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A further improvement of the Quantitative Subspace Theorem

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Abstract

In 2002, Evertse and Schlickewei obtained a quantitative version of the so-called Absolute Parametric Subspace Theorem. This result deals with a parametrized class of twisted heights. One of the consequences of this result is a quantitative version of the Absolute Subspace Theorem, giving an explicit upper bound for the number of subspaces containing the solutions of the Diophantine inequality under consideration.

In the present paper, we further improve Evertse's and Schlickewei's quantitative version of the Absolute Parametric Subspace Theorem and deduce an improved quantitative version of the Absolute Subspace Theorem. We combine ideas from the proof of Evertse and Schlickewei (which is basically a substantial refinement of Schmidt's proof of his Subspace Theorem from 1972), with ideas from Faltings' and Wüstholz' proof of the Subspace Theorem. A new feature is an "interval result," which gives more precise information on the distribution of the heights of the solutions of the system of inequalities considered in the Subspace Theorem.

1. Introduction

1.1. Let K be an algebraic number field. Denote by M_K its set of places and by $\|\cdot\|_v$ $(v \in M_K)$ its normalized absolute values, i.e., if v lies above $p \in M_{\mathbb{Q}} := \{\infty\} \cup \{\text{prime numbers}\}$, then the restriction of $\|\cdot\|_v$ to $\mathbb{Q}$ is $|\cdot|_p^{[K_v:\mathbb{Q}_p]/[K:\mathbb{Q}]}$. Define the norms and absolute height of $\mathbf{x} = (x_1, \dots, x_n) \in K^n$ by $\|\mathbf{x}\|_v := \max_{1 \leqslant i \leqslant n} \|x_i\|_v$ for $v \in M_K$ and $H(\mathbf{x}) := \prod_{v \in M_K} \|\mathbf{x}\|_v$.

Next, let S be a finite subset of M_K, n an integer $\geqslant 2$, and $\{L_1^{(v)}, \dots, L_n^{(v)}\}$ $(v \in S)$ linearly independent systems of linear forms from $K[X_1, \dots, X_n]$. The Subspace Theorem asserts that for every $\varepsilon > 0$, the set of solutions of

$$\prod_{v \in S} \prod_{i=1}^{n} \frac{\|L_i^{(v)}(\mathbf{x})\|_v}{\|\mathbf{x}\|_v} \leqslant H(\mathbf{x})^{-n-\varepsilon} \quad \text{in } \mathbf{x} \in K^n \tag{1.1}$$

lies in a finite union $T_1 \cup \cdots \cup T_{t_1}$ of proper linear subspaces of K^n. Schmidt [22] proved the Subspace Theorem in the case that S consists of the archimedean places of K and Schlickewei [17] extended this to the general case. Much work on the p-adization of the Subspace Theorem was done independently by Dubois and Rhin [8].

By an elementary combinatorial argument originating from Mahler (see [11, §21]), inequality (1.1) can be reduced to a finite number of systems of inequalities

$$\frac{\|L_i^{(v)}(\mathbf{x})\|_v}{\|\mathbf{x}\|_v} \leqslant H(\mathbf{x})^{d_{iv}} \ (v \in S,\ i = 1, \ldots, n) \ \text{ in } \mathbf{x} \in K^n, \tag{1.2}$$

where

$$\sum_{v \in S} \sum_{i=1}^{n} d_{iv} < -n.$$

Thus, an equivalent formulation of the Subspace Theorem is, that the set of solutions of (1.2) is contained in a finite union $T_1 \cup \cdots \cup T_{t_2}$ of proper linear subspaces of K^n. Notice that (1.2) may be viewed as an inequality over $\mathbb{P}^{n-1}(K)$. Making more precise earlier work of Vojta [30] and Schmidt [25], Faltings and Wüstholz [14, Th. 9.1] obtained the following refinement: *There exists a single, effectively computable proper linear subspace T of $\mathbb{P}^{n-1}(K)$ such that (1.2) has only finitely many solutions $\mathbf{x} \in \mathbb{P}^{n-1}(K) \setminus T$.*

(1.2) can be translated into a single twisted height inequality. Put

$$\delta := -1 - \frac{1}{n}\Big(\sum_{v \in S} \sum_{i=1}^{n} d_{iv}\Big), \qquad c_{iv} := d_{iv} - \frac{1}{n} \sum_{j=1}^{n} d_{jv} \ (v \in S,\ i = 1, \ldots, n).$$

Thus,

$$\delta > 0, \qquad \sum_{i=1}^{n} c_{iv} = 0 \text{ for } v \in S.$$

For $Q \geqslant 1$, $\mathbf{x} \in K^n$ define the twisted height

$$H_Q(\mathbf{x}) := \prod_{v \in S} \Big(\max_{1 \leqslant i \leqslant n} \|L_i^{(v)}(\mathbf{x})\|_v Q^{-c_{iv}} \Big) \cdot \prod_{v \notin S} \|\mathbf{x}\|_v. \tag{1.3}$$

(To our knowledge, this type of twisted height was used for the first time, but in a function field setting, by Dubois [7].)

Let $\mathbf{x} \in K^n$ be a solution to (1.2), and take $Q := H(\mathbf{x})$. Then

$$H_Q(\mathbf{x}) \leqslant Q^{-\delta}. \tag{1.4}$$

It is very useful to consider (1.4) with arbitrary reals c_{iv}, not just those arising from system (1.2), and with arbitrary reals Q not necessarily equal to $H(\mathbf{x})$. As will be explained in Section 2, the definition of H_Q can be extended to $\overline{\mathbb{Q}}^n$ (where it is assumed that $\overline{\mathbb{Q}} \supset K$), hence (1.4) can be considered for points $\mathbf{x} \in \overline{\mathbb{Q}}^n$. This leads to the following theorem.

THE ABSOLUTE PARAMETRIC SUBSPACE THEOREM. *Let* c_{iv} $(v \in S,\ i = 1,\dots,n)$ *be any reals with* $\sum_{i=1}^{n} c_{iv} = 0$ *for* $v \in S$, *and let* $\delta > 0$. *Then there are a real* $Q_0 > 1$ *and a finite number of proper linear subspaces* $T_1,\dots,T_{t_3}$ *of* $\overline{\mathbb{Q}}^n$, *defined over* K, *such that for every* $Q \geqslant Q_0$, *there is* $T_i \in \{T_1,\dots,T_{t_3}\}$ *with*

$$\{\mathbf{x} \in \overline{\mathbb{Q}}^n : H_Q(\mathbf{x}) \leqslant Q^{-\delta}\} \subset T_i.$$

Recall that a subspace of $\overline{\mathbb{Q}}^n$ is defined over K if it has a basis from K^n. In this general form, this result was first stated and proved in [11]. The nonabsolute version of the Parametric Subspace Theorem, with solutions $\mathbf{x} \in K^n$ instead of $\mathbf{x} \in \overline{\mathbb{Q}}^n$, was proved implicitly along with the Subspace Theorem.

1.2. In 1989, Schmidt was the first to obtain a quantitative version of the Subspace Theorem. In [24] he obtained, in the case $K = \mathbb{Q}$, $S = \{\infty\}$, an explicit upper bound for the number t_1 of subspaces containing the solutions of (1.1). This was generalized to arbitrary K, S by Schlickewei [18] and improved by Evertse [9]. Schlickewei observed that a good quantitative version of the Parametric Subspace Theorem, that is, with explicit upper bounds for Q_0 and t_3, would be more useful for applications than the existing quantitative versions of the basic Subspace Theorem concerning (1.1), and in 1996 he proved a special case of such a result. Then in 2002, Evertse and Schlickewei [11] proved a stronger, and fully general, quantitative version of the Absolute Parametric Subspace Theorem. This led to uniform upper bounds for the number of solutions of linear equations in unknowns from a multiplicative group of finite rank [12] and for the zero multiplicity of linear recurrence sequences [26], and more recently to results on the complexity of b-ary expansions of algebraic numbers [6], [3], to improvements and generalizations of the Cugiani-Mahler theorem [2], and approximation to algebraic numbers by algebraic numbers [4]. For an overview of recent applications of the Quantitative Subspace Theorem we refer to Bugeaud's survey paper [5].

1.3. In the present paper, we obtain an improvement of the quantitative version of Evertse and Schlickewei on the Absolute Parametric Subspace Theorem, with a substantially sharper bound for t_3. Our general result is stated in Section 2. In Section 3 we give some applications to (1.2) and (1.1).

To give a flavour, in this introduction we state special cases of our results. Let K, S be as above, and let c_{iv} $(v \in S,\ i = 1,\dots,n)$ be reals with

$$\sum_{i=1}^{n} c_{iv} = 0 \text{ for } v \in S, \qquad \sum_{v \in S} \max(c_{1v},\dots,c_{nv}) \leqslant 1; \tag{1.5}$$

the last condition is a convenient normalization. Further, let $L_i^{(v)}$ $(v \in S,\ i = 1,\dots,n)$ be linear forms such that for $v \in S$,

$$(1.6)\qquad \begin{cases} \{L_1^{(v)},\dots,L_n^{(v)}\}\subset\{X_1,\dots,X_n,\,X_1+\cdots+X_n\},\\ \{L_1^{(v)},\dots,L_n^{(v)}\}\text{ is linearly independent,}\end{cases}$$

and let H_Q be the twisted height defined by (1.3) and then extended to $\overline{\mathbb{Q}}$. Finally, let $0<\delta\leqslant 1$. Evertse and Schlickewei proved in [11] that in this case, the above stated Absolute Parametric Subspace Theorem holds with

$$Q_0:=n^{2/\delta},\quad t_3\leqslant 4^{(n+9)^2}\delta^{-n-4}.$$

This special case is the basic tool in the work of [12], [26], quoted above. We obtain the following improvement.

THEOREM 1.1. *Assume* (1.5), (1.6), *and let* $0<\delta\leqslant 1$. *Then there are proper linear subspaces* $T_1,\dots,T_{t_3}$ *of* $\overline{\mathbb{Q}}^n$, *all defined over* K, *with*

$$t_3\leqslant 10^6 2^{2n}n^{10}\delta^{-3}\big(\log(6n\delta^{-1})\big)^2,$$

such that for every Q *with* $Q\geqslant n^{1/\delta}$, *there is* $T_i\in\{T_1,\dots,T_{t_3}\}$ *with*

$$\{\mathbf{x}\in\overline{\mathbb{Q}}^n:\,H_Q(\mathbf{x})\leqslant Q^{-\delta}\}\subset T_i.$$

A new feature of our paper is the following *interval result.*

THEOREM 1.2. *Assume again* (1.5), (1.6), $0<\delta\leqslant 1$. *Put*

$$m:=\left[10^5 2^{2n}n^{10}\delta^{-2}\log(6n\delta^{-1})\right],\quad \omega:=\delta^{-1}\log 6n.$$

Then there are an effectively computable proper linear subspace T *of* $\overline{\mathbb{Q}}^n$, *defined over* K, *and reals* $Q_1,\dots,Q_m$ *with* $n^{1/\delta}\leqslant Q_1<\cdots<Q_m$, *such that for every* $Q\geqslant 1$ *with*

$$\{\mathbf{x}\in\overline{\mathbb{Q}}^n:\,H_Q(\mathbf{x})\leqslant Q^{-\delta}\}\not\subset T,$$

we have

$$Q\in\left[1,n^{1/\delta}\right)\cup[Q_1,Q_1^{\omega})\cup\cdots\cup[Q_m,Q_m^{\omega}).$$

The reals $Q_1,\dots,Q_m$ cannot be determined effectively from our proof. Theorem 1.1 is deduced from Theorem 1.2 and a gap principle. The precise definition of T is given in Section 2. We show that in the case considered here, i.e., with (1.6), the space T is the set of $\mathbf{x}=(x_1,\dots,x_n)\in\overline{\mathbb{Q}}^n$ with

$$(1.7)\qquad \sum_{j\in I_i}x_j=0\text{ for }i=1,\dots,p,$$

where $I_1,\dots,I_p$ ($p=n-\dim T$) are certain pairwise disjoint subsets of $\{1,\dots,n\}$ that can be determined effectively.

As an application, we give a refinement of the theorem of Faltings and Wüstholz on (1.2) mentioned above, again under assumption (1.6).

Corollary 1.3. *Let K, S be as above, let $L_i^{(v)}$ $(v \in S,\ i = 1, \dots, n)$ be linear forms with* (1.6), *and let d_{iv} $(v \in S,\ i = 1, \dots, n)$ be reals with*

$$d_{iv} \leqslant 0 \text{ for } v \in S,\ i = 1, \dots, n, \qquad \sum_{v \in S} \sum_{i=1}^{n} d_{iv} = -n - \varepsilon \text{ with } 0 < \varepsilon \leqslant 1.$$

Put

$$m' := \left[10^6 2^{2n} n^{12} \varepsilon^{-2} \log(6n\varepsilon^{-1})\right], \quad \omega' := 2n\varepsilon^{-1} \log 6n.$$

Then there are an effectively computable linear subspace T' of K^n and reals $H_1, \dots, H_{m'}$ with $n^{n/\varepsilon} \leqslant H_1 < H_2 < \cdots < H_{m'}$ such that for every solution $\mathbf{x} \in K^n$ of (1.2), *we have*

$$\mathbf{x} \in T' \quad \text{or } H(\mathbf{x}) \in \left[1, n^{n/\varepsilon}\right) \cup \left[H_1, H_1^{\omega'}\right) \cup \cdots \cup \left[H_{m'}, H_{m'}^{\omega'}\right).$$

Corollary 1.3 follows by applying Theorem 1.2 with

$$c_{iv} := \frac{n}{n+\varepsilon}\Big(d_{iv} - \frac{1}{n}\sum_{j=1}^{n} d_{jv}\Big) \ (v \in S,\ i = 1, \dots, n),$$

$$\delta := \frac{\varepsilon}{n+\varepsilon}, \quad Q := H(\mathbf{x})^{1+\varepsilon/n}.$$

The exceptional subspace T' is the set of $\mathbf{x} \in K^n$ with (1.7) for certain pairwise disjoint subsets $I_1, \dots, I_p$ of $\{1, \dots, n\}$.

It is an open problem to estimate from above the number of solutions $\mathbf{x} \in \mathbb{P}^{n-1}(K)$ of (1.2) outside $\mathbb{P}(T')$.

1.4. In Sections 2 and 3 we formulate our generalizations of the above stated results to arbitrary linear forms. In particular, in Theorem 2.1 we give our general quantitative version of the Absolute Parametric Subspace Theorem, which improves the result of Evertse and Schlickewei from [11], and in Theorem 2.3 we give our general interval result, dealing with points $\mathbf{x} \in \overline{\mathbb{Q}}^n$ outside an exceptional subspace T. Further, in Theorem 2.2 we give an "addendum" to Theorem 2.1, where we consider (1.4) for small values of Q. In Section 3 we give some applications to the Absolute Subspace Theorem; i.e., we consider absolute generalizations of (1.2), (1.1) with solutions $\mathbf{x}$ taken from $\overline{\mathbb{Q}}^n$ instead of K^n. Our central result is Theorem 2.3, from which the other results are deduced.

1.5. We briefly discuss the proof of Theorem 2.3. Recall that Schmidt's proof of his 1972 version of the Subspace Theorem [21], [23] is based on geometry of numbers and "Roth machinery," i.e., the construction of an auxiliary multi-homogeneous polynomial and an application of Roth's Lemma. The proofs of the quantitative versions of the Subspace Theorem and Parametric Subspace Theorem published since, including that of Evertse and Schlickewei, essentially follow the same lines. In 1994, Faltings and Wüstholz [14] came up with a very different proof of the Subspace Theorem. Their proof is an inductive argument, which involves constructions of auxiliary global line bundle sections on products of projective varieties of very large degrees, and an application of Faltings' Product Theorem. Ferretti observed that with their method, it is possible to prove quantitative results like ours, but with much larger bounds, due to the highly nonlinear projective varieties that occur in the course of the argument.

In our proof of Theorem 2.3 we use ideas from both Schmidt and Faltings and Wüstholz. In fact, similarly to Schmidt, we pass from $\overline{\mathbb{Q}}^n$ to an exterior power $\wedge^p\overline{\mathbb{Q}}^n$ by means of techniques from the geometry of numbers and apply the Roth machinery to the exterior power. But there, we replace Schmidt's construction of an auxiliary polynomial by that of Faltings and Wüstholz.

A price we have to pay is that our Roth machinery works only in the so-called *semistable case* (terminology from [14]) where the exceptional space T in Theorem 2.3 is equal to $\{\mathbf{0}\}$. Thus, we need an involved additional argument to reduce the general case where T can be arbitrary to the semistable case.

In this reduction we obtain, as a by-product of some independent interest, a result on the limit behaviour of the successive infima $\lambda_1(Q),\dots,\lambda_n(Q)$ of H_Q as $Q\to\infty$; see Theorem 16.1. Here, $\lambda_i(Q)$ is the infimum of all $\lambda>0$ such that the set of $\mathbf{x}\in\overline{\mathbb{Q}}^n$ with $H_Q(\mathbf{x})\leqslant\lambda$ contains at least i linearly independent points. Our limit result may be viewed as the "algebraic" analogue of recent work of Schmidt and Summerer [28].

1.6. Our paper is organized as follows. In Sections 2 and 3 we state our results. In Sections 4 and 5 we deduce from Theorem 2.3 the other theorems stated in Sections 2 and 3. In Sections 6 and 7 we have collected some notation and simple facts used throughout the paper. In Section 8 we state the semistable case of Theorem 2.3. This is proved in Sections 9–14. Here we follow [11], except that we use the auxiliary polynomial of Faltings and Wüstholz instead of Schmidt's. In Sections 15–18 we deduce the general case of Theorem 2.3 from the semistable case.

2. Results for twisted heights

2.1. All number fields considered in this paper are contained in a given algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$. Given a field F, we denote by $F[X_1,\dots,X_n]^{\mathrm{lin}}$ the F-vector space of linear forms $\alpha_1X_1+\cdots+\alpha_nX_n$ with $\alpha_1,\dots,\alpha_n\in F$.

Let $K\subset\overline{\mathbb{Q}}$ be an algebraic number field. Recall that the normalized absolute values $\|\cdot\|_v$ ($v\in M_K$) introduced in Section 1 satisfy the Product Formula

$$\prod_{v\in M_K}\|x\|_v=1\ \text{ for } x\in K^*. \tag{2.1}$$

Further, if E is any finite extension of K and we define normalized absolute values $\|\cdot\|_w$ ($w \in M_E$) in the same manner as those for K, we have for every place $v \in M_K$ and each place $w \in M_E$ lying above v,

$$\|x\|_w = \|x\|_v^{d(w|v)} \text{ for } x \in K, \text{ where } d(w|v) := \frac{[E_w : K_v]}{[E : K]} \tag{2.2}$$

and K_v, E_w denote the completions of K at v, E at w, respectively. Notice that

$$\sum_{w|v} d(w|v) = 1, \tag{2.3}$$

where '$w|v$' indicates that w is running through all places of E that lie above v.

2.2. We list the definitions and technical assumptions needed in the statements of our theorems. In particular, we define our twisted heights.

Let again $K \subset \overline{\mathbb{Q}}$ be an algebraic number field. Further, let n be an integer, $\mathcal{L} = (L_i^{(v)} : v \in M_K, i = 1, \ldots, n)$ a tuple of linear forms, and $\mathbf{c} = (c_{iv} : v \in M_K, i = 1, \ldots, n)$ a tuple of reals satisfying

$$n \geqslant 2, \quad L_i^{(v)} \in K[X_1, \ldots, X_n]^{\mathrm{lin}} \text{ for } v \in M_K, i = 1, \ldots, n, \tag{2.4}$$

$$\{L_1^{(v)}, \ldots, L_n^{(v)}\} \text{ is linearly independent for } v \in M_K, \tag{2.5}$$

$$\bigcup_{v \in M_K} \{L_1^{(v)}, \ldots, L_n^{(v)}\} =: \{L_1, \ldots, L_r\} \text{ is finite}, \tag{2.6}$$

$$c_{1v} = \cdots = c_{nv} = 0 \text{ for all but finitely many } v \in M_K, \tag{2.7}$$

$$\sum_{i=1}^{n} c_{iv} = 0 \text{ for } v \in M_K, \tag{2.8}$$

$$\sum_{v \in M_K} \max(c_{1v}, \ldots, c_{nv}) \leqslant 1. \tag{2.9}$$

In addition, let δ, R be reals with

$$0 < \delta \leqslant 1, \quad R \geqslant r = \#\left(\bigcup_{v \in M_K} \{L_1^{(v)}, \ldots, L_n^{(v)}\}\right), \tag{2.10}$$

and put

$$\Delta_{\mathcal{L}} := \prod_{v \in M_K} \|\det(L_1^{(v)}, \ldots, L_n^{(v)})\|_v, \tag{2.11}$$

$$H_{\mathcal{L}} := \prod_{v \in M_K} \max_{1 \leqslant i_1 < \cdots < i_n \leqslant r} \|\det(L_{i_1}, \ldots, L_{i_n})\|_v, \tag{2.12}$$

where the maxima are taken over all n-element subsets of $\{1, \ldots, r\}$.

For $Q \geqslant 1$, we define the twisted height $H_{\mathcal{L},\mathbf{c},Q} : K^n \to \mathbb{R}$ by

$$H_{\mathcal{L},\mathbf{c},Q}(\mathbf{x}) := \prod_{v\in M_K} \max_{1\leqslant i\leqslant n} \Big(\|L_i^{(v)}(\mathbf{x})\|_v \cdot Q^{-c_{iv}} \Big). \tag{2.13}$$

In case that $\mathbf{x} = \mathbf{0}$, we have $H_{\mathcal{L},\mathbf{c},Q}(\mathbf{x}) = 0$. If $\mathbf{x} \neq \mathbf{0}$, it follows from (2.4)–(2.7) that all factors in the product are nonzero and equal to 1 for all but finitely many v; hence, the twisted height is well defined and nonzero.

Now let $\mathbf{x} \in \overline{\mathbb{Q}}^n$. Then there is a finite extension E of K such that $\mathbf{x} \in E^n$. For $w \in M_E$, $i = 1, \ldots, n$, define

$$L_i^{(w)} := L_i^{(v)}, \quad c_{iw} := c_{iv} \cdot d(w|v) \tag{2.14}$$

if v is the place of K lying below w, and put

$$H_{\mathcal{L},\mathbf{c},Q}(\mathbf{x}) := \prod_{w\in M_E} \max_{1\leqslant i\leqslant n} \Big(\|L_i^{(w)}(\mathbf{x})\|_w \cdot Q^{-c_{iw}} \Big). \tag{2.15}$$

It follows from (2.14), (2.2), (2.3) that this is independent of the choice of E. Further, by (2.1), we have $H_{\mathcal{L},\mathbf{c},Q}(\alpha\mathbf{x}) = H_{\mathcal{L},\mathbf{c},Q}(\mathbf{x})$ for $\mathbf{x} \in \overline{\mathbb{Q}}^n$, $\alpha \in \overline{\mathbb{Q}}^*$.

To define $H_{\mathcal{L},\mathbf{c},Q}$, we needed only (2.4)–(2.7); properties (2.8), (2.9) are merely convenient normalizations.

2.3. Under the above hypotheses, Evertse and Schlickewei [11, Th. 2.1] obtained the following quantitative version of the Absolute Parametric Subspace Theorem.

There is a collection $\{T_1, \ldots, T_{t_0}\}$ of proper linear subspaces of $\overline{\mathbb{Q}}^n$, all defined over K, with

$$t_0 \leqslant 4^{(n+8)^2} \delta^{-n-4} \log(2R) \log\log(2R)$$

such that for every real $Q \geqslant \max(H_{\mathcal{L}}^{1/R}, n^{2/\delta})$, there is $T_i \in \{T_1, \ldots, T_{t_0}\}$ for which

$$\Big\{ \mathbf{x} \in \overline{\mathbb{Q}}^n : H_{\mathcal{L},\mathbf{c},Q}(\mathbf{x}) \leqslant \Delta_{\mathcal{L}}^{1/n} Q^{-\delta} \Big\} \subset T_i. \tag{2.16}$$

We improve this as follows.

THEOREM 2.1. *Let $n, \mathcal{L}, \mathbf{c}, \delta, R$ satisfy (2.4)–(2.10), and let $\Delta_{\mathcal{L}}, H_{\mathcal{L}}$ be given by (2.11), (2.12). Then there are proper linear subspaces $T_1, \ldots, T_{t_0}$ of $\overline{\mathbb{Q}}^n$, all defined over K, with*

$$t_0 \leqslant 10^6 2^{2n} n^{10} \delta^{-3} \log(3\delta^{-1}R) \log(\delta^{-1}\log 3R), \tag{2.17}$$

such that for every real Q with

$$Q \geqslant C_0 := \max\big(H_{\mathcal{L}}^{1/R}, n^{1/\delta} \big), \tag{2.18}$$

there is $T_i \in \{T_1, \ldots, T_{t_0}\}$ with (2.16).

Notice that in terms of n, δ, our upper bound for t_0 improves that of Evertse and Schlickewei from $c_1^{n^2}\delta^{-n-4}$ to $c_2^n\delta^{-3}(\log\delta^{-1})^2$, while it has the same dependence on R.

The lower bound C_0 in (2.18) still has an exponential dependence on δ^{-1}. We do not know of a method to reduce it in our general absolute setting. If we restrict to solutions $\mathbf{x}$ in K^n, the following can be proved.

THEOREM 2.2. *Let again $n, \mathcal{L}, \mathbf{c}, \delta, R$ satisfy* (2.4)–(2.10). *Assume in addition that K has degree d. Then there are proper linear subspaces $U_1, \ldots, U_{t_1}$ of K^n, with*

$$t_1 \leqslant \delta^{-1}\Big((90n)^{nd} + 3\log\log 3H_{\mathcal{L}}^{1/R}\Big)$$

such that for every Q with $1 \leqslant Q < C_0 = \max(H_{\mathcal{L}}^{1/R}, n^{1/\delta})$, there is $U_i \in \{U_1, \ldots, U_{t_1}\}$ with

$$\Big\{\mathbf{x} \in K^n : H_{\mathcal{L},\mathbf{c},Q}(\mathbf{x}) \leqslant \Delta_{\mathcal{L}}^{1/n} Q^{-\delta}\Big\} \subset U_i.$$

We mention that in various special cases, by an ad-hoc approach the upper bound for t_1 can be reduced. Recent work of Schmidt [27] on the number of "small solutions" in Roth's Theorem (essentially the case $n = 2$ in our setting) suggests that there should be an upper bound for t_1 with a polynomial instead of exponential dependence on d.

2.4. We now formulate our general interval result for twisted heights. We first define an exceptional vector space. We may view a linear form $L \in \overline{\mathbb{Q}}[X_1, \ldots, X_n]^{\mathrm{lin}}$ as a linear function on $\overline{\mathbb{Q}}^n$. Then its restriction to a linear subspace U of $\overline{\mathbb{Q}}^n$ is denoted by $L|_U$.

Let $n, \mathcal{L}, \mathbf{c}, \delta, R$ satisfy (2.4)–(2.10). Let U be a k-dimensional linear subspace of $\overline{\mathbb{Q}}^n$. For $v \in M_K$, we define $w_v(U) = w_{\mathcal{L},\mathbf{c},v}(U) := 0$ if $k = 0$ and

$$\begin{aligned}(2.19)\quad w_v(U) = w_{\mathcal{L},\mathbf{c},v}(U) := \min\Big\{&c_{i_1,v} + \cdots + c_{i_k,v} : \\ &L_{i_1}^{(v)}|_U, \ldots, L_{i_k}^{(v)}|_U \text{ are linearly independent}\Big\}\end{aligned}$$

if $k > 0$, where the minimum is taken over all k-tuples $i_1, \ldots, i_k$ such that $L_{i_1}^{(v)}|_U, \ldots, L_{i_k}^{(v)}|_U$ are linearly independent. Then the *weight of U with respect to* $(\mathcal{L}, \mathbf{c})$ is defined by

$$w(U) = w_{\mathcal{L},\mathbf{c}}(U) := \sum_{v \in M_K} w_v(U). \tag{2.20}$$

This is well defined since by (2.7), at most finitely many of the quantities $w_v(U)$ are nonzero.

By theory from, e.g., [14] (for a proof see Lemma 15.2 below), there is a unique, proper linear subspace $T = T(\mathcal{L}, \mathbf{c})$ of $\overline{\mathbb{Q}}^n$ such that

$$(2.21)\qquad \begin{cases} \dfrac{w(T)}{n-\dim T} \geqslant \dfrac{w(U)}{n-\dim U} \\ \qquad\qquad\qquad \text{for every proper linear subspace } U \text{ of } \overline{\mathbb{Q}}^n; \\ \text{subject to this condition, } \dim T \text{ is minimal.} \end{cases}$$

Moreover, this space T is defined over K.

In Proposition 17.5 below, we prove that

$$H_2(T) \leqslant \left(\max_{v,i} H_2(L_i^{(v)})\right)^{4^n}$$

with "Euclidean" heights H_2 for subspaces and linear forms defined in Section 6 below. Thus, T is effectively computable and it belongs to a finite collection depending only on $\mathcal{L}$. In Lemma 15.3 below, we prove that in the special case considered in Section 1, i.e.,

$$\{L_1^{(v)}, \ldots, L_n^{(v)}\} \subset \{X_1, \ldots, X_n,\ X_1 + \cdots + X_n\} \text{ for } v \in M_K,$$

we have

$$T = \{\mathbf{x} \in \overline{\mathbb{Q}}^n : \sum_{j \in I_i} x_j = 0 \text{ for } j = 1, \ldots, p\}$$

for certain pairwise disjoint subsets $I_1, \ldots, I_p$ of $\{1, \ldots, n\}$.

Now our interval result is as follows.

THEOREM 2.3. *Let $n, \mathcal{L}, \mathbf{c}, \delta, R$ satisfy (2.4)–(2.10), and let the vector space T be given by (2.21). Put*

$$(2.22)\qquad m_0 := \left[10^5 2^{2n} n^{10} \delta^{-2} \log(3\delta^{-1} R)\right], \quad \omega_0 := \delta^{-1} \log 3R.$$

Then there are reals $Q_1, \ldots, Q_{m_0}$ with

$$(2.23)\qquad C_0 := \max(H_{\mathcal{L}}^{1/R}, n^{1/\delta}) \leqslant Q_1 < \cdots < Q_{m_0}$$

such that for every $Q \geqslant 1$ for which

$$(2.24)\qquad \{\mathbf{x} \in \overline{\mathbb{Q}}^n : H_{\mathcal{L},\mathbf{c},Q}(\mathbf{x}) \leqslant \Delta_{\mathcal{L}}^{1/n} Q^{-\delta}\} \not\subset T,$$

we have

$$(2.25)\qquad Q \in [1, C_0) \cup [Q_1, Q_1^{\omega_0}) \cup \cdots \cup [Q_{m_0}, Q_{m_0}^{\omega_0}).$$

3. Applications to Diophantine inequalities

3.1. We state some results for "absolute" generalizations of (1.2), (1.1). We fix some notation. The absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$ of a number field $K \subset \overline{\mathbb{Q}}$ is denoted by G_K. The absolute height $H(\mathbf{x})$ of $\mathbf{x} \in \overline{\mathbb{Q}}^n$ is defined by choosing a number field K such that $\mathbf{x} \in K^n$ and taking $H(\mathbf{x}) := \prod_{v \in M_K} \|\mathbf{x}\|_v$. The *inhomogeneous* height of $L = \alpha_1 X_1 + \cdots + \alpha_n X_n \in \overline{\mathbb{Q}}[X_1, \ldots, X_n]^{\mathrm{lin}}$ is given by $H^*(L) := H(\mathbf{a})$, where $\mathbf{a} = (1, \alpha_1, \ldots, \alpha_n)$. Further, for a number field K, we define the field $K(L) := K(\alpha_1, \ldots, \alpha_n)$.

We fix an algebraic number field $K \subset \overline{\mathbb{Q}}$. Further, for every place $v \in M_K$, we choose and then fix an extension of $\| \cdot \|_v$ to $\overline{\mathbb{Q}}$. For $\mathbf{x} = (x_1, \ldots, x_n) \in \overline{\mathbb{Q}}^n$, $\sigma \in G_K$, $v \in M_K$, we put $\sigma(\mathbf{x}) := (\sigma(x_1), \ldots, \sigma(x_n))$, $\|\mathbf{x}\|_v := \max_{1 \leqslant i \leqslant n} \|x_i\|_v$.

3.2. We list some technical assumptions and then state our results. Let n be an integer $\geqslant 2$, R a real, S a finite subset of M_K, $L_i^{(v)}$ $(v \in S,\ i = 1, \ldots, n)$ linear forms from $\overline{\mathbb{Q}}[X_1, \ldots, X_n]^{\mathrm{lin}}$, and d_{iv} $(v \in S,\ i = 1, \ldots, n)$ reals such that

$$\{L_1^{(v)}, \ldots, L_n^{(v)}\} \text{ is linearly independent for } v \in S, \tag{3.1}$$

$$H^*(L_i^{(v)}) \leqslant H^*, \quad [K(L_i^{(v)}) : K] \leqslant D \text{ for } v \in S,\ i = 1, \ldots, n, \tag{3.2}$$

$$\#\Big(\bigcup_{v \in S} \{L_1^{(v)}, \ldots, L_n^{(v)}\}\Big) \leqslant R, \tag{3.3}$$

$$\sum_{v \in S} \sum_{i=1}^{n} d_{iv} = -n - \varepsilon \text{ with } 0 < \varepsilon \leqslant 1, \tag{3.4}$$

$$d_{iv} \leqslant 0 \text{ for } v \in S,\ i = 1, \ldots, n. \tag{3.5}$$

Further, put

$$A_v := \| \det(L_1^{(v)}, \ldots, L_n^{(v)}) \|_v^{1/n} \quad \text{for } v \in S. \tag{3.6}$$

3.3. We consider the system of inequalities

$$\max_{\sigma \in G_K} \frac{\|L_i^{(v)}(\sigma(\mathbf{x}))\|_v}{\|\sigma(\mathbf{x})\|_v} \leqslant A_v H(\mathbf{x})^{d_{iv}} \ (v \in S,\ i = 1, \ldots, n) \ \text{ in } \mathbf{x} \in \overline{\mathbb{Q}}^n. \tag{3.7}$$

According to [11, Th. 20.1], the set of solutions $\mathbf{x} \in \overline{\mathbb{Q}}^n$ of (3.7) with $H(\mathbf{x}) \geqslant \max(H^*, n^{2n/\varepsilon})$ is contained in a union of at most

$$2^{3(n+9)^2} \varepsilon^{-n-4} \log(4RD) \log\log(4RD) \tag{3.8}$$

proper linear subspaces of $\overline{\mathbb{Q}}^n$ that are defined over K. We improve this as follows.

THEOREM 3.1. *Assume (3.1)–(3.6). Then the set of solutions* $\mathbf{x} \in \overline{\mathbb{Q}}^n$ *of system (3.7) with*

$$H(\mathbf{x}) \geqslant C_1 := \max((H^*)^{1/3RD}, n^{n/\varepsilon}) \tag{3.9}$$

is contained in a union of at most

$$10^9 2^{2n} n^{14} \varepsilon^{-3} \log\left(3\varepsilon^{-1}RD\right) \log\left(\varepsilon^{-1}\log 3RD\right) \tag{3.10}$$

proper linear subspaces of $\overline{\mathbb{Q}}^n$ *that are all defined over* K.

Apart from a factor $\log \varepsilon^{-1}$, in terms of ε our upper bound has the same order of magnitude as the best known bound for the number of "large" approximants to a given algebraic number in Roth's Theorem (see, e.g., [27]).

Although for applications this seems to be of lesser importance now, for the sake of completeness we give without proof a quantitative version of an absolute generalization of (1.1). We keep the notation and assumptions from (3.1)–(3.6). In addition, we put

$$s := \#S, \quad \Delta := \prod_{v \in S} \| \det(L_1^{(v)}, \ldots, L_n^{(v)}) \|_v.$$

Consider

$$\prod_{v \in S} \prod_{i=1}^{n} \max_{\sigma \in G_K} \frac{\|L_i^{(v)}(\sigma(\mathbf{x}))\|_v}{\|\sigma(\mathbf{x})\|_v} \leqslant \Delta H(\mathbf{x})^{-n-\varepsilon}. \tag{3.11}$$

COROLLARY 3.2. *The set of solutions* $\mathbf{x} \in \overline{\mathbb{Q}}^n$ *of (3.11) with* $H(\mathbf{x}) \geqslant H_0$ *is contained in a union of at most*

$$\left(9n^2\varepsilon^{-1}\right)^{ns} \cdot 10^{10} 2^{2n} n^{15} \varepsilon^{-3} \log\left(3\varepsilon^{-1}D\right) \log\left(\varepsilon^{-1}\log 3D\right)$$

proper linear subspaces of $\overline{\mathbb{Q}}^n$ *that are all defined over* K.

Evertse and Schlickewei [11, Th. 3.1] obtained a similar result, with an upper bound for the number of subspaces that is about $\left(9n^2\varepsilon^{-1}\right)^{ns}$ times the quantity in (3.8). So in terms of n, their bound is of the order c^{n^2} whereas ours is of the order $c^{n \log n}$. Our Corollary 3.2 can be deduced by following the arguments of [11, §21], except that instead of Theorem 20.1 of that paper, one has to use our Theorem 3.1.

We now state our interval result, making more precise the result of Faltings and Wüstholz on (1.2).

THEOREM 3.3. *Assume again (3.1)–(3.6). Put*

$$m_1 := \left[10^8 2^{2n} n^{14} \varepsilon^{-2} \log\left(3\varepsilon^{-1}RD\right)\right],$$

$$\omega_1 := 3n\varepsilon^{-1} \log 3RD.$$

There is a proper linear subspace T of $\overline{\mathbb{Q}}^n$ defined over K that is effectively computable and belongs to a finite collection depending only on $\{L_i^{(v)} : v \in S, i = 1, \ldots, n\}$, and there are reals $H_1, \ldots, H_{m_1}$ with

$$C_1 := \max((H^*)^{1/3RD}, n^{n/\varepsilon}) \leqslant H_1 < \cdots < H_{m_1}$$

such that for every solution $\mathbf{x} \in \overline{\mathbb{Q}}^n$ of (3.7), we have

$$\mathbf{x} \in T \quad \text{or} \quad H(\mathbf{x}) \in [1, C_1) \cup [H_1, H_1^{\omega_1}) \cup \cdots \cup [H_{m_1}, H_{m_1}^{\omega_1}).$$

Our interval result implies that the solutions $\mathbf{x} \in \overline{\mathbb{Q}}^n$ of (3.7) outside T have bounded height. In particular, (1.2) has only finitely many solutions $\mathbf{x} \in \mathbb{P}^{n-1}(K) \setminus \mathbb{P}(T)$.

4. Proofs of Theorems 2.1 and 2.2

We deduce Theorem 2.1 from Theorem 2.3 and prove Theorem 2.2. For this purpose, we need some gap principles. We use the notation introduced in Section 2. In particular, K is a number field, $n \geqslant 2$, $\mathcal{L} = (L_i^{(v)} : v \in M_K, i = 1, \ldots, n)$ a tuple from $K[X_1, \ldots, X_n]^{\text{lin}}$, and $\mathbf{c} = (c_{iv} : v \in M_K : i = 1, \ldots, n)$ a tuple of reals. The linear forms $L_i^{(w)}$ and reals c_{iw}, where w is a place on some finite extension E of K, are given by (2.14).

We start with a simple lemma.

LEMMA 4.1. *Suppose that $\mathcal{L}, \mathbf{c}$ satisfy (2.4)–(2.7). Let $\mathbf{x} \in \overline{\mathbb{Q}}^n$, $\sigma \in G_K$, $Q \geqslant 1$. Then $H_{\mathcal{L},\mathbf{c},Q}(\sigma(\mathbf{x})) = H_{\mathcal{L},\mathbf{c},Q}(\mathbf{x})$.*

Proof. Let E be a finite Galois extension of K such that $\mathbf{x} \in E^n$. For any place v of K and any place w of E lying above v, there is a unique place w_σ of E lying above v such that $\|\cdot\|_{w_\sigma} = \|\sigma(\cdot)\|_w$. By (2.14) and $[E_{w_\sigma} : K_v] = [E_w : K_v]$, we have $L_i^{(w_\sigma)} = L_i^{(w)}$, $c_{i,w_\sigma} = c_{iw}$ for $i = 1, \ldots, n$. Thus,

$$\begin{aligned} H_{\mathcal{L},\mathbf{c},Q}(\sigma(\mathbf{x})) &= \prod_{v \in M_K} \prod_{w|v} \Big(\max_{1 \leqslant i \leqslant n} \|L_i^{(w)}(\sigma(\mathbf{x}))\|_w Q^{-c_{iw}} \Big) \\ &= \prod_{v \in M_K} \prod_{w|v} \Big(\max_{1 \leqslant i \leqslant n} \|L_i^{(w_\sigma)}(\mathbf{x})\|_w Q^{-c_{i,w_\sigma}} \Big) = H_{\mathcal{L},\mathbf{c},Q}(\mathbf{x}). \qquad \square \end{aligned}$$

We assume henceforth that $n, \mathcal{L}, \mathbf{c}, \delta, R$ satisfy (2.4)–(2.10). Let $\Delta_{\mathcal{L}}$, $H_{\mathcal{L}}$ be given by (2.11), (2.12). Notice that (2.2), (2.3), (2.14) imply that (2.4)–(2.9) remain valid if we replace K by E and the index $v \in M_K$ by the index $w \in M_E$. Likewise, in the definitions of $\Delta_{\mathcal{L}}, H_{\mathcal{L}}$ we may replace K by E and $v \in M_K$ by $w \in M_E$. This will be used frequently in the sequel.

We start with our first gap principle. For $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{C}^n$, we put $\|\mathbf{a}\| := \max(|a_1|, \ldots, |a_n|)$.

Proposition 4.2. *Let*

$$A \geqslant n^{1/\delta}. \tag{4.1}$$

Then there is a single proper linear subspace T_0 of $\overline{\mathbb{Q}}^n$, defined over K, such that for every Q with

$$A \leqslant Q < A^{1+\delta/2},$$

we have $\{\mathbf{x} \in \overline{\mathbb{Q}}^n : H_{\mathcal{L},\mathbf{c},Q}(\mathbf{x}) \leqslant \Delta_{\mathcal{L}}^{1/n} Q^{-\delta}\} \subset T_0$.

Proof. Let $Q \in [A, A^{1+\delta/2})$, and let $\mathbf{x} \in \overline{\mathbb{Q}}^n$ with $\mathbf{x} \neq \mathbf{0}$ and $H_{\mathcal{L},\mathbf{c},Q}(\mathbf{x}) \leqslant \Delta_{\mathcal{L}}^{1/n} Q^{-\delta}$. Take a finite extension E of K such that $\mathbf{x} \in E^n$. For $w \in M_E$, put

$$\theta_w := \max_{1 \leqslant i \leqslant n} c_{iw}.$$

By (2.14), (2.8), (2.9), we have

$$\sum_{i=1}^n c_{iw} = 0 \text{ for } w \in M_E, \quad \sum_{w \in M_E} \theta_w \leqslant 1. \tag{4.2}$$

Let $w \in M_E$ with $\theta_w > 0$. Using $A \leqslant Q < A^{1+\delta/2}$, we have

$$\max_{1 \leqslant i \leqslant n} \|L_i^{(w)}(\mathbf{x})\|_w Q^{-c_{iw}} > \Big(\max_{1 \leqslant i \leqslant n} \|L_i^{(w)}(\mathbf{x})\|_w A^{-c_{iw}} \Big) \cdot A^{-\theta_w \delta/2}.$$

If $w \in M_E$ with $\theta_w = 0$, then $c_{iw} = 0$ for $i = 1, \ldots, n$ and so we trivially have an equality instead of a strict inequality. By taking the product over w and using (4.2), we obtain

$$\begin{aligned} &H_{\mathcal{L},\mathbf{c},Q}(\mathbf{x}) > H_{\mathcal{L},\mathbf{c},A}(\mathbf{x}) A^{-\delta/2} \ \text{ if } \theta_w > 0 \text{ for some } w \in M_E, \\ &H_{\mathcal{L},\mathbf{c},Q}(\mathbf{x}) = H_{\mathcal{L},\mathbf{c},A}(\mathbf{x}) > H_{\mathcal{L},\mathbf{c},A}(\mathbf{x}) A^{-\delta/2} \ \text{ otherwise.} \end{aligned}$$

Hence,

$$H_{\mathcal{L},\mathbf{c},A}(\mathbf{x}) < \Delta_{\mathcal{L}}^{1/n} A^{-\delta/2}. \tag{4.3}$$

This is clearly true for $\mathbf{x} = \mathbf{0}$ as well.

Let T_0 be the $\overline{\mathbb{Q}}$-vector space spanned by the vectors $\mathbf{x} \in \overline{\mathbb{Q}}^n$ with (4.3). By Lemma 4.1, if $\mathbf{x}$ satisfies (4.3), then so does $\sigma(\mathbf{x})$ for every $\sigma \in G_K$. Hence T_0 is defined over K. Our proposition follows once we have shown that $T_0 \neq \overline{\mathbb{Q}}^n$, and for this, it suffices to show that $\det(\mathbf{x}_1, \ldots, \mathbf{x}_n) = 0$ for any $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \overline{\mathbb{Q}}^n$ with (4.3).

So take $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \overline{\mathbb{Q}}^n$ with (4.3). Let E be a finite extension of K with $\mathbf{x}_1, \ldots, \mathbf{x}_n \in E^n$. We estimate from above $\|\det(\mathbf{x}_1, \ldots, \mathbf{x}_n)\|_w$ for $w \in M_E$. For $w \in M_E$, $j = 1, \ldots, n$, put

$$\Delta_w := \|\det(L_1^{(w)}, \ldots, L_n^{(w)})\|_w, \quad H_{jw} := \max_{1 \leqslant i \leqslant n} \|L_i^{(w)}(\mathbf{x}_j)\|_w A^{-c_{iw}}.$$

First, let w be an infinite place of E. Put $s(w) := [E_w : \mathbb{R}]/[E : \mathbb{Q}]$. Then there is an embedding $\sigma_w : E \hookrightarrow \mathbb{C}$ such that $\|\cdot\|_w = |\sigma_w(\cdot)|^{s(w)}$. Put

$$\mathbf{a}_{jw} := \Big(A^{-c_{1w}/s(w)}\sigma_w(L_1^{(w)}(\mathbf{x}_j)), \dots, A^{-c_{nw}/s(w)}\sigma_w(L_n^{(w)}(\mathbf{x}_j))\Big) \tag{4.4}$$

for $j = 1, \dots, n$. Then $H_{jw} = \|\mathbf{a}_{jw}\|^{s(w)}$. So by Hadamard's inequality and (4.2),

$$\begin{aligned} \|\det(\mathbf{x}_1, \dots, \mathbf{x}_n)\|_w &= \Delta_w^{-1}\|\det\big(L_i^{(w)}(\mathbf{x}_j)\big)_{i,j}\|_w \\ &= \Delta_w^{-1}A^{c_{1w}+\cdots+c_{nw}}|\det(\mathbf{a}_{1w}, \dots, \mathbf{a}_{nw})|^{s(w)} \\ &\leqslant \Delta_w^{-1}n^{ns(w)/2}H_{1w}\cdots H_{nw}. \end{aligned} \tag{4.5}$$

Next, let w be a finite place of E. Then by the ultrametric inequality and (4.2),

$$\begin{aligned} \|\det(\mathbf{x}_1, \dots, \mathbf{x}_n)\|_w &= \Delta_w^{-1}\|\det\big(L_i^{(w)}(\mathbf{x}_j)\big)_{i,j}\|_w \\ &\leqslant \Delta_w^{-1}\max_\rho \|L_{\rho(1)}(\mathbf{x}_1)\|_w\cdots\|L_{\rho(n)}(\mathbf{x}_n)\|_w \\ &\leqslant \Delta_w^{-1}A^{c_{1w}+\cdots+c_{nw}}H_{1w}\cdots H_{nw} \\ &= \Delta_w^{-1}H_{1w}\cdots H_{nw}, \end{aligned} \tag{4.6}$$

where the maximum is taken over all permutations ρ of $1, \dots, n$.

We take the product over $w \in M_E$. Then using $\prod_{w\in M_E}\Delta_w = \Delta_{\mathcal{L}}$ (by (2.2), (2.14), (2.11)), $\sum_{w|\infty} s(w) = 1$ (sum of local degrees is global degree), (4.2), (4.3), and lastly our assumption $A \geqslant n^{1/\delta}$, we obtain

$$\prod_{w\in M_E}\|\det(\mathbf{x}_1, \dots, \mathbf{x}_n)\|_w \leqslant \Delta_{\mathcal{L}}^{-1}n^{n/2}\prod_{j=1}^n H_{\mathcal{L},\mathbf{c},A}(\mathbf{x}_j) < n^{n/2}A^{-n\delta/2} \leqslant 1.$$

Now the product formula implies that $\det(\mathbf{x}_1, \dots, \mathbf{x}_n) = 0$, as required. □

For our second gap principle, we need the following lemma.

Lemma 4.3. *Let $M \geqslant 1$. Then $\mathbb{C}^n$ is a union of at most $(20n)^n M^2$ subsets such that for any $\mathbf{y}_1, \dots, \mathbf{y}_n$ in the same subset,*

$$|\det(\mathbf{y}_1, \dots, \mathbf{y}_n)| \leqslant M^{-1}\|\mathbf{y}_1\|\cdots\|\mathbf{y}_n\|. \tag{4.7}$$

Proof. [10, Lemma 4.3]. □

Proposition 4.4. *Let $d := [K : \mathbb{Q}]$ and $A \geqslant 1$. Then there are proper linear subspaces $T_1, \dots, T_t$ of K^n, with*

$$t \leqslant (80n)^{nd},$$

such that for every Q with

$$A \leqslant Q < 2A^{1+\delta/2},$$

there is $T_i \in \{T_1, \dots, T_t\}$ *with*

$$\{\mathbf{x} \in K^n : H_{\mathcal{L},\mathbf{c},Q}(\mathbf{x}) \leqslant \Delta_{\mathcal{L}}^{1/n} Q^{-\delta}\} \subset T_i.$$

Proof. We use the notation from the proof of Proposition 4.2. Temporarily, we index places of K also by w. Similarly as in the proof of Proposition 4.2, we infer that if $\mathbf{x} \in K^n$ is such that there exists Q with $Q \in [A, 2A^{1+\delta/2})$ and $H_{\mathcal{L},\mathbf{c},Q}(\mathbf{x}) \leqslant \Delta_{\mathcal{L}}^{1/n} Q^{-\delta}$, then

$$H_{\mathcal{L},\mathbf{c},A}(\mathbf{x}) < 2\Delta_{\mathcal{L}}^{1/n} A^{-\delta/2}. \tag{4.8}$$

Put $M := 2^n$. Let $w_1, \dots, w_r$ be the infinite places of K, and for $i = 1, \dots, r$ take an embedding $\sigma_{w_i} : K \hookrightarrow \mathbb{C}$ such that $\| \cdot \|_{w_i} = |\sigma_{w_i}(\cdot)|^{s(w_i)}$.

For $\mathbf{x} \in K^n$ with (4.8) and $w \in \{w_1, \dots, w_r\}$, put

$$\mathbf{a}_w(\mathbf{x}) := \left(A^{-c_{1w}/s(w)} \sigma_w(L_1^{(w)}(\mathbf{x})), \dots, A^{-c_{nw}/s(w)} \sigma_w(L_n^{(w)}(\mathbf{x})) \right).$$

By Lemma 4.3, the set of vectors $\mathbf{x} \in K^n$ with (4.8) is a union of at most

$$((20n)^n M^2)^r \leqslant (80n)^{nd}$$

classes such that for any n vectors $\mathbf{x}_1, \dots, \mathbf{x}_n$ in the same class,

$$|\det(\mathbf{a}_w(\mathbf{x}_1), \dots, \mathbf{a}_w(\mathbf{x}_n))| \leqslant M^{-1} \text{ for } w = w_1, \dots, w_r. \tag{4.9}$$

We prove that the vectors $\mathbf{x} \in K^n$ with (4.8) belonging to the same class lie in a single proper linear subspace of K^n, i.e., that any n such vectors have zero determinant. This clearly suffices.

Let $\mathbf{x}_1, \dots, \mathbf{x}_n$ be vectors from K^n that satisfy (4.8) and lie in the same class. Let w be an infinite place of K. Then using (4.9) instead of Hadamard's inequality, we obtain, instead of (4.5),

$$\| \det(\mathbf{x}_1, \dots, \mathbf{x}_n)\|_w \leqslant \Delta_w^{-1} M^{-s(w)} H_{1w} \cdots H_{nw}.$$

For the finite places w of K, we still have (4.6). Then by taking the product over $w \in M_K$, we obtain, with a similar computation as in the proof of Proposition 4.2, employing our choice $M = 2^n$,

$$\prod_{w \in M_K} \| \det(\mathbf{x}_1, \dots, \mathbf{x}_n)\|_w < M^{-1}(2A^{-\delta/2})^n \leqslant 1.$$

Hence $\det(x_1, \dots, x_n) = 0$. This completes our proof. □

In the proofs of Theorems 2.1 and 2.3 we keep the assumptions (2.4)–(2.10).

Deduction of Theorem 2.1 from Theorem 2.3. Define

$$\mathcal{S}_Q := \{\mathbf{x} \in \overline{\mathbb{Q}}^n : H_{\mathcal{L},\mathbf{c},Q}(\mathbf{x}) \leqslant \Delta_{\mathcal{L}}^{1/n} Q^{-\delta}\}.$$

Theorem 2.3 implies that if Q is a real such that

$$Q \geqslant C_0 = \max(H_{\mathcal{L}}^{1/R}, n^{1/\delta}), \quad \mathcal{S}_Q \not\subset T,$$

then

$$Q \in \bigcup_{h=1}^{m_0} \bigcup_{k=1}^{s} \left[Q_h^{(1+\delta/2)^{k-1}}, Q_h^{(1+\delta/2)^k} \right),$$

where s is the integer with $(1+\delta/2)^{s-1} < \omega_0 \leqslant (1+\delta/2)^s$. Notice that we have a union of at most

$$m_0 s \leqslant m_0 \left(1 + \frac{\log \omega_0}{\log(1+\delta/2)}\right) \leqslant 3\delta^{-1} m_0 (1 + \log \omega_0)$$

intervals. By Proposition 4.2, for each of these intervals I, the set $\bigcup_{Q \in I} \mathcal{S}_Q$ lies in a proper linear subspace of $\overline{\mathbb{Q}}^n$, which is defined over K. Taking into consideration also the exceptional subspace T, it follows that for the number t_0 of subspaces in Theorem 2.1, we have

$$\begin{aligned} t_0 &\leqslant 1 + 3\delta^{-1} m_0 (1 + \log \omega_0) \\ &\leqslant 10^6 2^{2n} n^{10} \delta^{-3} \log(3\delta^{-1} R) \log(\delta^{-1} \log 3R). \end{aligned}$$

This proves Theorem 2.1. □

Proof of Theorem 2.2. We distinguish between $Q \in [n^{1/\delta}, C_0)$ and $Q \in [1, n^{1/\delta})$.

Completely similarly as above, we have

$$[n^{1/\delta}, C_0) \subseteq \bigcup_{j=1}^{s_1} [n^{(1+\delta/2)^{j-1}/\delta}, n^{(1+\delta/2)^j/\delta}) \quad (j = 1, \ldots, s_1),$$

where $n^{(1+\delta/2)^{s_1-1}/\delta} < C_0 \leqslant n^{(1+\delta/2)^{s_1}/\delta}$; i.e.,

$$s_1 = 1 + \left[\frac{\log(\delta \log C_0 / \log n)}{\log(1+\delta/2)} \right] \leqslant 2 + 3\delta^{-1} \log\log 3H_{\mathcal{L}}^{1/R}. \tag{4.10}$$

By Proposition 4.2, for each of the s_1 intervals I on the right-hand side, the set $\left(\bigcup_{Q \in I} \mathcal{S}_Q\right) \cap K^n$ lies in a proper linear subspace of K^n.

Next consider Q with $1 \leqslant Q < n^{1/\delta}$. Define $\gamma_0 := 0$, $\gamma_k := 1 + \gamma_{k-1}(1+\delta/2)$ for $k = 1, 2, \ldots$; i.e.,

$$\gamma_k := \frac{(1+\delta/2)^k - 1}{\delta/2} \quad \text{for } k = 0, 1, 2, \ldots .$$

Then

$$[1, n^{1/\delta}) \subseteq \bigcup_{k=1}^{s_2} [2^{\gamma_{k-1}}, 2^{\gamma_k})$$

where $(1+\delta/2)^{s_2-1} < \frac{\log(2n^{1/2})}{\log 2} \leqslant (1+\delta/2)^{s_2}$; i.e.,

$$s_2 = 1 + \left[\frac{\log\left(\log(2n^{1/2})/\log 2\right)}{\log(1+\delta/2)} \right] < 4\delta^{-1} \log\log 4n^{1/2}. \tag{4.11}$$

Applying Proposition 4.4 with $A = 2^{\gamma_{k-1}}$ ($k = 1, \dots, s_2$), we see that for each of the s_2 intervals I on the right-hand side, there is a collection of at most $(80n)^{nd}$ proper linear subspaces of K^n such that for every $Q \in I$, the set $\mathcal{S}_Q \cap K^n$ is contained in one of these subspaces.

Taking into consideration (4.10), (4.11), it follows that for the number of subspaces t_1 in Theorem 2.2, we have

$$\begin{aligned} t_1 &\leqslant s_1 + (80n)^{nd} s_2 \leqslant 2 + 3\delta^{-1} \log\log 3H_{\mathcal{L}}^{1/R} + (80n)^{nd} \cdot 4\delta^{-1} \log\log 4n^{1/2} \\ &< \delta^{-1}\big((90n)^{nd} + 3\log\log 3H_{\mathcal{L}}^{1/R}\big). \end{aligned}$$

This proves Theorem 2.2. $\square$

5. Proofs of Theorems 3.1 and 3.3

5.1. We use the notation introduced in Section 3 and keep the assumptions (3.1)–(3.6). Further, for $L = \sum_{i=1}^n \alpha_i X_i \in \overline{\mathbb{Q}}[X_1, \dots, X_n]^{\mathrm{lin}}$ and $\sigma \in G_K$, we put $\sigma(L) := \sum_{i=1}^n \sigma(\alpha_i) X_i$.

Fix a finite Galois extension $K' \subset \overline{\mathbb{Q}}$ of K such that all linear forms $L_i^{(v)}$ ($v \in S$, $i = 1, \dots, n$) have their coefficients in K'. Recall that for every $v \in M_K$, we have chosen a continuation of $\|\cdot\|_v$ to $\overline{\mathbb{Q}}$. Thus, for every $v' \in M_{K'}$, there is $\tau_{v'} \in \mathrm{Gal}(K'/K)$ such that $\|\alpha\|_{v'} = \|\tau_{v'}(\alpha)\|_v^{d(v'|v)}$ for $\alpha \in K'$, where v is the place of K lying below v'. Put

$$L_i^{(v)} := X_i,\ d_{iv} := 0 \text{ for } v \in M_K \setminus S,\ i = 1, \dots, n \tag{5.1}$$

and then,

$$L_i^{(v')} := \tau_{v'}^{-1}(L_i^{(v)}), \quad c_{i,v'} := d(v'|v) \cdot \frac{n}{n+\varepsilon}\left(d_{iv} - \frac{1}{n}\sum_{j=1}^n d_{jv}\right) \tag{5.2}$$
$$\text{for } v' \in M_{K'},\ i = 1, \dots, n,$$

$$\begin{cases} \mathcal{L} := (L_i^{(v')} : v' \in M_{K'},\ i = 1, \dots, n), \\ \mathbf{c} := (c_{i,v'} : v' \in M_{K'},\ i = 1, \dots, n), \end{cases} \tag{5.3}$$

and finally,

$$\delta := \frac{\varepsilon}{n+\varepsilon}. \tag{5.4}$$

Clearly,

$$\begin{aligned} &c_{1,v'} = \dots = c_{n,v'} = 0 \text{ for all but finitely many } v' \in M_{K'}, \\ &\sum_{j=1}^n c_{j,v'} = 0 \text{ for } v \in M_{K'}. \end{aligned}$$

Moreover, by (5.1), (5.2), (3.5), (3.4),

$$\Big(\sum_{v'\in M_{K'}} \max_{1\leqslant i\leqslant n} c_{i,v'}\Big) \leqslant 1. \tag{5.5}$$

By (5.1), (3.2), we have

$$\#\bigcup_{v'\in M_{K'}} \{L_1^{(v')},\dots,L_n^{(v')}\} \leqslant RD+n. \tag{5.6}$$

These considerations show that (2.4)–(2.10) are satisfied with K' in place of K, with the choices of $\mathcal{L},\mathbf{c},\delta$ from (5.1)–(5.4), and with $RD+n$ in place of R. Further,

$$\Delta_{\mathcal{L}} = \prod_{v'\in M_{K'}} \|\det(L_1^{(v')},\dots,L_n^{(v')})\|_{v'} = \prod_{v\in S} \|\det(L_1^{(v)},\dots,L_n^{(v)})\|_v, \tag{5.7}$$

$$H_{\mathcal{L}} = \prod_{v'\in M_{K'}} \max_{1\leqslant i_1<\cdots<i_n\leqslant r} \|\det(L_{i_1},\dots,L_{i_n})\|_{v'}, \tag{5.8}$$

where $\bigcup_{v'\in M_{K'}}\{L_1^{(v')},\dots,L_n^{(v')}\} =: \{L_1,\dots,L_r\}$.

By (3.2) and the fact that conjugate linear forms have the same inhomogeneous height, we have

$$\max_{1\leqslant i\leqslant r} H^*(L_i) = H^*. \tag{5.9}$$

For $v'\in M_{K'}$, $1\leqslant i_1<\cdots<i_n\leqslant r$ we have, by Hadamard's inequality if v' is infinite and the ultrametric inequality if v' is finite, that

$$\|\det(L_{i_1},\dots,L_{i_n})\|_{v'} \leqslant D_{v'}\prod_{i=1}^{r}\max(1,\|L_i\|_{v'}),$$

where $D_{v'} := n^{n[K'_{v'}:\mathbb{R}]/2[K':\mathbb{Q}]}$ if v' is infinite and $D_{v'} := 1$ if v' is finite. Taking the product over $v'\in M_{K'}$, noting that by (5.1), (5.9), the set $\{L_1,\dots,L_r\}$ contains $X_1,\dots,X_n$, which have inhomogeneous height 1, and at most DR other linear forms of inhomogeneous height $\leqslant H^*$, we obtain

$$H_{\mathcal{L}} \leqslant n^{n/2}H^*(L_1)\cdots H^*(L_r) \leqslant n^{n/2}(H^*)^{DR}. \tag{5.10}$$

The next lemma links system (3.7) to a twisted height inequality.

Lemma 5.1. *Let $\mathbf{x}\in\overline{\mathbb{Q}}^n$ be a solution of (3.7). Then with $\mathcal{L}$, $\mathbf{c}$, δ as defined by (5.1)–(5.4) and with*

$$Q := H(\mathbf{x})^{1+\varepsilon/n},$$

we have

$$H_{\mathcal{L},\mathbf{c},Q}(\sigma(\mathbf{x})) \leqslant \Delta_{\mathcal{L}}^{1/n}Q^{-\delta} \text{ for } \sigma\in G_K.$$

Proof. Let $\sigma \in G_K$. Put $A_v := 1$ for $v \in M_K \setminus S$. Pick a finite Galois extension E of K containing K' and the coordinates of $\sigma(\mathbf{x})$. Let $w \in M_E$ lie above $v' \in M_{K'}$ and the latter in turn above $v \in M_K$. In accordance with (2.14) we define $L_i^{(w)} := L_i^{(v')}$, $c_{iw} := d(w|v')c_{i,v'}$ for $i = 1, \dots, n$. Further, we put $d_{iw} := d(w|v)d_{iv}$, $A_w := A_v^{d(w|v)}$, and we choose $\tau_w \in \mathrm{Gal}(\overline{\mathbb{Q}}/K)$ such that $\tau_w|_{K'} = \tau_v$ and

$$\|\alpha\|_w = \|\tau_w(\alpha)\|_v^{d(w|v)} \text{ for } \alpha \in E. \tag{5.11}$$

Then (5.1), (5.2) imply for $i = 1, \dots, n$,

$$L_i^{(w)} = \tau_w^{-1}(L_i^{(v)}), \qquad c_{iw} = \frac{n}{n+\varepsilon}\left(d_{iw} - \frac{1}{n}\sum_{j=1}^n d_{jw}\right). \tag{5.12}$$

If $v \in S$, then from (3.7) it follows that

$$\frac{\|L_i^{(w)}(\sigma(\mathbf{x}))\|_w}{\|\sigma(\mathbf{x})\|_w} = \left(\frac{\|L_i^{(v)}(\tau_w\sigma(\mathbf{x}))\|_v}{\|\tau_w\sigma(\mathbf{x})\|_v}\right)^{d(w|v)} \leqslant A_w H(\mathbf{x})^{d_{iw}}, \tag{5.13}$$

while if $v \notin S$, we have $A_w = 1$ and $L_i^{(w)} = X_i$, $d_{iw} = 0$ for $i = 1, \dots, n$, and so the inequality is trivially true. Finally, (3.5), (3.6), (5.7) imply

$$\sum_{w \in M_E}\sum_{i=1}^n d_{iw} = -n - \varepsilon, \quad \prod_{w \in M_E} A_w = \Delta_{\mathcal{L}}^{1/n}. \tag{5.14}$$

By our choice of Q and by (5.12), (5.13), we have

$$\begin{aligned}\|L_i^{(w)}(\sigma(\mathbf{x}))\|_w Q^{-c_{iw}} &= \|L_i^{(w)}(\sigma(\mathbf{x}))\|_w H(\mathbf{x})^{-d_{iw} + \frac{1}{n}\sum_{j=1}^n d_{jw}} \\ &\leqslant A_w \|\sigma(\mathbf{x})\|_w H(\mathbf{x})^{\frac{1}{n}\sum_{j=1}^n d_{jw}}.\end{aligned}$$

By taking the product over w, using $H(\sigma(\mathbf{x})) = H(\mathbf{x})$, (5.14) and again our choice of Q we arrive at

$$H_{\mathcal{L},\mathbf{c},Q}(\sigma(\mathbf{x})) \leqslant \Delta_{\mathcal{L}}^{1/n} H(\mathbf{x})^{1-1-\varepsilon/n} = \Delta_{\mathcal{L}}^{1/n} Q^{-\delta}. \qquad \square$$

In addition, we need the following easy observation, which is stated as a lemma for convenient reference.

Lemma 5.2. *Let m, m' be integers and $A_0, B_0, \omega, \omega'$ reals with $B_0 \geqslant A_0 \geqslant 1$, $\omega' \geqslant \omega > 1$ and $m' \geqslant m > 0$, and let $A_1, \dots, A_m$ be reals with $A_0 \leqslant A_1 < \cdots < A_m$. Then there are reals $B_1, \dots, B_{m'}$ with $B_0 \leqslant B_1 < \cdots < B_{m'}$ such that*

$$[1, A_0) \cup \left(\bigcup_{h=1}^m [A_h, A_h^{\omega})\right) \subseteq [1, B_0) \cup \left(\bigcup_{h=1}^{m'} \left[B_h, B_h^{\omega'}\right)\right).$$

Proof. Let $S := \bigcup_{h=1}^{m} [A_h, A_h^{\omega}) \cup [A_m^{\omega}, \infty)$. It is easy to see that the lemma is satisfied with B_1 the smallest real in S with $B_1 \geqslant B_0$ and B_j the smallest real in S outside $\bigcup_{h=1}^{j-1} \left[B_h, B_h^{\omega'}\right)$ for $j = 2, \ldots, m'$. □

Proof of Theorem 3.1. We apply Theorem 2.1 with K' instead of K and with $\mathcal{L}, \mathbf{c}, \delta$ as in (5.1)–(5.4); according to (5.6) we could have taken $n + DR$, but instead we take $6(DR)^2$ instead of R. Then by (5.10) the quantity C_0 in Theorem 2.1 becomes

$$C_0' := \max(H_{\mathcal{L}}^{1/6(RD)^2}, n^{1/\delta}) \leqslant \max\left(\left(n^{n/2}(H^*)^{RD}\right)^{1/6(RD)^2}, n^{1/\delta}\right)$$
$$\leqslant \left(\max((H^*)^{1/3RD}, n^{n/\varepsilon})\right)^{1+\varepsilon/n} = H_0^{1+\varepsilon/n}$$

and the upper bound for the number of subspaces t_0 in Theorem 2.1 becomes

$$\begin{aligned} &10^6 2^{2n} n^{10}(1 + n\varepsilon^{-1})^3 \times \\ &\qquad \times \log\left(18(1 + n\varepsilon^{-1})(RD)^2\right) \log\left((1 + n\varepsilon^{-1}) \log(18(RD)^2)\right) \\ &\leqslant 10^9 2^{2n} n^{14} \varepsilon^{-3} \log\left(3\varepsilon^{-1} RD\right) \log\left(\varepsilon^{-1} \log 3RD\right)\Big), \end{aligned}$$

which is precisely the upper bound for the number of subspaces in Theorem 3.1.

Let $\mathbf{x} \in \overline{\mathbb{Q}}^n$ be a solution to (3.7) with $H(\mathbf{x}) \geqslant H_0$ and put $Q := H(\mathbf{x})^{1+\varepsilon/n}$. Then $Q \geqslant C_0'$. Moreover, by Lemma 5.1 and Theorem 2.1 we have

$$\{\sigma(\mathbf{x}) : \sigma \in G_K\} \subseteq \{\mathbf{y} \in \overline{\mathbb{Q}}^n : H_{\mathcal{L},\mathbf{c},Q}(\mathbf{y}) \leqslant \Delta_{\mathcal{L}}^{1/n} Q^{-\delta}\} \subset T_i$$

for some $T_i \in \{T_1, \ldots, T_{t_0}\}$. But then, in fact, we have that $\mathbf{x} \in T_i' := \bigcap_{\sigma \in G_K} \sigma(T_i)$, which is a proper linear subspace of $\overline{\mathbb{Q}}^n$ defined over K. We infer that the solutions $\mathbf{x} \in \overline{\mathbb{Q}}^n$ of (3.7) with $H(\mathbf{x}) \geqslant H_0$ lie in a union $T_1' \cup \cdots \cup T_{t_0}'$ of proper linear subspaces of $\overline{\mathbb{Q}}^n$, defined over K. This completes our proof. □

Proof of Theorem 3.3. We apply Theorem 2.3 with K' instead of K, with $\mathcal{L}, \mathbf{c}, \delta$ as in (5.1)–(5.4) and with $6(DR)^2$ instead of R. An easy computation shows that with these choices, the expressions for m_0, ω_0 in Theorem 2.3 are bounded above by the quantities m_1, ω_1 from the statement of Theorem 3.3. Further, C_0 becomes a quantity bounded above by $C_1^{1+\varepsilon/n}$. Now according to Theorem 2.3 and Lemmas 5.1, 5.2, there are reals $Q_1, \ldots, Q_{m_1}$ with $C_1^{1+\varepsilon/n} \leqslant Q_1 < \cdots < Q_{m_1}$ such that if $\mathbf{x} \in \overline{\mathbb{Q}}^n$ is a solution to (3.7) outside the subspace $T = T(\mathcal{L}, \mathbf{c})$ from Theorem 2.3, then

$$Q := H(\mathbf{x})^{1+\varepsilon/n} \in \left[1, C_1^{1+\varepsilon/n}\right) \cup \bigcup_{h=1}^{m_1} [Q_h, Q_h^{\omega_1}).$$

So with $H_i := Q_i^{(1+\varepsilon/n)^{-1}}$ $(i = 1, \ldots, m_1)$, we have

$$H(\mathbf{x}) \in [1, C_1) \cup \bigcup_{h=1}^{m_1} [H_h, H_h^{\omega_1}).$$

In fact, $H(\mathbf{x})$ belongs to the above union of intervals if $\sigma(\mathbf{x}) \notin T$ for any $\sigma \in G_K$ so, in fact, already if $\mathbf{x} \notin T' := \bigcap_{\sigma\in G_K} \sigma(T)$. Now T' is a proper $\overline{\mathbb{Q}}$-linear subspace of $\overline{\mathbb{Q}}^n$ defined over K and T' is effectively determinable in terms of T. The space T in turn is effectively determinable and belongs to a finite collection depending only on $\{L_i^{(v')} : v' \in M_{K'},\ i = 1,\dots,n\}$, so ultimately only on $\{L_i^{(v)} : v \in S,\ i = 1,\dots,n\}$. Hence the same must apply to T'. This completes our proof. $\square$

6. Notation and simple facts

We have collected some notation and simple facts for later reference. We fix an algebraic number field $K \subset \overline{\mathbb{Q}}$ and use v to index places on K. We have to deal with varying finite extensions $E \subset \overline{\mathbb{Q}}$ of K and sometimes with varying towers $K \subset F \subset E \subset \overline{\mathbb{Q}}$; then places on E are indexed by w and places on F by u. Completions are denoted by K_v, E_w, F_u, etc. We use notation $w|u$, $u|v$ to indicate that w lies above u, u above v. If $w|v$, we put

$$d(w|v) := \frac{[E_w : K_v]}{[E : K]}.$$

6.1. *Norms and heights.* Let E be any algebraic number field. If w is an infinite place of E, there is an embedding $\sigma_w : E \hookrightarrow \mathbb{C}$ such that $\|\cdot\|_w = |\sigma_w(\cdot)|^{[E_w:\mathbb{R}]/[E:\mathbb{Q}]}$. If w is a finite place of E lying above the prime p, then $\|\cdot\|_w$ is an extension of $|\cdot|_p^{[E_w:\mathbb{Q}_p]/[E:\mathbb{Q}]}$ to E.

To handle infinite and finite places simultaneously, we introduce

$$s(w) := \tfrac{[E_w:\mathbb{R}]}{[E:\mathbb{Q}]} \text{ if } w \text{ is infinite, } s(w) := 0 \text{ if } w \text{ is finite.} \tag{6.1}$$

Thus, for $x_1,\dots,x_n \in E$, $a_1,\dots,a_n \in \mathbb{Z}$, $w \in M_E$, we have

$$\|a_1x_1 + \cdots + a_nx_n\|_w \leqslant \Big(\sum_{i=1}^n |a_i|\Big)^{s(w)} \cdot \max(\|x_1\|_w,\dots,\|x_n\|_w). \tag{6.2}$$

Let $\mathbf{x} = (x_1,\dots,x_n) \in E^n$. Put

$$\|\mathbf{x}\|_w := \max(\|x_1\|_w,\dots,\|x_n\|_w) \text{ for } w \in M_E,$$

$$\left.\begin{aligned} \|\mathbf{x}\|_{w,1} &:= \Big(\sum_{i=1}^n |\sigma_w(x_i)|\Big)^{s(w)} \\ \|\mathbf{x}\|_{w,2} &:= \Big(\sum_{i=1}^n |\sigma_w(x_i)|^2\Big)^{s(w)/2} \end{aligned}\right\} \text{ for } w \in M_E,\ w \text{ infinite,}$$

$$\|\mathbf{x}\|_{w,1} = \|\mathbf{x}\|_{w,2} := \|\mathbf{x}\|_w \ \text{ for } w \in M_E,\ w \text{ finite.}$$

Now for $\mathbf{x} \in \overline{\mathbb{Q}}^n$, we define

$$H(\mathbf{x}) := \prod_{w\in M_E} \|\mathbf{x}\|_w,\quad H_1(\mathbf{x}) := \prod_{w\in M_E} \|\mathbf{x}\|_{w,1},\quad H_2(\mathbf{x}) := \prod_{w\in M_E} \|\mathbf{x}\|_{w,2},$$

where E is any number field such that $\mathbf{x} \in E^n$. This is independent of the choice of E. Then

$$n^{-1}H_1(\mathbf{x}) \leqslant n^{-1/2}H_2(\mathbf{x}) \leqslant H(\mathbf{x}) \leqslant H_2(\mathbf{x}) \leqslant H_1(\mathbf{x}) \text{ for } \mathbf{x} \in \overline{\mathbb{Q}}^n. \tag{6.3}$$

The standard inner product of $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{y} = (y_1, \dots, y_n) \in \overline{\mathbb{Q}}^n$ is defined by $\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^n x_i y_i$. Let again E be an arbitrary number field and $w \in M_E$. Then by the Cauchy-Schwarz inequality for the infinite places and the ultrametric inequality for the finite places,

$$\|\mathbf{x} \cdot \mathbf{y}\|_w \leqslant \|\mathbf{x}\|_{w,2} \cdot \|\mathbf{y}\|_{w,2} \text{ for } \mathbf{x}, \mathbf{y} \in E^n,\ w \in M_E. \tag{6.4}$$

If P is a polynomial with coefficients in a number field E or in $\overline{\mathbb{Q}}$, we define $\|P\|_w$, $\|P\|_{w,1}$, $\|P\|_{w,2}$, $H(P)$, $H_1(P)$, $H_2(P)$ by applying the above definitions to the vector $\mathbf{x}$ of coefficients of P. Then for $P_1, \dots, P_r \in E[X_1, \dots, X_m]$, $w \in M_E$, we have

$$\begin{cases} \|P_1 + \cdots + P_r\|_{w,1} \leqslant r^{s(w)} \max(\|P_1\|_{w,1}, \dots, \|P_r\|_{w,1}), \\ \|P_1 \cdots P_r\|_{w,1} \leqslant \|P_1\|_{w,1} \cdots \|P_r\|_{w,1}. \end{cases} \tag{6.5}$$

6.2. *Exterior products.* Let n be an integer $\geqslant 2$ and p an integer with $1 \leqslant p < n$. Put $N := \binom{n}{p}$. Denote by $C(n,p)$ the sequence of p-element subsets of $\{1, \dots, n\}$, ordered lexicographically, i.e., $C(n,p) = (I_1, \dots, I_N)$, where

$$\begin{aligned} I_1 &= \{1, \dots, p\},\ I_2 = \{1, \dots, p-1, p+1\}, \dots, \\ I_{N-1} &= \{n-p, n-p+2, \dots, n\},\ I_N = \{n-p+1, \dots, n\}. \end{aligned}$$

We use shorthand notation $I = \{i_1 < \cdots < i_p\}$ for a set $I = \{i_1, \dots, i_p\}$ with $i_1 < \cdots < i_p$.

We denote by $\det(a_{ij})_{i,j=1,\dots,p}$ the $p \times p$-determinant with a_{ij} on the i-th row and j-th column. The exterior product of $\mathbf{x}_1 = (x_{11}, \dots, x_{1n}), \dots, \mathbf{x}_p = (x_{p1}, \dots, x_{pn}) \in \overline{\mathbb{Q}}^n$ is given by

$$\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_p := (A_1, \dots, A_N),$$

where

$$A_l := \det(x_{i,i_j})_{i,j=1,\dots,p},$$

with $\{i_1 < \cdots < i_p\} = I_l$ the l-th set in the sequence $C(n,p)$, for $l = 1, \dots, N$.

Let $\mathbf{x}_1, \dots, \mathbf{x}_n$ be linearly independent vectors from $\overline{\mathbb{Q}}^n$. For $l = 1, \dots, N$, define $\widehat{\mathbf{x}_l} := \mathbf{x}_{i_1} \wedge \cdots \wedge \mathbf{x}_{i_p}$, where $I_l = \{i_1 < \cdots < i_p\}$ is the l-th set in $C(n,p)$. Then

$$\det(\widehat{\mathbf{x}}_1, \dots, \widehat{\mathbf{x}}_N) = \pm\Big(\det(\mathbf{x}_1, \dots, \mathbf{x}_n)\Big)^{\binom{n-1}{p-1}}. \tag{6.6}$$

Given a number field E such that $\mathbf{x}_1, \dots, \mathbf{x}_p \in E^n$ we have, by Hadamard's inequality for the infinite places and the ultrametric inequality for the finite places,

$$\|\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_p\|_{w,2} \leqslant \|\mathbf{x}_1\|_{w,2} \cdots \|\mathbf{x}_p\|_{w,2} \text{ for } w \in M_E. \tag{6.7}$$

Hence,

$$H_2(\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_p) \leqslant H_2(\mathbf{x}_1) \cdots H_2(\mathbf{x}_p) \text{ for } \mathbf{x}_1, \dots, \mathbf{x}_p \in \overline{\mathbb{Q}}^n. \tag{6.8}$$

The above definitions and inequalities are carried over to linear forms by identifying a linear form $L = \sum_{j=1}^n a_j X_j = \mathbf{a} \cdot \mathbf{X} \in \overline{\mathbb{Q}}[X_1, \dots, X_n]^{\mathrm{lin}}$ with its coefficient vector $\mathbf{a} = (a_1, \dots, a_n)$; e.g., $\|L\|_w := \|\mathbf{a}\|_w$, $H(L) := H(\mathbf{a})$. The exterior product of $L_i = \sum_{j=1}^n a_{ij} X_j = \mathbf{a}_i \cdot \mathbf{X} \in \overline{\mathbb{Q}}[X_1, \dots, X_n]^{\mathrm{lin}}$ $(i = 1, \dots, p)$ is defined by

$$L_1 \wedge \cdots \wedge L_p := A_1 X_1 + \cdots + A_N X_N,$$

where $(A_1, \dots, A_N) = \mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_p$. Analogously to (6.8) we have for any linear forms $L_1, \dots, L_p \in \overline{\mathbb{Q}}[X_1, \dots, X_n]^{\mathrm{lin}}$ $(1 \leqslant p \leqslant n)$,

$$H_2(L_1 \wedge \cdots \wedge L_p) \leqslant H_2(L_1) \cdots H_2(L_p). \tag{6.9}$$

Finally, for any $L_1, \dots, L_p \in \overline{\mathbb{Q}}[X_1, \dots, X_n]^{\mathrm{lin}}$, $\mathbf{x}_1, \dots, \mathbf{x}_p \in \overline{\mathbb{Q}}^n$, we have

$$(L_1 \wedge \cdots \wedge L_p)(\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_p) = \det(L_i(\mathbf{x}_j))_{1 \leqslant i,j \leqslant p}. \tag{6.10}$$

6.3. *Heights of subspaces.* Let T be a linear subspace of $\overline{\mathbb{Q}}^n$. The height $H_2(T)$ of T is given by $H_2(T) := 1$ if $T = \{\mathbf{0}\}$ or $\overline{\mathbb{Q}}^n$ and

$$H_2(T) := H_2(\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_p)$$

if T has dimension p with $0 < p < n$ and $\{\mathbf{x}_1, \dots, \mathbf{x}_p\}$ is any basis of T. This is independent of the choice of the basis. Thus, by (6.8), if $\{\mathbf{x}_1, \dots, \mathbf{x}_p\}$ is any basis of T, then

$$H_2(T) \leqslant H_2(\mathbf{x}_1) \cdots H_2(\mathbf{x}_p). \tag{6.11}$$

By a result of Struppeck and Vaaler [29], we have for any two linear subspaces T_1, T_2 of $\overline{\mathbb{Q}}^n$,

$$\begin{aligned} \max\big(H_2(T_1 \cap T_2),\, H_2(T_1 + T_2)\big) &\leqslant H_2(T_1 \cap T_2) H_2(T_1 + T_2) \\ &\leqslant H_2(T_1) H_2(T_2). \end{aligned} \tag{6.12}$$

Given a linear subspace V of $\overline{\mathbb{Q}}[X_1, \dots, X_n]^{\mathrm{lin}}$, we define $H_2(V) := 1$ if $V = \{\mathbf{0}\}$ or $\overline{\mathbb{Q}}[X_1, \dots, X_n]^{\mathrm{lin}}$ and $H_2(V) := H_2(L_1 \wedge \cdots \wedge L_p)$ otherwise, where $\{L_1, \dots, L_p\}$ is any basis of V.

Let T be a linear subspace of $\overline{\mathbb{Q}}^n$. Denote by $T^\perp$ the $\overline{\mathbb{Q}}$-vector space of linear forms $L \in \overline{\mathbb{Q}}[X_1, \dots, X_n]^{\mathrm{lin}}$ such that $L(\mathbf{x}) = 0$ for all $\mathbf{x} \in T$. Then ([19, p. 433])

$$H_2(T^\perp) = H_2(T). \tag{6.13}$$

We finish with the following lemma.

LEMMA 6.1. *Let T be a k-dimensional linear subspace of $\overline{\mathbb{Q}}^n$. Put $p := n-k$. Let $\{\mathbf{g}_1, \dots, \mathbf{g}_n\}$ be a basis of $\overline{\mathbb{Q}}^n$ such that $\{\mathbf{g}_1, \dots, \mathbf{g}_k\}$ is a basis of T. For $j = 1, \dots, N$, put $\widehat{\mathbf{g}}_j := \mathbf{g}_{i_1} \wedge \cdots \wedge \mathbf{g}_{i_p}$, where $\{i_1 < \cdots < i_p\} = I_j$ is the j-th set in the sequence $C(n,p)$. Let $\widehat{T}$ be the linear subspace of $\overline{\mathbb{Q}}^N$ spanned by $\widehat{\mathbf{g}}_1, \dots, \widehat{\mathbf{g}}_{N-1}$. Then*

$$H_2(\widehat{T}) = H_2(T).$$

Proof. Let $L_1, \dots, L_n \in \overline{\mathbb{Q}}[X_1, \dots, X_n]^{\mathrm{lin}}$ such that for $i, j = 1, \dots, n$, we have $L_i(\mathbf{g}_j) = 1$ if $i = j$ and 0 otherwise. Then $\{L_{k+1}, \dots, L_n\}$ is a basis of $T^\perp$. Moreover, by (6.10), we have

$$(L_{k+1} \wedge \cdots \wedge L_n)(\widehat{\mathbf{g}}_j) = 0$$

for $j = 1, \dots, N-1$. Hence $L_{k+1} \wedge \cdots \wedge L_n$ spans $\widehat{T}^\perp$. Now a repeated application of (6.13) gives

$$H_2(\widehat{T}) = H_2(\widehat{T}^\perp) = H_2(L_{k+1} \wedge \cdots \wedge L_n) = H_2(T^\perp) = H_2(T). \qquad \square$$

7. Simple properties of twisted heights

We fix tuples $\mathcal{L} = (L_i^{(v)} : v \in M_K, i = 1, \dots, n)$, $\mathbf{c} = (c_{iv} : v \in M_K, i = 1, \dots, n)$ satisfying the minimal requirements needed to define the twisted height $H_{\mathcal{L},\mathbf{c},Q}$; that is, (2.4)–(2.7). Further, $\Delta_{\mathcal{L}}$, $H_{\mathcal{L}}$ are defined by (2.11), (2.12), respectively. Write $\bigcup_{v \in M_K} \{L_1^{(v)}, \dots, L_n^{(v)}\} = \{L_1, \dots, L_r\}$, and let $d_1, \dots, d_t$ be the nonzero numbers among

$$\det(L_{i_1}, \dots, L_{i_n}) \quad (1 \leqslant i_1 < \cdots < i_n \leqslant r). \tag{7.1}$$

Then

$$\prod_{v \in M_K} \max(\|d_1\|_v, \dots, \|d_t\|_v) = H_{\mathcal{L}}. \tag{7.2}$$

Clearly,

$$\prod_{v \in M_K} \min(\|d_1\|_v, \dots, \|d_t\|_v) \geqslant \prod_{v \in M_K} \frac{\|d_1 \cdots d_t\|_v}{(\max(\|d_1\|_v, \dots, \|d_t\|_v))^{t-1}}$$

and so, invoking the product formula and $t \leqslant \binom{r}{n}$,

$$\prod_{v \in M_K} \min(\|d_1\|_v, \dots, \|d_t\|_v) \geqslant H_{\mathcal{L}}^{1-\binom{r}{n}}. \tag{7.3}$$

Consequently, for the quantity $\Delta_{\mathcal{L}}$ given by (2.15), we have

$$H_{\mathcal{L}}^{1-\binom{r}{n}} \leqslant \Delta_{\mathcal{L}} \leqslant H_{\mathcal{L}}. \tag{7.4}$$

LEMMA 7.1. *Put* $\theta := \sum_{v\in M_K}\max(c_{1v},\dots,c_{nv})$. *Let* $Q\geqslant 1$, $\mathbf{x}\in\overline{\mathbb{Q}}^n$, $\mathbf{x}\neq\mathbf{0}$. *Then*

$$H_{\mathcal{L},\mathbf{c},Q}(\mathbf{x})\geqslant n^{-1}H_{\mathcal{L}}^{-\binom{r}{n}}Q^{-\theta}.$$

Proof. Let E be a finite extension of K with $\mathbf{x}\in E^n$. Assume without loss of generality that $L_1,\dots,L_n$ (from $\{L_1,\dots,L_r\}$ defined above) are linearly independent, and put $\delta_w := \det(L_1^{(w)},\dots,L_n^{(w)})$ for $w\in M_E$. Note that also $\sum_{w\in M_E}\max_i c_{iw}=\theta$. We may write

$$L_i=\sum_{j=1}^n\gamma_{ijw}L_j^{(w)}\quad\text{for } w\in M_E,\ i=1,\dots,n,$$

with $\gamma_{ijw}\in K$. By Cramer's rule, we have $\gamma_{ijw}=\delta_{ijw}/\delta_w$, where δ_{ijw} is the determinant obtained from δ_w by replacing $L_j^{(w)}$ by L_i. So δ_{ijw} belongs to the set of numbers in (7.1). Further, $\prod_{w\in M_E}\|\delta_w\|_w=\Delta_{\mathcal{L}}$. Now (7.4) gives

$$\prod_{w\in M_E}\max_{1\leqslant i,j\leqslant n}\|\gamma_{ijw}\|_w\leqslant\Delta_{\mathcal{L}}^{-1}H_{\mathcal{L}}\leqslant H_{\mathcal{L}}^{\binom{r}{n}}.$$

Put $\mathbf{y}:=(L_1(\mathbf{x}),\dots,L_n(\mathbf{x}))$. Then, noting that $\mathbf{y}\neq\mathbf{0}$,

$$\begin{aligned}1\leqslant H(\mathbf{y})&\leqslant nH_{\mathcal{L}}^{\binom{r}{n}}\prod_{w\in M_E}\max_{1\leqslant i\leqslant n}\|L_i^{(w)}(\mathbf{x})\|_w\\&\leqslant nH_{\mathcal{L}}^{\binom{r}{n}}Q^{\theta}\prod_{w\in M_E}\max_{1\leqslant i\leqslant n}\|L_i^{(w)}(\mathbf{x})\|_wQ^{-c_{iw}}=nH_{\mathcal{L}}^{\binom{r}{n}}Q^{\theta}H_{\mathcal{L},\mathbf{c},Q}(\mathbf{x}).\end{aligned}$$

This proves our lemma. □

LEMMA 7.2. *Let* θ_v $(v\in M_K)$ *be reals, at most finitely many of which are nonzero. Put* $\Theta:=\sum_{v\in M_K}\theta_v$. *Define* $\mathbf{d}=(d_{iv}: v\in M_K,\ i=1,\dots,n)$ *by* $d_{iv}:=c_{iv}-\theta_v$ *for* $v\in M_K$, $i=1,\dots,n$.

(i) *Let* $\mathbf{x}\in\overline{\mathbb{Q}}^n$, $Q\geqslant 1$. *Then*

$$H_{\mathcal{L},\mathbf{d},Q}(\mathbf{x})=Q^{\Theta}H_{\mathcal{L},\mathbf{c},Q}(\mathbf{x}).$$

(ii) *Let* U *be a linear subspace of* $\overline{\mathbb{Q}}^n$. *Then*

$$w_{\mathcal{L},\mathbf{d}}(U)=w_{\mathcal{L},\mathbf{c}}(U)-\Theta\dim U.$$

(iii) $T(\mathcal{L},\mathbf{d})=T(\mathcal{L},\mathbf{c})$.

Proof. (i) Choose a finite extension E of K with $\mathbf{x}\in E^n$. In accordance with our conventions, we put $\theta_w:=d(w|v)\theta_v$ if $w\in M_E$ lies above $v\in M_K$; thus, $\sum_{w\in M_E}\theta_w=\sum_{v\in M_K}\theta_v$. The lemma now follows trivially by considering the factors of the twisted heights for $w\in M_E$ and taking the product.

(ii) is obvious, and (iii) is an immediate consequence of (ii). □

For $L \in \overline{\mathbb{Q}}[X_1, \dots, X_n]^{\mathrm{lin}}$ and a linear map

$$\varphi : \overline{\mathbb{Q}}^m \to \overline{\mathbb{Q}}^n : (x_1, \dots, x_m) \mapsto (\sum_{j=1}^m a_{1j}x_j, \dots, \sum_{j=1}^m a_{nj}x_j),$$

we define $L \circ \varphi \in \overline{\mathbb{Q}}[X_1, \dots, X_m]^{\mathrm{lin}}$ by

$$L \circ \varphi := L(\sum_{j=1}^m a_{1j}X_j, \dots, \sum_{j=1}^m a_{nj}X_j).$$

If $L \in K[X_1, \dots, X_n]^{\mathrm{lin}}$ and φ is defined over K, i.e., $a_{ij} \in K$ for all i, j, we have $L \circ \varphi \in K[X_1, \dots, X_m]^{\mathrm{lin}}$. More generally, for a system of linear forms $\mathcal{L} = (L_i^{(v)} : v \in M_K, i = 1, \dots, n)$, we put $\mathcal{L} \circ \varphi := (L_i^{(v)} \circ \varphi : v \in M_K, i = 1, \dots, n)$.

Lemma 7.3. *Let $(\mathcal{L}, \mathbf{c})$ be a pair satisfying (2.4)–(2.7), and let $\varphi : \overline{\mathbb{Q}}^n \to \overline{\mathbb{Q}}^n$ be an invertible linear map defined over K.*

(i) *Let $\mathbf{x} \in \overline{\mathbb{Q}}^n$, $Q \geqslant 1$. Then $H_{\mathcal{L}\circ\varphi,\mathbf{c},Q}(\mathbf{x}) = H_{\mathcal{L},\mathbf{c},Q}(\varphi(\mathbf{x}))$.*
(ii) *Let U be a proper linear subspace of $\overline{\mathbb{Q}}^n$. Then $w_{\mathcal{L}\circ\varphi,\mathbf{c}}(U) = w_{\mathcal{L},\mathbf{c}}(\varphi(U))$.*
(iii) *Let $T(\mathcal{L} \circ \varphi, \mathbf{c})$ be the subspace defined by (2.21), but with $\mathcal{L} \circ \varphi$ instead of φ. Then $T(\mathcal{L} \circ \varphi, \mathbf{c}) = \varphi^{-1}(T(\mathcal{L}, \mathbf{c}))$.*
(iv) $\Delta_{\mathcal{L}\circ\varphi} = \Delta_{\mathcal{L}}$, $H_{\mathcal{L}\circ\varphi} = H_{\mathcal{L}}$.

Proof. (i), (ii) are trivial. (iii) is a consequence of (ii). As for (iv), we have by the product formula that

$$\Delta_{\mathcal{L}\circ\varphi} = \prod_{v \in M_K} \Big(\| \det(\varphi) \|_v \cdot \| \det(L_1^{(v)}, \dots, L_n^{(v)} \|_v \Big) = \Delta_{\mathcal{L}}$$

and likewise, $H_{\mathcal{L}\circ\varphi} = H_{\mathcal{L}}$. □

Remark. A consequence of this lemma is that in order to prove Theorem 2.3, it suffices to prove it for $\mathcal{L} \circ \varphi$ instead of $\mathcal{L}$, where φ is any linear transformation of $\overline{\mathbb{Q}}^n$ defined over K. For instance, pick any $v_0 \in M_K$ and choose φ such that $L_i^{(v_0)} \circ \varphi = X_i$ for $i = 1, \dots, n$. Thus, we see that in the proof of Theorem 2.3 we may assume without loss of generality that $L_i^{(v_0)} = X_i$ for $i = 1, \dots, n$. It will be convenient to choose v_0 such that v_0 is non-archimedean and $c_{i,v_0} = 0$ for $i = 1, \dots, n$.

8. An interval result in the semistable case

We formulate an interval result like Theorem 2.3, but under some additional constraints. We keep the notation and assumptions from Section 2. Thus K is an algebraic number field, and n, $\mathcal{L} = (L_i^{(v)} : v \in M_K, i = 1, \dots, n)$, $\mathbf{c} = (c_{iv} : v \in M_K, i = 1, \dots, n)$, δ, R satisfy (2.4)–(2.10). Further, we add the condition as discussed in the above remark.

The weight $w(U) = w_{\mathcal{L},\mathbf{c}}(U)$ of a $\overline{\mathbb{Q}}$-linear subspace U of $\overline{\mathbb{Q}}^n$ is defined by (2.20). In addition to the above, we assume that the pair $(\mathcal{L}, \mathbf{c})$ is *semistable*; that is, the exceptional space $T = T(\mathcal{L}, \mathbf{c})$ defined by (2.21) is equal to $\{\mathbf{0}\}$.

For reference purposes, we have listed all our conditions below. Thus, K is an algebraic number field, n is a positive integer, δ, R are reals, $\mathcal{L} = (L_i^{(v)} : v \in M_K, i = 1, \ldots, n)$ is a tuple of linear forms and $\mathbf{c} = (c_{iv} : v \in M_K, i = 1, \ldots, n)$ is a tuple of reals satisfying the following conditions:

$$R \geqslant n \geqslant 2, \quad 0 < \delta \leqslant 1, \tag{8.1}$$

$$c_{1v} = \cdots = c_{nv} = 0 \text{ for all but finitely many } v \in M_K, \tag{8.2}$$

$$\sum_{i=1}^{n} c_{iv} = 0 \text{ for } v \in M_K, \tag{8.3}$$

$$\sum_{v \in M_K} \max(c_{1v}, \ldots, c_{nv}) \leqslant 1, \tag{8.4}$$

$$L_i^{(v)} \in K[X_1, \ldots, X_n]^{\mathrm{lin}} \text{ for } v \in M_K,\ i = 1, \ldots, n, \tag{8.5}$$

$$\{L_1^{(v)}, \ldots, L_n^{(v)}\} \text{ is linearly independent for } v \in M_K, \tag{8.6}$$

$$\# \bigcup_{v \in M_K} \{L_1^{(v)}, \ldots, L_n^{(v)}\} \leqslant R, \tag{8.7}$$

$$\begin{array}{l}\text{there is a non-archimedean place } v_0 \in M_K \text{ such that} \\ c_{i,v_0} = 0,\ L_i^{(v_0)} = X_i \text{ for } i = 1, \ldots, n,\end{array} \tag{8.8}$$

$$w(U) \leqslant 0 \text{ for every proper linear subspace } U \text{ of } \overline{\mathbb{Q}}^n. \tag{8.9}$$

Notice that (8.9) is equivalent to the assumption that the space T defined by (2.21) is $\{\mathbf{0}\}$.

THEOREM 8.1. *Assume* (8.1)–(8.9). *Put*

$$\begin{cases} m_2 := \left[61 n^6 2^{2n} \delta^{-2} \log(22 n^2 2^n R/\delta)\right], \\ \omega_2 := m_2^{5/2}, \quad C_2 := (2H_{\mathcal{L}})^{m_2^{2m_2}}. \end{cases} \tag{8.10}$$

Then there are reals $Q_1, \ldots, Q_{m_2}$ *with*

$$C_2 \leqslant Q_1 < \cdots < Q_{m_2}$$

such that for every $Q \geqslant 1$ *with*

$$\left\{\mathbf{x} \in \overline{\mathbb{Q}}^n : H_{\mathcal{L},\mathbf{c},Q}(\mathbf{x}) \leqslant Q^{-\delta}\right\} \neq \{\mathbf{0}\}, \tag{8.11}$$

we have $Q \in [1, C_2) \cup \bigcup_{h=1}^{m_2} \left[Q_h, Q_h^{\omega_2}\right)$.

The factor $\Delta_{\mathcal{L}}^{1/n}$ occurring in (2.24) has been absorbed into C_2. Theorem 8.1 may be viewed as an extension and refinement of a result of Schmidt on general Roth systems [20, Th. 2].

Theorem 8.1 is proved in Sections 9–14. In Sections 15–18 we deduce Theorem 2.3.

We outline how Theorem 2.3 is deduced from Theorem 8.1. Let again $T = T(\mathcal{L}, \mathbf{c})$ be the exceptional subspace for $(\mathcal{L}, \mathbf{c})$. Put $k := \dim T$. With the notation used in Sections 15–18, we construct a surjective homomorphism $\varphi'' : \overline{\mathbb{Q}}^n \to \overline{\mathbb{Q}}^{n-k}$ defined over K with kernel T, a tuple $\mathcal{L}'' := (L_i^{(v)\prime\prime} : v \in M_K, i = 1, \ldots, n-k)$ in $K[X_1, \ldots, X_{n-k}]^{\mathrm{lin}}$ and a tuple of reals $\mathbf{d} = (d_{iv} : v \in M_K, i = 1, \ldots, n-k)$ such that $(\mathcal{L}'', \mathbf{d})$ satisfies conditions analogous to (8.1)–(8.9) and

$$H_{\mathcal{L}'',\mathbf{d},Q'}(\varphi''(\mathbf{x})) \ll H_{\mathcal{L},\mathbf{c},Q}(\mathbf{x}) \ \text{ for } \mathbf{x} \in \overline{\mathbb{Q}}^n,\ Q \geqslant C_2,$$

where $Q' = Q^n$. Then Theorem 8.1 is applied with $\mathcal{L}''$ and $\mathbf{d}$.

An important ingredient in the deduction of Theorem 2.3 is an upper bound for the height $H_2(T)$ of T. In fact, in Sections 15 and 16 we prove a limit result for the successive infima for $H_{\mathcal{L},\mathbf{c},Q}$ (Theorem 16.1) where we need Theorem 8.1. We use this limit result in Section 17 to compute an upper bound for $H_2(T)$. In Section 18 we complete the proof of Theorem 2.3.

9. **Geometry of numbers for twisted heights**

We start with some generalities on twisted heights. Let K be a number field and $n \geqslant 2$. Let $(\mathcal{L}, \mathbf{c})$ be a pair for which for the moment we require only (2.4)–(2.7).

For $\lambda \in \mathbb{R}_{\geqslant 0}$, define $T(Q, \lambda) = T(\mathcal{L}, \mathbf{c}, Q, \lambda)$ to be the $\overline{\mathbb{Q}}$-vector space generated by

$$\{\mathbf{x} \in \overline{\mathbb{Q}}^n : H_{\mathcal{L},\mathbf{c},Q}(\mathbf{x}) \leqslant \lambda\}.$$

We define the successive infima $\lambda_i(Q) = \lambda_i(\mathcal{L}, \mathbf{c}, Q)$ $(i = 1, \ldots, n)$ of $H_{\mathcal{L},\mathbf{c},Q}$ by

$$\lambda_i(Q) := \inf\{\lambda \in \mathbb{R}_{\geqslant 0} : \dim T(Q, \lambda) \geqslant i\}.$$

Since we are working over $\overline{\mathbb{Q}}$, the successive infima need not be minima. For $i = 1, \ldots, n$, we define

$$T_i(Q) = T_i(\mathcal{L}, \mathbf{c}, Q) = \bigcap_{\lambda > \lambda_i(Q)} T(Q, \lambda).$$

We insert the following simple lemma.

Lemma 9.1. *Let $(\mathcal{L}, \mathbf{c})$ be any pair with (2.4)–(2.7), and let $Q \geqslant 1$.*

(i) *The spaces $T_1(Q), \ldots, T_n(Q)$ are defined over K.*

(ii) *Let $k \in \{1, \ldots, n-1\}$, and suppose that $\lambda_k(Q) < \lambda_{k+1}(Q)$. Then $\dim T_k(Q) = k$ and $T(Q, \lambda) = T_k(Q)$ for all λ with $\lambda_k(Q) < \lambda < \lambda_{k+1}(Q)$.*

Proof. (i) Lemma 4.1 implies that for any $\lambda \in \mathbb{R}_{>0}$ and any $\sigma \in G_K$, we have $\sigma(T(Q,\lambda)) = T(Q,\lambda)$. Hence $T(Q,\lambda)$ is defined over K. This implies (i) at once.

(ii) From the definition of the successive infima it follows at once that $\dim T(Q,\lambda) = k$ for all λ with $\lambda_k(Q) < \lambda < \lambda_{k+1}(Q)$. Since also $T(Q,\lambda) \subseteq T(Q,\lambda')$ if $\lambda \leqslant \lambda'$, this implies (ii). $\square$

The quantity $\Delta_{\mathcal{L}}$ is defined by (2.11). We recall the following analogue of Minkowski's Theorem.

Proposition 9.2. *Let again* $(\mathcal{L}, \mathbf{c})$ *be any pair with* (2.4)–(2.7). *Put*

$$\alpha := \sum_{v \in M_K} \sum_{i=1}^{n} c_{iv}.$$

Then for $Q \geqslant 1$, *we have*

$$n^{-n/2} \Delta_{\mathcal{L}} Q^{-\alpha} \leqslant \lambda_1(Q) \cdots \lambda_n(Q) \leqslant 2^{n(n-1)/2} \Delta_{\mathcal{L}} Q^{-\alpha}. \tag{9.1}$$

In particular, if $\alpha = 0$, *then*

$$n^{-n/2} \Delta_{\mathcal{L}} \leqslant \lambda_1(Q) \cdots \lambda_n(Q) \leqslant 2^{n(n-1)/2} \Delta_{\mathcal{L}}. \tag{9.2}$$

Proof. This is a reformulation of [11, Cor. 7.2]. In fact, this result is an easy consequence of an analogue over $\overline{\mathbb{Q}}$ of Minkowski's Theorem on successive minima, due to Roy and Thunder [16]. Using instead an Arakelov type result of S. Zhang [31], it is possible to improve $2^{n(n-1)/2}$ to $(cn)^n$ for some absolute constant c, but such a strengthening would not have any effect on our final result. $\square$

From now on, we assume that $n, \delta, R, \mathcal{L}, \mathbf{c}$ satisfy (8.1)–(8.9). We consider reals Q with

$$Q \geqslant C_2, \tag{9.3}$$

where C_2 is given by (8.10), and with (8.11); i.e.,

$$\lambda_1(Q) \leqslant Q^{-\delta}. \tag{9.4}$$

Our assumptions imply $\alpha = 0$, and so (9.2) holds. We deduce some consequences.

Lemma 9.3. *Suppose* $n, \delta, R, \mathcal{L}, \mathbf{c}$ *satisfy* (8.1)–(8.9) *and* Q *satisfies* (9.3), (9.4). *Let* $i_1, \dots, i_p$ *be distinct indices from* $\{1, \dots, n\}$. *Then*

$$Q^{-p-\frac{1}{2}} \leqslant \lambda_{i_1}(Q) \cdots \lambda_{i_p}(Q) \leqslant Q^{n-p+\frac{1}{2}}.$$

Proof. Write λ_i for $\lambda_i(Q)$. Lemma 7.1 and the conditions (8.7) (i.e., $r \leqslant R$), (8.4) and (9.3) imply

$$\lambda_1 \geqslant n^{-1} H_{\mathcal{L}}^{-\binom{R}{n}} Q^{-1} \geqslant Q^{-1-1/(3n)}.$$

This implies at once the lower bound for $\lambda_{i_1} \cdots \lambda_{i_p}$. Further, by (9.2), the upper bound for $\Delta_{\mathcal{L}}$ in (9.4) and again (9.3),

$$\lambda_{i_1} \cdots \lambda_{i_p} \leqslant 2^{n(n-1)/2} \Delta_{\mathcal{L}} \lambda_1^{p-n} \leqslant 2^{n(n-1)/2} H_{\mathcal{L}} \lambda_1^{p-n} \leqslant Q^{n-p+\frac{1}{2}}. \qquad \square$$

Lemma 9.4. *Suppose again that* $n, R, \delta, \mathcal{L}, \mathbf{c}$ *satisfy* (8.1)–(8.9) *and that* Q *satisfies* (9.3), (9.4). *Then there is* $k \in \{1, \dots, n-1\}$ *such that*

$$\lambda_k(Q) \leqslant Q^{-\delta/(n-1)} \lambda_{k+1}(Q).$$

Proof. Fix Q with (9.3), (9.4). Write λ_i for $\lambda_i(Q)$, for $i = 1, \dots, n$. Then by (9.2), the lower bound for $\Delta_{\mathcal{L}}$ in (7.4) and (9.4), (9.3),

$$\lambda_n \geqslant \left(n^{-n/2} \Delta_{\mathcal{L}} \lambda_1^{-1}\right)^{1/(n-1)} \geqslant \left(n^{-n/2} H_{\mathcal{L}}^{1-\binom{R}{n}} Q^{\delta}\right)^{1/(n-1)} \geqslant 1.$$

Take $k \in \{1, \dots, n-1\}$ such that $\lambda_k / \lambda_{k+1}$ is minimal. Then

$$\frac{\lambda_k}{\lambda_{k+1}} \leqslant \left(\frac{\lambda_1}{\lambda_n}\right)^{1/(n-1)} \leqslant \lambda_1^{1/(n-1)} \leqslant Q^{-\delta/(n-1)}. \qquad \square$$

10. A lower bound for the height of the k-th infimum subspace

Our aim is to deduce a useful lower bound for the height of the vector space $T_k(Q)$, where k is the index from Lemma 9.4. It is only at this point where we have to use our semistability assumption (8.9).

We need some lemmas, which are used also elsewhere. In the usual manner, we write

$$\bigcup_{v \in M_K} \{L_1^{(v)}, \dots, L_n^{(v)}\} = \{L_1, \dots, L_r\}.$$

The quantity $H_{\mathcal{L}}$ is given by (2.12).

Lemma 10.1. *Assume that* $\mathcal{L}$ *contains* $X_1, \dots, X_n$. *Let* $\{d_1, \dots, d_m\}$ *be the set consisting of* 1, *all determinants* $\det(L_{i_1}, \dots, L_{i_n})$ $(1 \leqslant i_1 < \cdots < i_n \leqslant r)$, *and all subdeterminants of order* $\leqslant n$ *of these determinants. Then*

$$\prod_{v \in M_K} \max(\|d_1\|_v, \dots, \|d_m\|_v) = H_{\mathcal{L}}.$$

Proof. Pick indices $1 \leqslant i_1 < \cdots < i_n \leqslant r$. Each of the subdeterminants of $\det(L_{i_1}, \dots, L_{i_n})$ can be expressed as a determinant of n linear forms from $L_{i_1}, \dots, L_{i_n}, X_1, \dots, X_n$. Since $X_1, \dots, X_n \in \{L_1, \dots, L_r\}$, these subdeterminants are up to sign in the set of determinants $\det(L_{i_1}, \dots, L_{i_n})$ $(1 \leqslant i_1 < \cdots < i_n \leqslant r)$. Now the lemma is clear from (2.12). $\square$

Lemma 10.2. *Let $\mathcal{L}$, $\mathbf{c}$ satisfy (2.4)–(2.7), and suppose in addition that $\mathcal{L}$ contains $X_1, \dots, X_n$. Let T be a k-dimensional linear subspace of $\overline{\mathbb{Q}}^n$ and $\{\mathbf{g}_1, \dots, \mathbf{g}_k\}$ a basis of T. Let E be a finite extension of K such that $\mathbf{g}_i \in E^n$ for $i = 1, \dots, k$.*

Let $\theta_1, \dots, \theta_u$ be the distinct nonzero numbers among

$$\det(L_{i_l}(\mathbf{g}_j))_{l,j=1,\dots,k} \quad (1 \leqslant i_1 < \cdots < i_k \leqslant r).$$

Then

$$\prod_{w \in M_E} \max(\|\theta_1\|_w, \dots, \|\theta_u\|_w) \leqslant \binom{n}{k}^{1/2} H_{\mathcal{L}} \cdot H_2(T), \tag{10.1}$$

$$\prod_{w \in M_E} \min(\|\theta_1\|_w, \dots, \|\theta_u\|_w) \geqslant \left(\binom{n}{k}^{1/2} H_{\mathcal{L}} \cdot H_2(T)\right)^{1-\binom{r}{k}}. \tag{10.2}$$

Proof. For $w \in M_E$, put

$$G_w := \|\mathbf{g}_1 \wedge \cdots \wedge \mathbf{g}_k\|_{w,2}, \quad H_w := \max(\|d_1\|_w, \dots, \|d_m\|_w),$$

where $\{d_1, \dots, d_m\}$ is the set from Lemma 10.1. Thus,

$$\prod_{w \in M_E} G_w = H_2(T), \quad \prod_{w \in M_E} H_w = H_{\mathcal{L}}. \tag{10.3}$$

Let $\{L_{i_1}, \dots, L_{i_k}\}$ be a k-element subset of $\{L_1, \dots, L_r\}$. Then the coefficients of $L_{i_1} \wedge \cdots \wedge L_{i_k}$ (being subdeterminants of order k) belong to $\{d_1, \dots, d_m\}$. Now (6.10), (6.4) imply for $w \in M_E$ that

$$\begin{aligned}
\| \det (L_{i_l}(\mathbf{g}_j)_{1 \leqslant l,j \leqslant k} \|_w &= \|(L_{i_1} \wedge \cdots \wedge L_{i_k}) \cdot (\mathbf{g}_1 \wedge \cdots \wedge \mathbf{g}_k)\|_w \\
&\leqslant \|L_{i_1} \wedge \cdots \wedge L_{i_k}\|_{w,2} \cdot \|\mathbf{g}_1 \wedge \cdots \wedge \mathbf{g}_k\|_{w,2} \\
&\leqslant \binom{n}{k}^{s(w)/2} H_w G_w.
\end{aligned}$$

By taking the maximum over all tuples $i_1, \dots, i_k$ and then the product over $w \in M_E$, and using (10.3), inequality (10.1) follows.

By the product formula,

$$\begin{aligned}
\prod_{w \in M_E} \min(\|\theta_1\|_w, \dots, \|\theta_u\|_w) &\geqslant \prod_{w \in M_E} \frac{\|\theta_1 \cdots \theta_u\|_w}{\max(\|\theta_1\|_w, \dots, \|\theta_u\|_w)^{u-1}} \\
&= \left(\prod_{w \in M_E} \max(\|\theta_1\|_w, \dots, \|\theta_u\|_w)\right)^{1-u},
\end{aligned}$$

and together with (10.1), $u \leqslant \binom{r}{k}$, this implies (10.2). □

We now deduce our lower bound for the height of the vector space $T_k(Q)$.

Lemma 10.3. *Let* $n, R, \delta, \mathcal{L}, \mathbf{c}$ *satisfy* (8.1)–(8.9), *and let* Q *satisfy* (9.3), (9.4). *Further, let* k *be the index from* Lemma 9.4. *Then*

$$H_2(T_k(Q)) \geqslant Q^{\delta/3R^n}.$$

Proof. Put $T := T_k(Q)$ and $\lambda_i := \lambda_i(Q)$ for $i = 1, \dots, n$. Let $v \in M_K$. Choose $\{i_1(v), \dots, i_k(v)\} \subset \{1, \dots, n\}$ such that the restrictions to T of the linear forms $L^{(v)}_{i_1(v)}, \dots, L^{(v)}_{i_k(v)}$ are linearly independent and

$$w_v(T) = \sum_{l=1}^{k} c_{i_l(v),v}.$$

Then by assumption (8.9),

$$\sum_{v \in M_K} \sum_{l=1}^{k} c_{i_l(v),v} = w(T) \leqslant 0.$$

Given any finite extension E of K and $w \in M_E$, define $i_l(w) := i_l(v)$ for $l = 1, \dots, k$, where v is the place of K below w. Then by (2.3), (2.14), we have

$$\sum_{w \in M_E} \sum_{l=1}^{k} c_{i_l(w),w} \leqslant 0. \tag{10.4}$$

Choose ε such that

$$0 < \varepsilon < 1, \qquad (1+\varepsilon)\lambda_k < \lambda_{k+1}. \tag{10.5}$$

Then there are linearly independent vectors $\mathbf{g}_1, \dots, \mathbf{g}_k \in T$ such that

$$H_{\mathcal{L},\mathbf{c},Q}(\mathbf{g}_j) \leqslant (1+\varepsilon)\lambda_j \ \text{ for } j = 1, \dots, k. \tag{10.6}$$

Let E be a finite extension of K such that $\mathbf{g}_j \in E^n$ for $j = 1, \dots, k$. Put

$$H_{jw} := \max_{1 \leqslant i \leqslant n} \|L^{(w)}_i(\mathbf{g}_j)\|_w Q^{-c_{iw}} \text{ for } w \in M_E,\ j = 1, \dots, k.$$

Thus,

$$\|L^{(w)}_i(\mathbf{g}_j)\|_w \leqslant H_{jw} Q^{c_{iw}} \text{ for } w \in M_E,\ i = 1, \dots, n,\ j = 1, \dots, k. \tag{10.7}$$

For $w \in M_E$, put

$$\theta_w := \det\left(L^{(w)}_{i_l(w)}(\mathbf{g}_j)\right)_{1 \leqslant l,j \leqslant k}.$$

We estimate from above and below $\prod_{w \in M_E} \|\theta_w\|_w$. We start with the upper bound. Let $w \in M_E$. From (10.7), using the triangle inequality if w is infinite and the ultrametric inequality if w is finite, we deduce

$$\|\theta_w\|_w \leqslant (k!)^{s(w)} H_{1w} \cdots H_{kw} Q^{\sum_{l=1}^{k} c_{i_l(w),w}}.$$

By taking the product over $w \in M_E$ and inserting (10.6), (10.4),

$$\prod_{w\in M_E} \|\theta_w\|_w \leqslant k! H_{\mathcal{L},\mathbf{c},Q}(\mathbf{g}_1)\cdots H_{\mathcal{L},\mathbf{c},Q}(\mathbf{g}_k) Q^{\sum_{w\in M_E}\sum_{l=1}^{k} c_{i_l(w),w}} \leqslant k!(1+\varepsilon)^k \lambda_1\cdots\lambda_k \leqslant 2^k k! \lambda_1\cdots\lambda_k. \tag{10.8}$$

By Lemma 9.4 and (10.5), we have

$$\lambda_1\cdots\lambda_k \leqslant (\lambda_1\cdots\lambda_k)^{k/n}\Big(Q^{-\delta/(n-1)}\lambda_{k+1}\Big)^{k(n-k)/n} \leqslant Q^{-k(n-k)\delta/n(n-1)}(\lambda_1\cdots\lambda_n)^{k/n}.$$

Applying (9.2) and using the upper bound in (7.4) for $\Delta_{\mathcal{L}}$, we obtain

$$\lambda_1\cdots\lambda_k \leqslant 2^{k(n-1)/2}\Delta_{\mathcal{L}}^{k/n} Q^{-k(n-k)\delta/n(n-1)} \leqslant 2^{k(n-1)/2} H_{\mathcal{L}}^{k/n} Q^{-k(n-k)\delta/n(n-1)},$$

and inserting the latter into (10.8) and using assumption (9.3) leads us to the upper bound

$$\prod_{w\in M_E} \|\theta_w\|_w \leqslant Q^{-\delta/2n}.$$

From (10.2) we conclude at once that

$$\prod_{w\in M_E} \|\theta_w\|_w \geqslant \left(\binom{n}{k}^{1/2} H_{\mathcal{L}}\cdot H_2(T)\right)^{1-\binom{R}{k}}.$$

A combination with the upper bound just established and again our assumption (9.3) gives $H_2(T) \geqslant Q^{\delta/3R^n}$, as required. □

11. **Inequalities in an exterior power**

Letting Q be a real with (9.3), (9.4), k the index from Lemma 9.4, and $N := \binom{n}{k}$, we construct $N-1$ linearly independent vectors $\widehat{\mathbf{h}}_1(Q),\ldots,\widehat{\mathbf{h}}_{N-1}(Q) \in \wedge^{n-k}\overline{\mathbb{Q}}^n \cong \overline{\mathbb{Q}}^N$ satisfying an appropriate system of inequalities. The construction is similar to that of [11]; the basic tool is Davenport's Lemma.

In the subsequent sections, Theorem 8.1 is proved by applying the Roth machinery to our system of inequalities. More precisely, we recall a nonvanishing result in Section 12, and construct a suitable auxiliary polynomial P in Section 13. Assuming Theorem 8.1 is false, we show that the nonvanishing result is applicable to P, and with the inequalities derived in the present section and the properties of P, we derive a contradiction.

We start with recalling [11, Lemma 6.3].

Lemma 11.1. *Let F be any algebraic number field and A_u ($u \in M_F$) positive reals such that*

$$A_u = 1 \textit{ for all but finitely many } u \in M_F; \quad \prod_{u\in M_F} A_u > 1.$$

Then there exist a finite extension E of F, and $\alpha \in E^*$, such that

$$\|\alpha\|_w \leqslant A_w \text{ for } w \in M_E,$$

where we have written $A_w := A_u^{d(w|u)}$, with u the place of F below w.

We keep the notation and assumptions from Sections 8, 9. Thus, $n \geqslant 2$, K is an algebraic number field and $\mathcal{L}$, $\mathbf{c}$, R, δ satisfy (8.1)–(8.9). We fix a real number $Q \geqslant 1$. Temporarily, we write λ_i for the i-th successive infimum $\lambda_i(Q)$ of $H_{\mathcal{L},\mathbf{c},Q}$ ($i = 1, \dots, n$). For a subset $\mathcal{S}$ of $\overline{\mathbb{Q}}^n$, we denote by $\operatorname{span}\mathcal{S}$ the $\overline{\mathbb{Q}}$-vector space generated by $\mathcal{S}$.

Let v_0 be the place from (8.7). Given a finite extension E of K, we write $w \in M_E$, $w|v_0$ to indicate that we let w run through all places of E lying above v_0, and we write $w \in M_E$, $w \nmid v_0$ to indicate that we let w run through all places of E not lying above v_0.

Choose $\varepsilon > 0$ such that

$$\begin{cases} (1+\varepsilon)^2\lambda_i < \lambda_{i+1} \text{ for each } i \text{ with } \lambda_i < \lambda_{i+1}, \\ (1+\varepsilon)^{n+1} \cdot n \cdot 2^{n^2} < 3^{n^2}. \end{cases} \tag{11.1}$$

Then choose linearly independent vectors $\mathbf{g}_1, \dots, \mathbf{g}_n$ of $\overline{\mathbb{Q}}^n$ such that

$$H_{\mathcal{L},\mathbf{c},Q}(\mathbf{g}_i) \leqslant (1 + \tfrac{1}{2}\varepsilon)\lambda_i \text{ for } i = 1, \dots, n. \tag{11.2}$$

Lemma 11.2. *There exist a finite extension E of K, and scalar multiples $\mathbf{g}_1', \dots, \mathbf{g}_n'$ of $\mathbf{g}_1, \dots, \mathbf{g}_n$, respectively, having their coordinates in E, such that*

$$\|L_i^{(w)}(\mathbf{g}_j')\|_w \leqslant n^{-s(w)}Q^{c_{iw}} \quad (i, j = 1, \dots, n,\ w \in M_E,\ w \nmid v_0), \tag{11.3}$$

$$\|L_i^{(w)}(\mathbf{g}_j')\|_w \leqslant \big((1+\varepsilon)n\lambda_j\big)^{d(w|v_0)} \quad (i, j = 1, \dots, n,\ w \in M_E,\ w \mid v_0). \tag{11.4}$$

Proof. Choose a finite extension F of K such that $\mathbf{g}_1, \dots, \mathbf{g}_n \in F^n$. For $j \in \{1, \dots, n\}$, put

$$A_{ju} := \begin{cases} n^{-s(u)} \cdot \Big(\max\limits_{1 \leqslant i \leqslant n} \|L_i^{(u)}(\mathbf{g}_j)\|_u Q^{-c_{iu}}\Big)^{-1} \quad (u \in M_F,\ u \nmid v_0), \\ \Big(\dfrac{n(1+\varepsilon)}{1+\frac{1}{2}\varepsilon} \cdot H_{\mathcal{L},\mathbf{c},Q}(\mathbf{g}_j)\Big)^{d(u|v_0)} \cdot \Big(\max\limits_{1 \leqslant i \leqslant n} \|L_i^{(u)}(\mathbf{g}_j)\|_u\Big)^{-1} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad (u \in M_F,\ u \mid v_0). \end{cases}$$

Notice that for $j = 1, \dots, n$, at most finitely many among the numbers A_{ju} ($u \in M_F$) are $\neq 1$, and $\prod_{u \in M_F} A_{ju} > 1$. So we can apply Lemma 11.1 and obtain that there are a finite extension E of F, and $\alpha_1, \dots, \alpha_n \in E^*$, such that

$$\|\alpha_j\|_w \leqslant A_{jw} \text{ for } w \in M_E,\ j = 1, \dots, n,$$

where we have written $A_{jw} := A_{ju}^{d(w|u)}$, with u the place of F below w. As is easily seen, for $j = 1, \dots, n$, we have that

$$A_{jw} := \begin{cases} n^{-s(w)} \cdot \Big(\max\limits_{1 \leqslant i \leqslant n} \| L_i^{(w)}(\mathbf{g}_j) \|_w Q^{-c_{iw}} \Big)^{-1} \quad (w \in M_E,\ w \nmid v_0), \\ \Big(\dfrac{n(1+\varepsilon)}{1+\frac{1}{2}\varepsilon} \cdot H_{\mathcal{L},\mathbf{c},Q}(\mathbf{g}_j) \Big)^{d(w|v_0)} \cdot \Big(\max\limits_{1 \leqslant i \leqslant n} \| L_i^{(w)}(\mathbf{g}_j) \|_w \Big)^{-1} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (w \in M_F,\ w \mid v_0). \end{cases}$$

Together with (11.2) this implies that $\mathbf{g}_j' := \alpha_j \mathbf{g}_j$ ($j = 1, \dots, n$) satisfies (11.3), (11.4). □

Lemma 11.3 (Davenport's Lemma). *There exist a finite extension E of K, a permutation π of $\{1, \dots, n\}$, and vectors $\mathbf{h}_j = \mathbf{h}_j(Q) \in E^n$ $(j = 1, \dots, n)$, with the following properties:*

(11.5) $\quad \operatorname{span}\{\mathbf{h}_1, \dots, \mathbf{h}_j\} = \operatorname{span}\{\mathbf{g}_1, \dots, \mathbf{g}_j\}$ *for* $j = 1, \dots, n$,

(11.6) $\quad \| L_i^{(w)}(\mathbf{h}_j) \|_w \leqslant n^{-s(w)} Q^{c_{iw}}$ $(i, j = 1, \dots, n,\ w \in M_E,\ w \nmid v_0)$,

(11.7) $\quad \| L_{\pi(i)}^{(w)}(\mathbf{h}_j) \|_w \leqslant \big(3^{n^2} \min(\lambda_i, \lambda_j) \big)^{d(w|v_0)}$

$$(i, j = 1, \dots, n,\ w \in M_E,\ w \mid v_0).$$

Proof. The proof is the same as that of [11, Lemma 9.2], except for some small modifications. In fact, starting with $\mathbf{g}_1, \dots, \mathbf{g}_n$, we construct scalar multiples $\mathbf{g}_1', \dots, \mathbf{g}_n'$ as in Lemma 11.2. Then [11, (9.17), (9.18)] hold, but with the vectors $\mathbf{g}_1, \dots, \mathbf{g}_n$ being replaced by $\mathbf{g}_1', \dots, \mathbf{g}_n'$ and the numbers $Q^{c_{iw}}$, $(1+\varepsilon)\lambda_j$ by $n^{-s(w)} Q^{c_{iw}}$ and $(1+\varepsilon)n\lambda_j$, respectively, for $i, j = 1, \dots, n$. We then copy the proof of [11, Lemma 9.2]. Here we have to use (11.1) instead of [11, (9.15)]. This yields vectors $\mathbf{h}_1, \dots, \mathbf{h}_n$ satisfying (11.5), (11.6) and (11.7) with $2^{n^2} n(1+\varepsilon)^{n+1}$ instead of 3^{n^2}. Together with our assumption (11.1) this implies our Lemma 11.3.

In the proof of [11, Lemma 9.2], the tuples $\mathcal{L} = (L_i^{(v)} : v \in M_K, i = 1, \dots, n)$ under consideration satisfy, in addition to (8.7), (8.8), the following conditions: $\| \det(L_1^{(v)}, \dots, L_n^{(v)}) \|_v = 1$ for $v \in M_K$ and $L_1^{(v)} = X_1, \dots, L_n^{(v)} = X_n$ for all but finitely many $v \in M_K$. But these conditions are not used anywhere. □

Let Q be a real with (9.3), (9.4), and let $k \in \{1, \dots, n-1\}$ be the index from Lemma 9.4. That is, Q satisfies

(11.8) $$Q \geqslant C_2, \quad \lambda_1(Q) \leqslant Q^{-\delta},$$

(11.9) $$\lambda_k(Q) \leqslant Q^{-\delta/(n-1)} \lambda_{k+1}(Q).$$

Put $N := \binom{n}{k}$. Let $C(n, n-k) = (I_1, \dots, I_N)$ be the sequence of $(n-k)$-elements subsets of $\{1, \dots, n\}$, arranged in lexicographical order. Thus, $I_1 =$

$\{1,\dots,n-k\}$, $I_2=\{1,\dots,n-k-1,n-k+1\},\dots,I_{N-1}=\{k,k+1,\dots,n\}$, $I_N=\{k+1,\dots,n\}$.

Let $\mathbf{h}_j=\mathbf{h}_j(Q)$ $(j=1,\dots,n)$ be the vectors from Lemma 11.3. For $v\in M_K$, $j=1,\dots,N$, define

$$\widehat{L}_j^{(v)}:=L_{i_1}^{(v)}\wedge\cdots\wedge L_{i_{n-k}}^{(v)},\quad \widehat{c}_{jv}:=c_{i_1,v}+\cdots+c_{i_{n-k},v}, \tag{11.10}$$

$$\widehat{\mathbf{h}}_j=\widehat{\mathbf{h}}_j(Q):=\mathbf{h}_{i_1}(Q)\wedge\cdots\wedge\mathbf{h}_{i_{n-k}}(Q), \tag{11.11}$$

$$\nu_j=\nu_j(Q):=\lambda_{i_1}(Q)\cdots\lambda_{i_{n-k}}(Q), \tag{11.12}$$

where $I_j=\{i_1<\cdots<i_{n-k}\}$. The permutation π from Lemma 11.3 induces a permutation $\widehat{\pi}$ of $\{1,\dots,N\}$ such that if $I_j=\{i_1,\dots,i_{n-k}\}$, then $I_{\widehat{\pi}(j)}=\{\pi(i_1),\dots,\pi(i_{n-k})\}$. In the usual manner, we write

$$\widehat{L}_j^{(w)}=\widehat{L}_j^{(v)},\quad \widehat{c}_{jw}=d(w|v)\widehat{c}_{jv} \tag{11.13}$$

for any place w of any finite extension of K, where v is the place of K below w.

Let E be the finite extension of K from Lemma 11.3. By (6.10), (11.6), (11.7) we have for $w\in M_E$, $i,j=1,\dots,N$,

$$\begin{aligned}\|\widehat{L}_i^{(w)}(\widehat{\mathbf{h}}_j)\|_w&=\|\det\big(L_{i_p}^{(w)}(\mathbf{h}_{j_q})\big)_{1\leqslant p,q\leqslant n-k}\|_w\\&\leqslant\begin{cases}((n-k)!)^{s(w)}n^{-(n-k)s(w)}Q^{\widehat{c}_{iw}}\leqslant Q^{\widehat{c}_{iw}} & \text{if } w\nmid v_0,\\ 3^{n^3}\min\big(\nu_{\widehat{\pi}^{-1}(i)},\,\nu_{\widehat{\pi}^{-1}(j)}\big) & \text{if } w\mid v_0,\end{cases}\end{aligned} \tag{11.14}$$

where $I_i=\{i_1<\cdots<i_{n-k}\}$, $I_j=\{j_1<\cdots<j_{n-k}\}$.

Our concern is about the points $\widehat{\mathbf{h}}_1,\dots,\widehat{\mathbf{h}}_{N-1}$. Define the quantities $\widehat{c}_{i,v_0}(Q)$ $(i=1,\dots,N)$ (so depending on Q (!)) by

$$Q^{\widehat{c}_{i,v_0}(Q)}:=\begin{cases}3^{n^3}\nu_{\widehat{\pi}^{-1}(i)}(Q) \text{ if } \widehat{\pi}^{-1}(i)\neq N,\\ 3^{n^3}\nu_{N-1}(Q) \text{ if } \widehat{\pi}^{-1}(i)=N.\end{cases} \tag{11.15}$$

Next, define

$$\widehat{c}_{iw}(Q):=d(w|v_0)\widehat{c}_{i,v_0}(Q) \tag{11.16}$$

if w is a place of some finite extension of K lying above v_0.

Now (11.14) implies for $i=1,\dots,N$, $j=1,\dots,N-1$,

$$\begin{cases}\|\widehat{L}_i^{(w)}(\widehat{\mathbf{h}}_j)\|_w\leqslant Q^{\widehat{c}_{iw}} \ (w\in M_E,\ w\nmid v_0),\\ \|\widehat{L}_i^{(w)}(\widehat{\mathbf{h}}_j)\|_w\leqslant Q^{\widehat{c}_{iw}(Q)} \ (w\in M_E,\ w\mid v_0).\end{cases} \tag{11.17}$$

We may take the same finite extension E of K as in (11.14) but, in fact, in view of (11.13), (11.16), we may take for E any finite extension of K that contains the coordinates of $\widehat{\mathbf{h}}_1,\dots,\widehat{\mathbf{h}}_{N-1}$. It is a feature of our new approach, as opposed to [11], that it allows us to handle exponents $\widehat{c}_{iw}(Q)$ that vary with Q.

We have collected some properties of the exponents $\widehat{c}_{iv}$, $\widehat{c}_{i,v_0}(Q)$.

LEMMA 11.4. *Let Q be a real with (11.8), (11.9). Put $N := \binom{n}{k}$. Then*

$$\sum_{i=1}^{N} \widehat{c}_{iv} = 0 \quad \textit{for } v \in M_K \setminus \{v_0\}, \tag{11.18}$$

$$\max_{1 \leqslant i \leqslant N} |\widehat{c}_{iv}| \leqslant (n-1) \max_{1 \leqslant i \leqslant n} c_{iv} \quad \textit{for } v \in M_K \setminus \{v_0\}, \tag{11.19}$$

$$\sum_{v \in M_K \setminus \{v_0\}} \max_{1 \leqslant i \leqslant N} |\widehat{c}_{iv}| \leqslant n-1, \tag{11.20}$$

$$\sum_{i=1}^{N} \widehat{c}_{i,v_0}(Q) \leqslant -\delta/n, \tag{11.21}$$

$$\max_{1 \leqslant i \leqslant N} |\widehat{c}_{i,v_0}(Q)| \leqslant n. \tag{11.22}$$

Proof. (11.18), (11.19) and (11.20) are easy consequences of (11.10), (8.2)–(8.4) and the choice of v_0. (11.18) is immediate. For (11.19), observe that

$$|\widehat{c}_{jv}| = \max\left(\sum_{i \in I_j} c_{iv}, \sum_{i \notin I_j} c_{iv}\right) \leqslant (n-1) \max_{1 \leqslant i \leqslant n} c_{iv},$$

and for (11.20), take the sum over v and apply (8.4). We prove (11.21). Write again λ_i, ν_j for $\lambda_i(Q)$, $\nu_j(Q)$, and put $N' := \binom{n-1}{n-k-1}$. Notice that by (11.12), $\nu_{N-1} = \lambda_k \lambda_{k+2} \cdots \lambda_N$, $\nu_N = \lambda_{k+1} \cdots \lambda_N$. Together with (11.15), (9.2), Lemma 9.4, (11.9), and (11.8), this implies

$$\begin{aligned} Q^{\sum_{i=1}^{N} \widehat{c}_{i,v_0}(Q)} &= 3^{n^3 N} \nu_1 \cdots \nu_N (\nu_{N-1}/\nu_N) \\ &= 3^{n^3 N} (\lambda_1 \cdots \lambda_n)^{N'} (\lambda_k/\lambda_{k+1}) \\ &\leqslant 3^{n^3 N} 2^{n(n-1)N'/2} Q^{-\delta/(n-1)} \leqslant Q^{-\delta/n}. \end{aligned}$$

We finish with proving (11.22). Let $i \in \{1, \ldots, N\}$. By (11.12), (11.15), we have

$$Q^{\widehat{c}_{i,v_0}(Q)} = 3^{n^3} \lambda_{i_1} \cdots \lambda_{i_{n-k}}$$

for certain distinct indices $i_1, \ldots, i_{n-k} \in \{1, \ldots, n\}$. Together with Lemma 9.3 and (11.8), this implies

$$Q^{|\widehat{c}_{i,v_0}(Q)|} \leqslant 3^{n^3} Q^{n - \frac{1}{2}} \leqslant Q^n. \qquad \square$$

Next, we prove some properties of the linear forms $\widehat{L}_i^{(v)}$. For $v \in M_K$, denote by $\widehat{A}_v$ the matrix of which the j-th row consists of the coefficients of $\widehat{L}_j^{(v)}$ for $j = 1, \ldots, N$. The inhomogeneous height of a set $\mathcal{S} = \{\alpha_1, \ldots, \alpha_s\} \subset \overline{\mathbb{Q}}$ is given by $H^*(\mathcal{S}) := \prod_{w \in M_E} \max(1, \|\alpha_1\|_w, \ldots, \|\alpha_s\|_w)$, where E is any number field containing $\mathcal{S}$. If $A_1, \ldots, A_m$ are matrices with elements from $\overline{\mathbb{Q}}$, we denote by $H^*(A_1,\dots,A_m)$ the inhomogeneous height of the set of elements of $A_1,\dots,A_m$.

LEMMA 11.5. *Let $\widehat{A}_1,\dots,\widehat{A}_s$ be the distinct matrices among $\widehat{A}_v$ $(v\in M_K)$. Then*

$$H^*(\widehat{A}_1^{-1},\dots,\widehat{A}_s^{-1})\leqslant H_{\mathcal{L}}^{R^n}.$$

Proof. Write $\bigcup_{v\in M_K}\{L_1^{(v)},\dots,L_n^{(v)}\}=\{L_1,\dots,L_r\}$; then $r\leqslant R$. For $i=1,\dots,s$, let $B_i:=(\det\widehat{A}_i)\widehat{A}_i^{-1}$. For $v\in M_K$, put $\delta_v:=\det(L_1^{(v)},\dots,L_n^{(v)})$, and let $\delta_1,\dots,\delta_u$ be the distinct numbers among δ_v $(v\in M_K)$.

Thanks to assumption (8.8), we can apply Lemma 10.1. For $v\in M_K$, the elements of the matrix $(\det\widehat{A}_v)\widehat{A}_v^{-1}$ are up to sign the coefficients of $L_{i_1}^{(v)}\wedge\cdots\wedge L_{i_k}^{(v)}$ for all k-element subsets $\{i_1<\cdots<i_k\}$ of $\{1,\dots,r\}$, and so are up to sign among the set $\{d_1,\dots,d_m\}$ from Lemma 10.1. Hence,

$$H^*(B_1,\dots,B_s)\leqslant H_{\mathcal{L}}.$$

By (6.6), we have $\det\widehat{A}_v=\delta_v^{N'}$ for $v\in M_K$, where $N':=\binom{n-1}{n-k-1}$. Now a combination of (7.3) and the inequality just established gives

$$\begin{aligned}H^*(\widehat{A}_1^{-1},\dots,\widehat{A}_s^{-1})&\leqslant H_{\mathcal{L}}\cdot\prod_{v\in M_K}\max_{i\leqslant i\leqslant u}\|\delta_i\|_v^{-N'}\\&\leqslant H_{\mathcal{L}}^{1+N'(\binom{r}{n}-1)}\leqslant H_{\mathcal{L}}^{R^n}.\end{aligned}$$

This proves our lemma. □

LEMMA 11.6. *Suppose Q satisfies (11.8), (11.9), and put $N:=\binom{n}{k}$. Let $\widehat{T}(Q)$ be the $\overline{\mathbb{Q}}$-vector space spanned by the vectors $\widehat{\mathbf{h}}_1(Q),\dots,\widehat{\mathbf{h}}_{N-1}(Q)$. Then*

$$H_2(\widehat{T}(Q))\geqslant Q^{\delta/3R^n}.$$

Proof. Put $T:=T_k(Q)$, $\widehat{T}:=\widehat{T}(Q)$. We have seen that T is spanned by $\mathbf{h}_1(Q),\dots,\mathbf{h}_k(Q)$. So we may apply Lemma 6.1. Now this lemma together with Lemma 10.3 gives $H_2(\widehat{T})=H_2(T)\geqslant Q^{\delta/3R^n}$. □

12. A nonvanishing result

Let N,m be integers $\geqslant 2$. Below, $\mathbf{i}$, $\mathbf{j}$ will denote mN-tuples $(i_{hl}:h=1,\dots,m,\ j=1,\dots,N)$, $(j_{hl}:h=1,\dots,m,\ j=1,\dots,N)$ of integers, and $\mathbf{i}\pm\mathbf{j}$ will denote their componentwise sum/difference.

We consider polynomials $P\in\overline{\mathbb{Q}}[\mathbf{X}_1,\dots,\mathbf{X}_m]=\overline{\mathbb{Q}}[X_{11},\dots,X_{mN}]$ in m blocks of N variables $\mathbf{X}_h=(X_{h1},\dots,X_{hN})$ $(h=1,\dots,m)$. Such a polynomial P is expressed as

$$P=\sum_{\mathbf{j}}a(\mathbf{j})\mathbf{X}^{\mathbf{j}}\quad\text{with }\mathbf{X}^{\mathbf{j}}:=\prod_{h=1}^{m}\prod_{l=1}^{N}X_{hl}^{j_{hl}},\tag{12.1}$$

where the sum is over a finite set of tuples $\mathbf{j} \in \mathbb{Z}_{\geqslant 0}^{mN}$ and where $a(\mathbf{j}) \in \overline{\mathbb{Q}}$. For a polynomial P as above and for $\mathbf{i} \in \mathbb{Z}_{\geqslant 0}^{mN}$, we define

$$P_{\mathbf{i}} := \left(\prod_{h=1}^{m} \prod_{l=1}^{N} \frac{1}{i_{hl}!} \frac{\partial^{i_{hl}}}{\partial X_{hl}^{i_{hl}}} \right) P.$$

Thus, if P is given by (12.1), then

$$P_{\mathbf{i}} = \sum_{\mathbf{j}} \binom{\mathbf{i}+\mathbf{j}}{\mathbf{i}} a(\mathbf{i}+\mathbf{j}) \mathbf{X}^{\mathbf{j}}, \quad \text{where } \binom{\mathbf{i}+\mathbf{j}}{\mathbf{i}} := \prod_{h=1}^{m} \prod_{l=1}^{N} \binom{i_{hl}+j_{hl}}{i_{hl}}. \tag{12.2}$$

We say that $P \in \overline{\mathbb{Q}}[\mathbf{X}_1, \dots, \mathbf{X}_m]$ is multihomogeneous of degree $(r_1, \dots, r_m)$ if it is homogeneous of degree r_h in block $\mathbf{X}_h$ for $h = 1, \dots, m$, i.e., if in (12.1) the sum is taken over tuples $\mathbf{j} \in \mathbb{Z}_{\geqslant 0}^{mN}$ with $\sum_{l=1}^{N} j_{hl} = r_h$ for $h = 1, \dots, m$.

We write points in $\overline{\mathbb{Q}}^{mN}$ as $(\mathbf{x}_1, \dots, \mathbf{x}_m)$, where $\mathbf{x}_1, \dots, \mathbf{x}_m \in \overline{\mathbb{Q}}^N$. The height $H_2(P)$ of $P \in \overline{\mathbb{Q}}[\mathbf{X}_1, \dots, \mathbf{X}_m]$ is defined as $H_2(\mathbf{a}_P)$, where $\mathbf{a}_P$ is a vector consisting of the nonzero coefficients of P.

Let T be a finite dimensional $\overline{\mathbb{Q}}$-vector space and B a positive integer. By a *grid of size B in T* we mean a set of the shape

$$\left\{ \sum_{i=1}^{d} x_i \mathbf{a}_i : \, x_i \in \mathbb{Z}, \, |x_i| \leqslant B \text{ for } i = 1, \dots, d \right\},$$

where $d = \dim T$ and $\{\mathbf{a}_1, \dots, \mathbf{a}_d\}$ is any basis of T.

We recall [9, Lemma 26]. We note that this result was deduced from a sharp version of Roth's Lemma and ultimately goes back to Faltings' Product Theorem [13].

Proposition 12.1. *Let m, N be integers $\geqslant 2$, ε a real with $0 < \varepsilon \leqslant 1$, and $r_1, \dots, r_m$ positive integers such that*

$$\frac{r_h}{r_{h+1}} \geqslant \frac{2m^2}{\varepsilon} \text{ for } h = 1, \dots, m-1. \tag{12.3}$$

Next, let P be a nonzero polynomial in $\overline{\mathbb{Q}}[\mathbf{X}_1, \dots, \mathbf{X}_m]$ that is homogeneous of degree r_h in the block $\mathbf{X}_h$ for $h = 1, \dots, m$, and let $T_1, \dots, T_m$ be $(N-1)$-dimensional linear subspaces of $\overline{\mathbb{Q}}^N$ such that

$$H_2(T_h)^{r_h} \geqslant \left(e^{r_1 + \dots + r_m} H_2(P) \right)^{(N-1)(3m^2/\varepsilon)^m}. \tag{12.4}$$

Finally, for $h = 1, \dots, m$, let Γ_h be a grid in T_h of size N/ε. Then there are $\mathbf{x}_h \in \Gamma_h$ with $\mathbf{x}_h \neq \mathbf{0}$ for $h = 1, \dots, m$ and $\mathbf{i} \in \mathbb{Z}_{\geqslant 0}^{mN}$ with

$$\sum_{h=1}^{m} \frac{1}{r_h} \left(\sum_{l=1}^{N} i_{hl} \right) \leqslant 2m\varepsilon \tag{12.5}$$

such that

$$P_{\mathbf{i}}(\mathbf{x}_1, \dots, \mathbf{x}_m) \neq 0. \tag{12.6}$$

13. Construction of the auxiliary polynomial

We start with recalling our main tools, which are a version of Siegel's Lemma due to Bombieri and Vaaler and Hoeffding's inequality from probability theory.

For an algebraic number field K, we denote by D_K the discriminant of K, and we put

$$C_K := |D_K|^{1/2[K:\mathbb{Q}]}.$$

LEMMA 13.1. *Let K be a number field, U, V integers with $V > U > 0$, and $L_1, \ldots, L_U$ nonzero linear forms from $K[X_1, \ldots, X_V]^{\mathrm{lin}}$. Then there exists $\mathbf{x} \in K^V \setminus \{\mathbf{0}\}$ such that*

$$L_1(\mathbf{x}) = 0, \ldots, L_U(\mathbf{x}) = 0, \tag{13.1}$$

$$H_2(\mathbf{x}) \leqslant V^{1/2} C_K \cdot \big(H_2(L_1) \cdots H_2(L_U)\big)^{1/(V-U)}. \tag{13.2}$$

Proof. This is a consequence of Bombieri and Vaaler [1, Th. 9]. □

In the lemma below, all random variables under consideration are defined on a given probability space with probability measure Prob. The expectation of a random variable $\mathbb{X}$ is denoted by $E(\mathbb{X})$.

LEMMA 13.2. *Let $\mathbb{X}_1, \ldots, \mathbb{X}_m$ be mutually independent random variables such that*

$$\mathrm{Prob}\big(\mathbb{X}_h \in [a_h, b_h]\big) = 1, \quad E(\mathbb{X}_h) = \mu_h \quad \textit{for } h = 1, \ldots, m,$$

where $a_h, b_h, \mu_h \in \mathbb{R}$, $a_h < b_h$ for $h = 1, \ldots, m$. Then for every $\varepsilon > 0$, we have

$$\mathrm{Prob}\left(\sum_{h=1}^{m} (\mathbb{X}_h - \mu_h) \geqslant m\varepsilon\right) \leqslant \exp\left(-\frac{2m^2\varepsilon^2}{\sum_{h=1}^{m}(b_h - a_h)^2}\right). \tag{13.3}$$

Proof. See W. Hoeffding [15, Th. 2]. □

For positive integers m, N and a tuple of positive integers $\mathbf{r} = (r_1, \ldots, r_m)$, define $\mathcal{U}(\mathbf{r})$ to be the set of tuples

$$\mathbf{j} = (j_{hl} : h = 1, \ldots, m, \, l = 1, \ldots, N) \in \mathbb{Z}_{\geqslant 0}^{mN}$$

such that

$$\sum_{l=1}^{N} j_{hl} = r_h \text{ for } h = 1, \ldots, m.$$

Put

$$V := \#\mathcal{U}(\mathbf{r}) = \prod_{h=1}^{m} \prod_{l=1}^{N} \binom{r_h + N - 1}{N - 1}. \tag{13.4}$$

Using the inequality

$$\binom{a+b}{b} \leqslant \frac{(a+b)^{a+b}}{a^a b^b} = \Big(1+\frac{b}{a}\Big)^a \Big(1+\frac{a}{b}\Big)^b \leqslant \Big(e\Big(1+\frac{b}{a}\Big)\Big)^a$$

for positive integers a, b, it follows that

$$V \leqslant (eN)^{r_1+\cdots+r_m}. \tag{13.5}$$

We deduce the following combinatorial lemma.

LEMMA 13.3. *Let N be a positive integer, $\mathbf{r} = (r_1, \dots, r_m)$ a tuple of positive integers, ε, γ reals with $0 < \varepsilon \leqslant 1$ and $\gamma > 0$, and $\widehat{\mathbf{c}}_h = (\widehat{c}_{h1}, \dots, \widehat{c}_{hN})$ $(h = 1, \dots, m)$ tuples of reals such that*

$$|\widehat{c}_{hl}| \leqslant \gamma \quad \text{for } h = 1, \dots, m,\ l = 1, \dots, N. \tag{13.6}$$

Then the number of tuples $\mathbf{j} = (j_{hl} : h = 1, \dots, m,\, l = 1, \dots, N) \in \mathcal{U}(\mathbf{r})$ such that

$$\sum_{h=1}^{m} \frac{1}{r_h} \left(\sum_{l=1}^{N} j_{hl} \widehat{c}_{hl} \right) \geqslant \frac{1}{N} \left(\sum_{h=1}^{m} \sum_{l=1}^{N} \widehat{c}_{hl} \right) + m\gamma\varepsilon \tag{13.7}$$

is at most

$$e^{-m\varepsilon^2/2} V. \tag{13.8}$$

Proof. We assume without loss of generality that $\gamma = 1$. We view $\mathbf{j}$ as a uniformly distributed random variable on $\mathcal{U}(\mathbf{r})$; i.e., each possible value of $\mathbf{j}$ is given probability $1/V$. Define random variables on $\mathcal{U}(\mathbf{r})$ by

$$\mathbb{X}_h := \frac{1}{r_h} \sum_{l=1}^{N} j_{hl} \widehat{c}_{hl} \quad (h = 1, \dots, m).$$

Notice that $\mathbb{X}_1, \dots, \mathbb{X}_m$ are mutually independent, and for $h = 1, \dots, m$,

$$\mathrm{Prob}\big(\mathbb{X}_h \in [-1, 1]\big) = 1, \quad (\text{by (13.6) and } \gamma = 1),$$

$$E(\mathbb{X}_h) = \mu_h := \frac{1}{N} \sum_{l=1}^{N} \widehat{c}_{hl}.$$

Now the number of tuples $\mathbf{j} \in \mathcal{U}(\mathbf{r})$ with (13.7) is precisely

$$V \cdot \mathrm{Prob} \left(\sum_{h=1}^{m} (\mathbb{X}_h - \mu_h) \geqslant m\varepsilon \right),$$

and by Lemma 13.2 this is at most $V \cdot e^{-m\varepsilon^2/2}$. □

Let K be an algebraic number field and $m, N, r_1, \dots, r_m$ integers $\geqslant 2$. We keep the notation introduced in Section 12. In particular, by $\mathbf{i}$ we denote an mN-tuple of nonnegative integers $\mathbf{i} = (i_{hl} : h = 1, \dots, m,\, l = 1, \dots, N)$, and similarly for $\mathbf{j}$, $\mathbf{k}$. Further, $K[\mathbf{X}_1, \dots, \mathbf{X}_m]$ denotes the ring of polynomials with coefficients in K in the blocks of variables $\mathbf{X}_h = (X_{h1},\dots,X_{hN})$ $(h = 1,\dots,m)$.

We consider polynomials in this ring that are homogeneous of degree r_h in $\mathbf{X}_h$ for $h = 1,\dots,m$. In analogy to (12.1), such a polynomial P can be expressed as

$$P = \sum_{\mathbf{j}\in\mathcal{U}(\mathbf{r})} a(\mathbf{j})\mathbf{X}^{\mathbf{j}} \text{ with } \mathbf{a}_P := \big(a(\mathbf{j}) : \mathbf{j}\in\mathcal{U}(\mathbf{r})\big) \in K^V. \tag{13.9}$$

We prove a simple auxiliary result.

Lemma 13.4. *Let P be a nonzero polynomial with* (13.9). *Further, let*

$$\widehat{L}_i = \sum_{j=1}^{N} \alpha_{ij} X_j \ (i = 1,\dots,N)$$

be linearly independent linear forms with coefficients in K, and let

$$\big(\beta_{ij}\big)_{i,j=1,\dots,N} = \Big(\big(\alpha_{ij}\big)_{i,j=1,\dots,N}\Big)^{-1}$$

be the inverse of the coefficient matrix of $\widehat{L}_1,\dots,\widehat{L}_N$. Lastly, put

$$C_v := \max_{i,j=1,\dots,N} \|\beta_{ij}\|_v \textit{ for } v\in M_K.$$

Then for every $\mathbf{i}\in\mathbb{Z}_{\geqslant 0}^{mN}$, we have

$$P_{\mathbf{i}} = \sum_{\mathbf{j}\in\mathcal{U}(\mathbf{r},\mathbf{i})} d_{\mathbf{i},\mathbf{j}}(\mathbf{a}_P) \prod_{h=1}^{m}\prod_{l=1}^{N} \widehat{L}_l(\mathbf{X}_h)^{j_{hl}} \tag{13.10}$$
$$\textit{with } \mathcal{U}(\mathbf{r},\mathbf{i}) := \{\mathbf{j}\in\mathbb{Z}_{\geqslant 0}^{mN} : \mathbf{j}+\mathbf{i}\in\mathcal{U}(\mathbf{r})\},$$

where $d_{\mathbf{i},\mathbf{j}}$ is a linear form with coefficients in K in V variables satisfying

$$\|d_{\mathbf{i},\mathbf{j}}\|_{v,1} \leqslant \Big((6N^2)^{s(v)}C_v\Big)^{r_1+\cdots+r_m} \quad \textit{for } \mathbf{j}\in\mathcal{U}(\mathbf{r}),\ v\in M_K. \tag{13.11}$$

Proof. Define new variables $Y_{hl} := \widehat{L}_l(\mathbf{X}_h)$ for $h = 1,\dots,m$, $l = 1,\dots,N$. Then by (12.2),

$$\begin{aligned} P_{\mathbf{i}} &= \sum_{\mathbf{j}\in\mathcal{U}(\mathbf{r},\mathbf{i})} \binom{\mathbf{i}+\mathbf{j}}{\mathbf{i}} a(\mathbf{i}+\mathbf{j})\mathbf{X}^{\mathbf{j}} \\ &= \sum_{\mathbf{j}\in\mathcal{U}(\mathbf{r},\mathbf{i})} a(\mathbf{i}+\mathbf{j}) \prod_{h=1}^{m}\prod_{l=1}^{N} \left(\binom{i_{hl}+j_{hl}}{i_{hl}}\Big(\sum_{j=1}^{N}\beta_{lj}Y_{lj}\Big)^{j_{hl}}\right) \\ &=: \sum_{\mathbf{j}\in\mathcal{U}(\mathbf{r},\mathbf{i})} a(\mathbf{i}+\mathbf{j}) D_{\mathbf{i},\mathbf{j}}(\mathbf{Y}). \end{aligned}$$

Let $v \in M_K$. Then by (6.5), we have for $\mathbf{j} \in \mathcal{U}(\mathbf{r},\mathbf{i})$, on noting $\binom{\mathbf{i}+\mathbf{j}}{\mathbf{i}} \leqslant 2^{\sum_{h,l}(i_{hl}+j_{hl})} = 2^{r_1+\cdots+r_m}$,

$$\|D_{\mathbf{i},\mathbf{j}}\|_{v,1} \leqslant \binom{\mathbf{i}+\mathbf{j}}{\mathbf{i}}^{s(v)} \left(N^{s(v)}C_v\right)^{\sum_{h,l} j_{hl}} \leqslant (2NC_v)^{r_1+\cdots+r_m}.$$

Together with (6.5), (13.5), this implies for $\mathbf{j} \in \mathcal{U}(\mathbf{r},\mathbf{i})$,

$$\|d_{\mathbf{i},\mathbf{j}}\|_{v,1} \leqslant V^{s(v)} \max_{\mathbf{k}\in\mathcal{U}(\mathbf{r})} \|D_{\mathbf{i},\mathbf{k}}\|_{v,1} \leqslant (6N^2C_v)^{r_1+\cdots+r_m}. \qquad \square$$

As before, let $\mathcal{L}, \mathbf{c}, n, R, \delta$ satisfy (8.1)–(8.9). We fix $k \in \{1,\ldots,n-1\}$ and consider all reals Q satisfying (11.8), (11.9).

Let v_0 be the place from (8.8) and $\widehat{L}_i^{(v)}$ ($v \in M_K$, $i = 1,\ldots,N$) be the linear forms and $\widehat{c_{iv}}$ ($v \in M_K \setminus \{v_0\}$, $i = 1,\ldots,N$), $\widehat{c}_{i,v_0}(Q)$ ($i = 1,\ldots,N$) the reals from Section 11.

We want to construct a suitable nonzero polynomial P of the shape (13.9). The next lemma is our first step. For $v \in M_K$, we write

$$P = \sum_{\mathbf{j}\in\mathcal{U}(\mathbf{r})} d_{\mathbf{j}}^{(v)}(\mathbf{a}_P) \prod_{h=1}^{m} \prod_{l=1}^{N} \widehat{L}_l^{(v)}(\mathbf{X}_h)^{j_{hl}}, \tag{13.12}$$

where $d_{\mathbf{j}}^{(v)}$ is a linear form with coefficients in K in V variables in the coefficient vector $\mathbf{a}_P$ of P.

LEMMA 13.5. *Let S_0 be a subset of*

$$S_1 := \{v \in M_K : \mathbf{c}_v := (c_{1v},\ldots,c_{nv}) \neq 0\},$$

and put $s_0 := \#S_0$. Let ε be a real with $0 < \varepsilon < 1$, m an integer with

$$m > 2\varepsilon^{-2}\log(2s_0+2) \tag{13.13}$$

and $r_1,\ldots,r_m$ positive integers. Lastly, let $Q_1,\ldots,Q_m$ be reals with (11.8), (11.9). Then there exists a nonzero polynomial P of the type (13.9) with the following properties:

(i) *For every $v \in S_0$ and each $\mathbf{j} \in \mathcal{U}(\mathbf{r})$ with*

$$\sum_{h=1}^{m} \frac{1}{r_h}\left(\sum_{l=1}^{N} j_{hl}\widehat{c}_{lv}\right) \geqslant mn\varepsilon \cdot \left(\max_{1\leqslant i\leqslant n} c_{iv}\right), \tag{13.14}$$

we have

$$d_{\mathbf{j}}^{(v)}(\mathbf{a}_P) = 0. \tag{13.15}$$

(ii) *For each $\mathbf{j} \in \mathcal{U}(\mathbf{r})$ with*

$$\sum_{h=1}^{m} \frac{1}{r_h}\left(\sum_{l=1}^{N} j_{hl}\widehat{c}_{l,v_0}(Q_h)\right) \geqslant -\frac{m\delta}{nN} + mn\varepsilon, \tag{13.16}$$

we have

$$d_{\mathbf{j}}^{(v_0)}(\mathbf{a}_P) = 0. \tag{13.17}$$

(iii) *For the height of P, we have*

$$H_2(P) \leqslant C_K\left(2^{3n} H_{\mathcal{L}}^{R^n}\right)^{r_1+\cdots+r_m}. \tag{13.18}$$

We recall here that by (8.2) the set S_1 is finite and that the place v_0 given by (8.8) does not belong to S_1.

Proof. We prove that there exists a nonzero polynomial P of the type (13.9) such that for every $v \in S_0$ and each $\mathbf{j} \in \mathbb{Z}_{\geqslant 0}^{mN}$ with

$$\sum_{h=1}^{m} \frac{1}{r_h}\left(\sum_{l=1}^{N} j_{hl}\widehat{c}_{lv}\right) \geqslant \left(\frac{m}{N}\sum_{l=1}^{N}\widehat{c}_{lv}\right) + mn\varepsilon\cdot\left(\max_{1\leqslant i\leqslant n} c_{iv}\right), \tag{13.19}$$

we have (13.15), and such that for each $\mathbf{j} \in \mathbb{Z}_{\geqslant 0}^{mN}$ with

$$\sum_{h=1}^{m} \frac{1}{r_h}\left(\sum_{l=1}^{N} j_{hl}\widehat{c}_{l,v_0}(Q_h)\right) \geqslant \left(\frac{1}{N}\sum_{h=1}^{m}\sum_{l=1}^{N}\widehat{c}_{l,v_0}(Q_h)\right) + mn\varepsilon, \tag{13.20}$$

we have (13.17). This suffices, since by (11.18), (11.21), the conditions (13.14), (13.16) imply (13.19), (13.20).

We may view (13.15) with (13.19) and (13.17) with (13.20) as a system of linear equations in the unknown vector $\mathbf{a}_P \in K^V$, where $V = \#\mathcal{U}(\mathbf{r})$. By (11.19), (11.22), Lemma 13.3, and assumption (13.13), the number of equations, i.e., the number of $\mathbf{j}$ with (13.19), (13.20), is

$$U \leqslant (s_0+1)Ve^{-m\varepsilon^2/2} \leqslant \tfrac{1}{2}V.$$

Combining Lemma 11.5 with Lemma 13.4 gives us

$$H_2(d_{\mathbf{j}}^{(v)}) \leqslant \left(6N^2 H_{\mathcal{L}}^{R^n}\right)^{r_1+\cdots+r_m}$$

for $v \in S_0 \cup \{v_0\}$, $\mathbf{j} \in \mathcal{U}(\mathbf{r})$. Now Lemma 13.1 implies that there is a nonzero $\mathbf{a}_P \in K^V$ with

$$H_2(\mathbf{a}_P) \leqslant C_K V^{1/2}\left(6N^2 H_{\mathcal{L}}^{R^n}\right)^{(r_1+\cdots+r_m)U/(V-U)}.$$

By inserting (13.5) and $N = \binom{n}{k} \leqslant 2^{n-1}$ we arrive at

$$\begin{aligned} H_2(P) = H_2(\mathbf{a}_P) &\leqslant C_K\left(6e^{1/2}N^{5/2}H_{\mathcal{L}}^{R^n}\right)^{r_1+\cdots+r_m} \\ &\leqslant C_K\left(2^{3n}H_{\mathcal{L}}^{R^n}\right)^{r_1+\cdots+r_m}. \end{aligned}$$

Our lemma follows. □

The next proposition lists the properties of our final auxiliary polynomial. For $v \in M_K$, $\mathbf{i} \in \mathbb{Z}_{\geqslant 0}^{mN}$, we write, analogously to (13.10),

$$P_{\mathbf{i}} = \sum_{\mathbf{j} \in \mathcal{U}(\mathbf{r},\mathbf{i})} d_{\mathbf{i},\mathbf{j}}^{(v)}(\mathbf{a}_P) \prod_{h=1}^{m} \prod_{l=1}^{N} \widehat{L}_l^{(v)}(\mathbf{X}_h)^{j_{hl}}, \tag{13.21}$$

where $\mathcal{U}(\mathbf{r},\mathbf{i}) = \{\mathbf{j} \in \mathbb{Z}_{\geqslant 0}^{mN} : \mathbf{i}+\mathbf{j} \in \mathcal{U}(\mathbf{r})\}$ and where $d_{\mathbf{i},\mathbf{j}}^{(v)}$ is a linear form in V variables with coefficients in K.

Proposition 13.6. *Let ε be a real with $0 < \varepsilon \leqslant 1$, m an integer with*

$$m \geqslant 2n\varepsilon^{-2} \log(4R/\varepsilon) \tag{13.22}$$

and $r_1, \ldots, r_m$ positive integers. Further, let $Q_1, \ldots, Q_m$ be reals with (11.8), (11.9). Then there exists a nonzero polynomial P of the type (13.9) with the following properties:

(i) *For every $v \in M_K \setminus \{v_0\}$, each tuple $\mathbf{i} \in \mathbb{Z}_{\geqslant 0}^{mN}$ with*

$$\sum_{h=1}^{m} \frac{1}{r_h} \left(\sum_{l=1}^{N} i_{hl} \right) \leqslant 2m\varepsilon \tag{13.23}$$

and each $\mathbf{j} \in \mathcal{U}(\mathbf{r},\mathbf{i})$ with

$$\sum_{h=1}^{m} \frac{1}{r_h} \left(\sum_{l=1}^{N} \widehat{c}_{lv} j_{hl} \right) > 4mn\varepsilon \max_{1 \leqslant i \leqslant n} c_{iv}, \tag{13.24}$$

we have

$$d_{\mathbf{i},\mathbf{j}}^{(v)}(\mathbf{a}_P) = 0. \tag{13.25}$$

(ii) *For each $\mathbf{i}$ with (13.23) and each $\mathbf{j} \in \mathcal{U}(\mathbf{r},\mathbf{i})$ with*

$$\sum_{h=1}^{m} \frac{1}{r_h} \left(\sum_{l=1}^{N} \widehat{c}_{l,v_0}(Q_h) j_{hl} \right) > -\frac{m\delta}{nN} + 4mn\varepsilon, \tag{13.26}$$

we have

$$d_{\mathbf{i},\mathbf{j}}^{(v_0)}(\mathbf{a}_P) = 0. \tag{13.27}$$

(iii) *For the height of P, we have*

$$H_2(P) \leqslant C_K \left(2^{3n} H_{\mathcal{L}}^{R^n} \right)^{r_1+\cdots+r_m}. \tag{13.28}$$

(iv) *For all $\mathbf{i} \in \mathbb{Z}_{\geqslant 0}^{mN}$, we have*

$$\prod_{v \in M_K} \left(\max_{\mathbf{j} \in \mathcal{U}(\mathbf{r},\mathbf{i})} \| d_{\mathbf{i},\mathbf{j}}^{(v)}(\mathbf{a}_P) \|_v \right) \leqslant C_K \left(2^{6n} H_{\mathcal{L}}^{2R^n} \right)^{r_1+\cdots+r_m}. \tag{13.29}$$

Proof. We construct a subset S_0 of

$$S_1 := \{v \in M_K : \mathbf{c}_v = (c_{1v}, \dots, c_{nv}) \neq \mathbf{0}\}$$

and apply Lemma 13.5 with this set. The set S_0 is obtained by dividing S_1 into subsets and picking one element from each subset. For $v \in M_K$, we put $\gamma_v := \max_{1 \leqslant i \leqslant n} c_{iv}$.

First, we divide S_1 into t_1 subsets $S_{11}, \dots, S_{1,t_1}$ in such a way that two places v_1, v_2 belong to the same subset if and only if

$$L_i^{(v_1)} = L_i^{(v_2)} \text{ for } i = 1, \dots, n.$$

By (8.7), we have $t_1 \leqslant R^n$.

We further subdivide the subsets S_{1j}. Let $j \in \{1, \dots, t_1\}$. Divide the cube $[-1, 1]^n$ into $t_2 := \big([2/\varepsilon] + 1\big)^n$ small subcubes of sidelength

$$\frac{2}{[2/\varepsilon] + 1} \leqslant \varepsilon.$$

Now divide $S_{1,j}$ into t_2 subsets $S_{1j1}, \dots, S_{1j,t_2}$ such that two places v_1, v_2 belong to the same subset if the two points

$$\left(\frac{c_{1,v_1}}{\gamma_{v_1}}, \dots, \frac{c_{n,v_1}}{\gamma_{v_1}}\right), \quad \left(\frac{c_{1,v_2}}{\gamma_{v_2}}, \dots, \frac{c_{n,v_2}}{\gamma_{v_2}}\right)$$

belong to the same small subcube. In this way, we have divided S_1 into

$$t_1 t_2 \leqslant R^n \left([2/\varepsilon] + 1\right)^n \leqslant (3R/\varepsilon)^n$$

subsets. Let S_0 consist of one element from each of the subsets. Thus,

$$s_0 := \#S_0 \leqslant (3R/\varepsilon)^n. \tag{13.30}$$

Further, for each $v \in S_1$, there is $v_1 \in S_0$ with

$$L_i^{(v)} = L_i^{(v_1)}, \qquad \left|\frac{c_{iv}}{\gamma_v} - \frac{c_{i,v_1}}{\gamma_{v_1}}\right| \leqslant \varepsilon \quad \text{for } i = 1, \dots, n.$$

This implies that for every $v \in S_1$, there is $v_1 \in S_0$ such that

$$\widehat{L}_l^{(v)} = \widehat{L}_l^{(v_1)} \text{ for } l = 1, \dots, N, \tag{13.31}$$

$$\left|\frac{\widehat{c}_{lv}}{n\gamma_v} - \frac{\widehat{c}_{l,v_1}}{n\gamma_{v_1}}\right| \leqslant \varepsilon \text{ for } l = 1, \dots, N. \tag{13.32}$$

We apply Lemma 13.5 with the subset S_0 constructed above. Condition (13.13) of this lemma is satisfied in view of our assumption (13.22) on m and in view of (13.30). Let P be the nonzero polynomial from Lemma 13.5. We show that this polynomial has all the properties listed in our proposition.

To prove (i), we first show that for every $\mathbf{j} \in \mathcal{U}(\mathbf{r})$, $v \in M_K \setminus \{v_0\}$ with

$$\sum_{h=1}^{m} \frac{1}{r_h} \left(\sum_{l=1}^{N} \widehat{c}_{lv} j_{hl}\right) > 2mn\varepsilon\gamma_v, \tag{13.33}$$

we have

$$d_{\mathbf{j}}^{(v)}(\mathbf{a}_P) = 0. \tag{13.34}$$

For $v \in M_K \setminus (S_1 \cup \{v_0\})$, we have $c_{iv} = 0$ for $i = 1, \ldots, n$, whence $\gamma_v = 0$ and $\widehat{c}_{lv} = 0$ for $l = 1, \ldots, N$, so there are no $\mathbf{j}$ with (13.33). For $v \in S_0$, we have (13.34) for all $\mathbf{j}$ with (13.14), and so certainly for all $\mathbf{j}$ with the weaker condition (13.33). Finally, let $v \in S_1 \setminus S_0$ and take $\mathbf{j} \in \mathcal{U}(\mathbf{r})$ with (13.33). Take $v_1 \in S_0$ with (13.31), (13.32). Condition (13.31) implies that $d_{\mathbf{j}}^{(v)}(\mathbf{a}_P) = d_{\mathbf{j}}^{(v_1)}(\mathbf{a}_P)$; hence, it suffices to show that $d_{\mathbf{j}}^{(v_1)}(\mathbf{a}_P) = 0$. Now condition (13.33) together with (13.32) implies

$$\sum_{h=1}^{m} \frac{1}{r_h} \left(\sum_{l=1}^{N} \frac{\widehat{c}_{l,v_1}}{n\gamma_{v_1}} \cdot j_{hl} \right) \geqslant \sum_{h=1}^{m} \frac{1}{r_h} \left(\sum_{l=1}^{N} \frac{\widehat{c}_{lv}}{n\gamma_v} \cdot j_{hl} \right) - \varepsilon \sum_{h=1}^{m} \sum_{l=1}^{N} \frac{j_{hl}}{r_h} > m\varepsilon.$$

Hence $\mathbf{j}, v_1$ satisfy (13.14), and so $d_{\mathbf{j}}^{(v_1)}(\mathbf{a}_P) = 0$ by Lemma 13.5. This shows (13.34) for $v \in M_K \setminus \{v_0\}$.

We now prove (i). Let $\mathbf{i} \in \mathbb{Z}_{\geqslant 0}^{mN}$ be a tuple with (13.23), and let $v \in M_K \setminus \{v_0\}$. Using expression (13.12) for P, we infer that $P_{\mathbf{i}}$ is a K-linear combination of polynomials

$$\sum_{\mathbf{j} \in \mathcal{U}(\mathbf{r})} d_{\mathbf{j}}^{(v)}(\mathbf{a}_P) \binom{\mathbf{j}}{\mathbf{k}} \prod_{h=1}^{M} \prod_{l=1}^{N} \widehat{L}_l(\mathbf{X}_h)^{j_{hl} - k_{hl}}$$

taken over tuples $\mathbf{k} \in \mathbb{Z}_{\geqslant 0}^{mN}$ with

$$\sum_{h=1}^{m} \frac{1}{r_h} \left(\sum_{j=1}^{N} k_{hl} \right) \leqslant 2m\varepsilon. \tag{13.35}$$

Hence, if $\mathbf{j} \in \mathcal{U}(\mathbf{r}, \mathbf{i})$, then $d_{\mathbf{i},\mathbf{j}}^{(v)}(\mathbf{a}_P)$ is a K-linear combination of terms $d_{\mathbf{j}+\mathbf{k}}^{(v)}(\mathbf{a}_P)$, over tuples $\mathbf{k}$ with (13.35). Now take $\mathbf{j} \in \mathcal{U}(\mathbf{r}, \mathbf{i})$ and suppose that $\mathbf{j}$ satisfies (13.24). Then for all $\mathbf{k}$ with (13.35), we have $\mathbf{j} + \mathbf{k} \in \mathcal{U}(\mathbf{r})$ and moreover, by (11.19),

$$\sum_{h=1}^{m} \frac{1}{r_h} \left(\sum_{l=1}^{N} \widehat{c}_{lv}(j_{hl} + k_{hl}) \right) > 4mn\varepsilon\gamma_v - 2mn\varepsilon\gamma_v = 2mn\varepsilon\gamma_v;$$

i.e., $\mathbf{j} + \mathbf{k}$ satisfies (13.33). So for all $\mathbf{k}$ with (13.35), we have that $\mathbf{j} + \mathbf{k}$ satisfies (13.34); i.e., $d_{\mathbf{j}+\mathbf{k}}^{(v)}(\mathbf{a}_P) = 0$. This implies that $d_{\mathbf{i},\mathbf{j}}^{(v)}(\mathbf{a}_P) = 0$. This proves (i).

The proof of (ii) follows the same lines, using part (ii) of Lemma 13.5 instead of (13.34). (iii) is merely a copy of part (iii) of Lemma 13.5.

It remains to prove (iv). Let $\mathbf{i}$ satisfy (13.23), and let $\mathbf{j} \in \mathcal{U}(\mathbf{r}, \mathbf{i})$. Then by Lemma 13.4,

$$\|d_{\mathbf{i},\mathbf{j}}^{(v)}\|_{v,1} \leqslant \left((6N^2)^{s(v)} C_v \right)^{r_1 + \cdots + r_m},$$

and so

$$\|d_{\mathbf{i},\mathbf{j}}^{(v)}(\mathbf{a}_P)\|_v \leqslant \|d_{\mathbf{i},\mathbf{j}}^{(v)}\|_{v,1} \cdot \|\mathbf{a}_P\|_v \leqslant \|\mathbf{a}_P\|_v \cdot \Big((6N^2)^{s(v)} C_v \Big)^{r_1+\cdots+r_m}$$

for $v \in M_K$, where by Lemma 11.5, we have

$$\prod_{v\in M_K} C_v \leqslant H_{\mathcal{L}}^{R^n}.$$

By taking the product over $v \in M_K$, using (iii), $N \leqslant 2^{n-1}$, we obtain

$$\prod_{v\in M_K} \max_{\mathbf{j}\in\mathcal{U}(\mathbf{r},\mathbf{i})} \|d_{\mathbf{i},\mathbf{j}}^{(v)}(\mathbf{a}_P)\|_v \leqslant C_K \Big(2^{3n} H_{\mathcal{L}}^{R^n} \cdot 6N^2 H_{\mathcal{L}}^{R^n}\Big)^{r_1+\cdots+r_m}$$
$$\leqslant C_K \Big(2^{6n} H_{\mathcal{L}}^{2R^n}\Big)^{r_1+\cdots+r_m}.$$

This proves (iv). □

14. **Proof of Theorem 8.1**

We keep the notation and definitions from the previous sections. Assume that Theorem 8.1 is false. Define the following parameters:

$$\varepsilon := \frac{\delta}{11n^2 2^{n-1}}, \quad m := \Big[2n\varepsilon^{-2}\log(4R/\varepsilon)\Big] + 1. \tag{14.1}$$

Notice that

$$nm \leqslant n + 2\cdot 11^2 n^6 2^{2n-2}\delta^{-2}\log(4\cdot 11n^2 2^{n-1}R/\delta) \leqslant m_2. \tag{14.2}$$

Hence by Lemma 9.4, there exist $k \in \{1,\dots,n-1\}$ and reals $Q_1,\dots,Q_m$ such that

$$Q_1 \geqslant C_2, \tag{14.3}$$
$$Q_{h+1} > Q_h^{\omega_2} \quad (h = 1,\dots,m-1), \tag{14.4}$$
$$\lambda_1(Q_h) \leqslant Q_h^{-\delta},\ \lambda_k(Q_h) \leqslant Q_h^{-\delta/(n-1)}\lambda_{k+1}(Q_h) \quad (h = 1,\dots,m). \tag{14.5}$$

Put

$$N := \binom{n}{k}.$$

For $h = 1,\dots,m$, let $\widehat{\mathbf{h}}_{h1} := \widehat{\mathbf{h}}_1(Q_h),\dots,\widehat{\mathbf{h}}_{h,N-1} := \widehat{\mathbf{h}}_{N-1}(Q_h)$ be linearly independent vectors from $\overline{\mathbb{Q}}^N$ satisfying (11.17) with $Q = Q_h$. By the remark following (11.17), we may take for the field E any finite extension of K containing the coordinates of $\widehat{\mathbf{h}}_{hj}$ for $h = 1,\dots,m$, $j = 1,\dots,N-1$. Thus, for $h = 1,\dots,m$, $l = 1,\dots,N$, $j = 1,\dots,N-1$, we have

$$\begin{cases} \|\widehat{L}_l^{(w)}(\widehat{\mathbf{h}}_{hj})\|_w \leqslant Q_h^{\widehat{c}_{lw}} \ (w \in M_E,\ w \nmid v_0), \\ \|\widehat{L}_l^{(w)}(\widehat{\mathbf{h}}_{hj})\|_w \leqslant Q_h^{\widehat{c}_{lw}(Q_h)} \ (w \in M_E,\ w \mid v_0). \end{cases} \tag{14.6}$$

For $h=1,\dots,m$, denote by $\widehat{T}_h$ the $\overline{\mathbb{Q}}$-vector space generated by $\widehat{\mathbf{h}}_{h1},\dots,\widehat{\mathbf{h}}_{h,N-1}$, and define the grid

$$\Gamma_h := \left\{ \sum_{j=1}^{N} x_j \widehat{\mathbf{h}}_{hj} \,:\, x_j \in \mathbb{Z},\ |x_j| \leqslant N/\varepsilon \text{ for } j = 1,\dots,N-1 \right\}. \tag{14.7}$$

Now choose a positive integer r_1 such that

$$r_1 > \frac{\varepsilon^{-1}\log Q_m}{\log Q_1}$$

and then integers $r_2,\dots,r_m$ such that

$$\frac{r_1 \log Q_1}{\log Q_h} \leqslant r_h < 1 + \frac{r_1 \log Q_1}{\log Q_h} \text{ for } h = 2,\dots,m.$$

Thus, $r_1,\dots,r_m$ are all positive integers with

$$Q_1^{r_1} \leqslant Q_h^{r_h} < Q_1^{r_1(1+\varepsilon)} \text{ for } h = 2,\dots,m. \tag{14.8}$$

Further, by choosing r_1 sufficiently large as we may, we can guarantee that

$$1.1^{r_1} > C_K. \tag{14.9}$$

With our choice of m in (14.1), there exists a nonzero polynomial P with the properties listed in Proposition 13.6. We apply our nonvanishing result Proposition 12.1 to P. We verify the conditions of that proposition. Condition (12.3) is satisfied since by (14.8), (14.4), (8.10), (14.2),

$$\frac{r_{h+1}}{r_h} \geqslant (1+\varepsilon)^{-1}\frac{\log Q_{h+1}}{\log Q_h} \geqslant (1+\varepsilon)^{-1} m_2^{5/2} \geqslant 2m^2/\varepsilon.$$

(12.4) follows by combining the lower bound for $H_2(\widehat{T}_h)$ from Lemma 11.6 with the lower bound $Q_1 \geqslant C_2$ from (14.3) and the upper bound for $H_2(P)$ from (13.28). More precisely, we have for $h = 1,\dots,m$,

$$\begin{aligned}
H_2(\widehat{T}_h)^{r_h} &\geqslant Q_h^{r_h\delta/3R^n} \geqslant Q_1^{r_1\delta/3R^n} && \text{by Lemma 11.6, (14.8)}\\
&\geqslant C_2^{r_1\delta/3R^n} = (2H_{\mathcal{L}})^{r_1\cdot m_2^{2m_2}\delta/3R^n} && \text{by (14.3), (8.10)}\\
&\geqslant \left((2H_{\mathcal{L}})^{(nm)^{2nm}\delta/3mR^n}\right)^{r_1+\cdots+r_m} && \text{by (14.2)}\\
&\geqslant \left(e\cdot 2^{4n}H_{\mathcal{L}}^{R^n}\right)^{(N-1)(3m^2/\varepsilon)^m(r_1+\cdots+r_m)} && \text{by (14.1)}\\
&\geqslant \left(e^{r_1+\cdots+r_m}H_2(P)\right)^{(N-1)(3m^2/\varepsilon)^m} && \text{by (13.28), (14.9),}
\end{aligned}$$

which is condition (12.4).

Now we conclude from Proposition 12.1 that there exist a tuple $\mathbf{i} \in \mathbb{Z}_{\geqslant 0}^{mN}$ such that

$$\sum_{h=1}^{m} \frac{1}{r_h}\left(\sum_{l=1}^{N} i_{hl}\right) \leqslant 2m\varepsilon$$

and nonzero points $\mathbf{x}_h \in \Gamma_h$ $(h = 1, \dots, m)$ such that

$$P_{\mathbf{i}}(\mathbf{x}_1, \dots, \mathbf{x}_m) \neq 0.$$

We finish by showing that $\prod_{w \in M_E} \|P_{\mathbf{i}}(\mathbf{x}_1, \dots, \mathbf{x}_m)\|_w < 1$. Then by the Product Formula, $P_{\mathbf{i}}(\mathbf{x}_1, \dots, \mathbf{x}_m) = 0$, which is against what we just proved. Thus, our assumption that Theorem 8.1 is false leads to a contradiction.

We express $P_{\mathbf{i}}$ as in (13.21) for $v \in M_K$. In the usual manner, where in all cases $w \in M_E$ and v is the place of K below w, we define

$$\begin{aligned}
\widehat{L}_l^{(w)} &:= \widehat{L}_l^{(v)} \quad (l = 1, \dots, N), \\
\widehat{c}_{lw} &:= d(w|v)\widehat{c}_{lv} \quad (w \nmid v_0,\ l = 1, \dots, N), \\
\widehat{c}_{lw}(Q_h) &:= d(w|v_0)\widehat{c}_{l,v_0}(Q_h) \quad (w|v_0,\ l = 1, \dots, N), \\
d_{\mathbf{i},\mathbf{j}}^{(w)}(\mathbf{a}_P) &:= d_{\mathbf{i},\mathbf{j}}^{(v)}(\mathbf{a}_P) \quad (\mathbf{j} \in \mathcal{U}(\mathbf{r}, \mathbf{i})),
\end{aligned}$$

and also

$$\gamma_w := \max_{1 \leqslant i \leqslant n} c_{iw}.$$

Then $\gamma_w = d(w|v) \max_{1 \leqslant i \leqslant n} c_{iv}$ if v is the place of K below w, and moreover, by (8.4) and $\sum_{w|v} d(w|v) = 1$ for $v \in M_K$,

$$\sum_{w \in M_E} \gamma_w \leqslant 1. \tag{14.10}$$

Now (13.21), (13.24), (13.26) imply that for $w \in M_E$, we have

$$P_{\mathbf{i}} = \sum_{\mathbf{j} \in \mathcal{U}_w} d_{\mathbf{i},\mathbf{j}}^{(w)}(\mathbf{a}_P) \prod_{h=1}^{m} \prod_{l=1}^{N} \widehat{L}_l^{(w)}(\mathbf{X}_h)^{j_{hl}}, \tag{14.11}$$

where for $w \in M_E$ with $w \nmid v_0$, $\mathcal{U}_w$ is the set of $\mathbf{j} \in \mathcal{U}(\mathbf{r}, \mathbf{i})$ with

$$\sum_{h=1}^{m} \frac{1}{r_h} \left(\sum_{l=1}^{N} \widehat{c}_{lw} j_{hl} \right) \leqslant 4mn\varepsilon\gamma_w \tag{14.12}$$

and for $w \in M_E$ with $w|v_0$, $\mathcal{U}_w$ is the set of $\mathbf{j} \in \mathcal{U}(\mathbf{r}, \mathbf{i})$ with

$$\sum_{h=1}^{m} \frac{1}{r_h} \left(\sum_{l=1}^{N} \widehat{c}_{l,w}(Q_h) j_{hl} \right) \leqslant d(w|v_0) \left(-\frac{m\delta}{nN} + 4mn\varepsilon \right). \tag{14.13}$$

Further, by (13.29), (14.9), we have

$$\prod_{w \in M_E} A_w \leqslant \left(2^{7n} H_{\mathcal{L}}^{2R^n} \right)^{r_1 + \cdots + r_m}, \tag{14.14}$$

$$\text{with } A_w := \max_{\mathbf{j} \in \mathcal{U}_w} \|d_{\mathbf{i},\mathbf{j}}^{w)}(\mathbf{a}_P)\|_w \text{ for } w \in M_E.$$

Finally, we observe that by (14.6), we have for the points $\mathbf{x}_h \in \Gamma_h$ ($h = 1, \dots, N$) and for $l = 1, \dots, N$,

$$(14.15) \qquad \begin{cases} \|\widehat{L}_l^{(w)}(\mathbf{x}_h)\|_w \leqslant N^{s(w)} Q_h^{\widehat{c}_{lw}} & (w \in M_E,\ w \nmid v_0), \\ \|\widehat{L}_l^{(w)}(\mathbf{x}_h)\|_w \leqslant Q_h^{\widehat{c}_{lw}(Q_h)} & (w \in M_E,\ w \mid v_0), \end{cases}$$

where we have used that w with $w|v_0$ is non-archimedean.

First, take $w \in M_E$ with $w \nmid v_0$. Then, in view of (14.11), (13.5), (14.8), (11.19), (14.12), we have

$$\begin{aligned} \|P_{\mathbf{i}}(\mathbf{x}_1, \dots, \mathbf{x}_m)\|_w &\leqslant V^{s(w)} A_w \cdot \max_{\mathbf{j} \in \mathcal{U}_w} \prod_{h=1}^m \prod_{l=1}^N \|\widehat{L}_l^{(w)}(\mathbf{x}_h)^{j_{hl}}\|_w \\ &\leqslant A_w (eN^2)^{s(w)(r_1 + \dots + r_m)} \prod_{h=1}^m Q_h^{\sum_{l=1}^N \widehat{c}_{lw} j_{lw}} \\ &\leqslant A_w (eN^2)^{s(w)(r_1 + \dots + r_m)} (Q_1^{r_1})^{\alpha_w} \end{aligned}$$

with

$$\begin{aligned} \alpha_w &\leqslant \sum_{h=1}^m \frac{1}{r_h} \left(\sum_{l=1}^N \widehat{c}_{lw} j_{lw} \right) + \varepsilon m \max_l |\widehat{c}_{lw}| \\ &\leqslant 5\gamma_w m n \varepsilon. \end{aligned}$$

So altogether, we have for $w \in M_E$ with $w \nmid v_0$,

$$(14.16) \qquad \|P_{\mathbf{i}}(\mathbf{x}_1, \dots, \mathbf{x}_m)\|_w \leqslant A_w (eN^2)^{s(w)(r_1 + \dots + r_m)} (Q_1^{m r_1})^{5\gamma_w n \varepsilon}.$$

In a similar fashion, we find for $w \in M_E$ with $w|v_0$, using (14.11), (14.8), (11.22), (14.13), noting that now we do not have a factor $(eN^2)^{s(w)(r_1 + \dots + r_m)}$ since w is non-archimedean,

$$\|P_{\mathbf{i}}(\mathbf{x}_1, \dots, \mathbf{x}_m)\|_w \leqslant A_w (Q_1^{r_1})^{\alpha_w}$$

with

$$\begin{aligned} \alpha_w &\leqslant \sum_{h=1}^m \frac{1}{r_h} \left(\sum_{l=1}^N \widehat{c}_{lw}(Q_h) j_{lw} \right) + \varepsilon m \max_{h,l} |\widehat{c}_{lw}(Q_h)| \\ &\leqslant d(w|v_0) \left(-\frac{m\delta}{nN} + 5mn\varepsilon \right). \end{aligned}$$

This gives for $w \in M_E$ with $w|v_0$,

$$(14.17) \qquad \|P_{\mathbf{i}}(\mathbf{x}_1, \dots, \mathbf{x}_m)\|_w \leqslant A_w \left(Q_1^{m r_1} \right)^{d(w|v_0)(-(\delta/nN) + 5n\varepsilon)}.$$

Now taking the product over $w \in M_E$, combining (14.16), (14.17), (14.14), (14.10), $\sum_{w|v_0} d(w|v_0) = 1$, we obtain

$$\prod_{w \in M_E} \|P_{\mathbf{i}}(\mathbf{x}_1, \dots, \mathbf{x}_m)\|_w \leqslant (eN^2 \cdot 2^{7n} H_{\mathcal{L}}^{2R^n})^{r_1 + \dots + r_m} \left(Q_1^{m r_1} \right)^{10n\varepsilon - \delta/nN}.$$

By our choice of ε in (14.1), and the inequalities $n \geqslant 2$, $N \leqslant 2^{n-1}$, the exponent on $Q_1^{mr_1}$ is $\leqslant -\delta/(11n \cdot 2^{n-1})$. Together with (14.3) this implies

$$\prod_{w \in M_E} \|P_{\mathbf{i}}(\mathbf{x}_1, \ldots, \mathbf{x}_m)\|_w \leqslant \left(2^{9n} H_{\mathcal{L}}^{2R^n} \cdot Q_1^{-\delta/11n \cdot 2^{n-1}}\right)^{mr_1} < 1,$$

as required. This completes the proof of Theorem 8.1.

15. Construction of a filtration

We construct a vector space filtration, which is an adaptation of the Harder-Narasimhan filtration constructed in [14].

Let $K \subset \overline{\mathbb{Q}}$ be an algebraic number field and n an integer that we now assume $\geqslant 1$ instead of $\geqslant 2$. Further, let $\mathcal{L} = (L_i^{(v)} : v \in M_K, i = 1, \ldots, n)$ be a tuple of linear forms and $\mathbf{c} = (c_{iv} : v \in M_K, i = 1, \ldots, n)$ a tuple of reals, satisfying (2.4)–(2.7).

Let $w_v = w_{\mathcal{L},\mathbf{c},v}$ $(v \in M_K)$ be the local weight functions on the collection of linear subspaces of $\overline{\mathbb{Q}}^n$, defined by (2.19). Then the global weight function is given by $w = w_{\mathcal{L},\mathbf{c}} = \sum_{v \in M_K} w_v$.

We give some convenient expressions for the local weights w_v. For $v \in M_K$, we reorder the indices $1, \ldots, n$ in such a way that

$$c_{1v} \leqslant \cdots \leqslant c_{nv} \text{ for } v \in M_K. \tag{15.1}$$

Let U be a k-dimensional linear subspace of $\overline{\mathbb{Q}}^n$. Let $v \in M_K$. Define

$$\begin{cases} I_v(U) := \emptyset \text{ if } k = 0, \\ I_v(U) := \{i_1(v), \ldots, i_k(v)\} \text{ if } k > 0, \end{cases} \tag{15.2}$$

where $i_1(v)$ is the smallest index $i \in \{1, \ldots, n\}$ such that $L_i^{(v)}|_U \neq 0$, and for $l = 2, \ldots, k$, $i_l(v)$ is the smallest index $i > i_{l-1}(v)$ in $\{1, \ldots, n\}$ such that $L_{i_1(v)}^{(v)}|_U, \ldots, L_{i_{l-1}(v)}^{(v)}|_U, L_i^{(v)}|_U$ are linearly independent. Then

$$w_v(U) = \sum_{i \in I_v(U)} c_{iv}. \tag{15.3}$$

It is not difficult to show that $I_v(U_1) \subseteq I_v(U_2)$ if U_1 is a linear subspace of U_2.

Define the linear subspaces of $\overline{\mathbb{Q}}^n$,

$$U_{0v} := \overline{\mathbb{Q}}^n,$$
$$U_{iv} := \{\mathbf{x} \in \overline{\mathbb{Q}}^n : L_1^{(v)}(\mathbf{x}) = \cdots = L_i^{(v)}(\mathbf{x}) = 0\} \quad (v \in M_K, \ i = 1, \ldots, n).$$

Then

$$w_v(U)=\sum_{i=1}^{n} c_{iv}\Big(\dim(U\cap U_{i-1,v})-\dim(U\cap U_{iv})\Big) \tag{15.4}$$
$$=c_{1v}\dim U+\sum_{i=1}^{n}(c_{i+1,v}-c_{iv})\dim(U\cap U_{iv}).$$

LEMMA 15.1. *For any two linear subspaces U_1, U_2 of $\overline{\mathbb{Q}}^n$, we have*

$$w(U_1\cap U_2)+w(U_1+U_2)\geqslant w(U_1)+w(U_2).$$

Proof. Let U_1, U_2 be two linear subspaces of $\overline{\mathbb{Q}}^n$. It clearly suffices to show that for any $v\in M_K$, we have

$$w_v(U_1\cap U_2)+w_v(U_1+U_2)\geqslant w_v(U_1)+w_v(U_2). \tag{15.5}$$

But this follows easily by combining (15.4) with $c_{i+1,v}-c_{iv}\geqslant 0$ for $i=1,\dots,n-1$ and

$$\dim(U_1\cap U_2)+\dim(U_1+U_2)=\dim U_1+\dim U_2,$$
$$U\cap(U_1+U_2)\supseteq(U\cap U_1)+(U\cap U_2)$$

for any three linear subspaces U, U_1, U_2 of $\overline{\mathbb{Q}}^n$. □

For any two linear subspaces U_1, U_2 of V with $\dim U_1<\dim U_2$, we define

$$\begin{cases} d(U_2,U_1):=\dim U_2-\dim U_1,\\ w(U_2,U_1)=w_{\mathcal{L},\mathbf{c}}(U_2,U_1):=w_{\mathcal{L},\mathbf{c}}(U_2)-w_{\mathcal{L},\mathbf{c}}(U_1),\\ \mu(U_2,U_1)=\mu_{\mathcal{L},\mathbf{c}}(U_2,U_1):=\dfrac{w(U_2,U_1)}{d(U_2,U_1)}. \end{cases} \tag{15.6}$$

We prove the following lemma.

LEMMA 15.2. *Let V be a linear subspace of $\overline{\mathbb{Q}}^n$, defined over K.*

(i) *There exists a unique proper linear subspace T of V such that*

$\mu(V,T)\leqslant\mu(V,U)$ *for every proper linear subspace U of V,*

subject to this constraint, T has minimal dimension.

This space T is defined over K.

(ii) *Let T be as in* (i) *and let U be any other proper linear subspace of V. Then $\mu(V,U\cap T)\leqslant\mu(V,U)$.*

Proof. Obviously, there exists a proper linear subspace T of V with (i) since $\mu(\cdot,\cdot)$ assumes only finitely many values. We prove first that T satisfies (ii) and then that T is uniquely determined and defined over K. Put $\mu:=$

$\mu(V,T)$. Then by Lemma 15.1 and since $\mu(V,W)\geqslant\mu$ for any proper linear subspace W of V,

$$\begin{aligned}w(V,U\cap T)&\leqslant w(V,U)+w(V,T)-w(V,T+U)\\&\leqslant w(V,U)+\mu d(V,T)-\mu d(V,T+U)\\&=\mu(V,U)d(V,U)+\mu d(T+U,T)\\&\leqslant\mu(V,U)(d(V,U)+d(T+U,T))\\&=\mu(V,U)d(V,U\cap T).\end{aligned}$$

This clearly proves (ii).

Now suppose that there exists another subspace T' with (i), i.e., $\mu(V,T')=\mu$ and $\dim T'=\dim T$. By (ii) we have $\mu(V,T\cap T')\leqslant\mu(V,T')=\mu$. By the definition of μ and the minimality of $\dim T$, we must have $T\cap T'=T=T'$.

It remains to prove that T is defined over K. Let $\sigma\in G_K$. Since V is defined over K and all linear forms $L_i^{(v)}$ have their coefficients in K, we have $\mu(V,\sigma(T))=\mu(V,T)=\mu$, while $\dim\sigma(T)=\dim T$. So by what we just proved, $\sigma(T)=T$. This holds for arbitrary σ; hence, T is defined over K. □

Remark. In the situation of Section 2, we have $V=\overline{\mathbb{Q}}^n$, $w(\overline{\mathbb{Q}}^n)=0$, and thus, the subspace $T=T(\mathcal{L},\mathbf{c})$ defined by (2.21) is precisely the subspace from (i). In a special case we can give more precise information about the subspace T.

LEMMA 15.3. *Let $V=\overline{\mathbb{Q}}^n$, and let T be the subspace from Lemma 15.2(i). Suppose that*

$$\bigcup_{v\in M_K}\{L_1^{(v)},\ldots,L_n^{(v)}\}\subseteq\{X_1,\ldots,X_n,X_1+\cdots+X_n\}. \tag{15.7}$$

Then there are nonempty, pairwise disjoint subsets $I_1,\ldots,I_p$ of $\{1,\ldots,n\}$ such that

$$T=\{\mathbf{x}\in\overline{\mathbb{Q}}^n:\ \sum_{j\in I_i}x_j=0\ \textit{for}\ j=1,\ldots,p\}. \tag{15.8}$$

Proof. Let $k:=\dim T$, $p:=n-k$. Define the $\overline{\mathbb{Q}}$-linear subspace of $\overline{\mathbb{Q}}^{n+1}$:

$$H:=\Big\{\mathbf{u}=(u_0,\ldots,u_n)\in\overline{\mathbb{Q}}^{n+1}:\ \sum_{j=1}^n u_jX_j\ -u_0\sum_{j=1}^n X_j\in T^{\perp}\Big\}.$$

Notice that $\dim H=p+1$ and $(1,\ldots,1)\in H$. We show that H is closed under coordinatewise multiplication; i.e., H is a sub-$\overline{\mathbb{Q}}$-algebra of $\overline{\mathbb{Q}}^{n+1}$. This being done, it is not difficult to show that there are pairwise disjoint subsets $I_0,\ldots,I_p$ of $\{0,\ldots,n\}$ such that H is the set of $\mathbf{u}\in\overline{\mathbb{Q}}^{n+1}$ with $u_i=u_j$ for each pair i,j for which there is $l\in\{0,\ldots,p\}$ with $i,j\in I_l$. This easily translates into (15.8).

Fix $\mathbf{a} = (a_0, \ldots, a_n) \in H$. Choose $c \in \overline{\mathbb{Q}}$ such that $b_i := a_i + c \neq 0$ for $i = 0, \ldots, n$. Then $\mathbf{b} := (b_0, \ldots, b_n) \in H$. Define the linear transformation

$$\varphi : \overline{\mathbb{Q}}^n \to \overline{\mathbb{Q}}^n : (x_1, \ldots, x_n) \mapsto (b_1 x_1, \ldots, b_n x_n).$$

In general, $\sum_{j=1}^n \xi_j X_j \in \varphi(T)^\perp$ if and only if $\sum_{j=1}^n b_j \xi_j X_i \in T^\perp$. Using this and $\mathbf{b} \in H$, it follows that for $(u_0, \ldots, u_n) \in \overline{\mathbb{Q}}^{n+1}$, we have

$$\begin{aligned} \text{(15.9)} \qquad & \sum_{j=1}^n u_j X_j - u_0 \sum_{j=1}^n X_j \in \varphi(T)^\perp \\ & \iff \sum_{j=1}^n b_j u_j X_j \; - u_0 \sum_{j=1}^n b_j X_j \in T^\perp \\ & \iff \sum_{j=1}^n b_j u_j X_j \; - b_0 u_0 \sum_{j=1}^n X_j \in T^\perp. \end{aligned}$$

This implies for any $v \in M_K$ and any subset $\{i_1, \ldots, i_k\}$ of $\{1, \ldots, n\}$ that $L_{i_1}^{(v)}|_{\varphi(T)}, \ldots, L_{i_k}^{(v)}|_{\varphi(T)}$ are linearly independent if and only if $L_{i_1}^{(v)}|_T, \ldots, L_{i_k}^{(v)}|_T$ are linearly independent. Consequently, $w(\varphi(T)) = w(T)$ and thus, $\mu(\overline{\mathbb{Q}}^n, \varphi(T)) = \mu(\overline{\mathbb{Q}}^n, T)$. Now Lemma 15.2(i) implies that $\varphi(T) = T$.

Combined with (15.9), this implies that if $\mathbf{u} \in H$, then $\mathbf{b} \cdot \mathbf{u} \in H$. But then, $\mathbf{a} \cdot \mathbf{u} = \mathbf{b} \cdot \mathbf{u} - c\mathbf{u} \in H$. This shows that H is closed under coordinatewise multiplication and proves our lemma. □

For every linear subspace U of $\overline{\mathbb{Q}}^n$, we define the point $P(U) = P_{\mathcal{L},\mathbf{c}}(U) := (\dim U, w(U)) \in \mathbb{R}^2$. In particular, $P(\{\mathbf{0}\}) = (0, 0)$. Notice that $\mu(U_2, U_1)$ defined by (15.6) is precisely the slope of the line segment from $P(U_1)$ to $P(U_2)$.

Let again V be a linear subspace of $\overline{\mathbb{Q}}^n$, defined over K. Denote by $C(V, \mathcal{L}, \mathbf{c})$ the upper convex hull of the points $P(U)$ for all linear subspaces U of V and by $B(V, \mathcal{L}, \mathbf{c})$ the upper boundary of $C(V, \mathcal{L}, \mathbf{c})$. Thus, $B(V, \mathcal{L}, \mathbf{c})$ is the graph of a piecewise linear, convex function from $[0, \dim V]$ to $\mathbb{R}$, and $C(V, \mathcal{L}, \mathbf{c})$ is the set of points on and below $B(V, \mathcal{L}, \mathbf{c})$.

As long as it is clear which are the underlying tuples $\mathcal{L}, \mathbf{c}$, we suppress the dependence on these tuples in our notation; i.e., we write w, μ, P for $w_{\mathcal{L},\mathbf{c}}, \mu_{\mathcal{L},\mathbf{c}}, P_{\mathcal{L},\mathbf{c}}$.

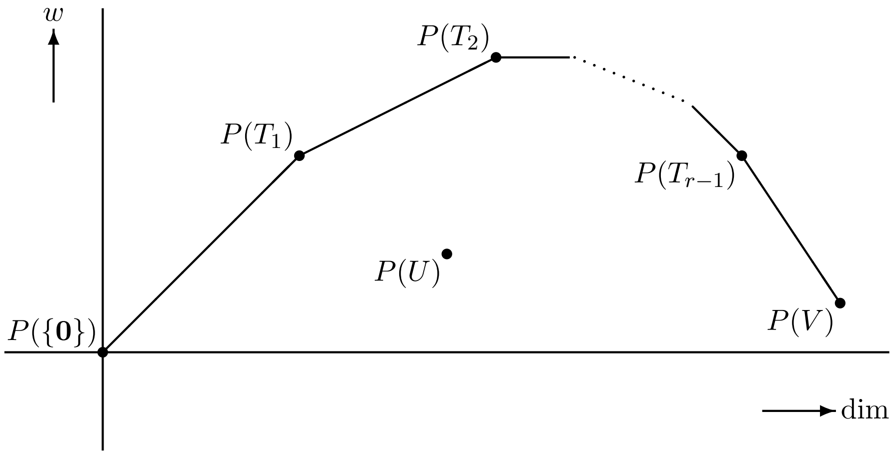
LEMMA 15.4. *There exists a unique filtration*

$$\text{(15.10)} \qquad \{\mathbf{0}\} \underset{\neq}{\subset} T_1 \underset{\neq}{\subset} \cdots \underset{\neq}{\subset} T_{r-1} \underset{\neq}{\subset} T_r = V$$

such that

$$P(\{\mathbf{0}\}),\, P(T_1), \ldots, P(T_{r-1}),\, P(V)$$

are precisely the vertices of $B(V, \mathcal{L}, \mathbf{c})$. *The spaces* $T_1, \ldots, T_{r-1}$ *are defined over* K.



Proof. (See the above figure.) The proof is by induction on $m := \dim V$. The case $m = 1$ is trivial. Let $m \geqslant 2$. There is only one candidate for the subspace in the filtration preceding V; it is the subspace T from Lemma 15.2(i). This space T is defined over K. By the induction hypothesis applied to T, there exists a unique filtration

$$\{\mathbf{0}\} \underset{\neq}{\subset} T_1 \underset{\neq}{\subset} \cdots \underset{\neq}{\subset} T_{r-1} = T$$

such that $P(\{\mathbf{0}\}), P(T_1), \ldots, P(T_{r-1})$ are precisely the vertices of $B(T, \mathcal{L}, \mathbf{c})$. Moreover, $T_1, \ldots, T_{r-2}$ are defined over K.

We have to prove that together with $P(V)$ these points are the vertices of $B(V, \mathcal{L}, \mathbf{c})$. We first note that since $T_{r-2} \underset{\neq}{\subset} T_{r-1}$, we have $\mu(V, T_{r-2}) > \mu(V, T_{r-1})$; hence,

$$\mu(T_{r-1}, T_{r-2}) = \frac{d(V, T_{r-2})\mu(V, T_{r-2}) - d(V, T_{r-1})\mu(V, T_{r-1})}{d(T_{r-1}, T_{r-2})} > \mu(V, T_{r-1}).$$

Therefore, $P(\{\mathbf{0}\}), P(T_1), \ldots, P(V)$ are the vertices of the graph of a piecewise linear convex function on $[0, m]$. Let C be the set of points on and below this graph. To prove that this graph is $B(V, \mathcal{L}, \mathbf{c})$, we have to show that C contains all points $P(U)$ with U a linear subspace of V.

If $U \subseteq T_{r-1}$, we have $P(U) \in C(T_{r-1}, \mathcal{L}, \mathbf{c}) \subset C$. Suppose that $U \not\subseteq T_{r-1}$. Then by Lemma 15.2(ii), we have $\mu(V, U \cap T_{r-1}) \leqslant \mu(V, U)$. Since $P(U \cap T_{r-1}) \in C$, $\dim U \geqslant \dim U \cap T_{r-1}$ and C is upper convex, this implies that $P(U) \in C$. This completes our proof. □

The filtration constructed above is called the *filtration of V* with respect to $(\mathcal{L}, \mathbf{c})$.

Remark. The Harder-Narasimhan filtration introduced by Faltings and Wüstholz in [14] is given by $\{\mathbf{0}\} \underset{\neq}{\subset} T'_{r-1} \underset{\neq}{\subset} \cdots \underset{\neq}{\subset} \mathrm{Hom}(V, \overline{\mathbb{Q}})$, where for a linear subspace T of V, we define T' as the set of linear functions from V to $\overline{\mathbb{Q}}$ that vanish identically on T.

16. The successive infima of a twisted height

As before, $K \subset \overline{\mathbb{Q}}$ is an algebraic number field, n an integer $\geqslant 1$, and $\mathcal{L}$ a tuple of linear forms and $\mathbf{c}$ a tuple of reals satisfying (2.4)–(2.7). We denote as usual by $\lambda_1(Q), \dots, \lambda_n(Q)$ the successive infima of $H_{\mathcal{L},\mathbf{c},Q}$. In this section, we prove a limit result for these successive infima as $Q \to \infty$.

Define

$$T_i(Q) := \bigcap_{\lambda > \lambda_i(Q)} \operatorname{span}\{\mathbf{x} \in \overline{\mathbb{Q}}^n : H_{\mathcal{L},\mathbf{c},Q}(\mathbf{x}) \leqslant \lambda\} \quad (i = 1, \dots, n). \tag{16.1}$$

Let

$$\{\mathbf{0}\} =: T_0 \underset{\neq}{\subsetneq} T_1 \underset{\neq}{\subsetneq} \cdots \underset{\neq}{\subsetneq} T_{r-1} \underset{\neq}{\subsetneq} T_r := \overline{\mathbb{Q}}^n$$

be the filtration of $\overline{\mathbb{Q}}^n$ with respect to $(\mathcal{L}, \mathbf{c})$, as defined in Lemma 15.4, and put $d_l := \dim T_l$ for $l = 0, \dots, r$. Given any two linear subspaces U, V of $\overline{\mathbb{Q}}^n$ with $\dim U < \dim V$, we define again $\mu(V, U) = \mu_{\mathcal{L},\mathbf{c}}(V, U) := \frac{w(V) - w(U)}{\dim V - \dim U}$.

Our general result on the successive infima of $H_{\mathcal{L},\mathbf{c},Q}$ is as follows.

Theorem 16.1. *For every $\delta > 0$, there exists Q_0 such that for every $Q \geqslant Q_0$, the following holds:*

$$Q^{-\mu(T_l, T_{l-1}) - \delta} \leqslant \lambda_i(Q) \leqslant Q^{-\mu(T_l, T_{l-1}) + \delta} \tag{16.2}$$
$$\text{for } l = 1, \dots, r,\ i = d_{l-1} + 1, \dots, d_l,$$

$$T_{d_l}(Q) = T_l \quad \text{for } l = 1, \dots, r. \tag{16.3}$$

We start with some preparations and lemmas. Fix a linear subspace T of $\overline{\mathbb{Q}}^n$ of dimension $k \in \{1, \dots, n-1\}$, which is defined over K. Choose an injective linear map

$$\varphi' : \overline{\mathbb{Q}}^k \hookrightarrow \overline{\mathbb{Q}}^n \ \text{ with } \varphi'(\overline{\mathbb{Q}}^n) = T$$

and a surjective linear map

$$\varphi'' : \overline{\mathbb{Q}}^n \twoheadrightarrow \overline{\mathbb{Q}}^{n-k} \ \text{ with } \operatorname{Ker}(\varphi'') = T,$$

both defined over K. Recall that for every linear form $L \in K[X_1, \dots, X_n]^{\mathrm{lin}}$ vanishing identically on T, there is a unique linear form $L'' \in K[X_1, \dots, X_{n-k}]^{\mathrm{lin}}$ such that $L = L'' \circ \varphi''$; we denote this L'' by $L \circ \varphi''^{-1}$.

We assume (15.1), which is no loss of generality. For $v \in M_K$, let the set $I_v(T)$ be given by (15.2), and define a tuple $\mathcal{L}'$ from $K[X_1, \dots, X_k]^{\mathrm{lin}}$ and a tuple of reals $\mathbf{c}'$ by

$$\begin{cases} \mathcal{L}' := (L_i^{(v)} \circ \varphi' : v \in M_K,\ i \in I_v(T)), \\ \mathbf{c}' := (c_{iv} : v \in M_K,\ i \in I_v(T)). \end{cases} \tag{16.4}$$

Let $v \in M_K$. Since $L_j^{(v)}|_T$ ($j \in I_v(T)$) form a basis of $\mathrm{Hom}(T, \overline{\mathbb{Q}})$, and since T is defined over K, there are unique $\alpha_{ijv} \in K$ such that $L_i^{(v)}|_T = \sum_{j \in I_v(T)} \alpha_{ijv} L_i^{(v)}|_T$ for $i \in I_v(T)^c := \{1, \dots, n\} \setminus I_v(T)$. By our definition of $I_v(T)$, we have $\alpha_{ijv} = 0$ for $i \in I_v(T)^c$, $j \in I_v(T)$, $j > i$. In other words, there are unique linear forms

$$\widetilde{L}_i^{(v)} = L_i^{(v)} - \sum_{\substack{j \in I_v(T) \\ j<i}} \alpha_{ijv} L_j^{(v)} \quad (i \in I_v(T)^c) \tag{16.5}$$

with $\alpha_{ijv} \in K$ that vanish identically on T. These linear forms are linearly independent, so they may be viewed as a basis of $\mathrm{Hom}(\overline{\mathbb{Q}}^n/T, \overline{\mathbb{Q}})$.

We now define a tuple $\mathcal{L}''$ in $K[X_1, \dots, X_{n-k}]^{\mathrm{lin}}$ and a tuple of reals $\mathbf{c}''$ by

$$\begin{cases} \mathcal{L}'' := (\widetilde{L}_i^{(v)} \circ \varphi''^{-1} : v \in M_K, i \in I_v(T)^c), \\ \mathbf{c}'' := (c_{iv} : v \in M_K, i \in I_v(T)^c). \end{cases} \tag{16.6}$$

Let U be a linear subspace of $\overline{\mathbb{Q}}^k$ of dimension u, say. Then $w_{\mathcal{L}',\mathbf{c}'}(U) = \sum_{v \in M_K} w_{\mathcal{L}',\mathbf{c}',v}(U)$ with, in analogy to (15.3),

$$w_{\mathcal{L}',\mathbf{c}',v}(U) = \begin{cases} 0 \text{ if } u = 0, \\ c_{i_1(v),v} + \cdots + c_{i_u(v),v} \text{ if } u > 0, \end{cases} \tag{16.7}$$

where $i_1(v)$ is the smallest index $i \in I_v(T)$ such that $L_i^{(v)} \circ \varphi'|_U \neq 0$ and for $l = 2, \dots, u$, $i_l(v)$ is the smallest index $i > i_{l-1}(v)$ in $I_v(T)$ such that $L_{i_1(v)}^{(v)} \circ \varphi'|_U, \dots, L_{i_{l-1}(v)}^{(v)} \circ \varphi'|_U$, $L_i^{(v)} \circ \varphi'|_U$ are linearly independent.

Likewise, if U is an u-dimensional linear subspace of $\overline{\mathbb{Q}}^{n-k}$, then $w_{\mathcal{L}'',\mathbf{c}''}(U) = \sum_{v \in M_K} w_{\mathcal{L}'',\mathbf{c}'',v}(U)$, with

$$w_{\mathcal{L}'',\mathbf{c}'',v}(U) = \begin{cases} 0 \text{ if } u = 0, \\ c_{i_1(v),v} + \cdots + c_{i_u(v),v} \text{ if } u > 0, \end{cases} \tag{16.8}$$

where $i_1(v)$ is the smallest index $i \in I_v(T)^c$ such that $\widetilde{L}_i^{(v)} \circ \varphi''^{-1}|_U \neq 0$ and for $l = 2, \dots, u$, $i_l(v)$ is the smallest index $i > i_{l-1}(v)$ in $I_v(T)^c$ such that $\widetilde{L}_{i_1(v)}^{(v)} \circ \varphi''^{-1}|_U, \dots, \widetilde{L}_{i_{l-1}(v)}^{(v)} \circ \varphi''^{-1}|_U$, $\widetilde{L}_i^{(v)} \circ \varphi''^{-1}|_U$ are linearly independent.

LEMMA 16.2. (i) *Let U be a linear subspace of $\overline{\mathbb{Q}}^k$. Then*

$$w_{\mathcal{L}',\mathbf{c}'}(U) = w_{\mathcal{L},\mathbf{c}}(\varphi'(U)).$$

(ii) *Let U be a linear subspace of $\overline{\mathbb{Q}}^{n-k}$. Then*

$$w_{\mathcal{L}'',\mathbf{c}''}(U) = w_{\mathcal{L},\mathbf{c}}(\varphi''^{-1}(U)) - w_{\mathcal{L},\mathbf{c}}(T).$$

Proof. (i) For $U=\{\mathbf{0}\}$ the assertion is true. Suppose U has dimension $u>0$. Let $v\in M_K$. The set $\{i_1(v),\dots,i_u(v)\}$ from (16.7) is precisely $I_v(\varphi'(U))$ since $I_v(\varphi'(U))\subseteq I_v(T)$. Therefore, $w_{\mathcal{L}',\mathbf{c}',v}(U)=w_{\mathcal{L},\mathbf{c},v}(\varphi'(U))$ for $v\in M_K$. Now (i) follows by summing over v.

(ii) Suppose U has dimension $u>0$. Let $v\in M_K$. Put $W:=\varphi''^{-1}(U)$. Recall that $I_v(W)=\{j_1(v),\dots,j_m(v)\}$, where $m:=\dim W$, $j_1(v)$ is the smallest index $j\in\{1,\dots,n\}$ such that $L_j^{(v)}|_W\neq 0$, etc. The indices $j_1(v),j_2(v),\dots$ do not change if we replace $L_j^{(v)}$ by $\widetilde{L}_j^{(v)}$ for $j\in I_v(T)^c$. This implies that the set $\{i_1(v),\dots,i_{n-k}(v)\}$ from (16.8) is $I_v(W)\setminus I_v(T)$, and so $w_{\mathcal{L}'',\mathbf{c}'',v}(U)=w_{\mathcal{L},\mathbf{c},v}(W)-w_{\mathcal{L},\mathbf{c},v}(T)$. By summing over v, we get (ii). $\square$

The pair $(\mathcal{L}',\mathbf{c}')$ gives rise to a class of twisted heights $H_{\mathcal{L}',\mathbf{c}',Q}:\overline{\mathbb{Q}}^k\to\mathbb{R}_{\geqslant 0}$ in the usual manner. That is, if $\mathbf{x}\in E^k$ for some finite extension E of K, then

$$H_{\mathcal{L}',\mathbf{c}',Q}(\mathbf{x})=\prod_{w\in M_E}\max_{i\in I_w(T)}\|L_i^{(w)}\circ\varphi'(\mathbf{x})\|_wQ^{-c_{iw}},\tag{16.9}$$

where $I_w(T):=I_v(T)$ if w lies above $v\in M_K$.

Likewise, we have twisted heights $H_{\mathcal{L}'',\mathbf{c}'',Q}:\overline{\mathbb{Q}}^{n-k}\to\mathbb{R}_{\geqslant 0}$, defined such that if $\mathbf{x}\in E^{n-k}$ for some finite extension E of K, then

$$H_{\mathcal{L}'',\mathbf{c}'',Q}(\mathbf{x})=\prod_{w\in M_E}\max_{i\in I_w^c(T)}\|\widetilde{L}_i^{(w)}\circ\varphi''^{-1}(\mathbf{x})\|_wQ^{-c_{iw}},\tag{16.10}$$

where $\widetilde{L}_i^{(w)}:=\widetilde{L}_i^{(v)}$ if w lies above $v\in M_K$.

In what follows, constants implied by $\ll$, $\gg$ depend only on $\mathcal{L},\mathbf{c}$ and T.

LEMMA 16.3. (i) *For $\mathbf{x}\in\overline{\mathbb{Q}}^k$, $Q\geqslant 1$, we have*

$$H_{\mathcal{L}',\mathbf{c}',Q}(\mathbf{x})\gg\ll H_{\mathcal{L},\mathbf{c},Q}(\varphi'(\mathbf{x})).$$

(ii) *For $\mathbf{x}\in\overline{\mathbb{Q}}^n$, $Q\geqslant 1$, we have*

$$H_{\mathcal{L}'',\mathbf{c}'',Q}(\varphi''(\mathbf{x}))\ll H_{\mathcal{L},\mathbf{c},Q}(\mathbf{x}).$$

Proof. (i) The inequality $H_{\mathcal{L}',\mathbf{c}',Q}(\mathbf{x})\leqslant H_{\mathcal{L},\mathbf{c},Q}(\varphi'(\mathbf{x}))$ for $\mathbf{x}\in\overline{\mathbb{Q}}^k$, $Q\geqslant 1$ is trivial. We prove the reverse inequality. Since the linear forms $\widetilde{L}_i^{(v)}$ $(i\in I_v(T)^c)$ defined in (16.5) vanish identically on T, there exist constants $C_v>0$ $(v\in M_K)$, all but finitely many of which are 1, such that for $\mathbf{x}\in K^k$, $v\in M_K$, $i\in I_v(T)^c$,

$$\|L_i^{(v)}(\varphi'(\mathbf{x}))\|_v\leqslant C_v\max_{\substack{j\in I_v(T)\\ j<i}}\|L_j^{(v)}(\varphi'(\mathbf{x}))\|_v.$$

Taking $Q\geqslant 1$ we obtain, in view of (15.1),

$$\|L_i^{(v)}(\varphi'(\mathbf{x}))\|_vQ^{-c_{iv}}\leqslant C_v\max_{\substack{j\in I_v(T)\\ j<i}}\|L_j^{(v)}(\varphi'(\mathbf{x}))\|_vQ^{-c_{jv}}.$$

This shows that for $\mathbf{x} \in K^k$, $Q \geqslant 1$, $v \in M_K$, we have

$$\max_{1\leqslant i\leqslant n} \|L_i^{(v)}(\varphi'(\mathbf{x}))\|_v Q^{-c_{iv}} \leqslant C_v \max_{j\in I_v(T)} \|L_j^{(v)}(\varphi'(\mathbf{x}))\|_v Q^{-c_{jv}}.$$

If instead we have $\mathbf{x} \in E^k$ for some finite extension E of K, we have the same inequalities for $w \in M_E$, but with constants $C_w := C_v^{d(w|v)}$, where $v \in M_K$ is the place below w. By taking the product over $w \in M_E$, we get (i).

The proof of (ii) is entirely similar. □

LEMMA 16.4. *Suppose that*

(16.11) $\quad \mu(\overline{\mathbb{Q}}^n, U) \geqslant \mu(\overline{\mathbb{Q}}^n, \{\mathbf{0}\})$ *for every proper linear subspace U of $\overline{\mathbb{Q}}^n$.*

Then for every $\delta > 0$, there is Q_0 such that for every $Q \geqslant Q_0$,

(16.12) $$Q^{-\mu(\overline{\mathbb{Q}}^n,\{\mathbf{0}\})-\delta} \leqslant \lambda_1(Q) \leqslant \cdots \leqslant \lambda_n(Q) \leqslant Q^{-\mu(\overline{\mathbb{Q}}^n,\{\mathbf{0}\})+\delta}.$$

Proof. We first assume that $n = 1$. In this case, $L_1^{(v)} = \alpha_v X$ with $\alpha_v \in K^*$ for $v \in M_K$, and $\mu(\overline{\mathbb{Q}}, \{0\}) = \sum_{v\in M_K} c_{1v}$. By the product formula, we have for $\mathbf{x} = x \in K^*$,

$$H_{\mathcal{L},\mathbf{c},Q}(x) = \prod_{v\in M_K} \|\alpha_v x\|_v Q^{-c_{1v}} = CQ^{-\mu(\overline{\mathbb{Q}},\{0\})}$$

for some nonzero constant C. This is true also for $x \notin K$. So for $n = 1$, our lemma is trivially true.

Next, we assume $n \geqslant 2$. We first make some reductions and then apply Theorem 8.1. By Lemma 7.2 there is no loss of generality if in the proof of our lemma, we replace c_{iv} by $c'_{iv} := c_{iv} - \frac{1}{n}\sum_{j=1}^n c_{jv}$ for $v \in M_K$, $j = 1, \ldots, n$. This shows that there is no loss of generality to assume that $\sum_{i=1}^n c_{iv} = 0$ for $v \in M_K$, i.e., condition (8.3). This being the case, suppose that $\sum_{v\in M_K} \max_{1\leqslant i\leqslant n} c_{iv} \leqslant \theta$ with $\theta > 0$. Then we can make a reduction to (8.4) by replacing Q by Q^θ and c_{iv} by c_{iv}/θ for $v \in M_K$, $i = 1, \ldots, n$. So we may also assume that (8.4) is satisfied. Finally, by Lemma 7.3 and the subsequent remark, there is no loss of generality to assume (8.8). Under assumption (8.3), condition (16.11) translates into (8.9). So we may assume without loss of generality that all conditions of Theorem 8.1 are satisfied. Notice that with these assumptions,

$$\mu(\overline{\mathbb{Q}}^n, \{\mathbf{0}\}) = \frac{1}{n} \sum_{v\in M_K} \sum_{i=1}^n c_{iv} = 0.$$

Let $0 < \delta \leqslant 1$. Theorem 8.1 implies that the set of Q with $\lambda_1(Q) \leqslant Q^{-\delta/2n}$ is bounded. Together with (9.2), this implies that for every sufficiently large Q, we have $\lambda_1(Q) \geqslant Q^{-\delta/2n}$, $\lambda_n(Q) \leqslant Q^\delta$. □

Proof of Theorem 16.1. We proceed by induction on r. For $r=1$, we can apply Lemma 16.4. Assume $r \geqslant 2$. We fix $\delta > 0$ and then $\delta' > 0$, which is a sufficiently small function of δ. We write w for $w_{\mathcal{L},\mathbf{c}}$ and μ for $\mu_{\mathcal{L},\mathbf{c}}$.

By Lemma 16.2(ii) with $T = T_{r-1}$, $k = d_{r-1} = \dim T$, we have for any two linear subspaces $U_1 \subsetneq U_2$ of $\overline{\mathbb{Q}}^{n-d_{r-1}}$ that

$$\mu_{\mathcal{L}'',\mathbf{c}''}(U_2, U_1) = \mu(\varphi''^{-1}(U_2), \varphi''^{-1}(U_1)).$$

Thus, the property of T_{r-1} that $\mu(\overline{\mathbb{Q}}^n, T_{r-1}) \leqslant \mu(\overline{\mathbb{Q}}^n, U)$ for any proper linear subspace U of $\overline{\mathbb{Q}}^n$ translates into

$$\mu_{\mathcal{L}'',\mathbf{c}''}(\overline{\mathbb{Q}}^{n-d_{r-1}}, \{\mathbf{0}\}) \leqslant \mu_{\mathcal{L}'',\mathbf{c}''}(\overline{\mathbb{Q}}^{n-d_{r-1}}, U)$$

for any proper linear subspace U of $\overline{\mathbb{Q}}^{n-d_{r-1}}$. So by Lemma 16.4, we have for every sufficiently large Q,

$$H_{\mathcal{L}'',\mathbf{c}'',Q}(\mathbf{y}) \geqslant Q^{-\mu(\overline{\mathbb{Q}}^n, T_{r-1}) - \delta'} \quad \text{for } \mathbf{y} \in \overline{\mathbb{Q}}^{n-d_{r-1}} \setminus \{\mathbf{0}\}.$$

Together with Lemma 16.3(ii), this implies for every sufficiently large Q,

$$H_{\mathcal{L},\mathbf{c},Q}(\mathbf{x}) \geqslant Q^{-\mu(\overline{\mathbb{Q}}^n, T_{r-1}) - 2\delta'} \quad \text{for } \mathbf{x} \in \overline{\mathbb{Q}}^n \setminus T_{r-1}. \tag{16.13}$$

Consequently, for every sufficiently large Q, we have

$$Q^{-\mu(\overline{\mathbb{Q}}^n, T_{r-1}) - 2\delta'} \leqslant \lambda_{d_{r-1}+1}(Q) \leqslant \cdots \leqslant \lambda_n(Q). \tag{16.14}$$

For $i = 1, \ldots, d_{r-1}$, denote by $\lambda'_i(Q)$ the i-th successive infimum of $H_{\mathcal{L},\mathbf{c},Q}$ restricted to T_{r-1}; i.e., the infimum of all $\lambda > 0$ such that the set of $\mathbf{x} \in T_{r-1}$ with $H_{\mathcal{L},\mathbf{c},Q}(\mathbf{x}) \leqslant \lambda$ contains at least i linearly independent points. By Lemma 16.3(i) with $T = T_{r-1}$, $k = d_{r-1}$ this is, apart from bounded multiplicative factors independent of Q, equal to the i-th successive infimum of $H_{\mathcal{L}',\mathbf{c}',Q}$. Further, by Lemma 16.2(i) with $T = T_{r-1}$, $k = d_{r-1}$, for any two subspaces $U_1 \subsetneq U_2$ of $\overline{\mathbb{Q}}^{d_{r-1}}$, we have $w_{\mathcal{L}',\mathbf{c}'}(U_2, U_1) = w(\varphi'(U_2), \varphi'(U_2))$. By applying the induction hypothesis to $(\mathcal{L}', \mathbf{c}')$ and then carrying it over to T_{r-1} by means of φ', we infer that for every sufficiently large Q, we have

$$Q^{-\mu(T_l, T_{l-1}) - \delta'} \leqslant \lambda'_i(Q) \leqslant Q^{-\mu(T_l, T_{l-1}) + \delta'} \tag{16.15}$$

for $l = 1, \ldots, r-1$, $i = d_{l-1}+1, \ldots, d_l$ and moreover,

$$\bigcap_{\lambda > \lambda'_{d_l}(Q)} \operatorname{span}\{\mathbf{x} \in T_{r-1} : H_{\mathcal{L},\mathbf{c},Q}(\mathbf{x}) \leqslant \lambda\} = T_l \tag{16.16}$$

for $l = 1, \ldots, r-1$. Clearly, we have $\lambda_i(Q) \leqslant \lambda'_i(Q)$ for $i = 1, \ldots, d_{r-1}$, and so

$$\lambda_{d_{r-1}}(Q) \leqslant Q^{-\mu(T_{r-1}, T_{r-2}) + \delta'}$$

for Q sufficiently large. Assuming δ' is sufficiently small, this is smaller than the lower bound $Q^{-\mu(\overline{\mathbb{Q}}^n, T_{r-1}) - 2\delta'}$ in (16.13). Hence for sufficiently large Q and sufficiently small ε, all vectors $\mathbf{x} \in \overline{\mathbb{Q}}^n$ with $H_{\mathcal{L},\mathbf{c},Q}(\mathbf{x}) \leqslant \lambda_{d_{r-1}}(Q) + \varepsilon$ lie in T_{r-1}. That is,

$$T_{d_{r-1}}(Q) = T_{r-1}, \;\; \lambda_i(Q) = \lambda_i'(Q) \text{ for } i = 1, \ldots, d_{r-1}.$$

Together with (16.16), this implies (16.3). Further, (16.15) becomes

$$Q^{-\mu(T_l,T_{l-1})-\delta'} \leqslant \lambda_i(Q) \leqslant Q^{-\mu(T_l,T_{l-1})+\delta'} \tag{16.17}$$

for $l = 1, \ldots, r-1$, $i = d_{l-1}+1, \ldots, d_l$. Using subsequently Proposition 9.2, the lower bounds in (16.17), (16.14), and that the quantity $\alpha = \sum_{v\in M_K}\sum_{i=1}^n c_{iv}$ from Proposition 9.2 equals

$$w(\overline{\mathbb{Q}}^n, \{\mathbf{0}\}) = \sum_{l=1}^{r} w(T_l, T_{l-1}) = \sum_{l=1}^{r} d_l \mu(T_l, T_{l-1}),$$

and taking δ' sufficiently small, we infer that for every sufficiently large Q,

$$\begin{aligned}\lambda_n(Q) &\leqslant 2^{n(n-1)/2}\Delta_{\mathcal{L}} Q^{-\alpha}\big(\lambda_1(Q)\cdots\lambda_{n-1}(Q)\big)^{-1} \\ &\leqslant Q^{-\alpha+\sum_{l=1}^{r} d_l\mu(T_l,T_{l-1})-\mu(\overline{\mathbb{Q}}^n,T_{r-1})+2n\delta'} \leqslant Q^{-\mu(\overline{\mathbb{Q}}^n,T_{r-1})+\delta}.\end{aligned}$$

As a consequence, (16.2) holds as well. This completes our proof. $\square$

17. A height estimate for the filtration subspaces

As before, K is a number field, n an integer $\geqslant 2$, and $(\mathcal{L}, \mathbf{c})$ a pair with (2.4)–(2.7). We derive an upper bound for the heights of the spaces occurring in the filtration of $(\mathcal{L}, \mathbf{c})$ in terms of the heights of the linear forms from $\mathcal{L}$. We start with some auxiliary results.

Let p be an integer with $1 < p < n$. Put $N := \binom{n}{p}$. Similarly as in Section 6, let $C(n, p) = (I_1, \ldots, I_N)$ be the lexicographically ordered sequence of p-element subsets of $\{1, \ldots, n\}$. For $j = 1, \ldots, N$, $v \in M_K$, define

$$\widehat{L}_j^{(v)} := L_{i_1}^{(v)} \wedge \cdots \wedge L_{i_p}^{(v)}, \;\; \widehat{c}_{jv} := c_{i_1,v} + \cdots + c_{i_p,v}, \tag{17.1}$$

where $I_j = \{i_1 < \cdots < i_p\}$ is the j-th set from $C(n, p)$, and put

$$\begin{cases} \widehat{\mathcal{L}} := (\widehat{L}_j^{(v)} : v \in M_K, \, j = 1, \ldots, N), \\ \widehat{\mathbf{c}} := (\widehat{c}_{jv} : v \in M_K, \, j = 1, \ldots, N). \end{cases} \tag{17.2}$$

Then $H_{\widehat{\mathcal{L}},\widehat{\mathbf{c}},Q} : \overline{\mathbb{Q}}^N \to \mathbb{R}_{\geqslant 0}$ is defined in a similar manner as $H_{\mathcal{L},\mathbf{c},Q}$; i.e., if $\widehat{\mathbf{x}} \in E^N$ for some finite extension E of K, then

$$H_{\widehat{\mathcal{L}},\widehat{\mathbf{c}},Q}(\widehat{\mathbf{x}}) := \prod_{w\in M_E} \max_{1\leqslant j\leqslant N} \|\widehat{L}_j^{(w)}(\widehat{\mathbf{x}})\|_w Q^{-\widehat{c}_{jw}},$$

where $\widehat{L}_j^{(w)} := \widehat{L}_j^{(v)}$, $\widehat{c}_{jw} := d(w|v)\widehat{c}_{jv}$ if w lies above $v \in M_K$.

LEMMA 17.1. *Let* $\mathbf{x}_1, \dots, \mathbf{x}_p \in \overline{\mathbb{Q}}^n$, $Q \geqslant 1$. *Then*

$$H_{\widehat{\mathcal{L}},\widehat{\mathbf{c}},Q}(\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_p) \leqslant p^{p/2} H_{\mathcal{L},\mathbf{c},Q}(\mathbf{x}_1) \cdots H_{\mathcal{L},\mathbf{c},Q}(\mathbf{x}_p).$$

Proof. Put $\widehat{\mathbf{x}} := \mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_p$. Let E be a finite extension of K such that $\mathbf{x}_1, \dots, \mathbf{x}_p \in E^n$. Let $I_j = \{i_1 < \cdots < i_p\}$ be one of the p-element subsets from $I_1, \dots, I_N$, and let $w \in M_E$. Then by an argument completely similar to the proofs of (4.5), (4.6), one shows

$$\begin{aligned}
\|\widehat{L}_j^{(w)}(\widehat{\mathbf{x}})\|_w Q^{-\widehat{c}_{jw}} &= \|\det\left(L_{i_k}^{(w)}(\mathbf{x}_l)\right)_{k,l=1,\dots,p}\|_w Q^{-\widehat{c}_{jw}} \\
&\leqslant p^{ps(w)/2} \prod_{l=1}^{p} \max_{1 \leqslant k \leqslant p} \|L_{i_k}^{(w)}(\mathbf{x}_l)\|_w Q^{-c_{i_k,w}} \\
&\leqslant p^{ps(w)/2} \prod_{l=1}^{p} \max_{1 \leqslant i \leqslant n} \|L_i^{(w)}(\mathbf{x}_l)\|_w Q^{-c_{iw}}.
\end{aligned}$$

By taking the maximum over $j = 1, \dots, N$ and then the product over $w \in M_E$, our lemma follows. $\square$

We keep the notation from above. For $Q \geqslant 1$, let $\lambda_1(Q), \dots, \lambda_n(Q)$ denote the successive infima of $H_{\mathcal{L},\mathbf{c},Q}$. Further, let $\nu_1(Q), \dots, \nu_N(Q)$ be the products $\lambda_{i_1}(Q) \cdots \lambda_{i_p}(Q)$ $(1 \leqslant i_1 < \cdots < i_p \leqslant n)$, ordered such that

$$\nu_1(Q) \leqslant \cdots \leqslant \nu_N(Q),$$

and let $\widehat{\lambda}_1(Q), \dots, \widehat{\lambda}_N(Q)$ denote the successive infima of $H_{\widehat{\mathcal{L}},\widehat{\mathbf{c}},Q}$.

LEMMA 17.2. *For* $Q \geqslant 1$, $j = 1, \dots, N$, *we have*

$$N^{-npN} \nu_j(Q) \leqslant \widehat{\lambda}_j(Q) \leqslant p^{p/2} \nu_j(Q).$$

Proof. Fix $Q \geqslant 1$, and write $\lambda_i, \widehat{\lambda}_j, \nu_j$ for $\lambda_i(Q), \widehat{\lambda}_j(Q), \nu_j(Q)$. Let $\varepsilon > 0$. Choose $\overline{\mathbb{Q}}$-linearly independent vectors $\mathbf{g}_1, \dots, \mathbf{g}_n \in \overline{\mathbb{Q}}^n$ such that $H_{\mathcal{L},\mathbf{c},Q}(\mathbf{g}_i) \leqslant \lambda_i(1+\varepsilon)$ for $i = 1, \dots, n$. Then the vectors $\mathbf{g}_{i_1} \wedge \cdots \wedge \mathbf{g}_{i_p}$ $(1 \leqslant i_1 < \cdots < i_p \leqslant n)$ are $\overline{\mathbb{Q}}$-linearly independent. Let $j \in \{1, \dots, N\}$, and let $i_1, \dots, i_p$ be the indices from $\{1, \dots, n\}$ such that $i_1 < \cdots < i_p$ and $\nu_j = \lambda_{i_1} \cdots \lambda_{i_p}$. Then by Lemma 17.1,

$$H_{\widehat{\mathcal{L}},\widehat{\mathbf{c}},Q}(\mathbf{g}_{i_1} \wedge \cdots \wedge \mathbf{g}_{i_p}) \leqslant p^{p/2}(1+\varepsilon)^p \nu_j. \tag{17.3}$$

So $\widehat{\lambda}_j \leqslant p^{p/2}(1+\varepsilon)^p \nu_j$. This holds for every $\varepsilon > 0$; hence,

$$\frac{\widehat{\lambda}_j}{\nu_j} \leqslant p^{p/2} \quad \text{for } j = 1, \dots, N. \tag{17.4}$$

Put

$$\widehat{\alpha} := \sum_{v \in M_K} \sum_{j=1}^{N} \widehat{c}_{jv}.$$

Notice that $\widehat{\alpha} = N'\alpha$, where $\alpha := \sum_{v\in M_K}\sum_{j=1}^{n} c_{iv}$, $N' := \binom{n-1}{p-1}$. Also, by (6.6), $\Delta_{\widehat{\mathcal{L}}} = \Delta_{\mathcal{L}}^{N'}$. These facts together with Proposition 9.2 imply

$$\nu_1\cdots\nu_N \leqslant 2^{n(n-1)N'/2}\Delta_{\widehat{\mathcal{L}}}Q^{-\widehat{\alpha}}.$$

On the other hand, Proposition 9.2 applied to $\widehat{\mathcal{L}}, \widehat{\mathbf{c}}$ gives

$$\widehat{\lambda}_1\cdots\widehat{\lambda}_N \geqslant N^{-N/2}\Delta_{\widehat{\mathcal{L}}}Q^{-\widehat{\alpha}},$$

and so

$$\prod_{j=1}^{N}\frac{\widehat{\lambda}_j}{\nu_j} \geqslant N^{-N/2}2^{-n(n-1)N'/2}.$$

Now our lemma follows by combining this with (17.4). □

Let $T_i(Q)$ $(i = 1,\ldots,n)$ be the spaces defined by (16.1). Further, define the linear subspaces of $\overline{\mathbb{Q}}^N$,

$$\widehat{T}_j(Q) := \bigcap_{\lambda>\widehat{\lambda}_j(Q)} \operatorname{span}\{\widehat{\mathbf{x}}\in\overline{\mathbb{Q}}^N : H_{\widehat{\mathcal{L}},\widehat{\mathbf{c}},Q}(\widehat{\mathbf{x}})\leqslant\lambda\}\quad (j=1,\ldots,N).$$

LEMMA 17.3. *Put $k := n-p$. Let $Q\geqslant 1$, and suppose that*

$$\lambda_{k+1}(Q) > 2^{2n^3 2^n}\lambda_k(Q). \tag{17.5}$$

Then

$$\frac{\widehat{\lambda}_{N-1}(Q)}{\widehat{\lambda}_N(Q)} \leqslant 2^{n^3 2^n}\frac{\lambda_k(Q)}{\lambda_{k+1}(Q)} < 2^{-n^3 2^n}, \tag{17.6}$$

$$H_2(\widehat{T}_{N-1}(Q)) = H_2(T_k(Q)). \tag{17.7}$$

Proof. Write again $\lambda_i, \widehat{\lambda}_j, \nu_j$ for $\lambda_i(Q), \widehat{\lambda}_j(Q), \nu_j(Q)$. Since

$$\nu_{N-1} = \lambda_k\lambda_{k+2}\cdots\lambda_N,\quad \nu_N = \lambda_{k+1}\cdots\lambda_N,$$

we have $\nu_{N-1}/\nu_N = \lambda_k/\lambda_{k+1}$. Together with Lemma 17.2, $N = \binom{n}{p}\leqslant 2^n$ and assumption (17.5), this implies (17.6).

As for (17.7), let $\varepsilon > 0$. Put $T := T_k(Q)$, $\widehat{T} := \widehat{T}_{N-1}(Q)$. Choose $\overline{\mathbb{Q}}$-linearly independent vectors $\mathbf{g}_1,\ldots,\mathbf{g}_n$ such that $H_{\mathcal{L},\mathbf{c},Q}(\mathbf{g}_i)\leqslant(1+\varepsilon)\lambda_i$ for $i=1,\ldots,n$. Write $\widehat{\mathbf{g}}_j := \mathbf{g}_{i_1}\wedge\cdots\wedge\mathbf{g}_{i_p}$, where $I_j = \{i_1<\cdots<i_p\}$ is the j-th set in $C(n,p)$. Then by (17.3),

$$H_{\widehat{\mathcal{L}},\widehat{\mathbf{c}},Q}(\widehat{\mathbf{g}}_j)\leqslant p^{p/2}(1+\varepsilon)^p\nu_{N-1}\ \text{for}\ j=1,\ldots,N-1.$$

Assuming ε is sufficiently small, $\{\mathbf{g}_1,\ldots,\mathbf{g}_k\}$ is a basis of T. Moreover, by Lemma 17.2 and (17.6), we have $p^{p/2}(1+\varepsilon)^p\nu_{N-1} < \widehat{\lambda}_N$. Hence by (17.3), $\{\widehat{\mathbf{g}}_1,\ldots,\widehat{\mathbf{g}}_{N-1}\}$ is a basis of $\widehat{T}$. Now $H_2(\widehat{T}) = H_2(T)$ follows from Lemma 6.1. □

We now make a first step towards estimating the heights of the subspaces in the filtration of $(\mathcal{L}, \mathbf{c})$. As usual, n is an integer $\geqslant 2$, K an algebraic number field, and $(\mathcal{L}, \mathbf{c})$ a pair satisfying (2.4)–(2.7). Put

$$H_2 := \max\{H_2(L_i^{(v)}) : v \in M_K,\, i = 1, \dots, n\}.$$

Lemma 17.4. *Assume that the subspace T_{r-1} preceding $\overline{\mathbb{Q}}^n$ in the filtration of $(\mathcal{L}, \mathbf{c})$ has dimension $n-1$. Then*

$$H_2(T_{r-1}) \leqslant H_2^{(n-1)^2}.$$

Proof. We assume without loss of generality that $c_{1v} \leqslant \cdots \leqslant c_{nv}$ for $v \in M_K$. Put $T := T_{r-1}$. By our choice of T, if T' is any other $(n-1)$-dimensional linear subspace of $\overline{\mathbb{Q}}^n$, then $\mu(\overline{\mathbb{Q}}^n, T) < \mu(\overline{\mathbb{Q}}^n, T')$, implying $w(T') < w(T)$.

Take $v \in M_K$. Let $i(v)$ be the smallest index i such that

$$U_{iv} := \{\mathbf{x} \in \overline{\mathbb{Q}}^n : L_1^{(v)}(\mathbf{x}) = \cdots = L_i^{(v)}(\mathbf{x}) = 0\} \subseteq T.$$

T is given by an up to a constant factor unique linear equation, which we may express as $\sum_{j=1}^n \alpha_{jv} L_j^{(v)}(\mathbf{x}) = 0$, where not all α_{jv} are 0. In fact, T is given by $\sum_{j=1}^{i(v)} \alpha_{jv} L_j^{(v)}(\mathbf{x}) = 0$, where $\alpha_{i(v),v} \neq 0$. It follows that $i(v)$ is the largest index i such that $\{L_i^{(v)}|_T : j \in \{1, \dots, n\} \setminus \{i\}\}$ is linearly independent. Hence,

$$w(T) = \sum_{v \in M_K} w_v(T) = \sum_{v \in M_K} \sum_{\substack{j=1 \\ j \neq i(v)}}^{n} c_{jv}. \tag{17.8}$$

Moreover,

$$\sum_{v \in M_K} U_{i(v),v} \subseteq T. \tag{17.9}$$

We prove that in (17.9) we have equality. Assume the contrary. Then there is an $(n-1)$-dimensional linear subspace $T' \neq T$ of $\overline{\mathbb{Q}}^n$ such that $\sum_{v \in M_K} U_{i(v),v} \subset T'$. Then if $j(v)$ denotes the smallest index i such that $U_{iv} \subseteq T'$, we have $j(v) \leqslant i(v)$ for $v \in M_K$. So

$$w(T') = \sum_{v \in M_K} \sum_{\substack{j=1 \\ j \neq j(v)}}^{n} c_{jv} \geqslant w(T),$$

contrary to what we observed above.

Knowing that we have equality in (17.9), there is a subset $\{v_1, \dots, v_s\}$ of M_K with $s \leqslant n-1$ such that $T = U_{i(v_1),v_1} + \cdots + U_{i(v_s),v_s}$. By (6.13), (6.11), we have

$$H_2(U_{i(v_l),v_l}) = H_2(U_{i(v_l),v_l}^{\perp}) \leqslant H_2^{n-1} \quad \text{for } l = 1, \dots, s,$$

and then by (6.12),

$$H_2(T) \leqslant \prod_{l=1}^{s} H_2(U_{i(v_l),v_l}) \leqslant H_2^{(n-1)^2}.$$

This completes our proof. □

Our final result is as follows.

PROPOSITION 17.5. *Let $T_1, \dots, T_{r-1}$ be the subspaces of $\overline{\mathbb{Q}}^n$ in the filtration of $(\mathcal{L}, \mathbf{c})$. Put $H_2 := \max\{H_2(L_i^{(v)}) : v \in M_K, i = 1, \dots, n\}$. Then*

$$H_2(T_i) \leqslant H_2^{4^n} \quad \text{for } i = 1, \dots, r-1.$$

Proof. Let $i \in \{1, \dots, r-1\}$, and put $T := T_i$, $k := \dim T$, $p := n - k$, $N := \binom{n}{p}$. Further, let $\widehat{\mathcal{L}}$, $\widehat{\mathbf{c}}$ be as in (17.1), (17.2). By (6.8), for the linear forms $\widehat{L}_j^{(v)}$ in $\widehat{\mathcal{L}}$, we have

$$H_2(\widehat{L}_j^{(v)}) \leqslant H_2^p \quad \text{for } v \in M_K,\ j = 1, \dots, N. \tag{17.10}$$

Let $0 < \theta < \mu(T_{i+1}, T_i) - \mu(T_{i+2}, T_{i+1})$. By Theorem 16.1, for every sufficiently large Q, we have that

$$T_k(Q) = T \tag{17.11}$$

and $\lambda_k(Q)/\lambda_{k+1}(Q) \leqslant Q^{-\theta}$. Together with Lemma 17.3(i), this implies that for Q sufficiently large, we have $\widehat{\lambda}_{N-1}(Q)/\widehat{\lambda}_N(Q) \leqslant Q^{-\theta/2}$, with a positive exponent $\theta/2$ independent of Q, and so $\dim \widehat{T}_{N-1}(Q) = N - 1$. Again from Theorem 16.1, but now applied with $\widehat{\mathcal{L}}, \widehat{\mathbf{c}}, N$ instead of $\mathcal{L}, \mathbf{c}, n$, it follows that there is a subspace $\widehat{T}$ of dimension $N - 1$ in the filtration of $(\widehat{\mathcal{L}}, \widehat{\mathbf{c}})$, such that

$$\widehat{T}_{N-1}(Q) = \widehat{T}$$

for every sufficiently large Q.

Now using subsequently (17.11), Lemma 17.3(ii), Lemma 17.4 (with $\widehat{\mathcal{L}}, \widehat{\mathbf{c}}, N$ instead of $\mathcal{L}, \mathbf{c}, n$), and (17.10), we obtain for Q sufficiently large,

$$H_2(T) = H_2(T_k(Q)) = H_2(\widehat{T}_{N-1}(Q)) = H_2(\widehat{T}) \leqslant (H_2^p)^{(N-1)^2} \leqslant H_2^{4^n},$$

where in the last step we have used $p(N-1)^2 \leqslant p\binom{n}{p}^2 \leqslant 4^n$. This completes our proof. □

18. **Proof of Theorem 2.3**

Let $n, \mathcal{L}, \mathbf{c}, \delta, R$ satisfy (2.4)–(2.10). Let $T = T(\mathcal{L}, \mathbf{c})$ be the subspace from (2.21). Recall that this space is defined over K. The hard core of our proof is to make explicit Lemma 16.3(ii).

Put $k := \dim T$. Choose a basis $\{\mathbf{g}_1, \dots, \mathbf{g}_k\}$ of T, contained in K^n. Write in the usual manner $\bigcup_{v \in M_K} \{L_1^{(v)}, \dots, L_n^{(v)}\} = \{L_1, \dots, L_r\}$, where $r \leqslant R$, and let $\theta_1, \dots, \theta_u$ be the distinct, nonzero numbers among

$$(\det(L_{i_l}(\mathbf{g}_j))_{l,j=1,\dots,k}\,, \quad 1 \leqslant i_1 < \cdots < i_k \leqslant r. \tag{18.1}$$

For $v \in M_K$, put

$$M_v := \max(\|\theta_1\|_v, \dots, \|\theta_u\|_v), \quad m_v := \min(\|\theta_1\|_v, \dots, \|\theta_u\|_v).$$

LEMMA 18.1. *We have*

$$\prod_{v \in M_K} \frac{M_v}{m_v} \leqslant (2H_{\mathcal{L}})^{(4R)^n}.$$

Proof. Let φ be a linear transformation of $\overline{\mathbb{Q}}^n$, defined over K. By Lemma 7.3, replacing $\mathcal{L}$ by $\mathcal{L} \circ \varphi$ has the effect that $T = T(\mathcal{L}, \mathbf{c})$ is replaced by $\varphi^{-1}(T)$. Taking the basis $\varphi^{-1}(\mathbf{g}_1), \dots, \varphi^{-1}(\mathbf{g}_k)$ of $\varphi^{-1}(T)$, we see that the quotients M_v/m_v $(v \in M_K)$ remain unchanged. This shows that to prove our lemma, we may replace $\mathcal{L}$ by $\mathcal{L} \circ \varphi$. Now choose linearly independent $L_1, \dots, L_n$ from $\mathcal{L}$, and then φ such that $L_i \circ \varphi = X_i$ for $i = 1, \dots, n$. Then $L \circ \varphi$ contains $X_1, \dots, X_n$.

So we may assume without loss of generality that $\mathcal{L}$ contains $X_1, \dots, X_n$ and then apply Lemma 10.2. Thus, we conclude that

$$\prod_{v \in M_K} \frac{M_v}{m_v} \leqslant \left(\binom{n}{k}^{1/2} H_{\mathcal{L}} \cdot H_2(T) \right)^{\binom{r}{k}}. \tag{18.2}$$

We estimate $H_2(T)$ from above by means of Proposition 17.5. The coefficients of $L_1, \dots, L_r$ belong to the set $\{d_1, \dots, d_m\}$ from Lemma 10.1. Hence,

$$H_2(L_i) \leqslant n^{1/2} \prod_{v \in M_K} \max(\|d_1\|_v, \dots, \|d_m\|_v) \leqslant n^{1/2} H_{\mathcal{L}}$$

for $i = 1, \dots, r$, and so $H_2(T) \leqslant (n^{1/2} H_{\mathcal{L}})^{4^n}$. By inserting this inequality together with $\binom{r}{k} \leqslant R^n/n!$ into (18.2), we infer

$$\prod_{v \in M_K} \frac{M_v}{m_v} \leqslant \left(\binom{n}{k}^{1/2} n^{4^n/2} \cdot H_{\mathcal{L}}^{4^n+1} \right)^{R^n/n!} \leqslant (2H_{\mathcal{L}})^{(4R)^n}. \qquad \square$$

In addition to (2.4)–(2.10), we assume that

$$c_{1v} \leqslant \cdots \leqslant c_{nv} \quad \text{for } v \in M_K, \tag{18.3}$$

which is no restriction.

By (15.3), we have

$$w(T) = w_{\mathcal{L},\mathbf{c}}(T) = \sum_{v \in M_K} \sum_{i \in I_v} c_{iv},$$

where $I_v = I_v(T) = \{i_1(v), \dots, i_k(v)\}$ is the set defined by (15.2). Put $I_v^c := \{1, \dots, n\} \setminus T$.

Let

$$\widetilde{L}_i^{(v)} := L_i^{(v)} - \sum_{\substack{j \in I_v \\ j<i}} \alpha_{ijv} L_j^{(v)} \quad (v \in M_K,\ i \in I_v^c) \tag{18.4}$$

be the linear forms from (16.5). Recall that these linear forms vanish identically on T. For $v \in M_K$, $i \in I_v$, put $\widetilde{L}_i^{(v)} := L_i^{(v)}$, and define the system

$$\widetilde{\mathcal{L}} := (\widetilde{L}_i^{(v)} : v \in M_K,\ i = 1, \dots, n).$$

Clearly, for every $v \in M_K$, the set $\{\widetilde{L}_i^{(v)} : i = 1, \dots, n\}$ is linearly independent.

LEMMA 18.2. *The system* $\widetilde{\mathcal{L}}$ *has the following properties*:

$$H_{\widetilde{\mathcal{L}},\mathbf{c},Q}(\mathbf{x}) \leqslant (2H_{\mathcal{L}})^{(8R)^n} H_{\mathcal{L},\mathbf{c},Q}(\mathbf{x}) \quad \textit{for } \mathbf{x} \in \overline{\mathbb{Q}}^n,\ Q \geqslant 1, \tag{18.5}$$

$$H_{\widetilde{\mathcal{L}}} \leqslant (nH_{\mathcal{L}})^{(8R)^n}. \tag{18.6}$$

Proof. Let $v \in M_K$. We find expressions for the coefficients α_{ijv} from the relations

$$L_i^{(v)}(\mathbf{g}_h) = \sum_{j \in I_v} \alpha_{ijv} L_j^{(v)}(\mathbf{g}_h) \quad \text{for } i \in I_v^c,\ h = 1, \dots, k$$

and Cramer's rule. Recall that $\alpha_{ijv} = 0$ for $j > i$ by the definition of I_v. In fact, each α_{ijv} is of the shape δ_{ijv}/δ_v, where $\delta_v = \det\Big((L_{i_l(v)}^{(v)}(\mathbf{g}_h))_{l,h=1,\dots,k}\Big)$, and δ_{ijv} is a similar sort of determinant, but with $L_j^{(v)}$ replaced by $L_i^{(v)}$. Clearly, δ_v and the numbers δ_{ijv} all occur among the numbers (18.1). Hence

$$\|\alpha_{ijv}\|_{v'} \leqslant \frac{M_{v'}}{m_{v'}} \quad \text{for } i \in I_v^c,\ j \in I_v,\ v' \in M_K. \tag{18.7}$$

We now prove (18.5). Let $\mathbf{x} \in \overline{\mathbb{Q}}^n$, $Q \geqslant 1$, and choose a finite extension E of K such that $\mathbf{x} \in E^n$. For $w \in M_E$ lying above $v \in M_K$, define in the usual manner $c_{iw}, L_i^{(w)}$ by (2.14) and similarly, $\widetilde{L}_i^{(w)} := \widetilde{L}_i^{(v)}$, $\alpha_{ijw} := \alpha_{ijv}$, $I_w := I_v$, $M_w := M_v^{d(w|v)}$, $m_w := m_v^{d(w|v)}$. Thus, (18.4), (18.7) and Lemma 18.1 hold with $w \in M_E$ instead of $v \in M_K$. It follows that for $w \in M_E$, we have

$$\max_{1 \leqslant i \leqslant n} \|\widetilde{L}_i^{(w)}(\mathbf{x})\|_w Q^{-c_{iw}} \leqslant n^{s(w)} \frac{M_w}{m_w} \cdot \max_{1 \leqslant i \leqslant n} \|L_i^{(w)}(\mathbf{x})\|_w Q^{-c_{iw}}.$$

By taking the product over $w \in M_E$, it follows that

$$H_{\widetilde{\mathcal{L}},\mathbf{c},Q}(\mathbf{x}) \leqslant n(2H_{\mathcal{L}})^{(4R)^n} H_{\mathcal{L},\mathbf{c},Q}(\mathbf{x}),$$

which implies (18.5).

We next prove (18.6). Let $d_1, \dots, d_t$ be the determinants of the n-element subsets of $\bigcup_{v \in M_K} \{L_1^{(v)}, \dots, L_n^{(v)}\}$, and let $\widetilde{d}_1, \dots, \widetilde{d}_s$ be the determinants of the

n-element subsets of $\bigcup_{v\in M_K}\{\widetilde{L}_1^{(v)},\dots,\widetilde{L}_n^{(v)}\}$. Then each $\widetilde{d}_i$ is a linear combination of elements from $d_1,\dots,d_t$ with at most n^n terms, each coefficient of which is a product of at most n elements from α_{ijv} ($v\in M_K$, $i\in I_v$, $j\in I_v^c$). So by (18.7),

$$\max_{1\leqslant i\leqslant s}\|\widetilde{d}_i\|_v\leqslant n^{ns(v)}\left(\frac{M_v}{m_v}\right)^n\cdot\max_{1\leqslant i\leqslant t}\|d_i\|_v$$

for $v\in M_K$. By taking the product over $v\in M_K$ and using Lemma 18.1, we obtain

$$H_{\widetilde{\mathcal{L}}}\leqslant n^n(2H_{\mathcal{L}})^{n(4R)^n}\cdot H_{\mathcal{L}}\leqslant(2H_{\mathcal{L}})^{(8R)^n},$$

which is (18.6). □

In the proof of Theorem 2.3, we assume

(18.8) there is a non-archimedean place $v_0\in M_K$ such that $c_{i,v_0}=0,\ \widetilde{L}_i^{(v_0)}=X_i$ for $i=1,\dots,n$,

(18.9) $$T=\{\mathbf{x}\in\overline{\mathbb{Q}}^n:\ x_1=\cdots=x_{n-k}=0\}.$$

We show that these are no restrictions. Let φ be a linear transformation of $\overline{\mathbb{Q}}^n$, defined over K. Lemma 7.3 says that $T(\mathcal{L}\circ\varphi,\mathbf{c})=\varphi^{-1}(T)$. Hence, if we construct a system of linear forms from $\mathcal{L}\circ\varphi$ and $T(\mathcal{L}\circ\varphi,\mathbf{c})$ in the same way as $\widetilde{\mathcal{L}}$ has been constructed from $\mathcal{L}$ and T, we obtain $\widetilde{\mathcal{L}}\circ\varphi$. Now choose φ such that $\{\widetilde{L}_1^{(v)}\circ\varphi,\dots,\widetilde{L}_n^{(v)}\circ\varphi\}=\{X_1,\dots,X_n\}$ and, moreover, $\{\widetilde{L}_i^{(v)}\circ\varphi:\ i\in I_v^c\}=\{X_1,\dots,X_{n-k}\}$. Then $\widetilde{\mathcal{L}}\circ\varphi$ contains $X_1,\dots,X_n$, and $T(\mathcal{L}\circ\varphi,\mathbf{c})$ is given by $X_1=\cdots=X_{n-k}=0$. Now Lemma 7.3 implies that in the proof of Theorem 2.3, we may replace $\mathcal{L}$ by $\mathcal{L}\circ\varphi$.

So henceforth, in addition to (2.4)–(2.10) and (18.3), we assume (18.8), (18.9).

The projection

(18.10) $$\varphi'':\ (x_1,\dots,x_n)\mapsto(x_1,\dots,x_{n-k})$$

has kernel T. We now define a tuple in $K[X_1,\dots,X_{n-k}]^{\mathrm{lin}}$,

(18.11) $$\begin{aligned}&\mathcal{L}''=(L_i^{(v)\prime\prime}:\ v\in M_K,\ i\in I_v^c)\\&\text{with } L_i^{(v)\prime\prime}:=\widetilde{L}_i^{(v)}\circ\varphi''^{-1}\quad(v\in M_K,\ i\in I_v^c)\end{aligned}$$

and a tuple of reals

(18.12) $$\begin{aligned}&\mathbf{d}=(d_{iv}:\ v\in M_K,\ i\in I_v^c)\\&\text{with } d_{iv}:=\frac{n-k}{n}\left(c_{iv}-\theta_v\right),\quad(v\in M_K,\ i\in I_v^c),\\&\text{where } \theta_v:=\frac{1}{n-k}\Big(\sum_{j\in I_v^c}c_{jv}\Big)\quad(v\in M_K).\end{aligned}$$

Notice that by Lemma 16.2(ii) and assumption (2.8), we have

$$\sum_{v\in M_K}\theta_v=\frac{w(\overline{\mathbb{Q}}^n)-w(T)}{n-k}=-\frac{w(T)}{n-k}. \tag{18.13}$$

The tuple $\mathcal{L}''$ is precisely that defined in (16.6), while $\mathbf{d}$ is a normalization of the tuple $\mathbf{c}''$ from (16.6). Eventually, we want to apply Theorem 8.1 to $(\mathcal{L}'',\mathbf{d})$, and to this end we have to verify that this pair satisfies the analogues of (8.2)–(8.9) with $\mathcal{L},\mathbf{c}$ replaced by $\mathcal{L}'',\mathbf{d}$; in fact, the tuple $\mathbf{d}$ has been chosen to satisfy (8.3), (8.4). Further, we need an estimate for $H_{\mathcal{L}''}$ in terms of $H_{\mathcal{L}}$. Finally, we have to relate the twisted height $H_{\mathcal{L}'',\mathbf{d},Q'}(\varphi''(\mathbf{x}))$ to $H_{\mathcal{L},\mathbf{c},Q}(\mathbf{x})$, where $Q':=Q^{n/(n-k)}$.

We start with the verification of (8.2)–(8.9), with $n-k,nR^n,\mathcal{L}'',\mathbf{d}$ replacing $n,R,\mathcal{L},\mathbf{c}$, and with indices i taken from I_v^c instead of $\{1,\dots,n\}$ for $v\in M_K$. It is clear that $\mathbf{d}$ satisfies (8.2), (8.3) and that $\mathcal{L}''$ satisfies (8.6). Further, from (18.8), (18.9) it follows easily that $\mathcal{L}''$ satisfies (8.8). In the lemma below we show that $\mathcal{L}'',\mathbf{d}$ has properties (18.14), (18.15), (18.16), which are precisely (8.4), (8.7), (8.9) with $n-k,nR^n,\mathcal{L}'',\mathbf{d}$ replacing $n,R,\mathcal{L},\mathbf{c}$. The weight $w_{\mathcal{L}'',\mathbf{d}}$ and twisted heights $H_{\mathcal{L}'',\mathbf{d},Q}$ are defined similarly as in Section 16, but with d_{iv} in place of c_{iv} in (16.8), (16.10).

LEMMA 18.3. *We have*

$$\sum_{v\in M_K}\max_{i\in I_v^c}d_{iv}\leqslant 1, \tag{18.14}$$

$$\#\left(\bigcup_{v\in M_K}\{L_i^{(v)\prime\prime}:\, i\in I_v^c\}\right)\leqslant nR^n, \tag{18.15}$$

$$w_{\mathcal{L}'',\mathbf{d}}(U)\leqslant 0\ \ \textit{for every linear subspace } U \textit{ of } \overline{\mathbb{Q}}^{n-k}. \tag{18.16}$$

Proof. We start with (18.14). Put $c'_{iv}:=c_{iv}-\frac{1}{n}\sum_{j=1}^n c_{jv}$ for $v\in M_K$, $i=1,\dots,n$. Then $\sum_{i=1}^n c'_{iv}=0$ for $v\in M_K$, while $\sum_{v\in M_K}\max_{1\leqslant i\leqslant n}c'_{iv}\leqslant 1$ by (2.9).

Consequently,

$$\begin{aligned}\sum_{v\in M_K}\max_{i\in I_v^c}d_{iv}&=\frac{n-k}{n}\sum_{v\in M_K}\left(\max_{i\in I_v^c}c'_{iv}-\frac{1}{n-k}\sum_{j\in I_v^c}c'_{jv}\right)\\&=\frac{n-k}{n}\cdot\sum_{v\in M_K}\left(\max_{i\in I_v^c}c'_{iv}+\frac{1}{n-k}\sum_{j\in I_v}c'_{jv}\right)\\&\leqslant\frac{n-k}{n}\cdot\left(1+\frac{k}{n-k}\right)\max_{1\leqslant i\leqslant n}c'_{iv}\leqslant 1.\end{aligned}$$

This proves (18.14).

Next, we prove (18.15). Let $v \in M_K$. The set $\{L_i^{(v)\prime\prime} : i \in I_v^c\}$ is determined by the linear forms $\widetilde{L}_i^{(v)}$ given by (18.4) and the latter by the ordered tuple $(L_1^{(v)}, \dots, L_n^{(v)})$. By (2.6) there are at most R^n distinct tuples among these as v runs through M_K. This proves (18.15).

We finish with proving (18.16). Take a linear subspace U of $\overline{\mathbb{Q}}^{n-k}$, and let $W := \varphi''^{-1}(U)$. By (18.12), (18.13), we have

$$w_{\mathcal{L}'',\mathbf{d}}(U) = \frac{n-k}{n}\left(w_{\mathcal{L}'',\mathbf{c}''}(U) - \dim U \sum_{v \in M_K} \theta_v\right) = \frac{n-k}{n}\left(w_{\mathcal{L}'',\mathbf{c}''}(U) + \dim U \cdot \frac{w(T)}{n-k}\right),$$

and then by Lemma 16.2(ii),

$$w_{\mathcal{L}'',\mathbf{d}}(U) = \frac{n-k}{n}\left(w(W) - w(T) + \dim U \cdot \frac{w(T)}{n-k}\right) = \frac{n-k}{n}\left(w(W) - \frac{w(T)}{n-k} \cdot (n - \dim W)\right).$$

Since this is $\leqslant 0$ by (2.21), this proves (18.16). □

Lemma 18.4. *We have*

$$H_{\mathcal{L}''} \leqslant (2H_{\mathcal{L}})^{(8R)^n}.$$

Proof. Let $\widetilde{d}_1, \dots, \widetilde{d}_s$ be the determinants of the n-element subsets of $\bigcup_{v \in M_K}\{\widetilde{L}_1^{(v)}, \dots, \widetilde{L}_n^{(v)}\} =: \{\widetilde{L}_1, \dots, \widetilde{L}_r\}$, and let $d_1'', \dots, d_u''$ be the determinants of the $(n-k)$-element subsets of $\bigcup_{v \in M_K}\{L_i^{(v)\prime\prime} : i \in I_v^c\}$. Pick one of the determinants d_i''. Then for some $i_1, \dots, i_{n-k}$, by (18.10), (18.11),

$$d_i'' = \det(\widetilde{L}_{i_1} \circ \varphi''^{-1}, \dots, \widetilde{L}_{i_{n-k}} \circ \varphi''^{-1}) = \det(\widetilde{L}_{i_1}, \dots, \widetilde{L}_{i_{n-k}}, X_{n-k+1}, \dots, X_n),$$

and then by (18.8), $\pm d_i'' \in \{\widetilde{d}_1, \dots, \widetilde{d}_s\}$. Consequently,

$$H_{\mathcal{L}''} = \prod_{v \in M_K} \max_{1 \leqslant i \leqslant u} \|d_i''\|_v \leqslant \prod_{v \in M_K} \max_{1 \leqslant i \leqslant s} \|\widetilde{d}_i\|_v = H_{\widetilde{\mathcal{L}}}.$$

Together with (18.6), this implies our lemma. □

Proposition 18.5. *Let Q be a real with*

$$Q \geqslant (2H_{\mathcal{L}})^{200(8R)^n/\delta} \tag{18.17}$$

and $\mathbf{x} \in \overline{\mathbb{Q}}^n$ with

$$H_{\mathcal{L},\mathbf{c},Q}(\mathbf{x}) \leqslant \Delta_{\mathcal{L}}^{1/n} Q^{-\delta}. \tag{18.18}$$

Put $Q' := Q^{n/(n-k)}$. Then

$$H_{\mathcal{L}'',\mathbf{d},Q'}(\varphi''(\mathbf{x})) \leqslant Q'^{-\frac{99}{100}\delta/n}. \tag{18.19}$$

Proof. We need the crucial observation that by (18.13), (2.21), (2.8),

$$\sum_{v\in M_K}\theta_v = -\frac{w(T)}{n-k} < -\frac{w(\overline{\mathbb{Q}}^n)}{n} = 0. \tag{18.20}$$

Let E be a finite extension of K with $\mathbf{x}\in E^n$. In accordance with our usual conventions, we put $L_i^{(w)\prime\prime} := L_i^{(v)\prime\prime}$, $d_{iw} := d(w|v)d_{iv}$, $I_w^c := I_v^c$ for places $w\in M_E$ lying above $v\in M_K$. Thus, (18.12), (18.13), (18.20) imply $d_{iw} := \frac{n-k}{n}(c_{iw}-\theta_w)$ for $w\in M_E, i\in I_w^c$ with $\sum_{w\in M_E}\theta_w<0$, and so

$$\begin{aligned}
H_{\mathcal{L}'',\mathbf{d},Q'}(\varphi''(\mathbf{x})) &= \prod_{w\in M_E}\max_{i\in I_w^c}\|\widetilde{L}_i^{(w)}(\mathbf{x})\|_w Q'^{-d_{iw}}\\
&= \prod_{w\in M_E} Q^{\theta_w}\max_{i\in I_w^c}\|\widetilde{L}_i^{(w)}(\mathbf{x})\|_w Q^{-c_{iw}}\\
&\leqslant \prod_{w\in M_E}\max_{1\leqslant i\leqslant n}\|\widetilde{L}_i^{(w)}(\mathbf{x})\|_w Q^{-c_{iw}}\\
&= H_{\widetilde{\mathcal{L}},\mathbf{c},Q}(\mathbf{x}).
\end{aligned}$$

Together with (18.5), (7.4), (18.20), this implies

$$\begin{aligned}
H_{\mathcal{L}'',\mathbf{d},Q'}(\varphi''(\mathbf{x})) &\leqslant (2H_{\mathcal{L}})^{(8R)^n} H_{\mathcal{L},\mathbf{c},Q}(\mathbf{x})\\
&\leqslant (2H_{\mathcal{L}})^{(8R)^n+R^n}\cdot\Delta_{\mathcal{L}}^{-1/n}H_{\mathcal{L},\mathbf{c},Q}(\mathbf{x}).
\end{aligned}$$

Now (18.19) follows easily from this last inequality and (18.17), (18.18). □

Proof of Theorem 2.3. We assume for the moment that $n-k\geqslant 2$. We intend to apply Theorem 8.1 with

$$n-k,\ nR^n,\ \frac{99}{100}\delta/n,\ \mathcal{L}'',\ \mathbf{d} \tag{18.21}$$

replacing $n,R,\delta,\mathcal{L},\mathbf{c}$, respectively. Clearly, with these replacements (8.1) holds, and we verified above that conditions (8.2)–(8.9) are satisfied as well.

Let m_2', ω_2' be the quantities m_2,ω_2 from Theorem 8.1, with the objects in (18.21) replacing $n,R,\delta,\mathcal{L},\mathbf{c}$, respectively. Further, let C_2' be the quantity obtained by applying the substitutions from (18.21) to C_2, but replacing $H_{\mathcal{L}''}$ by the upper bound $(2H_{\mathcal{L}})^{(8R)^n}$ from Lemma 18.4. Then Theorem 8.1 implies that there exist reals $Q_1',\dots,Q_{m_2'}'$ with $C_2'\leqslant Q_1'<\cdots<Q_{m_2'}'$ such that if $Q'\geqslant 1$ is a real with

$$\{\mathbf{y}\in\overline{\mathbb{Q}}^{n-k} : H_{\mathcal{L}'',\mathbf{d},Q'}(\mathbf{y})\leqslant Q'^{-\frac{99}{100}\delta/n}\}\neq\{\mathbf{0}\}, \tag{18.22}$$

then

$$Q'\in[1,C_2')\cup\bigcup_{h=1}^{m_2'}\left[Q_h',Q_h'^{\,\omega_2'}\right). \tag{18.23}$$

We proved (18.23) under the assumption $n-k \geqslant 2$. We now assume that $n-k=1$ and show that (18.23) is valid also in this case. The quantities m_2', ω_2', C_2' are defined as above, but with $n-k=1$ replacing n. We have $L_1^{(v)\prime\prime} = \alpha_v X$, $d_{1v}=0$ for $v \in M_K$, and so for $\mathbf{y} = y \in K^*$, by the product formula,

$$H_{\mathcal{L}'',\mathbf{d},Q'}(y) = \prod_{v \in M_K} \|\alpha_v y\|_v = \prod_{v \in M_K} \|\alpha_v\|_v.$$

This is valid also if $y \notin K$. Let $\{\alpha_v : v \in M_K\} = \{\alpha_1, \dots, \alpha_r\}$. By (18.15), we have $r \leqslant nR^n$. Moreover, by Lemma 18.4,

$$\prod_{v \in M_K} \max_{1 \leqslant i \leqslant r} \|\alpha_i\|_v = H_{\mathcal{L}''} \leqslant (2H_{\mathcal{L}})^{(8R)^n}.$$

Hence if $y \neq 0$, then

$$\begin{aligned} H_{\mathcal{L}'',\mathbf{d},Q'}(y) &\geqslant \prod_{v \in M_K} \min_{1 \leqslant i \leqslant r} \|\alpha_i\|_v \\ &\geqslant \prod_{v \in M_K} \frac{\|\alpha_1 \cdots \alpha_r\|_v}{(\max_{1 \leqslant i \leqslant r} \|\alpha_i\|_v)^{r-1}} \geqslant (2H_{\mathcal{L}})^{-(8R)^{2n}}. \end{aligned}$$

Now if y satisfies (18.22), then certainly $Q' \leqslant C_2'$, and so (18.23) is satisfied.

Let Q be one of the reals being considered in Theorem 2.3, i.e., with

$$\{\mathbf{x} \in \overline{\mathbb{Q}}^n : H_{\mathcal{L},\mathbf{c},Q}(\mathbf{x}) \leqslant \Delta_{\mathcal{L}}^{1/n} Q^{-\delta}\} \not\subset T.$$

Then by Proposition 18.5, either Q does not satisfy (18.17), or $Q' := Q^{n/(n-k)}$ satisfies (18.22). The first alternative implies $Q < {C_2'}^{n/(n-k)}$. So in either case,

$$Q \in \left[1, {C_2'}^{(n-k)/n}\right) \cup \bigcup_{h=1}^{{m_2}'} \left[Q_h^*, {Q_h^*}^{{\omega_2}'}\right),$$

where $Q_h^* := {Q_h'}^{(n-k)/n}$ for $h = 1, \dots, {m_2}'$.

To prove Theorem 2.3, we have to cut the intervals into smaller pieces. In general, any interval $[A, A^\theta)$ is contained in a union of at most $[\log\theta/\log\omega_0]+1$ intervals of the shape $[Q^*, Q^{*\omega_0})$. It follows that there are reals $Q_1, \dots, Q_m$, with $C_0 \leqslant Q_1 < \dots < Q_m$, such that

$$Q \in [1, C_0) \cup \bigcup_{h=1}^{m} [Q_h, Q_h^{\omega_0}),$$

where

$$m := 1 + \left[\frac{\log(\log {C_2'}^{(n-k)/n}/\log C_0)}{\log\omega_0}\right] + m_2'\left(1 + \left[\frac{\log\omega_2'}{\log\omega_0}\right]\right).$$

To finish our proof, we have to show that $m \leqslant m_0$.

We first estimate from above m_2'. Taking the definition of m_2 from (8.10) and the substitutions from (18.21), and using $R \geqslant n \geqslant 2$, we obtain

$$\begin{aligned} m_2' &\leqslant 61(n-k)^6 2^{2(n-k)}(100n/99\delta)^2 \log(22(n-k)^2 2^{n-k}\cdot nR^n \cdot 100n/99\delta) \\ &\leqslant 62n^8 2^{2n}\delta^{-2}\log(23n^4 2^n R^n \delta^{-1}) \leqslant 62n^{10}2^{2n}\delta^{-2}\log\left((3\delta^{-1}R)^{3n}\right) \\ &\leqslant 186n^9 2^{2n}\delta^{-2}\log(3\delta^{-1}R) \;=: m_*. \end{aligned}$$

Further,

$$\begin{aligned} 1 &+ \left[\frac{\log(\log {C_2'}^{(n-k)/n}/\log C_0)}{\log \omega_0}\right] \\ &\leqslant 1 + \left[\frac{\log\left(\log\left(2\times(2H_{\mathcal{L}})^{(8R)^n}\right)^{m_*^{2m_*}}/\log\max(H_{\mathcal{L}}^{1/R}, n^{1/\delta})\right)}{\log\omega_0}\right] \\ &\leqslant \frac{3m_*\log m_*}{\log(\delta^{-1}\log 3R)} \end{aligned}$$

and

$$1 + \left[\frac{\log\omega_2'}{\log\omega_0}\right] \leqslant 1 + \frac{5}{2}\cdot\frac{\log m_*}{\log\omega_0} \leqslant \frac{3\log m_*}{\log(\delta^{-1}\log 3R)}.$$

So altogether,

$$m \leqslant \frac{6m_*\log m_*}{\log(\delta^{-1}\log 3R)}.$$

Using $R \geqslant n \geqslant 2$, $186n^9 2^{2n} \leqslant 50^{2n}$, $\delta^{-2}\log(3\delta^{-1}R) \leqslant (\delta^{-1}\log 3R)^3$, this leads to

$$\begin{aligned} m &\leqslant 6m_* \times \frac{\log\left(186n^9 2^{2n}\delta^{-2}\log(3\delta^{-1}R)\right)}{\log(\delta^{-1}\log 3R)} \\ &\leqslant 6m_*\left(\frac{2n\log 50}{\log\log 6} + 3\right) \leqslant 100nm_* \\ &\leqslant 10^5 2^{2n} n^{10}\delta^{-2}\log(3\delta^{-1}R); \end{aligned}$$

i.e., $m \leqslant m_0$. This completes the proof of Theorem 2.3. □

References

[1] E. Bombieri and J. Vaaler, On Siegel's lemma, *Invent. Math.* **73** (1983), 11–32. MR 0707346. Zbl 0533.10030. http://dx.doi.org/10.1007/BF01393823.

[2] Y. Bugeaud, Extensions of the Cugiani-Mahler theorem, *Ann. Sc. Norm. Super. Pisa Cl. Sci.* **6** (2007), 477–498. MR 2370270. Zbl 1139.11032. Available at http://annaliscienze.sns.it/index.php?page=Article&id=104.

[3] Y. Bugeaud, An explicit lower bound for the block complexity of an algebraic number, *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei Mat. Appl.* **19** (2008), 229–235. MR 2439519. Zbl 05787954. http://dx.doi.org/10.4171/RLM/521.

[4] Y. Bugeaud, On the approximation to algebraic numbers by algebraic numbers, *Glas. Mat. Ser. III* **44(64)** (2009), 323–331. MR 2587305. Zbl 1194.11076. http://dx.doi.org/10.3336/gm.44.2.05.

[5] Y. Bugeaud, Quantitative versions of the subspace theorem and applications, *J. Théor. Nombres Bordeaux* **23** (2011), 35–57. MR 2780618. Zbl 06075032. http://dx.doi.org/10.5802/jtnb.749.

[6] Y. Bugeaud and J.-H. Evertse, On two notions of complexity of algebraic numbers, *Acta Arith.* **133** (2008), 221–250. MR 2434602. Zbl 1236.11062. http://dx.doi.org/10.4064/aa133-3-3.

[7] E. Dubois, Application de la méthode de W. M. Schmidt à l'approximation de nombres algébriques dans un corps de fonctions de caractéristique zéro, *C. R. Acad. Sci. Paris Sér. A-B* **284** (1977), A1527–A1530. MR 0441886. Zbl 0354.10029.

[8] E. Dubois and G. Rhin, Approximations rationnelles simultanées de nombres algébriques réels et de nombres algébriques p-adiques, in *Journées Arithmétiques de Bordeaux* (Conf., Univ. Bordeaux, 1974), *Astérisque* **24/25**, Soc. Math. France, Paris, 1975, pp. 211–227. MR 0374042. Zbl 0305.10031.

[9] J.-H. Evertse, An improvement of the quantitative subspace theorem, *Compositio Math.* **101** (1996), 225–311. MR 1394517. Zbl 0856.11030. Available at http://www.numdam.org/item?id=CM_1996__101_3_225_0.

[10] J.-H. Evertse, On the quantitative subspace theorem, *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)* **377** (2010), 217–240, 245. MR 2753657. http://dx.doi.org/10.1007/s10958-010-0185-6.

[11] J.-H. Evertse and H. P. Schlickewei, A quantitative version of the absolute subspace theorem, *J. Reine Angew. Math.* **548** (2002), 21–127. MR 1915209. Zbl 1026.11060. http://dx.doi.org/10.1515/crll.2002.060.

[12] J.-H. Evertse, H. P. Schlickewei, and W. M. Schmidt, Linear equations in variables which lie in a multiplicative group, *Ann. of Math.* **155** (2002), 807–836. MR 1923966. Zbl 1026.11038. http://dx.doi.org/10.2307/3062133.

[13] G. Faltings, Diophantine approximation on abelian varieties, *Ann. of Math.* **133** (1991), 549–576. MR 1109353. Zbl 0734.14007. http://dx.doi.org/10.2307/2944319.

[14] G. Faltings and G. Wüstholz, Diophantine approximations on projective spaces, *Invent. Math.* **116** (1994), 109–138. MR 1253191. Zbl 0805.14011. http://dx.doi.org/10.1007/BF01231559.

[15] W. Hoeffding, Probability inequalities for sums of bounded random variables, *J. Amer. Statist. Assoc.* **58** (1963), 13–30. MR 0144363. Zbl 0127.10602. http://dx.doi.org/10.1080/01621459.1963.10500830.

[16] D. Roy and J. L. Thunder, An absolute Siegel's Lemma, *J. Reine Angew. Math.* **476** (1996), 1–26. MR 1401695. Zbl 0860.11036. http://dx.doi.org/10.1515/crll.1996.476.1.

[17] H. P. Schlickewei, The $\mathfrak{p}$-adic Thue-Siegel-Roth-Schmidt theorem, *Arch. Math. (Basel)* **29** (1977), 267–270. MR 0491529. Zbl 0365.10026. http://dx.doi.org/10.1007/BF01220404.

[18] H. P. Schlickewei, The quantitative subspace theorem for number fields, *Compositio Math.* **82** (1992), 245–273. MR 1163217. Zbl 0751.11033. Available at http://www.numdam.org/item?id=CM_1992__82_3_245_0.

[19] W. M. Schmidt, On heights of algebraic subspaces and diophantine approximations, *Ann. of Math.* **85** (1967), 430–472. MR 0213301. Zbl 0152.03602. http://dx.doi.org/10.2307/1970352.

[20] W. M. Schmidt, Linear forms with algebraic coefficients. I, *J. Number Theory* **3** (1971), 253–277. MR 0308061. Zbl 0221.10034. http://dx.doi.org/10.1016/0022-314X(71)90001-1.

[21] W. M. Schmidt, Norm form equations, *Ann. of Math.* **96** (1972), 526–551. MR 0314761. Zbl 0226.10024. http://dx.doi.org/10.2307/1970824.

[22] W. M. Schmidt, Simultaneous approximation to algebraic numbers by elements of a number field, *Monatsh. Math.* **79** (1975), 55–66. MR 0364112. Zbl 0317.10042. http://dx.doi.org/10.1007/BF01533775.

[23] W. M. Schmidt, *Diophantine Approximation*, *Lecture Notes in Math.* **785**, Springer-Verlag, New York, 1980. MR 0568710. Zbl 0421.10019. http://dx.doi.org/10.1007/978-3-540-38645-2.

[24] W. M. Schmidt, The subspace theorem in Diophantine approximations, *Compositio Math.* **69** (1989), 121–173. MR 0984633. Zbl 0683.10027. Available at http://www.numdam.org/item?id=CM_1989__69_2_121_0.

[25] W. M. Schmidt, Vojta's refinement of the subspace theorem, *Trans. Amer. Math. Soc.* **340** (1993), 705–731. MR 1152325. Zbl 0794.11030. http://dx.doi.org/10.2307/2154673.

[26] W. M. Schmidt, The zero multiplicity of linear recurrence sequences, *Acta Math.* **182** (1999), 243–282. MR 1710183. Zbl 0974.11013. http://dx.doi.org/10.1007/BF02392575.

[27] W. M. Schmidt, Counting Diophantine approximations of small height, *J. Ramanujan Math. Soc.* **24** (2009), 171–190. MR 2543549. Zbl 05605540.

[28] W. M. Schmidt and L. Summerer, Parametric geometry of numbers and applications, *Acta Arith.* **140** (2009), 67–91. MR 2557854. Zbl 1236.11060. http://dx.doi.org/10.4064/aa140-1-5.

[29] T. Struppeck and J. D. Vaaler, Inequalities for heights of algebraic subspaces and the Thue-Siegel principle, in *Analytic Number Theory* (Allerton Park, IL, 1989), *Progr. Math.* **85**, Birkhäuser, Boston, MA, 1990, pp. 493–528. MR 1084199. Zbl 0722.11033.

[30] P. Vojta, A refinement of Schmidt's subspace theorem, *Amer. J. Math.* **111** (1989), 489–518. MR 1002010. Zbl 0662.14002. http://dx.doi.org/10.2307/2374670.

[31] S. Zhang, Positive line bundles on arithmetic varieties, *J. Amer. Math. Soc.* **8** (1995), 187–221. MR 1254133. Zbl 0861.14018. http://dx.doi.org/10.2307/2152886.

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