On geometric transfer in real twisted endoscopy

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Abstract

We prove the existence of a transfer of orbital integrals in endoscopy for real reductive groups when there is twisting by an automorphism defined over the reals and by a character on the real points of the group. Our proof contains a relatively short self-contained argument for the already known case of standard endoscopy.

1. Introduction

Endoscopy concerns conjugacy classes and irreducible representations for reductive groups: conjugacy classes within a stable class and irreducible representations within a packet. We consider just real groups. In the case of standard endoscopy, where there is no twisting, geometric and spectral transfer identities have been proved. These identities display structure on both packets and stable classes. In particular, the structure on a packet of discrete series representations reflects that on the set of conjugacy classes in a regular elliptic stable class. As is well known, this structure plays a role in various comparisons of trace formulas and in multiplicity formulas for automorphic representations. In the present paper we consider the broader setting of twisted endoscopy, again for real groups. Our purpose is to present a complete argument for the main geometric transfer identity. This identity shows that sums of integrals over suitably regular twisted conjugacy classes, when weighted by the transfer factors introduced in [KS99] (see also [KS12]), may be interpreted as integrals over stable conjugacy classes in an endoscopic group. The precise result has two immediate applications. First, locally (i.e., for real groups), it establishes the underlying structure for a functorial dual transfer of stable traces on a twisted endoscopic group to virtual twisted traces on the ambient group. In a separate paper [She11] we begin the description of an explicit form for the dual transfer via compatible spectral transfer factors. This extends the standard, or untwisted, case [She10], [She08b] and appears useful in the global theory; see, for example, [Art11, Th. 2.2.4]. Second, in the global picture,
our geometric transfer identity is of course one ingredient for stabilization of the geometric side of the general twisted version of the Arthur-Selberg trace formula.

Suppose $G$ is a connected reductive algebraic group defined over $\mathbb{R}$. There are two familiar types of twisting we will consider for an admissible representation $\pi$ of the reductive Lie group $G(\mathbb{R})$: composing $\pi$ with an $\mathbb{R}$-automorphism $\theta$ of $G$ and multiplying $\pi$ by a character $\varpi$ of $G(\mathbb{R})$. An isomorphism $A_\pi$ between $\pi \circ \theta$ and $\varpi \otimes \pi$, if it exists, provides us with a distribution $f \mapsto \text{Trace}(\pi(f) A_\pi)$, a $(\theta, \varpi)$-twisted character for $\pi$, on a suitable space of test functions $f$. Comparing these twisted traces with ordinary stable traces for a lower dimensional group, an endoscopic group $H_1(\mathbb{R})$ for $(G, \theta, \varpi)$, requires a correspondence on test functions. That is provided by the main geometric transfer identity which displays weighted sums of $(\theta, \varpi)$-twisted orbital integrals of test functions on $G(\mathbb{R})$ as stable orbital integrals of the corresponding test functions on $H_1(\mathbb{R})$. For the remainder of Section 1 we will discuss in some detail our setting for this and related results. The results themselves will then be described in more detail in Section 2.

Our setting is based on the constructions and results of [KS99] for the case of real groups. For the norm correspondence of [KS99, Chap. 3, §5.4] between points of $G(\mathbb{R})$ and points of an endoscopic group $H_1(\mathbb{R})$ for $(\theta, \varpi)$, it is an associated outer automorphism $\theta^*$ of a quasi-split inner form $G^*$ that is significant. If $\theta$ is inner, then $\theta^*$ is the identity, and we have a slight variant of the setting for standard endoscopy [LS87, §1.3]. To simplify the presentation we will carry a minor assumption on the norm correspondence for most of the paper. Fix an inner twist $\psi : G \to G^*$, where $G^*$ is quasi-split over $\mathbb{R}$. There is an $\mathbb{R}$-automorphism $\theta^*$ of $G^*$ which preserves a given $\mathbb{R}$-splitting of $G^*$ and for which $\theta^*$ and $\psi \circ \theta \circ \psi^{-1}$ differ by an inner automorphism of $G^*$. We then say $(G, \theta, \psi)$ is an inner twist of $(G^*, \theta^*)$, as in [KS99, App. B]. Start now with the pair $(G^*, \theta^*)$. We will consider those (isomorphism classes of) inner twists $(G, \theta, \psi)$ for which there is a norm correspondence from twisted conjugacy classes in $G(\mathbb{R})$ to the ordinary, i.e., untwisted, conjugacy classes in an endoscopic group $H_1(\mathbb{R})$. See Section 6 for a precise version of the assumption. If $\theta^*$ is the identity, this excludes certain inner automorphisms $\theta$. In these cases the twist $\theta$ persists to conjugacy classes in the endoscopic group according to the formalism of [KS99, §5.4]. The general excluded case is a variant of this, and we use a slightly twisted norm correspondence. It can be handled by a straightforward extension of our arguments, as we will describe in Section 12.

An endoscopic group $H_1$ comes with more data. First we assume that we are given, rather than the twisting character $\varpi$ itself, a 1-cocycle $a_\varpi$ (of the Weil group $W_{\mathbb{C}/\mathbb{R}}$ in the center of the connected complex dual group $G^\vee$ of $G$ or
an induced central torus in $G^*$ to which $\varpi$ is attached by Langlands’ construction [Bor79, 10.1]. A set $\epsilon$ of endoscopic data for $(G, \theta, a_\varpi)$ or $(G^*, \theta^*, a_\varpi)$ is a tuple $(H, \mathcal{H}, s, \xi)$ as in [KS99, §2.1]. There is no harm in assuming that $\xi$, an embedding of the group $\mathcal{H}$ in the $L$-group $L G = G^* \times W_{C/R}$, is the inclusion map $\text{incl}$, so that $\mathcal{H}$ is given as a subgroup of $L G$. We do so, and drop $\xi$ entirely from notation. This subgroup $\mathcal{H}$ is, by definition, a split extension of $W_{C/R}$ by $H^\vee$. In some cases, there is an $L$-isomorphism $\xi_1 : \mathcal{H} \to L H$. This provides us then with an $L$-embedding $\text{incl} \circ (\xi_1)^{-1}$ of $L H$ in $L G$, and $H$ itself may serve as an endoscopic group. The $L$-embedding $\text{incl} \circ (\xi_1)^{-1}$ determines both a term for geometric transfer factors and a shift in infinitesimal character for the dual spectral transfer from $H(\mathbb{R})$ to $G(\mathbb{R})$. The shift is necessary for the existence of a transfer identity satisfying the functoriality principle; for some examples, see [She, Part B, §2]. Existence of $\xi_1$ as isomorphism, however, excludes many cases; quick examples can be found for an outer automorphism of $SU(2,1)$ or for base change in $Sp_4$. (In standard endoscopy, examples are harder to find.) To avoid these exclusions, we add to the endoscopic data $\epsilon = (H, \mathcal{H}, s)$ a $z$-pair $(H_1, \xi_1)$ as in [KS99, Chap. 2], and then $H_1$, rather than $H$, serves as endoscopic group. This group $H_1$ is quasi-split over $\mathbb{R}$ with simply-connected derived group, and there is an exact sequence $1 \to Z_1 \to H_1 \to H \to 1$ defined over $\mathbb{R}$, where $Z_1$ is an induced central torus in $H_1$. Then $H_1(\mathbb{R}) \to H(\mathbb{R})$ is surjective and $L H$ is naturally embedded in $L H_1$. Now $\xi_1$ is an injective $L$-homomorphism of $\mathcal{H}$ in $L H_1$ (see [KS99, §2.2] for proof of existence), and $\xi_1$ determines, in particular, a character $\varpi_1$ on $Z_1(\mathbb{R})$. For example, in the $SU(2,1)$ case we may pass from the problematic $H = \text{PGL}(2)$ to the group $H_1 = \text{GL}(2)$ with sign character $\varpi_1$ on $Z_1(\mathbb{R}) = \mathbb{R}^\times$. Spectral transfer from $H_1(\mathbb{R})$ to $G(\mathbb{R})$ involves just those representations $\pi_1$ of $H_1(\mathbb{R})$ for which $Z_1(\mathbb{R})$ acts by $\varpi_1$. We will assume this property for a representation $\pi_1$ without further mention. The $L$-embedding $\text{incl} \circ (\xi_1)^{-1}$, now defined on a subgroup of $L H_1$, plays essentially the same role in spectral transfer as before, but of course with $H_1$ in place of $H$.

We will prove transfer for test functions on $G(\mathbb{R})$ that are smooth and either compactly supported or rapidly decreasing on $G(\mathbb{R})$. (Passage to functions with prescribed behavior under the action of twisted conjugation by the center is then routine.) In the case of smooth functions of compact support this provides a direct analogue of Waldspurger’s results in the nonarchimedean case [Wal08]. In particular, we use the same normalization of twisted orbital integrals [Wal08, §§1.5, 3.10]. We no longer need the technical assumption on the central behavior of $\theta$ from an earlier draft; see Lemma 8.1 and its preparation from Sections 6 and 7. Following the formalism of $z$-pairs [KS99, §2.2], we do prescribe behavior of test functions on $H_1(\mathbb{R})$ under translation by the central subgroup $Z_1(\mathbb{R}) = \text{Ker}(H_1(\mathbb{R}) \to H(\mathbb{R}))$. Namely, we require that a
test function \( f_1 \) on \( H_1(\mathbb{R}) \) satisfy
\[
f_1(z_1 h_1) = \varpi_1(z_1)^{-1} f_1(h_1)
\]
for all \( z_1 \in Z_1(\mathbb{R}) \), \( h_1 \in H_1(\mathbb{R}) \).

For our test functions we could go directly to \( C_c^\infty \)-spaces and then obtain, as a corollary of the geometric transfer, the dual transfer of stable admissible traces to twisted-invariant distributions. Instead we prefer to start with a more general space of functions of Harish-Chandra Schwartz type and then later pass to \( C_c^\infty \)-functions using a well-known result of Bouaziz [Bou94, Th. 6.2.1]. Thus from our main theorem we obtain first a dual spectral transfer of stable \textit{tempered} traces to \textit{tempered} twisted-invariant distributions. There has been recent progress by Mezo [Mez12] on identifying these distributions as weighted sums of tempered irreducible twisted traces. For standard endoscopy, this program has been completed [She10], with the weights identified as the predefined canonical spectral transfer factors of [She10]. Then, for standard endoscopy, we conclude from the existence of geometric transfer that a spectrally defined transfer identity for a pair \((f, f_1)\) of test functions of any type also yields a geometric transfer identity for the pair if and only if it is correct on the tempered spectrum, i.e., it has the spectral transfer factors as weights. For progress with twisted spectral factors and their relation to Mezo's constants, see [She11].

To define a \( \theta \)-Schwartz function \( f \) on \( G(\mathbb{R}) \) we consider, as usual, the manifold \( G(\mathbb{R})\theta \) within \( G(\mathbb{R}) \rtimes \text{Aut}_\mathbb{C}(G) \). On \( G(\mathbb{R})\theta \) there is an action of \( G(\mathbb{R}) \) by conjugation: \( x\theta, g = g^{-1}(x\theta)g = g^{-1} x \theta(g) \theta \). To a smooth complex-valued function \( f \) on \( G(\mathbb{R}) \) we attach the smooth function \( f_\theta \) on \( G(\mathbb{R})\theta \) given by \( f_\theta(x\theta) = f(x) \). We call \( f \) a \( \theta \)-Schwartz function on \( G(\mathbb{R}) \) if \( f_\theta \) is Schwartz on \( G(\mathbb{R})\theta \). This requires a straightforward generalization of Harish-Chandra's definition; see the appendix for details and references. Write \( \mathcal{C}(G(\mathbb{R}), \theta) \) for the space of all such functions. On \( H_1(\mathbb{R}) \) we consider the space \( \mathcal{C}(H_1(\mathbb{R}), \varpi_1) \) of functions that are \( \varpi_1 \)-Schwartz in the usual sense. As mentioned already, for the fully general case there is a twist also on \( H_1(\mathbb{R}) \) by an inner automorphism \( \theta_1 \). In that setting, \( H_1(\mathbb{R})\theta_1 \) may be replaced by an appropriate coset of \( H_1(\mathbb{R}) \) in \( H_1(\mathbb{C}) \) (see Section 12), and we again require that test functions transform by \( \varpi_1^{-1} \) under the translation action of \( Z_1(\mathbb{R}) \).

To specify a correspondence \((f, f_1)\) it will be sufficient to consider those twisted conjugacy classes of elements \( \delta \) in \( G(\mathbb{R}) \) that are strongly \( \theta \)-regular and have a (strongly \( G \)-regular) norm \( \gamma_1 \) in \( H_1(\mathbb{R}) \) in the sense of [KS99, §§3.3, 5.4]. Then the \( \theta \)-twisted centralizer \( \text{Cent}_\theta(\delta, G) \) of \( \delta \) is reductive and abelian, but is not necessarily connected (as complex group). Because \( \delta \) has a norm in \( H_1(\mathbb{R}) \), \( \varpi_1 \) is trivial on \( \text{Cent}_\theta(\delta, G)(\mathbb{R}) \); see [KS99], where we use Theorem 5.1.1D to strengthen Lemma 4.4.C. The ordinary centralizer \( \text{Cent}(\gamma_1, H_1) \) is a torus which we write as \( H_{\gamma_1} \). Recall we assume no twisting in \( H_1(\mathbb{R}) \) until Section 12.
There is a simple notion of compatibility for normalization of Haar measures on $\text{Cent}_\theta(\delta, G)(\mathbb{R})$ and $H_{\gamma_1}(\mathbb{R})$; see Section 11. We fix Haar measures $dg$ on $G(\mathbb{R})$ and $dh_1$ on $H_1(\mathbb{R})$. This choice can be avoided if we work instead with Schwartz measures $fdg$ and $f_1dh_1$. In any case, it plays no significant role provided we insist on compatible measures $dt_\delta$ and $dt_{\gamma_1}$ for $\text{Cent}_\theta(\delta, G)(\mathbb{R})$ and $H_{\gamma_1}(\mathbb{R})$ when $\gamma_1$ is a norm of strongly $\theta$-regular $\delta$. For $f \in \mathcal{C}(G(\mathbb{R}), \theta)$ and quotient measure $\frac{dg}{dt_\delta}$, we have the well-defined $(\theta, \varpi)$-twisted orbital integral

$$O^{\theta, \varpi}(\delta, f) = \int_{\text{Cent}_\theta(\delta, G)(\mathbb{R}) \setminus G(\mathbb{R})} f(g^{-1}\delta\theta(g))\varpi(g)\frac{dg}{dt_\delta}$$

(see the appendix). Finally, the familiar stable orbital integral $\text{SO}(\gamma_1, f_1)$ is defined for $f_1 \in \mathcal{C}(H_1(\mathbb{R}), \varpi_1)$ and the quotient measure $\frac{dh_1}{dt_{\gamma_1}}$. If strongly $\theta$-regular $\delta$ does not have a norm in $H_1(\mathbb{R})$, we may still define a $(\theta, \varpi)$-twisted orbital integral $O^{\theta, \varpi}(\delta, f)$, but it plays no role in the transfer to $H_1(\mathbb{R})$. There will be other endoscopic groups that do account for it [KS99, Chap. 6].

The last ingredient for our transfer identity is the transfer factor $\Delta(\gamma_1, \delta)$ from [KS99] (see also [KS12]). While its definition is complicated in general, it has the property that the relative factor

$$\Delta(\gamma_1, \delta)/\Delta(\gamma_1, \delta) = \Delta(\gamma_1, \delta; \gamma_1, \delta)$$

is canonical [KS99, Th. 4.6.A]. This means that the relative factor depends only on the data we have prescribed: the inner twist $(G, \theta, \psi)$, cocycle $a_{\varpi}$ defining the twisting character $\varpi$, endoscopic data $\epsilon$ with $z$-pair $(H_1, \xi_1)$ for $\epsilon$, and of course the pairs $(\gamma_1, \delta), (\gamma_1, \delta)$. When $\varpi$ is trivial, it is only the appropriate conjugacy classes of these pairs that matter: the stable (slightly twisted) conjugacy classes of $\gamma_1, \gamma_1$ in $H_1(\mathbb{R})$ and the ordinary $(G(\mathbb{R})$-) twisted conjugacy classes of $\delta, \delta$ in $G(\mathbb{R})$. In general, there is a further twist by $\varpi$ over twisted conjugacy classes in $G(\mathbb{R})$ in the sense of [KS99, Th. 5.1.D (2)].

The canonicity property motivates our approach to proving transfer and is critical to our arguments, reducing the difficulties in establishing the main transfer identity to simply stated problems at various walls in the endoscopic group. We are free to make convenient choices for the data determining the individual terms in transfer factors at each wall, and we thereby avoid the long consistency arguments for various local choices over on the ambient group in our original approach to the case of standard endoscopy for real groups [She82]. In particular, given the definitions of the transfer factors in [LS87] and the alternate characterization of stable orbital integrals we use here (see Section 4 and Theorem 12.1, where we may set $g_0$ to be the identity), the present paper offers a relatively short proof of the transfer for standard endoscopy. Indeed, we may go directly to Section 9 since the results of Sections 6–8 for ordinary conjugacy are known [She79a] and Section 5 is essentially just a statement of
the main jump formula from which the transfer follows quickly. The argument for this jump formula is a special case of the arguments in Sections 9–11. There we reduce easily to questions about the terms in transfer factors. Then, in loosely technical terms, our choice of $\alpha$-data from Section 3, which is different from but in the same spirit as that in [Kal11], makes the previously intractable term $\Delta_I$ easy to handle (Lemma 9.5). The term $\Delta_{II}$ is trivial to handle and so the burden is on $\Delta_{III}$. Our choice of $\chi$-data from Section 3 allows us to deal with this term in our main lemma (Lemma 9.3) by a sequence of cohomological calculations based on results in [LS87], [LS90] and [KS99], and we are done. In particular, we avoid the convoluted arguments needed in Section 13 of [She08a] for the proof of standard transfer sketched there.

In some cases there are particular normalizations for the absolute factor $\Delta(\gamma_1, \delta)$ which simplify its form, but these do not play a direct role in the arguments of the present paper. In fact, since the choice of normalization does not matter for existence of the transfer identity, in Section 5 we simply fix a pair $(\gamma_1, \delta)$ and specify $\Delta(\gamma_1, \delta)$ in a way that allows us to avoid carrying various constants in our calculations.

Finally, we note that Waldspurger has pointed out two corrections ([Wal09], personal communication) needed for the definition of twisted transfer factors in [KS99]. These have been addressed in [KS12]. The first does not affect our archimedean setting; see Remark 1 of Section 9. The second involves the choice of a sign in the Galois hypercohomology pairing of Appendix A of [KS99] used to define the term $\Delta_{III}$ in transfer factors. In the archimedean case we may simply invert the pairing without further change, as explained in Remark 2 of Section 9.

2. Statement of the main theorem

We fix a set $\varepsilon$ of endoscopic data, along with a $\varepsilon$-pair $(H_1, \xi_1)$ for $\varepsilon$, and study geometric transfer for $G(\mathbb{R})$ and $H_1(\mathbb{R})$ under the transfer factor $\Delta$. Until Section 12 we assume that the norm correspondence involves no twisting of the conjugacy classes in $H_1(\mathbb{R})$.

Suppose $f$ is a $\theta$-Schwartz function on $G(\mathbb{R})$, i.e., $f \in \mathcal{C}(G(\mathbb{R}), \theta)$. We have attached to $\varepsilon$ and $(H_1, \xi_1)$ the shift character $\varpi_1$ on the central subgroup $Z_1(\mathbb{R})$ of $H_1(\mathbb{R})$. Define the subset $\text{Trans}(f)$ of $\mathcal{C}(H_1(\mathbb{R}), \varpi_1)$ to consist of those $\varpi_1$-Schwartz functions $f_1$ on $H_1(\mathbb{R})$ whose strongly $G$-regular stable orbital integrals match, through the norm correspondence for $G(\mathbb{R})$ and $H_1(\mathbb{R})$ attached to $\theta$, $\Delta$-weighted combinations of $(\theta, \varpi)$-twisted orbital integrals of $f$:

$$\text{SO}(\gamma_1, f_1) = \sum_{\delta, \theta-\text{conj}} \Delta(\gamma_1, \delta)O^\theta,\varpi(\delta, f)$$
for all strongly $G$-regular $\gamma_1$ in $H_1(\mathbb{R})$. The summation is over $\theta$-conjugacy classes of strongly $\theta$-regular elements in $G(\mathbb{R})$; for fixed $\gamma_1$, the product $\Delta(\gamma_1,\delta)$ $O^{\theta,\varpi}(\delta,f)$ depends only on the $\theta$-conjugacy class of strongly $\theta$-regular $\delta$ and is nonvanishing on finitely many such classes (see Section 5).

This transfer identity for the pair $(f,f_1)$ says, in particular, that if strongly $G$-regular $\gamma_1$ is not a norm, then

$$SO(\gamma_1,f_1) = 0$$

since, by definition, we then have $\Delta(\gamma_1,\delta) = 0$ for all strongly $\theta$-regular $\delta$ in $G(\mathbb{R})$. Moreover, the stable orbital integrals of $f_1$ have relatively simple behavior around semiregular semisimple elements. One requirement of the identity is thus that the weights $\Delta$ provide a great deal of cancellation in the singularities of the individual $(\theta,\varpi)$-twisted orbital integrals of $f$.

Notice that $f_1 \in \text{Trans}(f)$ is determined uniquely modulo the annihilator in $\mathcal{C}(H_1(\mathbb{R}),\varpi_1)$ of the space of stable tempered characters on $H_1(\mathbb{R})$. The strongly $G$-regular elements are dense in the set of all regular semisimple elements in $H_1(\mathbb{R})$, and so functions $f_1$ and $f_2$ in $\text{Trans}(f)$ have the same stable orbital integrals on all regular semisimple elements. Then, by Harish-Chandra’s regularity theorem for characters (see [HC75, §11, Th. 1]) and a simple application of a stable Weyl integration formula, those integrals generate all stable tempered characters on $H_1(\mathbb{R})$. Hence, $f_1$ and $f_2$ agree on such characters, as asserted.

We may consider instead $f \in C_c^\infty(G(\mathbb{R}),\theta)$, by which we mean that $f_\theta$ lies in $C_c^\infty(G(\mathbb{R})\theta)$, and define the set $\text{Trans}_c(f)$ of functions $f_1 \in C_c^\infty(H_1(\mathbb{R}),\varpi_1)$ such that $f$ and $f_1$ have $\Delta$-matching orbital integrals in the same manner. Embedding $C_c^\infty(G(\mathbb{R}),\theta)$ in $\mathcal{C}(G(\mathbb{R}),\theta)$, we may adapt the argument above to see that $f_1 \in \text{Trans}_c(f)$ is determined uniquely modulo the annihilator in $C_c^\infty(H_1(\mathbb{R}),\varpi_1)$ of the space of all stable tempered characters on $H_1(\mathbb{R})$.

**Theorem 2.1 (Main Theorem).** For all $f \in \mathcal{C}(G(\mathbb{R}),\theta)$, the subset $\text{Trans}(f)$ of $\mathcal{C}(H_1(\mathbb{R}),\varpi_1)$ is nonempty.

We conclude from this theorem that the correspondence $(f,f_1)$, where $f \in \mathcal{C}(G(\mathbb{R}),\theta)$ and $f_1 \in \text{Trans}(f)$, is well defined. This correspondence determines a map from $\mathcal{C}(G(\mathbb{R}),\theta)$ to the quotient of $\mathcal{C}(H_1(\mathbb{R}),\varpi_1)$ by the annihilator of stable tempered characters on $H_1(\mathbb{R})$. If we switch from Schwartz functions to Schwartz measures, then the map is determined uniquely up to normalization of transfer factors. In standard endoscopy, where the dual tempered spectral transfer is available (see [She10] and [She08b] for the form needed), we may normalize the tempered spectral factors $\Delta(\pi_1,\pi)$ first if we wish. For example, for certain inner forms there is a common Whittaker normalization that has desirable properties [She08b, §§11, 13]. Then for simultaneous geometric and
spectral transfer identities the geometric factors must be normalized so that
\( \Delta(\pi_1, \pi) / \Delta(\gamma_1, \delta) \) coincides with a predefined, and canonical, compatibility
factor \( \Delta(\pi_1, \pi; \gamma_1, \delta) \) [She10, §12]. In the Whittaker case, this brings us back
to the geometric version of the Whittaker normalization in [KS99, §5.3] for
\( \Delta(\gamma_1, \delta) \) [She10, §12]. Similar results are expected for the twisted case; see
[She11].

There is an analogue for \( C_c^\infty \)-functions.

**Corollary 2.2.** For all \( f \in C_c^\infty(G(\mathbb{R}), \theta) \), the subset \( \text{Trans}_c(f) \) is non-empty.

**Proof.** Let \( f \in C_c^\infty(G(\mathbb{R}), \theta) \). Using the main theorem we first find \( f'_1 \)
in the subset \( \text{Trans}(f) \) of \( \mathcal{C}(H_1(\mathbb{R}), \varpi_1) \). Then because the stable orbital in-
tegrals of \( f'_1 \) vanish off the conjugacy classes meeting a set in \( H_1(\mathbb{R}) \) that is
bounded modulo \( Z_1(\mathbb{R}) \), Bouaziz’s characterization of stable orbital integrals
of \( C_c^\infty \)-functions shows that there exists \( f_1 \in C_c^\infty(H_1(\mathbb{R}), \varpi_1) \) such that
\[
\text{SO}(\gamma_1, f_1) = \text{SO}(\gamma_1, f'_1)
\]
for all strongly \( G \)-regular \( \gamma_1 \) in \( H_1(\mathbb{R}) \). Here, a slight extension of [Bou94,
Th. 6.2.1] is needed; see [Ren03, §5.3]. Then \( f_1 \in \text{Trans}_c(f) \). \( \square \)

Let \( K, K_1 \) be maximal compact subgroups of \( G(\mathbb{R}), H_1(\mathbb{R}) \) respectively.
If \( f \in C_c^\infty(G(\mathbb{R}), \theta) \) is \( K \)-finite, then spectral methods are expected to show
that there is \( K_1 \)-finite \( f_1 \) in \( \text{Trans}_c(f) \), as for standard endoscopy. In the
standard setting, if \( \Delta(\pi_1, \pi) \) is the spectral transfer factor compatible with
given geometric factor \( \Delta(\gamma_1, \delta) \), then the Paley-Wiener argument of Clozel in
an appendix to [CD84] shows that there is \( K_1 \)-finite \( f_1 \) satisfying tempered
spectral transfer for \( f \) with weights \( \Delta(\pi_1, \pi) \). Thus \( f_1 \in \text{Trans}_c(f) \).

Sections 3–11 are dedicated to a proof of the main theorem which, after
some preparation, hinges almost entirely on Theorem 5.1. In Sections 3 and 4,
we introduce a variant of Harish-Chandra’s \( 'F_f \) transform that fits better with
transfer factors. In particular, we obtain the limit formulas of Theorem 4.2 for
ordinary stable orbital integrals. These are simpler; for example, the trouble-
some fourth root of unity that appears in the jump formulas for stable \( 'F_f \) (see
[She08a, §3]) is gone. In Sections 5–10, our main goal is to prove Theorem 5.1,
which amounts to analogous limit formulas for the right side of the transfer
identity, i.e., for sums of twisted orbital integrals weighted by the transfer fac-
tors. At this stage we ignore the limit formulas for derivatives that will be
required later in the paper and focus instead on the needed analysis of terms
in the transfer factors.

The main lemma (Lemma 9.3) in the proof of Theorem 5.1 is a simple wall-
crossing property of the term \( \Delta_{III} \) in the transfer factor \( \Delta = \Delta_1 \Delta_{II} \Delta_{III} \Delta_{IV} \)
that we deduce from a detailed examination of constructions from [LS87],
Two features are crucial to the cancellations that yield this result: use of the $s$-compatible data sets introduced in Section 3 and precise control of data attached to the abstract norm map; see toral descent data at $\gamma_0$ in Section 7. The term $\Delta_{II}$ then contributes trivially at the wall, apart from the piece needed for descent to a neighborhood of the identity in a twisted centralizer of Dynkin type $A_1$, while analysis of $\Delta_I$ may be avoided if we use known results for standard endoscopy. Since we plan to deduce that case as well, we also give an independent analysis of $\Delta_I$ as an exercise with descent formulas from [LS90]. The term $\Delta_{IV}$ is, as usual, absorbed into the definition of normalized integrals.

Once we have finished the proof of Theorem 5.1, we extend the limit formulas to derivatives. Again, use of the alternative transform simplifies both statements and arguments. We then complete our proof of the main theorem in Section 11. In Section 12, the theorem is extended to the general case, i.e., to the case of slightly twisted norms.

Our notation will follow this pattern: $O$ for unnormalized integrals, $\Phi$ for normalized integrals, and $\Psi_{a,\chi}$ for our variant of the stabilized $'F_f$ transform.

We should mention the work of Renard [Ren97], [Ren03], which offers insight into the difficulties of local analysis for twisted transfer. In [Ren03], however, the focus is different from ours; certain choices are made there that we expressly exclude here by the symmetry ($s$-compatibility) requirements of the next section. Those choices are reminiscent of our initial approach to standard endoscopy [She82], and unfortunately the reference [Sh6] in [Ren03] consists only of some personal notes which make no attempt to address the remaining problems for making the method work. In the example of base change, we note that the consistency problems in [She81a] were resolved only by the new approach of [She84]. With the dual spectral transfer in mind (see [She11, §11]), we also need the slightly more general setting of [KS99], and we start with Schwartz functions to capture the dual tempered transfer first. Some of our early results from Section 6 have analogues in [Ren03], but our paths soon separate since we bundle transfer factors with the twisted integrals from the start and then focus on the space of (abstract) norms and the endoscopic group. This leads us to local problems for transfer factors directly related to descent arguments from [LS87] and [LS90]. Those are the problems we propose to describe and solve here since, as we have already mentioned for the special case of standard endoscopy, the desired transfer then follows quite quickly.

3. Generalized Weyl denominators

A stabilized version of Harish-Chandra’s $'F_f$ transform was introduced in [She79a] to characterize stable orbital integrals. We prepare in the present section to introduce a variant of this transform based on the generalized Weyl
denominators from [She08a, §9] (see also [She10, §7c]) that depend on the $a$-data and $\chi$-data of [LS87, §2] rather than on a choice of positive roots.

Let $G$ be a connected reductive algebraic group defined over $\mathbb{R}$, and let $T$ be a maximal torus in $G$ defined over $\mathbb{R}$. The familiar skew-symmetric Weyl denominator on the Lie algebra $\mathfrak{t}_R$ of $T(\mathbb{R})$ does not in general lift to $T(\mathbb{R})$. Harish-Chandra introduced the closely related function $\Delta'$ on $T(\mathbb{R})$ defined by

$$\Delta'(\gamma) = \left| \det(\text{Ad}(\gamma) - I)_{g/m} \right|^{1/2} \prod_{\alpha > 0, \text{imag}} (\alpha(\gamma) - 1),$$

where $m$ is Lie algebra of the centralizer $M$ in $G$ of the split component of $T$. The product is over those imaginary roots, i.e., roots of $T$ in $M$, which are positive for some specified ordering. See Section 17 of [HC75]; this paper has the final version of $\ell'$Fj. An earlier definition, which differs by a sign that depends on the ordering, is recognized by the presence of a term $\varepsilon_R$. Note also that we have modified the definition to accommodate the use of the right action of conjugation in prescribing orbital integrals. Following Harish-Chandra [HC75], we partition roots of $T$ in $G$ as real ($\sigma\alpha = \alpha$), imaginary ($\sigma\alpha = -\alpha$), or complex ($\sigma\alpha \neq \pm\alpha$). Here, and throughout, $\sigma$ denotes the action of the nontrivial element of $\Gamma = \text{Gal}(\mathbb{C}/\mathbb{R})$ on $T$, on the rational characters $X^*(T)$, etc. Then

$$\Delta'(\gamma) = \prod_{\alpha > 0, \text{imag}} (\alpha(\gamma) - 1) \prod_{\alpha \text{ real, complex}} \left| \alpha(\gamma)^{1/2} - \alpha(\gamma)^{-1/2} \right|^{1/2},$$

where $|\alpha(\gamma)^{1/2} - \alpha(\gamma)^{-1/2}|$ is convenient notation for $|(\alpha(\gamma) - 1)(\alpha(\gamma)^{-1} - 1)|^{1/2}$. If $\gamma$ is regular as an element of $M$, we may further write

$$\Delta'(\gamma) = \prod_{\alpha > 0, \text{imag}} \frac{(\alpha(\gamma) - 1)}{|\alpha(\gamma) - 1|} \prod_{\alpha} \left| \alpha(\gamma)^{1/2} - \alpha(\gamma)^{-1/2} \right|^{1/2}.$$

Let $O_\alpha$ denote the Galois orbit of the root $\alpha$ of $T$ in $G$. If $\alpha$ is imaginary, then $O_\alpha$ is symmetric: $O_\alpha = -O_\alpha = \{\pm\alpha\}$. Otherwise $O_\alpha$ is asymmetric. Then $O_\alpha$ and $-O_\alpha$ are disjoint and $O_\alpha$ consists of one or two roots according as $\alpha$ is real or complex. Recall that we define $a$-data $\{a_\alpha\}$ and $\chi$-data $\{\chi_\alpha\}$ as follows [LS87, 2.2 and 2.5]. For each root $\alpha$, $a_\alpha$ is a nonzero complex number and

$$a_{\sigma\alpha} = \overline{a}_\alpha, a_{-\alpha} = -a_\alpha.$$

In particular, if $\alpha$ is real, then $a_\alpha$ is a real number, while if $\alpha$ imaginary, then $a_\alpha$ is purely imaginary. Turning to $\chi$-data, if $\alpha$ is imaginary or complex, then $\chi_\alpha$ is a character on $\mathbb{C}^\times$. Further, if $\alpha$ is imaginary, then $\chi_\alpha$ must be an extension to $\mathbb{C}^\times$ of the sign character on $\mathbb{R}^\times$. Finally,

$$\chi_{\sigma\alpha} = \chi_\alpha \circ \sigma, \chi_{-\alpha} = \chi_\alpha^{-1}.$$

If $\alpha$ is real, then $\chi_\alpha$ is a character on $\mathbb{R}^\times$ and $\chi_{-\alpha} = \chi_\alpha^{-1}$.

If $O_\alpha$ is asymmetric, then $\chi_\alpha$ may be the trivial character, in which case the choice of $a_\alpha$ will not matter for the objects we construct (for the sake of
completeness, we will often pick \(a_\alpha = \pm 1 = -a_{-\alpha}\), and we say that such data are *trivial*.

The associated (right) generalized Weyl denominator is

\[
\Delta_{a,\chi,\text{right}}(\gamma) = \prod_\mathcal{O} \chi_\alpha \left( \frac{(\alpha(\gamma) - 1)}{a_\alpha} \right) \prod_\alpha \left| \alpha(\gamma)^{1/2} - \alpha(\gamma)^{-1/2} \right|^{1/2}
\]

\[
= \left| \det(\text{Ad}(\gamma) - I)_{g/t} \right|^{1/2} \prod_\mathcal{O} \chi_\alpha \left( \frac{(\alpha(\gamma) - 1)}{a_\alpha} \right),
\]

where the product is over all Galois orbits \(\mathcal{O}\), symmetric or not. Notice that the choice of representative \(\alpha\) for \(\mathcal{O}\) does not matter.

We may also define \(\Delta_{a,\chi,\text{left}}(\gamma)\) by replacing each term \(\chi_\alpha \left( \frac{(\alpha(\gamma) - 1)}{a_\alpha} \right)\) with the term

\[
\chi_\alpha(-a_\alpha(1 - \alpha(\gamma)^{-1})).
\]

A useful property for computing the dual transfer of characters is that the product

\[
\Delta_{a,\chi,\text{left}}(\gamma) \Delta_{a,\chi,\text{right}}(\gamma)
\]

coincides with the term \(\left| \det(\text{Ad}(\gamma) - I)_{g/t} \right|^{1/2} \prod_\mathcal{O} \chi_\alpha \left( \frac{(\alpha(\gamma) - 1)}{a_\alpha} \right)\) appearing in the Weyl integration formula [She10, Lemma 7.3]. In the present paper we are interested only in \(\Delta_{a,\chi,\text{right}}(\gamma)\) and will write it simply as \(\Delta_{a,\chi}(\gamma)\).

To return to the Harish-Chandra factor \(\Delta'(\gamma)\), we choose a positive system for the imaginary roots and then set

\[
\chi_\alpha(z) = \left( \frac{z}{|z|} \right)^{1/2} = \frac{z}{|z|}
\]

for \(\alpha\) positive imaginary. We also set \(\chi_\alpha\) trivial for all real roots and all complex roots. Then for any choice \(\{a_\alpha\}\) of \(a\)-data, we have

\[
\Delta'(\gamma) = \Delta_{a,\chi}(\gamma) \prod_{\alpha > 0, \text{imag}} \frac{a_\alpha}{|a_\alpha|},
\]

Notice that the product on the right is a fourth root of unity.

Suppose (arbitrarily chosen) \(\chi\)-data \(\{\chi_\alpha\}\) are replaced by another such set \(\{\chi'_\alpha = \eta_\alpha \chi_\alpha\}\). Then

\[
\Delta_{a,\chi'}(\gamma) = \Delta_{a,\chi}(\gamma) \prod_{\mathcal{O}, \text{symm}} \eta_\alpha \left( \frac{\alpha(\gamma) - 1}{a_\alpha} \right) \prod_{\pm \mathcal{O}, \text{asymm}} \eta_\alpha(\alpha(\gamma)).
\]

Suppose \(\alpha\) is imaginary, and choose a square root \(\alpha(\gamma)^{1/2}\) for \(\alpha(\gamma)\). Then \(\eta_\alpha(\alpha(\gamma)^{1/2})\) is independent of this choice, and the last formula may be rewritten as

\[
\Delta_{a,\chi'}(\gamma) = \Delta_{a,\chi}(\gamma) \prod_{\mathcal{O}, \text{symm}} \eta_\alpha(\alpha(\gamma)^{1/2}) \prod_{\pm \mathcal{O}, \text{asymm}} \eta_\alpha(\alpha(\gamma)),
\]

showing that the change is independent of the choice of \(a\)-data. Replacing \(\{a_\alpha\}\) by another set \(\{a'_\alpha = a_\alpha b_\alpha\}\) yields

\[
\Delta'_{a',\chi}(\gamma) = \Delta_{a,\chi}(\gamma) \prod_{\mathcal{O}, \text{symm}} \text{sign}(b_\alpha),
\]

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and then that change is independent of the choice of $\chi$-data.

Let $\alpha$ be an imaginary root of $T$. By a Cayley transform with respect to $\alpha$ we mean the restriction to $T$ of an inner automorphism of $G$, written $\gamma \to \gamma^s = s^{-1}\gamma s$ or $T \to T^s$, for which $s\sigma(s)^{-1}$ acts on $T$ as the Weyl reflection $\omega_\alpha$ with respect to $\alpha$. Then $T^s$ is defined over $\mathbb{R}$. This is a generalization of the usual Cayley transform (see [She79a], [She79b, §3], also a review in [She83, §2]) that works well for stable conjugacy. Such a transform exists if and only if the orbit of $\alpha$ under the imaginary Weyl group, i.e., the Weyl group of $T$ in $M$, contains a noncompact root (see [She79a, Prop. 4.11]). In the terminology of [She83, §2] this says that $\alpha$ is not totally compact. For each root $\beta$ of $T$, we denote by $\beta^s$ its transport by $s$ to a root of $T^s$.

Suppose that $\{a_\beta\}, \{\chi_\beta\}$ are $a$-data and $\chi$-data for $T$, and fix an imaginary root $\alpha$. Assume that $\alpha$ is not totally compact so that we may choose a Cayley transform $s$ with respect to $\alpha$. Then we call $\{a_\beta\}, \{\chi_\beta\}$ together with $a$-data and $\chi$-data $\{a_\beta^s\}, \{\chi_\beta^s\}$ for $T^s$ an $s$-compatible data set if

$$a_{\omega_\alpha(\beta)} = a_\beta, \chi_{\omega_\alpha(\beta)} = \chi_\beta$$

for all $\beta \neq \pm \alpha$, and

$$a_{\beta^s} = a_\beta, \chi_{\beta^s} = \chi_\beta$$

for all roots $\beta \neq \pm \alpha$ of $T$ except those complex $\beta$ for which $\beta^s$ is real, while for such $\beta$, we require

$$a_\beta = a_{\beta^s}, \chi_\beta = \chi_{\beta^s} \circ \text{Nm}_{\mathbb{R}}^C.$$ 

This definition places no additional restrictions on the data $a_\alpha, a_{\alpha^s}, \chi_\alpha$ or $\chi_{\alpha^s}$ corresponding to the Cayley roots $\alpha, \alpha^s$. On the other hand, we are not free to make the usual assumption that the data are trivial on all asymmetric orbits for $T^s$: the data must be nontrivial on those asymmetric (complex) orbits for $T^s$ which bifurcate into symmetric orbits on passage back to $T$; see the last step in the proof of Lemma 3.1. In the case of bifurcation of an asymmetric (complex) orbit for $T$ into asymmetric (real) orbits for $T^s$, mentioned in the definition, we may choose trivial data, but if we do not, then only real (Galois-invariant) $a_\beta, \chi_\beta$ are allowed. The requirements in this last case are made with the proofs of Lemmas 4.1 and 9.1 in mind.

**Lemma 3.1.** Suppose that $s$ is a Cayley transform. Then $s$-compatible data sets exist.

**Proof.** Write $\sigma, \sigma^s$ for the Galois actions on $T, T^s$ respectively. By construction, 

$$\sigma^s(\beta^s) = (\omega_\alpha \sigma \beta)^s$$

for all roots $\beta$ of $T$. Thus, as in the case of the standard Cayley transform, the roots $\pm \alpha^s$ are real. If $\beta$ is real, then so is $\beta^s$. If $\beta$ is complex, then either
ω_αβ ≠ ±σβ and β^s is complex or ω_αβ = σβ and β^s is real. Here the case ω_αβ = −σβ (equivalently, β^s imaginary) has been excluded since that implies β^s is orthogonal to α^s, so that β must be imaginary and orthogonal to α.

First we pick a-data and χ-data for T. Clearly we may adjust the data to satisfy the conditions that a_ωα(β) = a_β and χ_ωα(β) = χ_β for all imaginary β ≠ ±α. Suppose that β is real. Then we may take χ_β trivial and arrange that a_β = ±1 = −a_−β. Suppose that β is complex. Then we again take χ_β to be trivial and arrange that a_β = ±1 = −a_−β. We may also require that a_ωα(β) = a_β = a_σβ. For this we observe that the orbit of β under the group generated by σ and ω_α is asymmetric and, moreover, disjoint from its negative; if β^s is real, then ω_αβ = σβ and the orbit is {β, σβ}, whereas if β^s is complex, then ω_αβ ≠ ±σβ and the orbit is {β, σβ, ω_αβ, ω_ασβ}. The disjointness property is then clear.

To complete the proof of the lemma we show that we may define a-data and χ-data for T^s as follows. First use the formulas

\[a_{β^s} = a_β, \quad χ_{β^s} = χ_β\]

for all roots β of T except ±α and those complex β for which β^s is real. Suppose β is complex and β^s is real. We pick a_{β^s} = a_β and take χ_{β^s} trivial on \(R^\times\). We choose χ_{±α^s} trivial on \(R^\times\) and a_α^s = 1 = −a_−α^s.

There is nothing left to show for a_{β^s}, χ_{β^s} unless β is imaginary and β ≠ ±α. Then β^s is imaginary or complex according as β is orthogonal to α or not. If β is orthogonal to α, then \(σ^s(β^s) = (σβ)^s\) and so it is clear that our chosen \(a_{±β^s} = a_β, χ_{±β^s} = χ_β\) are appropriate. If β is not orthogonal to α, then

\[σ^s(β^s) = (−ω_αβ)^s.\]

Using the additional requirement

\[a_ωα(β) = a_β, χ_ωα(β) = χ_β,\]

we see that

\[a_{σ^s(β^s)} = a_{−ω_αβ} = a_{ω_αβ} = a_β = a_β^s\]

and

\[χ_{σ^s(β^s)} = χ_{−ω_αβ} = χ_{ω_αβ}σ = χ_βσ = χ_β^sσ.\]

Since clearly \(a_{−β^s} = −a_β^s\) and \(χ_{−β^s} = χ_β^1\), this finishes the proof. □

4. A limit formula for stable orbital integrals

We continue with the setting of the last section. Suppose that SO is an unnormalized stable orbital integral on the regular semisimple set of \(G(\mathbb{R})\), i.e., that there is a Schwartz function \(f\) on \(G(\mathbb{R})\) such that, for each regular semisimple \(γ\) in \(G(\mathbb{R})\), \(SO(γ, f)\) is the stable orbital integral \(SO(γ, f)\). Suppose
also that \( \gamma \) lies in \( T(\mathbb{R}) \). Then we use the factors \( \Delta' \) and \( \Delta_{a,\chi} \) from the last section to define the transforms

\[
\Psi(\gamma) = \Delta'(\gamma)\text{SO}(\gamma)
\]

for a given choice of positive imaginary roots for \( T \) and

\[
\Psi_{a,\chi}(\gamma) = \Delta_{a,\chi}(\gamma)\text{SO}(\gamma)
\]

for a given choice of \( a \)-data and \( \chi \)-data for \( T \). The choice of measures has been suppressed in notation; we follow [She79a] (see also Section 11). Our purpose in the present section is to deduce simple limit formulas for \( \Psi_{a,\chi} \) from the limit formulas for \( \Psi \); see [She79a] for a detailed proof of the latter.

We confine our attention to the behavior of orbital integrals near semiregular semisimple elements of \( G(\mathbb{R}) \), those elements with centralizer of type \( A_1 \). Suppose then that \( \gamma_0 \) is a semiregular element of \( T(\mathbb{R}) \), that \( \alpha(\gamma_0) = 1 \), where \( \alpha \) is an imaginary root which is not totally compact, and that \( s \) is a Cayley transform with respect to \( \alpha \). We may regard the coroot \( \alpha^\vee \) as an element of the Lie algebra of \( T \), and then \( a_\alpha \alpha^\vee \) lies in the real Lie algebra:

\[
\sigma(a_\alpha \alpha^\vee) = a_\alpha (-\alpha^\vee) = a_\alpha \alpha^\vee.
\]

For a sufficiently small nonzero real number \( \nu \), the element \( \gamma_0 = \exp(\nu a_\alpha \alpha^\vee) \) is a regular element in \( T(\mathbb{R}) \). Moreover, it is unchanged if \( \alpha \) is replaced by \(-\alpha \). At the same time, the element \( \gamma_s \) lies in \( T_s(\mathbb{R}) \) and is annihilated only by the real roots \( \pm \alpha^s \). Then \( \Psi_{a,s,\chi}(\gamma_s) \) is prescribed by smooth extension [HC75, §17, Th. 1]. In particular, if we set \( \gamma_{s,\nu} = \gamma_0 \exp(\nu a_\alpha (\alpha^s)^\vee) \), then

\[
\Psi_{a,s,\chi}(\gamma_0) = \lim_{\nu \to 0} \Psi_{a,s,\chi}(\gamma_{s,\nu}).
\]

We note first a lemma that simplifies our argument for the next theorem (and motivates the definition of \( s \)-compatibility).

**Lemma 4.1.** For any \( s \)-compatible data set \( \{a_\beta\}, \{\chi_\beta\}, \{a_{\beta^s}\}, \{\chi_{\beta^s}\} \), we have

\[
\prod_{O \neq O_\alpha} \chi_\beta \left( \frac{(\beta(\gamma_0) - 1)}{a_\beta} \right) = \prod_{O \neq \pm O_{\alpha^s}} \chi_{\beta^s} \left( \frac{(\beta^s(\gamma_0^s) - 1)}{a_{\beta^s}} \right).
\]

On the left, the product is over all Galois orbits \( O \) for \( T \) except \( O_\alpha = \{\pm \alpha\} \). Each term is independent of the choice of representative \( \beta \) for \( O \). The right side is defined by using all Galois orbits for \( T^s \) except \( \{\alpha^s\} \) and \( \{-\alpha^s\} \), and again the choice of representative has no effect on the terms.

**Proof.** If \( O \) is orthogonal to \( O_\alpha \), then we find immediately a matching term for \( O \) on the right side of the equation. For the remaining cases, if \( \beta \) is imaginary and \( \beta' = \omega_\alpha \beta \) is distinct from \( \beta \), then the contributions to the left from \( \{\pm \beta\} \) and \( \{\pm \beta'\} \) are equal and moreover they each equal the contribution to the right from each of the two orbits \( \{\beta^s, -(\beta')^s\} \) and \( \{-\beta^s, (\beta')^s\} \). If \( \beta \)
is complex and $\beta^s$ is complex, then we clearly have matching terms. If $\beta$
is complex and $\beta^s$ is real, then $(\sigma \beta)^s = \omega_{\alpha^s} \beta^s$. The product of the terms
for $\{\beta, \sigma \beta\}$ and $\{-\beta, -\sigma \beta\}$ is $\chi_\beta(\beta(\gamma_0))$. The product of the terms for $\{\beta^s\}$, $\{-\beta^s\}$ is $\chi_\beta^s(\beta^s(\gamma_0^s))$, which equals the product for $\{\omega_{\alpha^s} \beta^s\}$, $\{-\omega_{\alpha^s} \beta^s\}$. Since $\beta(\gamma_0) = \beta^s(\gamma_0^s)$ is real, $s$-compatibility ensures that

$$\chi_\beta(\beta(\gamma_0)) = \chi_\beta^s(\beta^s(\gamma_0^s)^2) = \chi_\beta^s(\beta^s(\gamma_0^s)) \cdot \chi_{\omega_{\alpha^s} \beta^s}(\omega_{\alpha^s} \beta^s(\gamma_0^s)),$$

and the lemma is proved. \qed

**Theorem 4.2.** For any $s$-compatible data set, we have

$$\lim_{\nu \to 0^-} \Psi_{a, \chi}(\gamma_\nu) = - \lim_{\nu \to 0^+} \Psi_{a, \chi}(\gamma_\nu)$$

and

$$\lim_{\nu \to 0^+} \Psi_{a, \chi}(\gamma_\nu) = \Psi_{a^s, \chi^s}(\gamma_0^s).$$

**Proof.** As a first step, we check that it is sufficient to verify these limits for one $s$-compatible data set. Suppose then that the result is true for the choice $\{a_\beta\}, \{\chi_\beta\}$ and $\{a_{\beta^s}\}, \{\chi_{\beta^s}\}$. We now use another set, which we write as

$\{a_\beta b_\beta\}, \{\chi_\beta \eta_\beta\}$ and $\{a_{\beta^s} b_{\beta^s}\}, \{\chi_{\beta^s} \eta_{\beta^s}\}$, and we consider the effect on $\Psi_{a, \chi}(\gamma_\nu)$ and $\Psi_{a^s, \chi^s}(\gamma_0^s)$. We may argue orbit by orbit.

Notice that only the data for $O_\alpha = \{\pm \alpha\}$ affect $\gamma_\nu$. The characters $\eta_{\pm \alpha} = \eta_{\pm \alpha}^{\pm 1}$ are trivial on $\mathbb{R}^\times$, while $b_\alpha = b_{-\alpha}$ may be any nonzero real number. Then $\gamma_\nu$ is replaced by $\gamma_{b_{a, \nu}}$ and $\Delta_{a, \chi}(\gamma_\nu)$ is multiplied by

$$\chi_\alpha(b_\alpha)^{-1} \eta_\alpha \left( \frac{\alpha(\gamma_\nu) - 1}{a_\alpha} \right) = \text{sign}(b_\alpha) \eta_\alpha (e^{i\nu a_\alpha})$$
since $\alpha(\gamma_\nu) = e^{i\nu a_\alpha}$ and $(e^{i\nu a_\alpha} - e^{-i\nu a_\alpha})/a_\alpha$ is real. Thus the first limit statement remains true (each side is replaced with the negative of the other if $b_\alpha$ is negative), and then the second limit statement follows also. Next we observe that $\eta_{\pm \alpha^s} = \eta_{\pm \alpha}^{\pm 1}$ and $b_{\alpha^s} = b_{-\alpha^s}$ contribute no change to $\Psi_{a^s, \chi^s}(\gamma_0^s)$ since

$$\eta_{\alpha^s} \left( \frac{\alpha^s(\gamma) \cdot 1}{a_{\alpha^s}} \right) \eta_{-\alpha^s} \left( \frac{\alpha^s(\gamma \cdot 1)}{a_{-\alpha^s}} \right) \chi_{\alpha^s}(b_{\alpha^s})^{-1} \chi_{-\alpha^s}(b_{-\alpha^s})^{-1} = \eta_{\alpha^s}(\alpha^s(\gamma))$$

for any regular $\gamma$ in $T^\alpha(\mathbb{R})$, and so has limit 1 as $\gamma$ approaches $\gamma_0^s$. Thus we are done with the orbits $O_\alpha, \pm O_{\alpha^s}$.

For the remaining orbits, we could do a calculation for each symmetric orbit $O$ and each asymmetric pair $\pm O$ individually. Instead we appeal to Lemma 4.1 to see that the (nonzero) total contribution can be cancelled from the limit formulas. This finishes the first step.

The second step in our proof is to compare the proposed limit formulas with the limit formulas for the stable version $\Psi = \Delta'.SO$ of Harish-Chandra’s $'Fj$ transform ([She79a], recalled in Section 3 of [She08a]). It is convenient to assume first that $\alpha$ itself is noncompact and then drop this assumption later.
We pick a system of positive imaginary roots for $T$ that is adapted to $\alpha$. This means that $\alpha$ is positive and that if $\beta$ is positive imaginary and not orthogonal to $\alpha$, then $\beta_1 = -\omega_{\alpha}(\beta)$ is also positive. For convenience, we will choose $\chi_\beta$ to be the standard character $z \to \frac{z}{|z|}$ if $\beta$ is positive imaginary and orthogonal to $\alpha$. This is also assumed for $\beta = \alpha$. In each of these cases we set $a_\beta = i$. For each pair of positive roots $\beta, \beta_1 = -\omega_{\alpha}(\beta)$ not orthogonal to $\alpha$ and distinct from $\alpha$, we pick one, labeling it $\beta$, and make $\chi_{\beta_1}$ the standard character. Then $\chi_{\beta_1}$ must be its inverse. Also we set $a_\beta = i$, so that $a_{\beta_1}$ must be $-i$. We assume that $\chi_\beta$ is the identity character if $\beta$ is real or complex.

Now we compare $\Delta_{\alpha, \chi}(\gamma)$ with $\Delta'(\gamma)$ at $\gamma_\nu = \gamma_0 \exp(i\nu\alpha^\vee)$, as well as $\Delta_{\alpha, \chi}(\gamma_0^s)$ with $\Delta'(\gamma_0^s)$. We proceed orbit by orbit, considering the contribution of $O$ to the change for $\Delta_{\alpha, \chi}$ and of $O^s$ to the change for $\Delta_{\alpha, \chi}s$. Real or complex orbits for $T$ contribute no change to either $\Delta'(\gamma_\nu)$ or $\Delta'(\gamma_0^s)$. Consider the imaginary orbits orthogonal to $\alpha$. Suppose there are $N$ such orbits. Then passage to $\Delta_{\alpha, \chi}(\gamma_\nu)$ multiplies $\Delta'(\gamma_\nu)$ by $(i)^{-N}$. Since $N$ is the number of imaginary orbits for $T^s$ and we use $s$-compatible data for $T^s$, the term $\Delta'(\gamma_0^s)$ is also multiplied by $(i)^{-N}$. Consider now the orbits of a pair of positive imaginary roots $\beta, \beta_1$ not orthogonal to $\alpha$ and distinct from $\alpha$. Then we replace

$$A(\gamma_\nu) = \frac{\beta(\gamma_\nu) - 1}{|\beta(\gamma_\nu) - 1|}\frac{\beta_1(\gamma_\nu) - 1}{|\beta_1(\gamma_\nu) - 1|}$$

by

$$B(\gamma_\nu) = \frac{\beta(\gamma_\nu) - 1}{|\beta(\gamma_\nu) - 1|}i\frac{|\beta_1(\gamma_\nu) - 1|}{-(\beta_1(\gamma_\nu) - 1)/i} = \frac{\beta(\gamma_\nu) - 1}{1 - \beta_1(\gamma_\nu)}\frac{|\beta_1(\gamma_\nu) - 1|}{|\beta(\gamma_\nu) - 1|}.$$ 

Because $\beta_1(\gamma_0) = \beta(\gamma_0)^{-1} = \overline{\beta(\gamma_0)}$, we have

$$\lim_{\nu \to 0^+} A(\gamma_\nu) = \lim_{\nu \to 0^-} A(\gamma_\nu) = 1,$$

whereas

$$\lim_{\nu \to 0^+} B(\gamma_\nu) = \lim_{\nu \to 0^-} B(\gamma_\nu) = \beta(\gamma_0).$$

Thus we have to multiply all limits by $\beta(\gamma_0)$. Consider now the change to $\Delta'(\gamma_0^s)$. This term is multiplied by

$$\chi_\beta^s \left( \frac{\beta_1(\gamma_0^s) - 1}{a_\beta^s} \right) \cdot \chi_{-\beta}^s \left( \frac{\beta_1(\gamma_0^s) - 1}{a_\beta^s} \right) = \chi_\beta^s \left( \frac{\beta_1(\gamma_0^s) - 1}{1 - \beta_1(\gamma_0^s)} \right) = \chi_\beta \left( \frac{\beta(\gamma_0) - 1}{1 - \beta_1(\gamma_0)} \right) = B(\gamma_0) = \beta(\gamma_0),$$

and so we are done with this case.

There is one remaining orbit, that of $\alpha$. Its contribution multiplies $\Delta'(\gamma_\nu)$ by $i^{-1}$, but there is no change to $\Delta'(\gamma_0^s)$. This is exactly what we need to deduce the claimed limits from the analogous limits for the Harish-Chandra
Suppose that \( \alpha \) is compact and that \( \omega \) is an element of the imaginary Weyl group for which \( \alpha^\dagger = \omega^{-1}\alpha \) is noncompact. Assume that \( \omega \) acts on \( T \) as \( \text{Int}(w) \). Then if \( s \) is a Cayley transform relative to \( \alpha \), \( s^\dagger = w^{-1}s \) is a Cayley transform relative to \( \alpha^\dagger \). Also if \( \gamma_0 \) is a semiregular element of \( T(\mathbb{R}) \) such that \( \alpha(\gamma_0) = 1 \), then \( \gamma_0^w \) is a semiregular element of \( T(\mathbb{R}) \), \( \alpha^\dagger(\gamma_0^w) = 1 \), and \( (\gamma_0^w)^s = \gamma_0^s \). Finally, to obtain an \( s^\dagger \)-compatible data set from an \( s \)-compatible data set \( \{a_\beta\}, \{\chi_\beta\} \) and \( \{a_\beta^\ast\}, \{\chi_\beta^\dagger\} \), we may replace \( \{a_\beta\}, \{\chi_\beta\} \) by \( \{a_\beta', \chi_\beta'\} \), where \( a_\beta' = a_\omega \beta \) and \( \chi_\beta' = \chi_\omega \beta \), and leave \( \{a_\beta^\ast\}, \{\chi_\beta^\dagger\} \) unchanged. Then

\[
\gamma_{\nu}^w = \gamma_0^w \exp \nu a_\alpha^\dagger(a^\dagger)^\vee,
\]

and because \( \text{SO} \) is stable, we have

\[
\Psi_{a,\chi}(\gamma_{\nu}) = \Psi_{a',\chi'}(\gamma_{\nu}^w).
\]

The limit formulas at \( \gamma_0 \) now follow immediately from those at \( \gamma_0^w \), and this completes the proof of Theorem 4.2.

Notice that Lemma 4.1 allows us to use \( \Delta_\alpha \) in place of \( \Delta_{a,\chi} \) in the statement of Theorem 4.2, where

\[
\Delta_\alpha(\gamma) = \chi_\alpha \left( \frac{(\alpha(\gamma) - 1)}{a_\alpha} \right) \left| \det(\text{Ad}(\gamma) - I)_{g/t} \right|^{1/2}.
\]

Here \( \Delta_{-\alpha} = \Delta_{-\alpha} = \Delta_\alpha \), and so only the (symmetric) orbit \( O \) of \( \alpha \) matters. We then write \( \Delta_O \) in place of \( \Delta_\alpha \).

We end this section with a remark on the normalized orbital integral

\[
\Phi(\gamma) = \left| \det(\text{Ad}(\gamma) - I)_{g/t} \right|^{1/2} \text{SO}(\gamma).
\]

Set

\[
\Psi_O(\gamma) = \Delta_O(\gamma) \text{SO}(\gamma) = \chi_\alpha \left( \frac{(\alpha(\gamma) - 1)}{a_\alpha} \right) \Phi(\gamma).
\]

Assume, as in the theorem, that \( \alpha \) is not totally compact. Notice that if we write \( a_\alpha \) as \( ib_\alpha \), where \( b_\alpha \) is real, then for \( |\nu| \) small and nonzero, we have

\[
\chi_\alpha \left( \frac{(\alpha(\gamma_\nu) - 1)}{a_\alpha} \right) = \chi_\alpha(e^{i\nu b_\alpha}) \chi_\alpha \left( \frac{e^{i\nu b_\alpha} - e^{-i\nu b_\alpha}}{ib_\alpha} \right) = \chi_\alpha(e^{i\nu b_\alpha}) \chi_\alpha \left( \frac{2\sin(\nu b_\alpha)}{\nu b_\alpha} \nu \right) = \chi_\alpha(e^{i\nu a_\alpha}) \text{sign}(\nu).
\]

Because \( s \) defines an inner twist between the identity components of their respective centralizers, the elements \( \gamma_0 \) and \( \gamma_0^s \) are stably conjugate in \( G(\mathbb{R}) \) in the sense introduced by Kottwitz in Section 3 of [Kot82] for the untwisted setting. Comparing limits for \( \Psi_O \) with limits for \( \Phi \), we see, by an argument along the lines of Section 2 that the assertions of Theorem 4.2 may be rephrased as the existence and equality of the limits of \( \Phi(\gamma) \) as (i) \( \gamma \) approaches \( \gamma_0 \),
through the regular elements of $T(R)$ and (ii) $\gamma$ approaches the stable conjugate $\gamma_0^s$ of $\gamma_0$ through the regular elements of $T^s(R)$ (see Section 2 of [She83]). This suggests another approach to the proof of transfer; we simply found our present approach quicker. Our preference for working with $\Psi_\Theta$ rather than $\Phi$ is explained by the formulas of Section 10 for derivatives.

It is now a short exercise to modify the characterization theorem for stable orbital integrals in [She79a] using the statement of Theorem 4.2 or, more precisely, its generalization to derivatives. As mentioned in Section 1, we will need eventually to introduce a slight twist in the stable integrals. Thus we will wait until Section 12, and then write a slightly more general characterization theorem (Theorem 12.1).

5. A limit formula for twisted orbital integrals

We return to the statement of the main theorem in Section 2 and follow the notation established in that setting. In particular, we will consider $(\theta, \varpi)$-twisted integrals for $G$, while the endoscopic group $H_1$ will now assume the role of the group of the last two sections. Recall that, because of our assumption on the inner twist $(G, \theta, \psi)$, we consider completely untwisted integrals on $H_1(R)$. To commence the proof of the main theorem, we assume that $f \in C(G(R), \theta)$ and define a function $\Phi_1$ on the strongly $G$-regular elements $\gamma_1$ of $H_1(R)$ by

$$\Phi_1(\gamma_1) = \left|\det(\text{Ad}(\gamma_1) - I)_{h_1/\gamma_1}\right|^{1/2} \sum_{\delta, \text{conj}} \Delta(\gamma_1, \delta)O^{\theta, \varpi}(\delta, f).$$

We must show $\Phi_1$ is a normalized stable orbital integral on $H_1(R)$. Our primary concern will be an analogue of the limit formulas of the last section.

Consider $\Phi_1$ near a semiregular element $\gamma_0$ in $H_1(R)$ annihilated by an imaginary root $\alpha_1$ of a maximal torus $T_1$ in $H_1$. Because $H_1$ is quasi-split over $R$, the root $\alpha_1$ is not totally compact [She79b, Lemma 9.2]. We then have a Cayley transform $s_1$ in the sense of Section 3 for $\alpha_1$, along with the semiregular element $\gamma_0^{s_1}$ in the adjacent Cartan subgroup $T_1^{s_1}(R)$ annihilated by the real root $\alpha_1^{s_1}$. We will choose an $s_1$-compatible data set in Section 9 based on compatible twisted data. We make the additional requirement that $\gamma_0$ be $G$-semiregular; see Section 6 for definition. For all nonzero real $\nu$ with $|\nu|$ sufficiently small, we will see that both $\gamma_\nu = \gamma_0 \exp(\nu a_{\alpha_1} a^{\gamma_1}_1)$ in $T_1(R)$ and $\gamma_{s_1, \nu} = \gamma_0^{s_1} \exp(\nu a_{\alpha_1^{s_1}}(a^{s_1}_1)^{\varpi})$ in $T_1^{s_1}(R)$ are $G$-regular and then that $\Phi_1(\gamma_\nu)$, $\Phi_1(\gamma_{s_1, \nu})$ are defined.

**Theorem 5.1.** All relevant limits exist, and the assertions of Theorem 4.2 are true for the group $H_1$ when $\Phi$ (normalized stable orbital integral on $H_1(R)$) is replaced by $\Phi_1$ (normalized transport of a weighted sum of twisted integrals)
on $G(\mathbb{R})$:

$$\lim_{\nu \to 0^-} \Psi_{a,\chi}(\gamma_\nu) = - \lim_{\nu \to 0^+} \Psi_{a,\chi}(\gamma_\nu)$$

and

$$\lim_{\nu \to 0^+} \Psi_{a,\chi}(\gamma_\nu) = \lim_{\nu \to 0^-} \Psi_{a^{s_1},\chi^{s_1}}(\gamma_{s_1,\nu}).$$

We will gather ingredients for a proof of the theorem over the next four sections, completing the argument in Section 10. Later the theorem will be strengthened to include derivatives (see Lemmas 10.1, 10.2) and all semiregular $\gamma_0$ (see Section 11). Often we will write $\gamma'_\nu$ for $\gamma_{s_1,\nu}$ and $a', \chi'$ for $a^{s_1}, \chi^{s_1}$.

To begin, we replace $O_{\theta, \varpi}(\delta, f)$ by the normalized integral

$$\Phi_{\theta, \varpi}(\delta, f) = \left| \det(\text{Ad}(\delta) \circ \theta - I)_{/\text{Cent}(\mathfrak{g}_0, \mathfrak{g})} \right|^{1/2} O^{\theta, \varpi}(\delta, f).$$

Assume strongly $G$-regular $\gamma_1$ is a norm of $\delta$. Then the term $\Delta_{IV}(\gamma_1, \delta)$ in the transfer factor is the quotient of the normalizing term above by that for ordinary orbital integrals on $H_1(\mathbb{R})$. Thus our proposed normalized stable orbital integral is given on $\gamma_1$ by

$$\Phi_1(\gamma_1) = \sum_{\delta, \theta-\text{conj}} \frac{\Delta(\gamma_1, \delta)}{\Delta_{IV}(\gamma_1, \delta)} \Phi^{\theta, \varpi}(\delta, f).$$

We may as well assume for the rest of the paper that there exists a strongly $G$-regular element in $H_1(\mathbb{R})$ that is a norm, for otherwise the zero function lies in $\text{Trans}(f)$ and the main theorem is proved. We then fix a pair $(\gamma, \delta)$, with strongly $G$-regular $\gamma \in H_1(\mathbb{R})$ a norm of strongly $\theta$-regular $\delta \in G(\mathbb{R})$, in order to normalize transfer factors as mentioned in Section 1. We gather all terms involving only $(\gamma, \delta)$ as

$$\Delta^*(\gamma, \delta) = \Delta(\gamma, \delta)[\Delta_I(\gamma)\Delta_{II}(\gamma)\Delta_{IV}(\gamma)]^{-1}.$$

Here we have dropped the second argument in our notation for $\Delta_I, \Delta_{II}, \Delta_{IV}$ since it plays no role. There is no harm for the proof of Theorem 5.1 in assuming that transfer factors are normalized so that

$$\Delta(\gamma, \delta) = \Delta_I(\gamma)\Delta_{II}(\gamma)\Delta_{IV}(\gamma),$$

and then

$$\Delta^*(\gamma, \delta) = 1.$$

This allows us to rewrite $\Phi_1(\gamma_1)$, for any strongly $G$-regular $\gamma \in H_1(\mathbb{R})$, as

$$\Delta_I(\gamma_1)\Delta_{II}(\gamma_1) \sum_{\delta, \theta-\text{conj}} \Delta_{III}(\gamma_1, \delta; \gamma, \delta) \Phi^{\theta, \varpi}(\delta, f),$$

where the summation is over $\theta$-conjugacy classes of strongly $\theta$-regular elements $\delta$ in $G(\mathbb{R})$. Here we declare the contribution of the class of $\delta$ to be zero if $\gamma_1$ is not a norm of $\delta$. 
If \( \gamma_1 \) is a norm of \( \delta \), then the torus \( \text{Cent}(\gamma_1, H_1) \) is a norm group (in the sense of the next section) which, as noted in Section 1, implies that the character \( \varpi \) is trivial on \( \text{Cent}_\theta(\delta, G)(\mathbb{R}) \). The transformation rule (2) of Theorem 5.1.D of [KS99] further allows us to write

\[
\int_{\text{Cent}_\theta(\delta, G)(\mathbb{R}) \setminus G(\mathbb{R})} \Delta_{III}(\gamma_1, g^{-1} \delta \sigma(g); \gamma, \varpi, \delta) f(g^{-1} \delta \sigma(g)) dg/dt.
\]

As a function of \( \delta \), this is constant on \( \theta \)-conjugacy classes, as is the normalizing factor \( \left| \det(\text{Ad}(\delta) \circ \theta - I)_{\text{Cent}(g_\theta, g)} \right|^{1/2} \) for \( \Phi^{\delta, \varpi}(\delta, f) \). The set of elements with \( \gamma_1 \) as norm forms a single stable \( \theta \)-conjugacy class of elements in \( G(\mathbb{R}) \), as will be reviewed in Sections 6 and 7. Thus the summation in \( \Phi_1(\gamma_1) \) may be taken over the (finite) set of \( \theta \)-conjugacy classes in this stable class.

In Section 7 we will define \( \Phi_1(\gamma_1) \) for \( G \)-regular elements \( \gamma_1 \) that are not strongly \( G \)-regular in the same way as for the untwisted case, i.e., by smooth extension. First, we need to describe our choice of stable \( \theta \)-conjugacy class with norm \( \gamma_1 \) in that setting. At the same time we prepare for the more delicate analysis of \( \Phi_1(\gamma_1) \) when \( \gamma_1 \) is near semiregular \( \gamma_0 \).

6. Norm groups and semiregular elements

To view semisimple elements of the endoscopic group \( H_1(\mathbb{R}) \) as norms, we adapt the definition of image in standard endoscopy (see (1.2) of [LS90]) to our twisted setting. Recall that we have made an assumption to avoid any twisting in \( H_1(\mathbb{R}) \). Namely, we have fixed quasi-split data \( (G^*, \theta^*) \) and inner twist \( (G, \theta, \psi) \):

\[
\psi \sigma(\psi)^{-1} = \text{Int}(u(\sigma))
\]

and

\[
\psi \circ \theta \circ \psi^{-1} = \text{Int}(g_\theta)^{-1} \circ \theta^*,
\]

where \( u(\sigma), g_\theta \) lie in \( G^*_{sa} \). We write \( u(\sigma), g_\theta \) also for the images of these two elements in \( G^* \) under the natural map \( G^*_{sa} \to G^* \). Define \( m : G \to G^* \) by

\[
m(\delta) = \psi(\delta) g_\theta^{-1}.
\]

Then our assumption is that we may choose \( u(\sigma), g_\theta \) so that

\[
\sigma(m)(\delta) = u(\sigma)^{-1} m(\delta) \theta^*(u(\sigma)).
\]

See Lemma 3.1.A and Appendix B of [KS99] for its (hyper)cohomological significance. It is not difficult to drop the assumption, as we will check in Section 12.

We start our discussion of norms with the correspondence of [KS99] between the set of stable conjugacy classes of strongly \( G \)-regular elements in \( H_1(\mathbb{R}) \) and the set of stable \( \theta \)-conjugacy classes of strongly \( \theta \)-regular elements in \( G(\mathbb{R}) \). Recall from the last section that we may as well assume this correspondence is nonempty. It is uniquely determined by the choice of \( g_\theta \); see [She11] for a related discussion. If the class of strongly \( \theta \)-regular \( \delta \) in \( G(\mathbb{R}) \) corresponds to the class of strongly \( G \)-regular \( \gamma_1 \) in \( H_1(\mathbb{R}) \), then \( \gamma_1 \) is a norm of \( \delta \). We will call a maximal torus \( T_1 \) over \( \mathbb{R} \) in \( H_1 \) a norm group for \( (G, \theta) \) if
$T_1(\mathbb{R})$ contains strongly $G$-regular elements that are norms of strongly $\theta$-regular elements in $G(\mathbb{R})$; this generalizes a definition in [KS99, §3.3].

Let $T_1$ be a maximal torus over $\mathbb{R}$ in $H_1$. Then by Lemma 3.3.B of [KS99] there exist a $\theta^*$-stable maximal torus $T$ in $G^*$ defined over $\mathbb{R}$ and an admissible homomorphism $T_1 \to T_{\theta^*}$ from $T_1$ to the coinvariants of $\theta^*$ in $T$. In more detail: there exist a $\theta^*$-stable maximal torus $T$ in $G^*$ defined over $\mathbb{R}$ and a $\theta^*$-stable Borel subgroup $B$ containing $T$, along with Borel subgroup $B_1$ containing $T_1$ such that the homomorphism

$$T_1 \to T_1/Z_1 \to T_{\theta^*}$$

attached to the pairs $(B_1, T_1)$ and $(B, T)$ is defined over $\mathbb{R}$. Here the map $T_1 \to T_1/Z_1$ is the natural projection, and the construction of $T_1/Z_1 \to T_{\theta^*}$ comes from the definition of endoscopic data. The strongly $\theta^*$-regular elements of $T(\mathbb{R})$, which include a dense subset of $T(\mathbb{R})^0$, have strongly $G^*$-regular norms in $T_1(\mathbb{R})$, and so the cited lemma shows that any maximal torus over $\mathbb{R}$ in $H_1$ is a norm group for the pair $(G^*, \theta^*)$.

Assume now that $T_1$ is a norm group for $(G, \theta)$. Suppose that $\gamma_1$ is a strongly $G$-regular element of $T_1(\mathbb{R})$ and that $\gamma_1$ is a norm of strongly $\theta$-regular $\delta$ in $G(\mathbb{R})$. First we take an admissible homomorphism $T_1 \to T_{\theta^*}$ mapping $\gamma_1$ to an element, say $\gamma^*$, of $T_{\theta^*}(\mathbb{R})$. Because $\gamma_1$ is a norm of $\delta$, there is also an associated isomorphism

$$\text{Int}(g) \circ \psi : G_\delta^0 \to (T^{\theta^*})^0$$

defined over $\mathbb{R}$, where $g$ is chosen in $G^*_{\text{sc}}$ so that

$$\delta^* = gm(\delta)\theta^*(g)^{-1}$$

lies in $T$ and $N(\delta^*) = \gamma^*$; see [KS99, §§3.3, 4.4]. Here, as in [KS99, §3.2], $N$ denotes the abstract norm map, i.e., the projection $T \to T_{\theta^*}$ to coinvariants, while $G_\delta^0$ denotes $\text{Cent}_\theta(\delta, G)^0$, a torus defined over $\mathbb{R}$. In the equation $\delta^* = gm(\delta)\theta^*(g)^{-1}$, the element $g$ has been identified with its image in $G^*$ (we will do this repeatedly, often without mention) and $m$ is the modification of the inner twist $\psi : G \to G^*$ defined in the first paragraph. Because of the strong regularity condition, $g$ is unique up to an element of $T_{\text{sc}}$ once $T_1 \to T_{\theta^*}$ has been fixed. Also, changing $T_1 \to T_{\theta^*}$ changes $g$ in a simple manner [KS99, §4.4].

In summary, if strongly $G$-regular $\gamma_1$ in $H_1(\mathbb{R})$ is a norm of strongly $\theta$-regular $\delta$ in $G(\mathbb{R})$, we identify the quotient of $\text{Cent}(\gamma_1, H_1) = T_1$ by $Z_1$ with the group of $\theta^*$-coinvariants in $T$. Here $T$ is provided by the data for an admissible homomorphism $T_1 \to T_{\theta^*}$. We also identify $G_\delta^0 = \text{Cent}_\theta(\delta, G)^0$ with the identity component of the group of $\theta^*$-invariants in $T$.

Recall that the strong $\theta$-regularity of $\delta$ ensures only that $\text{Cent}_\theta(\delta, G)$ is abelian and diagonalizable. The isomorphism $\text{Int}(g) \circ \psi$ above maps $\text{Cent}_\theta(\delta, G)$ onto the full group of $\theta^*$-invariants in $T$. 

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Now we drop the assumption of strong $G$-regularity on a semisimple element in $H_1(\mathbb{R})$. Then the ambient norm group is not unique unless the element is $G$-regular, and so we proceed torus by torus.

Suppose that $\gamma_0$ is an element in the norm group $T_1(\mathbb{R})$, and assume that $\delta_0$ is a $\theta$-semisimple element of $G(\mathbb{R})$. Then, by definition [KS99, §3.2], $\text{Int}(\delta_0) \circ \theta$ preserves some pair $(B^\dagger, T^\dagger)$. Write $T^{\delta_0}$ for the identity component of the fixed points of $\text{Int}(\delta_0) \circ \theta$ in $T^\dagger$. Then $T^{\delta_0}$ is a maximal torus in the reductive group $G^\theta_{\delta_0}$ defined over $\mathbb{R}$, and we may assume $T^{\delta_0}$ is defined over $\mathbb{R}$. (Otherwise replace $(B^\dagger, T^\dagger)$ by a suitable $G^\theta_{\delta_0}$-conjugate pair.) Fix an admissible homomorphism $T_1 \to T_\theta^*$, and write $\gamma_0^*$ for the image of $\gamma_0$. Then there is an isomorphism $\text{Int}(g) \circ \psi$ carrying $(B^\dagger, T^\dagger)$ to $(B, T)$, where $g \in G^\star_{\text{sc}}$. This implies that $\delta_0^* = gm(\delta_0)\theta^*(g)^{-1}$ lies in $T$.

**Definition.** We call $\gamma_0$ a $T_1$-norm of $\delta_0$ if we may choose $g \in G^\star_{\text{sc}}$ so that (i) $N(\delta_0^*) = \gamma_0^*$ and (ii) the isomorphism $\text{Int}(g) \circ \psi : T^{\delta_0} \to (T^{\theta^*})^\theta$ is defined over $\mathbb{R}$.

In the case that $\gamma_0$ is strongly $G$-regular (ii) follows from (i) [KS99, (3.3.6)]. In general, for given $T_1$, the choice of admissible homomorphism $T_1 \to T_\theta^*$ does not affect the existence of $g$.

Next, we consider together all elements in the $\gamma_0$-component $\gamma_0 T_1(\mathbb{R})^0$ of $T_1(\mathbb{R})$.

**Lemma 6.1.** The following are equivalent for $\gamma_0 \in T_1(\mathbb{R})$:

(i) $\gamma_0$ is a $T_1$-norm,
(ii) some strongly $G$-regular element in the $\gamma_0$-component is a norm,
(iii) every element of the $\gamma_0$-component is a $T_1$-norm.

**Proof.** Fix an admissible homomorphism $T_1 \to T_\theta^*$, and assume that $\gamma_0 \in T_1(\mathbb{R})$ is a $T_1$-norm of a $\theta$-semisimple $\delta_0 \in G(\mathbb{R})$. Choose elements $g, \delta_0^*$ as in the definition. Take $\varepsilon$ in the identity component of the Cartan subgroup $T^{\delta_0}(\mathbb{R})$ of $G^\theta_{\delta_0}(\mathbb{R})$, and consider $\delta = \varepsilon \delta_0$. Then $\delta$ is $\theta$-semisimple since $\text{Int}(\delta) \circ \theta$ preserves the same pair $(B^\dagger, T^\dagger)$ as $\text{Int}(\delta_0) \circ \theta$. Also, by results of Steinberg (see Theorem 1.1.A in [KS99]), $G^\theta_{\delta_0}(\mathbb{R})$ contains $T^{\delta_0}(\mathbb{R})$ as Cartan subgroup. Further we may choose $\varepsilon$ so that $\delta$ is strongly $\theta$-regular; the elements $\varepsilon$ with this property are dense in $T^{\delta_0}(\mathbb{R})^\theta$. Set

$$\delta^* = gm(\delta)\theta^*(g)^{-1} = gm(\varepsilon \delta_0)\theta^*(g)^{-1} = g\psi(\varepsilon)g^{-1} \cdot gm(\delta_0)\theta^*(g)^{-1} = \varepsilon^* \delta_0^* = \delta_0^* \varepsilon^*,$$

where $\varepsilon^* = g\psi(\varepsilon)g^{-1}$ lies in $T^{\theta^*}(\mathbb{R})^\theta$. The image of the $\gamma_0$-component in $T_1(\mathbb{R})$ under $T_1 \to T_\theta^*$ then contains $N(\delta^*) = \gamma_0^* N(\varepsilon^*)$, where $\gamma_0^*$ is, as before, the image of $\gamma_0$ under $T_1 \to T_\theta^*$. Since $\delta^*$ is strongly $\theta^*$-regular, each element in the $\gamma_0$-component which maps to $N(\delta^*)$ under $T_1 \to T_\theta^*$ is strongly $G$-regular, and (ii) now follows.
Assume (ii), and suppose strongly $G$-regular $\gamma_1$ in the $\gamma_0$-component of $T_1(\mathbb{R})$ is a norm of $\delta$. Choose $\delta^*, g$ as in the definition of norm for strongly $G$-regular elements. By our assumption that the restriction of $\theta$ to the center of $G$ is (strongly) semisimple, the homomorphism $N : T^\theta(\mathbb{R}) \to T_{\theta^*}(\mathbb{R})^0$ is surjective. Thus the image of $\gamma_1T_1(\mathbb{R})^0$ under $T_1 \to T_{\theta^*}$ coincides with the image under $N$ of $\delta^*T^\theta(\mathbb{R})^0$. We write an element $\gamma_2$ of $\gamma_1T_1(\mathbb{R})^0 = \gamma_0T_1(\mathbb{R})^0$ as the image under $N$ of some element $\delta_2^*$ in $\delta^*T^\theta(\mathbb{R})^0$. Then, as in Lemma 4.4.A of [KS99],

$$\sigma(\delta_2^*)\delta_2^{* - 1} = \sigma(\delta^*)\delta^{* - 1} = (\theta^* - 1)v(\sigma),$$

where the cochain $v(\sigma)$ is (the image in $T$ of) the cochain $g\delta(\sigma)\sigma(\delta)^{-1}$ in $T_{sc}$. Thus

$$\delta_2 = m^{-1}(g^{-1}\delta_2^*\theta^*(g))$$

is $\theta$-semisimple, lies in $G(\mathbb{R})$, and has norm $\gamma_2$, so that (iii) follows. The rest is immediate.

We now expand on the argument for (i) $\Rightarrow$ (ii) in the last lemma. Write the element $\varepsilon$ defined there as $exp Y$, where $Y$ belongs to the Cartan subalgebra $t^{\delta_0}(\mathbb{R})$ of the Lie algebra $g^{\delta_0}_0(\mathbb{R})$ of $G^{\delta_0}_0(\mathbb{R})$. Let $Y$ map to $Y^*$, where $Y^* \in t^{\theta^*}(\mathbb{R})$, under the bijection provided by $\text{Int}(g) \circ \psi$. Recall from the definition of $z$-pair that we have the exact sequence $1 \to Z_1 \to H_1 \to H \to 1$, with $Z_1$ central in $H_1$. We split the corresponding sequence for Lie algebras in the usual manner and identify, over $\mathbb{R}$, the Lie algebra $h$ as a subalgebra of $h_1$ complementary to $z_1$. Then the Lie algebra $t_H$ of $T_1/Z_1$ is a subspace of $t_1$ complementary to $z_1$. There is a linear isomorphism

$$t^{\theta^*}(\mathbb{R}) \to t_{\theta^*}(\mathbb{R}) \to t_H(\mathbb{R})$$

determined by the restriction of $N : T \to T_{\theta^*}$ to $\theta^*$-invariants and the chosen admissible isomorphism $T_{\theta^*} \to T_1/Z_1$. Write $Y_H$ for the image of $Y^*$, so that we have

$$t^{\delta_0}(\mathbb{R}) \ni Y \leftrightarrow Y^* \leftrightarrow N(Y^*) \leftrightarrow Y_H \in t_H(\mathbb{R}).$$

Write $Y_1 \in t_1(\mathbb{R})$ as $Y_1 = Y_H + Y_{z_1}$. Then the following is immediate.

**Lemma 6.2.** Assume that $\gamma_0$ is a $T_1$-norm of $\delta_0$ and $Y_1 \in t_1(\mathbb{R})$. Then the element

$$\gamma_0(Y_1) = \gamma_0 \cdot \exp Y_1 = \exp Y_1 \cdot \gamma_0$$

in the $\gamma_0$-component of $T_1(\mathbb{R})$ is a $T_1$-norm of the element

$$\delta_0(Y) = \exp Y \cdot \delta_0 = \delta_0 \cdot \exp \theta Y.$$

The cochain $v(\sigma)$ attached to $\gamma_0$ also serves for $\gamma_0(Y_1)$, while the attached element of $T$ is

$$\delta^*(Y) = \delta_0^* \cdot \exp Y^* = \exp Y^* \cdot \delta_0^*.$$
Now we consider all tori $T_1$ containing a given semisimple element $\gamma_0$ in $H_1(\mathbb{R})$. Let $\delta_0$ be a $\theta$-semisimple element of $G(\mathbb{R})$. We call $\gamma_0$ a norm of $\delta_0$ (or, for emphasis on the ambient group, a $G$-norm of $\delta_0$) if there exists a norm group $T_1$ such that $\gamma_0$ is a $T_1$-norm of $\delta_0$. Otherwise we say that $\gamma_0$ is not a (G-) norm. The following will be proved after Lemma 6.6.

**Lemma 6.3.** Let $\gamma_0$ be a semisimple element in $H_1(\mathbb{R})$ and let $\delta_0, \delta'_0$ be $\theta$-semisimple elements of $G(\mathbb{R})$. Then (i) if $\gamma_0$ is a $G$-norm of $\delta_0$, then so are all stable conjugates of $\gamma_0$ in $H_1(\mathbb{R})$, and (ii) if $\gamma_0$ is a $G$-norm of both $\delta_0$ and $\delta'_0$, then $\delta_0$ and $\delta'_0$ are stably $\theta$-conjugate.

**Remark.** By $\delta'_0$ is stably $\theta$-conjugate to $\delta_0$ we mean that we may write $\delta'_0 \in G(\mathbb{R})$ as $x\delta_0 \theta(x)^{-1}$, where $x \in G$ and $\text{Int}(x): G^\theta_{\delta_0} \rightarrow G^\theta_{\delta'_0}$ is an inner twist.

**Remark.** As pointed out by a referee, the converse statement for (ii) in Lemma 6.3 is false in general.

Assume now that semisimple $\gamma_0 \in H_1(\mathbb{R})$ is a $T_1$-norm of $\delta_0 \in G(\mathbb{R})$. Fix admissible $T_1 \rightarrow T^{\theta^*}$ and choose $g, \delta_0^*$ as in the definition of $T_1$-norm. Then $\text{Int}(g) \circ \psi$ is an isomorphism of $G^\theta_{\delta_0}$ with $(G^*)^*_{\delta_0^*}$. We will abbreviate (slightly) the notation for the latter group as $G^*_{\delta_0^*}$. We have required that $\text{Int}(g) \circ \psi$ maps, over $\mathbb{R}$, the maximal torus $T^{\delta_0}$ over $\mathbb{R}$ in $G^\theta_{\delta_0}$ to the maximal torus $(T^{\theta^*})^0$ in $G^*_{\delta_0^*}$. In the case that $G^\theta_{\delta_0}$ is of Dynkin type $A_1$ we claim that this requirement ensures first that $G^*_{\delta_0^*}$ is defined over $\mathbb{R}$ and then that $\text{Int}(g) \circ \psi: G^\theta_{\delta_0} \rightarrow G^*_{\delta_0^*}$ is an inner twist. Indeed, $\text{Int}(g) \circ \psi$ transports the two roots of $T^{\delta_0}$ in $G_0^\delta$, either both imaginary or both real, to the roots of $(T^{\theta^*})^0$ in $G^*_{\delta_0^*}$ which must be of the same type. An argument with root vectors then finishes the proof.

With no restriction on the Dynkin type of $G^\theta_{\delta_0}$, we will prove the next lemma at the end of this section.

**Lemma 6.4.** Suppose that semisimple $\gamma_0 \in H_1(\mathbb{R})$ is a $T_1$-norm of $\delta_0 \in G(\mathbb{R})$ and that $T_1 \rightarrow T^{\theta^*}$ is an admissible homomorphism. Then we may choose the elements $g, \delta_0^*$ so that (i) $\sigma(\delta_0^*)\delta_0^{*-1}$ is central in $G^*$, and (ii) $v(\sigma) = gu(\sigma)\sigma(g)^{-1}$ lies in the product of the torus $(T_{sc})^\theta_{sc}$ with the center of $G^*_{sc}$.

In particular, if $G$ is of adjoint type, then we may arrange that $\delta_0^*$ lies in $T(\mathbb{R})$. In general, for any $g, \delta_0^*$ as in this lemma, the group $G^*_{\delta_0^*}$ is defined over $\mathbb{R}$ and $\text{Int}(g) \circ \psi: G^\theta_{\delta_0} \rightarrow G^*_{\delta_0^*}$ is an inner twist.

Before continuing with the case that $G^\theta_{\delta_0}$ is of Dynkin type $A_1$, we record an explicit analysis of the roots of $G^\theta_{\delta_0}$ and $G^*_{\delta_0^*}$ following Steinberg (see [KS99, Chap. 1]). By a restricted root we will mean the restriction $\alpha_{res}$ of a root $\alpha$ of $T$ in $G^*$ to the torus $(T^{\theta^*})^0$. This torus is maximal in each of the reductive
groups \((G^{\theta*})^0\) and \(G_{\delta_0}^{\theta*}\). The set of all restricted roots forms a nonreduced root system in general. As in Section 1.3 of [KS99], we call \(\alpha\) of type \(R_1\) if neither \(2\alpha_{\text{res}}\) nor \(\frac{1}{2}\alpha_{\text{res}}\) is a restricted root, of type \(R_2\) if \(2\alpha_{\text{res}}\) is a restricted root, or of type \(R_3\) if \(\frac{1}{2}\alpha_{\text{res}}\) is a restricted root. Also following [KS99], we may identify a root \(\alpha_1 = ((\alpha^*)_{\text{res}})^{\vee}\) of \(T_1\) in \(H_1\), or of \(T_1/Z_1 \cong T_{\theta*}\) in \(H_1/Z_1\), as \(N\alpha\) or \(2N\alpha\). If \(\alpha\) is of type \(R_1, R_3\), then \(\alpha_1 = N\alpha\), and if \(\alpha\) is of type \(R_2\), then \(\alpha_1 = 2N\alpha\). Recall that \(N\alpha\) denotes the sum of all distinct roots in the \(\theta^2\)-orbit of \(\alpha\). Assume \(\alpha_1\) is a root of \(T_1\) in the identity component \((H_1)_{\gamma_0}\) of the centralizer of \(\gamma_0\) in \(H_1\). The identification of roots then implies that

\[
\alpha_1(\gamma_0) = N\alpha(\delta_0^*) = 1
\]

if \(\alpha\) is of type \(R_1\) or \(R_3\) and that

\[
\alpha_1(\gamma_0) = N\alpha(\delta_0^*)^2 = 1
\]

if \(\alpha\) is of type \(R_2\). Write this second case as \(R_{2,\pm}\) according as \(N\alpha(\delta_0^*) = \pm 1\).

We use \(\text{Int}(g) \circ \psi\) to identify roots of \(T^{\delta_0}\) in \(G_{\delta_0}^0\) with roots of \((T^{\theta*})^0\) in \(G_{\delta_0}^{\theta^*}\). Let \(\alpha\) be a root of \(T\) in \(G^\ast\). Then \(\alpha_{\text{res}}\) is a root of \(T^{\delta_0}\) in \(G_{\delta_0}^0\) if and only if \(N\alpha(\delta_0^*) = 1\) in the cases \(\alpha\) is of type \(R_1, R_2\), or if and only if \(N\alpha(\delta_0^*) = -1\) in the case \(\alpha\) is of type \(R_3\). We conclude the following.

**Lemma 6.5.** Assume that \(\alpha_1 = ((\alpha^*)_{\text{res}})^{\vee}\) is a root of \(T_1\) in \((H_1)_{\gamma_0}\), i.e., that \(\alpha_1(\gamma_0) = 1\). Then (i) \(\alpha_0 = r_\alpha\alpha_{\text{res}}\) is a root of \(T^{\delta_0}\) in \(G_{\delta_0}^0\), where \(r_\alpha = 1\) if \(\alpha\) is of type \(R_1\) or \(R_{2,\pm}\), \(r_\alpha = 2\) if \(\alpha\) is of type \(R_{2,-}\), and \(r_\alpha = \frac{1}{2}\) if \(\alpha\) is of type \(R_3\). Also, (ii) if \(\alpha\) is of any type except \(R_{2,-}\), then \(N\alpha(\delta_0^*) = 1\) and \(\alpha_0\) is a root of \((T^{\theta*})^0\) in \((G^{\theta*})^0\). Finally, (iii) if \(\alpha\) is of type \(R_{2,-}\), then \(N\alpha(\delta_0^*) = -1\) and \(\alpha_{\text{res}} = \frac{1}{2}\alpha_0\) is a root of \((T^{\theta*})^0\) in \((G^{\theta*})^0\).

**Remark.** We will often write \(N\alpha(\delta_0^*)\) for \(N\alpha(\delta_0^*)\). Notice we may make a definition of \(N\alpha\) that is intrinsic to \(G\) by using the automorphism \(\text{Int}(\delta_0) \circ \theta\) and the maximal torus \(T^1 = \text{Cent}(T^{\delta_0}, G)\).

Next, we assume also that \(\gamma_0\) is semiregular; i.e., \(\pm\alpha_1\) are the only roots of \(T_1\) in \((H_1)_{\gamma_0}\). We will say that \(\gamma_0\) is **\(G\)-semiregular** if \(\pm\alpha_0\) are the only roots of \(T^{\delta_0}\) in \(G_{\delta_0}^0\); i.e., both \((H_1)_{\gamma_0}\) and \(G_{\theta_0}^0\) are of Dynkin type \(A_1\). Explicitly, the extra condition is that if root \(\beta\) of \(T\) is not in the \(Q\)-span of the \(\theta^2\)-orbit of \(\alpha\), then \(N\beta(\delta_0^*) \neq 1\) if \(\beta\) is of type \(R_1\) or \(R_2\), and \(N\beta(\delta_0^*) \neq -1\) if \(\beta\) is of type \(R_3\). Notice that if \(\beta\) is of type \(R_2\), then \(N\beta(\delta_0^*) = -1\) implies that \(2\beta_{\text{res}}\) is a root of \(G_{\delta_0}^0\), and so we conclude that for \(\beta\) of type \(R_2\), the extra condition can be rewritten as \(\beta_1(\gamma_0) = N\beta(\delta_0^*)^2 \neq 1\), and then that the condition for \(\beta\) of type \(R_3\) is redundant. We may now write the \(G\)-semiregularity condition directly in terms of \(\gamma_0\) as
\[ \alpha_1(\gamma_0) = 1 \] and \( \beta_1(\gamma_0) \neq 1 \) for all roots \( \beta \) of type \( R_1 \) or \( R_2 \) not in the \( \mathbb{Q} \)-span of the \( \theta^* \)-orbit of \( \alpha \).

If semiregular \( \gamma_0 \in T_1(\mathbb{R}) \) is not a norm, we will use this condition as our definition of \( G \)-semiregularity (which coincides with the more natural definition using the map \( A_{G/H} \) of [KS99, Th. 3.3.A]).

We return to the setting of Theorem 5.1, where \( \alpha_1 \) is imaginary and \( s_1 \) is a Cayley transform with respect to \( \gamma \). Because of the stability of the transfer factor \( \Delta(\gamma_1, \delta) \) in its first argument \( \gamma_1 \) [KS99, Lemma 5.1.B], the argument of the last paragraph of the proof of Theorem 4.2 shows that there is no harm (for the proof of Theorem 5.1) in assuming \( \gamma_1 \) itself is noncompact and that \( s_1 \) is a Cayley transform within \( (H_1)_{\gamma_0} \). Then also \( \gamma_0^{s_1} = \gamma_0 \); i.e., \( \gamma_0 \) lies in \( T_1 \cap T_1^{s_1} \).

An element \( \gamma_1 \) in \( T_1(\mathbb{R}) \) is \( G \)-regular in the sense of [KS99] if and only if \( \beta_1(\gamma_1) \neq 1 \) for all roots \( \beta \) of type \( R_1 \) or \( R_2 \). Because \( \gamma_0 \) is assumed \( G \)-semiregular, the elements
\[
\gamma_\nu = \gamma_0 \exp(\nu a_{\alpha_1}(\alpha_1)^{\vee})
\]
in the \( \gamma_0 \)-component of \( T_1(\mathbb{R}) \) and the elements
\[
\gamma_{s_1,\nu} = \gamma_0 \exp(\nu a_{\alpha_1^1}(\alpha_1^1)^{\vee}),
\]
where \( \alpha_1^1 = \alpha_1^{s_1} \), in the \( \gamma_0 \)-component of \( T_1^{s_1}(\mathbb{R}) \) are easily checked to be \( G \)-regular for all real nonzero \( \nu \) with \( |\nu| \) sufficiently small. We gather the following observations with some special cases of Theorem 5.1 in mind (see Lemma 7.2).

**Lemma 6.6.** Suppose \( \gamma_0 \) is a \( G \)-semiregular element in a Cartan subgroup \( T_1(\mathbb{R}) \) of \( H_1(\mathbb{R}) \) annihilated by a noncompact imaginary root \( \alpha_1 \). Suppose that \( s_1 \) is a Cayley transform for \( \alpha_1 \) in \( (H_1)_{\gamma_0} \). Then (i) if \( \gamma_0 \) is not a \( G \)-norm, then the \( G \)-regular elements \( \gamma_\nu \) and \( \gamma_{s_1,\nu} \) are not norms, (ii) if \( T_1^{s_1} \) is a norm group for \( (G, \theta) \), then \( T_1 \) is also a norm group for \( (G, \theta) \), (iii) if \( \gamma_0 \) is a \( G \)-norm of \( \delta_0 \) in \( G(\mathbb{R}) \), then \( \gamma_0 \) is a \( T_1 \)-norm of \( \delta_0 \), (iv) if \( \gamma_0 \) is a \( G \)-norm of \( \delta_0 \) in \( G(\mathbb{R}) \), then \( \gamma_0 \) is a \( T_1^{s_1} \)-norm of \( \delta_0 \) if and only if \( G_{\delta_0}^{\theta} \) is split modulo center, and (v) if \( T_1^{s_1} \) is not a norm group for \( (G, \theta) \), then the group \( G_{\delta_0}^{\theta} \) is compact modulo center for each \( \delta_0 \) in \( G(\mathbb{R}) \) with \( T_1 \)-norm \( \gamma_0 \).

**Proof.** For (i), assume \( \gamma_0 \) is not a \( G \)-norm. We then apply Lemma 6.1 to \( \gamma_0 \) as element of \( T_1 \) to conclude that \( \gamma_\nu \) is not a norm, and to \( \gamma_0 \) as element of \( T_1^{s_1} \) to conclude that \( \gamma_{s_1,\nu} \) is not a norm.

For (ii), assume that \( T_1^{s_1} \) is a norm group for \( (G, \theta) \). By Lemma 6.1, there is a component of \( T_1^{s_1}(\mathbb{R}) \) consisting of \( T_1^{s_1} \)-norms. Choose a \( G \)-semiregular element \( \gamma_2 \) of this component annihilated by the real root \( \alpha_1^{s_1} \), and suppose it is a \( T_1^{s_1} \)-norm of \( \delta_2 \). There are \( \theta^* \)-stable maximal tori \( T, T' \) in \( G^* \) defined over \( \mathbb{R} \) and admissible homomorphisms \( T_1 \to T_{\theta^*}, T_1^{s_1} \to T_{\theta^*}' \). Since \( T_1^{s_1} \) is a
norm group for \((G, \theta)\), there is also an isomorphism \(\text{Int}(g_2) \circ \psi : T^{\delta_2} \to (T^{\theta_{\delta}})^0\) defined over \(\mathbb{R}\), where \(\delta_2^* = g_2m(\delta_2)\theta^*(g_2)^{-1}\) lies in \(T'\) and \(N(\delta_2)\) is the image of \(\gamma_2\) under \(T_1^{s_1} \to T_{\theta'}\). Recall that \(G_{\delta_2}^\theta\) is defined over \(\mathbb{R}\) and \(\text{Int}(g_2) \circ \psi : G_{\delta_2}^\theta \to G_{\delta_2}^{\theta^*}\) is an inner twist. The root \(\alpha_0\) of \(T^{\delta_2}\) in \(G_{\delta_2}^\theta\) corresponding to \(\alpha_1^* = \alpha_1^{s_1}\) is also real: \(\sigma\alpha_0'\) corresponds to \(\sigma\alpha_1' = \alpha_1'\) and so equals \(\alpha_0'\). Let \(t \in G_{sc}\) define an inverse Cayley transform in \((G_{\delta_2}^\theta)^{sc}\) for \(\alpha_0'\). On the other hand, the \(\theta^*\)-stable pairs \((B, T)\) and \((B', T')\) defining \(T_1 \to T_{\theta'}\), \(T_1^{s_1} \to T_{\theta'}^s\) are conjugate under \((G_{sc}^\theta)^{\theta_{sc}}\) (by Steinberg’s structure results; see [KS99, Th. 1.1.A]) and so they determine an element \(t^*\) of \((G_{sc}^{\theta_{sc}})^{\theta_{sc}}\) such that \(\text{Int}(t^*)\) maps \(T'\) to \(T_{\theta'}\) to \(T_{\theta'}^s\) and completes a commutative diagram with \(\text{Int}(s_1)^{-1} : T_1^{s_1} \to T_1\) and the admissible homomorphisms \(T_1 \to T_{\theta'}\), \(T_1^{s_1} \to T_{\theta'}^s\). Then \(t^*\) is an inverse Cayley transform for the real root \(r\alpha_0'\) of \((T^{\theta_{\delta}})^0\) in \((G^{\theta_{\delta}})^0\), where \(r = 1\) or \(\frac{1}{2}\) since the action of \(\sigma(t^*)^{-1}t^*\) on \((T^{\theta_{\delta}})^0\) coincides with the dual transport of \(\sigma(s_1)s_1^{-1}\) which acts on \(T_1^{s_1}\) as the Weyl reflection for \(\alpha_1'\); this dual transport coincides with the Weyl reflection for \(r\alpha_0'\). Here we define dual transport using the bijection (1.3.8) of [KS99]. We may arrange the choices so that \(t^*\) is standard; i.e., \(t^*\) lies in the image of \(SL_2\) in \((G_{sc}^{\theta_{sc}})^{\theta_{sc}}\) corresponding to the root \(r\alpha_0'\). The action of \(\sigma(t^*)^{-1}t^*\) on \((T^{\theta_{\delta}})^0\) coincides with the transport by \(\text{Int}(g_2) \circ \psi\) of the action of \(\sigma(t)^{-1}t\) on \(T^{\delta_2}\) (this is the inverse Cayley transform defined earlier in the present paragraph) since again each acts as the same Weyl reflection. Let \(T^{\delta_2}\) be the image of \(T^{\delta_2}\) under \(t\). This property of \(t, t^*\) (via our definition of Cayley transform in Section 3) implies that if \(g_3 = t^*g_2\psi(t^{-1})\), then the composition

\[\text{Int}(g_3) \circ \psi : T^{\delta_2} \to (T^{\theta_{\delta}})^0\]

is defined over \(\mathbb{R}\) and that

\[g_3m(\delta_2)\theta^*(g_3)^{-1} = \text{Int}(t^*)(\delta_2) = \delta_3^*\]

lies in \(T\). Finally, \(N(\delta_3)\) is the image of \((\gamma_2)^{s_1^{-1}} \in T_1^{1}(\mathbb{R})\), so that \((\gamma_2)^{s_1^{-1}}\) is a \(T_1\)-norm of \(\delta_3\). In particular, \(T_1\) is a norm group for \((G, \theta)\), and (ii) is proved.

For (iii), assume \(\gamma_0\) is a \(G\)-norm. Then because \((H_1)_{\gamma_0}\) is of type \(A_1\), we see that \(\gamma_0\) must be either a \(T_1\)-norm or \(T_1^{s_1}\)-norm. The argument for (ii) with \(\gamma_2 = \gamma_0\) shows that if \(\gamma_0\) is a \(T_1^{s_1}\)-norm of an element \(\delta_0\), then it is also a \(T_1\)-norm of \(\delta_0\).

For (iv), we return to the argument for (ii), except that now \(T_1\) in place of \(T_1^{s_1}\) is assumed a norm group for \((G, \theta)\). We replace the element \(\delta_2\) by \(\delta_0\) and, as usual, write \(T^{\delta_0}\) for the image of \((T^{\theta_{\delta}})^0\) under the embedding into \(G_{\delta_0}^\theta\). If \(G_{\delta_0}^\theta\) is split modulo center, which implies that the root \(\alpha_0\) of \(T^{\delta_0}\) is noncompact imaginary, then we may construct a Cayley transform \(s_1^1\) in \((G_{\delta_0}^\theta)^{sc}\) and argue along the same lines as (ii) to write \(\gamma_0\) as a \(T_1^{s_1}\)-norm of \(\delta_0\). For the converse,
assume $\gamma_0$ is also a $T_1^{\ast}$-norm of $\delta_0$. Then the argument for (ii) shows that $G_{\delta_0}^\theta$ contains a torus $T_{\delta_0}^{\ast}$ which has a real root and so is split modulo center.

Lemma 6.1 shows that (v) is a consequence of (iv) and the lemma follows.

Proof of Lemma 6.3. Suppose semisimple $\gamma_0$ is a $T_1$-norm of $\delta_0$, where $T_1$ is arbitrary. By definition, $T_1'$ lies in $(H_1)_{\gamma_0}$. If $T_1'$ is not fundamental in $(H_1)_{\gamma_0}$, then it has a real root. Now we argue similarly as for (ii) in Lemma 6.6, with $\gamma_0$ in place of $\gamma_2$ and $\delta_0$ in place of $\delta_2$, to display $\gamma_0$ as a $T_1$-norm of $\delta_0$, with $T_1$ of split rank one less than that of $T_1'$. Repeating this argument until real roots are exhausted, we conclude that if $\gamma_0$ is a $G$-norm of $\delta_0$, then $\gamma_0$ is a $T_1$-norm of $\delta_0$, where $T_1 = T_{\text{fund}}$ is fundamental in $(H_1)_{\gamma_0}$. Recall that a stable conjugate of $\gamma_0$ in $H_1(\mathbb{R})$ may be written as $w\gamma_0w^{-1}$, where the restriction of $\text{Int}(w)$ to $T_{\text{fund}}$ is defined over $\mathbb{R}$ [She83, Lemma 2.5.1]. Then (i) follows.

To prove (ii), let semisimple $\gamma_0$ be a $G$-norm of $\delta_0, \delta_0'$. Then by the last paragraph we may use an admissible homomorphism $T_1 \to T_{\theta^*}$, with $T_1$ fundamental in $(H_1)_{\gamma_0}$, to attach $g, g' \in G_{sc}^*$ and $\delta_0, (\delta_0')^* \in T$ to $\delta_0, \delta_0'$ respectively. Following the proof for (i) $\Rightarrow$ (ii) in Lemma 6.1, we use the elements $g, g'$ to define strongly $\theta$-regular $\delta_3, \delta_3'$ and corresponding elements $\delta_3^\ast, (\delta_3')^\ast$ in $\delta_0^*(T_{\theta^*})^0(\mathbb{R})$ and $(\delta_0')^*(T_{\theta^*})^0(\mathbb{R})$ respectively, such that $N(\delta_3^\ast) = N((\delta_3')^\ast)$. That construction allows us to assume $(\delta_3')^\ast = \delta_3^* t \theta^*(t)^{-1}$, where $t \in T$ satisfies $(\delta_0')^* = \delta_0^* t \theta^*(t)^{-1}$. Set $x = \psi^{-1}(g'^{-1}tg)$. Then $x \delta_3 \theta(x)^{-1} = \delta_3'$ and $x \delta_0 \theta(x)^{-1} = \delta_0'$. From the first of these two equations (the strongly regular case) we conclude that $\sigma(x)^{-1} x$ lies in the product of $G_{\delta_3}^\theta = T_{\delta_0}^{\ast}$ with $\theta$-invariants in the center of $G$, and so $\delta_0'$ is stably $\theta$-conjugate to $\delta_0$.

We turn now to the proof of Lemma 6.4. Our first remark is that the elements $v(\sigma) = gu(\sigma)\sigma(g)^{-1}$ and $\delta_0^* = gm(\delta_0)\theta^*(g)^{-1}$ from the statement of the lemma are unchanged when the inner twist $\psi : G \to G^\ast$ is replaced by $\text{Int}(x) \circ \psi$, where $x \in G_{sc}^*$, provided we replace $u(\sigma)$ by $xu(\sigma)\sigma(x)^{-1}$ and $g_\theta$ by $\theta^*(x)g_\theta x^{-1}$. Recall that $u(\sigma), g_\theta$ were discussed in the first paragraph of the present section. Notice also that the change in $\psi$ does not affect our assumption there about $u(\sigma), g_\theta$. We are thus free to choose $\psi$ as we wish within its inner class. Our choice will use fundamental splittings, as in [She11] but without the cuspidality assumption. The definitions are as follows.

Let $T_G$ be a fundamental maximal torus over $\mathbb{R}$ in $G$ and $B_G$ be a Borel subgroup of $G$ containing $T_G$. Then we call the pair $(B_G, T_G)$ fundamental if the set of $B_G$-simple roots of $T_G$ in $G$ is preserved by the action of $-\sigma_T$ on $X^*(T)$. Such pairs exist; see [Kot82, §10.4]. We will review this below as we use it. Consider a splitting $\text{spl}_G = (B_G, T_G, \{X_\alpha\})$ for $G$. Here $X_\alpha$ is a root vector for the $B_G$-simple root $\alpha$. Denote by $X_{-\alpha}$ the root vector for $-\alpha$ completing $X_\alpha$ and the coroot $H_\alpha$ to a simple triple. There are two possibilities: $\alpha$ is
complex and $|\{\pm \alpha, \pm \sigma \tau \alpha\}| = 4$ or $\alpha$ is imaginary and $|\{\pm \alpha, \pm \sigma \tau \alpha\}| = 2$. We call $\text{spl}_G$ fundamental if the pair $(B_G, T_G)$ is fundamental and $\sigma X_\alpha = X_{\sigma \tau \alpha}$ for all $B_G$-simple roots that are complex or noncompact imaginary, $\sigma X_\alpha = -X_{\sigma \tau \alpha}$ for all $B_G$-simple roots that are compact imaginary. A fundamental pair may be extended to a fundamental splitting; see [She11, §3] regarding imaginary roots. Suppose that $\eta$ is an automorphism of $G$ that preserves the fundamental splitting $\text{spl}_G$. If the restriction of $\eta$ to $T_G$ is defined over $\mathbb{R}$, then an argument with root vectors shows that $\eta$ is defined over $\mathbb{R}$ as automorphism of $G$.

The automorphism $\theta^*$ of $G^*$ preserves a (fixed) $\mathbb{R}$-splitting $(B^*, T^*, \{X_\alpha^*\})$. Here $T^*$ is a maximally split maximal torus defined over $\mathbb{R}$ and $B^*$ is also defined over $\mathbb{R}$. We may construct a $\theta^*$-stable fundamental pair $(B, T)$ for $G^*$ as follows. Consider the identity component $G^1$ of the group of fixed points of $\theta^*$ in $G^*$. Then $G^1$ has an $\mathbb{R}$-splitting that extends the pair $(G^1 \cap B^*, G^1 \cap T^*)$. Following Sections 10.3, 10.4 of [Kot86], we apply a rationality theorem of Steinberg to find a fundamental pair $(B^1, T^1)$ for $G^1$: choose $h \in (G^*_{sc})^0_{\text{ad}}$ such that $h \circ (h)^{-1}$ preserves $G^1 \cap T^*$ and acts on $G^1 \cap T^*$ as the longest element of the Weyl group of $G^1 \cap T^*$ in $G^1$, and then set $B^1 = h^{-1}B^* h, T^1 = h^{-1}T^* h$. Let $(B, T)$ be the corresponding $\theta^*$-stable pair for $G^*$. Then $T$ is fundamental since a real root would provide a real root for the fundamental torus $T^1$, and further the pair $(B, T)$ is fundamental, again by Steinberg’s structure theorem. We extend $(B, T)$ to a fundamental splitting spl. Then $\theta^*$ preserves spl up to an inner automorphism by an element of $T_{sc}$; this inner automorphism is defined over $\mathbb{R}$.

Returning to the inner twist $\psi : G \to G^*$, we adjust $\psi$ within its inner class so that the restriction of $\psi^{-1}$ to $T$ is defined over $\mathbb{R}$. Set $B_G = \psi^{-1}(B), T_G = \psi^{-1}(T)$. Then $(B_G, T_G)$ is a fundamental pair. We may further adjust $\psi$ by an inner automorphism by an element of $T_{sc}$ so that $\psi^{-1}$ transports the fundamental splitting spl of $G^*$ to a fundamental splitting $\text{spl}_G$ of $G$ extending $(B_G, T_G)$. With these adjustments to $\psi$ we now conclude that $\theta_G = \psi^{-1} \circ \theta^* \circ \psi$ is defined over $\mathbb{R}$. Then $\theta = \text{Int}(h_\theta) \circ \theta_G$, where $h_\theta = \psi_{sc}^{-1} (g_\theta^{-1})$. Both $\text{Int}(h_\theta)$ and $\text{Int}(g_\theta)$ are defined over $\mathbb{R}$, and we may take $u(\sigma)$ to be fixed by $\theta_{sc}^*_{\text{ad}}$ since $(T_{sc})^*_{sc} \to (T_{ad})^*_{sc}$ is surjective. Then the cocycle $z_\sigma$ of Lemma 3.1.A of [KS99] is simply $\psi_{sc}(h_\theta^{-1} \sigma(h_\theta))$. Returning to the assumption of the first paragraph of this section, we adjust the choice of $g_\theta, u(\sigma)$ by central elements in $G^*_{sc}$ to arrange that $z_\sigma = 1$ [KS99, p. 26].

Remark. Since $\theta_G$ has finite order, it follows that $\theta$ may be written as the product of an inner automorphism and an automorphism of finite order, where each automorphism is defined over $\mathbb{R}$. This result was pointed out by a referee who also supplied another proof.

We will also make use of connectivity properties of real points of fundamental tori. We continue with the same setting. From Sections 10.3, 10.4
of [Kot86] we see that \( T_{sc}(\mathbb{R}) \) is connected: because \((B, T)\) is a fundamental pair, \( X_*(T_{sc}) \) has a base preserved by \(-\sigma_T\), namely the coroots of the \( B \)-simple roots of \( T \), and so each \( \sigma_T \)-invariant element of \( X_*(T_{sc}) \) lies in \((1 + \sigma_T)X_*(T_{sc})\), which implies that \( T_{sc}(\mathbb{R}) \) has one component. The same argument for \( X_*(T_{ad}) \), using fundamental coweights in place of coroots, shows that \( T_{ad}(\mathbb{R}) \) is connected. Finally, recall that \((B, T)\) is \( \theta^* \)-stable. The image in \( X_*(T_{ad})/(1 - \theta^*_ad)X_*(T_{ad}) = X_*(T_{ad}\theta^*_ad) \) of the chosen base for \( X_*(T_{ad}) \) is a base for \( X_*(T_{ad}\theta^*_ad) \) since it has the correct cardinality, by Steinberg’s structure theorem. Thus \((T_{ad})\theta^*_ad(\mathbb{R}) \) is connected.

**Proof of Lemma 6.4.** First observe that (ii) follows once we have proved (i). The equation \( \sigma(\delta^*)\delta^{*-1} = (\theta^*−1)v(\sigma) \) from [KS99, Lemma 4.4.A] (see the proof of Lemma 6.1) implies that the image \( v(\sigma)_{\text{ad}} \) of \( v(\sigma) \) in \( T_{ad} \) is an element, in fact a cocycle, in \((T_{ad})\theta_{ad}^*\). Since \((T_{ad})\theta_{sc}^* \) and \((T_{ad})\theta_{ad}^* \) are both connected, the natural projection \( T_{sc} \to T_{ad} \) projects \((T_{sc})\theta_{sc}^* \) onto \((T_{ad})\theta_{ad}^* \), and (ii) follows.

For the proof of (i), it is sufficient to consider the case that the endoscopic group is basic, i.e., attached to the trivial endoscopic data \((G_1, G_1^\vee \times W_\mathbb{R}, 1)\) for the pair \((G, \theta)\), where \( G_1^\vee \) denotes the identity component of the fixed points of \( \theta^\vee \) in \( G^\vee \): if \( H_1 \) is any endoscopic group and \( H = H_1/Z_1 \), then an admissible embedding \( T_H \to T_{\theta^*} \) determines an admissible embedding \( T_{G_1} \to T_{\theta^*} \) (see [KS99, §3.3]), with same data \( g, \delta^*_0 \) attached to the same (strongly \( G \)-regular) element in \( T_{\theta^*}(\mathbb{R}) \).

Assume then that \( H_1 \) is basic. There exists an admissible embedding \( T_H \to T_{\theta^*} \), where \((B, T)\) is a \( \theta^* \)-stable fundamental pair, and thus there exist strongly \( G \)-regular \( T_1 \)-norms; here \( T_H = T_1/Z_1 \). Suppose strongly \( G \)-regular \( \gamma_1 \in T_1(\mathbb{R}) \) is a norm of \( \delta \in G(\mathbb{R}) \). Attach \( g \in G_{sc}^* \) and \( \delta^* \in T \) as usual. Then \( N\delta^* \in T_{\theta^*}(\mathbb{R}) \). Passing to the adjoint form \( G_{sc}^* \) of \( G^* \), we have that \( \delta_{ad}^* \) has image in \((T_{ad})\theta_{ad}^*(\mathbb{R}) \) under \( N_{ad} \). Since \( N_{ad} : T_{ad}(\mathbb{R}) \to (T_{ad})\theta_{ad}^*(\mathbb{R}) \) is surjective (domain and target are connected), we may then find \( \delta^* \in T \) such that \( \sigma(\delta^*)\delta^{*−1} \) is central in \( G^* \) and \( \delta^{**} = \delta^*(1 - \theta^*)T \). Multiplying \( \delta^{**} \) by a suitable central element allows us to replace \((1 - \theta^*)T \) by the image of \((1 - \theta^*_{sc})T_{sc} \). Then multiplying \( g \) by a suitable element of \( T_{sc} \), we obtain a replacement for the pair \( g, \delta^* \) with the desired property (i). Lemma 6.1 shows that the assumption of strongly \( G \)-regularity is unnecessary.

We remove the assumption that \( T \) is fundamental using induction on the split rank of \( T^{\theta^*} \). By Lemma 6.1 we may assume that \( \gamma_0, \delta_0 \) are the elements \( \gamma_2, \delta_2 \) of the proof of (ii) in Lemma 6.6, with attached \( g_2, \delta_2 \). We construct \( g_3, \delta_3^* \) and adjust them using the induction hypothesis, then replace \( g_2, \delta_2 \) accordingly. Then \( \delta_2 = \text{Int}(\delta^*)\delta_2^{-1}(\delta_3^*) \). Recall that \( \sigma(\delta^*)\delta^{*−1} \) is the image in \( G^* \) of the element \((\theta_{sc}^*−1)v_2(\sigma) \) in \( G_{sc}^* \) and \( \sigma(\delta_3^*)\delta_3^{*-1} \) is the image of \((\theta_{sc}^*−1)v_3(\sigma) \).

We claim that we can adjust \( g_2 \) again to arrange that \((\theta_{sc}^*−1)v_2(\sigma) \) and \((\theta_{sc}^*−1)v_3(\sigma) \) are the same central element of \( G_{sc}^* \). This will both complete our
inductive proof of (i) and provide a modification of the hypercycloidal property that is useful for the proof of the main lemma of Section 9.

To justify the claim, we return to Lemma 4.4.A of [KS99] and argue with $G_{sc}$ instead of $G$. We replace $\delta_0$ by $\delta_{sc} \in G_{sc}$ with image $\delta_0$ up to a central element. Then $\sigma(\delta_{sc}) = z_0 \delta_{sc}$, where $z_0$ is central in $G_{sc}$. Passing to a suitable strongly $G_{sc}$-regular element in each case, we find that

\[(\theta_{sc}^* - 1)v_2(\sigma) = \psi_{sc}(z_0)\sigma(\delta_{2,sc}^*)(\delta_{2,sc}^*)^{-1}\]

and

\[(\theta_{sc}^* - 1)v_3(\sigma) = \psi_{sc}(z_0)\sigma(\delta_{3,sc}^*)(\delta_{3,sc}^*)^{-1},\]

where $\delta_{2,sc}^* = g_2 m_{sc}(\delta_{sc})\delta_{sc}^*(g_2)^{-1}$ has image $\delta_2$ up to a central element in $G^*$ and $\delta_{2,sc}^* = g_3 m_{sc}(\delta_{sc})\delta_{sc}^*(g_3)^{-1}$ has image $\delta_3$ up to the same central element in $G^*$. Moreover, $\delta_{2,sc}^* = \text{Int}(t^*)^{-1}(\sigma(\delta_{3,sc}^*)^{-1})$ and $\sigma(\delta_{3,sc}^*)^{-1}$ is central. To prove the claim, we observe that $\text{Int}(t^*)^{-1}(\sigma(\delta_{3,sc}^*)^{-1})$ coincides with $\sigma(\delta_{2,sc}^*)(\delta_{2,sc}^*)^{-1}$ up to an element of $(1 - \theta_{sc}^*)T_{sc}$, so that we may adjust $g_2$ as desired. \[\Box\]

**Definition.** Choose $g$, $\delta_{sc}^*$ satisfying (i), and thus (ii), of Lemma 6.4. Then we will call $(T_1 \to T_{\theta^*}, g)$ toral data at $\gamma_0$.

7. **Application to Theorem 5.1**

We return to the normalized sum of twisted integrals

\[\Phi_1(\gamma_1) = \left| \det(\text{Ad}(\gamma_1) - I)_{b_1/t_1} \right|^{1/2} \sum_{\delta,\theta-\text{conj}} \Delta(\gamma_1, \delta) \ O^{\theta,\infty}(\delta, f)\]

for $\gamma_1$ strongly $G$-regular. This was rewritten in Section 5 as

\[\Delta_I(\gamma_1)\Delta_{II}(\gamma_1) \sum_{\delta,\theta-\text{conj}} \Delta_{III}(\gamma_1, \delta; \gamma, \bar{\gamma}) \Phi^{\theta,\infty}(\delta, f),\]

where the twisted integrals themselves are now normalized, and the terms $\Delta_I$, $\Delta_{II}$, and $\Delta_{III}$ come from the twisted transfer factor $\Delta$.

Fix a maximal torus $T_1$ over $\mathbb{R}$ in $H_1$, a $G$-semiregular element $\gamma_0$ in the Cartan subgroup $T_1(\mathbb{R})$ annihilated by an imaginary root $\alpha_1$, and a Cayley transform $s_1$ for $\alpha_1$.

Our next step is to write $\Phi_1(\gamma_1)$ for strongly $G$-regular $\gamma_1$ in the $\gamma_0$-component of $T_1(\mathbb{R})$ in a way that will be useful both for extending $\Phi_1$ to all $G$-regular elements and for jump analysis around $\gamma_0$.

If $\gamma_0$ is not a $T_1$-norm, then $\Phi_1(\gamma_1) = 0$ for all strongly $G$-regular $\gamma_1$ in the $\gamma_0$-component of $T_1(\mathbb{R})$, and so we define $\Phi_1(\gamma_1) = 0$ also for the remaining $G$-regular elements in the component. Assume then that $\gamma_0$ is a $T_1$-norm of the $\theta$-semisimple element $\delta_0$ of $G(\mathbb{R})$. Let $(T_1 \to T_{\theta^*}, g)$ be toral data at $\gamma_0$. As in Lemma 6.2, we have the element $\delta = \delta_0(Y) = (\exp Y)\delta_0$ with given norm.
\( \gamma_1 = \gamma_1(Y_1) = \gamma_0 \exp(Y_H + Y_3) \) in the \( \gamma_0 \)-component of \( T_1(\mathbb{R}) \). Suppose \( \gamma_1 \) is strongly \( G \)-regular, so that \( \delta \) is strongly \( \theta \)-regular. We fix representatives \( \delta' \) for the \( \theta \)-conjugacy classes in the stable \( \theta \)-conjugacy class of \( \delta \) and then define \( \text{inv}(\delta, \delta') \) and \( \kappa_\delta \) as in the preamble to Theorem 5.1.D of [KS99], which also describes how these two objects are paired. (For more on the definitions, see the proof of Lemma 9.6.) Then by (1) of that theorem, \( \Phi_1(\gamma_1) \) may be rewritten as

\[ \Delta_1(\gamma_1) \Delta_{II}(\gamma_1) \Delta_{III}(\gamma_1, \delta; \gamma, \overline{\gamma}) \sum_{\delta'} \langle \text{inv}(\delta, \delta'), \kappa_\delta \rangle \Phi^{\theta, \varpi}(\delta', f). \]

Suppose, slightly more generally, that \( \delta' \) is stably \( \theta \)-conjugate to strongly \( \theta \)-regular \( \delta = \varepsilon \delta_0 \), where \( \varepsilon \in T^{\delta_0}(\mathbb{R}) \). We may write

\[ \delta' = \delta(w) = w^{-1} \delta \theta(w) = w^{-1} \varepsilon w w^{-1} \delta_0 \theta(w), \]

where \( w \in G(\mathbb{C}) \) (we stress \( \mathbb{C} \) in notation just for this paragraph) and \( \sigma(w)w^{-1} \) lies in \( \text{Cent}_\theta(\delta, G(\mathbb{C})) \). As earlier, let \( T^1 = \text{Cent}(T^{\delta_0}, G) \). Then strong \( \theta \)-regularity implies that \( \text{Cent}_\theta(\delta, G) \) coincides with the group \( T^{\delta_0} \) of fixed points of \( \text{Int}(\delta_0) \circ \theta \) in \( T^1 \). Set

\[ \mathcal{A}_\theta(T^{\delta_0}) = \{ w \in G(\mathbb{C}) : \sigma(w)w^{-1} \in T_{\delta_0}(\mathbb{C}) \}. \]

Then, via the map \( w \to \delta(w) \),

\[ \mathcal{D}_\theta(T^{\delta_0}) = T_{\delta_0}(\mathbb{C}) \setminus \mathcal{A}_\theta(T^{\delta_0}) / G(\mathbb{R}) \]

parametrizes the \( \theta \)-conjugacy classes in the stable \( \theta \)-conjugacy class of \( \delta \).

If now we assume only that \( \delta = \delta_0(Y) \) is \( \theta \)-regular, then by definition (see the remark after Lemma 6.3),

\[ \{ \delta(w) : w \in \mathcal{A}_\theta(T^{\delta_0}) \} \]

is the stable \( \theta \)-conjugacy class of \( \delta \). We will define \( \Phi_1(\gamma_1) \) to be

\[ \left| \det(\text{Ad}(\gamma_1) - I)_{h_1/\iota_1} \right|^{1/2} \sum_w \Delta(\gamma_1, \delta(w)) O^{\theta, \varpi}(\delta(w), f), \]

where \( \sum_w \) indicates summation over a set of representatives \( w \) for \( \mathcal{D}_\theta(T^{\delta_0}) \) and

\[ \Delta(\gamma_1, \delta(w)) = \lim_{\gamma_1 \to \gamma_1} \Delta(\gamma_1^\dagger, \delta^\dagger(w)). \]

In this limit, the variable \( \gamma_1^\dagger = \gamma_1 \exp Y^\dagger \) is a strongly \( G \)-regular element in the \( \gamma_0 \)-component of \( T_1(\mathbb{R}) \). This element \( \gamma_1^\dagger \) is a norm of each (strongly \( \theta \)-regular) element \( \delta^\dagger(w) \), where \( \delta^\dagger = (\exp Y^\dagger) \delta \). Here \( Y^\dagger \leftrightarrow Y_H \), where \( Y^\dagger = Y_H + Y_3 \), as in Lemma 6.2. To see that the limit exists, we just have to recall how the term \( \Delta(\gamma_1^\dagger, \delta^\dagger(w)) \) depends on \( Y^\dagger \). First,

\[ \Delta(\gamma_1^\dagger, \delta^\dagger(w)) = \langle \text{inv}(\delta^\dagger, \delta^\dagger(w)), \kappa_\delta \rangle \Delta(\gamma_1^\dagger, \delta^\dagger). \]
The first term is a constant sign and so can be ignored. The term \( \Delta(\gamma_1^\dagger, \delta^\dagger) \) is a product

\[
\Delta_I(\gamma_1 \exp Y^\dagger)\Delta_{II}(\gamma_1 \exp Y^\dagger)\Delta_{III}(\gamma_1 \exp Y^\dagger, (\exp Y^\dagger)\delta; \tau, \delta)\Delta_{IV}(\gamma_1 \exp Y^\dagger).
\]

The new first term is a constant sign. The term \( \Delta_{II}\Delta_{IV} \) is a quotient of generalized Weyl denominators for \( G \) and \( H_1 \) (see [KS99, §4.3]). It is well defined, smooth, and nonzero on the subset of all \( G \)-regular elements in \( T_1(\mathbb{R}) \). It remains then to examine \( \Delta_{III}(\gamma_1 \exp Y^\dagger, (\exp Y^\dagger)\delta; \tau, \delta) \). A check of definitions shows that it is the product of a constant and a character on \( T^\delta(\mathbb{R}) \) evaluated at \( \exp Y^\dagger \); see the beginning of the proof of Lemma 9.3 where we introduce more detailed notation for an analysis of \( \Delta_{III} \). We conclude then that \( \lim_{\gamma_1^\dagger \to \gamma_1} \Delta(\gamma_1^\dagger, \delta^\dagger(w)) \) is well defined, which completes our (smooth) extension of \( \Phi_1 \) to the full \( G \)-regular set in \( T_1(\mathbb{R}) \).

Let \( w \in \mathfrak{A}(T^\delta_0) \), and write \( w^{-1}\delta \theta(w) \) as \( \delta_0(w) \). Then \( \text{Int}(w^{-1}) : G^\delta_0 \to G^\delta_{\delta_0(w)} \) is an inner twist and \( \delta_0(w) \) is stably conjugate to \( \delta_0 \). The inner type of the group \( G^\delta_{\delta_0(w)} \) of Dynkin type \( A_1 \), either split modulo center or compact modulo center, depends only on the double coset of \( w \) in \( \mathfrak{A}(T^\delta_0) \). We may ignore those \( w \) for which \( G^\delta_{\delta_0(w)} \) is compact modulo center, as they contribute nothing to the final limit formula (see Section 8). We have the following generalization of Lemma 4.2 of [She79a]. Again \( \alpha_0 \) denotes the multiple of \( \alpha_{\text{res}} \) that is a root of \( T^\delta_0 \) in \( G^\delta_{\delta_0} \).

**Lemma 7.1.** If both \( G^\delta_{\delta_0} \) and \( G^\delta_{\delta_0(w)} \) are split modulo center (i.e., both \( \alpha_0 \) and \( \omega_0 \) are noncompact imaginary roots), then there exists \( g \in G(\mathbb{R}) \) such that \( \text{Int}(g) \) maps \( G^\delta_{\delta_0} \) to \( G^\delta_{\delta_0(w)} \) and \( T^\delta_0 \) to \( T^\delta_{\delta_0(w)} \), and \( w^{-1}\alpha_0 = \pm g\alpha_0 \).

**Proof.** We follow the proof of Lemma 4.2 in [She79a]. First, a simple argument with root vectors shows that we can arrange that \( \text{Int}(w^{-1}) : G^\delta_{\delta_0} \to G^\delta_{\delta_0(w)} \) is defined over \( \mathbb{R} \); see the first paragraph of the cited proof. Let \( s \) be the standard Cayley transform in \((G^\delta_{\delta_0})_{sc} = \text{SL}(2)\) relative to the root \( \alpha_0 \) of \( T^\delta_0 \) in \( G^\delta_{\delta_0} \), and set \( T'_{\delta_0} = (T^\delta_0)^s \). We may argue in the untwisted setting with \( w \in \mathfrak{A}(T'_{\delta_0}) \), where \( T'_{\delta} \) is the maximal torus \( \text{Cent}(T^\delta_0, G) \) in \( G \), to choose \( g_1 \) in \( G(\mathbb{R}) \) so that \( \text{Int}(g_1) \) maps \( T^\delta_0 \) to \( w^{-1}T^\delta_0 w \) and acts on the maximal split torus in \( T^\delta_0 \) as \( \text{Int}(w^{-1}) \). Then \( \text{Int}(g_1^{-1}w^{-1}) \) normalizes the derived group of \( G^\delta_{\delta_0} \) (by another argument with root vectors) as well as \( T^\delta_0 \). Then \( \text{Int}(g_1^{-1}w^{-1}) \) normalizes \( G^\delta_{\delta_0} \) itself. Multiplying \( g_1 \) by a suitable element of \( G^\delta_{\delta_0}(\mathbb{R}) \), we obtain \( g \) in \( G(\mathbb{R}) \) such that \( \text{Int}(g^{-1}w^{-1}) \) normalizes both \( G^\delta_{\delta_0} \) and \( T^\delta_0 \). Then \( w^{-1}\alpha_0 \) coincides with \( \pm g\alpha_0 \). \( \square \)

We will need a twisted version of Proposition 4.6 of [She79a] in order to match the elements of \( \mathfrak{D}(T^\delta_0) \) contributing to jumps with the elements of \( \mathfrak{D}(T^\delta_{\delta_0}) \), where \( T'_{\delta_0} = (T^\delta_0)^s \). Assume \( \alpha_0 \) is noncompact and that \( s \) is
standard in \((G^\theta_{\delta_0})_{sc} = \text{SL}(2)\). Let \(w\) be an element of \(\mathcal{D}_\theta(T^{s_0})\) such that \(G^\theta_{\delta_0(w)}\) is split modulo center for some, and hence every, \(w\) representing \(w\). Then, following the last lemma, we may choose \(w\) so that \(w\) normalizes \(G^\theta_{\delta_0}\) and \(T^{s_0}\), and \(w^{-1}\alpha_0 = \pm\alpha_0\). Now consider those \(w\) with representative \(w\) such that \(w^{-1}\alpha_0 = \alpha_0\). Suppose \(w_0\) is an element of \(\text{Cent}_\theta(\delta_0, G)\) normalizing \(T^{s_0}\) for which the action of \(\text{Int}(w_0)\) on \(T^{s_0}\) realizes the Weyl reflection relative to \(\alpha_0\). Then \(w\) and \(ww_0\) represent the same element of \(\mathcal{D}_\theta(T^{s_0})\) if and only if we may choose \(w_0\) in \(G(\mathbb{R})\), i.e., in \(\text{Cent}_\theta(\delta_0, G)(\mathbb{R})\). If that is so, then we say that the Weyl reflection relative to \(\alpha_0\) is realized in \(G(\mathbb{R})\), keeping in mind that this notion depends on the choice of \(\delta_0\). The elements \(w\) of \(\mathcal{D}_\theta(T^{s_0})\) with a representative \(w\) such that \(w^{-1}\alpha_0 = \alpha_0\) are then exactly those \(w\) such that \(G^\theta_{\delta_0(w)}\) is split modulo center for each representative \(w\). We denote this subset of \(\mathcal{D}_\theta(T^{s_0})\) by \(\mathcal{D}_\theta(\alpha_0)\). On the other hand, if the Weyl reflection relative to \(\alpha_0\) is not realized in \(G(\mathbb{R})\), then for each element \(w\) of \(\mathcal{D}_\theta(T^{s_0})\) with representative \(w\) such that \(w^{-1}\alpha_0 = \alpha_0\), there is an element \(w_-\), distinct from \(w\), with representative \(w_- = ww_0\) such that \(w_-^{-1}\alpha_0 = -\alpha_0\). In this case, \(\mathcal{D}_\theta(\alpha_0)\) will denote the set of pairs \(\{w, w_-\}\).

Consider \(w'\) in \(\mathcal{D}_\theta(T^{s_0})\). Again following on from the proof of Lemm 7.1, since \(\alpha_0^s\) is a real root, we may find a representative \(w'\) for \(w'\) such that \(w'\) normalizes both \(T^{s_0}\) and \(G^\theta_{\delta_0}\), and \(w'^{-1}\alpha_0^s = \alpha_0^s\). We can then further arrange that \(w'\) centralizes \((G^\theta_{\delta_0})_{\text{der}}\). Thus \(w = s^{-1}w's = w'\) lies in \(\mathcal{D}_\theta(T^{s_0})\) and \(w^{-1}\alpha_0 = \alpha_0\). Let \(w\) be the class of \(w\) in \(\mathcal{D}_\theta(T^{s_0})\). Then another argument with root vectors shows that \(w' \rightarrow w\) is a well-defined bijective map of \(\mathcal{D}_\theta(T^{s_0})\) to those \(w \in \mathcal{D}_\theta(T^{s_0})\) with representative \(w\) such that \(w^{-1}\alpha_0 = \alpha_0\). This provides us with a bijection of \(\mathcal{D}_\theta(T^{s_0})\) with \(\mathcal{D}_\theta(\alpha_0)\).

Before continuing with the analysis, we finish the proof of Theorem 5.1 for some special cases.

**Lemma 7.2.** All limits in Theorem 5.1 are zero if

(i) \(\gamma_0\) is not a norm, or if

(ii) \(\gamma_0\) is a norm but \(T_1^{s_1}\) is not a norm group for \(G\), or if

(iii) \(\gamma_0\) is a norm, \(T_1^{s_1}\) is a norm group for \(G\), but \(\gamma_0\) is not a \(T_1^{s_1}\)-norm.

**Proof.** For (i), we have only to apply (i) of Lemma 6.6: \(\Phi_1(\gamma_\nu) = 0\) and \(\Phi_1(\gamma_{s_1, \nu}) = 0\) for \(|\nu|\) sufficiently small and nonzero. On the other hand, for (ii) and (iii) we have, in general, only that \(\Phi_1(\gamma_{s_1, \nu}) = 0\) for \(|\nu|\) sufficiently small. Thus it remains to show \(\lim_{\nu \rightarrow 0} \Phi_1(\gamma_\nu) = 0\). By Lemma 6.6, each group \(G^\theta_{\delta_0(w)}(\mathbb{R})\) is compact modulo center and so each unnormalized integral \(O^{\theta, \omega}(\delta(w), f)\) appearing in \(\Phi_1(\gamma_\nu)\) is bounded as \(\nu \rightarrow 0\) (see Section 8). Thus the limit of \(\Phi_1(\gamma_\nu)\) exists and is zero. \(\square\)
For our analysis of the limits in Theorem 5.1, we may now assume that both $T_1$ and $T_1^{s_1}$ are norm groups and that $\gamma_0$ is both a $T_1$-norm and a $T_1^{s_1}$-norm. Recall that we assume also that the root $\alpha_1$ of $T_1$ annihilating $\gamma_0$ is noncompact and that $s_1$ is a Cayley transform in $((H_1)_{\gamma_0})_{sc} = SL(2)$.

We return to the setting established at the end of the proof of Lemma 6.6. We may suppose $\gamma_0$ is both a $T_1^{s_1}$-norm and a $T_1$-norm of an element $\delta_0$ of $G(\mathbb{R})$ for which $G_{\delta_0}^g$ is split modulo center, as there. We have admissible homomorphisms $T_1 \to T^{\gamma_0}, T_1^{s_1} \to T^{\gamma_0^*}$, and inverse Cayley transform $t^*$ in $(G_{sc})_{sc}^{\delta_0}$ which maps $(T^{\gamma_0^*})^0$ to $(T^{\gamma_0^*})^0$, $T^t$ to $T, T^g$ to $T^g$ and completes a commutative diagram with $\text{Int}(s_1)^{-1}: T_1^{s_1} \to T_1$ and $T_1 \to T^{\gamma_0}, T_1^{s_1} \to T^{\gamma_0^*}$. Also $t \in G_{sc}$ defines an inverse Cayley transform in $(G_{sc}^{\delta_0})_{sc}$ for $\alpha_0^*$, where $\alpha_0^*$ is the root of $T^{\gamma_0^*}$ in $G_{\delta_0}^g$ corresponding to $\alpha_1^* = \alpha_1^{s_1}$. Then, with $g_2, g_3$ as at the end of the proof of Lemma 6.6, we choose $g = g_3$ and $g' = g_2$. There is another requirement that will be useful since it makes the limits we consider for $\Delta_I, \Delta_{II}$ in Lemmas 9.3 and 9.5 both equal to one. Namely we insist that if a complex root of $(T^{\gamma_0^*})^0$ is positive in the ordering determined by the toral data and our choice of $\mathbb{R}$-splitting for $(G^{\gamma_0^*})^0$, then its complex conjugate is also positive. That this is possible follows from a familiar argument using a suitable lexicographic ordering of roots for the $\mathbb{R}$-splitting. (Start with toral data for a maximally split torus in $H_1$, identify inverse Cayley transforms needed to reach $(T^{\gamma_0^*})^0$ through $H_1$, adjust the $\mathbb{R}$-splitting accordingly via Cayley transforms from the torus attached to the maximally split torus in $H_1$, and prescribe toral data for $T_1^{s_1}$ using the inverse transforms.)

We call the data of the last paragraph toral descent data at $\gamma_0$.

8. Jump analysis for twisted orbital integrals

The limit formulas for the individual twisted orbital integrals guide our analysis of the transfer factors, and so we write them next. Formulas of this type are well known. We need only to extend the setting and to write the results in a way that fits well with our transforms.

We continue with the toral descent data at $\gamma_0$ from the end of the last section; $\gamma_0$ is both a $T_1^{s_1}$-norm and a $T_1$-norm of an element $\delta_0$ of $G(\mathbb{R})$ for which $G_{\delta_0}^g$ is split modulo center and of Dynkin type $A_1$. Now $s$ will be the Cayley transform $t^{-1}$ in $(G_{\delta_0}^{g^*})_{sc}$. Fix an element of $\mathcal{D}_0(T^{g^*})$. Our choice in the last section of representative $w'$, along with $w$ and $w_0$, ensures that $G_{\delta_0}^g(w) = G_{\delta_0}^g(ww_0) = G_{\delta_0}^g$ and that the points $\delta_0(w), \delta_0(ww_0), \delta_0(w')$ all coincide. We will make a descent from $G(\mathbb{R})$ to $\text{Cent}_0(\delta_0, G)(\mathbb{R})$, then into $G_{\delta_0}^g(\mathbb{R})$, around $\delta_0(w)$. This generalizes the descent used in Section 4 of [She79a] for the untwisted case. Notice that because the twisting character $\tau$ is trivial on both Cartan subgroups $T_{\delta_0}(\mathbb{R}), T^{\delta_0}(\mathbb{R})$ in $G_{\delta_0}^g(\mathbb{R})$ [Kas99, Lemma 4.4.C] (more generally,
is trivial on both $T_{\delta_0}^1(\mathbb{R}), T_{\delta_0}^1(\mathbb{R})$ by [KS99, Th. 5.1.D]), we have that $\varpi$ is trivial on $G_{\delta_0}^0(\mathbb{R})$.

We may write $\alpha_0$ as $r_\alpha a_\alpha$, where $\alpha_1 = ((\alpha^\vee)_{\text{res}})^\vee$ and the coefficient $r_\alpha$ is described in Lemma 6.5. As in [KS99] (see Section 9 also), we use the same $\chi$-data and $\chi$-data for all multiples of $\alpha_{\text{res}}$ and write $\chi, a, \chi', a'$ for data $\chi_{\alpha_{\text{res}}}, a_{\alpha_{\text{res}}}, \chi_{\alpha_{\text{res}}}', a_{\alpha_{\text{res}}}'$.

Assume $\delta$ is a $\theta$-regular element in $T_{\delta_0}^0(\mathbb{R})$. For $\alpha_0$ of type $R_1$, set

$$\Delta_{\alpha_0}(\delta) = \chi\left(\frac{N\alpha(\delta) - 1}{a}\right) |N\alpha(\delta) - 1|^{1/2} |N\alpha(\delta) - 1|^{1/2},$$

which we abbreviate as

$$\chi\left(\frac{N\alpha(\delta) - 1}{a}\right) |N\alpha(\delta) - N\alpha(\delta)^{-1}|.$$

For $\alpha_0$ of type $R_2$ or $R_3$, we include the contribution from the orbits of all multiples of $\alpha_{\text{res}}$ to the numerators of $\Delta_{\nabla}, \Delta_{\nabla'}$:

$$\Delta_{\alpha_0}(\delta) = \chi\left(\frac{N\alpha(\delta) - 1}{a}\right) |N\alpha(\delta) - N\alpha(\delta)^{-1}|.$$

On the other hand, the roots $\pm a_0'$ of $T_{\delta_0}^0(\mathbb{R})$ form two Galois orbits and we include them both. Thus if $\delta'$ is a $\theta$-regular element in $T_{\delta_0}^0(\mathbb{R})$, then we define $\Delta_{\alpha_0'}(\delta')$ as we have $\Delta_{\alpha_0}(\delta)$, but using only the contribution from $+a_0'$ for the absolute value term. Set

$$\Delta_{\pm a_0'}(\delta') = \Delta_{a_0'}(\delta') \Delta_{-a_0'}(\delta')$$

$$= \chi'\left(\frac{N\alpha'(\delta')^{-r} - 1}{a'}\right) |N\alpha'(\delta')^{-r} - 1|^{1/2}$$

$$\chi'(\frac{N\alpha'(\delta')^{-r} - 1}{a'}) |N\alpha'(\delta')^{-r} - 1|^{1/2}$$

$$= \chi'(N\alpha'(\delta')^{-r} |N\alpha'(\delta')^{-r}|^{1/2}) |(N\alpha'(\delta')^{-r} - N\alpha'(\delta')^{-r/2}|,$$

where $r = 1$ if $\alpha_0$ is of type $R_1$ and $r = 2$ if $\alpha_0$ is of type $R_2$ or $R_3$.

For $\nu \in \mathbb{R}$, set $\delta_\nu = \exp(\nu Y(aa_0^\vee)). \delta_0$, where $Y(aa_0^\vee) \in t_{\delta_0}^0(\mathbb{R})$ corresponds under the bijection of Lemma 6.2 to the multiple $aa_0^\vee$ of the coroot $\alpha_0^\vee$ regarded as an element of $t_H(\mathbb{R})$. Then $\delta_\nu$ has as $T_1$-norm the element $\gamma_\nu$ from the statement of Theorem 5.1. Also,

$$\delta_\nu(w) = w^{-1} \delta_\nu \theta(w) = \exp(\nu Y(aa_0^\vee)).w^{-1} \delta_0 \theta(w)$$

since $w a_0 = \alpha_0$ implies that $w^{-1}Y(aa_0^\vee) = Y(aa_0^\vee)$. Again starting with $\delta_0$, define $\delta_{s,\nu}$ with $\gamma_{s,\nu}$ as $T_1'$-norm, and $\delta_{s,\nu}(w')$ similarly. For $|\nu|$ sufficiently small but nonzero, the elements $\delta_\nu(w), \delta_{s,\nu}(w')$ are $\theta$-regular.

Since $s$ is a Cayley transform mapping $T_{\delta_0}^0$ to $T_{\delta_0}^0$ within the group $G_{\delta_0}^0$, we require that the Haar measures on $T_{\delta_0}^0(\mathbb{R})$ and $T_{\delta_0}^0(\mathbb{R})$ are compatible in the
LEMMA 8.1. Let $f \in \mathcal{C}(G(\mathbb{R}), \theta)$. Then for any choice of $\chi, a, \chi_s$ and $a_s$ we have
\[
\lim_{\nu \to 0^+} \Delta_{\alpha_0}(\delta_\nu)O^{\theta, \omega}(\delta_\nu(w), f) - \lim_{\nu \to 0^-} \Delta_{\alpha_0}(\delta_\nu)O^{\theta, \omega}(\delta_\nu(w), f) = d(\alpha_0) \lim_{\nu \to 0} \Delta_{\pm \alpha_0}(\delta_{s, \nu})O^{\theta, \omega}(\delta_{s, \nu}(w'), f),
\]
where $d(\alpha_0) = 2$ if $w_0$ is realized in $G(\mathbb{R})$ in the sense of Section 7, and $d(\alpha_0) = 1$ otherwise.

For the proof, we first replace the version of Harish-Chandra’s compactness principle in Section 4 of [She79a] by the following.

LEMMA 8.2. If $C$ is a compact subset of $G(\mathbb{R})$, then there exist a neighborhood $\mathcal{V}$ of 0 in $g_{\delta_0}(\mathbb{R})$ and a compact subset $\overline{C}$ of $\text{Cent}_\theta(\delta_0, G(\mathbb{R}) \setminus G(\mathbb{R}))$ such that if $g \in G(\mathbb{R})$, $Y \in \mathcal{V}$, and $g^{-1}(\exp Y)\delta_0 \theta(g) \in C$, then $\text{Cent}_\theta(\delta_0, G(\mathbb{R})g) \in \overline{C}$.

Proof. We follow the argument for Theorem 8.1.4.1 of [War72] in our setting, noting the arguments for Proposition 3.1 of [Ren97].

Proof of Lemma 8.1. Notice that the choice of $\chi, a, \chi'$, and $a'$ does not matter, by an argument as in the first step of the proof of Theorem 4.2. In particular, there is no harm in taking $\chi'$ trivial and $a' = 1$.

By a continuity argument (see the appendix), it is enough to consider the case that $f \in C_c^\infty(G(\mathbb{R}), \theta)$. Using Lemma 8.2 with $\delta_0$ replaced by $\delta_0(w)$, we may then apply a variant of Harish-Chandra’s descent argument (specifically, we generalize step by step the arguments of [She79a, §4]) to write the normalized twisted integrals $\Phi^{\theta, \omega}(\delta_\nu(w), f)$ and $\Phi^{\theta, \omega}(\delta_{s, \nu}(w'), f)$ as the normalized ordinary orbital integrals of a function $\phi$ in $C_c^\infty(G(\delta_0)^+)$, evaluated at $\exp(\nu Y(\alpha_0^\sim))$ and $\exp(\nu Y(a'(\alpha_0^\sim)))$, respectively. Here $G(\delta_0)^+$ denotes the identity component of the derived group of $G(\delta_0)(\mathbb{R})$. In the descent we may replace $\delta_\nu(w)$ by an element $\delta = \varepsilon \delta_0$ with $\varepsilon$ sufficiently close to the identity in $\exp \delta_0(\mathbb{R})$ so that $\delta$ is strictly regular. There will be no harm in assuming further that $\delta$ is strongly $\theta$-regular, so that $\text{Cent}_\theta(\delta, G) = T_{\delta_0}$. (Otherwise we use $T_{\delta_0}(\mathbb{R})$ in place of $\text{Cent}_0(\delta, G)(\mathbb{R})$ in the definition of twisted orbital integral.) We may do the same in $T'_{\delta_0}$, replacing $\delta_{s, \nu}(w')$ by an appropriate element $\delta' = \varepsilon' \delta_0$.

The constant $d(\alpha_0)$ appears when we generalize Proposition 4.4 of [She79a]. We have $T_{\delta_0} = Z^\theta T_{\delta_0}$, $T'_{\delta_0} = Z^\theta T_{\delta_0}$, and an argument with root vectors shows that we also have $\text{Cent}_\theta(\delta_0, G) = Z^\theta G^\theta_{\delta_0}$. (Here $Z^\theta$ denotes the $\theta$-invariants in the center of $G$.) Denote the center of $G^\theta_{\delta_0}$ by $Z_{\delta_0}$, and write $G$ for the product $Z_{\delta_0}(\mathbb{R}), G(\delta_0)^+$. Then the three indices, all finite, that concern us are...
and identify the elements \( Y \) by the twist \( \text{Int}(\alpha) \). Rather than write down the evident general principle, we record explicit factors \( \Delta \) and so it remains to check that the (germs at the identity of the) normalizing factors \( \Delta \) realizes the Weyl reflection for \( \alpha_0 \). Either \( g \) or \( w_0g \) lies in \( T_{\delta_0} \), and \( T_{\delta_0} \cap G = T_{\delta_0}(\mathbb{R}) \). Suppose we cannot choose \( w_0 \) in \( \text{Cent}_0(\delta_0, G)(\mathbb{R}) \); i.e., \( w_0 \) is not realized in \( G(\mathbb{R}) \) in the sense of Section 7. Then we conclude that all three indices are the same. Suppose we may choose \( w_0 \) in \( \text{Cent}_0(\delta_0, G)(\mathbb{R}) \). Then the first index is twice the second, and further the first equals the third. Now we can proceed with the descent along the same lines as in Section 4 of [She79a], and the constant \( d(\alpha_0) \) will persist to the final jump formula in the statement of Lemma 8.1.

Let \( Y_0(a) = \alpha \omega \) \( \in t_{\delta_0}(\mathbb{R}) \) and \( Y'_0(a') = a'(\alpha_0)^\vee \in t_{\delta_0}(\mathbb{R}) \). (We could drop \( a' \) from notation since we have assumed \( a' = 1 \).) Then the familiar jump formula at the identity element for the ordinary orbital integrals of \( \phi \) may be rewritten as

\[
\lim_{\nu \to 0^+} \Delta(\exp \nu Y_0(a)) O(\exp \nu Y_0(a), \phi) - \lim_{\nu \to 0^-} \Delta(\exp \nu Y_0(a)) O(\exp \nu Y_0(a), \phi) = \lim_{\nu \to 0} \Delta'(\exp \nu Y'_0(a')) O(\exp \nu Y'_0(a'), \phi),
\]

where \( \Delta(\exp \nu Y_0(a)) \) is given by

\[
\chi \left( \frac{\alpha_0(\exp \nu Y_0(a)) - 1}{a} \right) \left| \left( \alpha_0(\exp \nu Y_0(a))^{1/2} - \alpha_0(\exp \nu Y_0(a))^{-1/2} \right) \right| = \chi \left( \frac{e^{2\nu a} - 1}{a} \right) |e^{\nu a} - e^{-\nu a}|,
\]

and

\[
\Delta'(\exp \nu Y'_0(a')) = |e^{\nu} - e^{-\nu}|.
\]

The vectors \( Y(\alpha \omega \omega) \), \( Y(a'(\alpha_1)^\vee) \) are positive multiples of \( Y_0(a) \), \( Y'_0(a') \), and so it remains to check that the (germs at the identity of the) normalizing factors \( \Delta, \Delta', \Delta_{\alpha_0}, \Delta_{\pm \alpha_0} \) behave correctly under a rescaling of the variable \( \nu \). Rather than write down the evident general principle, we record explicit calculations for each of the three types for \( \alpha \).

Assume first that \( \alpha_0 = \alpha_{res} \), where \( \alpha \) is a root of \( T \) in \( G^* \) of type \( R_1 \). Here, as in Section 6, we have transported the root \( \alpha_0 \) of \( T_{\delta_0} \) in \( G_{\delta_0}^* \) to \( (T^{\theta^*})^0 \) by the twist \( \text{Int}(g) \circ \psi \) from \( G_0 \) to \( G_{\delta_0}^{\theta^*} \), without change in notation. We similarly identify the elements \( Y \) and \( Y^* \) of Lemma 6.2. The coroot of \( \alpha_0 \) is \( N(\alpha)^\vee \), the sum of the coroots in the \( \theta^\vee \)-orbit of \( \alpha^\vee \), so that \( Y_0(a) = aN(\alpha^\vee) \). The root \( \alpha_1 \) of \( T_1 \) in \( H_1 \) has coroot \( (\alpha^\vee)_{res} \). In the Lie algebra \( t_1 \cong t_{\theta^*} = t/(\theta^* - 1)t \), we identify \( (\alpha^\vee)_{res} \) with the coroot of \( \alpha^\vee \in t \). Then \( Y(a\alpha_0^\vee) \) must be the real \( \theta^* \)-invariant \( \frac{1}{l_\alpha} N(\alpha^\vee) = \frac{1}{l_\alpha} Y_0(a) \), where \( l_\alpha \) is the cardinality of the \( \theta^\vee \)-orbit of
\(\alpha^\vee\) (or \(\theta^*\)-orbit of \(\alpha\)). Since \(N\alpha(\delta_0) = N\alpha(\delta_0^*) = 1\) and \(\langle N\alpha, N(\alpha^\vee) \rangle = 2l_\alpha\), we have that

\[
\Delta_{\alpha_0}(\delta_\nu) = \Delta_{\alpha_0}(\exp(\nu Y(a\alpha_1^\vee)))
= \chi \left( \frac{N\alpha(\exp(\nu^{1/l_\alpha} N(\alpha^\vee))) - 1}{a} \right)
\cdot \left| N\alpha \left( \exp\left(\frac{\nu}{l_\alpha} N(\alpha^\vee)\right) \right)^{1/2} - N\alpha \left( \exp\left(\frac{\nu}{l_\alpha} N(\alpha^\vee)\right) \right)^{-1/2} \right|
= \chi \left( \frac{e^{2\nu a} - 1}{a} \right) \left| e^{\nu a} - e^{-\nu a} \right|.
\]

By the same argument, \(Y'(a'(\alpha'_1)^\vee) = \frac{1}{l_\alpha} Y'_0(a')\) and

\[
\Delta_{\pm\alpha_0}(\delta_{s,\nu}) = \Delta_{\pm\alpha_0}(\exp(Y'(a'(\alpha'_1)^\vee))) = \left| e^\nu - e^{-\nu} \right|.
\]

We can now finish the proof for the case \(\alpha_0\) is of type \(R_1\). In the limit formula for the orbital integrals of \(\phi\), replace the variable \(\nu\) throughout by \(\frac{1}{l_\alpha} \nu\). Rewrite the quotient of

\[
\chi \left( \frac{e^{2\nu a} - 1}{a} \right) \left| e^{\nu a} - e^{-\nu a} \right|
\]

by

\[
\chi \left( \frac{e^{2\nu a/l_\alpha} - 1}{a} \right) \left| e^{\nu a/l_\alpha} - e^{-\nu a/l_\alpha} \right|,
\]

as

\[
\chi(e^{\nu a(1-1/l_\alpha)}) \chi \left( \frac{\sin(\nu b)}{\sin(\nu b/l_\alpha)} \right) \frac{\sin(\nu b)}{\sin(\nu b/l_\alpha)},
\]

where \(a = ib\). Since \(\chi\) is trivial on positive real numbers, the second term in this product is trivial, and so the quotient extends continuously at \(\nu = 0\) with nonzero value \(l_\alpha\). The same is true, with same value \(l_\alpha\), for the analogue

\[
\left| e^\nu - e^{-\nu} \right| \left| e^{\nu/l_\alpha} - e^{-\nu/l_\alpha} \right|^{-1} = \frac{\sinh(\nu)}{\sinh(\nu/l_\alpha)}
\]
on the other Cartan subgroup. This allows us to replace \(\Delta(\exp(\frac{1}{l_\alpha} \nu Y_0(a)))\) by \(\Delta_{\alpha_0}(\delta_\nu)\) and \(\Delta'(\exp(\frac{1}{l_\alpha} Y'_0(a')))\) by \(\Delta_{\pm\alpha_0}(\delta_{s,\nu})\) when computing limits, and so we get the desired formula.

Suppose that \(\alpha_0\) is of type \(R_3\), so that \(\alpha_0\) again has coroot \(N(\alpha^\vee)\), and \(Y_0(a) = aN(\alpha^\vee)\). Here the root \(\alpha_1\) of \(T_1\) in \(H_1\) has coroot \((\beta^\vee)_{\text{res}}\) in the notation of Section 1.3 of [KS99], where \((\alpha^\vee)_{\text{res}} = 2(\beta^\vee)_{\text{res}}\) (see Lemma 6.2). Thus

\[
Y^*(a\alpha_1^\vee) = \frac{a}{l_\beta} N(\beta^\vee) = \frac{a}{2l_\alpha} N(\alpha^\vee) = \frac{1}{2l_\alpha} Y_0(a).
\]

Again

\[
\Delta_{\alpha_0}(\delta_\nu) = \Delta_{\alpha_0}(\exp(\nu Y(a\alpha_1^\vee))),
\]
since $N\alpha(\delta_0)^2 = (-1)^2 = 1$. Also, $\langle N\alpha, N(\alpha^\vee) \rangle = 2l_\alpha$ and so we again get the formula

$$\Delta_{\alpha_0}(\delta_\nu) = \chi \left( \frac{e^{2\nu a} - 1}{a} \right) \left| e^{\nu a} - e^{-\nu a} \right|.$$ 

After the substitution of $\frac{1}{2 l_\alpha} \nu$ for $\nu$, we have to examine the quotient of

$$\chi \left( \frac{e^{\nu a / l_\alpha} - 1}{a} \right) \left| e^{\nu a / 2l_\alpha} - e^{-\nu a / 2l_\alpha} \right|,$$

by

$$\chi \left( \frac{e^{\nu a / l_\beta} - 1}{a} \right) \left| e^{\nu a / 2l_\beta} - e^{-\nu a / 2l_\beta} \right|,$$

and we may proceed as for $R_1$.

Suppose that $\alpha_0$ is of type $R_2$. In keeping with the notation of the last paragraph, we write the coroot of $\alpha_0$ as $2N(\beta^\vee)$ and $Y_0(a) = 2aN(\beta^\vee)$. Now the coroot $\alpha_1^\vee$ may be either $(\beta^\vee)_{\text{res}}$ or $(\alpha^\vee)_{\text{res}} = 2(\beta^\vee)_{\text{res}}$. Suppose $\alpha_1^\vee = (\beta^\vee)_{\text{res}}$. Then

$$Y^*(a\alpha_1^\vee) = \frac{a}{l_\beta}N(\beta^\vee) = \frac{1}{2l_\beta}Y_0(a).$$

Also,

$$\langle N\beta, N(\beta^\vee) \rangle = 2l_\alpha = l_\beta,$$

so that

$$\Delta_{\alpha_0}(\delta_\nu) = \chi \left( \frac{e^{2\nu a} - 1}{a} \right) \left| e^{\nu a} - e^{-\nu a} \right|.$$ 

Suppose $\alpha_1^\vee = (\alpha^\vee)_{\text{res}}$. Then

$$Y^*(a\alpha_1^\vee) = \frac{a}{l_\alpha}N(\alpha^\vee) = \frac{1}{2l_\alpha}Y_0(a) = \frac{1}{l_\beta}Y_0(a),$$

and once again we finish the argument the same way. □

For any $w \in \mathcal{A}_0(T^{\delta_0})$, we may also do a similar descent (i.e., find $\phi$ as in the proof above) around $\delta_0(w)$ in $\text{Cent}(\delta_0(w), G)(\mathbb{R})$. If $\text{Cent}(\delta_0(w), G)(\mathbb{R})$ is compact modulo center, then we conclude that $O^{R,\varpi}(\delta_\nu(w), f)$, like the ordinary orbital integral for $\phi$, is bounded as $\nu \to 0$ and so contributes nothing to the jump formula for $\Phi_1$. This remark also applies to the proof of Lemma 7.2 for the setting where every $O^{R,\varpi}(\delta_\nu(w), f)$ is of this type.
9. Twisted transfer factors

We now examine the various terms $\Delta_I, \ldots, \Delta_{IV}$ of the twisted transfer factor $\Delta(\gamma_1, \delta)$ in the setting of toral descent data at $\gamma_0$ (last paragraph of Section 7). For the relative analysis, we have three associated Cayley transforms. First, there is $s_1 : T_1 \to T^{s_1}_1$ associated with the root $\alpha_1$ in $H_1$. Second, there is $t^{s-1} = s^* : T \to T'$ in $(G^{\theta^*})^0$ associated with the least positive multiple of $\alpha_{res}$ that is a root, and, finally, there is $t^{-1} = s : T^{\delta_0} \to T'^{\delta_0}$ for the root $\alpha_0$ in $G^{\theta^*}_{\delta_0}$. Details of the construction of the terms $\Delta_I$ and $\Delta_{III}$ will be included where they are used in proofs. There is a last ingredient for our setting, a twisted analogue of the $s$-compatible data sets of Sections 5–7. The results for $\Delta_{II}$ and $\Delta_{IV}$ then follow quickly (Corollary 9.2), while the analysis for $\Delta_I$ and $\Delta_{III}$ takes longer. The proof of the main lemma, Lemma 9.3, will consist of several steps to remove parts (which we show to be trivial) of a particular $\Delta_{III}$ term until we arrive finally at a term we can compute explicitly and also show to be trivial.

We choose $a$-data and $\chi$-data following Section 1.3 of [KS99]. These are data for the system of restricted roots $\beta_{res}$ of $T$ in $G^\ast$. We use the same pair $a_{\beta_{res}}, \chi_{\beta_{res}}$ for any positive multiple of $\beta_{res}$ that is also a restricted root and the same data for coroots of the restrictions and for the restrictions of coroots

$$a_{\beta_{res}} = a(\beta_{res}) \chi_{\beta_{res}}$$

and

$$\chi_{\beta_{res}} = \chi(\beta_{res}) \chi_{\beta_{res}}.$$ 

This provides us then with data for the roots and coroots of $T_1$ in $H_1$. We make the same choices for the torus $T'$ and define $s^*$-compatibility for the twisted data set $\{a_{\beta_{res}}, \chi_{\beta_{res}}\}, \{a_{\beta_{res}}', \chi_{\beta_{res}}'\}$ as in Section 3. Our constructions ensure that $s^*$-compatible data (which we also call $s$-compatible) provide data for $T_1, T^{s_1}_1$ that are $s_1$-compatible.

Following p. 36 of [KS99], we write $\Delta_{II}$ in quotient form

$$\Delta_{II} = \Delta_{II}^{num} / \Delta_{II}^{denom},$$

where $\Delta_{II}^{num}$ is a term attached to $(G, \theta)$ and $\Delta_{II}^{denom}$ is from standard endoscopy for the group $H_1$. We now prefer to index the contributions to $\Delta_{II}^{num}$ by the orbits $O$ of reduced restricted roots $\alpha_{res}$. Thus the formulas of p. 36 of [KS99] yield

$$\Delta_{II}^{num}(\gamma_1, \delta) = \prod O \chi_{\alpha_{res}} \left( \frac{N(\alpha(\delta^s))^r - 1}{a_{\alpha_{res}}} \right),$$

where $\alpha_{res}$ represents $O$, and $r = 1$ or $r = 2$ according as $\alpha_{res}$ is of type $R_1$ or of type $R_2$. 
Remark 1. Waldspurger [Wal09] has pointed out that a correction is needed in the definition of twisted transfer factors in the nonarchimedean case and that it can be made by the insertion of 2 in certain contributions to $\Delta_{II}$ when the system of restricted roots $\alpha_{\text{res}}$ is not reduced. This has no effect in our present archimedean case; see [KS12, §1] for details. An alternate way of making the correction, which involves $\Delta_I$ instead and makes sense in all characteristics, is presented in [KS12]. It also has no effect on the definitions in the archimedean case [KS12, Prop. 3.5.2].

Remark 2. First we observe an error on p. 137 of [KS99] pointed out to us by Waldspurger. The exponent $-1$ in the formula (A.3.13) does not belong there. We emphasize that by the term Langlands’ pairing in the statement of (A.3.13) we mean the pairing from [Lan97]. The source of this error is on p. 131, where what is described as the Langlands map is the reciprocal of that defined in [Lan97]. To be explicit in the case at hand, if $T$ is a torus defined over $\mathbb{R}$, then the isomorphism $H_1(\mathbb{C}^\times, X_*(T)) \to X_*(T) \otimes \mathbb{C}^\times = T(\mathbb{C})$ defined in the middle of p. 131 of [KS99] has an exponent $-1$ not present in the isomorphism defined in [Lan97]. (An explicit formula is found on p. 243 of [Lan97] after the first commutative diagram.) We resolve this by inverting the formula for the pairing in (A.3.9). Then the formula (A.3.13) is true as stated in [KS99]. Now, in principle, we should insert an exponent $-1$ in the formula (A.3.14) involving Tate-Nakayama duality, but here in the archimedean case the term is simply a sign, and so we may use the formula as stated in [KS99]. Our resolution agrees with that suggested to us by Waldspurger for the general case; i.e., our $\Delta$ coincides with the term $\Delta'$ of (5.4.1) in [KS12]. It also gives the correct shift in infinitesimal character for Langlands functoriality of the dual spectral transfer [She11].

Returning now to our analysis of the various terms $\Delta_I, \ldots, \Delta_{IV}$, we observe the following generalization of Lemma 4.1.

**Lemma 9.1.** For any $s$-compatible twisted data set $\{a_{\beta_{\text{res}}}, \{\chi_{\beta_{\text{res}}}, \{a_{\beta_{\text{res}}}' \}}$, $\{\chi_{\beta_{\text{res}}}' \}$ we have

$$\prod_{O} \chi_{\beta_{\text{res}}} \left( \frac{N\beta(\delta_0)^r - 1}{a_{\beta_{\text{res}}}} \right) = \prod_{O'} \chi_{\beta_{\text{res}}} \left( \frac{N\beta'(\delta_0)^r - 1}{a_{\beta_{\text{res}}}'} \right).$$

On the left, the product is over all Galois orbits $O$ of reduced restricted roots for $T^{b_0}$ (i.e., of types $R_1$ or $R_2$) except those containing a multiple of $\alpha_0$. Each term is independent of the choice of representative $\beta_{\text{res}}$ for $O$; $r = 1$ if $\beta_{\text{res}}$ is of type $R_1$ and $r = 2$ if $\beta_{\text{res}}$ is of type $R_2$. The right side is defined analogously, using all Galois orbits $O'$ of reduced restricted roots for $T^{b_0}$ except those containing a multiple of $\alpha_0'$. For the precise meaning of $N\beta(\delta_0)$, see the remark after Lemma 6.5.
Proof. We match contributions to each side of the formula orbit by orbit as in the proof of Lemma 4.1.

Because we will eventually consider derivatives of the transforms $\Psi_{a,\lambda}$ and $\Psi_{a',\lambda'}$, we use the variables $\delta_0(Y), \gamma_0(Y_1)$, etc. from Lemma 6.2 in our limit formulas for terms of the transfer factors. Each of $\Delta_{II}$ and $\Delta_{IV}$ is defined as a quotient of a term associated with $G$ and a term associated with $H_1$ (Sections 4.3 and 4.5 of [KS99]). Each denominator cancels with an identical term in one of the transforms $\Psi_{a,\lambda}$ and $\Psi_{a',\lambda'}$ of Theorem 5.1. Denote the numerators as $\Delta_{III,\text{num}}$ and $\Delta_{IV,\text{num}}$. These numerators contribute the factors $\Delta_{\alpha_0}, \Delta_{\pm\alpha'_0}$ from the orbits in $Q_{\alpha_0}, Q_{\alpha'_0}$ for the twisted transforms in the jump formulas of the last section, and so these terms will also be removed. In the case of $\Delta_{II}$, what remains is each side of the equation in Lemma 9.1. There is a similar assertion for $\Delta_{IV}$.

Corollary 9.2. For an $s$-compatible data set and toral descent data at $\gamma_0$, we have

$$\lim_{Y_1 \to 0} \Delta_{III}(\gamma_0(Y_1), \delta_0(Y); \gamma_0(Y_1'), \delta_0(Y')) = 1.$$  

Lemma 9.3 (Main lemma). For an $s$-compatible data set and toral descent data at $\gamma_0$, we have

$$\lim_{Y_1, Y'_1 \to 0} \Delta_{III}(\gamma_0(Y_1), \delta_0(Y); \gamma_0(Y'_1), \delta_0(Y'_1)) = 1.$$  

Transitivity of the relative transfer factor (Lemma 5.1.A of [KS99]) then implies immediately the following about the terms of type $\Delta_{III}$ which appear in the limit formulas of Theorem 5.1 and Lemmas 10.1 and 10.2.

Corollary 9.4. In the same setting, we have

$$\lim_{Y_1 \to 0} \Delta_{III}(\gamma_0(Y_1), \delta_0(Y); \gamma_0(Y'_1), \delta_0(Y'_1)) = 1.$$  

Proof of Lemma 9.3. We start by showing that

$$\Delta_{III}(\gamma_0(Y_1), \delta_0(Y); \gamma_0(Y'_1), \delta_0(Y')),$$

defined as the term $\langle V_1, A_1 \rangle$ on p. 43 of [KS99], is the product of a term independent of $Y_1, Y'_1$, which we will denote

$$\Delta_{III}(\gamma_0, \delta_0; T_1, T'_1),$$

and a term which has limit 1 as $Y_1, Y'_1$ approach 0. A longer argument will then show that

$$\Delta_{III}(\gamma_0, \delta_0; T_1, T'_1) = 1.$$
Recall Remark 2 earlier in this section: the pairing $\langle \cdot, \cdot \rangle$ is now defined by
the reciprocal of the formula displayed on p. 135 of [KS99]. First we factor $V_1$ as $V_0.V(Y)$.

The tori $U, S$, and $S_1$ are attached to $T, T'$ in Section 4.4 of [KS99]. Notice that our $T_1, T'_1$ are labeled $T_{H_1}, T'_{H_1}$ there. The element $V_1$ belongs to the hypercohomology group denoted $H^1(\Gamma, U \rightarrow S_1)$. It is the class of the pair $(V, D_1)$, where $V = V(\sigma)$ is a Galois 1-cocycle in $U$ and $D_1$ is an element in $S_1$, and the hypercocycle identity $(1 - \theta)V = \sigma(D_1)D_1^{-1}$ is satisfied. We have defined 1-cochains $v(\sigma), v'(\sigma)$ in Section 6. The pair $(v(\sigma)^{-1}, v'(\sigma))$ lies in $T_{sc} \times T'_{sc}$. Its image under the projection to

$$U = T_{sc} \times T'_{sc}/\{(z^{-1}, z) : z \in Z_{sc}\}$$

is, by definition, $V$. (Our modification at the end of Section 6 does not affect $V$.) To describe $D_1$ we start with the elements $\delta^0_0(Y), (\delta^0_0)'(Y')$ of $T, T'$ (Lemma 6.2). To resolve a notational conflict with [KS99], we write the pull-back torus $T_1$ of $V_0$.

Remark 2 earlier in this section: the pairing $\langle \cdot, \cdot \rangle$ is now defined by the reciprocal of the formula displayed on p. 135 of [KS99]. First we factor $V_1$ as $V_0.V(Y)$.

The tori $U, S$, and $S_1$ are attached to $T, T'$ in Section 4.4 of [KS99]. Notice that our $T_1, T'_1$ are labeled $T_{H_1}, T'_{H_1}$ there. The element $V_1$ belongs to the hypercohomology group denoted $H^1(\Gamma, U \rightarrow S_1)$. It is the class of the pair $(V, D_1)$, where $V = V(\sigma)$ is a Galois 1-cocycle in $U$ and $D_1$ is an element in $S_1$, and the hypercocycle identity $(1 - \theta)V = \sigma(D_1)D_1^{-1}$ is satisfied. We have defined 1-cochains $v(\sigma), v'(\sigma)$ in Section 6. The pair $(v(\sigma)^{-1}, v'(\sigma))$ lies in $T_{sc} \times T'_{sc}$. Its image under the projection to

$$U = T_{sc} \times T'_{sc}/\{(z^{-1}, z) : z \in Z_{sc}\}$$

is, by definition, $V$. (Our modification at the end of Section 6 does not affect $V$.) To describe $D_1$ we start with the elements $\delta^0_0(Y), (\delta^0_0)'(Y')$ of $T, T'$ (Lemma 6.2). To resolve a notational conflict with [KS99], we write the pull-back torus $T_1$ of $V_0$ as $T_2$. Then $(\delta^0_0(Y), \gamma_0(Y_1))$ lies in $T_2$ and $((\delta^0_0)'(Y'), \gamma_0(Y'_1))$ lies in $T'_2$. The element

$$((\delta^0_0(Y), \gamma_0(Y_1))^{-1}, ((\delta^0_0)'(Y'), \gamma_0(Y'_1)))$$

of $T_2 \times T'_2$ factors as

$$((\delta^0_0, \gamma_0)^{-1}, ((\delta^0_0)'(Y'), \gamma_0(Y'_1)))(\exp Y^*, \exp Y_1)^{-1}, (\exp Y'^*, \exp Y'_1)).$$

This factoring persists for images in the quotient $S_1$ (defined on p. 42 of [KS99]), and we write the factoring in $S_1$ as $D_1 = D_0.D(Y_1, Y'_1)$. Because $Y^*, Y', \gamma_0, \gamma_1$, etc., lie in the real Lie algebras of the relevant tori, we also have a factoring of hypercocycles:

$$(V, D_1) = (V, D_0).(1, D(Y_1, Y'_1)).$$

Then $V_0$ will denote the class of $(V, D_0)$ and $V(Y_1, Y'_1)$ will denote the class of

$$(1, D(Y_1, Y'_1)).$$

Define

$$\Delta_{III}(\gamma_0, \delta_0; T_1, T'_1) = \langle V_0, A_1 \rangle,$$

so that

$$\langle V_1, A_1 \rangle = \Delta_{III}(\gamma_0, \delta_0; T_1, T'_1) \cdot \langle V(Y_1, Y'_1), A_1 \rangle.$$

To see that the complementary term $\langle V(Y_1, Y'_1), A_1 \rangle$ has limit 1 as $Y_1, Y'_1$ approach 0, we recall that the element $A_1$ in the hypercohomology group $H^1(W_{\mathbb{R}}, S_1^\vee \rightarrow U^\vee)$ is represented by the pair $(A^{-1}, s_U)$ specified on p. 45 of [KS99]. In particular, $A$ is a 1-cocycle of $W_{\mathbb{R}}$ in $S_1^\vee$. The pairing for hypercohomology is compatible with the Langlands parametrization of characters on $S_1(\mathbb{R})$ ([KS99, A.3.13], as corrected in Remark 2). This allows us to compute

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\langle V(Y_1, Y'_1), A_1 \rangle \) as the value of the character attached to the class of \( A^{-1} \) in \( H^1(W_\mathbb{R}, S_1^\vee) \) on the image \( D(Y_1, Y'_1) \) of
\[
((\exp Y^*, \exp Y_1), (\exp Y^{*'}, \exp Y'_1))
\]
in \( S_1(\mathbb{R}) \). The limit assertion is now immediate.

Thus it remains to show that \( \langle V_0, A_1 \rangle = 1 \). We factor each of \( (V, D_0) \) and \( (A^{-1}, s_U) \) further, and so reduce to calculations with familiar pairings in cohomology.

Recall from the end of Section 7 that we have arranged that the 1-cochains \( v(\sigma), v'(\sigma) \) are such that \( (\theta^* - 1)v(\sigma) \) and \( (\theta^* - 1)v'(\sigma) \) are the same central element, so that \( V(\sigma) \) is \( \theta^* \)-invariant. Thus \( V \) is a 1-cocycle in \( U^{\theta^*} \). We then have (or may check directly) that \( D_0 \in S_1(\mathbb{R}) \), so that \( (V, D_0) \) factors as \( (V, 1).(1, D_0) \). Turning to the dual side, we have from the hypercocycle equation that the element \( s_U \) determines a \( \Gamma \)-invariant element in \( U_\theta^\vee = U^\vee/(1 - \theta^\vee)U^\vee \), and hence an element \( s_{U,\theta} \) in \( \pi_0((U_\theta^\vee)^\Gamma) \). The group \( U^{\theta^*} \) is a torus since the usual isomorphism of \( U \) with \( T_{sc} \times T'_{ad} \) (see p. 38 of [KS99]) is \( \theta^* \)-equivariant and the invariants for each factor in the product torus are connected. The dual of \( U^{\theta^*} \) is \( U_\theta' \). Write \( \langle V, s_{U,\theta} \rangle \) for the Tate-Nakayama pairing of the class of \( V \) in \( H^1(\Gamma, U^{\theta^*}) \) with \( s_{U,\theta} \in \pi_0((U_\theta^\vee)^\Gamma) \) and \( \Lambda \) for the character on \( S_1(\mathbb{R}) \) attached to \( A^{-1} \) by the Langlands correspondence. Then we may compute \( \langle V_0, A_1 \rangle \) as the product \( \langle V, s_{U,\theta} \rangle .\Lambda(D_0) \) (see p. 135 of [KS99]). We check now that each term in this product equals 1.

The image \( v'_{ad}(\sigma) \) of the cochain \( v'(\sigma) \) in \( T'_{ad} \) is a cocycle in the torus \( (T'_{ad})^{\theta_{ad}} \). As usual, we identify the cocharacters of this torus with the \( \theta^* \)-invariant coweights of \( T'_{ad} \). Under the Tate-Nakayama isomorphism
\[
H^{-1}(\Gamma, X_*(T'_{ad})^{\theta_{ad}}) \to H^1(\Gamma, (T'_{ad})^{\theta_{ad}}),
\]
v'_{ad}(\sigma) is cohomologous to the cup product of the fundamental 2-cocycle for \( \mathbb{C}/\mathbb{R} \) with a \( \theta^* \)-invariant coweight \( x'_{cw} \) for \( T'_{ad} \) such that \( \sigma x'_{cw} = -x'_{cw} \), i.e., to \( (-1)^x_{cw} \). Write
\[
v'_{ad}(\sigma) = (-1)^x_{cw} (\sigma(t')t'^{-1})_{ad},
\]
where \( t' \) lies in the torus \( (T'_{sc})^{\theta_{sc}} \). Extend the root \( \alpha_0' \) trivially to \( Z_{sc} \). Then our assumptions on \( g' \) ensure that
\[
\alpha_0'(v'(\sigma)) = \alpha_0'(v'_{ad}(\sigma)) = 1.
\]
Thus \( \alpha_0'(\sigma(t')t'^{-1}) = 1 \). Apply the inverse Cayley transform \( t^* \) to \( x'_{cw} \) to obtain a \( \theta^* \)-invariant coweight \( x_{cw} \) for \( T_{ad} \). Then \( \sigma x_{cw} = -x_{cw} \) and a calculation shows that
\[
v_{ad}(\sigma) = (-1)^{x_{cw}}((-1)^{x_{cw}})(\sigma(t'')t''^{-1})_{ad},
\]
where \( \epsilon \in \{0, 1\} \) and \( t'' \in (T'_{sc})^{\theta_{sc}} \). To recall the characters and cocharacters of \( U \), we use \( t^* \) to identify \( T' \) with \( T \) over \( \mathbb{C} \). The characters may be identified as
pairs $(\lambda, \mu)$, where each of $\lambda$ and $\mu$ is a weight of $T_{\text{ad}}$ and $\lambda - \mu$ is an integral combination of roots, while the cocharacters may be identified as pairs $(\lambda^\vee, \mu^\vee)$ of coweights such that $\lambda^\vee + \mu^\vee$ is an integral combination of coroots. The canonical pairing is

$$\langle (\lambda, \mu), (\lambda^\vee, \mu^\vee) \rangle = \langle \lambda - \mu, \lambda^\vee \rangle + \langle \mu, \lambda^\vee + \mu^\vee \rangle.$$ 

Set

$$x = (-x_{cw} - \epsilon_0^\vee, x_{cw}).$$

(Recall $x'_{cw}$ has now been identified with $x_{cw}$.) Then $x$ lies in $X_*(U^\theta)$, $\sigma x = -x$, and, by evaluating characters on both sides of the following, we see that $(-1)^x = V(\sigma) \cdot u^{-1}$, where $u$ is the image in $U^\theta$ of $(t'', t')^{-1}$. Thus $\sigma \rightarrow (-1)^x$ is cohomologous to $V$.

We may now compute $\langle (V, \mathfrak{g}_{U', \theta}) \rangle$ by evaluating $x$, as character on $(U^\theta)^\vee = U_0'$, at the element $\mathfrak{g}_{U', \theta}$. In the notation of p. 39 of [KS99] where $\mathfrak{g}_T$ is defined, we have arranged that $\mathfrak{g}_T = \mathfrak{g}_T'$, so that to show $\langle (V, \mathfrak{g}_{U', \theta}) \rangle = 1$, it is enough to show that $\alpha_0^\vee(\mathfrak{g}_T) = 1$; i.e., $N(\alpha^\vee)(\mathfrak{g}_T) = 1$ if $\alpha_0$ is of type $R_1$ or $R_3$, or $N(\alpha^\vee)(\mathfrak{g}_T)^2 = 1$ if $\alpha_0$ is of type $R_2$. But if $\alpha_0$ is of type $R_1$ or $R_3$, then the corresponding root $\alpha_1$ of $H_1$ is of type $R_1$ or $R_2$ only, so that $N(\alpha^\vee)(\mathfrak{g}_T) = 1$, as desired. If $\alpha_0$ is of type $R_2$, then the corresponding root $\alpha_1$ is of type $R_2$ or $R_3$, and $N(\alpha^\vee)(\mathfrak{g}_T) = \pm 1$ accordingly. Since we need only $N(\alpha^\vee)(\mathfrak{g}_T)^2 = 1$, we are done. This remark, namely that $\alpha_0^\vee(\mathfrak{g}_T) = 1$, will be useful again. Also, a partial converse result (see the proof of Lemma 11.1) provides a crucial cancellation in the final steps of our proof of the main theorem.

It remains then to show that $\Lambda(D_0) = 1$. Here $s$-compatibility of the $\chi$-data plays a key role, along with an extension of the comparison arguments of Section 4 of [LS90] already used in the definition of $A$ in Section 4 of [KS99]. Our (second) argument for Lemma 9.5 below will have a similar structure, using the first combination of comparison from [LS90] in place of the second.

The element $D_0$ of $S_1(\mathbb{R})$ is the image of $((\delta_0^\vee, \gamma_0)^{-1}, ((\delta_0^\vee)'', \gamma_0))$ under $T_2 \times T_2' \rightarrow S_1$. As before, we will use $t^*$ to identify $T'$ with $T$, and then $T'_2$ with $T_2$, over $\mathbb{C}$. The element $((\delta_0^\vee)', \gamma_0)$ is thus identified with $(\delta_0^\vee, \gamma_0)$. As on p. 42 of [KS99], we identify $S_1$ as $T_2' \times T_{\text{ad}}$ ($T_2'$ is labelled $T_1$ in the reference) and then as $T_2 \times T_{\text{ad}}$. The Galois action on the first component is the transport $\sigma'$ of that on $T_2'$, while on the second component we use the twisted action

$$(1, t_{\text{ad}}) \rightarrow (\psi_{\tau_0}(\sigma'(t_{\text{ad}})), \sigma(t_{\text{ad}})).$$

Here $\psi_{\tau_0} : T_{\text{ad}} \rightarrow T_2$ is defined as follows. Pick $t_2 \in T_2$ in the inverse image of $t_{\text{ad}}$ under the surjection $T_2 \rightarrow T \rightarrow T_{\text{ad}}$. Then $\psi_{\tau_0}(t_{\text{ad}}) = w_0(t_2)t_2^{-1}$ is independent of the choice for $t_2$. The chosen Galois action makes

$$T_2 \times T_2' \rightarrow S_1 \rightarrow T_2 \times T_{\text{ad}} : (t_2, t_2') \rightarrow (t_2t_2', (t_2)_{\text{ad}})$$
defined over $\mathbb{R}$. Write $\delta_{ad}$ for the image of $((\delta_0^*, \gamma_0))^{-1}$ in $T_{ad}$. Then $\delta_{ad}$ is fixed by $\sigma$ and $\sigma'$ (recall our assumptions on $\delta_0^*$, $\gamma_0^*$), and $\psi_{w_0}(\delta_{ad}) = 1$, also because of our assumptions on $\delta_0^*$, $\gamma_0^*$. Notice that $D_0 \in S_1(\mathbb{R})$ is identified with $(1, \delta_{ad}) \in (T_2 \times T_{ad})(\mathbb{R})$.

As in [KS99], we identify $S_1^\vee$ as $T_2^\vee \times T_{sc}^\vee$, with Galois action

$$(t_2, t_{sc}) \mapsto (\sigma'(t_2), \varphi_{w_0}(\sigma'(t_2))\sigma(t_{sc})).$$

(Recall we have chosen $T_2^\vee$ rather than $T_2$ to be the torus $T_1$ in [KS99].) Here $\varphi_{w_0} : T_2^\vee \to T_{sc}^\vee$ is the dual of $\psi_{w_0}$. (This is the variant of the definition of $\alpha(w_0)$ in [KS99] needed when $U$ is replaced by $S_1$, and we will recall how to compute it when needed below.) The 1-cocycle $A(w)$ of $W_\mathbb{R}$ in $S_1^\vee$ is constructed as the element $(a_{T_2^\vee}(w), x_{sc}(w))$ of $T_2^\vee \times T_{sc}^\vee$, where $x_{sc}(w)$ is a product

$$\tau(w_0, \sigma'). \hat{b}(w_0)^{-1}. w_0(c'(w)). c(w)^{-1}. \varphi_{w_0}(a_{T_2^\vee}(w)).$$

To begin examining these terms, we observe that we may replace the cocycles $a_{T_2^\vee}(w), a_{T_2'}(w)$ by cocycles $a_-(w), a'_-(w)$ for which $\varphi_{w_0}(a'_-(w)) = 1$. Then $A(w)$ will be replaced by

$$A_-(w) = (a'_-(w), \hat{\tau}(w_0, \sigma'). \hat{b}(w_0)^{-1}. w_0(c'_-(w)). c_-(w)^{-1}),$$

and there is now no twist in the Galois action on the first component. This ensures that the second component is a cocycle for the action by $\sigma$. Our strategy then will be to examine that cocycle and see that the attached character on $T_{ad}(\mathbb{R})$ takes the value 1 at $\delta_{ad}$, which is sufficient to complete the proof of the lemma.

To define $a_-(w), a'_-(w)$, it is more convenient to view $S_1^\vee$ as a subtorus of $T_2^\vee \times T_{sc}^\vee$, with Galois actions $\sigma', \sigma$ on the first and second components respectively. The cocycle $A(w) = (a_{T_2^\vee}(w), a_{T_2'}(w))$ lies in $S_1^\vee$. By construction, $T^\vee \times T^\vee$ embeds in $T_2^\vee \times T_2'$, and $S_1^\vee$ contains the image of the standard homomorphism $T_{sc}^\vee \times T_{sc}^\vee \to T^\vee \times T^\vee$. Consider a cocycle in $S_1^\vee$ which is the image of a cocycle $(a'_+(w), a_+(w))$ in $T_{sc}^\vee \times T_{sc}^\vee$. We will write this image also as $(a'_+(w), a_+(w))$. To evaluate the corresponding character on $S_1(\mathbb{R})$ under the Langlands correspondence on the element $D_0$ we may, by functoriality of the correspondence, evaluate at $(\delta_{ad}, \delta_{ad})$ the character on $T_{ad}(\mathbb{R}) \times T_{ad}(\mathbb{R}) (\sigma'$ is the action for the first component, $\sigma$ for the second) attached to $(a'_+(w), a_+(w))$ as cocycle in $T_{sc}^\vee \times T_{sc}^\vee$. We will choose $(a'_+(w), a_+(w))$ so that the resulting value is 1, and thus $\Lambda(D_0)$ is unchanged when we divide $A(w)$ by $(a'_+(w), a_+(w))$. The cocycles $a'_+(w), a_+(w)$ will come from $c'(w), c(w)$.

The cochain $c'(w)$ is defined as a quotient $r'_1(w)/r'_2(w)$ of terms from constructions in Section 2.5 of [LS87]. First, $r'_1(w)$ is the term $r_p(w)$ for the group $G^\vee_\sigma = ((G^\vee)^0)^0$, Galois action $\sigma'$ and gauge $p$ associated to our choice
of positive roots (that determined by our fixed $\Gamma$-splitting of $G^\vee$ preserved by $\theta^\vee$ and our choice of toral data). Then

$$r'_i(w) = s_{p/p_0}(w) \prod r_{\pm \sigma'}(w),$$

where the product is over pairs $\pm O'$ of orbits for $\sigma'$ in the roots of $T^\vee = ((T^\vee)^{\theta^\vee})^0$ in $G^\vee_*$. The term $r'_i(w)$ is defined similarly, using the roots of $T^\vee_*$ in $H^\vee$. In the next paragraph, we will keep track of contributions after cancellation, using the pairs $\pm O'$ of orbits in $G^\vee_*$ (the reduced restricted roots) to index them.

We claim that there are nontrivial contributions from $\pm O', \pm 2O'$ to $c'(w)$ only in the following two cases: (i) neither $\pm O', \pm 2O'$ belongs to $H^\vee$, and (ii) $\pm 2O'$ belongs to $H^\vee$. Recall we have fixed the root $\alpha_0 = \alpha_{\text{res}}$ of $G^\theta_0$ and (reduced) $\alpha_*$ is the multiple of $\alpha_0$ that is a root of $(G^\theta)^0$. Now on the dual side, we set $\alpha_{**}$ to be the multiple of $(\alpha^\vee)^{\text{res}}$ that is a root of $T^\vee_*$ in $G^\vee_*$ and denote by $\beta_{**} = (\beta^\vee)^{\text{res}}$ a root of $T^\vee_*$ in $G^\vee_*$ distinct from $\pm \alpha_{**}$. The coroot of $\beta_{**}$ is $rN\beta$, where $r = 1$ unless $\beta$ (and hence also $\beta^\vee$) is of type $R_2$, in which case $r = 2$. The term $r_{\pm \sigma'}(w)$ is constructed in Section 2.5 of [LS87]. We will need its explicit form only for symmetric orbits. Then

$$r_{\pm \sigma'}(w) = \chi_{\beta_{**}}(u_0(w))^{rN\beta},$$

where $\beta_{**}$ represents $O'$ and $u_0(w)$ is defined in Section 2.5 of [LS87]. This applies also if $\beta_{**}$ is not reduced (as in case (ii)). Now to check the claim we examine the various possibilities as in the argument on p. 49 of [KS99]. We see that the contribution in case (i) is $r_{\pm \sigma'}(w)$, while in case (ii) it is $r_{\pm 2O'}(w)^{-1}$. In the remaining cases, it is 1, as asserted. We of course define $c(w)$ in the same way as $c'(w)$, using instead the action $\sigma$.

The terms $s_{p/p_0}(w)$ are, in general, different for $G^\vee_*$ and $H^\vee$. We have assumed that our toral data have the property that complex conjugates (relative to $\sigma'$ only) of positive complex roots are positive. Then both terms $s_{p/p_0}(w)$ contributing to $c'(w)$, but not necessarily those contributing to $c(w)$, are trivial (see Section 2.4 of [LS87] for their definition) and will be deleted in notation. We will deal with $s_{p/p_0}(w)$ for the action defined by $\sigma$ in the last paragraph of our proof.

Suppose $O' \neq \{\pm \alpha_{**}\}$ is asymmetric and not orthogonal to $\alpha_{**}$. Our plan is to remove a cocycle for each $\pm O'$ contributing to $c'(w)$ and then to remove a matching cocycle from $c(w)$. Because there exist trivial $\chi$-data for $\pm O'$, the contribution $r_{\pm \sigma'}(w)$ must be a cocycle (see also Corollary 2.5.B of [LS87]), and we may compute the corresponding character $\Lambda_{\pm O'}$ on $T_{\text{ad}}(\mathbb{R})$ as in Section 3.3 of [LS87]. Assume $\beta_{**}$ belongs to $\pm O'$. Suppose first that $\sigma'\beta_{**} = -u_0\beta_{**} \neq \pm \beta_{**}$ (i.e., $\beta_{**}$ is complex for $\sigma'$ and imaginary for $\sigma$). Then according to Lemma 3.3.D of [LS87], $\Lambda_{\pm O'}(\delta_{\text{ad}}) = \chi_{\beta_{**}}(N\beta(\delta_{\text{ad}})^*)$. To extract
a matching cocycle from $c(w)$ we may simply write down any cocycle that gives the correct character value. We will, however, take time to motivate our construction, as we will use the result later. Namely, we consider the (distinct, symmetric) $\sigma$-orbits $\mathcal{O}$ and $w_0\mathcal{O}$ of $\beta_{**}$ and $w_0\beta_{**}$. The contributions $r_{\pm\mathcal{O}}(w)$ and $r_{\pm w_0\mathcal{O}}(w)$ to $c(w)$ are not cocycles. However, because we use compatible $\chi$-data, $r_{\pm\mathcal{O}}(w)r_{\pm w_0\mathcal{O}}(w)$ is of the form

$$\chi_{\beta_{**}}(u_0(w))^{rN\beta} \chi_{w_0\beta_{**}}(u_0(w))^{r_{w_0}N\beta} = \chi_{\beta_{**}}(u_0(w))^{r(N\beta + w_0N\beta)}.$$ 

But

$$N\beta + w_0N\beta \equiv 2N\beta \mod N\alpha.$$ 

We extract $\chi_{\beta_{**}}(u_0(w))^{2rN\beta}$ from $c(w)$. This is a cocycle since $\chi_{\beta_{**}}^2$ is trivial on $\mathbb{R}^\times$ (Lemma 2.5.B of [LS87]). The value of the corresponding character at $\delta_{ad}$ is $\chi_{\beta_{**}}(x^2)$, where $x/\mathfrak{f} = N\beta(\delta_{ad})^r$; see the calculations of Section 3.3 of [LS87]. Since

$$\chi_{\beta_{**}}(x^2) = \chi_{\beta_{**}}(x/\mathfrak{f} \cdot x\mathfrak{f}) = \chi_{\beta_{**}}(x/\mathfrak{f}) = \Lambda_{\pm\mathcal{O}}(\delta_{ad}),$$

we have removed an appropriate pair of cocycles from $c'(w)$, $c(w)$.

In the next step of the definition of $a'_+(w)$, $a_+(w)$ we consider the asymmetric orbits $\mathcal{O}'$ not orthogonal to $\alpha_{**}$, for which the $\sigma$-orbit $\mathcal{O}$ of $\beta_{**}$ is also asymmetric. Then both $r_{\pm\mathcal{O}}(w)$ and $r_{\pm\mathcal{O}}(w)$ are cocycles. If $\mathcal{O}'$, $\mathcal{O}$ are of the same cardinality (i.e., both consist of a complex root and its conjugate), then the corresponding characters have the same value at $\delta_{ad}$, and so we remove $r_{\pm\mathcal{O}}(w), r_{\pm\mathcal{O}}(w)$ from $c'(w)$, $c(w)$ respectively. It remains to consider the case that $\sigma'\beta_{**} = \beta_{**}$ and $\beta_{**}$ is not orthogonal to $\alpha_{**}$. Then $w_0\beta_{**}$ also has this property, is distinct from $\beta_{**}$, and has same $\sigma$-orbit as $\beta_{**}$. Here we remove both $r_{\pm\mathcal{O}}(w)$ and $r_{\pm w_0\mathcal{O}}(w)$ from $c'(w)$, and $r_{\pm\mathcal{O}}(w)$ from $c(w)$. The requirement of $s$-compatibility that $\chi_{\beta_{**}} = \chi_{\beta_{**}}^s \circ Nm$ ensures that the corresponding characters match at $\delta_{ad}$ (see Section 3.3 of [LS87]), and so we are done.

The final step in the definition of $a'_+(w)$, $a_+(w)$ is needed only for the case where $\alpha_{**}$ is of type $R_2$ and $2\alpha_{**}$ is a root of $H'$, so that $\{\pm\alpha_{**}\}$ satisfies the requirements of case (ii) above. Then $r_{\pm 2\mathcal{O}}(w)^{-1}$ is a cocycle which we include in $a'_+(w)$, i.e., discard from $c'(w)$, since the method of Section 3.3 of [LS87] shows that the value of the corresponding character at $\delta_{ad}$ is 1.

We observe next that $c'_-(w) = c'(w)/a'_+(w)$ has contributions only from orbits which are orthogonal to $\alpha_{**}$, so that $c'_-(w)$ is fixed by $w_0$. Moreover, by construction [LS87], each contribution is the image of an $w_0$-invariant in $T_{sc}'$.

Turning now to $A_-(w)$, we verify that $\varphi_{w_0}(a'_-(w)) = 1$. The cocycle $a_{T_2}'(w)$ takes values in the torus $T_2'$, which is the quotient of $T_1' \times T_1'$ by the diagonally embedded torus $T_H'$. It may be written as the image of the pair $(t_1(w), t(w)^{-1})$ on p. 45 of [KS99]. To compute $\varphi_{w_0}$ on this image, we may choose an element
we construct root system consisting of those $\omega$ as $1$. We use the attached $\Gamma$-splittings for served by $\theta$. To find the Weyl group of $a$ we review the construction of $\xi$ (part of the endoscopic data $H$).

Write the action of $\sigma$ on $\xi$ the embedding $\omega$ and notice that $\sigma$ acts as $1$ on $\xi$.

Consider $\sigma$ as an abbreviation for $\sigma_T$. Then $\omega(w_H)$ lies in the Weyl group of $M^\vee$, and so we construct $n(\omega(w_H))$ in $M^\vee$ acting as $\omega(w_H)$ as in \cite{LS87}. Further, we may find $a^1_H(w)$ in $T^\vee_T \cap (M^\vee)^\text{der}$ so that $h(w) = t^1_H(w)n(\omega(w_H))w \in L^G$ lies in $H$ (part of the endoscopic data $e$) and acts on $H^\vee$ as $w_H$. Then for the embedding $\xi_1 : H \to L^H_1$ (part of the z-pair), we have $\xi_1(h(w)) = z_1(w)w$, where $z_1(w)$ is central in $H^\vee_1$. The embedding $\xi_{T^1_{T^1_T}} : L^T_{T^1} \to L^H_1$ has the property

$$
\xi_{T^1_{T^1_T}}(1 \times w) = r^*_s(w)n_s(\omega^*_H(w))w.
$$

Here $\sigma'$ acts as $\omega^*_H(\sigma) \cdot \omega_H$ on $T^\vee_H$, and $\omega^*_H(w) = \omega^*_H(\sigma)$ if $w \to \sigma$ under $W_\mathfrak{k} \to \Gamma$, while $\omega^*_H(w) = 1$ if $w \to 1$. Notice that $\omega'(\sigma)$ also lies in the Weyl group of $M^\vee$ (although we construct $n_s(\omega^*_H(w))$ in $H^\vee$). We have to compare $\xi_{T^1_{T^1_T}}$ with the embedding $\xi_{T^1_T} : L^T_{T^1} \to L^G_1$ which extends naturally to $\xi_{T^1_{T^1_T}} : L^T_{T^1} \to L^G$. Write the action of $\sigma'$ on $T^\vee_H = T^\vee_T$ as $\omega^*_G(\sigma) \cdot \omega_G$. Construct $n(\omega^*_G(w))$ in $M^\vee$, and notice that $\omega^*_G(w) = \omega^*_H(w)\omega(w_H)$. Then

$$
\xi_{T^1_T}(1 \times w) = r^*_s(w)n(\omega^*_G(w))w.
$$

We claim that

$$
n(\omega^*_G(\sigma)n(\omega(w_H))^{-1} = t^2_H(w)n_s(\omega^*_H(w)),
$$

where $t^2_H(w) \in T^\vee_H \cap (M^\vee)^\text{der}$. To prove this, we compare the left side to $n(\omega^*_H(w))$ using Lemma 2.1.A of \cite{LS87}. For the right side, there is a routine
generalization of Lemma 4.3.A of [LS90] to the twisted case that allows us to compare \(n_s(\omega'_H(w))\), an element in the Levi group of \(H^\vee\) corresponding to the appropriate multiples of roots in \(M^\vee_s\), to \(n(\omega'_H(w))\), an element of \(M^\vee_s\). The claim then follows. Thus
\[
\xi_{T^*_2}(1 \times w) = r'_1(w)\omega'_H(w)(t^1_H)^{-1}t^2_H(w).n_s(\omega'_H(w))h(w).
\]

Turning now to \(a_{T^*_2}(w)\), we set
\[
t_1(w) = z_1(w)^{-1}r'_g(w),
\]
and
\[
t(w) = r'_1(w)\omega'_H(w)(t^1_H)^{-1}t^2_H(w).
\]

Then
\[
\varphi_{\omega_0}(t(w)^{-1}t_H(w).a'_+(w)^{-1}) = \varphi_{\omega_0}(\omega'_H(w)(t^1_H)^{-1}t^2_H(w)) = 1
\]
since
\[
\omega'_H(w)(t^1_H)^{-1}t^2_H(w) \in T^\vee_s \cap (M^\vee_s)_{der}
\]
and \(\varphi_{\omega_0}\) is trivial on \(T^\vee_s \cap (M^\vee_s)_{der}\).

Our last step is to examine the second component
\[
\hat{\tau}(w_0, \sigma').\hat{b}(w_0)^{-1}.w_0(c'_-(w)).c_-(w)^{-1}
\]
of \(A_-(w)\). Consider
\[
w_0(c'_-(w)).c_-(w)^{-1} = c'_-(w)c_-(w)^{-1}.
\]
If \(\mathcal{O}'\) is orthogonal to \(\alpha_{ss}\), then \(\mathcal{O}'\) is also a \(\sigma\)-orbit \(\mathcal{O}\), and \(r_{\pm\mathcal{O}}(w) = r_{\pm\mathcal{O}}(w)\). Thus all that remains in \(w_0(c'_-(w)).c_-(w)^{-1}\) is a term in \((\mathbb{C} \times)^{r_{N\alpha}}\) and the term \(s_{p/p_0}(w)\) for the action \(\sigma\). We compare \(s_{p/p_0}(w)\) with \(\hat{\tau}(w_0, \sigma').\hat{b}(w_0)^{-1}\). Recall our assumption that if \(\beta_{ss} > 0\), then \(\sigma'\beta_{ss} > 0\) unless \(\sigma'\beta_{ss} = -\beta_{ss}\). Then \(\beta_{ss} > 0\) and \(\sigma\beta_{ss} > 0\) requires \(w_0\beta_{ss} = \sigma'\sigma\beta_{ss} > 0\). Thus the sum defining \(\hat{\tau}(w_0, \sigma')\) is empty, so that \(\hat{\tau}(w_0, \sigma') = 1\). Next, we use a routine generalization of Lemma 4.3.B in [LS90]. This shows that the term \(\hat{b}(w_0)\) is a product of an element of order two and an element in \((\mathbb{C} \times)^{r_{N\alpha}}\). The element of order two is of the form \(\prod \beta_{ss}(-1)^{\beta_{ss}}\), where the product is over representatives for the pairs \(\beta_{ss}, -w_0\beta_{ss}\) with the property that \(\beta_{ss} > 0\) and \(-w_0\beta_{ss} > 0\). If we consider just those pairs where \(\beta_{ss}, -w_0\beta_{ss}\) are also complex roots (if one is, the other is), then we obtain \(s_{p/p_0}(w)\). See Section 2.5 of [LS77], and cancel terms for \(G^\vee, H^\vee\) appropriately. Assume now that \(\beta_{ss}\) is imaginary. Then \((-1)^{\beta_{ss}}\) is a Galois cocycle which inflates to a cocycle of \(W_{G/\beta}\) of order at most two. To evaluate the corresponding character at \(\delta_{ad}\), we use the method of Section 3.2 of [LS77] to reduce the calculation to evaluation at the element \(N\beta(\delta_{ad})^r\) of a character of order at most two on the real points of a 1-dimensional torus \(T^{\beta_{ss}}\). Since \(T^{\beta_{ss}}(\mathbb{R})\) is compact, the (cocycle and) character must be trivial. Notice that here the canonical constructions of [LS77] have allowed us to avoid
the more complicated setting in Theorem 6.1.1 of [She81b], where case-by-case computations were needed. Thus we may discard the pairs \( \{ \beta_{ss}, -w_0\beta_{ss} \} \) for which \( \beta_{ss} \), and hence also \( -w_0\beta_{ss} \), is imaginary. Since no real roots contribute, we conclude that, after the discard, \( \hat{\tau}(w_0, \sigma'), \hat{b}(w_0)^{-1}, w_0(c_\varepsilon(w))_.c_\varepsilon(w)^{-1} \) is a cocycle with values in \( \mathbb{C}^{\delta T N_{\alpha}} \). It remains to evaluate the corresponding character at \( \delta_{ad} \). We again use the method of Section 3.2 of [LS87] to reduce this to the value of a character on a 1-dimensional torus \( T^{\delta^{**}} \) at the element \( N_\alpha(\delta_{ad})^r \). If \( \alpha_{ss} \) is of type \( R_1 \), then \( N_\alpha(\delta_{ad})^r = N_\alpha(\delta_{ad}) = 1 \), and if \( \alpha_{ss} \) is of type \( R_2 \), then \( N_\alpha(\delta_{ad})^r = N_\alpha(\delta_{ad})^2 = (\pm 1)^2 = 1 \) also. Thus the value is 1, and we have finished the proof of Lemma 9.3. Notice that we could have based our last calculation on the coroot of the root \( \alpha_1 \) of \( H_1 \) in place of the reduced \( \alpha_{ss} \). Then we arrive at the evaluation of a character at \( \alpha_1(\gamma_0) = 1 \). 

\begin{proof}

Lemma 9.5. For an s-compatible data set and toral descent data at \( \gamma_0 \), we have

\[ \Delta_I(\gamma_0(Y_1), \delta_0(Y)) = \Delta_I(\gamma_0(Y'_1), \delta_0(Y')) \]

for all \( Y_1 \in \mathfrak{t}_1(\mathbb{R}) \) and \( Y'_1 \in \mathfrak{t}'_1(\mathbb{R}) \).

Proof. Given our choices, the sign \( \Delta_I \) depends only on the torus \( T_1 \) or \( T'_1 \) to which the first argument, \( \gamma_0(Y_1) \) or \( \gamma_0(Y'_1) \), belongs. The lemma asserts that not even that matters. There are two ways we can argue this. The first is to observe that the proof in [She82], [LS90] of geometric transfer (with the transfer factors of [LS87]) for untwisted endoscopy avoids Lemma 9.5, using instead regular unipotent analysis and the local hypothesis. We deduce Lemma 9.5 in the untwisted case from the cited proof together with Corollaries 9.2 and 9.4 above: if transfer exists and all terms but \( \Delta_I \) are known to match correctly then \( \Delta_I \) must match correctly. We then prove Lemma 9.5 in the general case with the observation from Section 4.2 of [KS99] that \( \Delta_I \) for the twisted case may be interpreted as \( \Delta_I \) for a case of standard endoscopy.

Our second proof for Lemma 9.5 is a direct argument, allowing us to complete a proof for geometric transfer that works as well for, rather than assumes, standard endoscopy. The starting point is the observation cited above for twisted \( \Delta_I \). We consider standard endoscopy for the quasi-split group \( G_{sc}^{\alpha'} = (G_{sc}^{\alpha})_{<sc} \) (denoted \( G_{\alpha}^{\varepsilon} \) in [KS99]) and the datum \( \mathfrak{g}'_{T,\theta} \) defined on p. 32 of [KS99]. The two maximal tori \( T_{\Theta_{sc}}, T_{\Theta_{sc}}^{\varepsilon} \) in \( G_{sc}^{\Theta_{sc}} \) are norm (image) groups for the endoscopic group \( J \). Our toral data and \( a\)-data provide data for this setting also. Write \( \alpha_{ss} \) for the multiple of \( \alpha_0 \) that is a root of \( T_{\Theta_{sc}}^{\varepsilon} \) and define \( \alpha_{ss}' \) similarly. Recall that the inverse Cayley transform \( t^* \) carries \( \alpha_{ss}' \) to \( \alpha_{ss} \). Pick a \( G_{sc}^{\Theta_{sc}} \)-semiregular element \( \varepsilon \) of \( T_{\Theta_{sc}}^{\varepsilon} \) with image \( \varepsilon_J \) in \( J(\mathbb{R}) \). Then we make an endoscopic descent around the pair \( (\varepsilon, \varepsilon_J) \) as in [LS90]. By construction, the connected centralizers of \( \varepsilon, \varepsilon_J \) are isomorphic over \( \mathbb{R} \), so that the base endoscopy is trivial up to passage to \( \varepsilon \)-extensions. In particular, each
\[ \Delta_f \text{ term is trivial. Our setting satisfies the requirements for the comparison formulas of Section 3.3 of [LS90], including the condition (3.3.2). The formula of Lemma 9.5 is the same as the corresponding formula for } C^{\theta^*_c}_{\mathfrak{sc}} \text{ relative to the tori } T^{\theta^*_c}, T^{\theta^*_c}. \text{ Thus it is enough to show that the quotient of the two terms in the formula divided by the (trivial) quotient for the centralizers, or the quotient of the terms } \Theta_I \text{ of [LS90] for } T^{\theta^*_c} \text{ and } T^{\theta^*_c}, \text{ is trivial. Lemma 3.3.D of [LS90] describes a class } v \text{ in } H^1(\Gamma, T^{\theta^*_c}) \text{ with which we may pair } \sigma_{T, \theta}, \text{ by the Tate-Nakayama pairing, to obtain this quotient of the } \Theta_I. \text{ It remains thus to examine } v \text{ (which we will write as } v_s) \text{ and conclude that, because of our particular choice of } a\text{-data, this class is represented by a cocycle } (-1)^{\epsilon_\alpha} \text{, where } \epsilon \in \{0,1\}. \text{ Since } \alpha^*_\sigma \text{ is a root of } J^\vee, \text{ the pairing yields 1, and the lemma is then proved.}

We use, just for this paragraph, } \alpha \text{ to denote a reduced root of } T^{\theta^*_c} \text{ different from } \pm \alpha_\sigma. \text{ (We argue in } C^{\theta^*_c}_{\mathfrak{sc}} \text{ with no reference to } H^\vee \text{ or the endoscopic data.) Identify } T^{\theta^*_c} \text{ with } T^{\theta^*_c} \text{ via } t^*, \text{ and write } \sigma \text{ for the Galois action on } T^{\theta^*_c}, \text{ } \sigma' \text{ for the transport of the Galois action on } T^{\theta^*_c}, \text{ and } a'_\alpha \text{ for the } a\text{-data for } T^{\theta^*_c}. \text{ Then } \sigma = w_0 \sigma', \text{ where } w_0 \text{ is the Weyl reflection for } \alpha_\sigma, \text{ and}

\[ v_s(\sigma) = \tau(w_0, \sigma').b(w_0)^{-1}.w_0(y'(\sigma)).y(\sigma)^{-1}. \]

Here

\[ \tau(w_0, \sigma') = \prod_{\alpha > 0, \ w_0 \alpha < 0, \sigma \alpha > 0} (-1)^{\alpha^\vee}. \]

Up to multiplication by an element of } (\mathbb{C}^\times)^{\alpha^\vee}, \text{ the term } b(w_0) \text{ is } \prod (-1)^{\alpha^\vee}, \text{ where the product is over representatives for pairs } \{\alpha, -w_0 \alpha\} \text{ such that } \alpha > 0, \ w_0 \alpha < 0 \text{ (see Lemma 4.3.A of [LS90]). Here the order on the roots is obtained by transport of that determined by our choice of an } \mathbb{R}\text{-splitting. The choice of splitting does not affect the quotient of } \Delta_f \text{ terms, and there is no harm in our assumption that if } \alpha > 0 \text{ and } \sigma' \alpha = -\alpha, \text{ then } \sigma' \alpha > 0. \text{ (Or see Lemma 2.3.A of [LS87], and note that the assumption (3.3.2) of [LS90] is retained.) Finally,}

\[ y'(\sigma) = \prod_{\alpha > 0, \sigma' \alpha < 0} (a'_\alpha)^{\alpha^\vee}, \]

and

\[ y(\sigma) = \prod_{\alpha > 0, \sigma \alpha < 0} (a_\alpha)^{\alpha^\vee}. \]

Suppose } \alpha > 0, \sigma' \alpha < 0, \text{ so that } \alpha \text{ contributes to } w_0(y'(\sigma)). \text{ Then } \alpha = -\sigma' \alpha \text{ and } w_0 \alpha = \alpha = -\sigma' \alpha = -\sigma \alpha, \text{ so that } \alpha \text{ is imaginary for both } T^{\theta^*_c} \text{ and } T^{\theta^*_c}. \text{ By } t^*-compatibility \text{ of our } a\text{-data, we have } a'_\alpha = a_\alpha \text{, and the contribution from } \alpha \text{ to } w_0(y'(\sigma)) \text{ cancels that to } y(\sigma). \text{ There are two remaining types of contribution to } y(\sigma). \text{ The first is for } \alpha > 0, \text{ such that } \alpha = -\sigma \alpha \text{ and } w_0 \alpha \neq \alpha. \text{ Then } w_0 \alpha = -w_0 \sigma \alpha = -\sigma' \alpha < 0 \text{ since } -\sigma' \alpha \neq \alpha. \text{ Thus we also have } -w_0 \alpha > 0, \]
and \(-w_0\alpha\) is of same type as \(\alpha\). The contribution to \(y(\sigma)\) from \(\{\alpha, -w_0\alpha\}\) is

\[(a_\alpha)^{\sigma_\alpha}(a_{-w_0\alpha} - w_0\alpha) = (a_\alpha)^{\sigma_\alpha}(-w_0\alpha)(1)\]

since \(a_{-w_0\alpha} = -a_{w_0\alpha} = -a_\alpha\). The first term in the product lies in \((\mathbb{C}^\times)^{\sigma_\alpha}\) and the second cancels with a term in \(b(w_0)\) up to multiplication by an element of \((\mathbb{C}^\times)^{\alpha}.\) The second type of contribution to \(y(\sigma)\) is from \(\alpha > 0\) such that \(\sigma \alpha < 0\) and \(\sigma \alpha \neq -\alpha\). Then each of \(\alpha\) and \(-\sigma \alpha\) contributes and their joint contribution is

\[(a_\alpha)^{\sigma_\alpha}(a_{-sigma} - w_0\alpha) = (a_\alpha)^{\sigma_\alpha}(\sigma_\alpha)(-w_0\alpha)(1)\]

Since \((a_\alpha)^{\sigma_\alpha}(\sigma_\alpha)^{-\sigma_\alpha}\) is a coboundary, we may ignore it. Let \(\beta = -\sigma \alpha\). Then \(\beta > 0, \sigma \beta < 0\). Also \(w_0\beta = -\sigma \alpha < 0\) since \(\alpha > 0\) and \(\sigma \alpha \neq -\alpha\). Thus \((-1)^{-\sigma_\alpha} = (-1)^{\beta_\alpha}\) cancels with the corresponding term in \(b(w_0)\), and so we conclude that, up to coboundaries, the cocycle \(v_\ast(\sigma)\) lies in \((\mathbb{C}^\times)^{\alpha.}\) The lemma now follows. \(\square\)

Finally, the following equalities will be used in assembling the jump formulas in the next section. The terms were introduced in Section 7.

**Lemma 9.6.** Under the assumptions of the present section, we have

\[
\left< \text{inv}(\delta_0(Y), \delta_0(Y)(w)), \kappa_{\delta_0(Y)} \right> \\
= \left< \text{inv}(\delta_0(Y), \delta_0(Y)(ww_0)), \kappa_{\delta_0(Y)} \right> \\
= \left< \text{inv}(\delta_0(Y'), \delta_0(Y')(w')), \kappa_{\delta_0(Y')} \right>.
\]

**Proof.** The representatives \(w', w\) were defined in the paragraph before Lemma 7.2, and \(w_0\) lies in \(G_{\delta_0}.\) Write the three inv terms in the statement as \(\text{inv}(w), \text{inv}(ww_0), \text{inv}(w').\) To define \(\text{inv}(w)\) we start with the Galois cocycle \(\sigma(w)w^{-1}\) in the maximal torus \(A^{\delta_0} = \text{Cent}(T^{\delta_0}, G)\) of \(G.\) (Earlier we used the notation \(T^\dagger\) for \(A^{\delta_0}\).) Notice that \(A^{\delta_0}\) is preserved by \(\theta_0 = \text{Int}(\delta_0) \circ \theta\) and \(\text{Int}(\delta_0(Y)) \circ \theta\) acts as \(\theta_0\) on \(A^{\delta_0}\). Let \(A^{\delta_0}_{ac}\) be the corresponding torus in \(G_{ac}.\) Then we factor \(w\) in the usual manner, as the product of the image of an element \(w_{ac}\) of \(G_{ac}\) and a central element \(z.\) The pair \((\sigma(w_{ac})w_{ac}^{-1}, (\theta_0 - 1)z)\) represents \(\text{inv}(w)\), an element of \(H^1(\Gamma, A^{\delta_0}_{ac})\). Here \(B^{\delta_0}\) is the image of \(A^{\delta_0}\) under \(\theta_0 - 1\) and \(\varphi\) is the composition of \(\theta_0 - 1\) with the projection \(A^{\delta_0}_{ac} \rightarrow A^{\delta_0}.\) We have arranged that \(\sigma(ww_0)(ww_0)^{-1}\) coincides with \(\sigma(ww_0)^{-1}.(-1)^{\alpha_0^\dagger}(, 1)\) up to coboundaries in \(A^{\delta_0} \cap (G_{\delta_0})_{ac} = T^{\delta_0} \cap (G_{\delta_0})_{ac}\). Thus we can factor the corresponding hypercocycle as \((\sigma(w_{ac})w_{ac}^{-1}, (\theta_0 - 1)z),((-1)^{\alpha_0^\dagger}, 1)\). The usual argument (see the proof of Lemma 9.3) shows that the second term in the statement of the present lemma is \(\alpha_0^\dagger(\sigma_T)\) times the first. Since \(\alpha_0^\dagger(\sigma_T) = 1\) (see the proof of Lemma 9.3 again), we are done with the first equality.

Our choices ensure that \(\text{inv}(w)\) is represented by a hypercocycle \((a_{sc}(\sigma), \theta_0 - 1)z)\) and \(\text{inv}(w')\) is represented by \((s(a_{sc}(\sigma)), (\theta_0 - 1)z). Here, recall that
s is a Cayley transform in \((G_{\delta_0}^\theta)_{sc}\). Since we have also to analyze the dual data, we use our chosen toral data to pass from \(G\) to \(G^*\). Then in place of \(H^1(\Gamma, A_{sc}^\delta \xrightarrow{\delta} B^\delta)\) we consider \(H^1(\Gamma, T_{sc} \to (\theta^* - 1)T)\), etc., and we identify \(T'\) with \(T\) over \(\mathbb{C}\) using the inverse Cayley transform \(t^* = (s^*)^{-1}\). Consider the pair \((\text{inv}(w)^{-1}, \text{inv}(w'))\) in

\[
H^1(\Gamma, T_{sc} \times T_{sc}' \to (\theta^* - 1)(T \times T')).
\]

It is represented by

\[
((t_{sc}(\sigma), (\theta^* - 1)z)^{-1}, (t_{sc}(\sigma), (\theta^* - 1)z)),
\]

where \(t_{sc}(\sigma)\) is the image of \(a_{sc}(\sigma)\) under our identification of \(A^\delta_0\) with \(T\). To prove that the (equal) first and second terms in the statement of the lemma coincide with the third, we show that \((\text{inv}(w)^{-1}, \text{inv}(w'))\) pairs trivially with the class in

\[
H^1(W_{\mathbb{R}}, [(\theta^* - 1)(T \times T')]^\vee \to T_{ad}^\vee \times (T_{ad}'^\vee)),
\]

represented by

\[
((b_T(w)^{-1}, s_{ad}), (b_{T'}(w)^{-1}, s_{ad})),
\]

where \(b_T, b_{T'}\) are as constructed on p. 55 of [KS99]. (We will describe them in detail shortly.) Recall

\[
S = S(T, T') = T \times T'/\{(z^{-1}, z) : z \in Z(G^*)\}.
\]

The projection \((\theta^* - 1)(T \times T') \to (\theta^* - 1)S\) determines a map on hypercohomology groups under which the image of \((\text{inv}(w)^{-1}, \text{inv}(w'))\) is represented (in the obvious manner) by \(((t_{sc}(\sigma)^{-1}, 1), (t_{sc}(\sigma), 1))\). Thus by functoriality of the pairing, it is enough to show that \(((b_T(w)^{-1}, s_{ad}), (b_{T'}(w)^{-1}, s_{ad}))\) represents a class in the image of

\[
H^1(W_{\mathbb{R}}, [(\theta^* - 1)S]^\vee \to T_{ad}^\vee \times (T_{ad}'^\vee)),
\]

under the (dual) map on dual hypercohomology groups. Thus it is enough to show that the cocycle \((b_T(w), b_{T'}(w))\) in \([(\theta^* - 1)(T \times T')]^\vee\) lies in the subtorus \([(\theta^* - 1)S]^\vee\).

Recall the cocycle \((a_{T_2}(w), a_{T_2'}(w))\) of \(W_{\mathbb{R}}\) in \(T_2^\vee \times T_2'^\vee\) from the construction of \(\Delta_{III}\); see the proof of Lemma 9.3 and p. 45 of [KS99]. Also, the torus \([(\theta^* - 1)(T \times T')]^\vee\) may be identified with \(T_2^\vee \times T_2'^\vee / T_1^\vee \times T_1'^\vee\) (see p. 55 of [KS99]). Then \((b_T(w), b_{T'}(w))\) is, by definition, the image of \((a_{T_2}(w), a_{T_2'}(w))\) under the natural projection

\[
\text{proj} : T_2^\vee \times T_2'^\vee \to T_2^\vee \times T_2'^\vee / T_1^\vee \times T_1'^\vee.
\]

By construction, \((a_{T_2}(w), a_{T_2'}(w))\) lies in \(S_1^\vee\) (identified as a subtorus of \(T_2^\vee \times T_2'^\vee\)). We denote by \(\theta_2\) the extension of \(\theta^*\) to \(T_2, T_2'\) (p. 42 of [KS99]). The torus \((\theta^* - 1)T\) may be identified as the (isomorphic) image of \((\theta_2 - 1)T_2\) under the projection \(T_2 \to T\), and so \((\theta^* - 1)S\) may be identified with \((\theta_2 - 1)S_1\) under \(S_1 \to S\). Since \([(\theta_2 - 1)S_1]^\vee\) coincides with the image of \(S_1^\vee\) under \(\text{proj}\), we are done. □
10. Proof of Theorem 5.1 and extension to derivatives

To complete the proof of Theorem 5.1 we return to the formulas of Sections 7 and 8, and we combine them with the results of Section 9. We have only to consider the case that $\gamma_0$ is both a $T_1^{s_1}$-norm and a $T_1$-norm, and maintain the toral descent data attached to $\gamma_0$, along with the $s$-compatible data sets, in Section 7. Write $\Phi_1(\gamma_\nu)$ as

$$\det(\text{Ad}(\gamma_\nu) - I)_{b_1/11}^{1/2} \sum_w \Delta(\gamma_\nu, \delta_\nu(w)) \ O^{\theta, \omega}(\delta_\nu(w), f)$$

$$= \Delta_I(\gamma_\nu) \Delta_{II}(\gamma_\nu) \Delta_{III}(\gamma_\nu, \delta_\nu; \overline{\delta_0}, \delta_\nu) \Delta_{IV, \text{num}}(\delta_\nu)$$

$$\times \sum_w \langle \text{inv}(\delta_\nu, \delta_\nu(w)), \kappa_\delta_\nu \rangle \ O^{\theta, \omega}(\delta_\nu(w), f).$$

To pass to the transform $\Psi_{a, \chi}(\gamma_\nu)$, we simply replace $\Delta_{II}(\gamma_\nu)$ by $\Delta_{II, \text{num}}(\gamma_\nu)$.

Without changing notation, we drop the terms for those classes in $D$ for $\nu(\Omega)$, we may then replace the sum by a sum over representatives $\gamma_\nu$ to consider the case that $\gamma_\nu$ is regular in $\mathbb{R}$. Write $\Phi(\gamma_\nu)$ as

$$\Delta_I(\gamma_\nu) \Delta_{II, \text{num}}(\gamma_\nu) \Delta_{IV, \text{num}}(\delta_\nu) \Delta_{\alpha_0}(\delta_\nu)^{-1} \Delta_{III}(\gamma_\nu, \delta_\nu; \overline{\delta_0}, \delta_\nu)$$

$$\times (2/d(\alpha_0)) \sum_w \langle \text{inv}(\delta_\nu, \delta_\nu(w)), \kappa_\delta_\nu \rangle \ O^{\theta, \omega}(\delta_\nu(w), f).$$

On the other hand,

$$\Psi_{a', \chi'}(\gamma_\nu') = \Delta_I(\gamma_\nu') \Delta_{II, \text{num}}(\gamma_\nu') \Delta_{IV, \text{num}}(\delta_\nu') \Delta_{\alpha_0}(\delta_\nu')^{-1} \Delta_{III}(\gamma_\nu', \delta_\nu'; \overline{\delta_0}, \delta_\nu')$$

$$\times \sum_{w'} \langle \text{inv}(\delta_\nu', \delta_\nu'(w')), \kappa_\delta_\nu' \rangle \ O^{\theta, \omega}(\delta_\nu'(w'), f),$$

where the summation is over representatives $w'$ for the elements of $D_{\theta}(T^{s_0})$.

From Lemma 8.1, Corollaries 9.2, 9.4, and Lemmas 9.5, 9.6, we conclude that

$$\lim_{\nu \to 0^+} \Psi_{a, \chi}(\gamma_\nu) - \lim_{\nu \to 0^-} \Psi_{a, \chi}(\gamma_\nu) = 2 \lim_{\nu \to 0^+} \Psi_{a', \chi'}(\gamma_\nu).$$

For the final step in the proof of Theorem 5.1, we notice that the Weyl reflection $w_1$ for $\alpha_1$ provides a stable conjugation of $\gamma_\mu$ with $\gamma_\nu$. Since $\Phi_1$ is invariant under stable conjugacy, it is enough to examine the factor

$$\prod_{\Omega_1} \chi_\beta_1 \left( \frac{(\beta_1(\gamma_1) - 1)}{a_{\beta_1}} \right),$$

where $\gamma_1$ is regular in $T_1(\mathbb{R})$. There is no change in the total contribution from the orbits $\Omega_1 \neq \{\pm \alpha_1\}$ when $\gamma_1$ is replaced by $\gamma_1^{w_1}$ since the definition of $s$-compatible data ensures that $\chi_{w_1 \beta_1} = \chi_{\beta_1}$ and $a_{w_1 \beta_1} = a_{\beta_1}$; the contribution
from the orbit of $\beta_1$ is interchanged with that from the orbit of $w_1\beta_1$. For the case $O_1 = \{\pm \alpha_1\}$, we have
\[
\chi_{\alpha_1} \left( \frac{(\alpha_1^{\nu_1}(\gamma_1)) - 1}{\alpha_1} \right) = -\chi_{\alpha_1}(\alpha_1(\gamma_1))^{-1} \chi_{\alpha_1} \left( \frac{(\alpha_1(\gamma_1)) - 1}{\alpha_1} \right).
\]
Since $\chi_{\alpha_1}(\alpha_1(\gamma_0)) = \chi_{\alpha_1}(1) = 1$,
\[
\lim_{\nu \to 0^-} \Psi_{a,\chi}(\gamma_\nu) = - \lim_{\nu \to 0^+} \Psi_{a,\chi}(\gamma_\nu),
\]
and so we are done with the proof of Theorem 5.1. □

To consider limit formulas for derivatives, let $S(t_1)$ denote the symmetric algebra of $t_1$. Denote by $D \to \hat{D}$ the automorphism of $S(t_1)$ determined by the map $Y_1 \to Y_1 - n_1\alpha_1(Y_1)I$ of $t_1$ into $S(t_1)$, where $2n_1$ is the odd integer given by $\chi_{\alpha_1}(z) = (z/\bar{z})^{n_1} = (z/\bar{z})^{2n_1}$.

For $\gamma_1 = \gamma_0 \exp Y_1$ near $\gamma_0$, define $\chi_{\alpha_1}(1/\alpha_1(\gamma_1))^{1/2}$ to be $\chi_{\alpha_1}(\exp 1/2\alpha_1(Y_1))$. Then the function (germ)
\[
\hat{\Psi}_{a,\chi}(\gamma_1) = \chi_{\alpha_1}(1/\alpha_1(\gamma_1))^{-1/2} \Psi_{a,\chi}(\gamma_1)
\]
is defined for $G$-regular $\gamma_1$ near $\gamma_0$ in $T_1(\mathbb{R})$. (Recall the smooth extension from strongly $G$-regular elements to all $G$-regular elements in Section 7.) Moreover, this function is odd:
\[
\hat{\Psi}_{a,\chi}(1/\alpha_1(\gamma_1)) = -\hat{\Psi}_{a,\chi}(\gamma_1).
\]

**Lemma 10.1.** For all $D \in S(t_1)$, we have that both $\lim_{\nu \to 0^-} \hat{D}\Psi_{a,\chi}(\gamma_\nu)$ and $\lim_{\nu \to 0^+} \hat{D}\Psi_{a,\chi}(\gamma_\nu)$ exist. If $D^\nu_1 = -D$, then
\[
\lim_{\nu \to 0^-} \hat{D}\Psi_{a,\chi}(\gamma_\nu) = \lim_{\nu \to 0^+} \hat{D}\Psi_{a,\chi}(\gamma_\nu),
\]
and if $D^\nu_1 = D$, then
\[
\lim_{\nu \to 0^-} \hat{D}\Psi_{a,\chi}(\gamma_\nu) = - \lim_{\nu \to 0^+} \hat{D}\Psi_{a,\chi}(\gamma_\nu).
\]

**Proof.** Existence of each limit in the statement follows from the basic estimates (see the appendix). The twist $\hat{D}$ of $D$ was defined expressly to obtain the property
\[
\hat{D}\Psi_{a,\chi}(\gamma_0 \exp Y_1) = \chi_{\alpha_1} \left( \exp \frac{1}{2} \alpha_1(Y_1) \right) \cdot \hat{D}\Psi_{a,\chi}(\gamma_0 \exp Y_1).
\]
Thus we have
\[
\lim_{\nu \to 0^\pm} \hat{D}\Psi_{a,\chi}(\gamma_\nu) = \lim_{\nu \to 0^\pm} \hat{D}\Psi_{a,\chi}(\gamma_\nu).
\]
The desired equations are then immediate from the oddness of $\hat{\Psi}_{a,\chi}$. □

We may choose the $\chi'$-datum $\chi_{\alpha_1'}$ nontrivial. Because $\alpha_1'$ is real, we define $\hat{\Psi}_{a',\chi'}$ by
\[
\hat{\Psi}_{a',\chi'}(\gamma_1) = \chi_{\alpha_1'}(\alpha_1'(\gamma_1))^{-1} \Psi_{a',\chi'}(\gamma_1)
\]
for $G$-regular $\gamma'_1$ near $\gamma_0$ in $T'_1(\mathbb{R})$. The Cayley transform $s_1$ provides us with an isomorphism $D \to D'$ from $S(t_1)$ to $S(t'_1)$. We write $\hat{D}'$ for the image of $D'$ under the automorphism given by $Y'_1 \to Y'_1 - z\alpha'_1(Y'_1)I$, where $z$ is the complex number given by $\chi_{\alpha'_1}(x) = (\text{sgn } x)^\varepsilon |x|^z$. Then for all $D \in S(t_1)$, we have that

$$\lim_{\nu \to 0} \hat{\Delta}'_{a',\chi}(\gamma'_\nu) = \lim_{\nu \to 0} \hat{D}'\tilde{\Delta}_{a',\chi}(\gamma'_\nu)$$

exists.

**Lemma 10.2.** If $D^{u_1} = D$ then, for any $s$-compatible data set,

$$\lim_{\nu \to 0^+} \hat{D}\tilde{\Psi}_{a,\chi}(\gamma_\nu) = \lim_{\nu \to 0} \hat{D}'\tilde{\Psi}_{a',\chi}(\gamma'_\nu).$$

**Proof.** For this we return to the formulas obtained by descent in Section 8 and use Harish-Chandra descent for operators in the center of the universal enveloping algebra of the complex Lie algebra of $G$ as well, extending the arguments for Proposition 4.5 of [She79a] via results of Bouaziz (see Theorem 2.4.1 of [Bou87]). The formula then follows by repeating the steps at the start of this section. $\square$

This concludes then our extension of Theorem 5.1 to derivatives of $\Psi_{a,\chi}$. The extension applies, in particular, to the setting of Theorem 4.2.

**11. Completion of proof of the main theorem**

We recall once again that if

$$S(\gamma_1) = \sum_{\delta, \delta \text{-conj}} \Delta(\gamma_1, \delta)O^{\theta,\omega}(\delta, f),$$

then we have the normalized integral

$$\Phi(\gamma_1) = \frac{1}{2} \left| \det(\text{Ad}(\gamma_1) - I)_{h_1/t_1} \right|^1 S(\gamma_1)$$

and the transform

$$\Psi_{a,\chi}(\gamma_1) = \Delta_{a,\chi}(\gamma_1)S(\gamma_1).$$

Recall also that $S(\gamma_1)$, defined initially for strongly $G$-regular elements $\gamma_1$, was extended smoothly to all $G$-regular elements. Next, we extend $S$ to a smooth function around all regular elements in $T_1(\mathbb{R})$. Since $\Delta_{a,\chi}$ is nonvanishing and thus smooth on the regular set in $T_1(\mathbb{R})$, we may replace $S$ by $\Psi_{a,\chi}$ for this extension.

Assume that $\gamma_0$ is regular in $T_1(\mathbb{R})$, so that $(H_1)_{\gamma_0} = T_1$. If $\gamma_0$ is not a norm, then $\Psi_{a,\chi}(\gamma_1) = 0$ for $G$-regular $\gamma_1$ near $\gamma_0$ in $T_1(\mathbb{R})$ by Lemma 6.1, and so $S$ extends trivially. Suppose now that $\gamma_0$ is a $T_1$-norm of $\delta_0 \in G(\mathbb{R})$. We consider the case that $\delta_0$ is $\theta$-semiregular, by which we mean that $G^\theta_{\delta_0}$ is of type $A_1$. As before, we denote by $\pm \alpha_0$ the roots of $T_{\delta_0}$ in $G^\theta_{\delta_0}$. If the root $\alpha_0$ is real or totally compact, then we follow our earlier descent arguments (and include
derivatives) to see that $\Psi_{a,\chi}$ extends smoothly around $\gamma_0$. Suppose then that $\alpha_0$ is imaginary and not totally compact. By passage to a stable $\theta$-conjugate of $\delta_0$, we may assume that $\alpha_0$ itself is noncompact. Again we rely on the earlier descent arguments, except that Lemma 9.6 is replaced by the following.

**Lemma 11.1.** In the present setting, we have $\alpha_0^\vee(\mathfrak{s}_T) = -1$, and then
\[ \langle \operatorname{inv}(\delta_0(Y), \delta_0(Y)(w)), \kappa_{\delta_0}(Y) \rangle = - \langle \operatorname{inv}(\delta_0(Y), \delta_0(Y)(w w_0)), \kappa_{\delta_0}(Y) \rangle. \]

**Proof.** Since $\sigma \alpha_0 = -\alpha_0$, we also have that $\sigma(N\alpha^\vee) = -N\alpha^\vee$, and then $N\alpha^\vee(\mathfrak{s}_T)^2 = 1$ since $\mathfrak{s}_T$ is $\Gamma$-invariant. Suppose $\alpha$ is of type $R_2$. If $N\alpha^\vee(\mathfrak{s}_T) = 1$, then $\alpha_1$ is a root of $H_1$ and $\alpha_1(\gamma_0) = N\alpha(\delta_0)^2 = 1$ contradicting the regularity of $\gamma_0$. Thus $N\alpha^\vee(\mathfrak{s}_T) = -1$ is the only possibility. In fact, then the coroot $\beta_1$ of $2(\alpha^\vee)_{\operatorname{res}}$ is a root of $H_1$ and $\beta_1(\gamma_0) = N\beta(\delta_0) = N\alpha(\delta_0) = \pm 1$. Since $\beta_1(\gamma_0) \neq 1$, we must have $N\alpha(\delta_0) = -1$, a contradiction since $\alpha$ is of type $R_2$. We conclude that $\alpha$ cannot be of type $R_2$. Suppose $\alpha$ is of type $R_3$. If $N\alpha^\vee(\mathfrak{s}_T) = 1$, then $\beta_1$ is a root of $H_1$, where now $\beta_1$ denotes the coroot of $\frac{1}{2}(\alpha^\vee)_{\operatorname{res}}$. This implies $\beta_1(\gamma_0) = N\beta(\delta_0)^2 = N\alpha(\delta_0)^2 = 1$, which contradicts the regularity of $\gamma_0$. Thus $N\alpha^\vee(\mathfrak{s}_T) = -1 = \alpha_0^\vee(\mathfrak{s}_T)$. Finally if $\alpha$ is of type $R_1$, then $\alpha_0^\vee(\mathfrak{s}_T) = N\alpha^\vee(\mathfrak{s}_T) = -1$ since $\alpha_1$ cannot be a root of $H_1$. \hfill $\square$

The argument of Section 10.1 now shows that
\[ \lim_{\nu \to 0^-} \hat{D} \Psi_{\alpha,\chi}(\gamma_\nu) = \lim_{\nu \to 0^+} \hat{D} \Psi_{\alpha,\chi}(\gamma_\nu) \]
for all $D \in \mathcal{S}(t_1)$. Thus $\Psi_{a,\chi}$ extends smoothly around $\gamma_0$.

We then have that $\Psi_{a,\chi}$ extends smoothly around all regular elements $\gamma_0$ in $T_1(\mathbb{R})$ that are norms of $\theta$-semiregular elements in $G(\mathbb{R})$. Next, $\Psi_{a,\chi}$ extends to a smooth function around all regular elements $\gamma_0$ in $T_1(\mathbb{R})$ that are norms of $\theta$-semisimple elements in $G(\mathbb{R})$. For this, Lemma 6.2 implies immediately that we may apply a familiar principle of Harish-Chandra, which we call *semiregular is sufficient*; see, for example, [Var77, §6 of Part I, §13 of Part II] and also [War72, Lemma 8.4.4.6 and §8.5]. We conclude then that $\Psi_{a,\chi}$, and thus $S$ itself, extends to a smooth function on the full regular set of $T_1(\mathbb{R})$.

To finish the proof of the main theorem, Theorem 2.1, we check that $S$ satisfies all requirements of our characterization theorem for stable orbital integrals on $H_1(\mathbb{R})$, i.e., PropertiesI–IV of Theorem 12.1 with $G = H_1$, $g_0 = 1$. Recall that we use Haar measures attached to invariant differential forms of highest degree defined over $\mathbb{R}$, as in [She79a, §4] and [LS87, §1.4], and use provided inner twists or $\mathbb{R}$-isomorphisms to transport forms when needed (for example, in the formulation of Property I). We assume that the forms on $g, \mathfrak{h}_1$ are products corresponding to the Lie algebra decompositions $g = \mathfrak{g}^\theta + (1 - \theta)\mathfrak{g} + \mathfrak{g}_{\operatorname{der}}, \mathfrak{h}_1 = \mathfrak{z}_1 + \mathfrak{h}$. Suppose strongly $G$-regular $\gamma_1$ is a norm of strongly $\theta$-regular $\delta$. We require that Haar measures on $G_\mathfrak{o}^\theta(\mathbb{R})$ and $T_1(\mathbb{R})$ be compatible
in the following sense. First the underlying forms are to respect $g^0_δ + (g^0_δ \cap g_{\text{der}})$, $t_1 = 3_1 + t_H$. Because the constant $|\det((\text{Int}(\delta) \circ \theta - I)_{\text{Cent}(g^0_δ, \delta)} / g^0_δ)|$ or, more simply, $|\det(\theta^* - I)_1 / \psi^*|$ was omitted from the normalizing factor $\Delta_{IV}$, we include it now by requiring that the form on $t_H$ be obtained by transport of $|\det(\theta^* - I)_1 / \psi^*|^{-1}$ times the form on $g^0_δ$. For the Haar measure on $\text{Cent}(\delta, G)(\mathbb{R})$, we extend that on $G^0_3(\mathbb{R})$.

For Property III, it remains to consider $\Psi_{a,\chi}$ around a $T_1$-norm $\gamma_0$ annihilated only by real or complex roots. Again we use the semiregular is sufficient principle to assume that the root is real and unique up to sign and that both $(H_1)_{\gamma_0}$ and $G^0_{\gamma_0}$ are of Dynkin type $A_1$. Then descent finishes the argument. As in Section 14 of [She08a] for the standard (untwisted) case, an alternative proof that $\Psi_{a,\chi}$ extends to an $\infty_1$-Schwartz function on $T_1(\mathbb{R})_{\text{im-reg}}$ may be given via formulas for parabolic descent (see [Mez12], [She11]).

By Theorem 5.1 and its extension to derivatives, $S$ satisfies Property IV under the additional assumption that $\gamma_0$ is $G$-semiregular. Our (stronger) statements of limit formulas for transfer factors in Section 9 allow us to remove the assumption by an application of the semiregular is sufficient principle, and then we are done with the proof of the main theorem.

12. The general case: slightly twisted norms

Without the assumption at the beginning of Section 6, the norms of strongly $\theta$-regular elements in $G(\mathbb{R})$ lie in a certain coset of $H_1(\mathbb{R})$ in $H_1(\mathbb{C})$, rather than in $H_1(\mathbb{R})$ itself. This feature requires only a minor modification in the formulation of transfer, as we will recall from [KS99, §5.4]. We consider arbitrary $(G, \theta, a_{\omega_\emptyset})$, endoscopic data $\emptyset$, and $\omega_\emptyset$-pair $(H_1, \xi_1)$ (see Section 1).

We return to the first paragraph of Section 6. Recall that we work with the variant $m : G \to G^*$ of the inner twist $\psi$ defined by $m(\delta) = \psi(\delta)g_\emptyset^{-1}$. Without the assumption of the first paragraph, we have that

$$\sigma(m)(\delta) = z(\sigma)u(\sigma)^{-1}m(\delta)\theta^*(u(\sigma)),$$

where $z(\sigma)$ is a 1-cocycle of $\Gamma$ in $Z_{sc}^\times$. (As usual, we have used the same notation for the image of $z(\sigma)$ in $G^*$.) The image of $z(\sigma)$ in $(Z_{sc}^\times)_{\theta_{sc}}$ is a 1-cocycle $z_\emptyset(\sigma)$. As in (5.4) of [KS99], $z_\emptyset(\sigma)$ determines a 1-cocycle $z_H(\sigma)$ in the center of $H$ which we assume splits in $H$, since otherwise the transfer statement is empty. Let $z_H(\sigma) = h_{0}^{-1}\sigma(h_0)$. Then there is a 1-cocycle $z_1(\sigma) = h_{1}^{-1}\sigma(h_1)$ in the center of $H_1$ which projects to $z_H(\sigma)$ under $H_1 \to H$. Write $\theta_1$ for the automorphism $\text{Int}(h_1)$. We replace $H_1(\mathbb{R})$ by the coset $H_1(\mathbb{R})h_1$ in the formulation of transfer.

First we extend the definition of stable orbital integral to this setting and describe a characterization theorem. Until after Theorem 12.1, we return to $G$ as notation for the group on which we consider orbital integrals. Since it is enough for our purposes (i.e., for the case $G = H_1$), we also assume $G$ quasi-split over $\mathbb{R}$ and with simply-connected derived group. Then the complex
points of centralizers of semisimple elements are connected and there are no totally compact imaginary roots.

Fix an element \( g_0 \) in \( G(\mathbb{C}) \) such that \( \sigma(g_0)^{-1}g_0 \) is central, so that \( \theta = \text{Int}(g_0) \) lies in \( G_{\text{ad}}(\mathbb{R}) \) and \( G(\mathbb{R})g_0 \) lies in the inverse image of \( G_{\text{ad}}(\mathbb{R}) \) under the projection \( G \to G_{\text{ad}} \). There will be no harm in assuming that \( \theta \) preserves the pair \( (B_{\text{spl}}, T_{\text{spl}}) \), where \( B_{\text{spl}}, T_{\text{spl}} \) are from a chosen \( \mathbb{R} \)-splitting of \( G \). Then \( g_0 \) lies in the maximal torus \( T_{\text{spl}} \) of the splitting. There is also no harm in assuming \( g_0 \) lies in \( G_{\text{der}} \). Then \( \sigma(g_0)^{-1}g_0 = z(\sigma) \) lies in the center \( Z_{\text{der}} = Z_{\text{sc}} \) of \( G_{\text{der}} = G_{\text{sc}} \).

Let \( \gamma \in G(\mathbb{R})g_0 \subset G(\mathbb{C}) \). Then \( \text{Cent}(\gamma, G) \) is defined over \( \mathbb{R} \) since \( \sigma(\gamma)^{-1}\gamma = \sigma(g_0)^{-1}g_0 = z_\sigma \). Suppose \( \gamma \) is regular semisimple, so that \( T_\gamma = \text{Cent}(\gamma, G) \) is a maximal torus defined over \( \mathbb{R} \). If \( \gamma = \gamma'g_0 \), then right translation by \( g_0 \) maps bijectively the \( \text{Int}(g_0) \)-twisted conjugacy class of \( \gamma' \) to the \( G(\mathbb{R}) \)-conjugacy class of \( \gamma \). It also maps the intersection of \( G(\mathbb{R}) \) with the \( \text{Int}(g_0) \)-twisted conjugacy class of \( \gamma' \) in \( G(\mathbb{C}) \) to the intersection of \( G(\mathbb{R})g_0 \) with the conjugacy class of \( \gamma \) in \( G(\mathbb{C}) \). We will call this last set the stable conjugacy class of \( \gamma \) (again since \( G_{\text{der}} \) is simply-connected). The \( G(\mathbb{R}) \)-conjugacy classes in the stable conjugacy class of \( \gamma \) are parametrized by untwisted \( D(T_\gamma) \), as for the case \( \gamma \in G(\mathbb{R}) \).

Let \( T \) be a maximal torus over \( \mathbb{R} \) in \( G \). Then \( T \) contains an element \( \gamma \) in \( G(\mathbb{R})g_0 \) if and only if \( z_\sigma \) splits in \( H(\Gamma, T_{\text{der}}) = H(\Gamma, T_{\text{sc}}) \). In that case, \( T(\mathbb{R})\gamma \) also lies in \( G(\mathbb{R})g_0 \) and moreover \( T(\mathbb{R})\gamma = T \cap G(\mathbb{R})g_0 \). Write \( T(g_0) \) for the collection of all such \( T \). Clearly, \( T_{\text{spl}} \in T(g_0) \) and the set of regular semisimple elements in \( G(\mathbb{R})g_0 \) is the union over \( T \in T(g_0) \) of the (open, dense) regular set \( (T \cap G(\mathbb{R})g_0)_{\text{reg}} \) in \( T \cap G(\mathbb{R})g_0 \). Suppose \( T \in T(g_0) \), \( \gamma_0 \in T \cap G(\mathbb{R})g_0 \) is semiregular, and \( \alpha(\gamma_0) = 1 \), where \( \alpha \) is an imaginary root of \( T \). On replacing \( \gamma_0 \) by a stable conjugate we may assume that \( \alpha \) is noncompact, i.e., that \( \text{Cent}(\gamma_0, G) \) is split modulo center. If \( T' \) is a maximally split maximal torus over \( \mathbb{R} \) in \( \text{Cent}(\gamma_0, G) \), then clearly \( T' \in T(g_0) \). It then follows that if \( T \in T(g_0) \) and \( s \) is any Cayley transform relative to an imaginary root \( \alpha \) of \( T \), then \( T^s \in T(g_0) \). Also, if \( \gamma_0 \in T \cap G(\mathbb{R})g_0 \) is semiregular and \( \alpha(\gamma_0) = 1 \), then \( (\gamma_0)^s \in T^s \cap G(\mathbb{R})g_0 \). We denote by \( (T \cap G(\mathbb{R})g_0)_{\text{lim-reg}} \) the set of all elements in \( T \cap G(\mathbb{R})g_0 \) such that \( \alpha(\gamma_0) \neq 1 \) for all imaginary roots \( \alpha \) of \( T \).

Let \( S(\gamma, dt_\gamma, dg) \) be a complex-valued function defined for regular semisimple \( \gamma \) in \( G(\mathbb{R})g_0 \) and Haar measures \( dt_\gamma \) on \( T^{\gamma}(\mathbb{R}) = \text{Cent}(\gamma, G)(\mathbb{R}), dg \) on \( G(\mathbb{R}) \). Write \( \Phi \) for the normalized version of \( S \):

\[
\Phi(\gamma, dt_\gamma, dg) = \left| \det(\text{Ad}(\gamma) - I) \right|_{g/\Gamma}^{1/2} S(\gamma, dt_\gamma, dg).
\]

Since it is useful for our application, we assume that there is a central torus \( Z_1 \) and character \( \varpi_1 \) on \( Z_1(\mathbb{R}) \) such that

\[
S(z_1\gamma, dt_\gamma, dg) = \varpi_1(z_1)^{-1} S(\gamma, dt_\gamma, dg)
\]

for all \( z_1 \in Z_1(\mathbb{R}) \), regular semisimple \( \gamma \) in \( G(\mathbb{R})g_0 \), and all \( dt_\gamma, dg \).
Consider the following properties (I)–(IV):

(I) $S$ is invariant under stable conjugacy.

This means that if $w \in G(\mathbb{C})$ is such that $\gamma^w = w^{-1}\gamma w$ lies in $G(\mathbb{R})g_0$ and $dt_{\gamma^w}$ is obtained from $dt_{\gamma}$ by transport under $w$, then

$$S(\gamma^w, dt_{\gamma^w}, dg) = S(\gamma, dt_{\gamma}, dg).$$

(II) $S$ transforms under change of measures according to the rule

$$S(\gamma, \lambda dt_{\gamma}, \mu dg) = \frac{\mu}{\lambda} S(\gamma, dt_{\gamma}, dg).$$

Here $\lambda, \mu$ are positive real numbers.

Next, let $T \in \mathcal{T}(g_0)$. For $\gamma$ in $(T \cap G(\mathbb{R})g_0)_{\text{reg}}$ and fixed Haar measures $dt, dg$ on $T(\mathbb{R}), G(\mathbb{R})$ respectively, set $S^T(\gamma) = S(\gamma, dt, dg)$ and $\Phi^T(\gamma) = \Phi(\gamma, dt, dg)$.

(III) $\Phi^T$ is a $\varpi_1$-Schwartz function on $(T \cap G(\mathbb{R})g_0)_{\text{reg}}$ and extends to a $\varpi_1$-Schwartz function on $(T \cap G(\mathbb{R})g_0)_{\text{im-reg}}$.

Here the notion of $\varpi_1$-Schwartz function is clear since $T(\mathbb{R})\gamma_0$ lies in the inverse image of $T_{\text{ad}}(\mathbb{R})$ in $T(\mathbb{C})$.

The final property concerns behavior at the imaginary walls. It is simpler to state if we assume (I), (III). Suppose $T \in \mathcal{T}(g_0)$, $\gamma_0 \in T \cap G(\mathbb{R})g_0$ is semiregular and $\alpha(\gamma_0) = 1$, where $\alpha$ is an imaginary root of $T$. Let $s$ be a Cayley transform for $\alpha$ (in the sense of Section 3), and fix $s$-compatible $a$-data, $\chi$-data for $T, T^s = T'$. (We again use $'$ in place of $s$ in notation.) The Haar measure on $T'(\mathbb{R})$ is to be obtained by transport under $s$ from that on $T(\mathbb{R})$ in our earlier sense (Section 8). We have defined the generalized Weyl denominator $\Delta_{a,\chi}(\gamma)$ for $\gamma \in T(\mathbb{R})$. Notice that $\Delta_{a,\chi}(\gamma)$ depends on the image of $\gamma$ under the natural map $T \to T_{\text{ad}}$ rather than on $\gamma$ itself. We may therefore extend the definition of $\Delta_{a,\chi}$ to the inverse image of $T_{\text{ad}}(\mathbb{R})$ in $T(\mathbb{C})$ and so to $T'(\mathbb{R})\gamma_0$. We also extend $\Delta_{a',\chi'}$ to $T'(\mathbb{R})'\gamma'_0$. Thus we may define the transforms $\Psi_{a,\chi}, \Psi_{a',\chi'}$ on $(T'(\mathbb{R})\gamma_0)_{\text{reg}}, (T'(\mathbb{R})'\gamma'_0)_{\text{reg}}$ respectively, as before. For $\nu$ real and nonzero, set $\gamma_{\nu} = \exp(\nu a_\alpha \alpha^\vee), \gamma_{\nu}' = \exp(\nu a_{\alpha'} \alpha'^\vee), \gamma_{\nu}'$. Denote by $w$ the Weyl reflection for $\alpha$. To a differential operator $D$ in $S(t)$ attach $\hat{D}'$ in $S(t')$ and define the twists $\hat{D}, \hat{D}'$ as in Section 10.

(IV) If $D^w = D$, then

$$\lim_{\nu \to 0^+} \hat{D}\Psi_{a,\chi}(\gamma_{\nu}) = \lim_{\nu' \to 0} \hat{D}'\Psi_{a',\chi'}(\gamma'_{\nu'}).$$

With the assumption of (I) and (III), there will be no harm in assuming in (IV) that $\alpha$ is noncompact and that the Cayley transform comes from the simply-connected cover $\text{SL}(2)$ of $\text{Cent}(\gamma_0, G)$. Then $\gamma_0' = \gamma_0$. Also, the argument of Section 10, along with (I) and (III), shows that if $D^w = \pm D$, then

$$\lim_{\nu \to 0^-} \hat{D}\Psi_{a,\chi}(\gamma_{\nu}) = \mp \lim_{\nu \to 0^+} \hat{D}\Psi_{a,\chi}(\gamma_{\nu}).$$
Suppose now that $f$ is a $\varpi_1$-Schwartz function on $G(\mathbb{R})g_0$. (Again the notion is clear, or see the appendix.) Then the stable orbital integrals

$$\text{SO}(\gamma, f) = \text{SO}(\gamma, f, dt_\gamma, dg) = \sum_{\gamma' \in \mathcal{D}(T_\gamma)} \int_{T_\gamma'(\mathbb{R}) \backslash G(\mathbb{R})} f(g^{-1}\gamma') \frac{dg}{dt_\gamma}$$

transform by $\varpi_1^{-1}$ under translation by $Z_1(\mathbb{R})$ and satisfy (I)–(IV). A proof of this requires only a very minor variant of the standard argument; see the next proof or the appendix for more general results. Extension of our main theorem to the present setting rests on the converse theorem.

**Theorem 12.1.** Suppose $S$ transforms by $\varpi_1^{-1}$ under translation by $Z_1(\mathbb{R})$ and satisfies (I)–(IV). Then there exists $f \in C(G(\mathbb{R})g_0, \varpi_1)$ such that

$$S(\gamma, dt_\gamma, dg) = \text{SO}(\gamma, f, dt_\gamma, dg)$$

for all regular semisimple $\gamma$ in $G(\mathbb{R})g_0$, and all $dt_\gamma, dg$. If also $S$ vanishes off the orbits of some set $Z_1(\mathbb{R})B$, where $B$ is a bounded subset of the regular semisimple set of $G(\mathbb{R})g_0$, then $f$ may be chosen in $C_c^\infty(G(\mathbb{R})g_0, \varpi_1)$.

**Proof.** To find $f$ in $C(G(\mathbb{R})g_0, \varpi_1)$ we prove an analog of Lemma 4.8 of [She79a] in which $f$ is constructed satisfying a weaker condition, and we then finish by using the inductive argument for the proof of Theorem 4.7 in [She79a]. Assume $T \in \mathcal{T}(g_0)$. Then an argument shows that we may replace $g_0$ by an element of $G(\mathbb{R})g_0 \cap T$ if necessary and assume $g_0 \in T$. It is now straightforward to extend the wave packet construction in the proof of Lemma 4.8 to $G(\mathbb{R})g_0$ and thus find the desired $f$ in $C(G(\mathbb{R})g_0, \varpi_1)$. To pass to a $C_c^\infty$-function when the support is appropriate, we reduce to Bouaziz’s characterization theorem on $G_{ad}(\mathbb{R})$.

Finally, the extension of Theorem 2.1 requires a recasting of the norm correspondence and transfer factors. This again is straightforward (and described in Section 5.4 of [KS99]). First, for the norm correspondence we consider strongly $G$-regular elements $\gamma_1$ of $H_1(\mathbb{R})h_1$, assuming such elements exist, and strongly $\theta$-regular elements $\delta$ of $G(\mathbb{R})$. Then $\gamma_1$ is a norm of $\delta$ if the $\theta$-conjugacy class of $\delta$ in $G(\mathbb{C})$ is the image (under the canonical map) of the conjugacy class of $\gamma_1$ in $H_1(\mathbb{C})$. Let $T_1 = \text{Cent}(\gamma_1, H_1)$, a maximal torus over $\mathbb{R}$ in $H_1$. Then there are toral data $(T_1 \to T_{\delta^*}, g)$ as in Section 6 for which $\delta^* = gm(\delta)\theta^*(g)^{-1}$ has the property that $N(\delta^*)$ is the image of $\gamma_1$ under $T_1 \to T_{\delta^*}$. The cochain $v(\sigma)$ in $T_{sc}$ now has the extra term $z(\sigma)$ from $Z_{sc}$, but that does not affect the assertions of the lemmas in Section 6 when we now take semisimple $\gamma_0$ in $H_1(\mathbb{R})h_1$ instead of $H_1(\mathbb{R})$. Nor does it affect the definition of the relative term $\Delta_{II}$ in transfer factors, since $(z(\sigma)^{-1}, z(\sigma))$ represents the identity element of the torus $U$ of Section 4.4 of [KS99]. The results of Sections 6, 7 and 9 thus apply. After adjusting the definition of $\text{Trans}(f)$ and $\text{Trans}_c(f)$, we conclude then
Theorem 12.2. The assertions of the main theorem (Theorem 2.1) and corollary (Corollary 2.2) remain true in the general setting of Section 6.

13. Appendix: Harish-Chandra Schwartz functions

We return to the setting of Section 1, where \( f_\theta \) is a smooth function on \( G(\mathbb{R})\theta \). As pointed out by a referee, \( \theta \) is the product of an inner automorphism defined over \( \mathbb{R} \) and an automorphism of finite order also defined over \( \mathbb{R} \); see the remark near the end of Section 6. We may further assume that the inner automorphism is of the form \( \text{Int}(g) \), where \( g \in G_{sc}(\mathbb{R}) \). There will be no harm then in assuming that \( \theta \) itself is of finite order. Following [HC75], let \( V = \exp \mathfrak{v} \), where \( \mathfrak{v} \) is the maximal \( \mathbb{R} \)-split subalgebra of the Lie algebra \( \mathfrak{z}(\mathbb{R}) \) of \( Z(\mathbb{R}) \), so that we have \( G(\mathbb{R}) \) as a direct product \( (1 - \theta)V.\mathfrak{v}.G(\mathbb{R}) \), where \( \theta \) acts as automorphism of each factor. Then \( G(\mathbb{R})\theta \) is a direct product \( (1 - \theta)V.G_1 \), where \( G_1 = V^{\theta}.G(\mathbb{R})\theta \); \( G_1 \) embeds smoothly as an open subset of the Lie group \( V^{\theta}.G(\mathbb{R}) \times \{ \theta \} \) to which the results of [Bou87] apply. We will start with the space \( \mathcal{C}(G(\mathbb{R})\theta, \varpi) \), where we require \( f_\theta \) to transform by \( \varpi^{-1} \) under the twisted conjugacy action of \( V \) since we will need such a space for the twisted orbital integrals. Thus we require \( f_\theta(v\theta(v)^{-1}g\theta) = f_\theta(g\theta) \) for \( v \in V, g \in G(\mathbb{R}) \). (Since we assume a nonempty norm correspondence, the character \( \varpi \) is trivial on \( V^\theta \), the kernel of the action.) Call \( f_\theta \) a \( \varpi \)-Schwartz function if the restriction of \( f_\theta \) to \( G_1 \) is Schwartz in the Harish-Chandra sense [HC75]: the functions \( \sigma \) and \( \Xi \) appearing in Harish-Chandra’s seminorms are well defined on \( G_1 \) (see Sections 3.4, 3.5 of [Bou87]). We write \( \mathcal{C}(G(\mathbb{R})\theta, \varpi) \) for the Fréchet space of all such functions equipped with the Harish-Chandra seminorms. If \( \mathcal{O} \) is open in \( G(\mathbb{R})\theta \) and invariant under translation by \( (1 - \theta)V \), we define \( \mathcal{C}(\mathcal{O}, \varpi) \) analogously. It is clear also how to define the space \( \mathcal{C}(G(\mathbb{R})\theta) \) of (purely) Schwartz functions on \( G(\mathbb{R})\theta \).

We need a twisted analogue of Theorem 16.1 of [HC75], which asserts that \( f \to \, ^Ff \) is a well-defined continuous map on the appropriate Schwartz spaces. To shorten the presentation (but also make it clumsier than necessary), we take our \( (\theta, \varpi) \)-twisted transform to depend on the endoscopic data also, or more precisely, on endoscopic data and \( z \)-pair. To use pieces of the transfer factors in the definition, we will start with familiar constructions on \( G(\mathbb{R}) \) and then translate to \( G(\mathbb{R})\theta \). Fix a strongly \( \theta \)-regular element \( \delta_0 \) of \( G(\mathbb{R}) \) with norm \( \gamma_0 \) in \( H_1(\mathbb{R}) \), along with toral data, \( a \)-data and \( \chi \)-data associated with the torus \( \text{Cent}(\gamma_0, H_1) \). It will be enough for our purposes to define a transform \( \Psi^{\Phi}_{\delta_0} \) on the \( \theta \)-regular elements \( \delta \) in \( G^{\theta}_{\delta_0}(\mathbb{R})\delta_0 \), although extension to a larger set is easy. If \( \delta = \exp Y.\delta_0 \), then we set \( \gamma_1 = \exp Y.\gamma_0 \) (see Section 6) and define

\[
\Psi^{\Phi}_{\delta_0}(\delta) = \Delta_{III}(\gamma_0, \delta_0; \gamma, \delta) \Delta^{\text{num}}_{II}(\delta) \Phi^{\theta, \varpi}(\delta, f).
\]

We have omitted the term \( \Delta_I \) since fixed toral data and \( a \)-data guarantee that \( \Delta_I \) is a constant that plays no role here. The term \( \Delta^{\text{num}}_{II} \) (from Section 9) is a
twisted version of the Weyl denominator of Section 3. The presence of the constant \( \Delta_{III}(\gamma_0, \delta_0; \gamma, \delta) \) ensures that if \( g \in G(\mathbb{R}) \), then \( \Psi_{f_0}^{-1}(\delta_0 \theta(g)) = \Psi_{f_0}^{-1}(\delta) \), provided we follow the usual conventions in the choice of Haar measures. If we replace \( \delta_0 \) by (strongly) \( \theta \)-regular \( \delta_0' \) in \( G_{\theta_0}(\mathbb{R})^0 \delta_0 \), we obtain a translate of \( \Psi_{f_0} \) which does not matter for the Schwartz properties we seek. (For translation-invariance arguments, see, for example, Section 8.5 of [War72].) To pass to \( G(\mathbb{R}) \), we set \( \Phi(\delta \theta, \varphi) = \Phi(\delta, \varphi) \) and \( \Psi_{f_0}(\delta \theta) = \Psi_{f_0}(\delta) \) for all regular \( \delta \theta \) in \( \text{Conn}(\delta \theta, G(\mathbb{R})^0) \), a connected component of \( \text{Cent}(\delta \theta, G(\mathbb{R})^0) \). It is now routine to define \( \text{Conn}(\delta \theta, G(\mathbb{R})^0) \). Our assertion is that Theorem 16.1 of [HC75] together with the work of Bouaziz already cited implies that \( f_\theta \rightarrow \Psi_{f_0}^{\delta_0} \) is a well-defined continuous mapping from \( \mathcal{C}(G(\mathbb{R})^0) \) (or from \( \mathcal{C}(G(\mathbb{R})^0, \varphi) \)) to \( \mathcal{C}(\text{Conn}(\delta \theta)_{\text{im-reg}}, \varphi) \). Theorem 16.1 is proved in [Var77] following Harish-Chandra’s original argument. (The final steps are in Part II, Section 12.) An alternative argument not dependent on the construction of discrete series characters has been given by Wallach (see Chapter 7 of [Wal88]). Since an analogue for the otherwise needed discrete series result has not yet appeared, we follow step by step the arguments of [Wal88]. In particular, the crucial Lemma 7.4.3 extends to our setting by preparation from Sections 1–3 of [Bou87]. This is enough to finish the argument.

References


She, Examples in endoscopy for real groups (notes for BIRS summer school, August 2008). Available at http://andromeda.rutgers.edu/~shelstad.


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