheral circles O_p and M_p setwise. Hence this map is equal to the identity on S_p, which implies $f \circ R_D = R_D \circ f$. From this, we conclude in turn that f fixes the point $(1,1)$.

Let D' be the line $\{(x,y) \in \mathbb{R}^2 : x + y = 1\}$, and denote the reflection in D' by $R_{D'}$. Then the map $f^{-1} \circ R_{D'} \circ f \circ R_{D'}$ is an orientation-preserving quasisymmetry in G_p, fixes the point 0, and the peripheral circles O_p and M_p setwise. Hence this map is the identity on S_p, and so $f \circ R_{D'} = R_{D'} \circ f$. It follows that f fixes the points $(0,1)$ and $(1,0)$ or interchanges them. Since f is orientation-preserving, and fixes 0 and $(1,1)$, this map must fix $(0,1)$ and $(1,0)$. So f fixes all corners of the unit square.

By Theorem 1.4 the map f must be the identity and the carpets S_p and S_q are the same. Hence $p = q$.

Case 2. $f(0) = m$.

Then we get an induced normalized quasisymmetry $f_1\colon W_{\pi/2}(p) \to W_\pi(q)$. Moreover, by an argument as in Case 1, we have $f \circ R_D = R_V \circ f$. This implies that $f(S_p \cap D) = S_q \cap V$. Since we also have $f(M_p) = M_q$, the lower left corner c_p of M_p must be mapped to the midpoint of the bottom or the top side of M_q. At these points, S_q has a unique weak tangent isometric to $W_\pi(q)$. Hence we get an induced normalized quasisymmetry $f_2\colon W_{3\pi/2}(p) \to W_\pi(q)$. Considering $f_2^{-1} \circ f_1$ we get a contradiction to Proposition 7.3. So this case is impossible.

Case 3. $f(0) = m'_q$.

This is very similar to Case 2. We get an induced normalized quasisymmetry $f_1\colon W_{\pi/2}(p) \to W_\pi(q)$, and we have $f \circ R_D = R_V \circ f$. Since $f(M_p) = O_q$, this limits the possible image points of c_p under f to the midpoints of the top or bottom side of O_q. Again we get an induced normalized quasisymmetry $f_2\colon W_{3\pi/2}(p) \to W_\pi(q)$ and a contradiction by Proposition 7.3.

Case 4. $f(0) = c_q$.

We get an induced normalized quasisymmetry $f_1\colon W_{\pi/2}(p) \to W_{3\pi/2}(q)$. We also have $f \circ R_D = R_D \circ f$, and so $f(S_p \cap D) = S_q \cap D$. Pick a peripheral circle $C \neq O_p, M_p$ of S_p that is symmetric with respect to D, and let $v \in D \cap C$. Then v is a corner of the square C, and so S_p has a weak tangent at v that is isometric to $W_{3\pi/2}(p)$. Moreover, $C' = f(C)$ is a peripheral circle of S_q distinct from $O_q = f(M_p)$. It contains the point $v' = f(v)$ that lies on D. Hence v' is a corner of C', and so S_q has a weak tangent at v' isometric to $W_{3\pi/2}(q)$. We get an induced normalized quasisymmetry $f_2\colon W_{3\pi/2}(p) \to W_{3\pi/2}(q)$. Considering $f_2^{-1} \circ f_1$, we again get a contradiction to Proposition 7.3.

In sum, only Case 1 is actually possible, and we have $p = q$ as desired. □

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