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The Nash problem for surfaces

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Abstract

We prove that Nash mapping is bijective for any surface defined over an algebraically closed field of characteristic 0.

1. **Introduction**

The Nash problem [18] was formulated in the sixties (but published later) in the attempt to understand the relation between the structure of resolution of singularities of an algebraic variety X over a field of characteristic 0 and the space of arcs (germs of parametrized curves) in the variety. He proved that the space of arcs centered at the singular locus (endowed with an infinite-dimensional algebraic variety structure) has finitely many irreducible components and proposed to study the relation of these components with the essential irreducible components of the exceptional set of a resolution of singularities.

An irreducible component E_i of the exceptional divisor of a resolution of singularities is called *essential* if given any other resolution, the birational transform of E_i to the second resolution is an irreducible component of the exceptional divisor. Nash defined a mapping from the set of irreducible components of the space of arcs centered at the singular locus to the set of essential components of a resolution as follows. To each component W of the space of arcs centered at the singular locus he assigned the unique component of the exceptional set which meets the lifting of a generic arc of W to the resolution. Nash established the injectivity of this mapping. For the case of surfaces, it seemed plausible for him that the mapping is also surjective, and he posed the problem as an open question. He also proposed to study the mapping in the higher dimensional case. Nash resolved the question positively for the surface

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singularities of type A_k. As a general reference for the Nash problem, the reader may look at [18] and [10].

Besides the Nash problem, the study of arc spaces is interesting because it lays the foundations for motivic integration and because the study of its geometric properties reveals properties of the underlying varieties. (See papers of de Fernex, Denef, Ein, Ishii, Lazarsfeld, Loeser, Mustata, Yasuda and others.)

It is well known that birational geometry of surfaces is much simpler than in higher dimension. This fact reflects on the Nash problem: Ishii and Kollár showed in [10] a 4-dimensional example with nonbijective Nash mapping. In the same paper they showed the bijectivity of the Nash mapping for toric singularities of arbitrary dimension. Other advances in the higher dimensional case include [23], [6], [14]. Very recently there have appeared 3-dimensional counterexamples as well. The first ones are due to T. de Fernex [1]. Later J. Kollár showed even simpler counterexamples [11]: even the A_4-threefold singularity, defined by the equation $x^2+y^2+z^2+w^5=0$ is a counterexample. In the same paper he proposes a revised higher dimensional conjecture.

On the other hand, bijectivity of the Nash mapping has been shown for many classes of surfaces (see [6], [10], [8], [9], [12], [13], [17], [19], [20], [21], [22], [24], [25], [26]). The techniques leading to the proof of each of these cases are different in nature, and the proofs are often complicated. It is worth noting that even for the case of the rational double points not solved by Nash, a complete proof had to be awaited until 2010; see [19], where the problem is solved for any quotient surface singularity, and also [21] and [24] for the cases of D_n and E_6. In [4] it is shown that the Nash problem for surfaces only depends on the topological type of the singularity.

In this paper we resolve the Nash question for surfaces.

Main Theorem *Nash mapping is bijective for any surface defined over an algebraically closed field of characteristic* 0.

The core of the result is the case of normal surface singularities. After settling this case we deduce from it the general surface case.

The proof is based on the use of convergent wedges and topological methods. A wedge is a uniparametric family of arcs. The use of wedges in connection to the Nash problem was proposed by M. Lejeune-Jalabert [12]. Later A. Reguera [27], building onto the fundamental lemma of motivic integration by J. Denef and F. Loeser [2], proved a characterization of components which are at the image of the Nash map in terms of formal wedges defined over fields which are of infinite transcendence degree over the base field. In [4] the first author proves a characterization of the image of the Nash mapping for surfaces in terms of convergent (or even algebraic) wedges defined over the base field, which is the starting point of this article.

The present paper is partially inspired by the ideas of [19]; more concretely, by the use of representatives of wedges.

The idea of our proof is as follows. Let (X, O) be a normal surface singularity and

$$\pi : \tilde{X} \to (X, O)$$

be the minimal resolution of singularities. By a theorem of [4], if Nash mapping of (X, O) is not bijective, there exists a convergent wedge

$$\alpha : (\mathbb{C}^2, O) \to (X, O)$$

with certain precise properties (see Definition 1). As in [19], taking a suitable representative we may view α as a uniparametric family of mappings

$$\alpha_s : \mathcal{U}_s \to (X, O)$$

from a family of domains $\mathcal{U}_s$ to X with the property that each $\mathcal{U}_s$ is diffeomorphic to a disk. For any s, we consider the lifting

$$\tilde{\alpha}_s : \mathcal{U}_s \to \tilde{X}$$

to the resolution. Notice that $\tilde{\alpha}_s$ is the normalization mapping of the image curve.

On the other hand, if we denote by Y_s the image of $\tilde{\alpha}_s$ for $s \neq 0$, then we may consider the limit divisor Y_0 in $\tilde{X}$ when s approaches 0. This limit divisor consists of the union of the image of $\tilde{\alpha}_0$ and certain components of the exceptional divisor of the resolution whose multiplicities are easy to be computed. We prove an upper bound for the Euler characteristic of the normalization of any reduced deformation of Y_0 in terms of the following data: the topology of Y_0, the multiplicities of its components and the set of intersection points of Y_0 with the generic member Y_s of the deformation. Using this bound we show that the Euler characteristic of the normalization of Y_s is strictly smaller than one. This contradicts the fact that the normalization is a disk.

In the last section we deduce the general case from the normal case.

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2. **Preliminaries**

2.1. Let (X, O) be a complex analytic normal surface singularity. Let

$$\pi : (\tilde{X}, E) \to (X, O)$$

be the minimal resolution of singularities, which is an isomorphism outside the exceptional divisor $E := \pi^{-1}(O)$. Consider the decomposition $E = \cup_{i=0}^{r} E_i$ of

E into irreducible components. These irreducible components are the essential components of (X, O).

2.2. Given any irreducible component E_i we define by N_{E_i} the Zariski closure in the arc space of X of the set of nonconstant arcs whose lifting to the resolution is centered at E_i. These Zariski closed subsets are irreducible, and each irreducible component of the space of arcs is equal to some N_{E_i} for a certain component E_i. If N_{E_i} is an irreducible component of the space of arcs centered at the singular set, then its image by Nash mapping is the essential component E_i. Thus Nash mapping is not bijective if and only if there exist two different irreducible components E_i and E_j of the exceptional divisor of the minimal resolution such that we have the inclusion $N_{E_i} \subset N_{E_j}$ (see [18]). Such inclusions were called *adjacencies* in [4].

2.3. The germ (X, O) is embedded in an ambient space $\mathbb{C}^N$. Denote by B_ε the closed ball of radius ε centered at the origin and by $\mathbb{S}_\varepsilon$ its boundary sphere. Take a *Milnor radius* ε_0 for (X, O) in $\mathbb{C}^N$; that is, we choose $\varepsilon_0 > 0$ such that for a certain representative X and any radius $0 < \varepsilon \leq \varepsilon_0$ we have that all the spheres $\mathbb{S}_\varepsilon$ are transverse to X and $X \cap \mathbb{S}_\varepsilon$ is a closed subset of $\mathbb{S}_\varepsilon$. (See [16] for a proof of its existence.) In particular, $X \cap B_{\varepsilon_0}$ has conical structure. From now on we will denote by X_{ε_0} the *Milnor representative* $X \cap B_{\varepsilon_0}$ and by $\tilde{X}_{\varepsilon_0}$ the resolution of singularities $\pi^{-1}(X_{\varepsilon_0})$. In these conditions the space $\tilde{X}_{\varepsilon_0}$ admits the exceptional divisor E as a deformation retract. Hence the homology group $H_2(\tilde{X}_{\varepsilon_0}, \mathbb{Z})$ is free and generated by the classes of the irreducible components E_i. Since $\tilde{X}_{\varepsilon_0}$ is a smooth 4-manifold, there is a symmetric intersection product

$$\bullet : H_2(\tilde{X}_{\varepsilon_0}, \mathbb{Z}) \times H_2(\tilde{X}_{\varepsilon_0}, \mathbb{Z}) \to \mathbb{Z}.$$

The intersection product is negative definite since it is the intersection product of a resolution of a surface singularity.

2.4. We recall some terminology and results from [4]. Consider coordinates (t, s) in the germ $(\mathbb{C}^2, O)$. A *convergent wedge* is a complex analytic germ

$$\alpha : (\mathbb{C}^2, O) \to (X, O)$$

which sends the line $V(t)$ to the origin O. Given a wedge α and a parameter value s, the arc

$$\alpha_s : (\mathbb{C}, 0) \to (X, O)$$

is defined by $\alpha_s(t) = \alpha(t, s)$. The arc α_0 is called *the special arc* of the wedge. For small enough $s \neq 0$, the arcs α_s are called *generic arcs*.

Any nonconstant arc

$$\gamma : (\mathbb{C}, 0) \to (X, O)$$

admits a unique lifting to $(\tilde{X}, O)$, which we denote by $\tilde{\gamma}$.

Definition 1 ([4]). A convergent wedge α *realizes an adjacency* from E_j to E_i (with $j \neq i$) if and only if the lifting $\tilde{\alpha}_0$ of the special arc meets E_i transversely at a nonsingular point of E and the lifting $\tilde{\alpha}_s$ of a generic arc satisfies $\tilde{\alpha}_s(0) \in E_j$.

Our proof is based on the following theorem, which is the implication "(1) $\Rightarrow$ (a)" of Corollary B of [4].

THEOREM 2 ([4]). *An essential divisor E_i is in the image of the Nash mapping if there is no other essential divisor $E_j \neq E_i$ such that there exists a convergent wedge realizing an adjacency from E_j to E_i.*

The proof in [4] of this theorem has two parts. The first consists of proving that if there is an adjacency, then there exists a *formal wedge*

$$\alpha : \mathrm{Spec}(\mathbb{C}[[t,s]]) \to (X,O)$$

realising the adjacency. For that, firstly it is used a theorem of A. Reguera [27], which produces wedges defined over large fields. Then a specialisation argument is performed to produce a wedge defined over the base field $\mathbb{C}$. This was done independently in [14]. The second part is an argument based on D. Popescu's Approximation Theorem, which produces the convergent wedge from the formal one. In [5] the authors of the present paper give an alternative proof of the first part giving, in one step, a formal wedge defined over $\mathbb{C}$.

2.5. The previous theorem allows us to address the Nash question in the complex analytic case. Suppose that (X,O) is a singularity of a normal algebraic surface defined over an algebraically closed field $\mathbb{K}$ of characteristic 0. It is well known that (X,O) may be defined over a field $\mathbb{K}_1 \subset \mathbb{K}$ which is a finite extension of $\mathbb{Q}$, and hence admits an embedding into $\mathbb{C}$. Let $\bar{\mathbb{K}}_1$ be the algebraic closure of $\mathbb{K}_1$. We have then two field embeddings $\overline{\mathbb{K}}_1 \subset \mathbb{K}$ and $\overline{\mathbb{K}}_1 \subset \mathbb{C}$. In 7.1 and 7.2 of [4] it is shown that the bijectivity of the Nash mapping does not change by extension of algebraically closed fields. Therefore we deduce that if we prove the bijectivity of the Nash mapping for any complex analytic normal surface singularity, then it holds for any normal surface singularity defined over a field of characteristic equal to 0.

2.6. Following [19] we shall work with representatives rather than germs in order to get richer information about the geometry of the possible wedges. Recall that B_ε denotes the closed ball of $\mathbb{C}^N$ centered at the origin and $\mathbb{S}_\varepsilon$ its boundary sphere. We denote by $\dot{B}_\varepsilon$ the open ball. Remember that X_{ε_0} stands for a Milnor representative $X \cap B_{\varepsilon_0} \subset \mathbb{C}^N$ with ε_0 a Milnor radius for (X,O).

Consider the real analytic function $\rho : \mathbb{C}^N \to \mathbb{R}$ given by the square of the distance function to the origin in $\mathbb{C}^N$.

Lemma 3. *Given any nonconstant convergent arc γ, there exists a positive radius ε_0 such that the mapping γ is transverse to the sphere $\mathbb{S}_\varepsilon$ for any positive radius $\varepsilon \leq \varepsilon_0$.*

Proof. We follow the proof of the existence of Milnor representatives of analytic spaces given in [16]. The critical set C of the composition $\rho\circ\gamma$ is a real analytic subset of $\mathbb{C}$. We claim that the origin is an isolated point in C. Indeed, otherwise there is a 1-dimensional component of the germ $(C, 0)$, which admits a nonconstant parametrization $\theta : (\mathbb{R}, 0) \to \mathbb{C}$. The composition $\rho\circ\gamma\circ\theta$ is constant since, by the chain rule, its first derivative vanishes at any point. This implies that $\gamma\circ\theta(s)$ is always at distance 0 to the origin, and hence $\gamma\circ\theta$ is constant. Since 0 is an isolated point of $\gamma^{-1}(O)$ for being γ a nonconstant holomorphic arc, then $\theta(s)$ is constantly 0 and this is a contradiction. $\square$

Given any nonconstant convergent arc γ, since 0 is an isolated point of $\gamma^{-1}(O)$, we may consider a representative

$$\gamma|_D : D \to X$$

for an open bounded domain D in $\mathbb{C}$ such that 0 is the only point of $\gamma^{-1}(O)$ in D. We choose an open domain D' containing 0 and whose closure is contained in D.

Lemma 4. *There exists ε_0 small enough such that the restriction*

$$\gamma|_{D'\cap\gamma|_D^{-1}(B_{\varepsilon_0})} : D' \cap \gamma|_D^{-1}(B_{\varepsilon_0}) \to X_{\varepsilon_0}$$

is proper and for any positive radius $\varepsilon < \varepsilon_0$, the domain $D' \cap \gamma^{-1}(B_\varepsilon)$ is diffeomorphic to a closed disk.

Proof. Since 0 is the only point of $\gamma^{-1}(O)$ in D, the minimum of the function $\rho\circ\gamma$ in the compact set $\partial D'$ is a positive number η. We take ε_0 strictly smaller than $\sqrt{\eta}$ and such that the conclusion of Lemma 3 holds.

The inverse image by $\gamma|_D$ of the closed ball B_{ε_0} is a closed subset of D disjoint to $\partial D'$. Hence the connected components of $\gamma|_D^{-1}(B_{\varepsilon_0})$ contained in D' are compact. This shows the properness of $\gamma|_{D'\cap\gamma|_D^{-1}(B_{\varepsilon_0})}$.

Since the mapping $\gamma|_U$ is transverse to the sphere $\mathbb{S}_{\varepsilon_0}$, we obtain that the boundary of $D' \cap \gamma|_D^{-1}(B_{\varepsilon_0})$ is a disjoint union of differentiable circles.

Since $\gamma|_U$ is transverse to the sphere $\mathbb{S}_\varepsilon$ for any positive radius $\varepsilon \leq \varepsilon_0$, we have a smooth function

$$\rho\circ\gamma : D' \cap \gamma|_D^{-1}(B_{\varepsilon_0}) \to [0, \varepsilon_0]$$

such that the preimage of ε_0 defines the boundary of $D' \cap \gamma|_D^{-1}(B_{\varepsilon_0})$ and such that its only critical point is 0 and assumes the value 0. Therefore $D' \cap \gamma|_D^{-1}(B_{\varepsilon_0})$ is homeomorphic to the cone over the boundary of $D' \cap \gamma|_D^{-1}(B_{\varepsilon_0})$, and thus to the cone over a disjoint union of circles. Since $D' \cap \gamma|_D^{-1}(B_{\varepsilon_0})$ is a smooth manifold with boundary, we have that there is only one circle and therefore it is diffeomorphic to a closed disk. □

Definition 5. A *Milnor representative* of γ is a representative of the form

$$\gamma|_U : U \to X_{\varepsilon_0},$$

where U is diffeomorphic to a closed disk, we have the equality $\gamma|_U^{-1}(\partial X_{\varepsilon_0}) = \partial U$ and the mapping $\gamma|_U$ is transverse to any sphere $\mathbb{S}_\varepsilon$ for any $0 < \varepsilon \leq \varepsilon_0$. The radius ε_0 is called a *Milnor radius* for γ.

Remark 6. The union of Lemmata 3 and 4 gives that any nonconstant arc has a Milnor representative with U of the form $D' \cap \gamma|_D^{-1}(B_{\varepsilon_0})$ for some ε_0 and domains $D' \subset\subset D$ of $\mathbb{C}$.

2.7. Denote by D_δ the closed disk of radius δ centered at the origin of $\mathbb{C}$ and by $\dot{D}_\delta$ the open one. We denote with $\dot{A}$ the interior of a set A in the transcendent topology.

Given a wedge α with nonconstant special arc α_0, consider the mapping

$$\beta : (\mathbb{C}^2, (0,0)) \to (\mathbb{C}^N \times \mathbb{C}, (O,0))$$

given by $\beta(t,s) := (\alpha(t,s), s)$. Since α is defined in a neighbourhood of the origin in $\mathbb{C}^2$, we may consider a Milnor representative $\alpha_0|_U$ with Milnor radius ε_0 for α_0 such that for a positive and small enough δ, the mapping α is defined in $U \times D_\delta$. Consider the restriction

$$\beta|_{U \times D_\delta} : U \times D_\delta \to X \times D_\delta.$$

We denote by pr the projection of $U \times D_\delta$ onto the second factor.

LEMMA 7. *After possibly shrinking* δ, *we have that there exists* $\varepsilon > 0$ *such that, defining*

$$\mathcal{U} := \beta|_{U \times \dot{D}_\delta}^{-1}(X_\varepsilon \times \dot{D}_\delta),$$

we have that

(a) *the restriction* $\beta|_{\dot{\mathcal{U}}} : \dot{\mathcal{U}} \to \dot{X}_\varepsilon \times \dot{D}_\delta$ *is a proper and finite morphism of analytic spaces*
(b) *the set* $\beta(\dot{\mathcal{U}})$ *is a* 2*-dimensional closed analytic subset of* $\dot{X}_\varepsilon \times \dot{D}_\delta$;
(c) *the set* $\beta(\mathcal{U})$ *is contained in a bigger analytic* 2*-dimensional closed subset* $\mathcal{Y}$ *of* $X_{\varepsilon_1} \times \dot{D}_\delta$ *for some* $\varepsilon_1 > \varepsilon$ *such that* $\beta(\partial\mathcal{U}) = \mathcal{Y} \cap (\partial X_\varepsilon \times \dot{D}_\delta)$;
(d) *for any* $s \in D_\delta$, *the restriction* $\beta|_{U \times \{s\}}$ *is transverse to* $\mathbb{S}_\varepsilon \times \dot{D}_\delta$;
(e) *the set* $\mathcal{U}$ *is a smooth manifold with boundary* $\beta|_{\mathcal{U}}^{-1}(\partial X_\varepsilon \times \dot{D}_\delta)$;
(f) *for any* $s \in D_\delta$, *the intersection* $\mathcal{U} \cap (\mathbb{C} \times \{s\})$ *is diffeomorphic to a disk.*

Proof. By continuity, for any η there is a δ small enough so that for any $s \in D_\delta$, we have the inclusion

$$\alpha_s(\partial U) \subset B_{\varepsilon_0+\eta} \setminus B_{\varepsilon_0-\eta}.$$

For any positive ε_1 strictly smaller than $\varepsilon_0-\eta$ and any compact subset $K \in \dot{D}_\delta$, we have that $\beta|_{U\times D_\delta}^{-1}(B_{\varepsilon_1} \times K)$ is a closed subset of $U \times \dot{D}_\delta$ which is disjoint from the boundary $\partial(U \times \dot{D}_\delta)$. This easily implies that for any ε_1 strictly smaller than $\varepsilon_0 - \eta$, if we define

$$\dot{\mathcal{U}}_1 := \beta|_{U\times D_\delta}^{-1}(\dot{B}_{\varepsilon_1} \times \dot{D}_\delta),$$

then the restriction

$$\beta|_{\dot{\mathcal{U}}_1} : \dot{\mathcal{U}}_1 \to \dot{X}_{\varepsilon_1} \times \dot{D}_\delta$$

is a proper morphism of analytic spaces, which is in fact finite since each arc α_s is finite. Remmert's Finite Mapping Theorem ([7, Prop. 3.1.3, p. 65]) gives that the image $\beta(\dot{\mathcal{U}}_1)$ is a 2-dimensional closed analytic subset of $\dot{X}_{\varepsilon_1} \times \dot{D}_\delta$.

Fix a positive ε strictly smaller than ε_1, and define

$$\mathcal{U} := \beta|_{U\times \dot{D}_\delta}^{-1}(X_\varepsilon \times \dot{D}_\delta).$$

We have properties (a)–(b) for $\mathcal{U}$ for the same reason that we have them for $\mathcal{U}_1$ since $\varepsilon < \varepsilon_1 < \varepsilon_0 - \eta$. To get (c) we take $\mathcal{Y} := \beta(\dot{\mathcal{U}}_1)$.

Since transversality is an open property, we may fix a new δ small enough so that $\alpha_s|_U$ is transverse to $\mathbb{S}_\varepsilon$ for any $s \in \dot{D}_\delta$. This is (d) and this implies that $\mathcal{U}$ is a manifold with boundary, which gives (e). Denoting by pr the projection of $\mathbb{C}^2$ to the second factor, we also have that the restriction $\mathrm{pr}|_{\partial\mathcal{U}} : \partial\mathcal{U} \to \dot{D}_\delta$ is submersive. Therefore the mapping

$$\mathrm{pr}|_{\mathcal{U}} : \mathcal{U} \to \dot{D}_\delta$$

is a proper submersion which is also a submersion when restricted to the boundary $\partial\mathcal{U}$. Then Ehresmann Fibration Theorem gives (f). ☐

We will denote by $\mathcal{U}_s$ the fibre $\mathrm{pr}|_{\mathcal{U}}^{-1}(s)$. The fact that every $\mathcal{U}_s$ is a disk is a key in the proof as it was in the final step of the proof of the main result of [19].

From now on we only deal with wedges α, as the ones realising an adjacency (see Definition 1), whose special arc has a Milnor representative

$$\alpha_0|_U : U \to X_\varepsilon,$$

which is injective. For such a wedge, consider a representative $\beta : \mathcal{U} \to X_\varepsilon \times \dot{D}_\delta$ so small that it satisfies Lemma 7. We get the following

Lemma 8. *If δ is chosen small enough, then we have that $\alpha_s|_{\mathcal{U}_s}$ is generically one-to-one.*

Proof. Notice that

$$\alpha_s|_{\mathcal{U}_s} : \mathcal{U}_s \to X_\varepsilon$$

is a perturbation of the injective smooth mapping

$$\alpha_0|_{\mathcal{U}_0} : \mathcal{U}_0 \to X_\varepsilon.$$

Observe that $\beta^{-1}(\partial B_\varepsilon \times \{s\})$ is an $\mathbb{S}^1$ for any $s \in \dot{D}_\delta$. Since we cannot deform an embedding $\mathbb{S}^1 \to \mathbb{S}_\varepsilon$ to a noninjective mapping $\mathbb{S}^1 \to \mathbb{S}_\varepsilon$, we get that $\alpha_s|_{\mathcal{U}_s}$ is generically one-to-one. □

3. **Wedges and divisors**

3.1. Let α be a wedge realizing an adjacency. By the previous section, we may consider a representative $\beta : \mathcal{U} \to X_\varepsilon \times \dot{D}_\delta$ satisfying Lemmata 7 and 8. To simplify notation we take D_δ closed, redefining δ strictly smaller.

We consider the image $H := \beta(\mathcal{U})$. For every $s \in D_\delta$, the fibre H_s, by the natural projection onto D_δ, is the image of the representative

$$\alpha_s|_{\mathcal{U}_s} : \mathcal{U}_s \to X_\varepsilon.$$

Given the minimal resolution of singularities

$$\pi : \tilde{X}_\varepsilon \to X_\varepsilon,$$

we consider the mapping

$$\sigma : \tilde{X}_\varepsilon \times D_\delta \to X_\varepsilon \times D_\delta$$

defined by $\sigma(x, s) = (\pi(x), s)$. Note that the mapping σ is an isomorphism outside $E \times D_\delta$. We consider the strict transform of H by σ in $\tilde{X}_\varepsilon \times D_\delta$, which we denote by Y. We will explain this construction in detail, looking especially at the fibers of the restriction to Y of the projection of $\tilde{X}_\varepsilon \times D_\delta$ onto the second factor.

We define Y to be the analytic Zariski closure in $\tilde{X}_\varepsilon \times D_\delta$ of

$$\sigma^{-1}(H \setminus (\{O\} \times D_\delta)). \tag{1}$$

The space (1) is an irreducible surface; thus, so is its closure Y. Since $\tilde{X}_\varepsilon \times D_\delta$ is a smooth threefold, the surface Y considered with its reduced structure is a Cartier divisor (that is, a codimension 1 analytic subset whose sheaf of ideals is locally principal). We denote by Y_s the intersection $Y \cap (\tilde{X} \times \{s\})$.

The indeterminacy locus of the mapping $\sigma^{-1} \circ \beta|_{\mathcal{U}}$ has codimension 2. Hence reducing ε and δ if necessary, we can assume that the origin $(0, 0) \in \mathcal{U}$ is the only indeterminacy point. Denote by

$$\tilde{\beta} : \mathcal{U} \setminus \{(0, 0)\} \to \tilde{X}_\varepsilon \times D_\delta$$

the restriction of $\sigma^{-1}\circ\beta|_{\mathcal{U}}$ to its domain of definition $\mathcal{U}\setminus\{(0,0)\}$. Observe that we have the equality

$$\tilde{\beta}(\mathcal{U}\setminus\beta^{-1}(\{O\}\times D_\delta))=\sigma^{-1}(H\setminus(\{O\}\times D_\delta)).$$

Consequently Y is the analytic Zariski closure of $\tilde{\beta}(\mathcal{U}\setminus\{(0,0)\})$.

We claim that the morphism

$$\tilde{\beta}|_{\mathcal{U}\setminus\mathcal{U}_0}:\mathcal{U}\setminus\mathcal{U}_0\to\tilde{X}_\varepsilon\times(D_\delta\setminus\{0\})$$

is proper. Indeed, given any compact subset $K\subset\tilde{X}_\varepsilon\times(D_\delta\setminus\{0\})$, its preimage $\tilde{\beta}|^{-1}_{\mathcal{U}\setminus\mathcal{U}_0}(K)$ is equal to $\beta^{-1}(\sigma(K))$, which is compact because β is proper. Then the Remmert Direct Image Theorem shows that the image $\tilde{\beta}(\mathcal{U}\setminus\mathcal{U}_0)$ is a closed analytic subset of $\tilde{X}_\varepsilon\times(D_\delta\setminus\{0\})$. This immediately implies the equality

$$Y\cap(\tilde{X}_\varepsilon\times(D_\delta\setminus\{0\}))=\tilde{\beta}(\mathcal{U}\setminus\mathcal{U}_0). \tag{2}$$

For any $s\in D_\delta$, there exists a unique lifting

$$\tilde{\alpha}_s:\mathcal{U}_s\to\tilde{X}_\varepsilon$$

such that $\alpha_s=\pi\circ\tilde{\alpha}_s$. Obviously, for $s\neq 0$, we have the equality $\tilde{\beta}(t)=(\tilde{\alpha}_s(t),s)$ for any $t\in\mathcal{U}_s$. This, together with equality (2), implies the equality

$$Y_s=\tilde{\alpha}_s(\mathcal{U}_s).$$

Since Y is reduced, perhaps shrinking δ, we can assume that Y_s is reduced. Since α_s is proper and generically one-to-one, and $\mathcal{U}_s$ is smooth, we have that the mapping

$$\tilde{\alpha}_s:\mathcal{U}_s\to Y_s$$

is the normalisation of Y_s. We have obtained

LEMMA 9. *For any $s\in D_\delta\setminus\{0\}$, the divisor Y_s is reduced, the mapping*

$$\tilde{\alpha}_s:\mathcal{U}_s\to Y_s$$

is its normalisation and $\mathcal{U}_s$ is diffeomorphic to a disk.

The curve Y_0 does not need to be either reduced or irreducible. The set $Z_0:=\tilde{\alpha}_0(\mathcal{U}_0)$ is an irreducible component of Y_0. Since σ is an isomorphism outside $E\times D_\delta$ and H_0 is reduced out of the origin, we deduce that Y_0 is reduced at $Z_0\setminus E$. The rest of the irreducible components of Y_0 are components of the exceptional divisor E. We decompose the divisor Y_0 as a sum

$$Y_0=Z_0+\sum_{i=0}^{r}a_iE_i. \tag{3}$$

All the a_i's are nonnegative since the divisor Y_0 is effective.

3.2. If α is a wedge realizing the adjacency from E_j to E_0 with $j \neq 0$, then by definition, the lifting $\tilde{\alpha}_0$ meets E_0 transversely. In particular, $Z_0 \cdot E_0 = 1$ and $Z_0 \cdot E_i = 0$ for $i > 0$, where Z_0 is as in formula (3).

Since the divisor Y_s is a deformation of the divisor Y_0, we have the equality

$$Y_0 \cdot E_i = Y_s \cdot E_i \tag{4}$$

for any i. Denote by b_i the intersection product of $Y_s \cdot E_i$ and by M the matrix of the intersection form in $H_2(\tilde{X}_\varepsilon, \mathbb{Z})$ with respect to the basis $\{[E_0], \dots, [E_r]\}$. Then, (4) can be expressed as follows:

$$M(a_0, \dots, a_r)^t = (-1 + b_0, b_1, \dots, b_r)^t. \tag{5}$$

In the terminology of [19], the number b_i is the number of *returns* of the wedge *through the divisor* E_i; it is the number of points $p \in \alpha_s|_{\mathcal{U}_s}^{-1}(O)$ for which the lifting to $\tilde{X}$ of the germ at p of $\alpha_s|_{\mathcal{U}_s}$ meets E_i (counted with appropriate multiplicity).

Since α realizes an adjacency from E_j to E_0, we have more restrictions about b_i's and a_i's. They can be seen as consequences of the following lemma.

LEMMA 10. *All the entries of the inverse matrix* M^{-1} *are nonpositive.*

Proof. The matrix $-M$ is symmetric, positive definite and such that any nondiagonal entry is nonpositive. Hence, if endow $\mathbb{R}^r$ with the standard euclidean product, then there is a basis $v_1, \dots, v_r$ such that the angle formed by any two different vectors of the base is at least $\pi/2$, and the matrix $-M$ is the matrix of scalar products of pairs of vectors of the basis. Therefore the inverse matrix $-M^{-1}$ is the matrix of scalar products of pairs of vectors of a basis of vectors such that the angle formed by any two of the vectors is at most $\pi/2$. This implies that all the entries of $-M^{-1}$ are nonnegative. □

Hence, if we require in (5) that each b_i and each a_i be a nonnegative integer, then we get that b_0 has to be equal to 0 or to 1, and in this last case we get that $b_1 = \cdots = b_r = 0$.

Hence, we have the following immediate consequence.

COROLLARY 11. *If* α *is a wedge realizing an adjacency from* E_j *to* E_0 *(with* $j \neq 0$*) and* $(b_0, \dots, b_r)$ *are the intersection numbers* $Y_s \cdot E_i$ *associated with the generic member of a good wedge representative as in* (5)*, then* b_0 *is equal to* 0*. Moreover,* a_0 *is positive; that is, the divisor* E_0 *appears in the support of* Y_0*.*

Proof. Since α realizes an adjacency from E_j to E_0, we have $b_j \neq 0$. Then $b_0 = 0$. Now in the first row of system (5), in order to have the equality $b_0 = 0$, we need that $\sum_{j=0}^{r} a_j k_{0,j} = -1$. By definition, all a_j and all $k_{0,j}$ except $k_{0,0}$ are nonnegative. This implies that a_0 is different from 0. □

3.3. The equality (5) can be viewed as a linear system whose indeterminates are $a_0, \ldots, a_r$. It can be used to prove that wedges realizing certain adjacencies with certain prescribed returns do not exist. (We are using the terminology of [19].) The method is as follows: the adjacencies and the prescribed returns determine $b_0, b_1, \ldots, b_r$. The existence of the wedge is impossible if the solution of the linear system has either a negative or a nonintegral entry.

Using this method it is possible to prove the bijectivity of Nash mapping for many singularities (toric, dihedral . . .), but it does not suffice for all of them. It is interesting to compare this method with the methods of [19] for the E_8 singularity: the set of adjacencies with prescribed returns which this method is not able to rule out coincide precisely with the list of 25 adjacencies with prescribed returns that the second author is not able to rule out only with intersection multiplicity methods in [19].

4. **Euler characteristic estimates**

Let $\tilde{X}$ be a smooth compact domain with smooth boundary in a projective complex surface. Let

$$Y_0 = \sum_{i=0}^{m} c_i Z_i + \sum_{i=0}^{r} a_i E_i \tag{6}$$

be a divisor in $\tilde{X}$, where the E_i's are compact prime divisors contained in the interior of $\tilde{X}$ and the Z_i's are prime divisors meeting transversely the boundary of $\tilde{X}$. We denote by $(Y_0)^{\mathrm{red}}$ the reduced divisor associated with Y_0. In all this section we consider a deformation Y_s of the divisor Y_0 with the following two properties:

(I) the divisor Y_s is reduced;
(II) given any sequence $\{p_k\}_{k\in\mathbb{N}}$ of points that converges to a point $p \in Y_0 \cap \partial\tilde{X}$ and such that p_k belongs to Y_{s_k} for $s_k \neq 0$, we have that the limit of the tangent spaces $T_{p_k}Y_{s_k}$ converges to $T_p(Y_0)^{\mathrm{red}}$.

Notice that property (II) implies that Y_s is transverse to $\partial\tilde{X}$ for s small enough.

Let

$$n : \mathcal{U}_s \to Y_s$$

be the normalization of Y_s.

In this section we bound the Euler characteristic of the normalization $\mathcal{U}_s$ in terms of the topology of the reduced divisor associated with Y_0, the multiplicities c_i and a_i and the number of intersection points of Y_0 with Y_s, for $s \neq 0$.

First we do the case when Y_0 is a normal crossing divisor. We take a smaller s when necessary in the definition of Y_s.

4.1. Property (II) appears for free in the local context.

LEMMA 12. *The set $\mathcal{S}$ of smooth points $q \in (Y_0)^{\mathrm{red}}$ to which a sequence of points $q_n \in Y_{s_n}$ converges (with $s_n \neq 0$ and $\lim_{n\to 0} s_n = 0$) satisfying that $T_{q_n}Y_{s_n}$ does not converge to $T_q(Y_0)^{\mathrm{red}}$ is a discrete set in Y_0.*

Proof. We prove that $\mathcal{S}$ is contained in a 0-dimensional analytic subset of Y_0. Let Y be the total divisor $\cup_{s\in D_\delta}(Y_s \times \{s\})$ of $\tilde{X}_\varepsilon \times D_\delta$ giving rise to the deformation. Let $\mu : Y \to D_\delta$ be the restriction to Y of the projection of $\tilde{X}_\varepsilon \times D_\delta$ to the second factor. Let m be a positive integer. Take the mapping

$$\tau : D_{\delta^{1/m}} \to D_\delta$$

defined by $\tau(z) := z^m$. Consider the fibre product $Y \times_{D_\delta} D_{\delta^{1/m}}$ and its normalisation

$$n : Y' \to Y \times_{D_\delta} D_{\delta^{1/m}}.$$

Let

$$\theta_1 : Y' \to Y,$$
$$\theta_2 : Y' \to D_{\delta^{1/m}}$$

be the composition of the normalisation mapping and the natural projections of $Y \times_{D_\delta} D_{\delta^{1/m}}$ to each of the factors respectively.

For an adequate choice of m, the fibre $\theta_2^{-1}(0)$ is generically reduced. Thus the mapping θ_2 has a 0-dimensional analytic subset $\Sigma_1 \subset \theta_2^{-1}(0)$ of isolated critical points. Moreover outside a 0-dimensional subset $\Sigma_2 \subset Y_0$ the rank of the differential of $n|_{Y_0}$ is at least 1.

It is clear that $\mathcal{S}$ is contained in $\theta_1(\Sigma_1 \cup \Sigma_2)$, which is a 0-dimensional analytic subset since θ_1 is finite. □

The following is an immediate consequence.

LEMMA 13. *For every point p of $(Y_0)^{\mathrm{red}} \setminus \partial\tilde{X}$, we can choose a radius ε such that the family of divisors $Y_s \cap B(p,\varepsilon')$ satisfy property* (II) *for any $\varepsilon' \in (0,\varepsilon]$.*

4.2. *Local normal crossings case.* In this case $\tilde{X}$ is a ball B_ε centered at the origin of $\mathbb{C}^2$, and Y_0 is defined by $f_0 = x^a y^b = 0$, where x and y are the coordinates of $\mathbb{C}^2$. The divisor Y_s is defined by $f_s = 0$, where f_s is a 1-parameter holomorphic deformation of f_0 such that f_s is reduced for $s \neq 0$. Property (II) follows from Lemma 13. We have the following bound.

LEMMA 14. *If s is small enough, then the Euler characteristic of the normalization $\mathcal{U}_s$ of Y_s satisfies*

$$\chi(\mathcal{U}_s) \leq \sum_{p\in Y_s\cap Y_0} I_p(Y_s, (Y_0)^{\mathrm{red}}). \tag{7}$$

Proof. The only connected orientable surface with a boundary that has positive Euler characteristic is the disk. Hence $\chi(\mathcal{U}_s)$ is bounded above by the number of connected components of $\mathcal{U}_s$ that are disks.

Let W_s be an irreducible component of Y_s whose normalization is a disk. Its boundary $W_s \cap \mathbb{S}_\varepsilon$ is a circle that deforms to one of the components of $Y_0 \cap \mathbb{S}_\varepsilon$; that is, either to $V(x) \cap \mathbb{S}_\varepsilon$ or to $V(y) \cap \mathbb{S}_\varepsilon$. Both cases are symmetric. In the first case the equation g_s of W_s degenerates to x^c for a certain $c \leq a$; that is, $g_0 = x^c$. Thus the circle $W_s \cap \mathbb{S}_\varepsilon$ loops c times around the $V(y)$ and hence represents a nontrivial element in $\pi_1(B_\varepsilon \setminus V(y))$. The normalization of the component W_s is a mapping from a disk to W_s. If W_s does not meet $V(y)$, the circle $W_s \cap \mathbb{S}_\varepsilon$ would be a trivial element in $\pi_1(B_\varepsilon \setminus V(y))$, and this is not the case.

We conclude that each component of Y_s whose normalization is a disk has at least one intersection point with the union of the axis. This proves the lemma. □

4.3. *Global normal crossings case.* We assume Y_0 to be a normal crossings divisor. Define

$$\dot{E}_i = E_i \setminus \operatorname{Sing}((Y_0)^{\mathrm{red}}),$$
$$\dot{Z}_i = Z_i \setminus \operatorname{Sing}((Y_0)^{\mathrm{red}})$$

for any i. Given any point $p \in \tilde{X}$, we denote by $B(p, \varepsilon)$ the closed ball in $\tilde{X}$ of radius ε centered in p.

LEMMA 15. *If s is small enough, then the Euler characteristic of the normalization $\mathcal{U}_s$ of Y_s satisfies*

$$\chi(\mathcal{U}_s) \leq \sum_{i=0}^{m} c_i \chi(\dot{Z}_i) + \sum_{i=0}^{r} a_i \chi(\dot{E}_i) + \sum_{p \in Y_s \cap Y_0} I_p(Y_s, (Y_0)^{\mathrm{red}}). \tag{8}$$

Proof. Since $\tilde{X}$ is a domain in a projective surface we think of it embedded in some $\mathbb{P}^N$. Since $Y_0 \cap \partial\tilde{X}$ is compact, we may assume the existence of collar structure for the boundary of $\tilde{X}$ near $Y_0 \cap \partial\tilde{X}$. That is, there exists a neighbourhood $\mathcal{C}$ of $Y_0 \cap \partial\tilde{X}$ in $\tilde{X}$ and a smooth function

$$\kappa : \mathcal{C} \to (0, 1]$$

without critical points such that $\mathcal{C} \cap \partial\tilde{X} = \kappa^{-1}(1)$. Since property (II) is satisfied by the family Y_s in $\tilde{X}$, by Lemma 12 we have that, if $\mathcal{C}$ is chosen small enough, then we may ensure that it is also satisfied for the families $Y_s \cap \kappa^{-1}((0, t])$ for any $t \in (0, 1]$.

We choose a neighbourhood of Y_0 as the union of the following sets:

(i) Balls $B(p_1, \varepsilon_1), \ldots, B(p_R, \varepsilon_R)$ inside $\tilde{X}$ centered in each of the singular points $p_1, \ldots, p_R$ of $(Y_0)^{\mathrm{red}}$ with radii $\varepsilon_1, \ldots, \varepsilon_R$ as in Lemma 13.

(ii) Tubular neighbourhoods $\mathcal{A}_i$ for each E_i minus a finite number of disks, which we construct as follows. For every E_i, we take a pencil of hyperplanes in $\mathbb{P}^N$ such that none of them contains E_i. Given any point $x \in E_i$, we denote by H_x the unique hyperplane of the pencil meeting x. There is a finite number of points in $\dot{E}_i$ that are tangent to hyperplanes of the pencil. We denote them by $q_1^{E_i}, \ldots, q_{k(E_i)}^{E_i}$. For any j, consider a small disc $\Delta(q_j^{E_i}, \delta_j^{E_i})$ in E_i around $q_j^{E_i}$ of radius $\delta_j^{E_i}$. The discs are chosen mutually disjoint and disjoint to every $E_i \cap B(p_l, \varepsilon_l)$.

Fix $\varepsilon_l' < \varepsilon_l$ for any l. Define

$$E_i' := E_i \setminus \Big(\bigcup_{l=1}^{R} \dot{B}(p_l, \varepsilon_l') \cup \bigcup_{j=1}^{k} \dot{\Delta}(q_j^{E_i}, \delta_j^{E_i}) \Big).$$

We take a small tubular neighbourhood $\mathcal{A}_i'$ in $\tilde{X}$ of E_i such that given any point $x \in E_i'$, the unique connected component $\mathcal{A}_x$ of the intersection $H_x \cap \mathcal{A}_i'$ only meets E_i at x.

Define

$$\mathcal{A}_i := \bigcup_{x \in E_i'} \mathcal{A}_x.$$

The pencil defines a natural holomorphic projection

$$\zeta_{E_i} : \mathcal{A}_i \to E_i'.$$

We choose s small enough such that Y_s is transverse to $\zeta_{E_i}^{-1}(\partial E_i')$ and $\zeta_{E_i}|_{Y_s}$ is onto.

(iii) Tubular neighborhoods $\mathcal{D}_i$ around each Z_i minus the union of a finite number of discs and small annuli neighbouring $Z_i \cap \partial \tilde{X}$, whose construction is parallel to the one of the neighbourhoods $\mathcal{A}_i$. For further reference, we sketch the construction briefly. Choose a pencil of hyperplanes with properties as before. Define Z_i' as Z_i minus the union of $\kappa^{-1}((1/2, 1]) \cup \bigcup_{l=1}^{R} \dot{B}(p_l, \varepsilon_l')$, and a finite number of disks $\Delta(q_j^{Z_i}, \delta_j^{Z_i})$ centered at points $\{q_1^{Z_i}, \ldots, q_{k(Z_i)}^{Z_i}\}$ where the pencil is tangent. Define tubular neighbourhoods $\mathcal{D}_i$ of Z_i' in $\tilde{X}$ and a holomorphic projection $\zeta_{Z_i} : \mathcal{D}_i \to Z_i'$ imitating the construction in (iii). We choose s small enough such that Y_s is transversal to $\zeta_{Z_i}^{-1}(\partial Z_i')$ and $\zeta_{Z_i}|_{Y_s}$ is onto.

(iv) Sets $\mathcal{B}_j^{E_i}$ (respectively $\mathcal{B}_j^{Z_i}$) around each point $q_j^{E_i} \in E_i$ (respectively $q_j^{Z_i} \in Z_i$) given as a difference $B(q_j^{E_i}, \eta_j^{E_i}) \setminus \dot{\mathcal{A}}_i$ (respectively $B(q_j^{Z_i}, \eta_j^{Z_i}) \setminus \dot{\mathcal{D}}_i$), where the radius $\eta_j^{E_i}$ (respectively $\eta_j^{Z_i}$) is slightly larger than $\delta_j^{E_i}$ (respectively $\delta_j^{Z_i}$) and is such that the boundary of the Riemann surface $Y_s \cap \mathcal{B}_j^{E_i}$ (respectively $Y_s \cap \mathcal{B}_j^{Z_i}$) equals $Y_s \cap \zeta_{E_i}^{-1}(\partial \Delta(q_j^{E_i}, \eta_j^{E_i}))$ (respectively $Y_s \cap \zeta_{Z_i}^{-1}(\partial \Delta(q_j^{Z_i}, \eta_j^{Z_i})))$ for s small enough.

(v) The set $\mathcal{E} := \mathcal{C} \setminus (\cup_i \dot{\mathcal{D}}_i)$.

We compute an estimate for the Euler characteristic of the normalization of the intersection of Y_s with each of these pieces. Then, (8) is obtained as the sum of these estimates since, as we check later in the proof, the Euler characteristics of $n^{-1}(Y_s \cap \mathcal{A}_i \cap \mathcal{B}_j^{E_i})$, $n^{-1}(Y_s \cap \mathcal{D}_i \cap \mathcal{B}_j^{Z_i})$, $n^{-1}(Y_s \cap \mathcal{A}_i \cap B(p_l, \varepsilon_l))$, $n^{-1}(Y_s \cap \mathcal{D}_i \cap B(p_l, \varepsilon_l))$, $n^{-1}(Y_s \cap \mathcal{D}_i \cap \mathcal{E})$ are 0 for every i, j and l.

To estimate $\chi(n^{-1}(Y_s \cap B(p_l, \varepsilon_l))$ we use (7). To estimate $\chi(n^{-1}(Y_s \cap \mathcal{A}_i))$ we note that the composition

$$\zeta_{E_i} \circ n : Y_s \cap \mathcal{A}_i \to E_i'$$

is a holomorphic branched cover of Riemann surfaces of degree a_i. By the Riemann-Hurwitz formula, we get

$$\chi(Y_s \cap \mathcal{A}_i) \leq a_i \chi(E_i') = a_i \chi(\dot{E}_i) - k(E_i) a_i. \tag{9}$$

In a similar way, we obtain

$$\chi(Y_s \cap \mathcal{D}_i) \leq c_i \chi(Z_i') = c_i \chi(\dot{Z}_i) - k(Z_i) c_i. \tag{10}$$

For further use, notice that the inequality becomes an equality if there are no ramification points.

For a given divisor E_i, to estimate $\chi(n^{-1}(Y_s \cap (\cup_{j=1}^{k} \mathcal{B}_j^{E_i})))$ we observe that the boundary of the Riemann surface $n^{-1}(Y_s \cap \mathcal{B}_j^{E_i})$ equals $Y_s \cap \zeta_i^{-1}(\partial \Delta(q_j^{E_i}, \delta_j^{E_i}))$. It is an unramified cover of degree a_i over the circle $\partial \Delta(q_j^{E_i}, \delta_j^{E_i})$. We conclude that $\partial(n^{-1}(Y_s \cap \mathcal{B}_j^{E_i}))$ is a disjoint union of at most a_i circles. Since each connected component of $n^{-1}(Y_s \cap \mathcal{B}_j^{E_i})$ has boundary, we have at most a_i connected components, and since the Euler characteristic of each of these connected components is at most 1, we obtain the bound

$$\chi(n^{-1}(Y_s \cap (\cup_{j=1}^{k(E_i)} \mathcal{B}_j^{E_i})) \leq k(E_i) a_i. \tag{11}$$

Besides, we have obtained that $\chi(n^{-1}(Y_s \cap \mathcal{B}_j^{E_i} \cap \mathcal{A}_i)) = 0$ for all i and j. A similar procedure shows

$$\chi(n^{-1}(Y_s \cap (\cup_{j=1}^{k(Z_i)} \mathcal{B}_j^{Z_i})) \leq k(Z_i) c_i \tag{12}$$

and the equality $\chi(n^{-1}(Y_s \cap \mathcal{B}_j^{Z_i} \cap \mathcal{D}_i)) = 0$ for all i and j.

Now we prove that $\chi(n^{-1}(Y_s \cap \mathcal{A}_i \cap B(p_l, \varepsilon_l)) = 0$ (and that $\chi(n^{-1}(Y_s \cap \mathcal{D}_i \cap B(p_l, \varepsilon_l)) = 0)$. Let ρ_l denote the distance function to the point p_l. Since property (II) is satisfied for the family of divisors $Y_s \cap B(p_l, \varepsilon)$ for any $\varepsilon < \varepsilon_l$, we deduce that, fixing any $\varepsilon_l'' < \varepsilon_l'$, if s is chosen small enough, the analytic space $Y_s \cap B(p_l, \varepsilon_l) \setminus \dot{B}(p_l, \varepsilon_l'')$ is a smooth Riemann surface with boundary. Moreover, the restriction

$$\rho_l|_{Y_s \cap B(p_l, \varepsilon_l) \setminus \dot{B}(p_l, \varepsilon_l'')} : Y_s \cap B(p_l, \varepsilon_l) \setminus \dot{B}(p_l, \varepsilon_l'') \to [\varepsilon_l'', \varepsilon_l]$$

is a smooth function without critical points. We denote by $\mathcal{X}_s$ the gradient of $\rho_l|_{Y_s \cap B(p_l, \varepsilon_l) \setminus \dot{B}(p_l, \varepsilon_l'')}$.

The components of the boundary of $\mathcal{A}_i$ defined by

$$M := \zeta_{E_i}^{-1}(E_i \cap \partial B(p_l, \varepsilon_l'))$$

form a 3-dimensional real smooth submanifold M of $B(p_l, \varepsilon_l)$ which do not meet $B(p_l, \varepsilon_l'')$ if $\mathcal{A}_i$ is chosen small enough. Since the family of divisors $Y_s \cap B(p_l, \varepsilon_l)$ satisfies property (II), we have that, if s is small enough, the intersection of Y_s with M is transverse and moreover M divides $Y_s \cap B(p_l, \varepsilon_l) \setminus \dot{B}(p_l, \varepsilon_l'')$ into two pieces. Besides, the vector field $\mathcal{X}_s$ always points to the same side of M.

Consider the flow

$$\Phi : \big(Y_s \cap \partial B(p_l, \varepsilon_l)\big) \times [0, \varepsilon_l - \varepsilon_l''] \to Y_s \cap B(p_l, \varepsilon_l) \setminus \dot{B}(p_l, \varepsilon_l'')$$

associated to $-\mathcal{X}_s$. Since $\mathcal{X}_s$ always points at the same side of M we deduce that the flow line associated to any point $x \in Y_s \cap \partial B(p_l, \varepsilon_l)$ meets M at a unique time $t(x)$. This assignment is smooth. Therefore the set

$$\bigcup_{x \in Y_s \cap \partial B(p_l, \varepsilon_l)} \Phi(\{x\} \times [0, t(x)])$$

is a smooth manifold with boundary diffeomorphic to a union of annuli and it clearly coincides with $Y_s \cap B(p_l, \varepsilon_l) \cap \mathcal{A}_i$.

In the same way but considering the collar function κ instead of the distance function ρ_i, we get the equality $\chi(n^{-1}(Y_s \cap \mathcal{E})) = 0$. □

Remark 16. Observe that the sum of estimates (9) and (11) gives the estimate

$$\chi(n^{-1}(Y_s \cap U_i)) \leq a_i \chi(\dot{E}_i) \tag{13}$$

for a tubular neighbourhood U_i of $E_i \setminus \cup_l B(p_l, \varepsilon_l)$. The sum of (10) and (12) gives an analogous estimate for a tubular neighbourhood of the $Z_i \setminus \cup_l B(p_l, \varepsilon_l)$.

4.4. *The general local case.* In this case Y_0 is defined by $f_0 = \prod_{i=0}^m g_i^{c_i} = 0$, where the g_i are irreducible and reduced analytic function germs. We denote by μ_i the Milnor number of g_i at the origin. We take a Milnor ball B_ε for f_0 as the space $\tilde{X}$. The divisor Y_s is defined by $f_s = 0$, where f_s is a 1-parameter holomorphic deformation of f_s such that f_s is reduced for $s \neq 0$. We consider a sufficiently small δ so that $f_0^{-1}(\delta) \cap B_\varepsilon$ is the Milnor fibre of f_0. Property (II) follows from Lemma 13. We start in Lemma 17 by giving an alternative proof of an equality that was proved in [15]; Proposition 18 may be understood as a generalization of it.

LEMMA 17 ([15]). *The Euler characteristic of the Milnor fibre of f_0 is equal to*

$$\chi(f_0^{-1}(\delta) \cap B_\varepsilon) = \sum_{i=0}^{m} c_i \Big(1 - \mu_i - I_O\Big(g_i, \prod_{j \neq i} g_j\Big)\Big). \tag{14}$$

Proof. Given a vector v of $\mathbb{C}^2$, we denote by τ_v the translation of $\mathbb{C}^2$ associated with v. We choose m vectors $v_1, \dots, v_m$ in $\mathbb{C}^2$ such that for any t small enough and $i \neq j$, the curves $V(g_i \circ \tau_{tv_i} - t)$ and $V(\prod_{j\neq i} g_j \circ \tau_{tv_j} - t)$ meet transversely in B_ε.

Consider the deformation $F_t := \prod_{i=0}^m (g_i \circ \tau_{tv_i} - t)^{c_i}$. An easy local argument shows that for small enough t and any $s \in D_\delta \setminus \{0\}$, the set $F_t^{-1}(s)$ is smooth at the meeting points with ∂B_ε and transverse to it. This implies the existence of a finite subset of critical values Δ_t of D_δ such that the restriction

$$F_t : B_\varepsilon \cap F_t^{-1}(D_\delta \setminus \Delta_t) \to D_\delta \setminus \Delta_t$$

is a locally trivial fibration with fibre diffeomorphic to the Milnor fibre of f_0. See Theorem 2.2 of [3] for a proof of these facts in a much more general context.

Fix a small enough t different from 0. We view $F_t^{-1}(s)$ as a deformation of the normal crossings divisor $F_t^{-1}(0)$ inside B_ε and study it like in the global normal crossings case. The irreducible components of this divisor are $Z_i = V(g_i \circ \tau_{tv_i} - t)$ for $i = 0, \dots, m$. The component Z_i is a translation of the the Milnor fibre of g_i and, hence, its Euler characteristic is equal to $1 - \mu_i$. Consequently, using that the curve Z_i meets transversely the union $\cup_{j\neq i} Z_i$ and the conservativity of intersection multiplicity, we obtain

$$\chi(\dot{Z}_i) = 1 - \mu_i - I_O\Big(g_i, \prod_{j\neq i} g_j\Big).$$

Observe that the piece of the Milnor fibre contained in a neighbourhood of a singularity of $F_t^{-1}(0)$ is a union of cylinders because locally $F_t^{-1}(0)$ is normal crossings. Decompose the Milnor fibre as in Lemma 15. Observe that in this case inequality (10) becomes an equality because there is no ramification. Inequality (12) also becomes an equality; in the corresponding inequality for a component Z_i, for any point p_j, the Riemann surface $\mathcal{B} \cap B''_j$ is a union of c_i disjoint discs. Adding Euler characteristics we obtain the result. □

After this lemma we can prove the Euler characteristic bound that we want.

PROPOSITION 18. *If s is small enough, we have*

$$\chi(\mathcal{U}_s) \leq \sum_{i=0}^{m} c_i \bigg(1 - \mu_i - I_O\Big(g_i, \prod_{j\neq i} g_j\Big)\bigg) + \sum_{p \in Y_s \cap Y_0} I_p(Y_s, (Y_0)^{\mathrm{red}}). \tag{15}$$

Proof. A particular case: the divisor Y_s does not meet the origin for $s \neq 0$. In order to reduce the problem to the global normal crossings case, we consider the minimal embedded resolution

$$\pi : \tilde{X} \to B_\varepsilon$$

of $V(f_0)$. Let $\{E_i\}_{i=1}^r$ be the irreducible components of the exceptional divisor. For any $s \in D_\delta$, we denote by V_s the pullback of Y_s by π. Since the divisor Y_s does not meet the origin when $s \neq 0$, we have that it is isomorphic to V_s and that V_s does not meet the exceptional divisor of π. Then it is enough to prove the bound for the Euler characteristic of the divisor V_s for $s \neq 0$.

The divisor V_0 decomposes as $V_0 = \sum_{i=0}^m c_i Z_i + \sum_{i=0}^r a_i E_i$, where the c_i's appear on the equation of Y_0 and the a_i's are deduced from the c_i's solving the linear system derived from the identities $V_0 \cdot E_i = 0$. (Notice that V_s does not meet any E_i because Y_s does not meet the origin and then $V_s \cdot E_i = 0$ for all i.)

Using the bound obtained in Paragraph 4.2 and the fact that $\dot{Z}_i$ is a punctured disk for any i, we obtain

$$\chi(\mathcal{U}_s) \leq \sum_{i=0}^{r} a_i \chi(\dot{E}_i) + \sum_{p \in V_0 \cap V_s} I_p(V_s, (V_0)^{\mathrm{red}}). \tag{16}$$

Using the fact that the number of intersection points of Y_s and $(Y_0)^{\mathrm{red}}$ counted with multiplicity coincides with the number of intersection points of V_s and $(V_0)^{\mathrm{red}}$ counted with multiplicity, after Lemma 17, in order to prove the proposition it only rests to check that the first sum of the right side of (16) coincides with the Euler characteristic of the Milnor fibre of f_0.

For this we observe that the divisor $V_0 = \sum_{i=0}^m c_i Z_i + \sum_{i=0}^r a_i E_i$ is equal to the total transform of $V(f_0)$ by the modification π. This is because the coefficients a_i are also characterized by the equalities $V_0 \cdot E_i = 0$ for any i. The Euler characteristic of the Milnor fibre is given then by

$$\chi(f_0^{-1}(s)) = \sum_{i=0}^{r} a_i \chi(\dot{E}_i). \tag{17}$$

Indeed, if W_s is the pullback of the Milnor fibre $f_0^{-1}(s)$ by π we apply to W_s the procedure of the proof of Lemma 15 and the following easy facts:

- The piece of Milnor fibre contained at the balls neighbouring singular points of the total transform is a union of cylinders.
- The coverings associated to the part of Milnor fibre contained at the tubular neighbourhoods of $\dot{E}_i$ and $\dot{Z}_i$ are unramified.
- Each set $\dot{Z}_i$ is a punctured disk.

The general case. We reduce the proof to the previous particular case by a deformation argument. Recall that τ_v denotes the translation in the direction of a vector v. Let v_t be a holomorphic family of vectors in $\mathbb{C}^2$ with $v_0 = O$ and such that for t small enough, $V(f_0 \circ \tau_{v_t})$ does not meet the origin. It is easy to check that the 2-parameter family $F_{t,s} := f_s \circ \tau_{v_t}$ has the following properties:

(i) The set of parameters Δ such that $V(F_{t,s})$ meets the origin is a proper closed analytic subset in the parameter space.
(ii) There exist positive $\eta \ll \delta$ such that for any s with $0 < |s| \leq \delta$ and any t satisfying $0 \leq |t| < \eta$, the normalization of $V(F_{t,s}) \cap B_\varepsilon$ is diffeomorphic to the normalization of $V(F_{0,s}) = V(f_s)$.

Choose a parametrized curve in the parameter space of the family of the form $(t(s), s)$ with $t(0) = 0$ and such that for $s \neq 0$ small enough, $t(s)$ is nonzero and avoids Δ. Then, the normalization of $V(F_{t(s),s})$ is diffeomorphic to the normalization of $V(f_s)$ for any s. Applying the particular case to the family $V(F_{t(s),s})$, we prove the proposition for the general case. □

4.5. *General global case.* For any component E_i, we consider the set of irreducible components of the germ of E_i at each point of $\mathrm{Sing}((Y_0)^{\mathrm{red}}$. We denote these germs by $\{(\Gamma_k, p_k)\}_{k=1}^{d}$. We denote by μ_{E_i} the sum of Milnor numbers of these local branches and by ν_{E_i} the number of branches and define

$$\eta_{E_i} := \sum_{k=1}^{d} \sum_{l \neq k} I_{p_k}(\Gamma_k, \Gamma_l).$$

We also define the analogous numbers μ_{Z_i}, ν_{Z_i} and η_{Z_i} for any divisor Z_i.

For any i, we denote by $\dot{E}_i$ (respectively $\dot{Z}_i$) the set $E_i \setminus \mathrm{Sing}((Y_0)^{\mathrm{red}})$ (respectively $Z_i \setminus \mathrm{Sing}((Y_0)^{\mathrm{red}})$).

Proposition 19. *For nonzero and small enough s, we have*

$$\chi(\mathcal{U}_s) \leq \sum_{i=0}^{m} c_i(\chi(\dot{Z}_i) + \theta_{Z_i}) + \sum_{i=0}^{r} a_i(\chi(\dot{E}_i) + \theta_{E_i}) + \sum_{p \in Y_s \cap Y_0} I_p(Y_s, (Y_0)^{\mathrm{red}}), \tag{18}$$

where $\theta(Z_i)$ and $\theta(E_i)$ are defined by

$$\theta_{Z_i} := \nu_{Z_i} - \mu_{Z_i} - \eta_{E_i} - Z_i \cdot ((Y_0)^{\mathrm{red}} - Z_i),$$

$$\theta_{E_i} := \nu_{E_i} - \mu_{E_i} - \eta_{Z_i} - E_i \cdot ((Y_0)^{\mathrm{red}} - E_i).$$

Proof. The proof follows the scheme of the proof of Lemma 15. We consider small Milnor balls around the singular points of $(Y_0)^{\mathrm{red}}$ and small tubular neighbourhoods around the connected components of the complement of these balls in $(Y_0)^{\mathrm{red}}$. We split $\mathcal{U}_s$ into pieces, each being the part that maps into one of the neighbourhoods just defined. We bound the Euler characteristic of the parts corresponding to tubular neighbourhoods using Hurwitz formula as in the proof of Lemma 15. We bound the Euler characteristic of the pieces corresponding to the Milnor balls using Proposition 18. Summing up the contributions and rearranging terms, we get the desired expression. □

5. **Bijectivity of the Nash map for normal surface singularities**

THEOREM 20. *Nash mapping is bijective for any normal surface singularity defined over an algebraically closed field of characteristic equal to* 0.

Proof. The argument in Paragraph 2.5 shows that it is enough to deal with the complex case.

Let (X, O) be a complex normal surface singularity. If Nash mapping is not bijective then, by Theorem 2, there exists a wedge α realizing an adjacency from a component E_j of the exceptional divisor of the minimal resolution to a different component E_0. We take a representative $\alpha|_{\mathcal{U}}$ with $\mathcal{U}$ as in Lemma 7 and define the divisors Y_0 and Y_s as in Paragraph 3.1. As we stated in Lemma 9, the divisor Y_s is reduced, the domain $\mathcal{U}_s$ is a disk and the lifting

$$\tilde{\alpha}_s : \mathcal{U}_s \to \tilde{X}$$

is the normalization of Y_s. We will use the estimates of Section 4 to get a contradiction to the fact that the Euler characteristic of $\mathcal{U}_s$ is 1. In this way we show the nonexistence of α and, by Theorem 2, that the Nash mapping is bijective for normal surface singularities.

5.1. We are going to give an estimate for $\chi(\mathcal{U}_s)$ splitting $\mathcal{U}_s$ into three pieces, in the spirit of Section 4. Remember that the divisor Y_0 is as in (3), where $a_i \geq 0$ and $a_0 > 0$ (see Corollary 11), which means that E_0 is in the support of Y_0. In this case we have a single Z_i (comparing with the general case (6)) which has the topology of a disk and intersects transversely E_0 at a smooth point of E. Moreover the divisor Y_0 is reduced at the generic point of Z_0 and transverse to $\partial\tilde{X}$. Consequently it is clear that the family Y_s satisfies property (II) at the beginning of Section 4.

Let $\tilde{X}_1$ be a small ball in $\tilde{X}$ centered at the point $p = E_0 \cap Z_0$ so that the family $Y_s \cap \tilde{X}_1$ satisfies property (II). (Lemma 13 ensures its existence.) Let $\tilde{X}_3$ be a small compact tubular neighbourhood around the disk $Z_0 \setminus \tilde{X}_1$ in $\tilde{X}$. Define $\tilde{X}_2$ as the closure of the complement of $\tilde{X}_1 \cup \tilde{X}_3$ in $\tilde{X}$. For s nonzero and small enough, the divisor Y_s meets transversely the boundaries of the $\tilde{X}_i$. We define $\mathcal{U}_s^i$ as the normalization of $Y_s \cap \tilde{X}_i$. By Remark 16 we see that $\chi(\mathcal{U}_s^3) = 0$ since $Z_0 \setminus \tilde{X}_1$ is a topological annulus. Moreover, since the intersections $\mathcal{U}_s^1 \cap \mathcal{U}_s^2$ and $\mathcal{U}_s^1 \cap \mathcal{U}_s^3$ are a disjoint union of circles, we have that

$$\chi(\mathcal{U}_s) = \chi(\mathcal{U}_s^1) + \chi(\mathcal{U}_s^2). \tag{19}$$

5.2. First let us give a bound for the Euler characteristic of $\mathcal{U}_s^1$ improving the methods of Paragraph 4.2 for our special case. We may choose local coordinates (x, y) around p in $\tilde{X}_1$ so that we have $E_0 = V(y)$ and $Z_0 = V(x)$. Let g_s be the family of functions defining the divisor Y_s locally around p. We have, up to a unit, the equality $g_0 = xy^{a_0}$. Notice that in Corollary 11 we have proved that a_0 is positive. The Euler characteristic of $\mathcal{U}_s^1$ is bounded by the number of topological disks in the normalization of $V(g_s) \cap \tilde{X}_1$. In principle the number of circles in $\partial \tilde{X}_1 \cap Y_s$ is at most $a_0 + 1$. There certainly appears one circle K_s which is a small deformation of $V(x) \cap \partial \tilde{X}_1$. By the connectivity of $\mathcal{U}_s$, the boundary of the connected component of $\mathcal{U}_s^1$ containing K_s cannot consist only of K_s. This implies that the maximal number of disks that can appear in $\mathcal{U}_s^1$ is $a_0 - 1$, and hence

$$\chi(\mathcal{U}_s^1) \leq a_0 - 1. \tag{20}$$

5.3. The Euler characteristic of $\mathcal{U}_s^2$ is bounded using Proposition 19. Notice the following identities:

$$\nu_{E_0 \cap \tilde{X}_2} = \nu_{E_0} - 1,$$
$$\mu_{E_0 \cap \tilde{X}_2} = \mu_{E_0},$$
$$(E_0 \cap \tilde{X}_2) \centerdot ((Y_0)^{\mathrm{red}} - E_0 \cap \tilde{X}_2) = E_0 \centerdot ((Y_0)^{\mathrm{red}} - E_0) - 1,$$

which imply that

$$\theta_{E_0 \cap \tilde{X}_2} = \theta_{E_0}.$$

Then, by (19), we obtain

$$\chi(\mathcal{U}_s) \leq a_0 - 1 + \sum_{i=0}^{r} a_i(\chi(\dot{E}_i) + \theta_{E_i}) + \sum_{p \in Y_s \cap Y_0 \cap \tilde{X}_2} I_p(Y_s, (Y_0)^{\mathrm{red}}). \tag{21}$$

Note that the last term is the total number of *returns*. Defining $\delta_{a_j} = 1$ if $a_j \neq 0$ and $\delta_{a_j} = 0$ if $a_j = 0$, we have the obvious bound

$$\sum_{p \in Y_s \cap Y_0 \cap \tilde{X}_2} I_p(Y_s, (Y_0)^{\mathrm{red}}) \leq \sum_{j=0}^{r} \delta_{a_j} b_j. \tag{22}$$

If we denote by $k_{i,j}$ the intersection product $E_i \centerdot E_j$, by equation (5) we have that

$$\sum_{j=0}^{r} b_j = \sum_{j=0}^{r} \sum_{i=0}^{r} \delta_{a_j} a_i k_{i,j} + 1.$$

Regrouping and coming back to (22), we get the following:

$$\sum_{p \in Y_s \cap Y_0 \cap \tilde{X}_2} I_p(Y_s, (Y_0)^{\mathrm{red}}) \leq \sum_{i=0}^{r} a_i \Big(\sum_{j=0}^{r} \delta_{a_j} k_{i,j} \Big) + 1. \tag{23}$$

Now, on one hand, denoting by g_i the genus of the normalization of E_i, we have

$$\chi(\dot{E}_i) = 2 - 2g_i - \nu_{E_i}. \tag{24}$$

On the other hand, we have that

$$E_0 \cdot ((Y_0)^{\mathrm{red}} - E_0) = \sum_{j\neq 0} \delta_{a_j} k_{0,j} + E_0 \cdot Z_0 = \sum_{j\neq 0} \delta_{a_j} k_{0,j} + 1,$$
$$E_i \cdot ((Y_0)^{\mathrm{red}} - E_0) = \sum_{j\neq i} \delta_{a_j} k_{i,j} \quad \text{for any } 1 \le i \le r,$$

and hence

$$\theta_{E_0} = \nu_{E_0} - \mu_{E_0} - \eta_{E_0} - \sum_{j\neq 0} \delta_{a_j} k_{0,j} - 1, \tag{25}$$

$$\theta_{E_i} = \nu_{E_i} - \mu_{E_i} - \eta_{E_i} - \sum_{j\neq i} \delta_{a_j} k_{i,j} \quad \text{for any } 1 \le i \le r. \tag{26}$$

Performing substitutions (24)–(26) in (21) and using (23), we get to the following:

$$\chi(\mathcal{U}_s) \le \sum_{i=0}^{r} a_i(2 - 2g_i - \mu_{E_i} - \eta_{E_i} + k_{i,i}). \tag{27}$$

By negative definiteness, for any $0 \le i \le r$, the self-intersection $k_{i,i}$ is a negative integer. Observe that, since $\pi : \tilde{X} \to X$ is the minimal resolution, for any $0 \le i \le r$, if $k_{i,i}$ is equal to -1, then either the divisor E_i is singular or it has positive genus. (Otherwise it is a smooth rational divisor with self-intersection equal to -1 and the resolution is nonminimal.) If the divisor E_0 has an irreducible singularity then μ_{E_i} is at least 2. If the divisor E_i has a singular point with several irreducible branches, then η_{E_i} is at least 2. Therefore, we have

$$a_i(2 - 2g_i - \mu_i - \eta_i + k_{i,i}) \le 0$$

for any i. (Note that $a_i \ge 0$.) Hence we get that $\chi(\mathcal{U}_s) \le 0$. This is a contradiction because we know that $\mathcal{U}_s$ is a disk. □

6. **The nonnormal case**

In this section we deduce the bijectivity of the Nash mapping for any surface from the case of normal surface singularities proved in the previous section.

6.1. Consider a Hironaka resolution of singularities of an algebraic surface (a resolution which is an isomorphism outside the singular locus). Given any irreducible component C of the exceptional locus, we define the set N_C to be the Zariski closure of the arcs in the variety centered at the singular locus, not contained in it, and whose lifting to the resolution is centered at C.

As before the Lefschetz principle allows us to reduce the bijectivity of Nash mapping for nonnormal surfaces to the complex algebraic case.

6.2. Let X_1 be any reduced algebraic surface defined over $\mathbb{C}$. Let

$$n : X_2 \to X_1$$

be the normalization and

$$\pi : \tilde{X}_2 \to X_2$$

be the minimal resolution of the singularities of X_2.

Let $\cup_{i=1}^{r} E_i$ be a decomposition into irreducible components of the exceptional divisor of π. By the minimality of the resolution, all these components are essential.

Let $n^{-1}(\mathrm{Sing}(X_1)) = \cup_{i=1}^{s} A_i$ be a decomposition into irreducible components of the preimage of the singular set of X_1 by the normalization. Denote by B_i the strict transform of A_i by π. The decomposition into irreducible components of the exceptional divisor of the resolution $n \circ \pi$ is given by

$$(\cup_{i=1}^{s} B_i) \bigcup (\cup_{i=1}^{r} E_i). \tag{28}$$

All these components are essential.

6.3. As in Paragraph 2.2, Nash mapping is not bijective if and only if there exist two different irreducible components C_1 and C_2 among those in (28) such that we have the adjacency $N_{C_1} \subset N_{C_2}$ (see also [18]).

Suppose we have an adjacency of type $N_{B_i} \subset N_{B_j}$ when $i \neq j$, $N_{B_i} \subset N_{E_j}$ for any i, j, $N_{E_i} \subset N_{E_j}$ when $i \neq j$, or $N_{E_i} \subset N_{B_j}$ for any i, j. The proof of Theorem 2 works equally in the nonnormal case, and so we can find a convergent wedge

$$\alpha_1 : (\mathbb{C}^2, O) \to X_1$$

realising the adjacency.

6.4. Inclusions of type $N_{B_i} \subset N_{B_j}$ when $i \neq j$ and $N_{B_i} \subset N_{E_j}$ for any i, j cannot occur since this contradicts easily the continuity of α.

Notice that any wedge α_1 realising a nontrivial adjacency is a dominant map from $(\mathbb{C}^2, O)$ whose image is not contained in any proper analytic subset of X_1. Since $\mathbb{C}^2$ is normal, by the universal property of the normalization, it admits a lifting

$$\alpha_2 : (\mathbb{C}^2, O) \to X_2.$$

If the wedge α_1 realises an adjacency of the type $N_{E_i} \subset N_{E_j}$ in X_1, then its lifting to X_2 realises the corresponding adjacency in the normal surface X_2. This is impossible because the Nash problem is true for normal surfaces.

6.5. Assume we have an adjacency of type $N_{E_i} \subset N_{B_j}$. We consider a convergent wedge realizing the adjacency and consider its lifting

$$\alpha_2 : (\mathbb{C}^2, O) \to X_2.$$

The image $p := \alpha_2(O)$ is a normal singular point of X_2. In the next paragraph we will cut out the exterior of a neighbourhood U of p in X_2 and glue another piece of analytic surface instead so that the $B_j \cap U$'s extend to compact curves $\bar{B}_j$ such that $\bigcup_i E_i \cup \bigcup_j \bar{B}_j$ is in the exceptional divisor of a resolution of a new normal surface singularity (X, O). Then the push forward of α_2 to this new normal surface singularity is a wedge that realizes an adjacency between two essential divisors in the new normal surface. We use then that the Nash problem is true for normal surfaces and get a contradiction.

6.6. Consider a ball B_ε around p of sufficiently small radius so that it is a Milnor ball for X_2 and each B_i at p. Consider a resolution $\tilde{X}'_2$ of the pair $(X_2, \cup_i B_i)$ so that the preimage of $\cup B_i$ has strict normal crossings. Then on the one hand, we take a small tubular neighbourhood in $\tilde{X}'_2$ of the strict transform $\tilde{B}_j$ of each B_j. If ε and the radius of the tubular neighbourhood are small enough, we may assume that the tubular neighbourhood is biholomorphic to the product of $\tilde{B}_j$ and a disk. On the other hand, we consider a holomorphic embedding

$$\iota_j : \tilde{B}_j \to \mathbb{P}^1.$$

Consider the product $\mathbb{P}^1 \times D$, D being a small disk, and glue it with $\tilde{X}'_2$ identifying $\iota_j(\tilde{B}_j) \times D$ with the tubular neighbourhood of $\tilde{B}_j$ in $\tilde{X}'_2$. In this way we obtain a smooth surface Y which extends $\tilde{X}'_2$ and such that each disk $\tilde{B}_j$ extends to a compact $\bar{B}_j$ biholomorphic to $\mathbb{P}^1$ embedded in Y.

We perform sufficiently many blow ups in Y at points of $Y \setminus \tilde{X}'_2$ such that we obtain a new surface Y where the self-intersection of the strict transform of the $\bar{B}_j$'s are as a negative as we wish. Hence, the configuration in Y given by the union of the exceptional divisor of the resolution $\tilde{X}'_2$ and the strict transform of the $\bar{B}_j$'s has a negative-definite matrix. If the self intersections of the strict transforms of the $\bar{B}_j$'s in Y' are chosen to be negative enough, the blow down of this configuration in Y' gives a resolution of a new normal surface singularity (X, O) where the divisors E_i and the strict transforms of the $\bar{B}_j$'s are essential.

6.7. There is an obvious analytic morphism

$$\kappa : (X_2, p) \to (X, O).$$

The wedge $\kappa \circ \alpha_2$ realises an adjacency between two essential components of the resolution of (X, O). This contradicts the bijectivity of Nash mapping for normal surfaces.

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