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Values of certain L-series in positive characteristic

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Abstract

We introduce a class of deformations of the values of the Goss zeta function. We prove, with the use of the theory of deformations of vectorial modular forms as well as with other techniques, a formula for their value at 1, and some arithmetic properties of values at other positive integers. Our formulas involve Anderson and Thakur's function ω. We discuss how our formulas may be used to investigate the existence of a kind of functional equation for the Goss zeta function.

1. Introduction, results

Euler's formulas [11]

$$\zeta(-n) = (-1)^n \frac{B_{n+1}}{n+1}, \quad n \in \mathbb{Z}_{\geq 0}$$

and

$$\zeta(2k) = \frac{(-1)^{k+1}(2\pi)^{2k} B_{2k}}{2(2k)!}, \quad k \in \mathbb{Z}_{>0}$$

must have looked particularly intriguing before Riemann's discovery of the functional equation for *Riemann's zeta function*.

The double appearance of the Bernoulli numbers in these formulas and the occurrence of the constant π in the second prompted Euler to conjecture the existence of a potential functional equation [12]; a problem that Riemann was able to solve more than a century later.

There is yet another function, called the *Goss zeta function*, which bears similarities to Riemann's zeta function, of which the existence of functional equations is expected, although it seems difficult to even guess its appearance. The aim of this paper is to provide new functional identities involving values of this function.

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The zeta function of Goss. Let $q = p^e$ be a power of a prime number p with $e > 0$ an integer, and let $\mathbb{F}_q$ be the finite field with q elements. We consider, for an indeterminate θ, the polynomial ring $A = \mathbb{F}_q[\theta]$ and its fraction field $K = \mathbb{F}_q(\theta)$. On K, we use the absolute value $|\cdot|$ defined by $|a| = q^{\deg_\theta a}$, a being in K, so that $|\theta| = q$ ($\deg_\theta$ denotes the degree in θ). Let $K_\infty := \mathbb{F}_q((1/\theta))$ be the completion of K for this absolute value, let K_∞^{alg} be an algebraic closure of K_∞. We denote, by A^+, the subset of monic polynomials of A.

In his explicit class field theory for the field K, Carlitz was led to introduce the values of the following series:

$$\zeta(n) := \sum_{a \in A^+} a^{-n} \in K_\infty, \quad n > 0, \tag{1}$$

as analogues of the values of Riemann's zeta function at integers > 1;[1] see, for example, [7], [8], [9]. These "zeta-values" will play a fundamental role in the present paper.

The complete solution of the explicit class field theory problem for global fields of positive characteristic given by Hayes [24] gave impetus to the development of the theory. Shortly later, Goss [17] extended the study of of these series to the case of $A = \Gamma(X \setminus \{\infty\}, \mathcal{O}_X)$, where X is a smooth projective curve defined over $\mathbb{F}_q$ and ∞ is a closed point of X. Goss also provided the necessary analytic structure to study these series. Just as Riemann's zeta function interpolates meromorphically the Euler zeta-values mentioned at the beginning of the present paper, Goss' zeta function ζ is a kind of analytic interpolation obtained by Goss [17] of the zeta-values (1), whose construction can be extended to the more general class of rings A as above. In this paper however, we will only focus on the simplest case of $A = \mathbb{F}_q[\theta]$ (that is, $X = \mathbb{P}_1$ with its point at infinity).

No functional equation is known for these functions, in spite of evidence supporting their existence, collected in [22].

A new kind of interpolation. In this paper we introduce a new type of analytic interpolation, specializing in the zeta-values (1) with n varying in certain subsets of $\mathbb{Z}$. Some of the functions we introduce turn out to satisfy functional equations.

Let $\mathbb{C}_\infty$ be the completion of K_∞^{alg} for the unique extension of $|\cdot|$ to K_∞^{alg}, and let K^{sep} be the separable closure of K in K_∞^{alg}. We consider an element t of $\mathbb{C}_\infty$. We have the "evaluating at t" ring homomorphism

$$\chi_t : A \to \mathbb{F}_q[t]$$

[1] We will rarely mention Riemann's zeta function in this paper, so the double use of the symbol ζ will not bother us.

defined by $\chi_t(a) = a(t)$. In other words, $\chi_t(a)$ is the image of the polynomial $a(\theta)$ obtained by substituting θ with t in $a(\theta)$. For example, $\chi_t(1) = 1$ and $\chi_t(\theta) = t$. If we choose $t \in \mathbb{F}_q^{\mathrm{alg}}$, then χ_t factors through a Dirichlet character modulo the ideal generated by the minimal polynomial of t in A.

We can also consider t as an indeterminate. We have, for $\alpha > 0$ an integer, a formal series

$$L(\chi_t, \alpha) = \sum_{a \in A^+} \chi_t(a) a^{-\alpha} = \prod_{\mathfrak{p}} (1 - \chi_t(\mathfrak{p})\mathfrak{p}^{-\alpha})^{-1},$$

where the Eulerian product runs over the monic irreducible polynomials of A, which is well defined in $K_\infty[[t]]$, as it is easily verifiable by the reader. This formal series converges for $\log_q |t| < \alpha$, $\log_q$ being the logarithm in base q. It is easy to see, by using Goss' result [23, Th. 2], that it extends to an entire function on $\mathbb{C}_\infty$. This analytic continuation is given by ordering the summation according to the degrees. (See the proof of Corollary 3 and Remark 7.)

If $t = \theta^{q^k}$ for $k \in \mathbb{Z}$ and $\alpha > q^k$, then the series $L(\chi_{\theta^{q^k}}, \alpha)$ converges to the zeta-value

$$L(\chi_{\theta^{q^k}}, \alpha) = \zeta(\alpha - q^k) = \sum_{a \in A^+} a^{q^k - \alpha},$$

but with the above-mentioned result of Goss, the analytic extension takes, at the points $t = \theta^{q^k}$, the zeta-values $\zeta(\alpha - q^k)$ for $k \in \mathbb{Z}$. If, on the other hand, we consider $t \in \mathbb{F}_q^{\mathrm{alg}}$, then, for $\alpha > 0$, $L(\chi_t, \alpha)$ converges to the value at α of the L-series associated to a Dirichlet character, described in Goss' book [21].

For certain values of α, we show the existence of a functional equation for the function $L(\chi_t, \alpha)$ (of the variable t) which provides an alternative path to the analytic extension.

To describe this, we recall that Carlitz's module is the unique $\mathbb{F}_q$-linear algebra homomorphism

$$\phi_{\mathrm{Car}} : A \to \mathbf{End}_{\mathbb{F}_q-\mathrm{lin.}}(\mathbb{G}_a(\mathbb{C}_\infty)) = \mathbb{C}_\infty[\tau]$$

such that $\phi_{\mathrm{Car}}(\theta) = \theta + \tau$. Here, $\mathbb{C}_\infty[\tau]$ denotes the skew ring of polynomials $\sum_i c_i \tau^i$ in τ with coefficients in $\mathbb{C}_\infty$, with product defined by the commutation rule $\tau c = c^q \tau$.

We will need some classical tools related to Carlitz's module. The *exponential function* exp associated to ϕ_{Car} is defined, for all $\eta \in \mathbb{C}_\infty$, by the sum of the convergent series

$$\exp(\eta) = \sum_{n \geqslant 0} \frac{\eta^{q^n}}{d_n}, \tag{2}$$

where $d_0 := 1$ and $d_i := [i][i-1]^q \cdots [1]^{q^{i-1}}$, with $[i] = \theta^{q^i} - \theta$ if $i > 0$.

We choose once and for all a fundamental period $\widetilde{\pi}$ of exp. It is possible to show that $\widetilde{\pi}$ is equal, up to choice of a $(q-1)$-th root of $-\theta$, to the (value of the) convergent infinite product

$$\widetilde{\pi} := \theta(-\theta)^{\frac{1}{q-1}}\prod_{i=1}^{\infty}(1-\theta^{1-q^i})^{-1} \in (-\theta)^{\frac{1}{q-1}}K_\infty. \tag{3}$$

We will also need Anderson and Thakur's function ω, belonging to $K^{\mathrm{sep}}[[t]]$,

$$\omega(t) := \sum_{i=0}^{\infty}\exp\left(\frac{\widetilde{\pi}}{\theta^{i+1}}\right)t^i = \sum_{n=0}^{\infty}\frac{\widetilde{\pi}^{q^n}}{d_n(\theta^{q^n}-t)} \tag{4}$$

converging for $|t| < q$. This function was introduced in [5, Proof of Lemma 2.5.4, p. 177] and is denoted by ω_1 there. (The same function is denoted by s_{Car} in [34].) We shall prove

THEOREM 1. *The following identity holds*:

$$L(\chi_t, 1) = -\frac{\widetilde{\pi}}{(t-\theta)\omega(t)}.$$

We also obtained some information on the values $L(\chi_t, \alpha)$ with $\alpha > 1$ such that $\alpha \equiv 1 \pmod{q-1}$.

THEOREM 2. *Let α be a positive integer such that $\alpha \equiv 1 \pmod{q-1}$. Then, there exists a nonzero element $\lambda_\alpha \in \mathbb{F}_q(t,\theta)$ such that*

$$L(\chi_t, \alpha) = \lambda_\alpha\frac{\widetilde{\pi}^\alpha}{(t-\theta)\omega(t)}.$$

Theorem 1 implies that $\lambda_1 = -1$.

For our purposes, we need to recall three properties of the function ω. (See, for example, [32] for more details.)

1. *The function ω is solution of a difference equation.* More precisely, it generates the one-dimensional $\mathbb{F}_q(t)$-vector space of solutions of the τ-difference equation

$$\tau X = (t-\theta)X, \tag{5}$$

in the fraction field of the *Tate algebra* $\mathbb{T}$ of formal power series in t converging for $|t| \leq 1$ (that is, of series $\sum_{n\geq 0} c_n t^n \in \mathbb{C}_\infty[[t]]$ such that $\lim_{n\to\infty} c_n = 0$), where

$$\tau : \mathbb{C}_\infty((t)) \to \mathbb{C}_\infty((t))$$

is the operator defined by

$$\tau\sum c_i t^i = \sum c_i^q t^i.$$

2. *The function ω is defined and meromorphic over $\mathbb{C}_\infty$ (in the rigid sense).* The poles are simple, and their set is $\{\theta^{q^n} : n \geq 1\}$. Indeed, the function

$$\mathbf{\Omega} = \frac{1}{\tau\omega}$$

is entire on $\mathbb{C}_\infty$ with the only zeros at $t=\theta^q,\theta^{q^2},\dots$. This can be deduced from the τ-difference equation (5), which yields the identity

$$\mathbf{\Omega}(t)=-\theta^{-1}(-\theta)^{-\frac{1}{q-1}}\prod_{i=1}^{\infty}(1-t\theta^{-q^i})\in(-\theta)^{\frac{1}{q-1}}K_\infty[[t]].$$

3. *The residue of* ω *at* $t=\theta$ *is* $-\widetilde{\pi}$. The property is equivalent to the identity

$$\mathbf{\Omega}(\theta)=-\frac{1}{\widetilde{\pi}},$$

which can be verified by using (3).

The function $\mathbf{\Omega}$ was used in an essential way in [4, §4.4.11] in the study of the algebraic relations between values of the "geometric" gamma function of Thakur. Following the terminology of the authors, it is a *rigid analytic trivialization* of Carlitz's t-motive. The functions $\omega,\mathbf{\Omega}$ also appear, under several different notations, in the papers [1], [5], [4], [31]. In [1], the function ω is related to the theory of *scattering matrices* (see Section 3.1 of loc. cit.). An overview of further properties of ω and $\mathbf{\Omega}$ is contained in [32].

Before presenting the other results of this paper, we explain the interest of Theorems 1 and 2. The identity

$$L(\chi_t,1)=-\widetilde{\pi}\mathbf{\Omega}(t)$$

holds, and we notice that the product is independent of the choice of a fundamental period $\widetilde{\pi}$ of exp. But since the function $\mathbf{\Omega}$ is entire, so is the map $t\mapsto L(\chi_t,1)$, with the only zeros at the points $t=\theta^q,\theta^{q^2},\dots$. The properties above imply that

$$\lim_{t\to\theta}L(\chi_t,1)=1 \tag{6}$$

and, for $k>0$,

$$\lim_{t\to\theta^{q^k}}L(\chi_t,1)=0. \tag{7}$$

These limits are useful in proving the following corollary of Theorem 1 and rediscovering, in a new way, known results about the function ζ of Goss.

Corollary 3. *We have* $\zeta(0)=1$ *and* $\zeta(1-q^k)=0$ *for* $k>0$.

The interest of the corollary lies in its proof. Indeed, we are going to show that the zeros are located where the poles of $\tau\omega$ are, just as the trivial zeros of Riemann's zeta function are located at the poles of Euler's gamma function.

Proof of Corollary 3. We need to use Goss' works [21], [23]. Let $\mathbb{S}_\infty=\mathbb{C}_\infty\times\mathbb{Z}_p$ be the "complex plane" over which he introduced his continuous-analytic extensions of the zeta-values (1). For (t,x,y) varying in the space

$\mathbb{C}_\infty\times\mathbb{C}_\infty^\times\times\mathbb{Z}_p=\mathbb{C}_\infty\times\mathbb{S}_\infty$, we recall the definition of the series $L(\chi_t,x,y)$ in [23]:

$$L(\chi_t,x,y)=\sum_{d\geq 0}x^{-d}\left(\sum_{a\in A^+(d)}\chi_t(a)\langle a\rangle^{-y}\right),$$

where $A^+(d)$ denotes the set of monic polynomials of A of degree d in θ and the $\langle\cdot\rangle^y$ denotes p-adic exponentiation of the 1-unit $\langle a\rangle=a\theta^{-d}$ ($d=\deg_\theta a$) as defined in [23] and in [21, §8.5].[2] In [23, Th. 2], Goss shows that the function $f(t,x,y):=L(\chi_t,x,y)$ is a *continuous family of entire functions* in t,x^{-1}, parametrized by $y\in\mathbb{Z}_p$. This means that we can write

$$f(t,x,y)=\sum_{n\geq 0}\sum_{i+j=n}t^ix^{-j}c_{i,j}(y)$$

with $c_{i,j}:\mathbb{Z}_p\to\mathbb{C}_\infty$ continuous and, setting

$$m_n=\max_{i+j=n}\max_{y\in\mathbb{Z}_p}|c_{i,j}(y)|,$$

we have $m_nr^n\to 0$ as $n\to\infty$, for all $r\geq 0$ (compare with [21, Def. 8.5.1]). In particular, the map

$$t\mapsto f(t,\theta^j,j)$$

is entire for all $j\in\mathbb{Z}$ and must then coincide with $L(\chi_t,1)=-\widetilde{\pi}\boldsymbol{\Omega}(t)$ for $j=1$, for all $t\in\mathbb{C}_\infty$, because it coincides with it for $|t|$ small. In particular, we obtain from (6) and (7) that $f(\theta,\theta^{-1},1)=1$ and $f(\theta^{q^k},\theta^{-1},1)=0$ for $k>0$. Theorem 2 of [23] also implies, now fixing $t_0\in\mathbb{C}_\infty$ and varying $(x,y)\in\mathbb{S}_\infty$, that the map

$$(x,y)\mapsto f(t_0,x,y)$$

is *analytic* over the space $\mathbb{S}_\infty$ (after Definition 8.5.1 of loc. cit.).[3] If $t_0=\theta^{q^k}$ ($k\geq 0$), we get $f(\theta^{q^k},x,y)=\zeta(x\theta^{q^k},y-q^k)$, Goss' continuous-analytic extension of ζ connecting to (1) through $\zeta(\theta^{-n},n)=\zeta(n)$. In particular, we obtain, evaluating at $(x,y)=(\theta^{-1},1)\in\mathbb{S}_\infty$,

$$\zeta(0)=\zeta(1,0)=1$$

and

$$\zeta(1-q^k)=\zeta(\theta^{q^k-1},1-q^k)=0,\quad k>0. \qquad \square$$

The vanishing of $\zeta(n)$ at negative integers n divisible by $q-1$ is a well-known result due to Goss [17], and we only obtain fragments of this result. What is really important here is that the function $\tau\omega$ plays a role analogue

[2]Notice the choice of uniformizer $\pi=\theta^{-1}$ of K (in Goss' notations).

[3]See also Theorem 1 of loc. cit., where it is shown that the function $\alpha\mapsto L(\chi_t^\beta,\alpha)$ extends analytically to the whole plane $\mathbb{S}_\infty$ for any choice of t and β positive integer.

to that of Euler's gamma function in the classical functional equation of Riemann's zeta function; its poles provide the trivial zeroes of Goss' ζ at the points $1-q^k$, $k>0$, and this is the first known interplay between certain values of ζ at positive and negative multiples of $q-1$. The trivial zeros of ζ are simple; it is not yet known whether our method implies this property. See Remark 7 for a discussion about a generalization of the function $L(\chi_t,x,y)$.

In Section 1.1 we introduce a function of two variables $f_q(t,X)$, satisfying the identity

$$\frac{1}{\tau\omega(t)}=(-\theta)^{-q/(q-1)}f_q(t,\theta)$$

(with the appropriate choice of $(q-1)$-th root of $-\theta$). We will describe three classes of functional equations for this function which can be considered as analogues of the classical translation, multiplication, reflection formulas for Euler's gamma function, hence providing yet another similarity between ω and Euler's gamma function. Transcendence and algebraic independence issues will also be discussed.

Corollary 4. *We have, for $\alpha\equiv 1 \pmod{q-1}$ positive and $k>0$, the formulas*

$$\zeta(\alpha q^k-1)=\sum_{a\in A^+}a^{1-\alpha q^k}=(-1)^{k-1}(\tau^k\lambda_\alpha)(\theta)\frac{\widetilde{\pi}^{\alpha q^k-1}}{[k][k-1]\cdots[1]}.$$

The identities of this corollary are also well known, especially for the case $\alpha=1$ when $\lambda_\alpha=-1$ (see Carlitz [6] and Goss [16]). But here again, we have a glimpse of something like a functional equation behind the proof. Theorem 2 yields these formulas thanks to the identity $\mathbf{\Omega}(\theta)=-1/\widetilde{\pi}$. Similarly, in the computation of the value of Riemann's zeta function at, e.g., two, by means of its functional equation, one is led to use the formula $\Gamma(1/2)=\sqrt{\pi}$.

Proof of Corollary 4. From the definition of $L(\chi_t,\alpha)$ we deduce $\tau^kL(\chi_t,\alpha)=L(\chi_t,\alpha q^k)$ (use $k\geq 0$). Apply τ^k to both the left- and the right-hand sides of the identity of Theorem 2 (or Theorem 1 if $\alpha=1$), and compute the limit for $t\to\theta$ in both left- and right-hand sides of the obtained identity. Observing the relation

$$\tau^k((t-\theta)\omega(t))=(t-\theta^{q^k})\cdots(t-\theta^q)(t-\theta)\omega(t), \tag{8}$$

which can be deduced from (5), and the limit $\lim_{t\to\theta}(t-\theta)\omega(t)=\mathbf{\Omega}(\theta)^{-1}=-\widetilde{\pi}$, the corollary follows. □

In particular, we notice the formula

$$\tau^k\lambda_\alpha=(t-\theta^{q^k})\cdots(t-\theta^q)\lambda_{q^k\alpha}$$

and the fact that $\lambda_\alpha(\theta^{q^{-k}})$ is well-defined nonzero for all k nonnegative (due to the fact that $\zeta(\alpha q^k-1)$ is well defined and nonzero). Apart from this, the explicit computation of the λ_α's is difficult and very little is known about these coefficients which could encode, we hope, an interesting generalization in $\mathbb{F}_q(t,\theta)$ of the theory of Bernoulli-Carlitz' numbers.

If $t=\xi\in\mathbb{F}_q^{\mathrm{alg}}$, Theorems 1 and 2 imply that for $\alpha\equiv 1 \pmod{q-1}$ positive, $L(\chi_\xi,\alpha)$, the value of an L-function associated to a Dirichlet character is a multiple of $\widetilde{\pi}^\alpha$ by an algebraic element of $\mathbb{C}_\infty$. More precisely, we obtain the following result.

Corollary 5. *For $\xi\in\mathbb{F}_q^{\mathrm{alg}}$ with $\xi^{q^r}=\xi$ $(r>0)$ and $\alpha>0$ with $\alpha\equiv 1$ $\pmod{q-1}$, we have*

$$L(\chi_\xi,\alpha)=\lambda_\alpha(\xi)\frac{\widetilde{\pi}^\alpha}{\rho_\xi},$$

where $\rho_\xi\neq 0$ is the root $(\tau\omega)(\xi)\in\mathbb{C}_\infty$ of the polynomial equation

$$X^{q^r-1}=(\xi-\theta^{q^r})\cdots(\xi-\theta^q).$$

Proof. Consider (8) with $k=r$, and then apply Theorem 2, noticing that $(\tau^r(\tau\omega))(\xi)=((\tau\omega)(\xi))^{q^r}$. Of course, since the series $L(\chi_\xi,\alpha)$ converges, the value $\lambda_\alpha(\xi)$ is well defined. □

Some cases of the above corollary are also covered by the work [29] (when $\alpha=1$); see also [10]. One of the new features of Corollary 5 is to highlight that several such results on values at one of L-series in positive characteristic belong to the special family described in Theorem 1.

Remark 6. The convolution algebra of v-adic measures with v a place of K, particularly important in the study of the values of ζ at negative integers, is canonically isomorphic to the ring of formal higher derivatives acting on the Tate algebra $\mathbb{T}$, where our deformations of $\zeta(n)$ lie; see [20]. The author owes this remark to David Goss.

Remark 7. The result of Goss [23, Th. 2] can be generalized to the following setting. Consider s variables $t_1,\ldots,t_s$ and, for $\beta_1,\ldots,\beta_s$ nonnegative integers and (x,y) an element of $\mathbb{S}_\infty$, the series

$$L(\chi_{t_1}^{\beta_1},\ldots,\chi_{t_s}^{\beta_s},x,y)=\sum_{d\geq 0}x^{-d}\sum_{a\in A^+(d)}\chi_{t_1}^{\beta_1}(a)\cdots\chi_{t_s}^{\beta_s}(a)\langle a\rangle^{-y}.$$

It can be proved that the above series again defines a continuous family of entire functions in $t_1,\ldots,t_s$ and x^{-1}, parametrized by $y\in\mathbb{Z}_p$. Indeed, one can argue by specialization in the above type of series (for various s) with $\beta_1=\cdots=\beta_s=1$, case accessible by using Goss' method of proof of [23, Th. 2]; we skip the details of this verification.

In particular, for all $\alpha > 0$ and $\beta_1, \dots, \beta_s \geq 0$, the series

$$\sum_{a \in A^+} \chi_{t_1}^{\beta_1}(a) \cdots \chi_{t_s}^{\beta_s}(a) a^{-\alpha} \in K_\infty[[t_1, \dots, t_s]]$$

is an element of the Tate algebra $\mathbb{T}_s$.[4] But, in fact, it extends to an entire function.

Each of these functions interpolates infinitely many zeta-values. To see this, it suffices to evaluate at points such as

$$(t_1, \dots, t_s) = (\theta^{q^{n_1}}, \dots, \theta^{q^{n_s}}), \quad n_1, \dots, n_s \in \mathbb{Z}.$$

We get the zeta-values

$$\zeta(\alpha - \beta_1 q^{n_1} - \cdots - \beta_s q^{n_s}), \quad n_1, \dots, n_s \in \mathbb{Z}.$$

Methods of proof of Theorems 1 *and* 2. We will provide two independent proofs of Theorem 1. The first one uses *deformations of vectorial modular forms* (see Section 2). The second one, written in Section 4, is obtained from a variant of Anderson's so-called *log-algebraic power series identities* for *twisted harmonic sums*; see [2], [3]. We expect, in principle, that this method will be useful in computing the values at $\alpha = 1$ of a variant of our series $L(\chi_t, \alpha)$ associable to general Carlitz-Hayes' modules. (This method is also used in [29].) Unfortunately, it does not seem to be flexible enough to handle values $L(\chi_t, \alpha)$ with $\alpha > 1$.

The proof of Theorem 2 that we propose relies again on certain properties of deformations of vectorial modular forms. We will present a result, Theorem 8, immediately implying Theorem 1, and Theorem 2 will be obtained from a simple modification of the techniques introduced to prove Theorem 8.

A fundamental identity for deformations of vectorial modular forms. To present Theorem 8, we need to introduce more tools. Let Ω be the set $\mathbb{C}_\infty \setminus K_\infty$. It is well known [13] that it has a structure of geometrically connected rigid analytic space. Goss' paper [18] provides the background for the related theory. We recall that the group $\mathbf{GL}_2(K_\infty)$ acts on Ω by homographies in a way compatible with the rigid structure. The group $\Gamma = \mathbf{GL}_2(A)$ is a discrete subgroup of $\mathbf{GL}_2(K_\infty)$ acting discontinuously on Ω, and the quotient $\Gamma \backslash \Omega$ can be given a rigid analytic structure as well.

For $z \in \Omega$, we denote by Λ_z the A-module $A + zA$, free of rank 2. The evaluation at $\eta \in \mathbb{C}_\infty$ of the *exponential function* $\exp_{\Lambda_z}$ associated to the lattice

[4] The Tate algebra $\mathbb{T}_s$ is the ring of series in monomials in $t_1, \dots, t_s$ converging in the polydisk $\{(t_1, \dots, t_s) \in \mathbb{C}_\infty^s, |t_i| \leq 1 \text{ for all } i\}$.

Λ_z is given by the series

$$\exp_{\Lambda_z}(\eta) = \sum_{i=0}^{\infty} \alpha_i(z)\eta^{q^i} \tag{9}$$

for functions $\alpha_i : \Omega \to \mathbb{C}_\infty$ with $\alpha_0 = 1$. We recall that for $i > 0$, α_i is a *Drinfeld modular form* of weight $q^i - 1$ and type 0 in the sense of [13]; see also [19].

We also recall from [34] the series

$$\boldsymbol{s}_1(z,t) = \sum_{i=0}^{\infty} \frac{\alpha_i(z)z^{q^i}}{\theta^{q^i} - t},$$
$$\boldsymbol{s}_2(z,t) = \sum_{i=0}^{\infty} \frac{\alpha_i(z)}{\theta^{q^i} - t}.$$

Let $\mathbf{Hol}(\Omega)$ be the ring of holomorphic functions $\Omega \to \mathbb{C}_\infty$. The series $\boldsymbol{s}_1, \boldsymbol{s}_2$ define formal series $\boldsymbol{s}_i(z,t) = \sum_{n\geq 0} f_{i,n}(z)t^n$ $(i = 1,2)$ of $\mathbf{Hol}(\Omega)[[t]]$ (i.e., the coefficients are holomorphic functions) such that for all $z \in \Omega$, the series obtained by specialization of the coefficients converges for $|t| < q$. In particular, $\boldsymbol{s}_1, \boldsymbol{s}_2$ define two functions $\Omega \to \mathbb{T}$. With the use of the recursive formulas for the coefficients α_i and Lemma 4 given in [34], one can show that for any $t \in B_1$, the functions $\boldsymbol{s}_1(\cdot,t)$ and $\boldsymbol{s}_2(\cdot,t)$ are holomorphic functions $\Omega \to \mathbb{C}_\infty$.

We point out that for a fixed choice of $z \in \Omega$, the matrix function ${}^t(\boldsymbol{s}_1(z,t), \boldsymbol{s}_2(z,t))$ is the *canonical rigid analytic trivialisation* of the *t-motive associated* to the lattice Λ_z discussed in [32]. We set, for $i = 1,2$,

$$\boldsymbol{d}_i(z,t) := \widetilde{\pi}\omega(t)^{-1}\boldsymbol{s}_i(z,t).^5$$

The advantage of using these functions instead of the $\boldsymbol{s}_i$'s is that $\boldsymbol{d}_2$ has a *u-expansion* defined over $\mathbb{F}_q[t,\theta]$ (see Proposition 16), u being the *local parameter at infinity* of $\Gamma\backslash\Omega$:

$$u(z) = \frac{1}{\exp(\widetilde{\pi}z)}, \quad z \in \Omega.$$

Moreover, for all $z \in \Omega$, $\boldsymbol{d}_1(z,t)$ and $\boldsymbol{d}_2(z,t)$ are entire functions of the variable $t \in \mathbb{C}_\infty$ (see Corollary 18).

On the other hand, both the series

$$\boldsymbol{e}_1(z,t) = {\sum_{c,d\in A}}' \frac{\chi_t(c)}{cz+d}, \quad \boldsymbol{e}_2(z,t) = {\sum_{c,d\in A}}' \frac{\chi_t(d)}{cz+d}$$

converge for $(z,t) \in \Omega \times \mathbb{C}_\infty$ with $|t| \leq 1$, where the dash $'$ denotes a sum avoiding the pair $(c,d) = (0,0)$. The series $\boldsymbol{e}_1, \boldsymbol{e}_2$ define functions $\Omega \to \mathbb{T}$.

[5] In the notations of [34], we have $\boldsymbol{d}_2 = \boldsymbol{d}$.

Moreover, for every $t \in \mathbb{C}_\infty$ such that $|t| \leq 1$, both the functions $z \mapsto \boldsymbol{e}_i(z,t)$ are holomorphic on Ω (see Proposition 22).

The Frobenius $\mathbb{F}_q$-linear map τ acts on $\mathbf{Hol}(\Omega)$, the ring of holomorphic functions $\Omega \to \mathbb{C}_\infty$: if $f \in \mathbf{Hol}(\Omega)$, then $\tau f = f^q$. We consider the unique $\mathbb{F}_q((t))$-linear extension of τ:

$$\mathbf{Hol}(\Omega) \otimes_{\mathbb{F}_q} \mathbb{F}_q((t)) \to \mathbf{Hol}(\Omega) \otimes_{\mathbb{F}_q} \mathbb{F}_q((t)),$$

again denoted by τ.

We shall prove the fundamental theorem

THEOREM 8. *The following identities hold for $z \in \Omega, t \in \mathbb{C}_\infty$ such that $|t| \leq 1$:*

$$\begin{aligned} L(\chi_t,1)^{-1}\boldsymbol{e}_1(z,t) &= -(t-\theta)\omega(t)(\tau\boldsymbol{d}_2)(z,t)h(z), \\ L(\chi_t,1)^{-1}\boldsymbol{e}_2(z,t) &= (t-\theta)\omega(t)(\tau\boldsymbol{d}_1)(z,t)h(z). \end{aligned} \tag{10}$$

In the statement of the theorem, h is the opposite of the unique normalized Drinfeld cusp form of weight $q+1$ and type 1 for $\Gamma = \mathbf{GL}_2(A)$ as in Gekeler's paper [13]. This function was the object of extensive investigation. It is known that it is proportional to the unique Poincaré series of weight $q+1$ and type 1 [13, (5.11)]. Gekeler [13, Th. (6.1)] also showed that h has the following infinite product expansion:

$$h = -u \prod_{a \in A^+} f_a^{q^2-1}(u),$$

where $f_a(X)$ is the polynomial $\phi_{\mathrm{Car}}(a)(X^{-1})X^{|a|}$, making it explicit that h has the coefficients of its u-expansion in A and carries analogies with the unique normalized cusp form of weight 12 for $\mathbf{SL}_2(\mathbb{Z})$.

It can be observed that both right-hand sides in (10) are well defined for $t \in \mathbb{C}_\infty$ with $|t| < q^q$ so that these identities provide analytic extensions of the functions $\boldsymbol{e}_1, \boldsymbol{e}_2$ in terms of the variable t. In Section 3 we show how to deduce Theorem 1 from Theorem 8. But the formula of Theorem 1 can be jointly applied with Theorem 8 and gives the identities

$$\boldsymbol{e}_1(z,t) = \widetilde{\pi}^{-1}(\tau\boldsymbol{d}_2)(z,t)h(z), \quad \boldsymbol{e}_2(z,t) = -\widetilde{\pi}^{-1}(\tau\boldsymbol{d}_1)(z,t)h(z).$$

Corollary 18 then implies that $\boldsymbol{e}_1(z,t), \boldsymbol{e}_2(z,t)$ extend to entire functions of the variable t for every choice of $z \in \Omega$.

1.1. *Gamma phenomenology and transcendence properties.* We have seen how the function ω in Corollaries 3 and 4 played a role similar to that of Euler's gamma function in the functional equation of Riemann's zeta function. Here, we mention yet another connection between these two functions via the functions f_{q^k} defined below.

Thanks to Theorem 1, the function $L(\chi_t, 1)$ can be written as

$$L(\chi_t, 1) = \frac{f_q(t, \theta)}{f_q(\theta, \theta)}, \tag{11}$$

where f_q denotes the formal series

$$f_q(t, X) = \prod_{n>0} \left(1 - \frac{t}{X^{q^n}}\right) \in \mathbb{F}_q[[t, X^{-1}]],$$

converging for $t, X \in \mathbb{C}_\infty$ with $|X| > 1$. It now becomes important to consider a more general family of functions $(f_{q^k})_{k\geq 1}$ defined by

$$f_{q^k}(t, X) = \prod_{i>0} \left(1 - \frac{t}{X^{q^{ki}}}\right).$$

Here we describe three kinds of functional equations allowing us to trace analogies with corresponding functional equations of Euler's gamma function. At the end of this section, we discuss certain issues in transcendence and algebraic independence of values of the function $L(\chi_t, \alpha)$.

Analogues of the translation formula for Γ. Writing

$$\tau_1 : \mathbb{F}_q((t))((X^{-1})) \to \mathbb{F}_q((t))((X^{-1}))$$

for the unique $\mathbb{F}_q((t))$-endomorphism associating X to X^q, the vector

$$F = {}^t(f_{q^k}, \tau_1 f_{q^k}, \ldots, \tau_1^{k-1} f_{q^k})$$

is a solution, as it can be checked easily, of the system of τ_1-difference equations of order 1 (with component-wise action of the operator τ_1):

$$\tau_1 F = B_k F, \tag{12}$$

where

$$B_k = \begin{pmatrix} 0 & I_{k-1} \\ 1 - \frac{t}{X^{q^k}} & 0 \end{pmatrix}$$

is a block matrix of $\mathbf{GL}_k(\mathbb{F}_q(t, X))$ with an identity block of size $k - 1$. In particular, we have

$$f_{q^k}(t, X) = \left(1 - \frac{t}{X^{q^k}}\right) \tau_1^k f_{q^k}(t, X). \tag{13}$$

This functional equation can be considered as an analogue of the translation formula $\Gamma(s + 1) = s\Gamma(s)$ of Euler's gamma function.

Analogue of the multiplication formula for Γ. For $k > 0$ a given integer, the functional equation

$$f_q(t, X) = \prod_{l=0}^{k-1} \tau_1^l f_{q^k}(t, X)$$

can be verified directly by using (12).

Analogue of the reflection formula for Γ. Let us fix an integer $k>0$. The functional equation

$$f_{q^k}(t^{q-1},X^{q-1})=\prod_{\lambda\in\mathbb{F}_q^\times}f_{q^k}(\lambda t,X)$$

holds and can be verified in the following way. The right and the left-hand sides satisfy the same functional equation

$$F=\left(1-\frac{t^{q-1}}{X^{q^k(q-1)}}\right)\tau_1^kF.$$

The above is a homogeneous, linear τ_1^k-difference equation of order 1. Since the subfield of τ_1^k-invariant elements of $\mathbb{F}_q((t))((X^{-1}))$ is $\mathbb{F}_q((t))$, the set of solutions of the above equation in $\mathbb{F}_q((t))((X^{-1}))$ is a $\mathbb{F}_q((t))$-vector space of dimension one, so the two solutions are multiples of each other and the factor of proportionality is equal to one.

Remark 9. Of these functional equations, only the first one transfers directly to the function ω and to the values of the series $L(\chi_t,\alpha)$ for $\alpha\equiv 1 \pmod{q-1}$ positive. However, Anderson, Brownawell and Papanikolas [4] gave evidence of the fact that the function ω is intimately related to the arithmetic of the values of Thakur's *geometric* gamma function, which satisfies yet three other classes of functional equations in analogy with the three shown above. At the time of writing the present paper, the exact connection between the values of the general functions $f_{q^k}(t,X)$ and the values of Thakur's function is not completely elucidated.

Transcendence and algebraic independence of values. Our results can be used to obtain transcendence properties of values $L(\chi_t,1)$ (or more generally, of values $L(\chi_t,\alpha)$ with $\alpha\equiv 1 \pmod{q-1}$). With Corollary 5, we notice that if t belongs to $\mathbb{F}_q^{\text{alg}}$, then $L(\chi_t,1)$ is transcendental by the well-known transcendence of $\widetilde{\pi}$. Moreover, we have the following result.

COROLLARY 10. *Let t be an element of $K^{\text{alg}}\setminus\mathbb{F}_q^{\text{alg}}$. The quantities $\widetilde{\pi}$ and $L(\chi_t,1)$ are algebraically independent over K if and only if t is not of the form θ^{q^k} with $k\in\mathbb{Z}$.*

Sketch of proof. By (3), (11) and Corollary 4, we see that if t is of the form θ^{q^k} for $k\in\mathbb{Z}$, then $\widetilde{\pi}$ and $L(\chi_t,1)$ are algebraically dependent. Assume conversely that $\widetilde{\pi}$ and $L(\chi_t,1)$ are algebraically dependent. Then $f_q(t,\theta)$ and $f_q(\theta,\theta)$ are algebraically dependent. By [33, Th. 11], the functions $f_q(t,X)$ and $f_q(\theta,X)$ are algebraically dependent over $\mathbb{C}_\infty(X)$. An appropriate variant of [33, Prop. 21] allows us to show that, for some $\alpha,\beta\in\mathbb{Z}$ not both vanishing, there exists an algebraic function $g(X) \in \mathbb{C}_\infty(X)^{\mathrm{alg}}$ such that

$$f_q(t,X)^\alpha f_q(\theta,X)^\beta = g(X).$$

We notice now that the functions $f_q(t,X)$ and $f_q(\theta,X)$ of the variable X have infinitely many zeros, respectively located at the sets $\{t^{q^k}, k \in \mathbb{Z}\}$ and $\{\theta^{q^k}, k \in \mathbb{Z}\}$. This is compatible with the existence of g as above (algebraic and nonzero, therefore with finitely many poles and zeroes) only if $t = \theta^{q^k}$ for some $k \in \mathbb{Z}$. □

The above considerations can be extended to give a necessary and sufficient condition on $t_1, \ldots, t_s$ algebraic for the algebraic independence of $\mathbf{\Omega}(t_1), \ldots, \mathbf{\Omega}(t_s)$, from which one deduces the corresponding condition for $L(\chi_{t_1}, 1), \ldots, L(\chi_{t_s}, 1)$. The details will be described in another work.

1.2. *Structure of proof of Theorem* 8. We adopt the notations

$$\mathcal{E} = L(\chi_t, 1)^{-1}(\boldsymbol{e}_1, \boldsymbol{e}_2)$$

and

$$\mathcal{F} = \begin{pmatrix} \boldsymbol{d}_1 \\ \boldsymbol{d}_2 \end{pmatrix},$$

and we consider the representation $\rho_t : \mathbf{GL}_2(A) \to \mathbf{GL}_2(\mathbb{F}_q[t])$ defined by

$$\rho_t(\gamma) = \begin{pmatrix} \chi_t(a) & \chi_t(b) \\ \chi_t(c) & \chi_t(d) \end{pmatrix}$$

for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathbf{GL}_2(A)$. Then, for any such a choice of γ, we have the functional equations (see Propositions 16 and 22)

$$\begin{aligned} \mathcal{F}(\gamma(z), t) &= (cz+d)^{-1} \rho_t(\gamma) \cdot \mathcal{F}(z,t), \\ {}^t\mathcal{E}(\gamma(z), t) &= (cz+d)\ {}^t\rho_t^{-1}(\gamma) \cdot {}^t\mathcal{E}(z,t). \end{aligned}$$

Here and in the following, $\gamma(z)$ denotes the homography action of γ at z: $\gamma(z) = (az+b)/(cz+d)$. This immediately puts the functions ${}^t\mathcal{E}$ and $\mathcal{F}$ in the framework of *deformations of vectorial modular forms*, a topic that will be developed in Section 2 (see Definition 12 and the papers [27], [30]): the parameter of the deformation is t.

Let B_1 be the set of $t \in \mathbb{C}_\infty$ such that $|t| \leq 1$. We will make use of a remarkable sequence $\mathcal{G} = (\mathcal{G}_k)_{k \in \mathbb{Z}}$ of functions $\Omega \times B_1 \to \mathbb{C}_\infty$ defined by the scalar product (with component-wise action of τ):

$$\mathcal{G}_k = (\tau^k \mathcal{E}) \cdot \mathcal{F}.$$

For $k \geq 0$, $\mathcal{G}_k$ turns out to be an element of $M_{q^k-1,0} \otimes \mathbb{F}_q[t,\theta]$, where $M_{w,m}$ denotes the $\mathbb{C}_\infty$-vector space of Drinfeld modular forms of weight w and type m. In fact, we need only examine the functions $\mathcal{G}_0, \mathcal{G}_1$ to prove Theorem 1. Once their explicit computation is accomplished (see Proposition 26), the proof of

Theorem 8 is only a matter of solving a nonhomogeneous system in two equations and two indeterminates $\boldsymbol{e}_1, \boldsymbol{e}_2$. We need, to compute $\mathcal{G}_0$ and $\mathcal{G}_1$, a *deformation of Legendre's identity* (proved in [34]):

$$h^{-1}(\tau\omega)^{-1} = \boldsymbol{d}_1(\tau\boldsymbol{d}_2) - \boldsymbol{d}_2(\tau\boldsymbol{d}_1)$$

(see Proposition 16, eq. (17)).

It turns out that this formula is equivalent to the identity $\mathcal{G}_0 = -1$, obtained in Proposition 26. Theorem 1 will be deduced with the computation of a limit in the identity

$$\boldsymbol{e}_1 = -L(\chi_t, 1)\tau(\omega\boldsymbol{d}_2)h,$$

and a similar path will be followed for Theorem 2, this time using the more general sequences $\mathcal{G}_{\alpha,0,k} = (\tau^k\mathcal{E}_{\alpha,0}) \cdot \mathcal{F}$ defined later, for which we will have $\tau^k\mathcal{E}_{1,0} = \mathcal{E}_{q^k,0} = \tau^k\mathcal{E}$ and $\mathcal{G}_{1,0,k} = \mathcal{G}_k$.

Next, we must explain why we use deformation of vectorial modular forms in our setting. The terminology reflects that if we specialize the parameter t to some value in B_1, we get some close variant of a vectorial modular form (as in Definition 11). In fact, we consider the parameter t as varying in B_1 only. This simplifies many computations and is not restrictive for our results since at the end we need formulas valid in a neighborhood of $t = 0$ that can be later extended analytically so that it will be possible to evaluate at $t = \theta$. But in many cases, the analysis of the interesting value $t = \theta$ can be made directly; let us give an example.

Let g_k be the normalization of the Eisenstein series of weight $q^k - 1$ and type 0 of [13]. By definition, for $k > 0$ (see loc. cit.), we have

$$g_k(z) = {\sum_{c,d\in A}}'(cz+d)^{1-q^k} = -\zeta(q^k-1)^{-1}{\sum_{c,d\in A}}'(cz+d)^{-q^k}(c,d)\cdot\begin{pmatrix} z \\ 1 \end{pmatrix}, \tag{14}$$

a scalar product of vectorial modular forms of weights q^k and -1 associated respectively to the transpose of the inverse of the identity representation and the identity representation. We have equality of convergent series

$$(\tau^k\mathcal{E})(z,\theta) = -\zeta(q^k-1)^{-1}{\sum_{c,d\in A}}'(cz+d)^{-q^k}(c,d)$$

and we also have a well-defined limit (see [34]):

$$\lim_{t\to\theta}\mathcal{F}(z,t) = \begin{pmatrix} z \\ 1 \end{pmatrix}.$$

The above formula for g_k can be rewritten in the following way:

$$g_k(z) = -\lim_{t\to\theta}\mathcal{G}_k(z,t) = -\lim_{t\to\theta}(\tau^k\mathcal{E})(z,t)\cdot\mathcal{F}(z,t).$$

It is precisely the use of the parameter t that allows us to extract new interesting arithmetic information from such an identity. For $k=0$, the series $\sum'_{c,d\in A}(cz+d)^{-q^k}(c,d)$ is only conditionally convergent, but the limit $t\to\theta$ in the formula above is still well defined and results in the identity $1=g_0=-\lim_{t\to\theta}\mathcal{G}_0(z,t)$. Thanks to [14, Th. 6.2], this can also be viewed as an analogue of the classical *Legendre period relation*. Considering again our deformation of Legendre's period relation (17) and comparing it this time with Theorem 8, we complete showing how the former can be deduced from the latter.

2. **Vectorial modular forms and their deformations**

We denote by J_γ the factor of automorphy $(\gamma,z)\mapsto cz+d$ if $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$. We recall the local parameter at infinity of $\Gamma\backslash\Omega$, $u=u(z)=1/\exp(\widetilde{\pi}z)$. For all w,m, we have an embedding of $M_{w,m}$ in $\mathbb{C}_\infty[[u]]$ associating to every Drinfeld modular form its u-expansion; see [13], [19]. We will often identify modular forms with their u-expansions.

In all of the following, t will be considered either as a new indeterminate or as a parameter varying in $\mathbb{C}_\infty$, and we will freely switch from formal series to functions. We will say that a series $\sum_{i\geq i_0}c_iu^i$ (with the coefficients c_i in some field extension of $\mathbb{F}_q(t,\theta)$) is *normalized* if $c_{i_0}=1$. We will also say that the series is *of type* $m\in\mathbb{Z}/(q-1)\mathbb{Z}$ if $i\not\equiv m \pmod{q-1}$ implies $c_i=0$. This definition is obviously compatible with the notion of *type* of a Drinfeld modular form already mentioned in the introduction; see [13].

The following definition is a simple adaptation of the notion of vectorial modular form for $\mathbf{SL}_2(\mathbb{Z})$ investigated in works by Knopp and Mason such as [27], [30]. Let $\rho:\Gamma\to\mathbf{GL}_s(\mathbb{C}_\infty)$ be a representation of Γ.

Definition 11. A *vectorial modular form of weight* w, *type* m *and dimension* s *associated to* ρ is a holomorphic function $f:\Omega\to\mathbf{Mat}_{s\times1}(\mathbb{C}_\infty)$ with the following two properties. Firstly, for all $\gamma\in\Gamma$,

$$f(\gamma(z))=\det(\gamma)^{-m}J_\gamma^w\rho(\gamma)\cdot f(z).$$

Secondly, the vectorial function $f={}^t(f_1,\ldots,f_s)$ is *tempered* at infinity in the following way. There exists $\nu\in\mathbb{Z}$ such that for all $i\in\{1,\ldots,s\}$, the following limit holds:

$$\lim_{|z|=|z|_i\to\infty}u(z)^\nu f_i(z)=0.$$

($|z|_i$ denotes, for $z\in\mathbb{C}_\infty$, the infimum $\inf_{a\in K_\infty}\{|z-a|\}$.)

We denote by $\mathcal{M}^s_{w,m}(\rho)$ the $\mathbb{C}_\infty$-vector space generated by these functions. For $s=1$ and $\rho=\mathbf{1}$ the constant representation, our space $\mathcal{M}^1_{w,m}(\mathbf{1})$ is the space of meromorphic modular forms of weight w and type m that are holomorphic on Ω, which we denote by $M^!_{w,m}$. This is the $\mathbb{C}_\infty$-vector space (of infinite dimension) generated by the holomorphic functions $f:\Omega\to\mathbb{C}_\infty$ satisfying, for all $z\in\Omega$ and $\gamma\in\Gamma$, $f(\gamma(z))=\det(\gamma)^{-m}J_\gamma^w f(z)$ and meromorphic at infinity. This vector space is generated by the functions $h^{-i}m_i$, where m_i is a Drinfeld modular form of weight $w+i(q+1)$ and type i.

For the vectorial modular forms occurring in (14), we have, if ρ is the identity representation, ${}^t\mathcal{E}_{q^k}(z,\theta)\in\mathcal{M}^2_{q^k,0}({}^t\rho^{-1})$ and $\mathcal{F}(z,\theta)\in\mathcal{M}^2_{-1,0}(\rho)$.[6]

We recall that B_1 is the disk $\{t\in\mathbb{C}_\infty$ such that $|t|\leq 1\}$. We will work with certain functions $f:\Omega\times B_1\to\mathbb{C}_\infty$ with the property that for all $z\in\Omega$, $f(z,t)$ can be identified with a series of $\mathbb{T}$ converging for all $t_0\in B_1$ to $f(z,t_0)$. For such functions, we will then also write $f(z)$ to stress the dependence on $z\in\Omega$ when we want to consider them as functions $\Omega\to\mathbb{T}$. Sometimes, we will not specify the variables z,t and write f instead of $f(z,t)$ or $f(z)$ to lighten our formulas just as we did in some parts of the introduction. Moreover, z will denote a variable in Ω throughout this work.

Let us denote by $\mathcal{R}$ the integral ring whose elements are the formal series $f=\sum_{i\geq 0}f_it^i$ such that

(1) For all i, f_i is a map $\Omega\to\mathbb{C}_\infty$ belonging to $\mathbf{Hol}(\Omega)$, the ring of holomorphic functions on Ω.
(2) For all $z\in\Omega$, $\sum_{i\geq 0}f_i(z)t^i$ is an element of $\mathbb{T}$.

The ring $\mathcal{R}$ is endowed with the injective endomorphism τ acting on formal series as follows:

$$\tau\sum_{i\geq 0}f_i(z)t^i=\sum_{i\geq 0}f_i(z)^qt^i.$$

2.1. *Deformations of vectorial modular forms.* Let us consider a representation

$$\rho:\Gamma\to\mathbf{GL}_s(\mathbb{F}_q[t]). \tag{15}$$

We assume that the determinant representation $\det(\rho)$ is the μ-th power of the determinant character for some $\mu\in\mathbb{Z}/(q-1)\mathbb{Z}$.

Definition 12. A *deformation of vectorial modular forms* of weight w, dimension s and type m associated with a representation ρ as in (15) is a column matrix $\mathcal{F}={}^t(f_1,\ldots,f_s)\in\mathbf{Mat}_{s\times 1}(\mathcal{R})$ such that the following two properties hold. Firstly, considering $\mathcal{F}$ as a map $\Omega\to\mathbf{Mat}_{s\times 1}(\mathbb{T})$ we have, for all $\gamma\in\Gamma$,

$$\mathcal{F}(\gamma(z))=J_\gamma^w\det(\gamma)^{-m}\rho(\gamma)\cdot\mathcal{F}(z).$$

[6] More generally, one speaks about matrix modular forms associated to left and right actions of two representations of Γ (or $\mathbf{SL}_2(\mathbb{Z})$). Then, $\mathcal{E}_{q^k}(\cdot,\theta)$ is a row matrix modular form associated to the right action of ρ^{-1}.

Secondly, the entries of $\mathcal{F}$ are *tempered*: there exists $\nu \in \mathbb{Z}$ such that, for all $t \in B_1$ and $i \in \{1, \ldots, s\}$, $\lim_{|z|=|z|_i \to \infty} u(z)^\nu f_i(z) = 0$.

The set of deformations of vectorial modular forms of weight w, dimension s and type m associated to a representation ρ is a $\mathbb{T}$-module, which we will denote by $\mathcal{M}^s_{w,m}(\rho)$. (We use the same notations as for the spaces of vectorial modular forms.) It is easy to see that we can endow the space $\mathcal{M}^s(\rho) = \oplus_{w,m}\mathcal{M}^s_{w,m}(\rho)$ (the sum is direct) with the structure of a graded module over the graded ring $M^! \otimes \mathbb{T}$, where $M^! = \oplus_{w,m} M^!_{w,m}$.

If $\mathcal{F} \in \mathcal{M}^s_{w,m}(\rho)$ is a deformation of vectorial modular forms and if t_0 is an element of B_1, we have that $\mathcal{F}(\cdot, t_0)$ is a vectorial modular form of the space $\mathcal{M}^s_{w,m}(\rho|_{t=t_0})$, where $\rho|_{t=t_0}$ is the representation $\mathbf{GL}_s(A) \to \mathbf{GL}_s(\mathbb{C}_\infty)$ obtained by evaluating, for each $\gamma \in \Gamma$, the coefficients of $\rho(\gamma)$ at $t = t_0$.

Lemma 13. *Let* $\mathbf{1} : \Gamma \to (1)$ *be the trivial representation. We have* $\mathcal{M}^1_{w,m}(\mathbf{1}) = M^!_{w,m} \otimes \mathbb{T}$.

Proof. It suffices to show that $\mathcal{M}^1_{w,m}(\mathbf{1}) \subset M^!_{w,m} \otimes \mathbb{T}$, the other inclusion being evident. Let $\mathcal{G}$ be in $\mathcal{M}^1_{w,m}(\mathbf{1})$. Then, for all $t \in B_1$, $\mathcal{G}(\cdot, t)$ is an element of $M^!_{w,m}$ and, by the fact that $\mathcal{G}$ is tempered, there exists an integer n such that $h^n\mathcal{G}(\cdot, t) \in M_{w+n(q+1),m+n}$. Let $(b_1, \ldots, b_s)$ be a basis of the latter vector space. Then, we have functions $c_1, \ldots, c_s$ of the variable $t \in B_1$ such that, for all $(z, t) \in \Omega \times B_1$,

$$\mathcal{G}(z, t) = h(z)^{-n}(c_1(t)b_1(z) + \cdots + c_s(t)b_s(z)).$$

We can find $z_1, \ldots, z_s \in \Omega$ such that the matrix $B = (h(z_j)^{-n} b_i(z_j))_{1 \leq i,j \leq s}$ is invertible, from which we deduce that

$$(c_1(t), \ldots, c_s(t)) = (\mathcal{G}(z_1, t), \ldots, \mathcal{G}(z_s, t)) \cdot B^{-1}.$$

But for all $z \in \Omega$, $\mathcal{G}(z, \cdot) \in \mathbb{T}$. Therefore, $c_i \in \mathbb{T}$ for $i = 1, \ldots, s$. Finally, $\mathcal{M}^1_{w,m}(\mathbf{1}) = M^!_{w,m} \otimes \mathbb{T}$. $\square$

Lemma 14. *Let k be a nonnegative integer. If $\mathcal{F}$ is in $\mathcal{M}^s_{w,m}(\rho)$, then $\tau^k\mathcal{F} \in \mathcal{M}^s_{wq^k,m}(\rho)$. If we choose nonnegative integers $k_1, \ldots, k_s$, then*

$$\det(\tau^{k_1}\mathcal{F}, \ldots, \tau^{k_s}\mathcal{F}) \in M^!_{w(q^{k_1}+\cdots+q^{k_s}),sm-\mu} \otimes \mathbb{T}.$$

In particular,

$$W_\tau(\mathcal{F}) = \det(\tau^0\mathcal{F}, \ldots, \tau^{s-1}\mathcal{F}) \in M^!_{w(1+q+q^2+\cdots+q^{s-1}),sm-\mu} \otimes \mathbb{T}.$$

Proof. From the definition, and for all $k' \in \mathbb{Z}$,

$$(\tau^{k'}\mathcal{F})(\gamma(z)) = J_\gamma^{wq^{k'}} \det(\gamma)^{-m}\rho(\gamma)(\tau^{k'}\mathcal{F})(z)$$

because $\tau(\rho(\gamma)) = \rho(\gamma)$. Moreover, τ is an endomorphism of $\mathcal{R}$, and it is obvious that $\mathcal{F}$ being tempered, also implies that $\tau^k\mathcal{F}$ is tempered.

Now define the matrix function

$$\mathbf{M}_{k_1,\ldots,k_s} = (\tau^{k_1}\mathcal{F},\ldots,\tau^{k_s}\mathcal{F}).$$

After the first part of the lemma we have, for $\gamma \in \mathbf{GL}_2(A)$,

$$\mathbf{M}_{k_1,\ldots,k_s}(\gamma(z)) = \det(\gamma)^{-m}\rho(\gamma)\cdot\mathbf{M}_{k_1,\ldots,k_s}(z)\cdot\mathbf{Diag}(J_\gamma^{wq^{k_1}},\cdots,J_\gamma^{wq^{k_s}}),$$

and we conclude the proof taking determinants of both sides. □

Lemma 15. *Let us consider $\mathcal{F}$ in $\mathcal{M}^s_{w,m}(\rho)$ and let $\mathcal{E}$ be such that ${}^t\mathcal{E}$ is in $\mathcal{M}^s_{w',m'}({}^t\rho^{-1})$. Let us denote by $\mathcal{G}_k$ the scalar product $(\tau^k\mathcal{E})\cdot\mathcal{F}$. Then, for nonnegative k,*

$$\mathcal{G}_k \in M^!_{wq^k+w',m+m'}\otimes\mathbb{T}.$$

Furthermore, we have

$$\tau^k\mathcal{G}_{-k} \in M^!_{w+w'q^k,m+m'}\otimes\mathbb{T}.$$

Proof. By Lemma 14, $\tau^k({}^t\mathcal{E})$ is in $\mathcal{M}^s_{w'q^k,m'}({}^t\rho^{-1})$ and $\tau^k\mathcal{F}$ is in $\mathcal{M}^s_{wq^k,m}(\rho)$. Let γ be in $\mathbf{GL}_2(A)$. We have, after transposition, and for all $k\in\mathbb{Z}$,

$$(\tau^k\mathcal{E})(\gamma(z)) = J_\gamma^{w'q^k}\det(\gamma)^{-m'}\mathcal{E}(z)\cdot\rho^{-1}(\gamma),$$

and since $\tau^k\mathcal{G}_{-k} = {}^t\mathcal{E}\cdot(\tau^k\mathcal{F})$,

$$(\tau^k\mathcal{F})(\gamma(z)) = J_\gamma^{wq^k}\det(\gamma)^{-m}\rho(\gamma)\cdot(\tau^k\mathcal{F}(z)).$$

Hence, for $k\geq 0$,

$$\mathcal{G}_k(\gamma(z)) = J_\gamma^{w'q^k+w}\det(\gamma)^{-m-m'}\mathcal{G}_k(z),$$

and

$$\tau^k\mathcal{G}_{-k}(\gamma(z)) = J_\gamma^{w+w'q^k}\det(\gamma)^{-m-m'}\mathcal{G}_{-k}(z).$$

On the other hand, $\mathcal{G}_k$ and $\tau^k\mathcal{G}_{-k}$ are tempered for all $k\geq 0$. The lemma follows by using Lemma 13. □

From now on, we will use the representation $\rho=\rho_t$, defined in Section 1.2, and the transpose of its inverse.

2.2. *The function $\mathcal{F}$.* The function of the title of this subsection is the vector valued function $\binom{\boldsymbol{d}_1}{\boldsymbol{d}_2}$. As we will see in (24), we may write it, with the obvious extension of the notations, as

$$\mathcal{F}(z,t) = \begin{pmatrix}\boldsymbol{d}_1(z,t)\\ \boldsymbol{d}_2(z,t)\end{pmatrix} = \widetilde{\pi}(t-\theta)\mathbf{\Omega}(t)\left(\begin{array}{c} E\left(\frac{-z}{t-\theta}\right)\\ \\ E\left(\frac{-1}{t-\theta}\right)\end{array}\right),$$

where $E=E_{\exp_{\Lambda_z}}$ (see Section 4).

In the next proposition, containing the properties of $\mathcal{F}$ of interest for us, we write g for the unique normalized Drinfeld modular form of weight $q-1$ and type 0 for Γ (proportional to an Eisenstein series) and Δ for the cusp form $-h^{q-1}$.

PROPOSITION 16. *We have the following four properties for $\mathcal{F}$ and the functions $\boldsymbol{d}_1, \boldsymbol{d}_2$*:

(1) *We have $\mathcal{F} \in \mathcal{M}^2_{-1,0}(\rho_t)$.*

(2) *The functions $\boldsymbol{d}_1, \boldsymbol{d}_2$ span the $\mathbb{F}_q(t)$-vector space of dimension 2 of solutions of the following τ-linear difference equation*:

$$X = (t - \theta^q)\Delta\tau^2 X + g\tau X, \tag{16}$$

in the field $\cup_{n\geq 0}\tau^{-n}\mathcal{K}$, where $\mathcal{K}$ is the field of fractions of $\mathcal{R}$.

(3) *Define*

$$\Psi(z,t) := \begin{pmatrix} \boldsymbol{d}_1(z,t) & \boldsymbol{d}_2(z,t) \\ (\tau\boldsymbol{d}_1)(z,t) & (\tau\boldsymbol{d}_2)(z,t) \end{pmatrix}.$$

Then, for all $z \in \Omega$ and t with $|t| < q$, we have

$$\det(\Psi) = (t-\theta)^{-1}\omega(t)^{-1}h(z)^{-1}. \tag{17}$$

(4) *We have the series expansion*

$$\boldsymbol{d}_2 = \sum_{i\geq 0} c_i(t)u^{(q-1)i} \in 1 + u^{q-1}\mathbb{F}_q[t,\theta][[u^{q-1}]], \tag{18}$$

convergent for t, u such that $|t|, |u|$ are sufficiently small.

The field $\tau^{-n}\mathcal{K}$ above is the fraction field of the ring $\tau^{-n}\mathcal{R}$ whose elements are the series $\sum_{i\geq 0} f_i t^i$ with $f_i^{q^n} \in \mathbf{Hol}(\Omega)$ and, for all $z \in \Omega$, $\sum_{i\geq 0} f_i(z)t^i \in \mathbb{T}$. The identity (17) is our deformation of Legendre's period relation. Both left- and right-hand sides have the well-defined limit $t \to \theta$. The identity we then obtain is that of [14, Th. 6.2], an analogue of the classical Legendre's period relation.

Proof of Proposition 16. All the properties but one follow immediately from the results of [34], where some of them are stated in slightly different, but equivalent formulations. The only property we have to justify here is that $\mathcal{F}$ is tempered. After (18), we are led to check that there exists $\nu \in \mathbb{Z}$ such that $u(z)^\nu \boldsymbol{d}_1 \to 0$ for $z \in \Omega$ such that $|z| = |z|_i \to \infty$. For this, we have the following lemma, which concludes the proof of the Proposition (and will also be used later).

LEMMA 17. *The following limits hold for all $t \in \mathbb{C}_\infty$ such that $|t| \leq 1$:*

$$\lim_{|z|=|z|_i\to\infty} u(z)\boldsymbol{d}_1(z,t) = 0, \qquad \lim_{|z|=|z|_i\to\infty} u(z)(\tau\boldsymbol{d}_1)(z,t) = 1.$$

Proof. We recall the series expansion

$$\boldsymbol{d}_1(z) = \frac{\widetilde{\pi}}{\omega(t)}\boldsymbol{s}_1(z) = \frac{\widetilde{\pi}}{\omega(t)}\sum_{n\geq 0}\exp_{\Lambda_z}\left(\frac{z}{\theta^{n+1}}\right)t^n,$$

which converges for all t such that $|t| < q$ and for all $z \in \Omega$.

By a simple modification of the proof of [14, Lemma 5.9 p. 286], we have

$$\lim_{|z|_i=|z|\to\infty} u(z)t^n \exp_{\Lambda_z}(z/\theta^{n+1})^q = 0,$$

uniformly in $n > 0$, for all t such that $|t| \leq 1$ (in fact, even if $|t| \leq q$).

Moreover, it is easy to show that

$$\lim_{|z|_i=|z|\to\infty} u(z)\exp_{\Lambda_z}(z/\theta)^q = \widetilde{\pi}^{-q} \lim_{|z|_i=|z|\to\infty} \exp(\widetilde{\pi}z/\theta)^q/\exp(\widetilde{\pi}z) = 1,$$

which gives the second limit, from which we deduce the first limit as well. □

COROLLARY 18. *For all $z \in \Omega$, the series $\boldsymbol{d}_1(z,t), \boldsymbol{d}_2(z,t) \in \mathbb{C}_\infty[[t]]$ have infinite radii of convergence and define entire functions.*

Proof. We assume that z is fixed as before. We know from the definitions that the radius of convergence in $\mathbb{R}_{\geq 0} \cup \{\infty\}$ of the series is $r > 1$. Replacing X by $\boldsymbol{d}_i(z,t)$ in (16), we get an identity of formal series in which the right-hand side has radius of convergence r^q and the left-hand side has radius of convergence r. Therefore, $r = \infty$ and the corollary follows easily. □

2.3. *Structure of* $\mathcal{M}^2$. Let us denote by $\mathcal{F}^*$ the function $\binom{-\boldsymbol{d}_2}{\boldsymbol{d}_1}$, which is easily seen to be an element of $\mathcal{M}^2_{-1,-1}({}^t\rho_t^{-1})$. In this subsection we give some information on the structure of the spaces $\mathcal{M}^2_{w,m}({}^t\rho_t^{-1})$.

PROPOSITION 19. *We have*

$$\mathcal{M}^2_{w,m}({}^t\rho_t^{-1}) = (M^!_{w+1,m+1} \otimes \mathbb{T})\mathcal{F}^* \oplus (M^!_{w+q,m+1} \otimes \mathbb{T})(\tau\mathcal{F}^*).$$

More precisely, for all $\mathcal{E}$ with ${}^t\mathcal{E} \in \mathcal{M}^2_{w,m}({}^t\rho_t^{-1})$, we have

$${}^t\mathcal{E} = (\tau\omega)h((\tau\mathcal{G}_{-1})\mathcal{F}^* + \mathcal{G}_0(\tau\mathcal{F}^*)),$$

where we have written $\mathcal{G}_k = (\tau^k\mathcal{E}) \cdot \mathcal{F}$ for all $k \in \mathbb{Z}$.

The first part of the proposition is equivalent to the equality

$$\mathcal{M}^2_{w,m}(\rho_t) = (M^!_{w+1,m} \otimes \mathbb{T})\mathcal{F} \oplus (M^!_{w+q,m} \otimes \mathbb{T})(\tau\mathcal{F}).$$

In the rest of this paper, we will only discuss the structure of $\mathcal{M}^2_{w,m}({}^t\rho_t^{-1})$.

Proof of Proposition 19. Let us temporarily write $\mathcal{M}'$ for

$$(M^!_{w+1,m+1} \otimes \mathbb{T})\mathcal{F}^* + (M^!_{w+q,m+1} \otimes \mathbb{T})(\tau\mathcal{F}^*).$$

It is easy to show, thanks to the results of [34], that the sum is direct. Indeed, in loc. cit. it is proved that $\boldsymbol{d}_2$, $\tau\boldsymbol{d}_2$, g and h are algebraically independent over $\mathbb{C}_\infty((t))$. Moreover, $\mathcal{M}'$ clearly embeds in $\mathcal{M}^2_{w,m}({}^t\rho_t^{-1})$. It remains to show the opposite inclusion.

By Proposition 16, the matrix $M=(\mathcal{F},\tau^{-1}\mathcal{F})$ is invertible. From (17) we deduce that

$$\tau M^{-1}=(t-\theta)\omega h\begin{pmatrix} -\boldsymbol{d}_2 & \boldsymbol{d}_1 \\ \tau\boldsymbol{d}_2 & -\tau\boldsymbol{d}_1\end{pmatrix}.$$

Let $\mathcal{E}$ be such that ${}^t\mathcal{E}\in\mathcal{M}^2_{w,m}({}^t\rho_t^{-1})$. Thanks to the above expression for τM^{-1}, the identity

$$\begin{pmatrix}\mathcal{E}\\ \tau\mathcal{E}\end{pmatrix}\cdot\mathcal{F}=\begin{pmatrix}\mathcal{G}_0\\ \mathcal{G}_1\end{pmatrix},$$

product of a 2×2 matrix with a one-column matrix (which is the definition of $\mathcal{G}_0,\mathcal{G}_1$), yields the formulas

(19)

$$\begin{aligned}\mathcal{E}&=(\mathcal{G}_0,\tau^{-1}\mathcal{G}_1)\cdot M^{-1}\\ &=(t-\theta^{1/q})h^{1/q}(\tau^{-1}\omega)(\mathcal{G}_0,\tau^{-1}\mathcal{G}_1)\cdot\begin{pmatrix}-\tau^{-1}\boldsymbol{d}_2 & \tau^{-1}\boldsymbol{d}_1\\ \boldsymbol{d}_2 & -\boldsymbol{d}_1\end{pmatrix}\\ &=(t-\theta^{1/q})h^{1/q}(\tau^{-1}\omega)((\tau^{-1}\mathcal{G}_1)\boldsymbol{d}_2-\mathcal{G}_0(\tau^{-1}\boldsymbol{d}_2),-(\tau^{-1}\mathcal{G}_1)\boldsymbol{d}_1+\mathcal{G}_0(\tau^{-1}\boldsymbol{d}_1)).\end{aligned}$$

Now, we observe that we have, from the second part of Proposition 16 and for all $k\in\mathbb{Z}$,

$$\mathcal{G}_k=g(\tau\mathcal{G}_{k-1})+\Delta(t-\theta^q)\tau^2\mathcal{G}_{k-2}.$$

Applying this formula for $k=1$, we obtain

$$\tau\mathcal{G}_{-1}=\frac{\tau^{-1}\mathcal{G}_1-g^{1/q}\mathcal{G}_0}{(t-\theta)\Delta^{1/q}}, \tag{20}$$

and by using part two of Proposition 16 again, we eliminate $\tau^{-1}\mathcal{G}_1$ and $\tau^{-1}\boldsymbol{d}_i$

$$(\tau^{-1}\mathcal{G}_1)\boldsymbol{d}_i-\mathcal{G}_0\tau^{-1}\boldsymbol{d}_i=\Delta^{1/q}(t-\theta)((\tau\mathcal{G}_{-1})\boldsymbol{d}_i-\mathcal{G}_0(\tau\boldsymbol{d}_i)),\quad i=1,2.$$

Replacing this in (19) and using $\Delta^{1/q}h^{1/q}=-h$ and $(t-\theta^{1/q})\tau^{-1}\omega=\omega$, we get the formula:

$$\mathcal{E}=(t-\theta)\omega h(-(\tau\mathcal{G}_{-1})\boldsymbol{d}_2+\mathcal{G}_0(\tau\boldsymbol{d}_2),(\tau\mathcal{G}_{-1})\boldsymbol{d}_1-\mathcal{G}_0(\tau\boldsymbol{d}_1)). \tag{21}$$

By Lemma 15, we have $\mathcal{G}_0\in M^!_{w-1,m}\otimes\mathbb{T},(\tau\mathcal{G}_{-1})\in M^!_{w-q,m}\otimes\mathbb{T}$, and the proposition follows from the fact that $h\in M_{q+1,1}$. ☐

Remark 20. Let us choose any $\mathcal{E}=(f_1,f_2)$ such that ${}^t\mathcal{E}\in\mathcal{M}^2_{w,m}({}^t\rho_t^{-1})$. Proposition 19 implies that there exists $\mu\in\mathbb{Z}$ such that

$$h^\mu f_1\in\mathbb{M}^\dagger_{\mu(q+1)+w,1,\mu+m},$$

where $\mathbb{M}^\dagger_{\alpha,\beta,m}$ is a certain sub-module of the module of *almost A-quasi-modular forms* as introduced in [34].

2.4. *Deformations of vectorial Eisenstein and Poincaré series.* The aim of this subsection is to construct nontrivial elements of $\mathcal{M}^2_{w,m}({}^t\rho_t^{-1})$.

Following Gekeler [13, §3], we recall that for all $\alpha > 0$, there exists a polynomial $G_\alpha(u) \in \mathbb{C}_\infty[u]$, called the *$\alpha$-th Goss polynomial*, such that, for all $z \in \Omega$, the function $G_\alpha(u(z))$ equals the sum of the convergent series

$$\widetilde{\pi}^{-\alpha} \sum_{a \in A} \frac{1}{(z+a)^\alpha}.$$

Several properties of these polynomials are collected in [13, Prop. (3.4)]. Here, we will need to recall that for all α, G_α is of type α as a formal series of $\mathbb{C}_\infty[[u]]$. Namely,

$$G_\alpha(\lambda u) = \lambda^\alpha G_\alpha(u) \quad \text{for all } \lambda \in \mathbb{F}_q^\times.$$

We also recall, for $a \in A$, the function

$$u_a(z) := u(az) = \exp(\widetilde{\pi} a z)^{-1} = u^{|a|} f_a(u)^{-1} = u^{|a|} + \cdots \in A[[u]],$$

where $f_a \in A[[u]]$ is the *a-th inverse cyclotomic polynomial* defined in [13, (4.6)]. Obviously, we have

$$u_{\lambda a} = \lambda^{-1} u_a \quad \text{for all } \lambda \in \mathbb{F}_q^\times.$$

We will use the following lemma.

Lemma 21. *Let α be a positive integer such that $\alpha \equiv 1 \pmod{q-1}$. We have, for all $t \in \mathbb{C}_\infty$ such that $|t| \leq 1$ and $z \in \Omega$, convergence of the series below, and equality*

$$\sideset{}{'}\sum_{c,d \in A} \frac{\chi_t(c)}{(cz+d)^\alpha} = -\widetilde{\pi}^\alpha \sum_{c \in A^+} \chi_t(c) G_\alpha(u_c(z)),$$

from which it follows that the series in the left-hand side is not identically zero.

Proof. Convergence is ensured by Lemma 23 (or Proposition 22) and the elementary properties of Goss' polynomials. On the other hand, the series on the right-hand side converges for all $t \in \mathbb{C}_\infty$. We then compute

$$\begin{aligned}
\sideset{}{'}\sum_{c,d} \frac{\chi_t(c)}{(cz+d)^\alpha} &= \sum_{c \neq 0} \chi_t(c) \sum_{d \in A} \frac{1}{(cz+d)^\alpha} \\
&= \widetilde{\pi}^\alpha \sum_{c \neq 0} \chi_t(c) \sum_{d \in A} \frac{1}{(c\widetilde{\pi} z + d\widetilde{\pi})^\alpha} \\
&= \widetilde{\pi}^\alpha \sum_{c \neq 0} \chi_t(c) G_\alpha(u_c) \\
&= \widetilde{\pi}^\alpha \sum_{c \in A^+} \chi_t(c) G_\alpha(u_c) \sum_{\lambda \in \mathbb{F}_q^\times} \lambda^{1-\alpha} \\
&= -\widetilde{\pi}^\alpha \sum_{c \in A^+} \chi_t(c) G_\alpha(u_c).
\end{aligned}$$

The nonvanishing of the series comes from the nonvanishing contribution of the term $G_\alpha(u)$ in the last series. Indeed, the order of vanishing at $u = 0$ of the right-hand side is equal to $\min_{c\in A^+}\{\mathrm{ord}_{u=0}G_\alpha(u_c)\} = \mathrm{ord}_{u=0}G_\alpha(u)$, which is $< \infty$. $\square$

Following [13], we consider the subgroup

$$H = \left\{\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}\right\}$$

of $\Gamma = \mathbf{GL}_2(A)$ and its left action on Γ. For $\delta = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, the map $\delta \mapsto (c, d)$ induces a bijection between the orbit set $H\backslash\Gamma$ and the set of $(c, d) \in A^2$ with c, d relatively prime.

We consider the factor of automorphy

$$\mu_{\alpha,m}(\delta, z) = \det(\delta)^{-m} J_\delta^\alpha,$$

where m and α are positive integers. (At the same time, m will also determine a type; that is, a class modulo $q - 1$.)

Let $V_1(\delta)$ be the row matrix $(\chi_t(c), \chi_t(d))$. It is easy to show that the row matrix

$$\mu_{\alpha,m}(\delta, z)^{-1} u^m(\delta(z))V_1(\delta)$$

only depends on the class of $\delta \in H\backslash\Gamma$, so that we can consider the following expression:

$$\mathcal{E}_{\alpha,m}(z) = \sum_{\delta\in H\backslash\Gamma} \mu_{\alpha,m}(\delta, z)^{-1} u^m(\delta(z))V_1(\delta),$$

which is a row matrix whose two entries are formal series.

Let $\mathcal{V}$ be the set of functions $\Omega \to \mathbf{Mat}_{1\times 2}(\mathbb{C}_\infty[[t]])$. We introduce, for α, m integers, $f \in \mathcal{V}$ and $\gamma \in \Gamma$, the Petersson slash operator

$$f|_{\alpha,m}\gamma = \det(\gamma)^m(cz + d)^{-\alpha} f(\gamma(z)) \cdot \rho_t(\gamma).$$

This will be used in the next proposition, where we denote by $\log_q^+(x)$ the maximum between 0 and $\log_q(x)$, the logarithm in base q of $x > 0$. We point out that we will not apply this proposition in full generality. Indeed, in this paper, we essentially consider the case $m = 0$ of the proposition. The proposition is presented in this way for the sake of completeness.

Proposition 22. *Let α, m be nonnegative integers with $\alpha \geq 2m + 1$, and write $r(\alpha, m) = \alpha - 2m - 1$. We have the following properties*:

(1) *For $\gamma \in \Gamma$, the map $f \mapsto f|_{\alpha,m}\gamma$ induces a permutation of the subset of $\mathcal{V}$*:

$$\mathcal{S} = \{\mu_{\alpha,m}(\delta, z)^{-1} u^m(\delta(z))V_1(\delta); \delta \in H\backslash\Gamma\}.$$

(2) *If $t \in \mathbb{C}_\infty$ and α, m are chosen so that $r(\alpha, m) > \log_q^+ |t|$, then the components of $\mathcal{E}_{\alpha,m}(z,t)$ are series of functions of $z \in \Omega$ that converge absolutely and uniformly on every compact subset of Ω to holomorphic functions.*

(3) *If $|t| \leq 1$, then the components of $\mathcal{E}_{\alpha,m}(z,t)$ converge absolutely and uniformly on every compact subset of Ω also if $\alpha - 2m > 0$.*

(4) *For any choice of α, m, t submitted to the convergence conditions above, the function ${}^t\mathcal{E}_{\alpha,m}(z,t)$ belongs to the space $\mathcal{M}^2_{\alpha,m}({}^t\rho_t^{-1})$.*

(5) *If $\alpha - 1 \not\equiv 2m \pmod{(q-1)}$, the matrix function $\mathcal{E}_{\alpha,m}(z,t)$ is identically zero.*

(6) *If $\alpha - 1 \equiv 2m \pmod{(q-1)}$, $\alpha \geq (q+1)m + 1$ so that $\mathcal{E}_{\alpha,m}$ converges, then $\mathcal{E}_{\alpha,m}$ is not identically zero in its domain of convergence.*

The elements of the set $\mathcal{S}$ as in the proposition are easily viewed as couples of polynomials in t with coefficients holomorphic on Ω.

Proof of Proposition 22. 1. We choose $\delta \in H\backslash\Gamma$ corresponding to a pair $(c,d) \in A^2$ with c, d relatively prime, and we set

$$f_\delta = \mu_{\alpha,m}(\delta, z)^{-1} u^m(\delta(z)) V_1(\delta) \in \mathcal{S}.$$

We have

$$\begin{aligned} f_\delta(\gamma(z)) &= \mu_{\alpha,m}(\delta, \gamma(z))^{-1} u^m(\delta(\gamma(z))) V_1(\delta) \\ &= \mu_{\alpha,m}(\gamma, z) \mu_{\alpha,m}(\delta\gamma, z)^{-1} u^m(\delta\gamma(z)) V_1(\delta) \\ &= \mu_{\alpha,m}(\gamma, z) \mu_{\alpha,m}(\delta\gamma, z)^{-1} u^m(\delta\gamma(z)) V_1(\delta\gamma) \cdot \rho_t(\gamma)^{-1} \\ &= \mu_{\alpha,m}(\gamma, z) \mu_{\alpha,m}(\delta', z)^{-1} u^m(\delta'(z)) V_1(\delta') \cdot \rho_t(\gamma)^{-1} \\ &= \mu_{\alpha,m}(\gamma, z) f_{\delta'} \cdot \rho_t(\gamma)^{-1}, \end{aligned}$$

with $\delta' = \delta\gamma$ and $f_{\delta'} = \mu_{\alpha,m}(\delta', z)^{-1} u^m(\delta'(z)) V_1(\delta')$, from which part 1 of the proposition follows.

2. Convergence and holomorphy are ensured by simple modifications of [13, (5.5)], or by the arguments in [15, Ch. 10]. More precisely, let us choose $0 \leq s \leq 1$ and look at the component at the place $s+1$,

$$\mathcal{E}_s(z,t) = \sum_{\delta \in H\backslash\Gamma} \mu_{\alpha,m}(\delta, z)^{-1} u(\delta(z))^m \chi_t(c^s d^{1-s}),$$

of the vector series $\mathcal{E}_{\alpha,m}$. Writing $\alpha = n(q-1) + 2m + l'$ with n nonnegative integer and $l' \geq 1$ we see, following Gerritzen and van der Put, [15, pp. 304–305] and taking into account the inequality $|u(\delta(z))| \leq |cz+d|^2/|z|_i$, that the term of the series $\mathcal{E}_s$

$$\mu_{\alpha,m}(\delta, z)^{-1} u^m(\delta(z)) \chi_t(c^s d^{1-s}) = (cz+d)^{-n(q-1)-l'-2m} u(\delta(z))^m \chi_t(c^s d^{1-s})$$

(where δ corresponds to (c,d)) has absolute value bounded from above by

$$|z|_i^{-m}\left|\frac{\chi_t(c^sd^{1-s})}{(cz+d)^{n(q-1)+l'}}\right|.$$

Applying the first part of the proposition, to check convergence, we can freely substitute z with $z+a$ ($a\in A$) and we may assume, without loss of generality, that $|z|=|z|_i$. We verify that, either $\lambda=\deg_\theta z\in\mathbb{Q}\setminus\mathbb{Z}$, or $\lambda\in\mathbb{Z}$, a case in which for all $\rho\in K_\infty$ with $|\rho|=|z|$, we have $|z-\rho|=|z|$. In both cases, for all c,d, $|cz+d|=\max\{|cz|,|d|\}$. Then, the series defining $\mathcal{E}_s$ can be decomposed as follows:

$$\mathcal{E}_s=\left(\sideset{}{'}\sum_{|cz|<|d|}+\sideset{}{'}\sum_{|cz|\geq|d|}\right)\mu_{\alpha,m}(\delta,z)^{-1}u^m(\delta(z))\chi_t(c^sd^{1-s}).$$

We now look for upper bounds for the absolute values of the terms of the series above, separating the two cases in a way similar to that of Gerritzen and van der Put in loc. cit.

Assume first that $|cz|<|d|$; that is, $\deg_\theta c+\lambda<\deg_\theta d$. Then

$$\left|\frac{\chi_t(c^sd^{1-s})}{(cz+d)^{n(q-1)+l'}}\right|\leq\kappa\max\{1,|t|\}^{\deg_\theta d}|d|^{-n(q-1)-l'}\leq\kappa q^{\deg_\theta d(\log_q^+|t|-n(q-1)-l')},$$

where κ is a constant depending on λ, and the corresponding sub-series converges with the imposed conditions on the parameters, because $\log_q^+|t|-n(q-1)-l'<0$.

If on the other side $|cz|\geq|d|$ (that is, $\deg_\theta c+\lambda\geq\deg_\theta d$), then

$$\left|\frac{\chi_t(c^sd^{1-s})}{(cz+d)^{n(q-1)+l'}}\right|\leq\kappa'\max\{1,|t|\}^{\deg_\theta d}|c|^{-n(q-1)-l'}\leq\kappa' q^{\deg_\theta c(\log_q^+|t|-n(q-1)-l')},$$

with a constant κ' depending on λ, again because $\log_q^+|t|-n(q-1)-l'<0$. This completes the proof of the second part of the Proposition.

3. This property can be deduced from the proof of the second part because if $|t|\leq1$, then $|\chi_t(c^sd^{1-s})|\leq1$.

4. The property is obvious by the first part of the proposition, because $\mathcal{E}_{\alpha,m}=\sum_{f\in\mathcal{S}}f$, and because the functions are obviously tempered thanks to the estimates we used in the proof of part two.

5. We consider $\gamma=\mathbf{Diag}(1,\lambda)$ with $\lambda\in\mathbb{F}_q^\times$; the corresponding homography, multiplication by λ^{-1}, is equal to that defined by $\mathbf{Diag}(\lambda^{-1},1)$. Hence, we have

$$\begin{aligned}\mathcal{E}_{\alpha,m}(\gamma(z))&=\lambda^{\alpha-m}\mathcal{E}_{\alpha,m}(z)\cdot\mathbf{Diag}(1,\lambda^{-1})\\&=\lambda^m\mathcal{E}_{\alpha,m}(z)\cdot\mathbf{Diag}(\lambda,1),\end{aligned}$$

from which it follows that $\mathcal{E}_{\alpha,m}$ is identically zero if $\alpha-1\not\equiv 2m\pmod{q-1}$.

6. If $m = 0$ and $\alpha = 1$, we simply appeal to Lemma 21. Assuming now that either $m > 0$ or $\alpha > 1$, we have that $\mathcal{E}_{\alpha,m}$ converges at $t = \theta$ and

$$\begin{aligned} z\mathcal{E}_0(z,\theta) + \mathcal{E}_1(z,\theta) &= \sum_{\delta \in H\backslash\Gamma} \det(\delta)^m (cz+d)^{1-\alpha} u(\delta(z))^m \\ &= P_{\alpha-1,m}, \end{aligned}$$

where $P_{\alpha-1,m} \in M_{\alpha-1,m}$ is the Poincaré series of weight $\alpha - 1$ and type m so that [15, Prop. 10.5.2] suffices for our purposes. $\square$

Let α, m be nonnegative integers such that $\alpha - 2m > 1$ and $\alpha - 1 \equiv 2m \pmod{(q-1)}$. We have constructed functions

$$\begin{aligned} \mathcal{E}_{\alpha,m} &: \Omega \to \mathbf{Mat}_{1\times 2}(\mathcal{R}), \\ \mathcal{F} &: \Omega \to \mathbf{Mat}_{2\times 1}(\mathcal{R}), \end{aligned}$$

with ${}^t\mathcal{E}_{\alpha,m} \in \mathcal{M}^2_{\alpha,m}({}^t\rho_t^{-1})$, $\mathcal{F} \in \mathcal{M}^2_{-1,0}(\rho_t)$. Therefore, after Lemma 15, the functions

$$\mathcal{G}_{\alpha,m,k} = (\tau^k \mathcal{E}_{\alpha,m}) \cdot \mathcal{F} = \mathcal{E}_{q^k\alpha,m} \cdot \mathcal{F} : \Omega \to \mathbb{T}$$

satisfy $\mathcal{G}_{\alpha,m,k} \in M^!_{q^k\alpha-1,m} \otimes \mathbb{T}$.

A special case. After Proposition 22, if $\alpha > 0$ and $\alpha \equiv 1 \pmod{q-1}$, then $\mathcal{E}_{\alpha,0} \neq 0$. We call these series *deformations of vectorial Eisenstein series.*

Lemma 23. *With $\alpha > 0$ such that $\alpha \equiv 1 \pmod{q-1}$, the following identity holds for all $t \in \mathbb{C}_\infty$ such that $|t| \leq 1$:*

$$\mathcal{E}_{\alpha,0}(z,t) = L(\chi_t,\alpha)^{-1} {\sum_{c,d}}' (cz+d)^{-\alpha} V_1(c,d)$$

and $\mathcal{E}_{\alpha,0}$ is not identically zero.

Proof. We recall the notation

$$V_1(c,d) = (\chi_t(c), \chi_t(d)) \in \mathbf{Mat}_{1\times 2}(\mathbb{F}_q[t]).$$

We have

$$\begin{aligned} {\sum_{c,d}}' (cz+d)^{-\alpha} V_1(c,d) &= \sum_{(c',d')=1} \sum_{a \in A^+} a^{-\alpha} (c'z+d')^{-\alpha} V_1(ac',ad') \\ &= L(\chi_t,\alpha)\mathcal{E}_{\alpha,0}(z,t), \end{aligned}$$

where the first sum is over pairs of A^2 distinct from $(0,0)$, while the second sum is over the pairs (c',d') of relatively prime elements of A^2. Convergence features are easy to deduce from Proposition 22. Indeed, we have convergence if $\log_q^+ |t| < r(\alpha,m) = \alpha - 1$ (that is, $\max\{1,|t|\} \leq q^{\alpha-1}$ if $\alpha > 1$) and we have convergence for $\alpha = 1$ and $|t| \leq 1$. In all cases, convergence holds for $|t| \leq 1$. Nonvanishing of the function also follows from Proposition 22. $\square$

3. **Proof of the theorems**

We will need two auxiliary lemmas.

LEMMA 24. *Let $\alpha > 0$ be an integer such that $\alpha \equiv 1 \pmod{q-1}$. For all $t \in \mathbb{C}_\infty$ such that $|t| \leq 1$, we have*

$$\lim_{|z|_i=|z|\to\infty} \boldsymbol{d}_1(z) \sum_{c,d}{}' \frac{\chi_t(c)}{(cz+d)^\alpha} = 0.$$

Proof. By Lemma 17, we have that $\lim_{|z|=|z|_i\to\infty} f(z)\boldsymbol{d}_1(z,t) = 0$ for all $t \in B_1$ and for all f of the form $f(z) = \sum_{n=1}^\infty c_n u(z)^n$, with $c_i \in \mathbb{C}_\infty$, locally convergent at $u = 0$. But after Lemma 21, $\sum_{c,d}' \frac{\chi_t(c)}{(cz+d)^\alpha}$ is equal, for $|z|_i$ big enough, to the sum of the series $f(z) = -\widetilde{\pi}^\alpha \sum_{c\in A^+} \chi_t(c) G_\alpha(u_c(z))$, which is of the form $-\widetilde{\pi}^\alpha \kappa_\alpha u^{\nu_\alpha} + o(u^{\nu_\alpha})$, where $G_\alpha(X) = \kappa_\alpha X^{\nu_\alpha} + o(X^{\nu_\alpha})$, and the lemma follows. (In general, it is very difficult to compute κ_α and ν_α explicitly.) □

LEMMA 25. *Let $\alpha > 0$ be an integer such that $\alpha \equiv 1 \pmod{q-1}$. For all $t \in \mathbb{C}_\infty$ such that $|t| \leq 1$, we have*

$$\lim_{|z|_i=|z|\to\infty} \sum_{c,d}{}' \frac{\chi_t(d)}{(cz+d)^\alpha} = -L(\chi_t, \alpha).$$

Proof. It suffices to show that

$$\lim_{|z|_i=|z|\to\infty} \sum_{c\neq 0} \sum_{d\in A} \frac{\chi_t(d)}{(cz+d)^\alpha} = 0.$$

Assuming that $z' \in \Omega$ is such that $|z'| = |z'|_i$, we see that for all $d \in A$, $|z'+d| \geq |z'|_i$. Now, consider $c \in A \setminus \{0\}$ and $z' = cz$ with $|z| = |z|_i$. Then, $|cz+d| \geq |cz|_i = |cz|$ so that, for $|t| \leq 1$,

$$\left|\frac{\chi_t(d)}{(cz+d)^\alpha}\right| \leq |cz|^{-\alpha}.$$

This implies that

$$\left|\sum_{c\neq 0} \sum_{d\in A} \frac{\chi_t(d)}{(cz+d)^\alpha}\right| \leq |z|^{-\alpha},$$

from which the Lemma follows. □

The next step is to prove the following proposition.

PROPOSITION 26. *For all $\alpha > 0$ with $\alpha \equiv 1 \pmod{q-1}$, then $\mathcal{G}_{\alpha,0,0} \in M_{\alpha-1,0} \otimes \mathbb{T}$, and we have the limit $\lim_{|z|=|z|_i\to\infty} \mathcal{G}_{\alpha,0,0} = -1$.*

Moreover, if $\alpha \leq q(q-1)$, then

$$\mathcal{G}_{\alpha,0,0} = -E_{\alpha-1},$$

where $E_{\alpha-1}$ is the normalized Eisenstein series of weight $\alpha - 1$ for Γ.

Proof. Let us write

$$F_\alpha(z,t) := \boldsymbol{d}_1(z) \sum_{c,d}{}' \frac{\chi_t(c)}{(cz+d)^\alpha} + \boldsymbol{d}_2(z) \sum_{c,d}{}' \frac{\chi_t(d)}{(cz+d)^\alpha}$$

as series converging for all $(z,t) \in \Omega \times \mathbb{C}_\infty$ with $|t| \leq 1$. By Lemma 23, we have

$$F_\alpha(z,t) = L(\chi_t,\alpha)\mathcal{E}_{\alpha,0}(z,t) \cdot \mathcal{F}(z,t) = L(\chi_t,\alpha)\mathcal{G}_{\alpha,0,0},$$

so that $F_\alpha \in M^!_{\alpha-1,0} \otimes \mathbb{T}$. After (18), we verify that for all t with $|t| \leq 1$, $\lim_{|z|_i=|z|\to\infty} \boldsymbol{d}_2(z) = 1$. From Lemmas 24 and 25,

$$\lim_{|z|_i=|z|\to\infty} F_\alpha(z,t) = -L(\chi_t,\alpha).$$

Therefore, for all t such that $|t| \leq 1$, $F_\alpha(z,t)$ converges to a holomorphic function on Ω and is endowed with a u-expansion holomorphic at infinity. In particular, $F_\alpha(z,t)$ is a family of modular forms of $M_{\alpha-1,0} \otimes \mathbb{T}$.

Since for the selected values of α, $M_{\alpha-1,0} = \langle E_{\alpha-1}\rangle$, we obtain that $F_\alpha = -L(\chi_t,\alpha)E_{\alpha-1}$. □

Proof of Theorem 8. Let us consider, for given $\alpha > 0$, the form $\mathcal{E} = \mathcal{E}_{\alpha,0}$ and the scalar product form $\mathcal{G}_{\alpha,0,k} = (\tau^k\mathcal{E}) \cdot \mathcal{F}$. The general computation of $\mathcal{G}_0 = \mathcal{G}_{\alpha,0,0}$ and $\tau\mathcal{G}_{-1} = \tau\mathcal{G}_{\alpha,0,-1}$ as in Proposition 19 is difficult, but for $\alpha = 1$, we can apply Proposition 26. We have $\mathcal{G}_{1,0,0} = -1$ and $\mathcal{G}_{1,0,1} = \mathcal{G}_{q,0,0} = -g = -E_{q-1}$. Therefore, $\mathcal{G}_{\alpha,0,-1} = 0$ by (20) and Theorem 8 follows. □

Proof of Theorem 1. By Proposition 19 (or (21)) with $\mathcal{E} = \mathcal{E}_{\alpha,0}$, we see that the first component of this function is

$$(t-\theta)\omega h(\mathcal{G}_{\alpha,0,0}(\tau\boldsymbol{d}_2) - (\tau\mathcal{G}_{\alpha,0,-1})\boldsymbol{d}_2).$$

On the other hand, by Lemma 21, this component is also equal to

$$L(\chi_t,\alpha)^{-1}\widetilde{\pi}^\alpha \sum_{c\in A^+} \chi_t G_\alpha(u_c).$$

Therefore, the following identity holds:

$$L(\chi_t,\alpha) = \frac{\widetilde{\pi}^\alpha \sum_{c\in A^+}\chi_t(c)G_\alpha(u_c)}{(\tau\omega)h((\tau\mathcal{G}_{\alpha,0,-1})\boldsymbol{d}_2 - \mathcal{G}_{\alpha,0,0}(\tau\boldsymbol{d}_2))}. \tag{22}$$

In particular, the numerator and the denominator of the above fraction are proportional to each other.

For $\alpha = 1$ we can replace, by the above discussion, $\mathcal{G}_{\alpha,0,-1} = 0$ and $\mathcal{G}_{\alpha,0,0} = -1$. Thanks to the fact that $h = -u + o(u)$ and $\sum_{c\in A^+}\chi_t(c)u_c = u + o(u)$, we obtain

$$L(\chi_t,1) = \frac{\widetilde{\pi}\sum_{c\in A^+}\chi_t(c)u_c}{(\tau(\omega\boldsymbol{d}_2))h} = -\frac{\widetilde{\pi}}{\tau\omega} + o(1),$$

from which we deduce Theorem 1 and even some additional information; namely, the formula

$$(\tau \boldsymbol{d}_2)h = -\sum_{c\in A^+} \chi_t(c)u_c. \tag{23}$$ □

Proof of Theorem 2. For general α, we set

$$\lambda_\alpha = \frac{\sum_{c\in A^+} \chi_t(c)G_\alpha(u_c)}{h((\tau\mathcal{G}_{\alpha,0,-1})\boldsymbol{d}_2 - \mathcal{G}_{\alpha,0,0}(\tau\boldsymbol{d}_2))}.$$

By (22), λ_α is an element of $\mathbb{L}$ ($\mathbb{L}$ being the fraction field of $\mathbb{T}$) and

$$L(\chi_t,\alpha) = \lambda_\alpha \frac{\widetilde{\pi}^\alpha}{\tau\omega}.$$

We are going to show that λ_α belongs to $\mathbb{F}_q(t,\theta)$.

Let us write f for the series $\sum_{c\in A^+} \chi_t(c)G_\alpha(u_c)$, ϕ for $\lambda_\alpha h\tau\mathcal{G}_{\alpha,0,-1}$ and ψ for $-\lambda_\alpha h\mathcal{G}_{\alpha,0,0}$, so that

$$f = \phi\boldsymbol{d}_2 + \psi\tau\boldsymbol{d}_2.$$

Proposition 19 then tells us that $\phi \in M^!_{\alpha+1,1}\otimes\mathbb{L}$ and $\psi \in M^!_{\alpha+q,1}\otimes\mathbb{L}$.

Let L be an algebraically closed field containing $\mathbb{L}$, hence containing also $\mathbb{F}_q(t,\theta)$. As for any choice of w,m, $M^!_{w,m}$ embeds in $\mathbb{C}_\infty((u))$, and since there is a basis of this space with u-expansions defined over K, we have that $\mathbf{Aut}(L/\mathbb{F}_q(t,\theta))$ acts on $M^!_{w,m}\otimes L$ through the coefficients of the u-expansions. Let σ be an element of $\mathbf{Aut}(L/\mathbb{F}_q(t,\theta))$ and, for $\mu \in M^!_{w,m}\otimes L$, let us denote by $\mu^\sigma \in M^!_{w,m}\otimes L$ the form obtained applying σ to every coefficient of the u-expansion of μ.

Since $f,\boldsymbol{d}_2$ and $\tau\boldsymbol{d}_2$ are defined over $\mathbb{F}_q[t,\theta]$, we obtain $f = \phi^\sigma\boldsymbol{d}_2 + \psi^\sigma\tau\boldsymbol{d}_2$, so that

$$(\phi-\phi^\sigma)\boldsymbol{d}_2 + (\psi-\psi^\sigma)\tau\boldsymbol{d}_2 = 0.$$

We cannot have $\phi^\sigma \neq \phi$ or $\psi^\sigma \neq \psi$ by Proposition 19. Hence, for all σ, $\phi^\sigma = \phi$ and $\psi^\sigma = \psi$. This means that the u-expansions of ϕ,ψ are both defined over $\mathbb{F}_q(t^{1/q^s},\theta^{1/q^s})$ for some $s \geq 0$. By the fact that $\mathcal{G}_{\alpha,0,0} = -1 + o(1)$ (this follows from the first part of Proposition 26), we get that $\lambda_\alpha \in \mathbb{F}_q(t^{1/q^s},\theta^{1/q^s})$.

We have proven that $\lambda_\alpha = \widetilde{\pi}^{-\alpha}L(\chi_t,\alpha)(t-\theta)\omega \in \mathbb{F}_q(t^{1/q^s},\theta^{1/q^s})$. But we already know that $L(\chi_t,\alpha) \in K_\infty[[t]]$, $\omega \in K^{\mathrm{sep}}[[t]]$ and $\widetilde{\pi} \in K_\infty^{\mathrm{sep}}$ (the separable closure of K_∞). Therefore,

$$\lambda_\alpha \in \mathbb{F}_q(t^{1/q^s},\theta^{1/q^s}) \cap K_\infty^{\mathrm{sep}}((t)) = \mathbb{F}_q(t,\theta).$$ □

Remark 27. Proposition 26 tells us that $\phi \in M_{\alpha+1,1}\otimes\mathbb{L}$, which is more precise than $\phi \in M^!_{\alpha+1,1}\otimes\mathbb{L}$, following Proposition 19. We also have $\psi = (f-\phi\boldsymbol{d}_2)/\tau\boldsymbol{d}_2$. Since $\boldsymbol{d}_2 = 1+\cdots$ has u-expansion in $\mathbb{F}_q[t,\theta][[u]]$ by the fourth part of Proposition 16, the same property holds for $\tau\boldsymbol{d}_2$ and ψ also belongs to $M_{\alpha+q,1} \otimes \mathbb{L}$. We observe the additional information that both $\mathcal{G}_{\alpha,0,0}$ and $\tau\mathcal{G}_{\alpha,0,-1}$ are defined over $\mathbb{F}_q(t^{1/q^s}, \theta^{1/q^s})$; this also follows from Proposition 19.

Remark 28. Structures similar to the above emerge in the classical setting too, but in a more fragmentary way. Let us consider, for z in the complex upper-half plane $\mathcal{H}$, the *basic quasi-periods* $\eta_1(z)$ and $\eta_2(z)$ of the lattice $z\mathbb{Z}+\mathbb{Z}$. (See Lang's book [28, Ch. 18].) Now look at the holomorphic vectorial function $\mathcal{F}_1' : \mathcal{H} \to \mathbb{C}^2$ defined by $\mathcal{F}_1'(z) = \binom{\eta_1(z)}{\eta_2(z)}$, which is a vectorial modular form of weight 1 associated to the identity representation.

Choose an odd integer $\alpha \geq 5$, and consider the vectorial series

$$\mathcal{E}_\alpha'(z) = \sum_{m,n\in\mathbb{Z}}{}' (mz+n)^{-\alpha}(m,n).$$

It is easy to show that the transpose of the series above converges to a nonzero vectorial modular form of weight α associated to the representation $\gamma \mapsto {}^t\gamma^{-1}$. More specifically, $\mathcal{E}_\alpha'$ is a *vectorial Poincaré series* after [27].

The function $\mathcal{G}_\alpha' = \mathcal{E}_\alpha' \cdot \mathcal{F}_1'$ is a meromorphic modular form of weight $\alpha+1$ for $\mathbf{SL}_2(\mathbb{Z})$ (which is holomorphic on $\mathcal{H}$); we shall compute it.

For this purpose, we appeal to the classical *Legendre's period relation* (as in Lang's book [28]), from which one can also deduce the formula $\eta_2 = (3/\pi^2)E_2$, where E_2 is the normalized logarithmic derivative of the unique normalized cusp form of weight 12. We get, for $G_{2k} = \sum_{m,n}'(mz+n)^{-2k}$, the Eisenstein series of weight $2k$ (take $k \geq 2$) and eliminating η_1,

$$\begin{aligned}\mathcal{G}_\alpha' &= \sum_{m,n\in\mathbb{Z}}{}' (mz+n)^{-\alpha}(m\eta_1+n\eta_2)\\ &= \eta_2 \sum_{m,n\in\mathbb{Z}}{}' (mz+n)^{1-\alpha} - 2\pi i \sum_{m,n\in\mathbb{Z}}{}' m(mz+n)^{-\alpha}\\ &= \frac{4\pi^2}{\alpha-1}\left(DG_{\alpha-1} - \frac{\alpha-1}{12}E_2G_{\alpha-1}\right),\end{aligned}$$

where $D = (2\pi i)^{-1}d/dz$. In other words, $\mathcal{G}_\alpha'$, up to the factor $4\pi^2/(\alpha-1)$, is proportional to the so-called *Ramanujan's modular derivative* of the Eisenstein series $G_{\alpha-1}$.[7]

This simple computation can be used to explicitly determine the vectorial form $\mathcal{E}_\alpha'$. Indeed, we may also consider the vectorial form $\mathcal{F}_{-1}' = \binom{z}{1}$ of weight -1 and, from the obtained relations, compute the entries of $\mathcal{E}_\alpha'$ solving a linear system with coefficients given by the matrix $(\mathcal{F}_{-1}', \mathcal{F}_1')$. The computation of the limit $\Im(z) \to \infty$ in the first component of $\mathcal{E}_\alpha'$ agrees with the second set of Euler formulas in the introduction. This very much resembles our scheme of

[7] The above computation is actually meaningful for $\alpha = 3$ by dealing in appropriate way with the conditionally convergent series.

proof of Theorem 8, but there is no obvious analogue of the deformations we used above.

Contiguity relations for Gauss' hypergeometric functions seem analogous to part of the theory of τ-difference equations used here. Evidence of this analogy can be observed, for example, by reading the papers [25], [26], from which one will perhaps be able to extract an analogue of our deformations in the classical setting. More work is needed to reach a good understanding of the problem.

4. **Alternative proof of Theorem 1**

In this section we shall prove Theorem 1 by using techniques related to "log-algebraic" power series identities essentially introduced by Anderson in [2], [3]. We recall that the Tate algebra $\mathbb{T}$ is a $\mathbb{C}_\infty$-Banach algebra with respect to the norm $\|\cdot\|$ defined, for $f=\sum_{n\geq 0}c_nt^n\in\mathbb{T}$, by $\|f\|=\sup_{n\geq 0}|c_n|$, that it is complete, Noetherian and factorial.

The map $\tau:\mathbb{C}_\infty((t))\to\mathbb{C}_\infty((t))$ induces an $\mathbb{F}_q[t]$-automorphism of $\mathbb{T}$. The set

$$\mathbb{T}[[\tau]]=\left\{\sum_{n\geq 0}a_n\tau^n,\ \text{with } a_n\in\mathbb{T}\right\}$$

is endowed with the evident ring structure determined by the noncommutative product

$$\sum_{i\geq 0}a_i\tau^i\cdot\sum_{j\geq 0}a'_j\tau^j=\sum_{k\geq 0}\left(\sum_{i+j=k}a_i\tau^i(a'_j)\right)\tau^k,$$

where $\sum_{i\geq 0}a_i\tau^i$ and $\sum_{j\geq 0}a'_j\tau^j$ are elements of $\mathbb{T}[[\tau]]$.

Let $\mathfrak{a}=\sum_{i\geq 0}a_i\tau^i$ be an element of $\mathbb{T}[[\tau]]$, let $R(\mathfrak{a})$ be the supremum of the set of the real numbers $r\geq 0$ such that the limit $\lim_{i\to\infty}\|a_i\|r^{q^i}$ is well defined and zero. (We call this number the *radius of convergence* of $\mathfrak{a}$.) If $f=\sum_{n\geq 0}c_nt^n\in\mathbb{T}$ is such that $\|f\|<R(\Phi)$, the series (*evaluation of* $\mathfrak{a}$ *at* f)

$$E_{\mathfrak{a}}(f)=\sum_{n\geq 0}\left(\sum_{m\geq 0}\phi_m\tau^m(c_n)\right)t^n$$

converges to an element of $\mathbb{T}$ and defines an $\mathbb{F}_q[t]$-linear map

$$E_{\mathfrak{a}}:\mathbb{B}_r\to\mathbb{T},$$

where $\mathbb{B}_r$ is the set (and $\mathbb{F}_q[t]$-sub-module) of elements f of $\mathbb{T}$ such that $\|f\|<r$, which can be called *disk of convergence* of $E_{\mathfrak{a}}$.

Let $\mathfrak{e}$ be the formal series of $\mathbb{T}[[\tau]]$ associated to Carlitz' exponential function

$$\mathfrak{e}=\sum_{i\geq 0}\frac{\tau^i}{d_i}\in\mathbb{C}_\infty[[\tau]].$$

Since $R(\mathfrak{e})=\infty$, we can associate to it the $\mathbb{F}_q[t]$-linear map $E_{\mathfrak{e}}:\mathbb{T}\to\mathbb{T}$ (exponential operator) agreeing with Carlitz' exponential function exp over $\mathbb{C}_\infty$. The series $E_{\mathfrak{e}}\left(-\frac{\widetilde{\pi}}{t-\theta}\right)$ is well defined in $\mathbb{T}$ and, in fact, comparing with (4) and the identity

$$\widetilde{\pi}\sum_{i\geq 0}\frac{t^i}{\theta^{i+1}}=-\frac{\widetilde{\pi}}{t-\theta},$$

we see that it coincides with ω:

$$\omega(t)=E_{\mathfrak{e}}\left(-\frac{\widetilde{\pi}}{t-\theta}\right). \tag{24}$$

From this identity it is apparent that ω is a $(t-\theta)$-torsion point for the $\mathbb{F}_q[t]$-linear extension of the Carlitz module

$$\phi_{\mathrm{Car}}:\mathbb{F}_q[t,\theta]\to\mathbb{T}[[\tau]],$$

uniquely determined by $\phi_{\mathrm{Car}}(\theta)=\theta+\tau$ and $\phi_{\mathrm{Car}}(t)=t$. This is an equivalent way of expressing the fact that ω is a solution of (5).

We also have the following fundamental and elementary property.

LEMMA 29. *For all $a\in A$, the function ω is eigenfunction of the operator $\phi_{\mathrm{Car}}(a)$ with eigenvalue $\chi_t(a)$:*

$$\phi_{\mathrm{Car}}(a)\omega=\chi_t(a)\omega. \tag{25}$$

Proof. This can be easily deduced from (5) and $\mathbb{F}_q$-linearity. However, thanks to (24), we can argue as follows. Observe that for all $a\in A$, $a-\chi_t(a)$ is divisible by $t-\theta$. By the identity $\phi_{\mathrm{Car}}(a)\mathfrak{e}=\mathfrak{e}a$, the series ω belongs to the $(a-\chi_t(a))$-torsion as well; that is, $\phi_{\mathrm{Car}}(a-\chi_t(a))\omega=0$. (Notice that for $f\in\mathbb{T}$, there is no need to distinguish between $E_{\phi_{\mathrm{Car}}(a)}(f)$ and $\phi_{\mathrm{Car}}(a)f$.) $\square$

Remark 30. Similar identities hold for the so-called *Baker's function* in the framework of Krichever modules, See, for example, [35, Prop. 5.11]. The exponential expression for ω in (24) is also useful to associate good variants of the series $L(\chi_t^\beta,\alpha)$ to more general rings A. This topic will be developed in another work.

Proof of Theorem 1. We provide a useful identity between $\mathbb{F}_q[t]$-linear operators as above (more precisely, we will furnish two different series expansions for the same operator) in the skew ring $\mathbb{T}[[\tau]]$. To define the first operator, we recall the element $\mathfrak{l}$ of $\mathbb{T}[[\tau]]$ associated to the *Carlitz logarithm*:

$$\mathfrak{l}=\sum_{n\geq 0}\frac{\tau^n}{l_n},$$

where the sequence $(l_n)_{n\geq 0}$ is defined inductively by $l_0=1$ and $l_n=-[n]l_{n-1}$. It is easy to show that $R(\mathfrak{l})=q^{q/(q-1)}$.

The first operator is then, for $X \in \mathbb{C}_\infty$ such that $|X| < 1$,

$$\mathfrak{l}_X := \sum_{n \geq 0} \frac{X^{q^n} \tau^n}{l_n}.$$

By $R(\mathfrak{a} \cdot X) = |X|^{-1} R(\mathfrak{a})$ (for general $\mathfrak{a} \in \mathbb{T}[[\tau]]$), we see that $\mathfrak{l}_X$ has radius of convergence $|X|^{-1} q^{q/(q-1)}$. For $X \in \mathbb{C}_\infty$ such that $|X| < 1$ and $z \in \mathbb{C}_\infty$ in the disk of convergence of $\mathfrak{l}$, the following identity holds:

$$E_{\mathfrak{l}_X}(z) = \log(Xz),$$

where log is Carlitz' logarithm.

Let us introduce the second operator, at first sight distinct from the first. For $X \in \mathbb{C}_\infty$ with $|X| < 1$, the series $\mathfrak{j}_X \in \mathbb{C}_\infty[[\tau]]$,

$$\mathfrak{j}_X = \sum_{d \geq 0} X^{q^d} \sum_{a \in A^+(d)} a^{-1} \phi_{\mathrm{Car}}(a),$$

is well defined. Indeed, let us recall the formula (see for example Thakur's book [36])

$$\phi_{\mathrm{Car}}(a) = \sum_{j=0}^{\deg_\theta(a)} \left\{ \begin{matrix} a \\ q^j \end{matrix} \right\} \tau^j, \tag{26}$$

where

$$\left\{ \begin{matrix} a \\ q^j \end{matrix} \right\} = \sum_{k=0}^{j} \frac{a^{q^k}}{d_k l_{j-k}^{q^k}}$$

is a polynomial of degree $(\deg_\theta(a) - j) q^j$.

The coefficient of τ^j in $\mathfrak{j}_X$ is then the sum of the series

$$\sum_{d \geq 0} X^{q^d} \sum_{a \in A^+(d)} a^{-1} \left\{ \begin{matrix} a \\ q^j \end{matrix} \right\},$$

which is convergent as the absolute value of its d-th term is $\leq |X|^{q^d} q^{(d-j)q^j - d}$.

We shall show

Lemma 31. *For $|X| < 1$, we have the identity*

$$\mathfrak{j}_X = \mathfrak{l}_X.$$

Proof. We denote by $S_d(m)$ the sum $\sum_{a \in A^+(d)} a^m$, and we recall, again from Thakur's book [36, Cor. 5.6.4], that if $d > k$, then $S_d(q^k - 1) = 0$, and that, for $k \geq d$,

$$S_d(q^k - 1) = \frac{d_k}{l_d d_{k-d}^d}. \tag{27}$$

Next, using (26) and (27), for $|X|<1$, we compute

$$\begin{aligned}
\mathfrak{j}_X &= \sum_{d\geq 0} X^{q^d} \sum_{a\in A^+(d)} a^{-1} \sum_{j=0}^{d} \left\{ \begin{matrix} a \\ q^j \end{matrix} \right\} \tau^j \\
&= \sum_{d\geq 0} X^{q^d} \sum_{j=0}^{d} \sum_{a\in A^+(d)} \sum_{k=0}^{j} \frac{a^{q^k-1}}{d_k l_{j-k}^{q^k}} \tau^j \\
&= \sum_{d\geq 0} X^{q^d} \sum_{j=0}^{d} \sum_{k=0}^{j} \frac{1}{d_k l_{j-k}^{q^k}} S_d(q^k-1)\tau^j \\
&= \sum_{d\geq 0} X^{q^d} \sum_{j=0}^{d} \sum_{k=d}^{j} \frac{1}{d_k l_{j-k}^{q^k}} \frac{d_k}{l_d d_{k-d}^d} \tau^j \\
&= \sum_{d\geq 0} X^{q^d} \frac{1}{l_d} \tau^d. \qquad \square
\end{aligned}$$

Since $\|\omega\| = |\exp(\widetilde{\pi}/\theta)| = q^{\frac{1}{q-1}}$, for all X with $|X|<q$, we have that $E_{\mathfrak{l}_X}(\omega)$ is well defined and

$$E_{\mathfrak{l}_X}(\omega) = E_{\mathfrak{l}_X}\left(\sum_{i\geq 0} \exp\left(\frac{\widetilde{\pi}}{\theta^{i+1}}\right) t^i\right) = \sum_{i\geq 0} t^i \log\left(X \exp\left(\frac{\widetilde{\pi}}{\theta^{i+1}}\right)\right). \tag{28}$$

To proceed further, we need the following elementary lemma, where log temporarily denotes the classical logarithm, $\log_q$ denotes the logarithm in base q and e is Euler's number.

Lemma 32. *We have, for all $a\in A$ with $\deg_\theta(a) = d \geq 0$,*

$$\log_q \left\| E_{\mathfrak{e}}\left(-\frac{a\widetilde{\pi}}{t-\theta}\right)\right\| \leq (e\log(q))^{-1} q^{d+\frac{1}{q-1}}.$$

Proof. Since

$$E_{\mathfrak{e}}\left(-\frac{a\widetilde{\pi}}{t-\theta}\right) = \sum_{n\geq 0} \exp\left(\frac{a\widetilde{\pi}}{\theta^{n+1}}\right) t^n$$

and

$$\exp\left(\frac{a\widetilde{\pi}}{\theta^{n+1}}\right) = \sum_{m\geq 0} \frac{1}{d_m}\left(\frac{a\widetilde{\pi}}{\theta^{n+1}}\right)^{q^m},$$

we have

$$\left\| E_{\mathfrak{e}}\left(-\frac{a\widetilde{\pi}}{t-\theta}\right)\right\| = \sup_{n\geq 0}\sup_{m\geq 0} \left|\frac{1}{d_m}\left(\frac{a\widetilde{\pi}}{\theta^{n+1}}\right)^{q^m}\right|.$$

Now, we compute

$$\left|\frac{1}{d_m}\left(\frac{a\widetilde{\pi}}{\theta^{n+1}}\right)^{q^m}\right| = q^{q^m(-(n+1)+d+\frac{q}{q-1}-m)},$$

and the lemma follows from the elementary inequality, valid for $x \geq 0$:

$$q^x \left(-(n+1) + d + \frac{q}{q-1} - x\right) \leq q^{d-n+\frac{1}{q-1}} (e \log(q))^{-1}. \qquad \square$$

We observe that for all $a \in A$, $\phi_{\mathrm{Car}}(a)(\omega) = E_{\mathfrak{e}}\left(-\frac{a\widetilde{\pi}}{t-\theta}\right)$. By Lemma 32, if $X \in \mathbb{C}_\infty$ and $a \in A$ is of degree $d \geq 0$, then

$$\|X^{q^d} a^{-1} \phi_{\mathrm{Car}}(a)(\omega)\| \leq q^{q^d(\log_q |X| + (e\log(q))^{-1})} q^{\frac{1}{(e\log(q))(q-1)} - d}.$$

Therefore, if X is such that $|X| < q^{-1/(e\log(q))}$, the series

$$\sum_{d \geq 0} X^{q^d} \sum_{a \in A^+(d)} a^{-1} \phi_{\mathrm{Car}}(a)(\omega)$$

converges. By Lemma 31, we have, for X submitted to the condition above, the identity

$$E_{\mathfrak{l}_X}(\omega) = \sum_{d \geq 0} X^{q^d} \sum_{a \in A^+(d)} a^{-1} \phi_{\mathrm{Car}}(a)(\omega).$$

By Lemma 29, the latter sum is equal to $\sum_{d\geq 0} X^{q^d} \sum_{a \in A^+(d)} a^{-1} \chi_t(a)\omega$, yielding the identity

$$E_{\mathfrak{l}_X}(\omega) = \left(\sum_{d \geq 0} X^{q^d} \sum_{a \in A^+(d)} a^{-1} \chi_t(a)\right)\omega. \tag{29}$$

In principle, we have obtained an identity of series that holds for $|X| < q^{-1/(e\log(q))}$ only. However, the left-hand side of (29) clearly converges for $|X| < q$ and the right-hand side converges at least for $|X| \leq 1$. The value at $X = 1$ of the left-hand side is

$$E_{\mathfrak{l}} E_{\mathfrak{e}}\left(-\frac{\widetilde{\pi}}{t-\theta}\right) = -\frac{\widetilde{\pi}}{t-\theta}.$$

By making the limit $X \to 1$ on the right-hand side, we obtain

$$\lim_{X \to 1} \left(\sum_{d \geq 0} X^{q^d} \sum_{a \in A^+(d)} a^{-1} \chi_t(a)\right)\omega = L(\chi_t, 1)\omega.$$

Therefore, $-\frac{\widetilde{\pi}}{t-\theta} = L(\chi_t, 1)\omega$ and our Theorem 1 follows. $\square$

Remark 33. The well-known identity

$$\log(1) = \zeta(1),$$

first obtained by Carlitz [6] (see also [3]), can be easily deduced from our proof.

Notice also that the identity in $\mathbb{T}[[\tau]]$,

$$\mathfrak{l}\phi_{\mathrm{Car}}(a) = a\mathfrak{l}, \quad a \in A \setminus \{0\},$$

yields an identity $E_{\mathfrak{l}} E_{\phi_{\mathrm{Car}}(a)} = aE_{\mathfrak{l}}$ only on the disk $\{f \in \mathbb{T}, \|f\| < |a|^{-1} q^{\frac{q}{q-1}}\}$. Since ω lies on the boundary of this disk when $a = \theta$, there is no reason for having identities between $(E_{\mathfrak{l}} E_{\phi_{\mathrm{Car}(a)}})(\omega)$ and $aE_{\mathfrak{l}}(\omega)$. In fact, the identities are false, and by (25), the identities

$$(E_{\mathfrak{l}} E_{\phi_{\mathrm{Car}(a)}})(\omega) = \chi_t(a) E_{\mathfrak{l}}(\omega), \quad a \in A \setminus \{0\}$$

hold instead.

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