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# Linearization of generalized interval exchange maps

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*To the memory of Gérard Rauzy*

## Abstract

A standard interval exchange map is a one-to-one map of the interval that is locally a translation except at finitely many singularities. We define for such maps, in terms of the Rauzy-Veech continuous fraction algorithm, a diophantine arithmetical condition called restricted Roth type, which is almost surely satisfied in parameter space. Let $T_0$ be a standard interval exchange map of restricted Roth type, and let $r$ be an integer $\geqslant 2$. We prove that, amongst $C^{r+3}$ deformations of $T_0$ that are $C^{r+3}$ tangent to $T_0$ at the singularities, those that are conjugated to $T_0$ by a $C^r$-diffeomorphism close to the identity form a $C^1$-submanifold of codimension $(g-1)(2r+1)+s$. Here, $g$ is the genus and $s$ is the number of marked points of the translation surface obtained by suspension of $T_0$. Both $g$ and $s$ can be computed from the combinatorics of $T_0$.

## Contents

## 1. Introduction

1.1. *Presentation of the main result.* Many problems of stability in the theory of dynamical systems face the difficulty of small divisors. The most famous example is probably given by the Kolmogorov-Arnold-Moser theory on the persistence of quasi-periodic solutions of Hamilton's equations for quasi-integrable Hamiltonian systems (both finite and infinite-dimensional, like non-linear wave equations). This is a very natural situation with many applications to physics and astronomy. What all these different problems have in common, roughly speaking, is the following: one can associate some "frequencies" to the orbits under investigation and some arithmetical condition is needed to prove their existence and stability.

The simplest example of quasiperiodic dynamics is given by irrational rotations of the circle. Poincaré asked under which condition is a given homeomorphism of the circle equivalent (in some sense, e.g., topologically or smoothly) to some rotation and proved that any orientation-preserving homeomorphism of the circle with no periodic orbit is semiconjugate to an irrational rotation. Denjoy proved that when the rotation number is irrational, adding regularity to a given homeomorphism $f$ (namely, requiring $f$ to be piecewise $C^1$ with $Df$ of bounded variation) is enough to guarantee topological conjugacy to a rotation. The step to higher order differentiability for the conjugacy $h$ requires new techniques and additional hypotheses on the rotation number. A small divisor problem must be overcome, and this was first achieved (in the circle case) by Arnold in [Arn65]. He proved that if the rotation number verifies a diophantine condition and if the analytic diffeomorphism $f$ is close enough to a rotation, then the conjugation is analytic. At the same time, examples of analytic diffeomorphisms, with irrational rotation number, for which the conjugation is not even absolutely continuous were given. Later Herman ([Her79]) proved a global result: there exists a full Lebesgue measure set of rotation numbers for which a $C^\infty$ (resp. $C^\omega$) diffeomorphism is $C^\infty$ (resp. $C^\omega$) conjugated to a rotation. In the finitely differentiable case one can prove a similar result, but the conjugacy is less regular than the diffeomorphism. This phenomenon of loss of differentiability is typical of small divisors problems.

The suspension of circle rotations produces linear flows on the two-dimensional torus. When analyzing the recurrence of rotations or the suspended flows, the modular group $\mathrm{GL}(2,\mathbb{Z})$ is of fundamental importance, providing the renormalization scheme associated to the continued fraction of the rotation number.

A generalization of the linear flows on the two-dimensional torus is obtained by considering linear flows on translation surfaces of higher genus (see, e.g., [Zor06] for a nice introduction to the subject). By a Poincaré section their dynamics can be reduced to (standard) interval exchange maps (i.e.m.), which generalize rotations of the circle.

A (standard) i.e.m. $T$ on an interval $I$ (of finite length) is a one-to-one map which is locally a translation except at a finite number of discontinuities. Thus, $T$ is orientation-preserving and preserves Lebesgue measure. By asking only that $T$ is locally an orientation-preserving homeomorphism one obtains the definition of a generalized i.e.m. Let $d$ be the number of intervals of continuity of $T$. When $d = 2$, by identifying the endpoints of $I$, standard i.e.m. correspond to rotations of the circle and generalized i.e.m. to homeomorphisms of the circle. Standard i.e.m. can be suspended following the construction of Veech [Vee78] to give rise to translation surfaces.

Typical standard i.e.m. are minimal ([Kea75]), but note that ergodic properties of minimal standard i.e.m. can differ substantially from those of circle rotations. They need not be ergodic ([Kea77], [KN76]), but almost every standard i.e.m. (both in the topological sense [KR80] and in the measure-theoretical sense [Mas82], [Vee82]) is ergodic. Moreover, the typical nonrotational standard i.e.m. is weakly mixing [AF07].

Rauzy and Veech have defined an algorithm that generalizes the classical continued fraction algorithm (corresponding to the choice $d = 2$) and associates to an i.e.m. another i.e.m. that is its first return map to an appropriate subinterval [Rau79], [Vee82]. Both Rauzy-Veech "continued fraction" algorithm and its accelerated version due to Zorich [Zor96] are ergodic with respect to an absolutely continuous invariant measure in the space of normalized standard i.e.m. However, in the case of the Rauzy-Veech algorithm the measure has infinite mass whereas the invariant measure for the Zorich algorithm has finite mass. The ergodic properties of these renormalisation dynamics in parameter space have been studied in detail ([Vee84a], [Vee84b], [Zor97], [Zor99], [AGY06], [Buf06], [AB07], [Yoc10]).

The possible combinatorial data for an i.e.m. (standard or generalized) are the vertices of *Rauzy diagrams*; the arrows of these diagrams correspond to the possible transitions under the Rauzy-Veech algorithm.

The Rauzy-Veech algorithm, which makes sense for generalized i.e.m., stops if and only if the i.e.m. has a *connection*, i.e., a finite orbit that starts and ends at a discontinuity. When the i.e.m. has no connection, the algorithm associates to it an infinite path in a Rauzy diagram that can be viewed as a "rotation number."

One can characterize the infinite paths associated to standard i.e.m. with no connections ($\infty$–complete paths; see Section 2.3). One says that a generalized i.e.m. $T$ is *irrational* if its associated path is $\infty$–complete; then $T$ is semiconjugated to any standard i.e.m. with the same rotation number [Yoc].

This generalization of Poincaré's theorem suggests the following very natural question: *What part of the theory of circle homeomorphisms and diffeomorphisms generalizes to interval exchange maps?*

All translation surfaces obtained by suspension from standard i.e.m. with a given Rauzy diagram have the same genus $g$ and the same number $s$ of marked points; these numbers are related to the number $d$ of intervals of continuity by the formula $d = 2g + s - 1$.

Regarding Denjoy's theorem, partial results ([CG97], [BHM10], [MMY10]) go in the negative direction, suggesting that topological conjugacy to a standard i.e.m. has positive codimension in genus $g \geq 2$.

A first step in the direction of extending small divisor results beyond the torus case was achieved by Forni's important paper ([For97], see also [For]) on the cohomological equation associated to linear flows on surfaces of higher genus. In [MMY05] we considered the cohomological equation $\psi \circ T_0 - \psi = \varphi$ for a standard i.e.m. $T_0$. We found explicitly in terms of the Rauzy-Veech algorithm a full measure class of standard i.e.m. (which we called Roth type i.e.m.) for which the cohomological equation has bounded solution. In order for this to hold, the datum $\varphi$ must belong to a finite codimension subspace of the space of functions having on each continuity interval a continuous derivative with bounded variation. The improved loss of regularity (with respect to [For97]) will be decisive for the proof of our main result.

The cohomological equation is the linearization of the conjugacy equation $T \circ h = h \circ T_0$ for a generalized i.e.m. $T$ close to the standard i.e.m. $T_0$. We say that a generalized i.e.m. $T$ is a *simple deformation of class $C^r$* of a standard i.e.m. $T_0$ if

- $T$ and $T_0$ have the same discontinuities,
- $T$ and $T_0$ coincide in the neighborhood of the endpoints of $I$ and of each discontinuity,
- $T$ is a $C^r$-diffeomorphism on each continuity interval onto its image.

Our main result is a local conjugacy theorem, which is stated in full generality in Section 5. For simple deformations, the result can be summarized as follows.

THEOREM. *For almost all standard* i.e.m. $T_0$ *and for any integer $r \geq 2$ amongst the $C^{r+3}$-simple deformations of $T_0$, those that are $C^r$-conjugate to $T_0$ by a diffeomorphism $C^r$-close to the identity form a $C^1$-submanifold of codimension $d^* = (g-1)(2r+1) + s$.*

The standard i.e.m. $T_0$ considered in the theorem are the Roth type i.e.m. for which the Lyapunov exponents of the KZ-cocycle (see Section 2.6) are nonzero (we call this *restricted Roth type*). They still form a full measure set by Forni's theorem [For02].

The tangent space at $T_0$ to the $C^1$-submanifold of $C^{r+3}$-simple deformations that are $C^r$-conjugate to $T_0$ is formed of $C^{r+3}$-functions $\varphi$ that vanish in a neighborhood of the singularities of $T_0$ and can be written as

$$\varphi = \psi \circ T_0 - \psi,$$

where $\psi$ is a $C^r$-function vanishing at the singularities of $T_0$.

To extend this result to generalized i.e.m. $T$ of class $C^r$ that are not simple deformations of a standard i.e.m. $T_0$, there are gluing problems of the derivatives of $T$ at the discontinuities. Indeed, there is a conjugacy invariant which is an obstruction to linearization (see Section 4).

An earlier result is presented in an unpublished manuscript of de la Llave and Gutierrez [dlLG], which was recently communicated to us by P. Hubert. They consider standard i.e.m. with periodic paths for the Rauzy-Veech algorithm. (For $d = 2$, this corresponds to rotations by a quadratic irrational.) They prove that, amongst piecewise analytic generalized i.e.m., the bi-Lipschitz conjugacy class of such a standard i.e.m. contains a submanifold of finite codimension. They also prove that bi-Lipschitz conjugacy implies $C^1$-conjugacy.

The proof of our theorem is based on an adaptation of Herman's Schwarzian derivative trick. In [Her85] Herman gave simple proofs of local conjugacy theorems for diffeomorphisms $f$ of the circle. Let $\omega$ denote the rotation number, assumed to satisfy a diophantine condition $|\omega - p/q| \geq \gamma q^{-2-\tau}$ for some $\gamma > 0$, $\tau < 1$, and let $R_\omega$ be the corresponding rotation of the circle. Taking Schwarzian derivatives, the conjugacy equation $f \circ h = h \circ R_\omega$ becomes $(Sh) \circ R_\omega - Sh = ((Sf) \circ h)(Dh)^2$, a linear difference equation in the Schwarzian derivative $Sh$ of the conjugacy (but the right-hand side depends also on $h$). Given a diffeomorphism $h$, one computes the right-hand side $((Sf) \circ h)(Dh)^2$, then solves the equation $\psi \circ R_\omega - \psi = ((Sf) \circ h)(Dh)^2$ and thus finds a diffeomorphism $\tilde{h} = \Phi(h)$ as smooth as $h$ with $S\tilde{h} = \psi$. Herman now uses the Schauder-Tychonov theorem to find a fixed point of $\Phi$ and thus the required conjugacy. He was aware of the possibility of using the contraction principle (at the cost of one more derivative for $f$) as we do in our proof. Herman's method is presented in more detail in Appendix B.1.

In Section 8 of the paper we explain how to adapt our result to the setting of perturbations of linear flows on translation surfaces. Indeed, we prove the following corollary of the main theorem. (We refer the reader to Section 8 and to Appendix C for the definition of Roth-type translation surface and of simple deformation of a vertical vector field.)

COROLLARY. *Given a translation surface of restricted Roth type and any integer $r \geq 2$, let us consider the $C^{r+3}$-simple deformations of the vertical vector field. Those that are $C^r$-equivalent to the vertical vector field by a diffeomorphism $C^r$-close to the identity form a $C^1$-submanifold of codimension $d^* = (g-1)(2r+1) + s$.*

1.2. *Open problems.*

1. *Prove the theorem for* $r = 1$: *For almost all standard* i.e.m. $T_0$, *amongst the* $C^4$*-simple deformations of* $T_0$, *those that are* $C^1$*-conjugate to* $T_0$ *by a diffeomorphism* $C^1$*-close to the identity form a* $C^1$*-submanifold of codimension* $d^* = 3g - 3 + s$.

A *rationale* for this conjecture comes from the following argument. Note that here $d^*$ is equal to $(d-1)+(g-1)$. The integer $d-1$ is the dimension of the space of standard i.e.m. up to affine conjugacy. In order to have a $C^1$-conjugacy between a generalized i.e.m. $T$ and a standard i.e.m. $T_0$ with the same rotation number, a necessary condition is that the Birkhoff sums of $\mathrm{Log} DT$ (equal to $\mathrm{Log} DT^n$) are bounded. The integral of $\mathrm{Log} DT$ with respect to the unique invariant measure is automatically zero, taking care of the largest exponent of the KZ-cocycle; killing the components with respect to the remaining $g - 1$ positive exponents leads to the expected value of $d^*$.

On the other hand, when the derivatives of the iterates $DT^n$ are allowed to grow exponentially fast, one could expect to have wandering intervals (see [MMY10]). This suggests the existence of a dichotomy between being $C^1$-conjugated to a standard i.e.m. and having wandering intervals. One can therefore ask whether the following is true.

2. *For almost all standard* $T_0$, *any generalized* i.e.m. $T$ *of class* $C^4$ *that is a simple deformation of* $T_0$ *and is topologically conjugated to* $T_0$ *is also* $C^1$*-conjugated to* $T_0$.

The two conjectures above can be formulated in a slightly more general setting (not restricted to simple deformations) using the conjugacy invariant introduced in Section 4.

3. The local $C^r$-conjugacy class of a standard i.e.m. $T_0$ (of restricted Roth type) exhibited by our theorem can be considered as a *local stable manifold* for the renormalization operator $\mathcal{R}$ defined by the Rauzy-Veech induction (with rescaling) on generalized i.e.m. in a suitable functional space. By standard techniques this local stable manifold extends to a global stable manifold

$$W^s(T_0) = \cup_{n\geq 0}\mathcal{R}^{-n}(W^s_{loc}(\mathcal{R}^n T_0),$$

which is the full $C^r$-conjugacy class of $T_0$.

*Is this stable manifold "properly embedded" in parameter space*?

More precisely, given a sequence of diffeomorphisms $h_n$ in $\mathrm{Diff}^r(\overline{I})$ such that $h_n \to \infty$,

> *Is it possible that* $h_n \circ T_0 \circ h_n^{-1} \to T_0$ *in the* $C^{r+3}$ *topology*? *Is it possible that* $h_n \circ T_0 \circ h_n^{-1}$ *stays bounded in the* $C^{r+3}$ *topology*?

In the case $d = 2$, the answer to both questions is no. For the second question, this is a consequence of Herman's global conjugacy theorem for circle diffeomorphisms.

4. *Describe the set of generalized $C^r$ interval exchange maps that are semiconjugate to a given standard* i.e.m. $T_0$ (*with no connections*).

In the circle case, for a diophantine rotation number, one has a $C^\infty$-submanifold of codimension 1. In the Liouville case one still has a topological manifold of codimension 1 that is transverse to all 1-parameter strictly increasing families. One can therefore dare to ask

(1) *Is the above set a topological submanifold of codimension $d-1$?*
(2) *If the answer is positive, does there exist a* (*smooth*) *field of "transversal" subspaces of dimension $d-1$?*

The questions make sense for any $T_0$, but the answer could depend on the diophantine properties of $T_0$.

5. *In a generic smooth family of generalized* i.e.m., *is the rotation number irrational with positive probability?*

In the circle case the answer is affirmative, thanks to Herman's theorem. This is not very likely in higher genus.

6. *Let $r \geq 1$. Describe exactly* (*in terms of the Rauzy-Veech renormalization algorithm*) *the set of rotation numbers such that the $C^r$-conjugacy class of $T_0$ has finite codimension in the space of $C^\infty$ generalized* i.e.m. *Does this set depend on $r$?*

In the circle case, this set is (for any $r \geqslant 1$) the set of diophantine rotation numbers ([Yoc84], [Her79]). In higher genus, our theorem (in the stronger form stated in Section 5) guarantees that this set contains the restricted Roth type rotation numbers and therefore has full measure. It looks as if our methods extend to prove that the unrestricted Roth type rotation numbers also belong to this set (but the codimension of the $C^r$-conjugacy class of $T_0$ is different). Of course the codimension of the $C^r$-conjugacy class will depend on $r$, but the point here is that we only require the codimension to be finite. Note that the answer is not known even at the level of the cohomological equation!

A related question is the optimal loss of differentiability, for instance for restricted Roth type rotation numbers. A careful reading of the proof (and of Appendix A) will convince the reader that we may consider $C^{r+2+\tau}$ (for any $\tau > 0$) simple deformations of $T_0$ instead of $C^{r+3}$-simple deformations and still get the same conclusion. On the other hand, the cohomological equation suggests that some form of the result could be true for $C^{r+1+\tau}$ simple deformations of $T_0$ (for any $\tau > 0$). This is certainly true in genus 1. This is, however, beyond the reach of our method.

1.3. *Summary of the paper.* In the next section we introduce standard and generalized interval exchange maps. We recall the definition and the main properties of the Rauzy-Veech continued fraction algorithm and explain how it allows us to define, in a very natural way, a "rotation number" for certain generalized i.e.m. The algorithm generates a dynamical system in parameter space, equipped with a very important cocycle, the Kontsevich-Zorich cocycle. The notation and the presentation of this section closely follow the expository paper [Yoc06] (see also [Yoc], [Yoc10]).

Section 3 is devoted to the study of the cohomological equation. We introduce a boundary operator on the space of piecewise-continuous functions which vanishes on coboundaries, and we take care of the neutral component of the KZ-cocycle. We review the results of [MMY05] (Theorem 3.10) recalling, in particular, the definition of Roth type i.e.m. We actually improve on the results of [MMY05] by showing that under the same assumptions one can obtain a continuous (instead of bounded) solution. We also reformulate the results in higher smoothness using the boundary operator.

In Section 4 we introduce, for any integer $r \geqslant 1$, an invariant for $C^r$-conjugacy with values in the conjugacy classes of the group $J^r$ of $r$-jets of orientation-preserving diffeomorphisms of $(\mathbb{R}, 0)$. We show that it is also preserved by the renormalization operator defined by the Rauzy-Veech algorithm. We explain the relation of this conjugacy invariant with the boundary operator.

Section 5 contains the precise formulation of our main result (Theorem 5.1): $C^1$ parameter families of generalized i.e.m. of class $C^{r+3}$ through a standard i.e.m. $T_0$ of restricted Roth type are considered. It is assumed that the $C^{r+3}$-conjugacy invariant vanishes and an appropriate transversality hypothesis (related to the cohomological equation) is satisfied. The theorem then states that the local $C^r$-conjugacy class of $T_0$ intersects the family along a submanifold whose tangent space at $T_0$ is given by the cohomological equation. We also show how the hypothesis on the conjugacy invariant allows us to reduce the proofs to the case of simple families.

Section 6 contains the proof of Theorem 5.1 when $r \geqslant 3$; following Herman, we use Schwartzian derivatives to construct a map whose fixed point is a candidate for the conjugating map. In the circle case, this fixed point is always the conjugating map. In the present case, some extra equations representing gluing conditions have to be satisfied; these equations define the local conjugacy class in parameter space.

Section 7 deals with the case $r = 2$ of Theorem 5.1. Indeed, a $C^2$-diffeomorphism does not have in general a Schwartzian derivative. We need a little improvement of Herman's Schwarzian derivative trick. We show how one can effectively use the primitive of the Schwarzian derivative to construct a contracting map whose fixed point will turn out to be the conjugacy, under appropriate gluing conditions.

In Section 8 we explain how to adapt our result to the simple deformations of linear flows on translation surfaces. After a brief introduction to translation surfaces, we study the action of the boundary operator at the level of the surface and we prove that the conjugacy invariant is trivial for simple deformations of the vertical vector field. We then introduce *restricted Roth type translation surfaces* and prove the corollary stated at the end of Section 1.1.

In Appendix A we show that the main result of [MMY05] (in the improved version of Theorem 3.10) is also valid with data whose first derivatives are Hölder continuous instead of having bounded variation.

Appendix B is devoted to the case of circle diffeomorphisms. In Section B.1 we deal with $C^r$-conjugacy, $r \geqslant 3$. Herman's original result (through the Schauder-Tychonov fixed point theorem) gives a stronger conclusion in this setting; however, the simple variant based on the fixed point theorem for contracting maps is a better preparation for the more difficult case of Section 6. In the same way, Section B.2 introduces the main idea of Section 7 in a simpler setting.

Finally, Appendix C is devoted to the study of Roth-type translation surfaces. Proposition C.1 gives several equivalent formulations of condition (a) in the definition of a Roth-type i.e.m. (see Section 3.3). This is then used in order to prove that the i.e.m. obtained as first return maps on an open bounded segment (in good position) of the vertical flow on a (restricted) Roth-type translation surface are of (restricted) Roth-type.

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## 2. **Background**

2.1. *Interval exchange maps.* Let $I$ be an open bounded interval. A generalized interval exchange map (g.i.e.m.) $T$ on $I$ is defined by the following data. Let $\mathcal{A}$ be an alphabet with $d \geqslant 2$ symbols. Consider two partitions mod.0 of $I$ into $d$ open subintervals indexed by $\mathcal{A}$ (the *top* and *bottom* partitions):

$$I = \sqcup I^t_\alpha = \sqcup I^b_\alpha.$$

The map $T$ is defined on $\sqcup I^t_\alpha$, and its restriction to each $I^t_\alpha$ is an orientation-preserving homeomorphism onto the corresponding $I^b_\alpha$.

The g.i.e.m. $T$ is *standard* if $|I^t_\alpha| = |I^b_\alpha|$ for each $\alpha \in \mathcal{A}$, and the restriction of $T$ to each $I^t_\alpha$ is a translation.

Let $r$ be an integer $\geqslant 1$ or $\infty$. The g.i.e.m. $T$ is *of class* $C^r$ if the restriction of $T$ to each $I^t_\alpha$ extends to a $C^r$-diffeomorphism from the closure of $I^t_\alpha$ onto the closure of $I^b_\alpha$. For finite $r$, it is easy to see that the g.i.e.m. with fixed $\mathcal{A}$ form a Banach manifold.

The points $u^t_1 < \cdots < u^t_{d-1}$ separating the $I^t_\alpha$ are called the *singularities* of $T$. The points $u^b_1 < \cdots < u^b_{d-1}$ separating the $I^b_\alpha$ are called the singularities of $T^{-1}$. We also write $I = (u_0, u_d)$, $u^t_0 = u^b_0 = u_0$, $u^t_d = u^b_d = u_d$.

The *combinatorial data* of $T$ is the pair $\pi = (\pi_t, \pi_b)$ of bijections from $\mathcal{A}$ onto $\{1, \ldots, d\}$ such that

$$I^t_\alpha = (u^t_{\pi_t(\alpha)-1}, u^t_{\pi_t(\alpha)}), \quad I^b_\alpha = (u^b_{\pi_b(\alpha)-1}, u^b_{\pi_b(\alpha)})$$

for each $\alpha \in \mathcal{A}$.

We always assume that the combinatorial data are *irreducible*: for $1 \leqslant k < d$, we have

$$\pi_t^{-1}(\{1, \ldots, k\}) \neq \pi_b^{-1}(\{1, \ldots, k\}).$$

2.2. *The elementary step of the Rauzy-Veech algorithm.* A *connection* is a triple $(u^t_i, u^b_j, m)$, where $m$ is a nonnegative integer such that

$$T^m(u^b_j) = u^t_i.$$

Keane has proved [Kea75] that a *standard* i.e.m. with no connection is minimal.

Let $T$ be a g.i.e.m. with no connection. Then we have $u^t_{d-1} \neq u^b_{d-1}$. Set $\widehat{u}_d := \max(u^t_{d-1}, u^b_{d-1})$, $\widehat{I} := (u_0, \widehat{u}_d)$, and denote by $\widehat{T}$ the first return map of $T$ in $\widehat{I}$. The return time is 1 or 2.

One checks that $\widehat{T}$ is a g.i.e.m. on $\widehat{I}$ whose combinatorial data $\widehat{\pi}$ are canonically labeled by the same alphabet $\mathcal{A}$ as $\pi$ (cf. [MMY05, p. 829]). Moreover, $\widehat{T}$ has no connection; this allows one to iterate the algorithm.

We say that $\widehat{T}$ is deduced from $T$ by an elementary step of the Rauzy-Veech algorithm. We say that the step is of *top* (resp. *bottom*) *type* if $u^t_{d-1} < u^b_{d-1}$ (resp. $u^t_{d-1} > u^b_{d-1}$). One then writes $\widehat{\pi} = R_t(\pi)$ (resp. $\widehat{\pi} = R_b(\pi)$).

2.3. *Rauzy diagrams.* A *Rauzy class* on the alphabet $\mathcal{A}$ is a nonempty set of irreducible combinatorial data that is invariant under $R_t, R_b$ and minimal with respect to this property. A *Rauzy diagram* is a graph whose vertices are the elements of a Rauzy class and whose arrows connect a vertex $\pi$ to its images $R_t(\pi)$ and $R_b(\pi)$. Each vertex is therefore the origin of two arrows. As $R_t, R_b$ are invertible, each vertex is also the endpoint of two arrows.

An arrow connecting $\pi$ to $R_t(\pi)$ (respectively $R_b(\pi)$) is said to be of *top type* (resp. *bottom type*). The *winner* of an arrow of top (resp. bottom) type starting at $\pi = (\pi_t, \pi_b)$ with $\pi_t(\alpha_t) = \pi_b(\alpha_b) = d$ is the letter $\alpha_t$ (resp. $\alpha_b$) while the *loser* is $\alpha_b$ (resp. $\alpha_t$).

A path $\gamma$ in a Rauzy diagram is *complete* if each letter in $\mathcal{A}$ is the winner of at least one arrow in $\gamma$; it is $k$*–complete* if $\gamma$ is the concatenation of $k$ complete paths. An infinite path is $\infty$*–complete* if it is the concatenation of infinitely many complete paths.

2.4. *The Rauzy-Veech algorithm.* Let $T = T^{(0)}$ be an i.e.m. with no connection. We denote by $\mathcal{A}$ the alphabet for the combinatorial data $\pi^{(0)}$ of $T^{(0)}$ and by $\mathcal{D}$ the Rauzy diagram on $\mathcal{A}$ having $\pi^{(0)}$ as a vertex.

The i.e.m. $T^{(1)}$, with combinatorial data $\pi^{(1)}$, deduced from $T^{(0)}$ by the elementary step of the Rauzy-Veech algorithm has also no connection. It is therefore possible to iterate this elementary step indefinitely and get a sequence $T^{(n)}$ of i.e.m. with combinatorial data $\pi^{(n)}$, acting on a decreasing sequence $I^{(n)}$ of intervals and a sequence $\gamma(n, n{+}1)$ of arrows in $\mathcal{D}$ from $\pi^{(n)}$ to $\pi^{(n+1)}$ associated to the successive steps of the algorithm. For $m < n$, we also write $\gamma(m, n)$ for the path from $\pi^{(m)}$ to $\pi^{(n)}$ made of the concatenation of the $\gamma(l, l{+}1)$, $m \leqslant l < n$.

We write $\gamma(T)$ for the infinite path starting from $\pi^{(0)}$ formed by the $\gamma(n, n{+}1)$, $n \geqslant 0$. If $T$ is a standard i.e.m. with no connection, then $\gamma(T)$ is $\infty$-complete ([MMY05, p. 832]). Conversely, an $\infty$-complete path is equal to $\gamma(T)$ for some standard i.e.m. with no connection. On the other hand, for a generalized i.e.m. $T$ with no connection, the path $\gamma(T)$ is not always $\infty$-complete.

*Definition* 2.1. A generalized i.e.m. $T$ is *irrational* if it has no connection and $\gamma(T)$ is $\infty$-complete. We then call $\gamma(T)$ the *rotation number* of $T$.

In the circle case $d = 2$, the Rauzy diagram has one vertex and two arrows. If the rotation number of a circle homeomorphism $T$ has a continued fraction expansion $[a_1, a_2, \ldots]$, the associated $\infty$-complete path takes $a_1$ times the first arrow, then $a_2$ times the second arrow, $a_3$ times the first arrow, … .

From the definition, a standard i.e.m. is irrational if and only if it has no connection. Two standard i.e.m. with no connection are topologically conjugated if and only if they have the same rotation number [Yoc]. More generally, if $T$ is an irrational g.i.e.m. with the same rotation number as a standard i.e.m. $T_0$, then there is, as in the circle case, a semiconjugacy from $T$ to $T_0$, i.e., a continuous nondecreasing surjective map $h$ from the interval $I$ of $T$ onto the interval $I_0$ of $T_0$ such that $T_0 \circ h = h \circ T$ (cf. [Yoc]).

2.5. *Suspension and genus.* Let $T$ be a standard i.e.m. with combinatorial data $\pi = (\pi_t, \pi_b)$. For $\alpha \in \mathcal{A}$, let

$$\lambda_\alpha = |I^t_\alpha| = |I^b_\alpha|, \quad \tau_\alpha = \pi_b(\alpha) - \pi_t(\alpha), \quad \zeta_\alpha = \lambda_\alpha + i\tau_\alpha.$$

In the complex plane, draw a top (resp. bottom) polygonal line from $u_0$ to $u_d$ through $u_0+\zeta_{\pi_t^{-1}(1)}, u_0+\zeta_{\pi_t^{-1}(1)}+\zeta_{\pi_t^{-1}(2)}, \dots$ (resp. $u_0+\zeta_{\pi_b^{-1}(1)}, u_0+\zeta_{\pi_b^{-1}(1)}+\zeta_{\pi_b^{-1}(2)}, \dots$). These two polygonal lines bound a polygon. Gluing the $\zeta_\alpha$ bottom and top sides of the polygon produces a translation surface $M_T$ ([Zor06]). The vertices of the polygon form a set of marked points $\Sigma$ on $M_T$. The cardinality $s$ of $\Sigma$, the genus $g$ of $M_T$, and the number $d$ of intervals are related by

$$d = 2g + s - 1.$$

The genus $g$ can be computed directly from the combinatorial data as follows. Define an antisymmetric matrix $\Omega = \Omega(\pi)$ by

$$\Omega_{\alpha\,\beta} = \begin{cases} +1 & \text{if } \pi_t(\alpha) < \pi_t(\beta), \pi_b(\alpha) > \pi_b(\beta), \\ -1 & \text{if } \pi_t(\alpha) > \pi_t(\beta), \pi_b(\alpha) < \pi_b(\beta), \\ \ \ 0 & \text{otherwise.} \end{cases}$$

Then the rank of $\Omega$ is $2g$. Actually ([Yoc06], [Yoc10]), if one identifies $\mathbb{R}^{\mathcal{A}}$ with the relative homology group $H_1(M_T, \Sigma, \mathbb{R})$ via the basis defined by the sides $\zeta_\alpha$ of the polygon, the image of $\Omega$ coincides with the absolute homology group $H_1(M_T, \mathbb{R})$. Another way to compute $s$ (and thus $g$) consists in going around the marked points, as explained in Section 3.1.

2.6. *The* (*discrete time*) *Kontsevich-Zorich cocycle*. Let $\mathcal{D}$ be a Rauzy diagram on an alphabet $\mathcal{A}$. To each arrow $\gamma$ of $\mathcal{D}$, we associate the matrix $B_\gamma \in \mathrm{SL}(\mathbb{Z}^{\mathcal{A}})$:

$$B_\gamma = \mathbb{I} + E_{\alpha\,\beta},$$

where $\alpha$ is the loser of $\gamma$, $\beta$ is the winner of $\gamma$, and $E_{\alpha\,\beta}$ is the elementary matrix whose only nonzero coefficient is in position $\alpha\,\beta$. For a path $\gamma$ in $\mathcal{D}$ made of the successive arrows $\gamma_1 \cdots \gamma_l$, we associate the product $B_\gamma = B_{\gamma_l} \cdots B_{\gamma_1}$. It belongs to $\mathrm{SL}(\mathbb{Z}^{\mathcal{A}})$ and has nonnegative coefficients.

Let $T$ be a g.i.e.m. with no connection, whose combinatorial data is a vertex of $\mathcal{D}$. Let $\widehat{T}$ be deduced from $T$ by a certain number of steps of the Rauzy-Veech algorithm, and let $\gamma$ be the associated path of $\mathcal{D}$. Let $\Gamma$ be the space of functions on $\sqcup I_\alpha^t$ that are constant on each $I_\alpha^t$, and let $\widehat{\Gamma}$ be the corresponding subspace for $\widehat{T}$. Both $\Gamma$ and $\widehat{\Gamma}$ are canonically identified with $\mathbb{R}^{\mathcal{A}}$. Then $B_\gamma$ is the matrix of the following operator $S$ from $\Gamma$ to $\widehat{\Gamma}$. For $\chi \in \Gamma$,

$$S\chi(x) = \sum_{0 \leqslant i < r(x)} \chi(T^i(x)),$$

where $x$ belongs to the domain $\widehat{I}$ of $\widehat{T}$ and $r(x)$ is the return time of $x$ in $\widehat{I}$.

Let $\mathcal{R}$ be the Rauzy class associated to $\mathcal{D}$. Restricted to standard i.e.m. (considered up to affine conjugacy), the Rauzy-Veech algorithm defines a map

$Q_{RV}$ on the parameter space $\mathcal{R} \times \mathbb{P}(\mathbb{R}^{\mathcal{A}})$. The operator $S$ define a cocycle over these dynamics called *the (extended) Kontsevich-Zorich cocycle.*

## 3. The cohomological equation revisited

3.1. *The boundary operator.* Let $T$ be a generalized i.e.m. on an interval $I$, $I = \sqcup I^t_\alpha = \sqcup I^b_\alpha$ the associated partitions (mod.0), $\pi = (\pi_t, \pi_b)$ the combinatorial data of $T$ on an alphabet $\mathcal{A}$. We denote by ${}_b\alpha, {}_t\alpha, \alpha_b, \alpha_t$ the elements of $\mathcal{A}$ such that $\pi_b({}_b\alpha) = \pi_t({}_t\alpha) = 1$, $\pi_b(\alpha_b) = \pi_t(\alpha_t) = d$.

We denote by $\mathcal{A}^{(2)}$ the union of two disjoint copies of $\mathcal{A}$. Elements of $\mathcal{A}^{(2)}$ are denoted by $(\alpha, L)$ or $(\alpha, R)$ and are associated to the left and right endpoints of the intervals $I^t_\alpha$ (or $I^b_\alpha$). More precisely, for $\upsilon \in \mathcal{A}^{(2)}$, we denote by $u^t(\upsilon)$, $u^b(\upsilon)$ the left endpoints of $I^t_\alpha$, $I^b_\alpha$ respectively if $\upsilon = (\alpha, L)$, and by $u^t(\upsilon)$, $u^b(\upsilon)$ the right endpoints of $I^t_\alpha$, $I^b_\alpha$ respectively if $\upsilon = (\alpha, R)$.

Given combinatorial data $\pi = (\pi_t, \pi_b)$, the set $\mathcal{A}^{(2)}$ is endowed with a permutation $\sigma$ defined as follows:

$$\begin{aligned}
\sigma(\alpha, R) &= (\beta, L), && \text{when } \alpha \neq \alpha_t,\ \pi_t(\beta) = \pi_t(\alpha) + 1,\\
\sigma(\alpha_t, R) &= (\alpha_b, R), &&\\
\sigma(\alpha, L) &= (\beta, R), && \text{when } \alpha \neq {}_b\alpha,\ \pi_b(\beta) = \pi_b(\alpha) - 1,\\
\sigma({}_b\alpha, L) &= ({}_t\alpha, L). &&
\end{aligned}$$

The cycles of $\sigma$ are canonically associated to the marked points of any translation surface constructed by suspension from an i.e.m. having $\pi$ as combinatorial data. We denote by $\Sigma$ the set of cycles of $\sigma$, by $s$ the cardinality of $\Sigma$. We have $d = 2g + s - 1$.

Let $r \geqslant 0$ be an integer. We denote by $C^r(\sqcup I^t_\alpha)$ the space of functions $\varphi$ on $\sqcup I^t_\alpha$ such that, for each $\alpha \in \mathcal{A}$, the restriction of $\varphi$ to $I^t_\alpha$ extends to a $C^r$-function on the closure of $I^t_\alpha$.

For a function $\varphi$ in $C^0(\sqcup I^t_\alpha)$ and $\upsilon \in \mathcal{A}^{(2)}$, we make a slight abuse of notation by writing $\varphi(\upsilon)$ for the limit of $\varphi$ at the left (resp. right) endpoint of $I^t_\alpha$ if $\upsilon = (\alpha, L)$ (resp. $\upsilon = (\alpha, R)$). We also write $\varepsilon(\upsilon) = -1$ if $\upsilon = (\alpha, L)$, $\varepsilon(\upsilon) = +1$ if $\upsilon = (\alpha, R)$.

*Definition* 3.1. The *boundary operator* $\partial : C^0(\sqcup I^t_\alpha) \to \mathbb{R}^\Sigma$ is defined by

$$(\partial\varphi)_C = \sum_{\upsilon \in C} \varepsilon(\upsilon)\, \varphi(\upsilon),$$

where $C$ is any cycle of $\sigma$. The kernel of the boundary operator is denoted by $C^0_\partial(\sqcup I^t_\alpha)$.

Note that

$$\sum_{C \in \Sigma} (\partial\varphi)_C = \sum_{\alpha \in \mathcal{A}} (\varphi(\alpha, R) - \varphi(\alpha, L)). \tag{3.1}$$

When $\varphi$ belongs to $C^1(\sqcup I_\alpha^t)$, this gives

$$\sum_{C\in\Sigma}(\partial\varphi)_C = \int_I D\varphi(x)\,dx. \tag{3.2}$$

The following proposition summarizes the properties of the boundary operator. Recall that $\Gamma \subset C^0(\sqcup I_\alpha^t)$ is the set of functions that are constant on each $I_\alpha^t$. We denote by $\mathbb{R}_0^\Sigma$ the hyperplane of $\mathbb{R}^\Sigma$ formed by the vectors for which the sum of the coordinates vanishes.

Let $M$ be a translation surface constructed by suspension from a standard i.e.m. $T_0$ having $\pi$ as combinatorial data. Then we can identify $\Sigma$ with the set of marked points on $M$, $\Gamma$ with the relative homology group $H_1(M,\Sigma,\mathbb{R})$ (the characteristic function of $I_\alpha^t$ corresponds to oriented parallel sides with label $\alpha$ of the polygon that gives rise to $M$ after the gluing). It is then clear that the operator $\partial$ restricted to $\Gamma$ is indeed the boundary operator

$$\partial : H_1(M,\Sigma,\mathbb{R}) \to H_0(\Sigma,\mathbb{R}) = \mathbb{R}^\Sigma.$$

Proposition 3.2. (1) *For a* g.i.e.m. $T$ *with combinatorial data* $\pi$, *and* $\psi \in C^0(\bar I)$, *one has* $\partial\psi = \partial(\psi\circ T)$.

(2) *The kernel* $\Gamma_\partial$ *of the restriction of* $\partial$ *to* $\Gamma$ *is the image of* $\Omega(\pi)$, *and the image is* $\mathbb{R}_0^\Sigma$.

(3) *The boundary operator* $\partial : C^0(\sqcup I_\alpha^t) \to \mathbb{R}^\Sigma$ *is onto.*

(4) *Let* $T$ *be a* g.i.e.m. *with combinatorial data* $\pi$, *and let* $\widetilde T$, *acting on a subinterval* $\widetilde I \subset I$, *be obtained from* $T$ *by one or several steps of the Rauzy-Veech algorithm. For* $\varphi \in C^0(\sqcup I_\alpha^t)$, *denote by* $S\varphi \in C^0(\sqcup \widetilde I_\alpha^t)$ *the special Birkhoff sums corresponding to the first return in* $\widetilde I$. *Then we have*

$$\partial(S\varphi) = \partial\varphi,$$

*where the left-hand side boundary operator is defined using the combinatorial data* $\widetilde\pi$ *of* $\widetilde T$.

*Proof.* Let $\psi \in C^0(\bar I)$, $C \in \Sigma$. For $\upsilon = (\alpha, R) \in C$ with $\alpha \neq \alpha_t$, we have $u^t(\upsilon) = u^t(\sigma(\upsilon))$ with $\varepsilon(\upsilon) = -\varepsilon(\sigma(\upsilon))$. Therefore, $(\partial\psi)_C = \varepsilon_1\psi(1) - \varepsilon_0\psi(0)$, where $\varepsilon_0$ (resp. $\varepsilon_1$) is 1 or 0, depending whether or not $({}_t\alpha, L)$ (resp. $(\alpha_t, R)$) belongs to $C$.

Similarly, for $\upsilon = (\alpha, L) \in C$ with $\alpha \neq {}_b\alpha$, we have $u^b(\upsilon) = u^b(\sigma(\upsilon))$ with $\varepsilon(\upsilon) = -\varepsilon(\sigma(\upsilon))$. Therefore, $(\partial(\psi\circ T))_C = \varepsilon'_1\psi(1) - \varepsilon'_0\psi(0)$, where $\varepsilon'_0$ (resp. $\varepsilon'_1$) is 1 or 0, depending whether or not $({}_b\alpha, L)$ (resp. $(\alpha_b, R)$) belongs to $C$.

As $\sigma({}_b\alpha, L) = ({}_t\alpha, L)$ and $\sigma(\alpha_t, R) = (\alpha_b, R)$, we have $\varepsilon_0 = \varepsilon'_0$ and $\varepsilon_1 = \varepsilon'_1$. This proves (1).

The restriction of the operator $\partial$ to $\Gamma$ was described in homological terms just before the proposition. It follows from this description that the image of $\Gamma$ by $\partial$ is indeed $\mathbb{R}_0^\Sigma$. As the image of $\Omega(\pi)$ is identified with the image of the absolute homology group $H_1(M,\mathbb{R})$ in $H_1(M,\Sigma,\mathbb{R})$, the proof of (2) is complete.

Let $\phi^*(x) = x$. Then $\sum_{C\in\Sigma}(\partial\phi^*)_C = 1$. Thus, the image of $\partial$ is strictly bigger than $\mathbb{R}_0^\Sigma$, which proves (3).

To prove (4), it is sufficient to consider the case where $\widetilde{T}$ is obtained from $T$ by one step of the Rauzy-Veech algorithm. We assume that this step is of top type, the case of bottom type being symmetric. Denote by $\widetilde{\sigma}$ the permutation of $\mathcal{A}^{(2)}$ defined from the combinatorial data $\widetilde{\pi}$ of $\widetilde{T}$, by $\alpha'_t$ the element of $\mathcal{A}$ such that $\pi_b(\alpha'_t) = \pi_b(\alpha_t)+1$, by $\widetilde{\alpha}_b$ the element of $\mathcal{A}$ such that $\pi_b(\widetilde{\alpha}_b) = d-1$. We have $\sigma(\upsilon) = \widetilde{\sigma}(\upsilon)$, except for

$$\begin{aligned} \sigma(\alpha'_t, L) &= (\alpha_t, R), & \widetilde{\sigma}(\alpha'_t, L) &= (\alpha_b, R),\\ \sigma(\alpha_t, R) &= (\alpha_b, R), & \widetilde{\sigma}(\alpha_t, R) &= (\widetilde{\alpha}_b, R),\\ \sigma(\alpha_b, L) &= (\widetilde{\alpha}_b, R), & \widetilde{\sigma}(\alpha_b, L) &= (\alpha_t, R). \end{aligned}$$

Let $\varphi \in C^0(\sqcup I^t_\alpha)$. For $\upsilon \in \mathcal{A}^{(2)}$, we have $S\varphi(\upsilon) = \varphi(\upsilon)$ except for

$$\begin{aligned} S\varphi(\alpha_b, L) &= \varphi(\alpha_b, L) + \varphi(u^b_{d-1}),\\ S\varphi(\alpha_b, R) &= \varphi(\alpha_b, R) + \varphi(\alpha_t, R),\\ S\varphi(\alpha_t, R) &= \varphi(u^b_{d-1}). \end{aligned}$$

From these formulas, it is easy to see that $\partial(S\varphi) = \partial\varphi$. □

*Remark* 3.3. Let $\varphi \in C^0_\partial(\sqcup I^t_\alpha)$ such that $\varphi(\upsilon) = 0$ for all $\upsilon \in \mathcal{A}^{(2)}$. Assume also that there exists $\psi \in C(\overline{I})$ such that $\varphi = \psi \circ T - \psi$. Then, given such a function $\psi$, there is a family $(\psi_C)_{C\in\Sigma}$ such that

$$\psi(u^t(\upsilon)) = \psi(u^b(\upsilon)) = \psi_C$$

for all $\upsilon \in C$, all $C \in \Sigma$. The function $\psi$, hence also the family $(\psi_C)_{C\in\Sigma}$, is only well defined up to an additive constant by $\varphi$. We will denote by $\nu(\varphi)$ the image in $\mathbb{R}^\Sigma/\mathbb{R}$ of the family $(\psi_C)_{C\in\Sigma}$.

3.2. *Continuity of the solutions of the cohomological equation.* The main tool in [MMY05] to obtain bounded solutions of the cohomological equations is the Gottschalk-Hedlund theorem ([GH55], [Her79]).

THEOREM 3.4. *Let $f$ be a minimal homeomorphism of a compact metric space $X$, and let $\varphi$ be a continuous function on $X$. The following properties are equivalent*:

(1) $\varphi = \psi \circ f - \psi$ *for some continuous function $\psi$ on $X$.*
(2) $\varphi = \psi \circ f - \psi$ *for some bounded function $\psi$ on $X$.*
(3) *There exists $C > 0$ such that the Birkhoff sums of $\varphi$ satisfy $|S_n\varphi(x)| < C$ for all $n \in \mathbb{Z}$, $x \in X$.*

(4) *There exist* $C > 0$, $x_0 \in X$ *such that the Birkhoff sums of* $\varphi$ *satisfy* $|S_n\varphi(x_0)| < C$ *for all* $n \geqslant 0$.

Let $T$ be a (standard) i.e.m. with no connection. Let

$$Z := \{T^{-m}(u_i^t), T^n(u_j^b);\ 0 < i, j < d,\ m \geqslant 0,\ n \geqslant 0\}$$

be the union of the orbits of the singularities of $T$ and $T^{-1}$. In the interval $\overline{I}$ where $T$ is acting, we split the points of $Z$ into right and left limits to get a compact metric space $\widehat{I}$ (homeomorphic to a Cantor set) on which $T$ induces a minimal homeomorphism $\widehat{T}$. Denote by $p$ the canonical projection from $\widehat{I}$ onto $\overline{I}$ so that $p \circ \widehat{T} = T \circ p$. Let $\varphi \in C^0(\sqcup I_\alpha^t)$; then $\widehat{\varphi} := \varphi \circ p$ is continuous on $\widehat{I}$. Assume that the Birkhoff sums $(S_n\varphi)_{n\geqslant 0}$ of $\varphi$ for $T$ are bounded. Then the same is true for the Birkhoff sums of $\widehat{\varphi}$ for $\widehat{T}$, and we conclude from Gottschalk-Hedlund's theorem that there exists a continuous function $\widehat{\psi}$ on $\widehat{I}$ such that $\widehat{\varphi} = \widehat{\psi} \circ \widehat{T} - \widehat{\psi}$. For an arbitrary continuous function $\widehat{\psi}$ on $\widehat{I}$, there is *a priori* no continuous function $\psi$ on $\overline{I}$ such that $\widehat{\psi} = \psi \circ p$. However, we have the following elementary result, which was not observed in [MMY05].

Proposition 3.5. *Let* $\widehat{\psi}$ *be a continuous function on* $\widehat{I}$. *Assume that* $\widehat{\varphi} = \widehat{\psi} \circ \widehat{T} - \widehat{\psi}$ *is induced by a function* $\varphi \in C^0(\sqcup I_\alpha^t)$. *Then* $\widehat{\psi}$ *is induced by a continuous function on* $\overline{I}$.

*Proof.* The continuous function $\widehat{\psi}$ on $\widehat{I}$ is induced by a continuous function on $\overline{I}$ if and only if for every $z \in Z$, the values of $\widehat{\psi}$ on the two points $z_l$, $z_r$ of $\widehat{I}$ sitting over $z$ are equal. For $z \in Z$, let $\delta\psi(z) := \widehat{\psi}(z_r) - \widehat{\psi}(z_l)$. If $z \neq u_i^t$, the values of $\widehat{\psi} - \widehat{\psi} \circ \widehat{T}$ at $z_l$ and $z_r$ are the same; hence, $\delta\psi(z) = \delta\psi(T(z))$. Therefore, for every $0 < i, j < d,\ m, n \geqslant 0$, we have $\delta\psi(T^{-m}(u_i^t)) = \delta\psi(u_i^t)$ and $\delta\psi(T^n(u_j^b)) = \delta\psi(u_j^b)$. As $\widehat{\psi}$ is continuous on $\widehat{I}$ and every half orbit $\{T^n(u_j^b);\ n \geqslant 0\}$ or $\{T^n(u_i^t);\ n \leqslant 0\}$ is dense, we must have $\delta\psi(z) = 0$ for all $z \in Z$, and the conclusion of the proposition follows. □

Corollary 3.6. *Let* $T$ *be a* (*standard*) i.e.m. *with no connection*, $\varphi \in C^0(\sqcup I_\alpha^t)$. *If the Birkhoff sums* $(S_n\varphi)_{n\geqslant 0}$ *of* $\varphi$ *for* $T$ *are bounded, there exists* $\psi \in C^0(\overline{I})$ *such that* $\varphi = \psi \circ T - \psi$.

3.3. *Interval exchange maps of Roth type.* We recall the diophantine condition on the rotation number of an i.e.m. introduced in [MMY05].

Let $\underline{\gamma}$ be an $\infty$-complete path in a Rauzy diagram $\mathcal{D}$. Write $\underline{\gamma}$ as an infinite concatenation

$$\underline{\gamma} = \gamma(1) * \cdots * \gamma(n) * \cdots$$

of finite complete paths of minimal length. Then for $n > 0$, define

$$Z(n) := B_{\gamma(n)}, \quad B(n) := B_{\gamma(1)*\cdots*\gamma(n)} = Z(n)\cdots Z(1).$$

We introduce three conditions.

(a) For all $\tau > 0$, $||Z(n+1)|| = \mathcal{O}(||B(n)||^{\tau})$.
(b) There exists $\theta > 0$ and a hyperplane $\Gamma_0 \subset \Gamma = \mathbb{R}^{\mathcal{A}}$ such that

$$||B(n)_{|\Gamma_0}|| = \mathcal{O}(||B(n)||^{1-\theta}).$$

(c) Define

$$\Gamma_s = \{\chi \in \Gamma, \exists \tau > 0,\ ||B(n)\chi|| = \mathcal{O}(||B(n)||^{-\tau})\}$$

and $\Gamma_s^{(n)} := B(n)\Gamma_s$ for $n \geqslant 0$. For $k < \ell$, denote by $B_s(k,\ell)$ the restriction of $B(k,\ell) := B_{\gamma(k+1)*\cdots*\gamma(\ell)}$ to $\Gamma_s^{(k)}$ and by $B_\flat(k,\ell)$ the operator from $\Gamma/\Gamma_s^{(k)}$ to $\Gamma/\Gamma_s^{(\ell)}$ induced by $B(k,\ell)$. We ask that, for all $\tau > 0$,

$$||B_s(k,\ell)|| = \mathcal{O}(||B(\ell)||^{\tau}), \quad ||(B_\flat(k,\ell))^{-1}|| = \mathcal{O}(||B(\ell)||^{\tau}).$$

*Remark* 3.7. (1) The definition of $Z(n)$ is slightly different from the definition in [MMY05], but an elementary computation shows that condition (a) with the present definition is equivalent to condition (a) with the definition of [MMY05].

(2) Condition (b) is formulated in a slightly different way from that in [MMY05], in order to depend only on the rotation number and not of the length data. But actually condition (b) implies that there exists exactly one normalized standard i.e.m. $T$ with rotation number $\underline{\gamma}$, that $T$ is uniquely ergodic, and that the hyperplane $\Gamma_0$ of condition (b) must be formed on functions in $\Gamma$ with mean 0 on $I$.

(3) For any combinatorial data, the set of length data for which the associated i.e.m. has a rotation number satisfying (a), (b), (c) has full measure. For (c), this is an immediate consequence of Oseledets theorem. For (b), it is a consequence of the fact that the larger Lyapunov of the KZ-cocycle is simple (Veech). For (a), a proof is provided in [MMY05], but much better diophantine estimates were later obtained in [AGY06].

(4) It follows from Forni's theorem [For02] on the hyperbolicity of the KZ-cocycle that, for almost all rotation numbers, one has $\dim \Gamma_s = g$.

*Definition* 3.8. A rotation number $\underline{\gamma}$ (or a standard i.e.m. $T$ having $\underline{\gamma}$ as rotation number) is of *Roth type* if the three conditions (a), (b), (c) are satisfied. It is of *restricted Roth type* if, moreover, one has $\dim \Gamma_s = g$.

*Remark* 3.9. Let $T$ be a standard i.e.m. of restricted Roth type. Then $\Gamma_s$ is exactly equal to the subspace $\Gamma_T \subset \Gamma$ of functions $\chi \in \Gamma$, which can be written as $\psi \circ T - \psi$, for some $\psi \in C^0(\overline{I})$. Indeed, we have $\Gamma_s \subset \Gamma_T$ from [Zor96] or [MMY05]. On the other hand, $\Gamma_T$ is contained in the subspace $\Gamma^\sharp \subset \Gamma$ of functions that go to 0 under the KZ-cocycle. As the KZ-cocycle acts trivially on $\Gamma/\Gamma_\partial$, $\Gamma^\sharp$ is contained in $\Gamma_\partial$ and is actually an isotropic subspace of this symplectic space. As $\Gamma_s \subset \Gamma^\sharp$ and $\dim \Gamma_s = g$, we must have $\Gamma_T = \Gamma_s = \Gamma^\sharp$.

Let $T$ be a standard i.e.m. of restricted Roth type. Choose a $g$-dimensional subspace $\Gamma_u \subset \Gamma_\partial$ such that $\Gamma_\partial = \Gamma_s \oplus \Gamma_u$. We recall the main result of [MMY05], in the form that is convenient for our purpose. We denote by $C^{1+BV}(\sqcup I^t_\alpha)$ the space of functions $\varphi \in C^1(\sqcup I^t_\alpha)$ such that $D\varphi$ is a function of bounded variation. We write

$$|D\varphi|_{BV} = \sum_\alpha \mathrm{Var}_{I^t_\alpha} D\varphi, \quad ||\varphi||_{1+BV} = ||\varphi||_0 + ||D\varphi||_0 + |D\varphi|_{BV}.$$

We denote by $C^{1+BV}_\partial(\sqcup I^t_\alpha)$ the intersection of $C^{1+BV}(\sqcup I^t_\alpha)$ with the kernel of the boundary operator $\partial$.

THEOREM 3.10. *Let $T$ be a standard* i.e.m. *of restricted Roth type. There exist bounded linear operators $L_0 : \varphi \mapsto \psi$ from $C^{1+BV}_\partial(\sqcup I^t_\alpha)$ to $C^0(\bar{I})$ and $L_1 : \varphi \mapsto \chi$ from $C^{1+BV}_\partial(\sqcup I^t_\alpha)$ to $\Gamma_u$ such that, for all $\varphi \in C^{1+BV}_\partial(\sqcup I^t_\alpha)$, we have*

$$\varphi = \chi + \psi \circ T - \psi.$$

*Remark* 3.11. In [MMY05] the result was formulated in the following weaker way. For every $\varphi \in C^{1+BV}(\sqcup I^t_\alpha)$ with $\int_I D\varphi(x)\,dx = 0$, there exists $\chi \in \Gamma$ and a bounded function $\psi$ on $I$ such that $\varphi = \chi + \psi \circ T - \psi$. To obtain the present stronger form, we observe that

- By Proposition 3.5 or Corollary 3.6, the solution $\psi$ is automatically continuous on $\bar{I}$.
- The condition $\int_I D\varphi(x)\,dx = 0$ means that we ask that the sum of the components of $\partial\varphi$ is 0. In view of Proposition 3.2(2), it is then possible to subtract $\chi \in \Gamma$ in order to have $\partial(\varphi - \chi) = 0$. However, in view of Proposition 3.2(1), it is more natural to start with $\varphi \in C^{1+BV}_\partial(\sqcup I^t_\alpha)$. Then, the correction $\chi$ must belong to $\Gamma_\partial$. As $\Gamma_s = \Gamma_T$ from Remark 3.9, there is a unique way to find the correction $\chi \in \Gamma_u$ in order to have $\varphi - \chi = \psi \circ T - \psi$ for some $\psi \in C^0(\bar{I})$.
- That the operator $\varphi \mapsto \psi$ (and consequently also the operator $\varphi \mapsto \chi$) is bounded follows from the proof in [MMY05]. One shows that, for some $\chi \in \Gamma$, the Birkhoff sums of $\varphi - \chi$ satisfy

  $$||S_n(\varphi - \chi)||_0 \leqslant C||\varphi||_{1+BV},$$

  and then Gottschalk-Hedlund's theorem implies that $||\psi||_0 \leqslant C\,||\varphi||_{1+BV}$.

In Appendix A, we show that it is possible to deal in the same way with functions $\varphi \in C^{1+\tau}(\sqcup I^t_\alpha)$ for any $\tau > 0$.

3.4. *The cohomological equation in higher smoothness.* This subsection is a slight modification of the corresponding subsection in [MMY05], taking the boundary operator into account. We assume that $T$ is a standard i.e.m. with no connection.

For $r \geqslant 1$, we denote by $\Gamma(r)$ the set of functions $\chi \in C^\infty(\sqcup I^t_\alpha)$ such that the restriction of $\chi$ to each $I^t_\alpha$ is a polynomial of degree $< r$, by $\Gamma_\partial(r)$ the subspace of functions $\chi \in \Gamma(r)$ that satisfy $\partial D^i \chi = 0$ for all $0 \leqslant i < r$, by $\Gamma_T(r)$ the subspace of functions $\chi \in \Gamma(r)$, which can be written as $\psi \circ T - \psi$ for some $\psi \in C^{r-1}(\bar{I})$. We observe that for $\psi \in C^{r-1}(\bar{I})$, we have $\partial D^i(\psi \circ T - \psi) = 0$ for all $0 \leqslant i < r$; hence, $\gamma_T(r) \subset \Gamma_\partial(r)$.

Proposition 3.12. *One has*

$$\dim\Gamma(r) = rd, \quad \dim\Gamma_\partial(r) = (2g-1)r + 1, \quad \dim\Gamma_T(r) = \dim\Gamma_T + r - 1.$$

*Proof.* The first assertion is obvious. For $r \geqslant 1$, the derivation operator $D$ sends $\Gamma(r+1)$ into $\Gamma(r)$, $\Gamma_T(r+1)$ into $\Gamma_T(r)$, $\Gamma_\partial(r+1)$ into $\Gamma_\partial(r)$.

Let $\chi \in \Gamma_T(r)$. Write $\chi = \psi \circ T - \psi$ with $\psi \in C^{r-1}(\bar{I})$. Let $\psi_1 \in C^r(\bar{I})$ be a primitive of $\psi$ and $\chi_1 := \psi_1 \circ T - \psi_1$. Then $\chi_1$ belongs to $\Gamma_T(r+1)$ and $D\chi_1 = \chi$. Therefore, $D : \Gamma_T(r+1) \to \Gamma_T(r)$ is onto. If $\chi_1 \in \Gamma_T(r+1)$ satisfies $D\chi_1 = 0$, we write $\chi_1 = \psi_1 \circ T - \psi_1$ with $\psi_1 \in C^r(\bar{I})$. Then $\psi := D\psi_1$ is continuous and $T$-invariant, hence constant (as T is minimal), which implies that $\varphi_1 \in \mathbb{R}\delta$. Conversely, $\mathbb{R}\delta$ is contained in the kernel of $D : \Gamma_T(r+1) \to \Gamma_T(r)$, hence equal to this kernel. We conclude that $\dim \Gamma_T(r) = \dim \Gamma_T + r - 1$.

Let $\chi \in \Gamma_\partial(r)$. Let $\chi_1 \in \Gamma(r+1)$ with $D\chi_1 = \chi$. We have $\chi_1 \in \Gamma_\partial(r+1)$ if and only if $\partial\chi_1 = 0$. The sum of the components of $\partial\chi_1$ is equal to $\int_I \chi(x)\,dx$; hence, $\int_I \chi(x)\,dx = 0$ is a necessary condition for $\chi$ to be in the image by $D$ of $\Gamma_\partial(r+1)$. On the other hand, the condition is also sufficient by Proposition 3.2(2). Also by Proposition 3.12(2), the kernel of $D$ in $\Gamma_\partial(r)$ is $\Gamma_\partial = \mathrm{Im}\Omega(\pi)$, which is of dimension $2g$. We conclude by induction on $r$ that $\dim\Gamma_\partial(r) = (2g-1)r + 1$. □

We define $C^{r+BV}(\sqcup I^t_\alpha)$ as the space of functions $\varphi \in C^r(\sqcup I^t_\alpha)$ such that $D^r\varphi$ is of bounded variation. We endow this space with its natural norm. We denote by $C^{r+BV}_\partial(\sqcup I^t_\alpha)$ the subspace of $\varphi \in C^{r+BV}(\sqcup I^t_\alpha)$ such that $\partial D^i \varphi = 0$ for all $0 \leqslant i < r$.

Theorem 3.13. *There exist a bounded operator* $\Pi : C^{r+BV}_\partial(\sqcup I^t_\alpha) \to \Gamma_\partial(r)/\Gamma_T(r)$, *extending the canonical projection from* $\Gamma_\partial(r)$ *to* $\Gamma_\partial(r)/\Gamma_T(r)$, *and a bounded operator* $\varphi \mapsto \psi$ *from the kernel of* $\Pi$ *to* $C^{r-1}(\bar{I})$ *such that if* $\varphi \in C^{r+BV}_\partial(\sqcup I^t_\alpha)$ *satisfies* $\Pi(\varphi) = 0$, *then we have*

$$\varphi = \psi \circ T - \psi.$$

In other terms, if we choose a subspace $\Gamma_u(r) \subset \Gamma_\partial(r)$ such that $\Gamma_\partial(r) = \Gamma_T(r) \oplus \Gamma_u(r)$ and identify the quotient $\Gamma_\partial(r)/\Gamma_T(r)$ with $\Gamma_u(r)$, we can write any $\varphi \in C^{r+BV}_\partial(\sqcup I^t_\alpha)$ as $\varphi = \Pi(\varphi) + \psi \circ T - \psi$, with $\psi \in C^{r-1}(\bar{I})$.

*Proof.* The proof is by induction on $r$, the case $r = 1$ being the theorem above. Assume that $r > 1$ and the result is true for $r-1$. Let $\varphi \in C^{r+BV}_\partial(\sqcup I^t_\alpha)$.

According to the induction hypothesis, we can write

$$D\varphi = \chi_1 + \psi_1 \circ T - \psi_1,$$

where $\chi_1 \in \Gamma_\partial(r-1)$ and $\psi_1 \in C^{r-2}(\bar{I})$. Let $\psi \in C^{r-1}(\bar{I})$ be a primitive of $\psi_1$. Then there exists a primitive $\chi$ of $\chi_1$ such that

$$\varphi = \chi + \psi \circ T - \psi.$$

As $\partial\varphi = 0$, we must also have $\partial\chi = 0$, and thus $\chi$ belongs to $\Gamma_\partial(r)$. This completes the proof of the induction step. □

## 4. **A conjugacy invariant**

4.1. *Definition of the invariant.* Let $r$ be an integer $\geqslant 1$ or $\infty$. We denote by $J^r$ the group of $r$-jets at 0 of orientation-preserving diffeomorphisms of $\mathbb{R}$ fixing 0.

Let $\pi = (\pi_t, \pi_b)$ an element of a Rauzy class $\mathcal{R}$ on an alphabet $\mathcal{A}$, and let $T$ be a generalized i.e.m. of class $C^r$ with combinatorial data $\pi$. For each $\upsilon \in \mathcal{A}^{(2)}$, we define an element $j(T,\upsilon) \in J^r$ as the $r$-jet at 0 of

$$x \mapsto T(u^t(\upsilon) + x) - u^b(\upsilon),$$

where $x$ varies in an interval of the form $(0, x_0)$ when $\upsilon = (\alpha, L)$, $(-x_0, 0)$ when $\upsilon = (\alpha, R)$, and the $r$-jet at 0 exists by definition of a generalized i.e.m. of class $C^r$.

For each cycle $C$ of $\sigma$, we choose an element $\upsilon_0 \in C$ and we write $C = \{\upsilon_0, \upsilon_1 = \sigma(\upsilon_0), \ldots, \upsilon_\kappa\}$. We then define

$$J(T,C) := j(T,\upsilon_0)^{\varepsilon(\upsilon_0)} j(T,\upsilon_1)^{\varepsilon(\upsilon_1)} \cdots j(T,\upsilon_\kappa)^{\varepsilon(\upsilon_\kappa)} \quad \in J^r,$$

where, as in Section 3.1, we have $\varepsilon(\upsilon) = -1$ if $\upsilon = (\alpha, L)$, $\varepsilon(\upsilon) = +1$ if $\upsilon = (\alpha, R)$.

*Definition* 4.1. The invariant $J(T)$ of $T$ is the family, parametrized by the cycles $C \in \Sigma$ of $\sigma$, of the conjugacy classes in $J^r$ of the $J(T,C)$.

It is clear that the conjugacy class of $J(T,C)$ does not depend on the choice of the element $\upsilon_0 \in C$. When $d = 2$, the invariant $J(T)$ is the obstruction for $T$ to be $C^r$-conjugated to a $C^r$-diffeomorphism of the circle.

4.2. *Conjugacy classes in $J^r$.* The classification of elements in $J^\infty$ up to conjugacy is well known and is a simple exercise. The classification in $J^r$ for finite $r$ is an obvious consequence, truncating to order $r$ the Taylor developments. It is not used in the rest of the paper.

Let $j$ be an element of $J^\infty$. If $j$ is distinct from the neutral element of $J^\infty$, its contact with the identity is an integer $k \geqslant 1$. If $k = 1$, i.e., the linear part of $j$ is distinct from the identity, then $j$ is conjugate to its linear part. If $k > 1$, there exists in the conjugacy class of $j$ a unique element of the form $x \mapsto x \pm x^k + ax^{2k-1}$ $(a \in \mathbb{R})$.

4.3. *Invariance under conjugacy.* Let $r$ be an integer $\geqslant 1$ or $\infty$. Let $\pi = (\pi_t, \pi_b)$ be an element of a Rauzy class $\mathcal{R}$ on an alphabet $\mathcal{A}$, and let $T$ be a generalized i.e.m. of class $C^r$ with combinatorial data $\pi$.

Let $h$ be a $C^r$-orientation-preserving diffeomorphism from the interval $I = (u_0, u_d)$ for $T$ to some other open bounded interval $\widehat{I} = (\widehat{u}_0, \widehat{u}_d)$, which extends to a $C^r$-diffeomorphism from the closure of $I$ to the closure of $\widehat{I}$. Let $\widehat{T} = h \circ T \circ h^{-1}$, acting on $\widehat{I}$.

Proposition 4.2. *The invariants of $T$ and $\widehat{T}$ are the same.*

*Proof.* For $x_0$ in the closure of $I$, let $j(h, x_0)$ be the $r$-jet at 0 of $x \mapsto h(x_0 + x) - h(x_0)$. For $\upsilon \in \mathcal{A}^{(2)}$, we have

$$j(\widehat{T}, \upsilon) = j(h, u^b(\upsilon)) j(T, \upsilon) j(h, u^t(\upsilon))^{-1}.$$

Writing $j(\widehat{T}, \upsilon)^{\varepsilon(\upsilon)} = j(h, x_+(\upsilon)) j(T, \upsilon)^{\varepsilon(\upsilon)} j(h, x_-(\upsilon))^{-1}$, we check that for all $\upsilon \in \mathcal{A}^{(2)}$, we have $x_-(\upsilon) = x_+(\sigma(\upsilon))$:

- if $\upsilon = (\alpha, R)$, $\alpha \neq \alpha_t$, then $x_-(\upsilon) = x_+(\sigma(\upsilon)) = u^t(\upsilon)$;
- if $\upsilon = (\alpha, L)$, $\alpha \neq {}_b\alpha$, then $x_-(\upsilon) = x_+(\sigma(\upsilon)) = u^b(\upsilon)$;
- if $\upsilon = (\alpha_t, R)$, then $x_-(\upsilon) = x_+(\sigma(\upsilon)) = 1$;
- if $\upsilon = ({}_b\alpha, L)$, then $x_-(\upsilon) = x_+(\sigma(\upsilon)) = 0$.

Therefore, for $C \in \Sigma$, $\upsilon_0$ as in the definition of $J(T, C)$, we obtain

$$J(\widehat{T}, C) = j(h, x_+(\upsilon_0)) J(T, C) j(h, x_+(\upsilon_0))^{-1}.$$

The proof of the proposition is complete. □

4.4. *Invariance under renormalization.* Let $r$ be an integer $\geqslant 1$ or $\infty$. Let $\pi = (\pi_t, \pi_b)$ an element of a Rauzy class $\mathcal{R}$ on an alphabet $\mathcal{A}$, and let $T$ be a generalized i.e.m. of class $C^r$ with combinatorial data $\pi$.

We assume that $u^t_{d-1} \neq u^b_{d-1}$, so we can perform one step of the Rauzy-Veech algorithm to obtain a generalized i.e.m. $\widetilde{T}$, which is also of class $C^r$. We denote by $\widetilde{\pi}$ the combinatorial data for $\widetilde{T}$. As in Proposition 3.2(4), the set of cycles of the permutation $\widetilde{\sigma}$ of $\mathcal{A}^{(2)}$ induced by $\widetilde{\pi}$ is naturally identified with $\Sigma$.

Proposition 4.3. *The invariants of $T$ and $\widetilde{T}$ are the same.*

*Proof.* We assume that the step of the Rauzy-Veech algorithm from $T$ to $\widetilde{T}$ is of top type, the case of bottom type being symmetric. Denote by $\alpha'_t$ the element of $\mathcal{A}$ such that $\pi_b(\alpha'_t) = \pi_b(\alpha_t) + 1$, by $\widetilde{\alpha}_b$ the element of $\mathcal{A}$ such that $\pi_b(\widetilde{\alpha}_b) = d - 1$. We have $\sigma(\upsilon) = \widetilde{\sigma}(\upsilon)$, except for

$$\begin{aligned} \sigma(\alpha'_t, L) &= (\alpha_t, R), & \widetilde{\sigma}(\alpha'_t, L) &= (\alpha_b, R), \\ \sigma(\alpha_t, R) &= (\alpha_b, R), & \widetilde{\sigma}(\alpha_t, R) &= (\widetilde{\alpha}_b, R), \\ \sigma(\alpha_b, L) &= (\widetilde{\alpha}_b, R), & \widetilde{\sigma}(\alpha_b, L) &= (\alpha_t, R). \end{aligned}$$

On the other hand, we have $j(\widetilde{T}, \upsilon) = j(T, \upsilon)$, except for

$$j(\widetilde{T}, (\alpha_t, R)) = j^*,$$
$$j(\widetilde{T}, (\alpha_b, L)) = j^* j(T, (\alpha_b, L)),$$
$$j(\widetilde{T}, (\alpha_b, R)) = j(T, \alpha_t, R)) j(T, (\alpha_b, R)),$$

where $j^*$ is the $r$-jet of $T$ at $u^t_{d-1}$. Thus, we have

$$j(\widetilde{T}, (\alpha_b, L))^{-1}\, j(\widetilde{T}, (\alpha_t, R)) = j(T, (\alpha_b, L))^{-1}.$$

In view of the formulas for $\widetilde{\sigma}$, we obtain the cancellations that prove the proposition. □

4.5. *Relation with the boundary operator.* Let $r$ be an integer $\geqslant 1$ or $\infty$. Let $\pi = (\pi_t, \pi_b)$ an element of a Rauzy class $\mathcal{R}$ on an alphabet $\mathcal{A}$, let $T_0$ be a standard i.e.m. with combinatorial data $\pi$, and let $(T_t)_{t\in[-t_0,t_0]}$ be a family of g.i.e.m. of class $C^r$ through $T_0$ with the same combinatorial data $\pi$.

We assume that $t \mapsto T_t$ is of class $C^1$ in the following sense: denote by $u^t_1(t) < \cdots < u^t_{d-1}(t)$ the singularities of $T_t$, by $u^b_1(t) < \cdots < u^b_{d-1}(t)$ those of $T_t^{-1}$; then the functions $t \mapsto u^t_i(t)$, $t \mapsto u^b_j(t)$ are of class $C^1$. Moreover, for each $\alpha \in \mathcal{A}$, each $0 \leqslant i \leqslant r$, the partial derivative $\partial_t \partial^i_x T_t(x)$ should be defined on $\{(t, x);\, t \in [-t_0, t_0],\, x \in I^t_\alpha(t)\,\}$ and extend to a continuous function on the closure of this set (i.e., including the endpoints of $I^t_\alpha(t)$). The function $\varphi(x) := \frac{d}{dt}_{|t=0} T_t(x)$ is then an element of $C^r(\sqcup I^t_\alpha)$.

Proposition 4.4. (1) *One has $\partial\varphi = 0$. Conversely, for any $\varphi \in C^r_\partial(\sqcup I^t_\alpha)$, there exists a $C^1$-family of* g.i.e.m. *of class $C^r$ such that $\varphi = \frac{d}{dt}_{|t=0} T_t$.*

(2) *Assume that, for some $1 \leqslant k \leqslant r$, the conjugacy invariant of $T_t$ in $J^k$ is trivial for all $t \in [-t_0, t_0]$. Then one has $\partial D^\ell \varphi = 0$ for all $1 \leqslant \ell \leqslant k$.*

*Proof.* (1) For $\upsilon \in \mathcal{A}^{(2)}$, let $\delta u^t(\upsilon) = \frac{d}{dt}_{|t=0} u^t(\upsilon, t), \delta u^b(\upsilon) = \frac{d}{dt}_{|t=0} u^b(\upsilon, t)$. Differentiating at $t = 0$, the relation $T_t(u^t(\upsilon, t)) = u^b(\upsilon, t)$ gives $\delta u^t(\upsilon) + \varphi(u^t(\upsilon)) = \delta u^b(\upsilon)$, from which $\partial\varphi = 0$ follows easily.

Conversely, for $\varphi \in C^r_\partial(\sqcup I^t_\alpha)$, one can choose the $u^t_i(t), u^b_j(t)$ such that $\delta u^t(\upsilon) + \varphi(u^t(\upsilon)) = \delta u^b(\upsilon)$. Then it is easy to complete the construction to get a family $(T_t)$ with the required property.

(2) Writing the $k$-jet of a germ of $C^r$-diffeomorphism $f$ of $(\mathbb{R}, 0)$ as $(Df(0), \ldots, D^k f(0))$, we have, for $\upsilon \in \mathcal{A}^{(2)}$,

$$\frac{d}{dt}_{|t=0} j(T_t, \upsilon) = (D\varphi(u^t(\upsilon)), \ldots, D^k\varphi(u^t(\upsilon))).$$

As the product close to the identity in any Lie group (like $J^k$) is commutative up to second order terms, the assertion of the proposition follows. □

*Remark* 4.5. The conjugacy invariant in $J^1$ (which is commutative) can be defined directly from the boundary operator. Identifying $J^1$ with $\mathbb{R}$ by associating to a germ $f$ the logarithm of its derivative at 0, we have indeed that the invariant in $J^1$ of a g.i.e.m. $T$ of class $C^1$ is $\partial \log DT$, where $\log DT$ is considered as a function in $C(\sqcup I^t_\alpha)$.

*Remark* 4.6. For an affine i.e.m. $T$, the invariant in $J^\infty$ coincides with the invariant in $J^1$. The function $\log DT$ belongs to $\Gamma$, and the invariant takes its values in $\mathbb{R}^\Sigma_0$.

## 5. **The main theorem: statement and reduction to the simple case**

5.1. *The setting.* Let $\pi = (\pi_t, \pi_b)$ an element of a Rauzy class $\mathcal{R}$ on an alphabet $\mathcal{A}$, and let $T_0$ be a standard i.e.m. *of restricted Roth type* with combinatorial data $\pi$.

We fix an integer $r \geqslant 2$. We will consider a smooth family $(T_t)$ through $T_0$ of generalized i.e.m., acting on the same interval $I = (u_0, u_d)$. Our main theorem will describe the set of parameters for which $T_t$ is conjugated to $T_0$ by a $C^r$-diffeomorphism of the closure of $I$ which is $C^r$-close to the identity.

We set

$$d^* = (2r+1)(g-1) + s.$$

Let $\ell$ be an integer $\geqslant 0$. The parameter $t$ runs in a neighborhood $V := [-t_0, t_0]^{\ell+d^*}$ of 0 in $\mathbb{R}^{\ell+d^*}$. We write $t = (t', t'')$ with $t' \in [-t_0, t_0]^\ell$ and $t'' \in [-t_0, t_0]^{d^*}$. We also assume that

- Each $T_t$ is a generalized i.e.m. (with the same combinatorial data as $T_0$) of class $C^{r+3}$.
- The map $t \mapsto T_t$ is of class $C^1$ in the following sense. Denote by $u^t_1(t) < \cdots < u^t_{d-1}(t)$ the singularities of $T_t$, by $u^b_1(t) < \cdots < u^b_{d-1}(t)$ those of $T_t^{-1}$. Then the functions $t \mapsto u^t_i(t)$, $t \mapsto u^b_j(t)$ are of class $C^1$. Moreover, for each $\alpha \in \mathcal{A}$, each $0 \leqslant i \leqslant r+3$, each $1 \leqslant i \leqslant \ell + d^*$, the partial derivative $\partial_{t_j} \partial^i_x T_t(x)$ should be defined on $\{(t,x);\, t \in V,\, x \in I^t_\alpha(t)\}$ and extend to a continuous function on the closure of this set (i.e., including the endpoints of $I^t_\alpha(t)$).

As we look for g.i.e.m. that are $C^R$-conjugated to standard i.e.m., it is certainly natural and necessary to assume that the conjugacy invariant in $J^r$ of $T_t$ is trivial for all $t \in V$. We will actually need the stronger assumption

- For all $t \in V$, the conjugacy invariant of $T_t$ in $J^{r+3}$ is trivial.

Consider the derivative with respect to $t$ of $T_t$ at $t = 0$. It can be viewed as a linear map $\Delta T$ from $\mathbb{R}^{\ell+d^*}$ to $C^{r+3}(\sqcup I^t_\alpha)$ (where we write $I^t_\alpha$ instead of $I^t_\alpha(0)$). Because the $J_{r+3}$ invariant is trivial for all $t \in V$, it follows from

Proposition 4.4 that any function $\varphi$ in the image of $\Delta T$ satisfies

$$\partial D^\ell \varphi = 0, \qquad \forall 0 \leqslant \ell \leqslant r+3.$$

In particular, the image of $\Delta T$ is contained in the space $C_\partial^{r+1+BV}(\sqcup I_\alpha^t)$ of Section 3.4, and we can compose $\Delta T$ with the operator $\Pi : C_\partial^{r+1+BV}(\sqcup I_\alpha^t) \to \Gamma_\partial(r+1)/\Gamma_T(r+1)$ of Theorem 3.13 to obtain a map $\overline{\Delta T} : \mathbb{R}^{\ell+d^*} \to \Gamma_\partial(r+1)/\Gamma_T(r+1)$. Observe that, according to Proposition 3.12, the dimension of $\Gamma_\partial(r+1)/\Gamma_T(r+1)$ is $g + r(2g-2) = d^* - s + 1$. We will make the following transversality assumption:

(Tr1) The restriction of $\overline{\Delta T}$ to $\{0\} \times \mathbb{R}^{d^*}$ is a homomorphism *onto*

$$\Gamma_\partial(r+1)/\Gamma_T(r+1).$$

After a linear change of variables in parameter space, we can and will also assume that $\mathbb{R}^\ell \times \{0\}$ is contained in the kernel of $\overline{\Delta T}$.

When $s = 1$, $d^*$ is equal to the dimension of $\Gamma_\partial(r+1)/\Gamma_T(r+1)$; then (Tr1) means that the restriction of $\overline{\Delta T}$ to $\{0\} \times \mathbb{R}^{d^*}$ is an isomorphism onto $\Gamma_\partial(r+1)/\Gamma_T(r+1)$.

When $s > 1$, we will ask for one more tranversality condition. Let $t \in \mathrm{Ker}\overline{\Delta T}$, $\varphi := \Delta T(t) \in C^{r+3}(\sqcup I_\alpha^t) \cap C_\partial^{r+1+BV}(\sqcup I_\alpha^t)$. The image $\Pi(\varphi)$ in $\Gamma_\partial(r+1)/\Gamma_T(r+1)$ is equal to 0. On the other hand, let $\widehat{\psi} \in C^{r+3}(\bar{I})$ be a function such that $\widehat{\psi}(0) = \widehat{\psi}(1) = 0$, and let $\widehat{\psi}(u_i^t) = \frac{d}{d\tau} u_i^t(\tau t)_{|\tau=0}$, $\widehat{\psi}(u_j^b) = \frac{d}{d\tau} u_j^b(\tau t)_{|\tau=0}$ for all $0 < i, j < d$. Then $\varphi_1 := \varphi + \widehat{\psi} - \widehat{\psi} \circ T$ satisfies $\Pi(\varphi_1) = 0$ and $\varphi_1(\upsilon) = 0$ for all $\upsilon \in \mathcal{A}^{(2)}$. Writing $\varphi_1 = \psi_1 \circ T_0 - \psi_1$ and considering the values of $\psi_1$ on the cycles of $\sigma$, we define as in Remark 3.3 an element $\nu(\varphi_1) \in \mathbb{R}^\Sigma/\mathbb{R}$. It is obvious that this vector only depends on $t$ (not on the choice of $\widehat{\psi}$), and we denote it by $\overline{\nu}(t)$. We ask that

(Tr2) The restriction of $\overline{\nu}$ to the intersection of the kernel of $\overline{\Delta T}$ with $\{0\} \times \mathbb{R}^{d^*}$ is an isomorphism onto $\mathbb{R}^\Sigma/\mathbb{R}$.

After a linear change of variables in parameter space, we can and will assume that $\mathbb{R}^\ell \times \{0\}$ is equal to the kernel of $\overline{\nu}$.

5.2. *Statement of the theorem.* Under the hypotheses of the last subsection, we have

THEOREM 5.1. *There exist $t_1 \leqslant t_0$ and a neighborhood $W$ of the identity in* $\mathrm{Diff}^r(\bar{I})$ *with the following properties*:

(1) *For every $t' \in [-t_1, t_1]^\ell$, there exist a unique $t'' =: \theta(t') \in [-t_1, t_1]^{d^*}$ and a unique $h =: h_{t'} \in W$ such that, with $t = (t', t'')$*

$$T_t = h \circ T_0 \circ h^{-1}.$$

(2) *The maps $t' \mapsto t'' = \theta(t')$ and $t' \mapsto h_{t'}$ are of class $C^1$; moreover, $\theta(0) = 0$ and $D\theta_{|t'=0} = 0$.*

The theorem thus states that, amongst $C^{r+3}$ g.i.e.m. close to $T_0$ with trivial conjugacy invariant in $J^{r+3}$, those that are conjugated to $T_0$ by a $C^r$-diffeomorphism close to the identity form a $C^1$-submanifold of codimension $d^* = (g-1)(2r+1)+s$. The theorem also describes the tangent space to this submanifold at $T_0$, in terms of the cohomological equation.

As we look for a $C^r$-conjugacy to a standard i.e.m., it is natural to restrict our attention to generalized i.e.m. with trivial $C^r$-conjugacy invariant in $J^r$. It is unclear whether it is necessary to assume, as we do, that the $C^{r+3}$-conjugacy invariant is trivial. In the circle case ($d = 2$), a linearization theorem still holds if one only assumes that the $C^{r+1}$-conjugacy invariant is trivial; the situation is unclear when only the $C^r$-conjugacy invariant is assumed to be trivial.

5.3. *Simple families.*

*Definition* 5.2. We say that a family $(T_t)$ as above is *simple* if $u_i^t(t)$ is, for all $0 < i < d$, independent of $t$, and if, for all $\alpha \in \mathcal{A}$ and all $t$, $T_t$ coincides with $T_0$ in the neighborhood of each endpoint of $I_\alpha^t$.

The aim of this section is to show the following proposition.

PROPOSITION 5.3. *There exists $t_2 < t_0$ and a $C^1$-family $(\widetilde{h}_t)_{t\in[-t_2,t_2]^{\ell+d^*}}$ in $\mathrm{Diff}^{r+3}(\overline{I})$ such that the family $(\widetilde{T}_t) := (\widetilde{h}_t^{-1} \circ T_t \circ \widetilde{h}_t)$ is simple and still satisfies the hypotheses of the last section.*

*Proof.* Write $u_i^t$, $u_j^b$ for $u_i^t(0)$, $u_j^b(0)$. A first step is to choose a $C^1$-family $(\widehat{h}_t)_{t\in[-t_2,t_2]^{\ell+d^*}}$ in $\mathrm{Diff}^\infty(\overline{I})$ such that $\widehat{h}_t(u_i^t) = u_i^t(t)$, $\widehat{h}_t(u_j^b) = u_j^b(t)$ for all $t \in [-t_2, t_2]^{\ell+d^*}$. This is possible, after taking $t_2 < t_0$ sufficiently small, since the $u_i^t, u_j^b$ are all distinct (as $T_0$ has no connection). Then, for the family $(\widehat{T}_t) := (\widehat{h}_t^{-1} \circ T_t \circ \widehat{h}_t)$, we have that the $u_i^t(t)$ and $u_j^b(t)$ are independent of $t$.

Next, for $\upsilon \in \mathcal{A}^{(2)}$, $t \in [-t_2, t_2]^{\ell+d^*}$, we introduce the $(r+3)$-jet $j(\widehat{T}_t, \upsilon)$ of Section 4.1. For every cycle $C = \{\upsilon_0, \dots, \upsilon_\kappa\}$ of $\sigma$, every $t \in [-t_2, t_2]^{\ell+d^*}$, we have

$$J(\widehat{T}_t, C) := j(\widehat{T}_t, \upsilon_0)^{\varepsilon(\upsilon_0)} \cdots j(\widehat{T}_t, \upsilon_\kappa)^{\varepsilon(\upsilon_\kappa)} = 1.$$

We look now for a $C^1$-family $(\overline{h}_t)_{t\in[-t_2,t_2]^{\ell+d^*}}$ in $\mathrm{Diff}^{r+3}(\overline{I})$ such that

(1) $\overline{h}_t(u_i^t) = u_i^t$, $\overline{h}_t(u_j^b) = u_j^b$ for all $t \in [-t_2, t_2]^{\ell+d^*}$, $0 < i, j < d$;
(2) $\widehat{T}_t \circ \overline{h}_t(u^t(\upsilon) + x) = \overline{h}_t(u^b(\upsilon) + x)$ for all $\upsilon \in \mathcal{A}^{(2)}$ of the form $(\alpha, L)$, $x > 0$ small enough;
(3) $\widehat{T}_t \circ \overline{h}_t(u^t(\upsilon) - x) = \overline{h}_t(u^b(\upsilon) - x)$ for all $\upsilon \in \mathcal{A}^{(2)}$ of the form $(\alpha, R)$, $x > 0$ small enough.

These conditions obviously imply that $\widetilde{T}_t := \overline{h}_t^{\,-1} \circ \widehat{T}_t \circ \overline{h}_t$ is simple (and that it still satisfies the hypotheses of the last subsection). It is possible to solve (1)–(3) for $\overline{h}_t \in \mathrm{Diff}^{r+3}(\overline{I})$, since (2) and (3) connect values of $\overline{h}_t$ on different small intervals bounded by the singularities. Compatibility conditions then occur on the product of $r$-jets along cycles of $\sigma$; they are fulfilled as soon as the conjugacy invariant is trivial. □

According to Proposition 5.3, it is sufficient to prove Theorem 5.1 for simple families. This will be done in the next section for $r \geqslant 3$ and in Section 7 for $r = 2$.

## 6. **Proof:** $C^r$-conjugacy, $r \geqslant 3$

In this section we assume that $r \geqslant 3$ and will prove the theorem in this case. The case $r = 2$ will be dealt with in the next section. Therefore, let $(T_t)$ be a $C^1$-family of $C^{r+3}$ g.i.e.m. satisfying the hypotheses of Section 5.1. According to Proposition 5.3, we can and will also assume that the family is simple.

Recall that the Schwarzian derivative of a $C^3$-orientation-preserving diffeomorphism $f$ is defined by

$$Sf := D^2\mathrm{Log}Df - \frac{1}{2}(D\mathrm{Log}Df)^2.$$

The composition rule for Schwarzian derivatives is

$$S(f \circ g) = Sf \circ g\ (Dg)^2 + Sg.$$

6.1. *Smoothness of the composition map.* The tangent space at id to $\mathrm{Diff}^r(\overline{I})$ is the space $C^r_{0,0}(\overline{I})$ of $C^r$-functions on $\overline{I}$ vanishing at $u_0$ and $u_d$.

Lemma 6.1. *The composition map*

$$C^r(\overline{I}) \times \mathrm{Diff}^r(\overline{I}) \to C^{r-1}(\overline{I})$$
$$(\varphi, h) \mapsto \varphi \circ h$$

*is of class $C^1$. Its differential at $(0, \mathrm{id})$ is the map $(\delta\varphi, \delta h) \mapsto \delta\varphi$ from $C^r(\overline{I}) \times C^r_{0,0}(\overline{I})$ to $C^{r-1}(\overline{I})$.*

The map $(\varphi, h) \mapsto \varphi \circ h$ *valued in* $\mathbf{C}^r(\overline{I})$ is only continuous. It becomes $C^1$ when seen as taking its values in $C^{r-1}(\overline{I})$. The formula for the derivative at $(0, \mathrm{id})$ is elementary.

We denote by $C^k_{\mathrm{comp}}(\sqcup I^t_\alpha)$ the space of functions $\varphi \in C^k(\sqcup I^t_\alpha)$ that vanish in the neighborhood of the endpoints of each $I^t_\alpha$. Obviously, a map $\varphi \in C^k_{\mathrm{comp}}(\sqcup I^t_\alpha)$ satisfies $\partial D^\ell \varphi = 0$ for $0 \leqslant \ell \leqslant k$.

Lemma 6.2. *The map*

$$\Phi : [-t_0, t_0]^{\ell+d^*} \times \mathrm{Diff}^r(\overline{I}) \to C^{r-1}_{\mathrm{comp}}(\sqcup I^t_\alpha)$$
$$(t, h) \mapsto ST_t \circ h(Dh)^2$$

*is of class $C^1$. Its differential at $(0, \mathrm{id})$ is the map $(\delta t, \delta h) \mapsto D^3\delta\varphi$ from $\mathbb{R}^{\ell+d^*} \times C^r_{0,0}(\overline{I})$ to $C^{r-1}_{\mathrm{comp}}(\sqcup I^t_\alpha)$, with $\delta\varphi = \Delta T(\delta t)$.*

*Proof.* From the formula above for $ST_t$, the derivative of $t \mapsto ST_t$ at $t = 0$ is $\delta t \mapsto D^3\Delta T(\delta t)$. The lemma then follows from Lemma 6.1 and an elementary computation. □

6.2. *The cohomological equation.* In the following we fix a subspace $\Gamma_u$ in $\Gamma_\partial(r-2)$ such that

$$\Gamma_\partial(r-2) = \Gamma_T(r-2) \oplus \Gamma_u \ \oplus \mathbb{R}1.$$

According to Proposition 3.12, we have

$$\dim \Gamma_u = (2r-5)(g-1).$$

From Theorem 3.13, there exist bounded linear operators $L_0 : \ C_\partial^{r-1}(\sqcup I_\alpha^t) \to C_0^{r-3}(I)$, $L_1 : \ C_\partial^{r-1}(\sqcup I_\alpha^t) \to \Gamma_u$ such that, for $\varphi \in C_\partial^{r-1}(\sqcup I_\alpha^t)$, we have

$$\varphi = \int_I \varphi(x)dx + L_1(\varphi) + L_0(\varphi) \circ T_0 - L_0(\varphi).$$

Here, $C_0^{r-3}(I)$ is the space of $C^{r-3}$-functions on $I$ that vanish at $u_0$.

Lemma 6.3. *The map*

$$\Psi : [-t_0, t_0]^{\ell+d^*} \times \mathrm{Diff}^r(\bar{I}) \to C_0^{r-3}(\bar{I})$$
$$(t,h) \mapsto L_0(\Phi(t,h))$$

*is of class* $C^1$. *Its differential at* $(0, \mathrm{id})$ *is the map* $(\delta t, \delta h) \mapsto L_0(D^3\delta\varphi)$ *from* $\mathbb{R}^{\ell+d^*} \times C_{0,0}^r(\bar{I})$ *to* $C_0^{r-3}(\bar{I})$, *with* $\delta\varphi = \Delta T(\delta t)$.

*Proof.* Indeed, $L_0$ is linear and the derivative of $\Phi$ has been computed in Lemma 6.2. □

6.3. *Relation between a diffeomorphism and its Schwarzian derivative.* The next three lemmas present the Schwarzian derivative operator as a composition of a first-order quasilinear operator with a second-order nonlinear differential operator, which is sometimes called the nonlinearity operator.

Lemma 6.4. *The map*

$$\mathcal{N} : \mathrm{Diff}^r(\bar{I}) \to C^{r-2}(\bar{I})$$
$$h \mapsto D\mathrm{Log}Dh$$

*is a* $C^\infty$*-diffeomorphism. Its differential at* $\mathrm{id} \in \mathrm{Diff}^r(\bar{I})$ *is the map* $\delta h \mapsto D^2\delta h$ *from* $C_{0,0}^r(\bar{I})$ *to* $C^{r-2}(\bar{I})$.

*Proof.* That $\mathcal{N}$ is a $C^\infty$ map and the formula for its differential at id is elementary. Given $N \in C^{r-2}(\bar{I})$, let $N_1 \in C^{r-1}(\bar{I})$ be the primitive of $N$ such that the mean value over $\bar{I}$ of $\exp(N_1)$ is 1. Then define

$$h(x) = \int_{u_0}^x \exp(N_1(t))dt.$$

It is clear that $h \in \mathrm{Diff}^r(\overline{I})$, that it is the only element of $\mathrm{Diff}^r(\overline{I})$ such that $\mathcal{N}(h) = N$, and that $N \mapsto h$ is $C^\infty$. $\square$

Lemma 6.5. *The map*

$$\mathcal{Q} : C^{r-2}(\overline{I}) \to C_0^{r-3}(\overline{I}) \times \mathbb{R}^2$$
$$N \mapsto \Big(\psi = DN - \frac{1}{2}N^2 - c_0, c_0 = DN(u_0) - \frac{1}{2}N^2(u_0), c_1 = N(u_0)\Big)$$

*is of class $C^\infty$. Its differential at $0$ is given by*

$$\delta\psi = D\delta N - \delta c_0, \delta c_0 = D\delta N(u_0), \delta c_1 = \delta N(u_0),$$

*which is an isomorphism from $C^{r-2}(\overline{I})$ onto $C_0^{r-3}(\overline{I}) \times \mathbb{R}^2$. Therefore, the restriction of $\mathcal{Q}$ to an appropriate neighborhood of $0 \in C^{r-2}(\overline{I})$ is a $C^1$-diffeomorphism onto a neighborhood of $(0,0,0) \in C_0^{r-3}(\overline{I}) \times \mathbb{R}^2$.*

*Proof.* The first two statements are obtained by elementary computation. The last one is a consequence of the inverse function theorem in Banach spaces. $\square$

Lemma 6.6. *The map*

$$\mathcal{S} := \mathcal{Q} \circ \mathcal{N} : \mathrm{Diff}^r(\overline{I}) \to C_0^{r-3}(\overline{I}) \times \mathbb{R}^2$$
$$h \mapsto (\psi, c_0, c_1) = (Sh - Sh(u_0), Sh(u_0), D\mathrm{Log}Dh(u_0))$$

*is of class $C^\infty$, and its restriction to an appropriate neighborhood of* $\mathrm{id} \in \mathrm{Diff}^r(\overline{I})$ *is a $C^\infty$-diffeomorphism onto a neighborhood of* $(0,0,0) \in C_0^{r-3}(\overline{I}) \times \mathbb{R}^2$. *The differential of $\mathcal{S}$ at* $\mathrm{id} \in \mathrm{Diff}^r(\overline{I})$ *is the isomorphism*

$$\delta h \mapsto (D^3\delta h - D^3\delta h(u_0), D^3\delta h(u_0), D^2\delta h(u_0))$$

*from $C_{0,0}^r(\overline{I})$ to $C_0^{r-3}(\overline{I}) \times \mathbb{R}^2$.*

*Proof.* This is a direct consequence of the last two lemmas. $\square$

We denote by $W_0$, $W_1$ neighborhoods of id in $\mathrm{Diff}^r(\overline{I})$ and of $(0,0,0)$ in $C_0^{r-3}(\overline{I}) \times \mathbb{R}^2$ respectively such that $\mathcal{S}$ defines a $C^\infty$-diffeomorphism from $W_0$ onto $W_1$. We denote by $\mathcal{P} : W_1 \to W_0$ the inverse diffeomorphism and by $P$ the differential of $\mathcal{P}$ at $(0,0,0)$.

6.4. *The fixed point theorem.*

Lemma 6.7. *The map*

$$(t, h, c_0, c_1) \mapsto \mathcal{P}(\Psi(t,h), c_0, c_1)$$

*is defined and of class $C^1$ in a neighborhood of $(0, \mathrm{id}, 0, 0)$ in $[-t_0, t_0]^{\ell+d^*} \times$ $\mathrm{Diff}^r(\overline{I}) \times \mathbb{R}^2$, with values in $W_0$. Its differential at $(0, \mathrm{id}, 0, 0)$ is the map $(\delta t, \delta h, \delta c_0, \delta c_1) \mapsto P(L_0(D^3\delta\varphi), \delta c_0, \delta c_1)$, with $\delta\varphi = \Delta T(\delta t)$, from $\mathbb{R}^{\ell+d^*} \times C_{0,0}^r(\overline{I}) \times \mathbb{R}^2$ to $C_{0,0}^r(\overline{I})$.*

*Proof.* This is a consequence of Lemmas 6.3 and 6.6. □

LEMMA 6.8. *There exist an open neighborhood $W_2$ of* $\mathrm{id} \in \mathrm{Diff}^r(\bar{I})$ *and an open neighborhood $W_3$ of* $(0,0,0) \in [-t_0, t_0]^{\ell+d^*} \times \mathbb{R}^2$ *such that, for each* $(t, c_0, c_1) \in W_3$, *the map*

$$h \mapsto \mathcal{P}(\Psi(t,h), c_0, c_1)$$

*has exactly one fixed point in $W_2$, which we denote by $\mathcal{H}(t, c_0, c_1)$. Moreover, the map $\mathcal{H}$ is of class $C^1$ on $W_3$, and its differential at $(0,0,0)$ is the map* $(\delta t, \delta c_0, \delta c_1) \mapsto P(L_0(D^3\delta\varphi), \delta c_0, \delta c_1)$, *with $\delta\varphi = \Delta T(\delta t)$, from $\mathbb{R}^{\ell+d^*} \times \mathbb{R}^2$ to* $C^r_{0,0}(\bar{I})$.

*Proof.* This is a consequence of the implicit function theorem applied to the fixed point equation $\mathcal{P}(\Psi(t,h), c_0, c_1) = h$. □

Let $(t, c_0, c_1) \in W_3$, $h = \mathcal{H}(t, c_0, c_1)$. Then $h$ satisfies

$$\Phi(t,h) = ST_t \circ h (Dh)^2 = L_1(\Phi(t,h)) + \int_0^1 \Phi(t,h)(x)\, dx \quad + Sh \circ T_0 - Sh.$$

For $(t, c_0, c_1) \in W_3$, we write $H := T_t \circ h \circ T_0^{-1}$. We have $H = h$ if and only if $T_t = h \circ T_0 \circ h^{-1}$.

6.5. *Conditions for $H$ to be a diffeomorphism.*

LEMMA 6.9. *For $(t, c_0, c_1) \in W_3$, the following are equivalent*:

(1) $h(u_i^t) = u_i^t$ *for all* $0 < i < d$;

(2) *$H$ is a homeomorphism of $\bar{I}$ satisfying $H(u_j^b) = u_j^b$ for all* $0 < j < d$.

*Proof.* The map $H$ is one-to-one (mod.0), fixes $u_0$ and $u_d$, and is continuous except perhaps at the $u_j^b$, $0 < j < d$. Looking at the left and right limits of $H$ at these points gives the lemma. □

When the equivalent conditions of the lemma are satisfied, $H$ is in fact a piecewise $C^r$-diffeomorphism of $\bar{I}$, possibly with discontinuities of the derivatives of order $\leqslant 2$ at the $u_j^b$.

LEMMA 6.10. *Let $(t, c_0, c_1) \in W_3$ such that the equivalent conditions of the last lemmas are satisfied. Then $H$ is a $C^2$-diffeomorphism of $\bar{I}$ if and only if for all $\upsilon = (\alpha, L) \in \mathcal{A}^{(2)}$ with $\alpha \neq {}_b\alpha$, one has*

$$\mathrm{Log}Dh(u^t(\upsilon)) = \mathrm{Log}Dh(u^t(\sigma(\upsilon))), \quad D\mathrm{Log}Dh(u^t(\upsilon)) = D\mathrm{Log}Dh(u^t(\sigma(\upsilon))).$$

*Proof.* Indeed, these relations express that the left and right limits of the first two derivatives of $H$ at the $u_j^b$ are the same. □

*Remark* 6.11. When $\upsilon = (\alpha, R)$, $\alpha \neq \alpha_t$, one has $u^t(\upsilon) = u^t(\sigma(\upsilon))$; hence,

$$\mathrm{Log}Dh(u^t(\upsilon)) = \mathrm{Log}Dh(u^t(\sigma(\upsilon))), \quad D\mathrm{Log}Dh(u^t(\upsilon)) = D\mathrm{Log}Dh(u^t(\sigma(\upsilon)))$$

is always true.

Lemma 6.12. *Let* $(t, c_0, c_1) \in W_3$ *such that the equivalent conditions of the last two lemmas are satisfied. Assume also that* $L_1(\Phi(t, \mathcal{H}(t, c_0, c_1))) = 0$. *Then* $H$ *is a* $C^r$*-diffeomorphism of* $\overline{I}$.

*Proof.* We have to prove that the derivative of order $3+k$ of $H$ is continuous at each $u_j^b$ for all $0 \leqslant k \leqslant r-3$, $0 < j < d$. This is equivalent to showing that for all $0 \leqslant k \leqslant r-3$, all $\upsilon = (\alpha, L) \in \mathcal{A}^{(2)}$ with $\alpha \neq {}_b\alpha$,

$$D^k Sh(u^t(\upsilon)) = D^k Sh(u^t(\sigma(\upsilon))),$$

with $h = \mathcal{H}(t, c_0, c_1)$ as above.

As $L_1(\Phi(t, \mathcal{H}(t, c_0, c_1))) = 0$, we have

$$ST_t \circ h (Dh)^2 = \int_0^1 \Phi(t,h)(x)\, dx \quad + Sh \circ T_0 - Sh,$$

and for $0 < k \leqslant r-3$,

$$D^k(ST_t \circ h(Dh)^2) = D^k Sh \circ T_0 - D^k Sh.$$

As $D^k ST_t$ vanishes at the $u_i^t$ for $0 \leqslant i \leqslant d$, $0 \leqslant k \leqslant r-3$, and $D^k Sh$ is continuous at $u_j^b$, the required equalities follow. □

6.6. *Equations for the conjugacy class of* $T_0$.

Proposition 6.13. *Let* $(t, c_0, c_1) \in W_3$ *such that* $h = \mathcal{H}(t, c_0, c_1)$ *satisfies*

$$\begin{aligned}
h(u_i^t) &= u_i^t && \textit{for all } 0 < i < d,\\
\mathrm{Log} Dh(u^t(\upsilon)) &= \mathrm{Log} Dh(u^t(\sigma(\upsilon))) && \textit{for all } \upsilon = (\alpha, L) \in \mathcal{A}^{(2)}, \alpha \neq {}_b\alpha,\\
D\mathrm{Log} Dh(u^t(\upsilon)) &= D\mathrm{Log} Dh(u^t(\sigma(\upsilon))) && \textit{for all } \upsilon = (\alpha, L) \in \mathcal{A}^{(2)}, \alpha \neq {}_b\alpha,\\
L_1(\Phi(t,h)) &= 0.
\end{aligned}$$

*Then if* $(t, c_0, c_1)$ *is close enough to* $(0,0,0)$, *we have* $T_t \circ h = h \circ T_0$.

*Conversely, let* $t \in [-t_0, t_0]^{\ell + d^*}$ *and* $h \in \mathrm{Diff}^r(\overline{I})$ *such that* $T_t \circ h = h \circ T_0$. *Let* $c_0 = Sh(0)$, $c_1 = D\mathrm{Log}Dh(0)$. *If* $t$ *is close enough to* $0$ *and* $h$ *is close enough to the identity, then* $h = \mathcal{H}(t, c_0, c_1)$ *and the relations above are satisfied.*

*Proof.* We first prove the second part of the proposition.

Let $t \in [-t_0, t_0]^{\ell + d^*}$ close to $(0,0,0)$, $h \in \mathrm{Diff}^r(\overline{I})$ close to the identity, such that $T_t \circ h = h \circ T_0$. Then we have

$$ST_t \circ h(Dh)^2 = Sh \circ T_0 - Sh.$$

Let $c_0 = Sh(0)$, $c_1 = D\mathrm{Log}Dh(0)$. Then we have $h = \mathcal{P}(\Psi(t,h), c_0, c_1)$ and, therefore, $h = \mathcal{H}(t, c_0, c_1)$. Moreover, $L_1(\Phi(t, \mathcal{H}(t, c_0, c_1))) = 0$ holds. Finally, $H := T_t \circ h \circ T_0^{-1}$ is equal to $h$, hence it follows from Lemmas 6.9 and 6.10 that the other relations in the proposition are satisfied. This concludes the proof of the second part of the proposition.

For the proof of the first part, the argument is slightly different, depending whether or not $(\alpha_t, R)$ and $({}_b\alpha, L)$ belong to the same cycle of $\sigma$ in $\mathcal{A}^{(2)}$. Let $t, c_0, c_1, h$ as in the proposition. From Lemma 6.12, we already know that $h$ and $H$ belong to $\mathrm{Diff}^r(\overline{I})$.

• We first assume that $(\alpha_t, R)$ and $({}_b\alpha, L)$ belong to the same cycle of $\sigma$. By assumption (and Remark 6.11), we have $\mathrm{Log}Dh(u^t(\upsilon)) = \mathrm{Log}Dh(u^t(\sigma(\upsilon)))$ for all $\upsilon \in \mathcal{A}^{(2)}$ except $(\alpha_t, R)$ and $({}_b\alpha, L)$. In particular, this gives

$$\mathrm{Log}Dh(u_0) = \mathrm{Log}Dh(u^t({}_t\alpha, L)) = \mathrm{Log}Dh(u^t(\alpha_t, R)) = \mathrm{Log}Dh(u_d),$$
$$\mathrm{Log}DH(u_0) = \mathrm{Log}Dh(u^t({}_b\alpha, L)) = \mathrm{Log}Dh(u^t(\alpha_b, R)) = \mathrm{Log}DH(u_d).$$

The same argument applies to $D\mathrm{Log}Dh$ and to $D^kSh$ for $0 \leqslant k \leqslant r-3$, according to the proof of Lemma 6.12. This allows us to conclude that both $h$ and $H$ are induced by $C^r$-diffeomorphisms of the circle $\mathbb{T}$ obtained by identifying the endpoints of $\overline{I}$. Moreover, the relation $ST_t \circ h(Dh)^2 = c + Sh \circ T_0 - Sh$ implies $SH = Sh + c$, with $c = \int_0^1 \Phi(t,h)(x)\, dx$. The following lemma allows us to conclude that $h = H$.

Lemma 6.14. *Let* $\mathrm{Diff}^r(\mathbb{T}, 0)$ *be the group of orientation-preserving* $C^r$-*diffeomorphisms of the circle fixing* $0$, *and let* $C_0^{r-3}(\mathbb{T})$ *be the space of* $C^{r-3}$-*functions on the circle vanishing at* $0$. *The map*

$$\mathrm{Diff}^r(\mathbb{T}, 0) \to C_0^{r-3}(\mathbb{T})$$
$$h \mapsto Sh - Sh(0)$$

*is of class* $C^\infty$ *and its restriction to an appropriate neighborhood of the identity is a* $C^\infty$-*diffeomorphism onto a neighborhood of* $0$ *in* $C_0^{r-3}(\mathbb{T})$.

*Proof.* The first assertion is trivial, the differential at the identity being the map $\delta h \mapsto D^3\delta h - D^3\delta h(0)$ from $C_0^r(\mathbb{T})$ (the space of $C^r$-functions on the circle vanishing at 0) to $C_0^{r-3}(\mathbb{T})$. This is clearly an isomorphism; hence, the lemma follows by the implicit function theorem. □

• We now assume that $(\alpha_t, R)$ and $({}_b\alpha, L)$ do not belong to the same cycle of $\sigma$. By assumption (and Remark 6.11), we still have $\mathrm{Log}Dh(u^t(\upsilon)) = \mathrm{Log}Dh(u^t(\sigma(\upsilon)))$ for all $\upsilon \in \mathcal{A}^{(2)}$ except $(\alpha_t, R)$ and $({}_b\alpha, L)$. This now gives

$$\mathrm{Log}Dh(u_0) = \mathrm{Log}Dh(u^t({}_t\alpha, L)) = \mathrm{Log}Dh(u^t({}_b\alpha, L)) = \mathrm{Log}DH(u_0),$$
$$\mathrm{Log}Dh(u_d) = \mathrm{Log}Dh(u^t(\alpha_t, R)) = \mathrm{Log}Dh(u^t(\alpha_b, R)) = \mathrm{Log}DH(u_d).$$

The same argument applies to $D\mathrm{Log}Dh$ and to $D^kSh$ for $0 \leqslant k \leqslant r-3$, according to the proof of Lemma 6.12. This allows us to conclude that the $r$-jets at $u_d$ of $h$ and $H$ are the same. Moreover, the relation $ST_t \circ h(Dh)^2 = c + Sh \circ T_0 - Sh$ implies $SH = Sh + c$, with $c = \int_0^1 \Phi(t,h)(x)\, dx$. As $SH(u_d) =$

$Sh(u_d)$, we must have $c = 0$. As $Sh = SH$ and the 3-jets of $h$ and $H$ at $u_d$ are equal, we conclude also in this case that $h = H$. □

6.7. *End of the proof of Theorem* 5.1 *for* $r \geqslant 3$. From the proposition above, we have to determine in a neighborhood of $0 \in [-t_0, t_0]^{\ell+d^*}$ the set of $t$ for which, for some $(c_0, c_1)$ close to $(0, 0)$, the diffeomorphism $h = \mathcal{H}(t, c_0, c_1)$ satisfies

$$h(u_i^t) = u_i^t \text{ for all } 0 < i < d, \tag{6.1}$$

$$\mathrm{Log}Dh(u^t(\upsilon)) = \mathrm{Log}Dh(u^t(\sigma(\upsilon))) \quad \text{for all } \upsilon = (\alpha, L) \in \mathcal{A}^{(2)}, \alpha \neq {}_b\alpha, \tag{6.2}$$

$$D\mathrm{Log}Dh(u^t(\upsilon)) = D\mathrm{Log}Dh(u^t(\sigma(\upsilon))) \quad \text{for all } \upsilon = (\alpha, L) \in \mathcal{A}^{(2)}, \alpha \neq {}_b\alpha, \tag{6.3}$$

$$L_1(\Phi(t, h)) = 0. \tag{6.4}$$

We will see that there are exactly $(d^*+2)$ independent equations for $t, c_0, c_1$ in the system above. Looking at the linearized system at $(0, 0, 0)$ will allow to apply the implicit function theorem and conclude. We deal separately with the same two cases that appeared in the proof of Proposition 6.13.

• We first assume that $(\alpha_t, R)$ and $({}_b\alpha, L)$ belong to the same cycle of $\sigma$. There are $(d-1)$ equations in (6.1), $(2r-5)(g-1)$ equations in (6.4) (the dimension of $\Gamma_u$). In (6.2), for each cycle of $\sigma$ which does not contain $({}_b\alpha, L)$, there is one redundant equation. So the number of equations in (6.2) is really $(d-1) - (s-1) = (2g-1)$. Similarly, there are $(2g-1)$ equations in (6.3).

Thus, the total number of equations in the system (6.1)–(6.4) is $(d-1) + (2r-5)(g-1) + (2g-1) + (2g-1) = d^* + 2$, as claimed.

Consider now the linearized system obtained from (6.1)–(6.4) at $(0, 0, 0)$. Writing as before $\delta\varphi = \Delta T(t)$, from Lemma 6.8 we have

$$\delta h = P(L_0(D^3\delta\varphi), \delta c_0, \delta c_1).$$

From the definition of $L_0$ and $P$ (cf. Lemma 6.6), this is equivalent to

$$D^3\delta\varphi = D^3\delta h \circ T_0 - D^3\delta h + L_1(D^3\delta\varphi),$$

$$D^3\delta h(0) = \delta c_0, \quad D^2\delta h(0) = \delta c_1,$$

where we have used in the first equation that $\int_0^1 D^3\delta\varphi(x)\, dx = 0$.

Now, the linearized version of equation (6.4) is

$$L_1(D^3\delta\varphi) = 0. \tag{6.5}$$

If this holds, we have

$$D^3\delta\varphi = D^3\delta h \circ T_0 - D^3\delta h$$

and then, by integration,

$$D^2\delta\varphi = D^2\delta h \circ T_0 - D^2\delta h + \chi_2$$

for some $\chi_2 \in \Gamma(1)$. But the linearized version of (6.3) is

$$D^2\delta h(u^t(\upsilon)) = D^2\delta h(u^t(\sigma(\upsilon))) \quad \text{for all } \upsilon = (\alpha, L) \in \mathcal{A}^{(2)}, \alpha \neq {}_b\alpha. \tag{6.6}$$

If this holds, $\chi_2$ has to be constant. (At each $u_j^b$, the left and right values of $\chi_2$ are the same.) As $\int_0^1 D^2\delta\varphi(x)\,dx = 0$, we must have $\chi_2 = 0$. Observe that the equation $D^2\delta\varphi = D^2\delta h \circ T_0 - D^2\delta h$ determines $\delta c_0$. One more integration then gives

$$D\delta\varphi = D\delta h \circ T_0 - D\delta h + \chi_1$$

for some $\chi_1 \in \Gamma(1)$. Using now the linearized version of (6.2),

$$D\delta h(u^t(\upsilon)) = D\delta h(u^t(\sigma(\upsilon))) \quad \text{for all } \upsilon = (\alpha, L) \in \mathcal{A}^{(2)}, \alpha \neq {}_b\alpha, \tag{6.7}$$

we proceed in the same way to conclude that if (6.5)–(6.7) holds, one has $\chi_1 = 0$, $D\delta\varphi = D\delta h \circ T_0 - D\delta h$ and $\delta c_1$ is determined. One last integration gives

$$\delta\varphi = \delta h \circ T_0 - \delta h + \chi_0$$

for some $\chi_0 \in \Gamma(1)$. The linearized version of (6.1) is

$$\delta h(u_i^t) = 0. \tag{6.8}$$

If this holds, then as above, one obtains first that $\chi_0$ is constant; as $\delta h(u_d) = 0 = \delta h(u^t(\alpha_b, R))$, we have $\chi_0 = 0$.

Recalling the definition of $\nu$ in Remark 3.3 and $\Pi$ in Theorem 3.13, we conclude that if (6.5)–(6.8) holds, then

$$\Pi(\delta\varphi) = 0, \quad \nu(\delta\varphi) = 0.$$

Going backwards, we see that these relations are actually equivalent to (6.5)–(6.8). In view of the transversality hypotheses (Tr1), (Tr2) of Section 5.1, the theorem in this case now follows from the implicit function theorem.

We now assume that $(\alpha_t, R)$ and $({}_b\alpha, L)$ do not belong to the same cycle of $\sigma$. There are still $(d-1)$ equations in (6.1), $(2r-5)(g-1)$ equations in (6.4). In (6.2), for each cycle of $\sigma$ that contains neither $({}_b\alpha, L)$ nor $(\alpha_t, R)$, there is one redundant equation. This would give $(d-1)-(s-2) = 2g$ for the number of equations in (6.2), and similarly in (6.3), leading to a grand total of $(d^*+4)$ equations. However, we will now see that the equations

$$\mathrm{Log}Dh(u^t({}_t\alpha, L)) = \mathrm{Log}Dh(u^t(\sigma({}_t\alpha, L))), \tag{6.9}$$

$$D\mathrm{Log}Dh(u^t({}_t\alpha, L)) = D\mathrm{Log}Dh(u^t(\sigma({}_t\alpha, L))) \tag{6.10}$$

are also redundant.

Indeed, assume that (6.1)–(6.4) holds, with the exception of (6.9)–(6.10). Let $H = T_t \circ h \circ T_0^{-1}$ as above. Following Lemmas 6.9, 6.10, 6.12, $H$ is a homeomorphism of $\overline{I}$ fixing $u_0, u_d$ and each $u_j^b$; moreover, the restrictions of $H$ to $[u_0, u^b({}_t\alpha, L)]$ and $[u^b({}_t\alpha, L), u_d]$ are $C^r$-diffeomorphisms.

As in the end of the proof of Proposition 6.13, we obtain that the $r$-jets of $h$ and $H$ at $u_d$ are the same and that $Sh = SH$. This implies that $h = H$ on $[u^b({}_t\alpha, L), u_d]$. Comparing the $r$-jets of $h$ and $H$ at $u^b({}_t\alpha, L)$ shows that the $r$-jets of $h$ at $u_0$ and $u^b({}_t\alpha, L)$ are the same.

It is also true that the $r$-jets of $H$ at $u_0$ and $u^b({}_t\alpha, L)$ are the same, or, equivalently, that the $r$-jets of $h$ at $u^t({}_b\alpha, L)$ and $u^t(\sigma({}_t\alpha, L))$ are the same. For the first two derivatives, it follows from (6.2), (6.3) (without using (6.9)–(6.10)); for the higher derivatives, the argument is the same as that in Lemma 6.12. Therefore, the restrictions to $[u_0, u^b({}_t\alpha, L)]$ of both $h$ and $H$ satisfy periodic boundary conditions, and we conclude by Lemma 6.14 that $h = H$ on the full interval $\overline{I}$.

This proves that (6.9), (6.10) are redundant, and we are left with $(d^* + 2)$ equations, as in the first case.

We now consider the linearized system, as in the first case. We still write

$$\delta\varphi = \Delta T(t),$$
$$\delta h = P(L_0(D^3\delta\varphi), \delta c_0, \delta c_1);$$

hence,

$$D^3\delta\varphi = D^3\delta h \circ T_0 - D^3\delta h + L_1(D^3\delta\varphi),$$
$$D^3\delta h(0) = \delta c_0, \quad D^2\delta h(0) = \delta c_1.$$

Assuming (6.5), we get

$$D^3\delta\varphi = D^3\delta h \circ T_0 - D^3\delta h$$

and then, by integration,

$$D^2\delta\varphi = D^2\delta h \circ T_0 - D^2\delta h + \chi_2$$

for some $\chi_2 \in \Gamma(1)$. The linearized version of (6.3) minus (6.10) is

$$\text{(6.11)} \quad D^2\delta h(u^t(\upsilon)) = D^2\delta h(u^t(\sigma(\upsilon))) \quad \text{for all } \upsilon = (\alpha, L) \in \mathcal{A}^{(2)}, \alpha \neq {}_b\alpha, {}_t\alpha.$$

This implies that $\chi_2 \circ T_0^{-1}$ is constant on $(u_0, u^b({}_t\alpha, L))$ and on $(u^b({}_t\alpha, L), u_d)$. Moreover, as $D^2\delta h(u^t(\alpha_b, R)) = D^2\delta h(u_d)$, the value of $\chi_2$ on $(u^b({}_t\alpha, L), u_d)$ is 0. But we also have $\int_I \chi_2 = \int_I D^2\delta\varphi = 0$; hence, $\chi_2 = 0$ everywhere.

One more integration then gives

$$D\delta\varphi = D\delta h \circ T_0 - D\delta h + \chi_1$$

for some $\chi_1 \in \Gamma(1)$.

The linearized version of (6.2) minus (6.9) is

$$\text{(6.12)} \quad D\delta h(u^t(\upsilon)) = D\delta h(u^t(\sigma(\upsilon))) \quad \text{for all } \upsilon = (\alpha, L) \in \mathcal{A}^{(2)}, \alpha \neq {}_b\alpha, {}_t\alpha.$$

This now implies $\chi_1 = 0$. One last integration gives

$$\delta\varphi = \delta h \circ T_0 - \delta h + \chi_0$$

for some $\chi_0 \in \Gamma(1)$. If we finally assume (6.8), as in the first case we get

$$\Pi(\delta\varphi) = 0, \quad \nu(\delta\varphi) = 0.$$

Going backwards, we see that these relations are actually equivalent to the conjunction of (6.5), (6.8), (6.11), (6.12). Therefore we conclude, as in the first case, by the implicit function theorem.

The proof of the theorem for $r \geqslant 3$ is now complete. □

## 7. **Proof: $C^2$-conjugacy**

In this section we prove the theorem in the case $r = 2$. It may help the reader to look first at Appendix B.2, where the main idea is presented in the simpler setting of circle diffeomorphisms. Therefore, let $(T_t)$ be a $C^1$-family of $C^5$ g.i.e.m. satisfying the hypotheses of Section 5.1 with $r = 2$. According to Proposition 5.3, we can and will also assume that the family is simple.

In this section we state the intermediate steps in the proof of the conjugacy theorem with $r = 2$, but only comment on the parts that substantially differ from the case $r \geq 3$, which was presented in Section 6.

7.1. *Smoothness of the composition map.* This section is identical to Section 6.1. The tangent space at id to $\mathrm{Diff}^2(\overline{I})$ is the space $C^2_{0,0}(\overline{I})$ of $C^2$-functions on $\overline{I}$ vanishing at $u_0$ and $u_d$.

LEMMA 7.1. *The composition map*

$$C^2(\overline{I}) \times \mathrm{Diff}^2(\overline{I}) \to C^1(\overline{I})$$
$$(\varphi, h) \mapsto \varphi \circ h$$

*is of class $C^1$. Its differential at $(0, \mathrm{id})$ is the map $(\delta\varphi, \delta h) \mapsto \delta\varphi$ from $C^2(\overline{I}) \times C^2_{0,0}(\overline{I})$ to $C^1(\overline{I})$.*

LEMMA 7.2. *The map*

$$\Phi : [-t_0, t_0]^{\ell+d^*} \times \mathrm{Diff}^2(\overline{I}) \to C^1_{\mathrm{comp}}(\sqcup I^t_\alpha)$$
$$(t, h) \mapsto ST_t \circ h (Dh)^2$$

*is of class $C^1$. Its differential at $(0, \mathrm{id})$ is the map $(\delta t, \delta h) \mapsto D^3\delta\varphi$ from $\mathbb{R}^{\ell+d^*} \times C^2_{0,0}(\overline{I})$ to $C^1_{\mathrm{comp}}(\sqcup I^t_\alpha)$, with $\delta\varphi = \Delta T(\delta t)$.*

7.2. *The cohomological equation.* We denote by $P^*$ the operator

$$\varphi \mapsto \int_{u_0}^{x} (\varphi(y) - \int_I \varphi)\, dy$$

from $C^1_{\mathrm{comp}}(\sqcup I^t_\alpha)$ to the space $C^2_c(\sqcup I^t_\alpha)$ of functions in $C^2(\sqcup I^t_\alpha)$ that are continuous on $\overline{I}$ and vanish at $u_0$ and $u_d$.

We choose subspaces $\Gamma_c \subset \Gamma$, $\Gamma_u \subset \Gamma_\partial$ such that

$$\Gamma = \Gamma_c \oplus \Gamma_\partial, \quad \Gamma_\partial = \Gamma_u \oplus \Gamma_s.$$

From Theorem 3.10, there exist bounded operators

$$L_c : C^2_c(\sqcup I^t_\alpha) \to \Gamma_c, \quad L_u : C^2_c(\sqcup I^t_\alpha) \to \Gamma_u, \quad L_0 : C^2_c(\sqcup I^t_\alpha) \to C^0_0(\overline{I})$$

such that for $\varphi \in C^2_c(\sqcup I^t_\alpha)$,

$$\varphi + L_c(\varphi) + L_u(\varphi) = L_0(\varphi) \circ T_0 - L_0(\varphi).$$

Here, $C^0_0(\overline{I})$ denotes the space of continuous functions on $\overline{I}$ that vanish at $u_0$. We write $L$ for the bounded operator from $\Gamma_s$ to $C^0_0(\overline{I})$ such that $v = L(v) \circ T_0 - L(v)$.

LEMMA 7.3. *The map*

$$\Psi_1 : (t, h, v) \mapsto L(v) + L_0(P^*(\Phi(t, h)))$$

*from* $[-t_0, t_0]^{\ell+d^*} \times \mathrm{Diff}^2(\overline{I}) \times \Gamma_s$ *to* $C^0_0(\overline{I})$ *is of class* $C^1$. *Its differential at* $(0, \mathrm{id}, v)$ *is*

$$(\delta t, \delta h, \delta v) \mapsto L(\delta v) + L_0(D^2 \delta\varphi).$$

7.3. *Relation between a diffeomorphism and the primitive of its Schwarzian derivative.*

LEMMA 7.4. *The map*

$$\mathcal{Q}_1 : C^0(\overline{I}) \to C^0_0(\overline{I}) \times \mathbb{R}$$
$$N \mapsto (\psi_1(x) = N(x) - N(u_0) - \frac{1}{2}\int_{u_0}^{x} N^2(y)\, dy, c_1 = N(u_0))$$

*is of class* $C^\infty$. *Its differential at* $0$ *is given by*

$$\delta\psi_1 = D\delta N - \delta c_1, \delta c_1 = \delta N(u_0),$$

*which is an isomorphism from* $C^0(\overline{I})$ *onto* $C^0_0(\overline{I}) \times \mathbb{R}$. *Therefore, the restriction of* $\mathcal{Q}_1$ *to an appropriate neighborhood of* $0 \in C^0(\overline{I})$ *is a* $C^\infty$*-diffeomorphism onto a neighborhood of* $(0, 0) \in C^0_0(\overline{I}) \times \mathbb{R}$.

Combining the last lemma with Lemma 6.4, which is still valid for $r = 2$, we obtain

LEMMA 7.5. *The map*

$$\mathcal{S}_1 := \mathcal{Q}_1 \circ \mathcal{N} : \mathrm{Diff}^2(\overline{I}) \to C^0_0(\overline{I}) \times \mathbb{R}$$
$$h \mapsto (\psi_1, c_1) = (D\mathrm{Log}Dh - D\mathrm{Log}Dh(u_0) - \frac{1}{2}\int_{u_0}(D\mathrm{Log}Dh)^2, D\mathrm{Log}Dh(u_0))$$

*is of class* $C^\infty$, *and its restriction to an appropriate neighborhood of* $\mathrm{id} \in \mathrm{Diff}^2(\overline{I})$ *is a* $C^\infty$*-diffeomorphism onto a neighborhood of* $(0, 0, 0) \in C^0_0(\overline{I}) \times \mathbb{R}$.

*The differential of* $\mathcal{S}_1$ *at* $\mathrm{id} \in \mathrm{Diff}^2(\bar{I})$ *is the isomorphism* $\delta h \mapsto (D^2\delta h - D^2\delta h(u_0), D^2\delta h(u_0))$ *from* $C^2_{0,0}(\bar{I})$ *to* $C^0_0(\bar{I}) \times \mathbb{R}$.

We denote by $W_0$, $W_1$ neighborhoods of id in $\mathrm{Diff}^2(\bar{I})$ and of $(0,0)$ in $C^0_0(\bar{I}) \times \mathbb{R}$ respectively such that $\mathcal{S}_1$ defines a $C^\infty$-diffeomorphism from $W_0$ onto $W_1$. We denote by $\mathcal{P}_1 : W_1 \to W_0$ the inverse diffeomorphism and by $P_1$ the differential of $\mathcal{P}_1$ at $(0,0)$.

7.4. *The fixed point theorem.*

LEMMA 7.6. *The map*

$$(t, h, v, c_1) \mapsto \mathcal{P}_1(\Psi_1(t, h, v), c_1)$$

*is defined and of class* $C^1$ *in a neighborhood of* $(0, \mathrm{id}, 0, 0)$ *in* $[-t_0, t_0]^{\ell+d^*} \times \mathrm{Diff}^2(\bar{I}) \times \Gamma_s \times \mathbb{R}$, *with values in* $W_0$. *Its differential at* $(0, \mathrm{id}, 0, 0)$ *is the map* $(\delta t, \delta h, \delta v, \delta c_1) \mapsto P_1(L_0(D^2\delta\varphi) + L(v), \delta c_1)$, *with* $\delta\varphi = \Delta T(\delta t)$, *from* $\mathbb{R}^{\ell+d^*} \times C^2_{0,0}(\bar{I}) \times \Gamma_s \times \mathbb{R}$ *to* $C^2_{0,0}(\bar{I})$.

LEMMA 7.7. *There exist an open neighborhood* $W_2$ *of* $\mathrm{id} \in \mathrm{Diff}^2(\bar{I})$ *and an open neighborhood* $W_3$ *of* $(0,0,0) \in [-t_0, t_0]^{\ell+d^*} \times \Gamma_s \times \mathbb{R}$ *such that, for each* $(t, v, c_1) \in W_3$, *the map*

$$h \mapsto \mathcal{P}_1(\Psi_1(t, h, v), c_1)$$

*has exactly one fixed point in* $W_2$, *which we denote by* $\mathcal{H}(t, v, c_1)$. *Moreover, the map* $\mathcal{H}$ *is of class* $C^1$ *on* $W_3$, *and its differential at* $(0,0,0)$ *is the map* $(\delta t, \delta v, \delta c_1) \mapsto P_1(L_0(D^2\delta\varphi) + L(v), \delta c_1)$, *with* $\delta\varphi = \Delta T(\delta t)$, *from* $\mathbb{R}^{\ell+d^*} \times \Gamma_s \times \mathbb{R}$ *to* $C^2_{0,0}(\bar{I})$.

Let $(t, v, c_1) \in W_3$, $h = \mathcal{H}(t, v, c_1)$. Then $h$ satisfies

$$P^*(\Phi(t,h)) + L_c(P^*(\Phi(t,h))) + L_u(P^*(\Phi(t,h))) + v = N_1 h \circ T_0 - N_1 h,$$

with

$$N_1 h(x) = D\mathrm{Log}Dh(x) - \frac{1}{2}\int_{u_0}^{x} (D\mathrm{Log}Dh(y))^2\, dy.$$

For $(t, v, c_1) \in W_3$, we write $H = \mathcal{H}(t, v, c_1) := T_t \circ h \circ T_0^{-1}$. We have $H = h$ if and only if $T_t = h \circ T_0 \circ h^{-1}$.

7.5. *Conditions for* $H$ *to be a diffeomorphism.* Lemma 6.9 is still valid in our present setting.

LEMMA 7.8. *For* $(t, v, c_1) \in W_3$, *the following are equivalent*:

(1) $h(u_i^t) = u_i^t$ *for all* $0 < i < d$;
(2) $H$ *is a homeomorphism of* $\bar{I}$ *satisfying* $H(u_j^b) = u_j^b$ *for all* $0 < j < d$.

When the equivalent conditions of the lemma are satisfied, $H$ is in fact a piecewise $C^2$-diffeomorphism of $\overline{I}$, with possibly discontinuities of the derivatives of order $\leqslant r$ at the $u_j^b$. We will replace Lemma 6.10 by the next two lemmas.

LEMMA 7.9. *Let $(t, v, c_1) \in W_3$ such that the equivalent conditions of the last lemmas are satisfied. Then $H$ is a $C^1$-diffeomorphism of $\overline{I}$ if and only if for all $\upsilon = (\alpha, L) \in \mathcal{A}^{(2)}$ with $\alpha \neq {}_b\alpha$, one has*

$$\mathrm{Log}Dh(u^t(\upsilon)) = \mathrm{Log}Dh(u^t(\sigma(\upsilon))).$$

LEMMA 7.10. *Let $(t, v, c_1) \in W_3$. The function $D\mathrm{Log}Dh \circ T_0$ is continuous on $\overline{I}$ if and only if one has, for all $\upsilon = (\alpha, R) \in \mathcal{A}^{(2)}$ with $\alpha \neq \alpha_t$*

$$D\mathrm{Log}Dh(u^b(\upsilon)) = D\mathrm{Log}Dh(u^b(\sigma(\upsilon))).$$

*Remark* 7.11. When $\upsilon = (\alpha, R)$, $\alpha \neq \alpha_t$, one has $u^t(\upsilon) = u^t(\sigma(\upsilon))$; hence,

$$\mathrm{Log}Dh(u^t(\upsilon)) = \mathrm{Log}Dh(u^t(\sigma(\upsilon)))$$

is always true. Similarly, when $\upsilon = (\alpha, L)$, $\alpha \neq {}_b\alpha$, one has $u^b(\upsilon) = u^b(\sigma(\upsilon))$; hence,

$$D\mathrm{Log}Dh(u^b(\upsilon)) = D\mathrm{Log}Dh(u^b(\sigma(\upsilon)))$$

is always true.

7.6. *Equations for the conjugacy class of $T_0$.*

PROPOSITION 7.12. *Let $(t, v, c_1) \in W_3$ such that $h = \mathcal{H}(t, v, c_1)$ satisfies*

$$\begin{aligned} h(u_i^t) &= u_i^t && \text{for all } 0 < i < d, \\ \mathrm{Log}Dh(u^t(\upsilon)) &= \mathrm{Log}Dh(u^t(\sigma(\upsilon))) && \text{for all } \upsilon = (\alpha, L) \in \mathcal{A}^{(2)}, \alpha \neq {}_b\alpha, \\ D\mathrm{Log}Dh(u^b(\upsilon)) &= D\mathrm{Log}Dh(u^b(\sigma(\upsilon))) && \text{for all } \upsilon = (\alpha, R) \in \mathcal{A}^{(2)}, \alpha \neq \alpha_t. \end{aligned}$$

*Then if $(t, v, c_1)$ is close enough to $(0, 0, 0)$, we have $T_t \circ h = h \circ T_0$.*

*Conversely, let $t \in [-t_0, t_0]^{\ell + d^*}$ and $h \in \mathrm{Diff}^2(\overline{I})$ such that $T_t \circ h = h \circ T_0$. Let $c_1 = D\mathrm{Log}Dh(0)$. If $t$ is close enough to $0$ and $h$ is close enough to the identity, then there exists $v \in \Gamma_s$ such that $(t, v, c_1)$ belongs to $W_3$, $h = \mathcal{H}(t, v, c_1)$, and the relations above are satisfied.*

*Proof.* We first prove the second part of the proposition. Let $t \in [-t_0, t_0]^{\ell + d^*}$ close to 0, $h \in \mathrm{Diff}^2(\overline{I})$ close to the identity, such that $T_t \circ h = h \circ T_0$. Then we have

$$(D\mathrm{Log}DT_t \circ h)\, Dh = D\mathrm{Log}Dh \circ T_0 - D\mathrm{Log}Dh. \tag{7.1}$$

Let $\mathcal{S}_1(h) = (\psi_1, c_1)$. Then we have

$$\psi_1 \circ T_0 - \psi_1 = (D\mathrm{Log}DT_t \circ h)\, Dh - R,$$

with

$$\begin{aligned} DR &= \frac{1}{2}((D\mathrm{Log}Dh \circ T_0)^2 - (D\mathrm{Log}Dh)^2) \\ &= \frac{1}{2}(D\mathrm{Log}DT_t \circ h)^2 (Dh)^2 + (D\mathrm{Log}DT_t \circ h)Dh(D\mathrm{Log}Dh). \end{aligned}$$

It follows that

$$D(\psi_1 \circ T_0 - \psi_1) = (ST_t \circ h)(Dh)^2.$$

As we have $\partial(\psi_1 \circ T_0 - \psi_1) = 0$, the function $(ST_t \circ h)(Dh)^2$ has zero mean value and we have

$$\psi_1 \circ T_0 - \psi_1 = P^*((ST_t \circ h)(Dh)^2) + \chi$$

for some $\chi \in \Gamma$. But this means that we have $\psi_1 = \Psi_1(t, h, v)$ for some $v \in \Gamma_s$. This implies $h = \mathcal{H}(t, v, c_1)$. Moreover, $h = H$; hence, the first two sets of relations in the proposition are satisfied from Lemmas 7.8 and 7.9. We also have from (7.1) that the function $D\mathrm{Log}Dh \circ T_0$ is continuous on $\overline{I}$. Then the third set of relations in the proposition follows from Lemma 7.10. This concludes the proof of the second part of the proposition.

We now assume that $(t, v, c_1), h = \mathcal{H}(t, v, c_1)$ satisfy the three sets of relations in the proposition. We will prove below that relation (7.1) is satisfied. With $H$ as above, we then have $D\mathrm{Log}DH = D\mathrm{Log}Dh$. But $H$ is a $C^1$-diffeomorphism of $\overline{I}$ (piecewise $C^2$) by Lemmas 7.8 and 7.9. The relation $D\mathrm{Log}DH = D\mathrm{Log}Dh$ then implies that $H$ is a $C^2$ diffeomorphism and $h = H$.

To see that (7.1) is satisfied, we will use the following lemma, where $C^0(\overline{I}, 0)$ denotes the space of continuous functions on $\overline{I}$ with mean value 0.

Lemma 7.13. *The map*

$$\begin{gathered} (\varphi, N) \mapsto \mathcal{T}(\varphi, N) = \varphi - \Delta\varphi \\ C^0(\overline{I}, 0) \times C^0(\overline{I}) \longrightarrow C^0(\overline{I}, 0) \end{gathered}$$

*with*

$$D\Delta\varphi = \frac{1}{2}\varphi^2 + \varphi N - \int_I (\frac{1}{2}\varphi^2 + \varphi N), \qquad \int_I \Delta\varphi = 0$$

*is of class $C^1$. Its differential at $(0,0)$ is $(\delta\varphi, \delta N) \mapsto \delta\varphi$. Thus, for $N$ close enough to $0$, the map $\varphi \mapsto \mathcal{T}(\varphi, N)$ is a $C^1$-diffeomorphism from a neighborhood of $0 \in C^0(\overline{I}, 0)$ to another neighborhood of $0$.*

Let $N = D\mathrm{Log}Dh$. Take first $\varphi_0 = (D\mathrm{Log}DT_t \circ h)Dh$. This function does belong to $C^0(\overline{I}, 0)$.

One has $\mathcal{T}(\varphi_0, N) = \varphi_0 - \Delta\varphi_0$ with

$$D\Delta\varphi_0 = \frac{1}{2}(D\mathrm{Log}DT_t \circ h)^2 (Dh)^2 + (D\mathrm{Log}DT_t \circ h)Dh(D\mathrm{Log}Dh) - c_0;$$

hence,

$$D(\varphi_0 - \Delta\varphi_0) = (ST_t \circ h)(Dh)^2 - c_0,$$

where the constant $c_0$ is the mean value of $(ST_t \circ h)(Dh)^2$ (as $D\varphi_0$ has mean value 0). Therefore, $\mathcal{T}(\varphi_0, N) = P^*(\Phi(t,h)) - c$, with $c$ equal to the mean value of $P^*(\Phi(t,h))$.

Next take $\varphi_1 = D\mathrm{Log}Dh \circ T_0 - D\mathrm{Log}Dh$. This function has mean value 0, and is continuous from Lemma 7.10. Therefore, it belongs to $C^0(\overline{I}, 0)$. One has $\mathcal{T}(\varphi_1, N) = \varphi_1 - \Delta\varphi_1$ with

$$D\Delta\varphi_1 = \frac{1}{2}[(D\mathrm{Log}Dh \circ T_0)^2 - (D\mathrm{Log}Dh)^2].$$

Let $\mathcal{S}_1 h = (\psi_1, c_1)$. Therefore, we have

$$\varphi_1 - \Delta\varphi_1 = \psi_1 \circ T_0 - \psi_1 + \chi$$

for some $\chi \in \Gamma$. As $\varphi_1 - \Delta\varphi_1$ is continuous with zero mean value, it must also be equal to $P^*(\Phi(t,h)) - c$.

We conclude that $\mathcal{T}(\varphi_0, N) = \mathcal{T}(\varphi_1, N)$; hence, $\varphi_0 = \varphi_1$ by the lemma. This is (7.1), and the proof of the proposition is complete. □

7.7. *End of the proof of Theorem* 5.1 *for* $r = 2$. From the proposition above, we have to determine in a neighborhood of $0 \in [-t_0, t_0]^{\ell + d^*}$ the set of $t$ for which, for some $(v, c_1)$ close to $(0,0)$, the diffeomorphism $h = \mathcal{H}(t, v, c_1)$ satisfies

$$h(u_i^t) = u_i^t \qquad \text{for all } 0 < i < d, \tag{7.2}$$

$$\mathrm{Log}Dh(u^t(v)) = \mathrm{Log}Dh(u^t(\sigma(v))) \qquad \text{for all } v = (\alpha, L) \in \mathcal{A}^{(2)}, \alpha \neq {}_b\alpha, \tag{7.3}$$

$$D\mathrm{Log}Dh(u^b(v)) = D\mathrm{Log}Dh(u^b(\sigma(v))) \qquad \text{for all } v = (\alpha, R) \in \mathcal{A}^{(2)}, \alpha \neq \alpha_t. \tag{7.4}$$

We will see that there are exactly $(d^* + g + 1)$ independent equations for $t, v, c_1$ in the system above. Looking at the linearized system at $(0,0,0)$ will allow us to apply the implicit function theorem and conclude. We deal separately with the same two cases that appeared in Section 6.

• We first assume that $(\alpha_t, R)$ and $({}_b\alpha, L)$ belong to the same cycle of $\sigma$. There are $(d-1)$ equations in (7.2). In (7.3), for each cycle of $\sigma$ that does not contain $({}_b\alpha, L)$, there is one redundant equation. So the number of equations in (7.3) is really $(d-1) - (s-1) = (2g-1)$. Similarly, there are $(2g-1)$ equations in (7.4).

Therefore, the total number of equations in the system (7.2)–(7.4) is $(d-1) + (2g-1) + (2g-1) = d^* + g + 1$, as claimed.

Consider now the linearized system obtained from (7.2)–(7.4) at $(0,0,0)$. Writing as before $\delta\varphi = \Delta T(t)$, we have, from Lemma 7.7

$$\delta h = P_1(L_0(D^2\delta\varphi) + L(v), \delta c_1).$$

From the definition of $L_0$, $P^*$ and $P_1$, this is equivalent to

$$D^2\delta h \circ T_0 - D^2\delta h = D^2\delta\varphi + \chi_2, \quad D^2\delta h(u_0) = \delta c_1,$$

with $\chi_2 = \delta v + L_c(D^2\delta\varphi) + L_u(D^2\delta\varphi)$. The linearized version of (7.3) is

$$D^2\delta h(u^b(\upsilon)) = D^2\delta h(u^b(\sigma(\upsilon))) \quad \text{for all } \upsilon = (\alpha, R) \in \mathcal{A}^{(2)}, \alpha \neq \alpha_t. \tag{7.5}$$

This implies that $\chi_2$ is continuous at each $u_i^t$, hence constant. As it has mean value 0, we have $\chi_2 = 0$. Integrating $D^2\delta h \circ T_0 - D^2\delta h = D^2\delta\varphi$ gives

$$D\delta h \circ T_0 - D\delta h = D\delta\varphi + \chi_1$$

for some $\chi_1 \in \Gamma(1)$. The linearized version of (7.2) is

$$D\delta h(u^t(\upsilon)) = D\delta h(u^t(\sigma(\upsilon))), \quad \text{for all } \upsilon = (\alpha, L) \in \mathcal{A}^{(2)}, \alpha \neq {}_b\alpha. \tag{7.6}$$

If it holds, $\chi_1$ has to be constant. (At each $u_j^b$, the left and right values of $\chi_1$ are the same.) As $\int_0^1 D\delta\varphi(x)\,dx = 0$, we must have $\chi_1 = 0$. Observe that the equation $D\delta\varphi = D\delta h \circ T_0 - D\delta h$ determines $\delta c_1$. One more integration then gives

$$\delta\varphi = \delta h \circ T_0 - \delta h + \chi_0$$

for some $\chi_0 \in \Gamma(1)$. The linearized version of (7.1) is

$$\delta h(u_i^t) = 0. \tag{7.7}$$

If this holds, then as above, one first obtains that $\chi_0$ is constant; as $\delta h(u_d) = 0 = \delta h(u^t(\alpha_b, R))$, we have $\chi_0 = 0$.

Recalling the definition of $\nu$ in Remark 3.3 and $\Pi$ in Theorem 3.13, we conclude that if (7.5)–(7.7) holds, then

$$\Pi(\delta\varphi) = 0, \quad \nu(\delta\varphi) = 0.$$

Going backwards, we see that these relations are actually equivalent to (7.5)–(7.7). In view of the transversality hypotheses (Tr1), (Tr2) of Section 5.1, the theorem in this case now follows from the implicit function theorem.

• We now assume that $(\alpha_t, R)$ and $({}_b\alpha, L)$ do not belong to the same cycle of $\sigma$. There are still $(d-1)$ equations in (7.1). In (7.2), for each cycle of $\sigma$ that contains neither $({}_b\alpha, L)$ nor $(\alpha_t, R)$, there is one redundant equation. This would give $(d-1) - (s-2) = 2g$ for the number of equations in (7.2), and similarly in (7.3), leading to a grand total of $(d^* + g + 3)$ equations. However, we will now see that the equations

$$\mathrm{Log}Dh(u^t({}_t\alpha, L)) = \mathrm{Log}Dh(u^t(\sigma({}_t\alpha, L))), \tag{7.8}$$

$$D\mathrm{Log}Dh(u^b(\alpha_b, R)) = D\mathrm{Log}Dh(u^b(\sigma(\alpha_b, R))) \tag{7.9}$$

are also redundant.

Indeed, assume that (7.2)–(7.4) holds, with the exception of (7.8)–(7.9). Let $H = T_t \circ h \circ T_0^{-1}$ as above. We first show that (7.1) (in Section 7.6) holds. We will use a variant of Lemma 7.13.

Denote by $C^0_*(\overline{I})$ the space of functions on $\overline{I}$ that vanish at $u_0$ and are continuous on $\overline{I}$ except possibly at $u^t(\alpha_b, R)$, where they have a right and left limit. Let $C^0_*(\overline{I},0) = \{\varphi \in C^0_*(\overline{I}),\ \int_I \varphi = 0\}$, and let $\pi : C^0_*(\overline{I}) \to C^0_*(\overline{I},0)$ be the projection operator such that for $\varphi \in C^0_*(\overline{I})$, $\varphi - \pi(\varphi)$ is constant on $(u^t(\alpha_b, R), u_d)$ and 0 on $(u_0, u^t(\alpha_b, R))$. We define a map

$$(\varphi, N) \mapsto \mathcal{T}(\varphi, N) := \pi(\varphi - \Delta\varphi)$$

from $C^0_*(\overline{I},0) \times C^0(\overline{I})$ to $C^0_*(\overline{I},0)$ by the formulas

$$D\Delta\varphi = \frac{1}{2}\varphi^2 + \varphi N - \int_I (\frac{1}{2}\varphi^2 + \varphi N), \quad \Delta\varphi(u_0) = 0.$$

LEMMA 7.14. *The map $\mathcal{T}$ is of class $C^1$ and satisfies $\mathcal{T}(0,N) = 0$ for all $N \in C^0(\overline{I})$. Its differential at $(0,0)$ is $(\delta\varphi, \delta N) \mapsto \delta\varphi$. Thus, for $N$ close enough to 0, the map $\varphi \mapsto \mathcal{T}(\varphi, N)$ is a $C^1$-diffeomorphism from a neighborhood of 0 in $C^0_*(\overline{I},0)$ to another neighborhood of 0 in $C^0_*(\overline{I},0)$.*

Let $N := D\mathrm{Log}Dh \in C^0(\overline{I})$. The function $\varphi_0 := (D\mathrm{Log}DT_t \circ h)Dh$ belongs to $C^0_*(\overline{I},0)$. (It is actually continuous at $u^t(\alpha_b, R)$.) A small computation gives $\mathcal{T}(\varphi_0, N) = \pi(P^*(\Phi(t,h)))$.

Let $\varphi_1 := D\mathrm{Log}Dh \circ T_0 - D\mathrm{Log}Dh$. As (7.4) is satisfied with the exception of (7.9), the function $\varphi_1$ belongs to $C^0_*(\overline{I})$. Moreover, it clearly has mean value 0; hence, we have $\varphi_1 \in C^0_*(\overline{I},0)$. Writing $\mathcal{S}_1(h) = (\psi_1, c_1)$, after a short computation we obtain that

$$\mathcal{T}(\varphi_1, N) = \psi_1 \circ T_0 - \psi_1 + \chi$$

for some $\chi \in \Gamma$. Observe that it follows from (7.4) minus (7.9) that $\varphi_1(u_0) = 0$. Therefore, $\mathcal{T}(\varphi_1, N)(u_0) = 0$. As $\mathcal{T}(\varphi_1, N)$ belongs to $C^0_*(\overline{I},0)$, one must have $\mathcal{T}(\varphi_1, N) = \pi(P^*(\Phi(t,h)))$. We then conclude from the lemma that $\varphi_0 = \varphi_1$, which is (7.1).

From (7.2) minus (7.8), the function $\mathrm{Log}DH$ is continuous on $\overline{I}$, except perhaps at $u^b({}_t\alpha, L)$. From (7.1), we deduce by integration that $\mathrm{Log}DH - \mathrm{Log}Dh$ is constant on $(u_0, u^b({}_t\alpha, L))$ and $(u^b({}_t\alpha, L), u_d)$. From (7.2), we also have

$$\mathrm{Log}DH(u_d) = \mathrm{Log}Dh(u^t(\alpha_b, R)) = \mathrm{Log}Dh(u^t(\alpha_t, R) = \mathrm{Log}Dh(u_d).$$

As $\int_I Dh = \int_I DH$, we conclude that $Dh = DH$ and finally (as $h(u_0) = u_0 = H(u_0)$) that $h = H$. We have thus proven that (7.8) and (7.9) are redundant.

We now consider the system linearized from (7.2)–(7.4) minus (7.8)–(7.9). As in the first case, with $\delta\varphi = \Delta T(t)$, we have

$$\delta h = P_1(L_0(D^2\delta\varphi) + L(v), \delta c_1),$$

which is equivalent to

$$D^2\delta h \circ T_0 - D^2\delta h = D^2\delta\varphi + \chi_2, \quad D^2\delta h(u_0) = \delta c_1,$$

with $\chi_2 = \delta v + L_c(D^2\delta\varphi) + L_u(D^2\delta\varphi)$.

The linearized version of (7.3) minus (7.9) is

(7.10) $$D^2\delta h(u^b(\upsilon)) = D^2\delta h(u^b(\sigma(\upsilon))) \quad \text{for all } \upsilon = (\alpha, R) \in \mathcal{A}^{(2)}, \alpha \neq \alpha_t, \alpha_b.$$

This implies that $\chi_2$ is continuous on $(u_0, u^t(\alpha_b, R))$ and $(u^t(\alpha_b, R), u_d)$. From (7.10), we also have that $D^2\delta h(u_0) = D^2\delta h(u^b({}_t\alpha, L))$; hence, $\chi_2(u_0) = 0$. As $\chi_2$ has mean value 0, we obtain $\chi_2 = 0$.

Integrating $D^2\delta h \circ T_0 - D^2\delta h = D^2\delta\varphi$ gives

$$D\delta h \circ T_0 - D\delta h = D\delta\varphi + \chi_1$$

for some $\chi_1 \in \Gamma(1)$. The linearized version of (7.2) is

(7.11) $$D\delta h(u^t(\upsilon)) = D\delta h(u^t(\sigma(\upsilon))) \quad \text{for all } \upsilon = (\alpha, L) \in \mathcal{A}^{(2)}, \alpha \neq {}_b\alpha, {}_t\alpha.$$

If it holds, $\chi_1 \circ T_0^{-1}$ has to be constant on $(u_0, u^b({}_t\alpha, L))$ and $(u^b({}_t\alpha, L), u_d)$. Also, from (7.11), we have $D\delta h(u_d) = D\delta h(u^t(\alpha_b, R))$; hence, $\chi_1 \circ T_0^{-1}(u_d) = 0$. As $\int_0^1 D\delta\varphi(x)\,dx = 0$, we must have $\int_I \chi_1 = 0$ and $\chi_1 = 0$.

Observe that the equation $D\delta\varphi = D\delta h \circ T_0 - D\delta h$ determines $\delta c_1$. One more integration then gives

$$\delta\varphi = \delta h \circ T_0 - \delta h + \chi_0$$

for some $\chi_0 \in \Gamma(1)$. The linearized version of (7.1) is

(7.12) $$\delta h(u_i^t) = 0.$$

If this holds, then as above one first obtains that $\chi_0$ is constant; as $\delta h(u_d) = 0 = \delta h(u^t(\alpha_b, R))$, we have $\chi_0 = 0$.

Recalling the definition of $\nu$ in Remark 3.3 and $\Pi$ in Theorem 3.13, we conclude that if (7.10)–(7.12) holds, then

$$\Pi(\delta\varphi) = 0, \quad \nu(\delta\varphi) = 0.$$

Going backwards, we see that these relations are actually equivalent to (7.10)–(7.12). In view of the transversality hypotheses (Tr1), (Tr2) of Section 5.1, the theorem in this case now follows from the implicit function theorem. The proof of the theorem for $r = 2$ is now complete. □

## 8. **Simple deformations of linear flows on translation surfaces**

### 8.1. *Translation surfaces.*

*Definition* 8.1. Let $M$ be a compact connected orientable surface and $\Sigma = \{A_1, \dots, A_s\}$ be a nonempty finite subset of $M$. A *structure of translation surface* on $(M, \Sigma)$ is a maximal atlas $\zeta$ for $M - \Sigma$ of charts by open sets of $\mathbb{C} \simeq \mathbb{R}^2$, which satisfies the two following properties:

(1) Any coordinate change between two charts of the atlas is locally a translation of $\mathbb{R}^2$.

(2) For every $1 \leqslant i \leqslant s$, there exist an integer $\kappa_i \geqslant 1$, a neighborhood $V_i$ of $A_i$, a neighborhood $W_i$ of 0 in $\mathbb{R}^2$, and a ramified covering $\pi : (V_i, A_i) \to (W_i, 0)$ of degree $\kappa_i$ such that every injective restriction of $\pi$ is a chart of $\zeta$.

It is equivalent to equip $M$ with a complex structure and a holomorphic 1-form that does not vanish on $M - \Sigma$ and has at $A_i$ a zero of order $\kappa_i - 1$.

For a structure of translation surface $\zeta$ on $(M, \Sigma)$ and $g \in \mathrm{GL}(2, \mathbb{R})$, one defines a new structure $g.\zeta$ by postcomposing the charts of $\zeta$ by $g$.

*Definition* 8.2. Let $\zeta$ be a structure of translation surface on $(M, \Sigma)$. The *vertical vector field* is the vector field on $M - \Sigma$ that reads as $\frac{\partial}{\partial y}$ in the charts of $\zeta$. The associated flow is the *vertical flow*. An orbit of the vertical flow that ends (resp. starts) at a point of $\Sigma$ is called an *ingoing* (resp. *outgoing*) *vertical separatrix*. A *vertical connection* is an orbit of the vertical flow that both starts and ends at a point of $\Sigma$.

More generally, a *linear flow* on $(M, \Sigma, \zeta)$ is a flow on $M - \Sigma$ that is vertical for $g.\zeta$, for some $g \in \mathrm{GL}(2, \mathbb{R})$.

*Definition* 8.3. A $C^r$ *simple deformation* of the vertical vector field $X_0$ of $(M, \Sigma, \zeta)$ is a nonvanishing $C^r$-vector field $X$ on $M - \Sigma$ that coincides with $X_0$ in a neighborhood of $\Sigma$ and is appropriately $C^r$-close to $X_0$ on $M - \Sigma$.

*Definition* 8.4. Let $\zeta$ be a structure of translation surface on $(M, \Sigma)$. An open bounded horizontal segment $I$ is in *good position* if

(1) $I$ meets every vertical connection,
(2) the endpoints of $I$ are distinct and either belong to $\Sigma$ or are connected to a point of $\Sigma$ by a vertical segment not meeting $I$.

If there is no vertical connection, or no horizontal connection, then such segments always exist. One may even ask that the left endpoint of $I$ be in $\Sigma$ ([Yoc10, Prop. 5.7, p. 16]). In particular, one can always find $g \in \mathrm{GL}(2, \mathbb{R})$, preserving the vertical direction, and a segment in good position that is horizontal for $g.\zeta$.

When $I$ is in good position, the return map $T_I$ of the vertical flow on $I$ is an i.e.m. and the translation surface $(M, \Sigma, \zeta)$ can be recovered from $T_I$ and the appropriate *suspension data* via Veech's *zippered rectangles construction.*

8.2. *The boundary operator and the conjugacy invariant.* Let $(M, \Sigma, \zeta)$ be a translation surface, $(V^\tau)$ be its vertical flow, $I$ a horizontal segment in good position, and $T_I$ the associated return map. Let $I = \sqcup I^t_\alpha = \sqcup I^b_\alpha$ be the partitions defining the i.e.m. $T_I$. As usual, we denote by $u^t_1 < \cdots < u^t_{d-1}$ the singularities of $T_I$ and by $u^b_1 < \cdots < u^b_{d-1}$ those of $T_I^{-1}$.

Let $r$ be an integer $\geqslant 0$. Denote by $C^r_c(M-\Sigma)$ the functions of class $C^r$ with compact support in $M-\Sigma$. For a function $\Phi \in C^r_c(M-\Sigma)$, one defines a function $\varphi := I(\Phi)$ on $\sqcup I^t_\alpha$ by

$$\varphi(x) = \int_0^{r(x)} \Phi(V^\tau(x))\, d\tau,$$

where $r(x)$ is the return time of $x$ to $I$. Observe that $I$ commutes with horizontal partial derivatives. If $r \geqslant 1$ and $\Phi \in C^r_c(M-\Sigma)$, then one has $\frac{\partial}{\partial x}\Phi \in C^{r-1}_c(M-\Sigma)$ and $I(\frac{\partial}{\partial x}\Phi) = D[I(\Phi)]$.

Proposition 8.5. *The operator $I$ sends $C^r_c(M-\Sigma)$ continuously into $C^r(\sqcup I^t_\alpha)$. Its image is the subspace of functions $\varphi \in C^r(\sqcup I^t_\alpha)$ satisfying $\partial D^i\varphi = 0$ for all $0 \leqslant i \leqslant r$.*

*Proof.* Regarding the first assertion, the case $r=0$ is clear and the assertion for higher $r$ follows from the commutation with horizontal partial derivatives.

Let $\Phi \in C^0_c(M-\Sigma), \varphi := I(\Phi) \in C^0(\sqcup I^t_\alpha)$. For $1 \leqslant j \leqslant d-1$, define

$$L_\Phi(u^t_j) = \int_0^{\rho^t_j} \Phi(V^\tau(u^t_j))\, d\tau,$$

$$L_\Phi(u^b_j) = \int_{\rho^b_j}^0 \Phi(V^\tau(u^b_j))\, d\tau,$$

where $\rho^b_j < 0 < \rho^t_j$ are the times such that $V^{\rho^t_j}(u^t_j)$ and $V^{\rho^b_j}(u^b_j)$ belong to $\Sigma$.

We use the notation of Section 3.1. Let $\upsilon \in \mathcal{A}^{(2)}$; if neither $u^t(\upsilon)$ nor $u^b(\upsilon)$ is an endpoint of $I$, one has

$$\varphi(\upsilon) = L_\Phi(u^t(\upsilon)) + L_\Phi(u^b(\upsilon)).$$

For the remaining elements of $\mathcal{A}^{(2)}$, we have

$$\varphi({}_t\,\alpha, L) + \varphi({}_b\,\alpha, L) = L_\Phi(u^t({}_b\,\alpha, L) + L_\Phi(u^b({}_t\,\alpha, L)),$$

$$\varphi(\alpha_t, R) + \varphi(\alpha_b, R) = L_\Phi(u^t(\alpha_b, R) + L_\Phi(u^b(\alpha_t, R)).$$

In view of the definitions of the boundary operator $\partial$ and the permutation $\sigma$ of $\mathcal{A}^{(2)}$, there is a total cancellation of the terms in the formula for $\partial\varphi$.

We have proven that $\partial I(\Phi) = 0$ for $\Phi \in C^0_c(M-\Sigma)$. The case of higher $r$ follows from the commutation of $I$ with horizontal partial derivatives. □

The following proposition may be seen as a nonlinear version of the previous proposition. Let $X_*$ be a $C^r$ simple deformation of the vertical vector field $X_0$ of $\zeta$. We denote by $V^\tau_*$ the flow of $X_*$. We assume that $X$ and $X_0$ coincide on the vertical separatrices segments connecting $\Sigma$ to the endpoints of $I$. This warrants that the return map $T_*$ of $V^\tau_*$ on $I$ is a generalized i.e.m. of class $C^r$ with the same combinatorics as $T_0 := T_I$. We can therefore consider the conjugacy invariant $J(T_*) \in J^r$ introduced in Section 4.1.

PROPOSITION 8.6. *The invariant $J(T_*)$ is trivial.*

*Proof.* The proof is essentially the same as the proof of the relation $\partial I(\Phi) = 0$ in the previous proposition. Let $u^t_{1,*} < \cdots < u^t_{d-1,*}$ be the singularities of $T_*$ and let $u^b_{1,*} < \cdots < u^b_{d-1,*}$ be those of $T_*^{-1}$. For each $1 \leqslant j \leqslant d-1$, let $I^t_j$ be a small horizontal segment transverse to the separatrix from $u^t_j$ to $\Sigma$ and very close to $\Sigma$, with coordinate $x^t_j$ centered at the intersection with the separatrix. Similarly, define $I^b_j$ with coordinate $x^b_j$. Let $J_X(u^t_{j,*})$ be the $r$-jet at $u^t_{j,*}$ of the transition map from $I$ to $I^t_j$ along $X_*$. Let $J_X(u^b_{j,*})$ be the $r$-jet at $x^b_j = 0$ of the transition map from $I^b_j$ to $I$ along $X_*$.

Let $\upsilon \in \mathcal{A}^{(2)}$. If neither $u^t(\upsilon)$ nor $u^b(\upsilon)$ is an endpoint of $I$, one has

$$j(T_*, \upsilon) = J_X(u^b_*(\upsilon))J_X(u^t_*(\upsilon)).$$

For the remaining elements of $\mathcal{A}^{(2)}$, we have

$$j(T_*, ({}_t\,\alpha, L))j(T_*, ({}_b\,\alpha, L)) = J_X(u^b({}_t\,\alpha, L))J_X(u^t({}_b\,\alpha, L),$$
$$j(T_*, (\alpha_t, R))j(T_*, (\alpha_b, R)) = J_X(u^b(\alpha_t, R))J_X(u^t(\alpha_b, R).$$

The same cancellation takes place. Therefore, the conjugacy invariant is trivial. □

8.3. *Statement of the result.*

*Definition* 8.7. Let $(M, \Sigma, \zeta)$ be a translation surface. It is of *Roth type* (resp. *restricted Roth type*) if there exists *some* open bounded horizontal segment $I$ in good position such that the return map $T_I$ of the vertical flow on $I$ is an i.e.m. of Roth type (resp. restricted Roth type).

Actually, in Appendix C we show that for a (restricted) Roth type translation surface, $T_I$ will be of (restricted) Roth type for *any* horizontal segment $I$ in good position.

Recall that two vector fields $X, Y$ are said to be $C^r$-equivalent if there exist a $C^r$-diffeomorphism $H$ sending the time-oriented orbits of the flow of $X$ on the time-oriented orbits of the flow of $Y$. An equivalent formulation is that $H^*X$ is a positive scalar multiple of $Y$, or that $Y$ is obtained from $H^*X$ by time reparametrization.

COROLLARY OF THE MAIN THEOREM. *Let $(M, \Sigma, \zeta)$ be a translation surface of restricted Roth type, and let $r$ be an integer $\geqslant 2$. Amongst the $C^{r+3}$-simple deformations of the vertical vector field $X_0$, those that are $C^r$-equivalent to $X_0$ by a diffeomorphism $C^r$-close to the identity form a $C^1$-submanifold of codimension $d^* = (g-1)(2r+1) + s$.*

Concerning conjugacy between $X_t$ and $X_0$ instead of equivalence, see the final remark after the proof.

8.4. *Proof of the corollary.* Let $(X_t)_{t\in[-t_0,t_0]^{\ell+d^*}}$ be a $C^1$-family of $C^{r+3}$-simple deformations of the vertical vector field $X_0$. We assume that the family is in general position. At various stages we will allow to restrict $t$ to a smaller neighborhood of 0. We divide the proof into several steps.

(1) Choose an open bounded horizontal segment $I$ in good position such that the return map $T_0$ of $X_0$ on $I$ is an i.e.m. of restricted Roth type. By slightly shifting vertically $I$ if necessary, we may assume that the endpoints $u_0$, $u_d$ of $I$ are not in $\Sigma$. By definition of good position, there are vertical segments $J_0$, $J_d$ disjoint from $I$ connecting these endpoints to points of $\Sigma$.

There exists a $C^1$-family $(k_t)$ of $C^{r+3}$-diffeomorphisms of $M$, supported on a compact set of $M-\Sigma$, such that $k_0$ is the identity and for all $t$, the vector field $k_t^* X_t$ coincides with $X_0$ on $J_0$ and $J_d$. Replacing $X_t$ by $k_t^* X_t$, we assume from now on that $X_t$ coincides with $X_0$ on $J_0$ and $J_d$.

(2) The singularities $u_1^t < \cdots < u_{d-1}^t$ of $T_0$ are the last intersections with $I$ of the ingoing vertical separatrices of $X_0$, while the singularities $u_1^b < \cdots < u_{d-1}^b$ of $T_0^{-1}$ are the first intersections with $I$ of the outgoing vertical separatrices of $X_0$. As $X_t$ coincides with $X_0$ in the neighborhood of $\Sigma$, we can also define ingoing and outgoing separatrices for $X_t$. By the implicit function theorem, the ingoing separatrices will have as last intersection with $I$ points $u_1^t(t) < \cdots < u_{d-1}^t(t)$, which are $C^1$-functions of $t$. Notice here that the fact that $X_t$ coincides with $X_0$ on $J_0$ and $J_d$ is crucial to guarantee that these are *last* intersections.

Having the separatrices under control, we know that the first return map $T_t$ for $X_t$ on $I$ is a generalized i.e.m. of class $C^{r+3}$ with the same combinatorics as $T_0$ and that $(T_t)$ is a $C^1$-family of such g.i.e.m. Moreover, every infinite half-orbit of $X_t$ (in the past or in the future) intersects $I$. This is a consequence of the implicit function theorem, taking into account that the return times to $I$ for $X_0$ are bounded.

(3) From Section 8.2, for all $t$ close to 0, the conjugacy invariant of $T_t$ in $J^{r+3}$ is trivial. On the other hand, this is the only restriction on $T_t$. The map $X \mapsto T_X$, which associates the return map to $I$ to a $C^{r+3}$ simple deformation $X$ of $X_0$ such that $X = X_0$ on $J_0 \cup J_d$, is a *submersion* onto g.i.e.m. with trivial $J^{r+3}$-invariant.

It follows that the family $(T_t)$ itself will be in general position (amongst g.i.e.m. with trivial $J^{r+3}$-invariant). By our main theorem, there is a $C^1$-submanifold $\mathcal{C}$ of codimension $d^*$ through 0 that consists exactly of the parameters $t$ such that $T_t$ is conjugated to $T_0$ by a $C^r$-diffeomorphism of $I$ that is $C^r$-close to the identity.

(4) We will promote, for $t \in \mathcal{C}$, the conjugacy $h_t$ between $T_0$ and $T_t$ (i.e., $h_t \circ T_0 = T_t \circ h_t$) to a $C^r$-equivalence $H_t$ between $X_0$ and $X_t$. This is best done in two steps.

First, consider a small neighborhood $U$ of $I$ in $M - \Sigma$ and a $C^1$-family $(H_t^\sharp)_{t\in\mathcal{C}}$ of $C^r$-diffeomorphisms of $M$, $C^r$-close to the identity, with the following properties:

- For each $t \in \mathcal{C}$, $H_t^\sharp$ has support in $U$ and is the identity on $J_0$ and $J_d$.
- For each $t \in \mathcal{C}$, $H_t^\sharp$ preserves $I$ and the restriction of $H_t$ to $I$ is equal to $h_t^{-1}$.
- $H_0^\sharp$ is the identity.

Then for each $t \in \mathcal{C}$, the vector field $X_t^\sharp := (H_t^\sharp)^* X_t$ is a $C^r$ simple deformation of $X_0$ for which the return map to $I$ is equal to $T_0$. We also still have the property that every infinite half-orbit of $X_t^\sharp$ intersects $I$. Observe also that $X_0^\sharp = X_0$.

(5) Denote by $(V_t^\tau)$ the flow of $X_t^\sharp$. For $x \in I$, not a singularity of $T_0$, let $r_t(x)$ be the return time to $I$ of $x$ under $X_t^\sharp$.

The $C^r$-equivalence $H_t$ we are looking for will be

$$H_t = \widetilde{H}_t \circ (H_t^\sharp)^{-1},$$

where $\widetilde{H}_t$ is a $C^r$-equivalence between $X_0$ and $X_t^\sharp$ satisfying

$$\widetilde{H}_t(V_0^\tau(x)) = V_t^{g_{x,t}(\tau)}(x) \quad \text{for } x \in I,\ \tau \in [0, r_0(x)].$$

Here $g_{x,t}$ is a diffeomorphism from $[0, r_0(x)]$ onto $[0, r_t(x)]$. However, we have to be careful in the choice of $g_{x,t}$ when $x$ gets close to the endpoints of $I$ or the singularities of $T_0$ because we want $\widetilde{H}_t$ to preserve $\Sigma$ and be of class $C^r$ on the whole of $M$. We will actually define not only $g_{x,t}$ but also the right and left limits $g_{\upsilon,t}$ for $\upsilon \in \mathcal{A}^{(2)}$ (cf. Section 3.1). Observe that for any $\alpha \in \mathcal{A}$, the restriction of the return time function $r_t$ to $I_\alpha^t$ extends to a $C^r$-function on the closure of $I_\alpha^t$. In particular, the values $r_t(\upsilon)$ are well defined.

(6) For $1 \leqslant j \leqslant d-1$, let $A_j \in \Sigma$ be the endpoint of the ingoing $X_t^\sharp$-separatrix from $u_j^t$ and let $\rho_{j,t}$ the time span of this separatrix.

We will only consider the case where both $J_0$ and $J_d$ are *outgoing* separatrices. The other cases are dealt in the same manner, with minor modifications. Under this assumption, for each $1 \leqslant j \leqslant d-1$, we have

$$0 < \rho_{j,t} < \min(r_t(\upsilon_{j,-}), r_t(\upsilon_{j,+})),$$

where $\upsilon_{j,\pm}$ are the elements of $\mathcal{A}^{(2)}$ adjacent to $u_j^t$.

(7) Let $1 \leqslant j \leqslant d-1$, and let $\varepsilon > 0$ be small enough so that $X_t$ and $X_0$ are equal in a $3\varepsilon$-neighborhood of $\Sigma$. The image by $V_0^{\rho_{j,0}}$ of the segment

$(u_j^t - 2\varepsilon, u_j^t + 2\varepsilon) \subset I$ is a horizontal segment $I_j(2\varepsilon)$ through $A_j$. Let

$$C(2\varepsilon) = \left\{ z \in \mathbb{C},\, 0 < |z| < 2\varepsilon,\, -\frac{3\pi}{2} < \arg z < \frac{\pi}{2} \right\},$$

and let $z_j = x_j + iy_j : C_j(2\varepsilon) \to C(2\varepsilon)$ be the chart of $\zeta$ such that the equation of $I_j(\varepsilon)$ is $y_j = 0$. The domain $C_j(2\varepsilon)$ is a circular open cone of radius $2\varepsilon$, aperture $2\pi$ at $A_j$.

For $t$ close enough to 0, there is a $C^r$-map $G_{j,t}$ defined on $(-\varepsilon, \varepsilon)$ (and depending in a $C^1$ way on $t$) such that for $s \in (-\varepsilon, \varepsilon)$, the point $V_t^\tau(u_j^t + s)$ belongs to $I_j(2\varepsilon)$ at a time $\tau$ close to $\rho_{j,0}$ and its $x_j$-coordinate is $G_{j,t}(s)$. Observe that

$$G_{j,0}(s) \equiv s, \qquad G_{j,t}(0) = 0.$$

Write $G_{j,t}(s) = s w_{j,t}(s)$, with $w_{j,t}(s)$ close to 1. Define then a map $G^*_{j,t} : C_j(\varepsilon) \to C_j(2\varepsilon)$ given in the $z_j$-coordinate by

$$G^*_{j,t}(z_j) = z_j w_{j,t}(x_j).$$

Observe that $G^*_{j,0}$ is the identity, that $G^*_{j,t}$ preserves the vertical foliation, and that its restriction to the segment $I_j(\varepsilon)$ is $(x_j, 0) \mapsto (G_{j,t}(x_j), 0)$.

(1) We now choose the $g_{x,t}$ in order to satisfy the following properties:

• For each $x \in I$ that is not a singularity of $T_0$, $g_{x,t}$ is a diffeomorphism from $[0, r_0(x)]$ onto $[0, r_t(x)]$.

• For each $\upsilon \in \mathcal{A}^{(2)}$, $\upsilon \neq ({}_t\alpha, L), (\alpha_t, R)$, $g_{\upsilon,t}$ is a diffeomorphism from $[0, r_0(\upsilon)]$ onto $[0, r_t(\upsilon)]$.

• For $\upsilon_0 := ({}_t\alpha, L)$, $g_{\upsilon_0,t}$ is a diffeomorphism from $[-|J_0|, r_0(\upsilon_0)]$ onto $[-|J_0|, r_t(\upsilon_0)]$; for $\upsilon_d := (\alpha_t, R)$, $g_{\upsilon_d,t}$ is a diffeomorphism from $[-|J_d|, r_0(\upsilon_d)]$ onto $[-|J_d|, r_t(\upsilon_d)]$.

• For $\tau \in [0, \varepsilon)$, any $x$ or $\upsilon$, we have

$$g_{x,t}(\tau) = g_{\upsilon,t}(\tau) = \tau,$$
$$g_{x,t}(r_0(x) - \tau) = r_t(x) - \tau, \qquad g_{\upsilon,t}(r_0(\upsilon) - \tau) = r_t(\upsilon) - \tau.$$

• For $1 \leqslant j \leqslant d-1$, the diffeomorphisms $g_{\upsilon_{j,-},t}$ and $g_{\upsilon_{j,+},t}$ coincide on $[0, \rho_{j,0}]$ and send this interval onto $[0, \rho_{j,t}]$; we denote by $g_{u_j^t,t}$ this restriction.

• The similar condition for the outgoing separatrices is as follows. Let $1 \leqslant j \leqslant d-1$, and let $\upsilon'_{j,\pm}$ be the elements of $\mathcal{A}^{(2)}$ adjacent to $u_j^b$. Then we must have

$$r_t(\upsilon'_{j,-}) - g_{\upsilon'_{j,-},t}(r_0(\upsilon'_{j,-}) - \tau) = r_t(\upsilon'_{j,+}) - g_{\upsilon'_{j,+},t}(r_0(\upsilon'_{j,+}) - \tau)$$

for $0 \leqslant \tau \leqslant \rho'_{j,0}$; here $\rho'_{j,0}$ is the time span from $\Sigma$ to $u_j^b$ for $X_0$.

• Let $R_t$ be the open subset of $I \times \mathbb{R}$ formed of the pairs $(x, \tau)$ such that $0 < \tau < r_t(x)$ if $x$ is not one of the $u_j^t$, $0 < \tau < \rho_{j,t}$ if $x = u_j^t$. Then the map

$G_t : (x, \tau) \mapsto (x, g_{x,t}(\tau))$ is a $C^r$-diffeomorphism from $R_0$ onto $R_t$, and $t \mapsto G_t$ is $C^1$.

- We also ask for a similar condition along the outgoing separatrices.
- If $(x, \tau) \in R_0$ satisfies $V_0^{\tau(x)} \in C_j(\varepsilon)$ (for some $1 \leqslant j \leqslant d-1$), then

$$V_t^{g_{x,t}(\tau)}(x) = G^*_{j,t}(V_0^{\tau(x)}).$$

It is fastidious but not difficult to check that these conditions are compatible and that one can indeed satisfy all conditions.

As mentioned earlier, from the $g_{x,t}$, we obtain an equivalence $\widetilde{H}_t$ between $X_0$ and $X_t^\sharp$ by the formula

$$\widetilde{H}_t(V_0^\tau(x)) = V_t^{g_{x,t}(\tau)}(x).$$

The properties required along the ingoing and outgoing separatrices guarantee that $\widetilde{H}_t$ is a $C^r$-diffeomorphism of $M - \Sigma$. Then, the properties of the $G^*_{j,t}$ warrant that $\widetilde{H}_t$ is also $C^r$ in the neighborhood of $\Sigma$. Indeed, if $k$ is the ramification index of $\zeta$ at $A_j$ and we write $z_j = Z_j^k$, we will have in the $Z_j$-coordinate that

$$G^*_{j,t}(Z_j) = Z_j(w_{j,t}(\Re Z_j^k))^{\frac{1}{k}}.$$

We conclude that for $t \in \mathcal{C}$, $X_t$ and $X_0$ are indeed $C^r$-equivalent.

(9) On the other hand, if $X_t$ and $X_0$ are $C^r$-equivalent, the restriction of the $C^r$-equivalence to $I$ is a $C^r$-conjugacy between $T_t$ and $T_0$; hence, $t \in \mathcal{C}$. □

*Remark* 8.8. One could look for a *conjugacy* (respecting time) rather than an equivalence between $X_t$ and $X_0$. To transform an equivalence into a conjugacy, one needs that the return times to $I$ of $X_0$ and $X_t$ differ by the coboundary of an appropriately smooth function on $I$. Therefore, from the results on the cohomological equation (using also a transversality argument), one finds a submanifold $\mathcal{C}^*$ of $\mathcal{C}$ of codimension $g$ such that $X_t$ and $X_0$ are $C^{r-2}$-conjugated for $t \in \mathcal{C}^*$. However, it is not clear at all that $d^* + g$ is the right codimension for the $C^{r-2}$-conjugacy class of $X_0$ amongst $C^{r+3}$ (or $C^\infty$) simple deformations of $X_0$.

## Appendix A. The cohomological equation with $C^{1+\tau}$ data

In this appendix we show that Theorem 3.10 is also valid with $C^{1+\tau}$ data. Let $\tau \in (0,1)$. We denote by $C^{1+\tau}_\partial(\sqcup I^t_\alpha)$ the space of functions $\varphi \in C^1_\partial(\sqcup I^t_\alpha)$ whose restrictions to each $I^t_\alpha$ is of class $C^{1+\tau}$. Let $T$ be a standard i.e.m. of Roth type. We choose a subspace $\Gamma_u \subset \Gamma_\partial$ complementing $\Gamma_T$.

Theorem A.1. *There exist bounded linear operators* $L_0 : \varphi \mapsto \psi$ *from* $C^{1+\tau}_\partial(\sqcup I^t_\alpha)$ *to* $C^0(\overline{I})$ *and* $L_1 : \varphi \mapsto \chi$ *from* $C^{1+\tau}_\partial(\sqcup I^t_\alpha)$ *to* $\Gamma_u$ *such that, for all* $\varphi \in C^{1+\tau}_\partial(\sqcup I^t_\alpha)$, *we have*

$$\varphi = \chi + \psi \circ T - \psi.$$

*Proof.* We use the notation of Section 3.3. Associated to any initial subpath $\gamma(1) * \cdots * \gamma(n)$ of the "rotation number" $\underline{\gamma}$ of $T$, there is an i.e.m. $T^{(n)}$ defined on an interval $I^{(n)}$ with the same left endpoint $u_0$ as $I$. $T^{(n)}$ is the first return map of $t$ on $I^{(n)}$ and is deduced from $T$ by the steps of the Rauzy-Veech algorithm represented by $\gamma(1) * \cdots * \gamma(n)$. For $\ell < n$, we have a "special Birkhoff sum" operator $S(\ell, n)$ defined as follows. If $\varphi$ is a function on $\sqcup I^{t,(\ell)}_\alpha$, then $S(\ell, n)\varphi$ is defined on $\sqcup I^{t,(n)}_\alpha$ by

$$S(\ell, n)\varphi(x) = \sum_{0 \leqslant i < r(x)} \varphi((T^{(\ell)})^i(x)),$$

where $r(x)$ is the return time of $x$ in $I^{(n)}$ under $T^{(\ell)}$. There are three steps in the proof of the theorem:

- One first obtains, for some $\delta > 0$, and any function $\varphi \in C^\tau(\sqcup I^t_\alpha)$ with $\int_I \varphi = 0$,

  $$||S(0, n)\varphi||_{C^0} \leqslant C\, ||B(n)||^{1-\delta}\, ||\varphi||_{C^\tau}.$$

  Here, only conditions (a) and (b) in the definition of Roth type are used.
- One then obtain by integration (also using condition (c) in the definition of Roth type) that there exists $\delta' > 0$ such that for any $\varphi \in C^{1+\tau}_\partial(\sqcup I^t_\alpha)$, one can find a unique $\chi \in \Gamma_u$ such that

  $$||S(0, n)(\varphi - \chi)||_{C^0} \leqslant C\, ||B(n)||^{-\delta'}\, ||\varphi||_{C^{1+\tau}}.$$
- This last estimate easily implies (using condition (a)) that the ordinary Birkhoff sums of $\varphi - \chi$ are bounded. It follows then, as explained in Section 3, that $\varphi - \chi = \psi \circ T - \psi$ for some $\psi \in C^0(\overline{I})$.

The last two steps are done in exactly the same way in the present setting as in the setting of Theorem 3.10. We will therefore only indicate how to prove the estimate of the first step.

Therefore let $\varphi \in C^\tau(\sqcup I^t_\alpha)$ with $\int_I \varphi = 0$. The method is as in [MMY05]. We write

$$\varphi = \varphi_0 + \chi_0,$$

with $\varphi_0$ of mean value 0 on each $I^t_\alpha$ and $\chi_0 \in \Gamma$ (of mean value 0 as $\int_I \varphi = 0$). For $0 < \ell \leqslant n$, in the same way we write

$$S(\ell - 1, \ell)\varphi_{\ell-1} = \varphi_\ell + \chi_\ell$$

with $\varphi_\ell$ of mean value 0 on each $I^{t,(\ell)}_\alpha$ and $\chi_\ell \in \Gamma^{(\ell)}$ (of mean value 0).

Then, we have

$$S(0,n)\varphi = \varphi_n + \sum_0^n S(\ell,n)\chi_\ell.$$

For $0 \leqslant \ell \leqslant n$, $\alpha \in \mathcal{A}$, $x, y \in I_\alpha^{t,(\ell)}$, one has

$$\begin{aligned} |\varphi_\ell(x) - \varphi_\ell(y)| &= |S(0,\ell)\varphi(x) - S(0,\ell)\varphi(y)| \\ &\leqslant r(x)|I_\alpha^{t,(\ell)}|^\tau ||\varphi||_{C^\tau}. \end{aligned}$$

Here $r(x)$ is the sum of the $\alpha$-column of $B(\ell)$. From condition (a), we have (cf. [MMY05, Prop., p. 835]) $|I_\alpha^{t,(\ell)}| \leqslant C||B(\ell)||^{-\frac{1}{2}}$; hence, we obtain

$$|\varphi_\ell(x) - \varphi_\ell(y)| \leqslant C||B(\ell)||^{1-\frac{\tau}{2}}||\varphi||_{C^\tau}.$$

As $\varphi_\ell$ vanishes in each $I_\alpha^{t,(\ell)}$, this implies

$$||\varphi_\ell||_{C^0} \leqslant C||B(\ell)||^{1-\frac{\tau}{2}}||\varphi||_{C^\tau}.$$

For $0 < \ell \leqslant n$, this gives

$$\begin{aligned} ||\varphi_\ell + \chi_\ell||_{C^0} &\leqslant ||Z(\ell)||\;||\varphi_{\ell-1}||_{C^0} \\ &\leqslant C||B(\ell)||^{1-\frac{\tau}{3}}||\varphi||_{C^\tau}, \\ ||\chi_\ell||_{C^0} &\leqslant C||B(\ell)||^{1-\frac{\tau}{3}}||\varphi||_{C^\tau}. \end{aligned}$$

Putting these estimates in the expression for $S(0,n)\varphi$ above, we have to bound from above the sum

$$\sum_0^n ||B(\ell)||^{1-\frac{\tau}{3}}||B_0(\ell,n)||, \tag{A.1}$$

where $B_0(\ell,n)$ is the restriction of $B(\ell,n)$ to the hyperplane $\Gamma_0^{(\ell)}$ (of functions with mean value 0 on $I^{(\ell)}$, constant on each $I_\alpha^{t,(\ell)}$). To estimate the sum in (A.1), we deal separately with the terms with small $\ell$ and large $\ell$.

- When $||B(\ell)|| < ||B(n)||^{\frac{\theta}{3}}$, we write

$$B_0(\ell,n) = B_0(n)\,B_0(\ell)^{-1}$$

and get from condition (b) of Roth type (as $B(\ell)$ is symplectic)

$$\begin{aligned} ||B_0(\ell,n)|| &\leqslant ||B_0(n)||\;||B(\ell)^{-1}|| \\ &\leqslant C||B(n)||^{1-\theta}\,||B(\ell)|| \\ ||B(\ell)||^{1-\frac{\tau}{3}}||B_0(\ell,n)|| &\leqslant ||B(n)||^{1-\frac{\theta}{3}}. \end{aligned}$$

When $||B(\ell)|| \geqslant ||B(n)||^{\frac{\theta}{3}}$, we just bound $||B_0(\ell,n)||$ by $||B(\ell,n)||$.

Claim. *For every $\eta > 0$, there exists $C(\eta)$ such that for all $0 \leqslant \ell \leqslant n$, one has*

$$||B(n)|| \leqslant ||B(\ell)|| \, ||B(\ell, n)|| \leqslant C(\eta)||B(n)||^{1+\eta}.$$

In this case the claim gives the following bound:

$$||B(\ell)||^{1-\frac{\tau}{3}} ||B_0(\ell, n)|| \leqslant C||B(n)||^{1-\frac{\tau\theta}{10}}.$$

As $||B(n)||$ grows at least exponentially fast, one obtains that the sum in (A.1) is indeed bounded by $C\,||B(n)||^{1-\delta}$ for $\delta < \frac{\tau\theta}{10}$.

*Proof of the claim.* The left-hand inequality is trivial. If $m - \ell \geqslant 2d - 3$, all coefficients of $B(\ell, m)$ are $\geqslant 1$ ([MMY05, lemma on p. 833]. Therefore, for $n \geqslant m \geqslant \ell \geqslant 0$ with $m - \ell \geqslant 2d - 3$, we have $||B(n)|| \geqslant ||B(\ell)|| \, ||B(m, n)||$. The right-hand inequality in the claim now follows from condition (a) in the definition of Roth type. □

The proof of the inequality for special Birkhoff sums of $C^\tau$-functions is now complete. A mentioned above, the rest of the proof of the theorem is the same as for Theorem 3.10. □

## Appendix B. The case of circle diffeomorphisms

B.1. *The $C^r$-case*, $r \geqslant 3$. Let $F$ be a $C^{r+3}$-orientation-preserving diffeomorphism of the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ that is $C^{r+3}$-close to a rotation $R_\omega$. We assume that $\omega$ satisfies a diophantine condition $CD(\gamma, \tau)$ with $\tau < 1$:

$$\forall \frac{p}{q}, \quad |\omega - \frac{p}{q}| \geqslant \gamma q^{-2-\tau}.$$

Following Herman [He], we show that one can write

$$F = R_t \circ h \circ R_\omega \circ h^{-1}$$

for some unique $t$ close to 0 and some unique $h \in \mathrm{Diff}^r_+(\mathbb{T})$ normalized by $\int_{\mathbb{T}}(h - \mathrm{id}) = 0$. Both $t$ and $h$ are $C^1$-functions of $F$.

We denote by $\mathrm{Diff}^r_{+,0}(\mathbb{T})$ the set of $h \in \mathrm{Diff}^r_+(\mathbb{T})$ satisfying $\int_{\mathbb{T}}(h - \mathrm{id}) = 0$, by $C^r_0(\mathbb{T})$ the space of $C^r$-functions on $\mathbb{T}$ with zero mean value.

Lemma B.1. *The map $(F, h) \mapsto \Phi(F, h) := (SF \circ h)(Dh)^2$ from $\mathrm{Diff}^{r+3}_+(\mathbb{T}) \times \mathrm{Diff}^r_{+,0}(\mathbb{T})$ to $C^{r-1}(\mathbb{T})$ is of class $C^1$. Its differential at $(R_\omega, \mathrm{id})$ is the map $(\delta F, \delta h) \mapsto D^3\delta F$.*

Lemma B.2. *The map $h \mapsto Sh - \int_{\mathbb{T}} Sh$ from $\mathrm{Diff}^r_{+,0}(\mathbb{T})$ to $C^{r-3}_0(\mathbb{T})$ is of class $C^\infty$. Its differential at* id *is $\delta h \mapsto D^3\delta h$. Therefore, its restriction to a neighborhood of the identity in $\mathrm{Diff}^r_{+,0}(\mathbb{T})$ is a $C^\infty$-diffeomorphism onto a neighborhood of* 0 *in $C^{r-3}_0(\mathbb{T})$.*

Let us write $\mathcal{P}$ for the inverse diffeomorphism, $P$ for its differential at 0 (consisting in taking thrice a primitive with mean value zero).

As $\omega$ satisfies $CD(\gamma, \tau)$ with $\tau < 1$, there exists a bounded operator $L$ from $C^{r-1}(\mathbb{T})$ to $C_0^{r-3}(\mathbb{T})$ such that for every $\varphi \in C^{r-1}(\mathbb{T})$,

$$\varphi = \int_{\mathbb{T}} \varphi + L(\varphi) \circ R_\omega - L(\varphi).$$

From the two lemmas above, we see that the map

$$(F, h) \mapsto \mathcal{P}(L(\Phi(F, h)))$$

is defined and of class $C^1$ in a neighborhhood of $(R_\omega, \mathrm{id})$ in $\mathrm{Diff}_+^{r+3}(\mathbb{T}) \times \mathrm{Diff}_{+,0}^r(\mathbb{T})$, with values in $\mathrm{Diff}_{+,0}^r(\mathbb{T})$. The differential at $(R_\omega, \mathrm{id})$,

$$(\delta F, \delta h) \mapsto P(L(D^3 \delta F)),$$

does not involve $\delta h$. Therefore, if $F$ is close enough to $R_\omega$, this map will have a unique fixed point $h = \mathcal{H}(F)$ close to the identity. This fixed point satisfies, with $c = \int_{\mathbb{T}} \Phi(F, h)$,

$$S(F \circ h) = S(h \circ R_\omega) + c.$$

One then concludes from Lemma B.2 that $F \circ h = R_t \circ h \circ R_\omega$ for some $t$ close to 0.

B.2. *The $C^2$-case.* We now show how to adapt the argument when $h$ is only of class $C^2$. The Schwarzian derivative of $h$ no longer exists but its primitive can still be used!

Let $F \in \mathrm{Diff}_+^5(\mathbb{T})$ be close to $R_\omega$, with $\omega$ still satisfying $CD(\gamma, \tau)$ for some $\gamma > 0, \tau < 1$. Lemma B.1 with $r = 2$ is still valid. For $h \in \mathrm{Diff}_{+,0}^2(\mathbb{T})$, we define $N_1 h \in C_0^0(\mathbb{T})$ by

$$N_1 h(x) = D\mathrm{Log}Dh(x) - \frac{1}{2} \int^x ((D\mathrm{Log}Dh)^2(y) - c_1) dy,$$

where $c_1 = \int_{\mathbb{T}} (D\mathrm{Log}Dh)^2(y)\, dy$ and the primitive is taken in order to have $\int_{\mathbb{T}} N_1 h(x)\, dx = 0$.

LEMMA B.3. *The map $h \mapsto N_1 h$ from $\mathrm{Diff}_{+,0}^2(\mathbb{T})$ to $C_0^0(\mathbb{T})$ is of class $C^\infty$. Its differential at* id *is $\delta h \mapsto D^2 \delta h$. Therefore, its restriction to a neighborhood of the identity in $\mathrm{Diff}_{+,0}^2(\mathbb{T})$ is a $C^\infty$-diffeomorphism onto a neighborhood of* 0 *in $C_0^0(\mathbb{T})$.*

Let us write $\mathcal{P}_1$ for the inverse diffeomorphism, $P_1$ for its differential at 0 (consisting in taking twice a primitive with mean value zero).

Let us also write $P^*$ for the operator from $C^1(\mathbb{T})$ to $C_0^2(\mathbb{T})$:

$$\varphi \mapsto \int^x (\varphi(y) - \int_{\mathbb{T}} \varphi)\, dy,$$

the primitive being taken in order to have mean value 0.

Now consider the map

$$(F, h) \mapsto \mathcal{P}_1(L(P^*(\Phi(F, h)))).$$

It is defined and of class $C^1$ in a neighborhood of $(R_\omega, \mathrm{id})$ in $\mathrm{Diff}^5_+(\mathbb{T}) \times \mathrm{Diff}^2_{+,0}(\mathbb{T})$, with values in $\mathrm{Diff}^2_{+,0}(\mathbb{T})$, sending $(R_\omega, \mathrm{id})$ to id. The differential at $(R_\omega, \mathrm{id})$,

$$(\delta F, \delta h) \mapsto P_1(L(D^2 \delta F)),$$

does not involve $\delta h$. Therefore, if $F$ is close enough to $R_\omega$, this map will have a unique fixed point $h = \mathcal{H}(F)$ close to the identity. This fixed point satisfies

$$P^*(\Phi(F, h)) = N_1 h \circ R_\omega - N_1 h. \tag{B.1}$$

We will see below that this implies

$$(D\mathrm{Log}DF \circ h)(Dh) = D\mathrm{Log}Dh \circ R_\omega - D\mathrm{Log}Dh. \tag{B.2}$$

From (B.2), we get $\mathrm{Log}D(F \circ h) = \mathrm{Log}D(h \circ R_\omega) + c_0$ by integration. As the integral over $\mathbb{T}$ of both $D(F \circ h)$ and $D(h \circ R_\omega)$ is equal to 1, the constant $c_0$ must be equal to 0. We conclude that $F \circ h = R_t \circ h \circ R_\omega$ for some $t$ close to 0.

To see that (B.1) indeed implies (B.2), we introduce the map

$$(\psi, h) \mapsto \psi - \Delta\psi$$

from $C^0_0(\mathbb{T}) \times \mathrm{Diff}^2_{+,0}(\mathbb{T})$ to $C^0_0(\mathbb{T})$ defined by

$$D\Delta\psi = \frac{1}{2}\psi^2 + \psi D\mathrm{Log}Dh - c(\psi, h),$$
$$c(\psi, h) = \int_{\mathbb{T}} (\frac{1}{2}\psi^2 + \psi D\mathrm{Log}Dh), \quad \int_{\mathbb{T}} \Delta\psi = 0.$$

This map is of class $C^1$. The differential with respect to $\psi$ at $\psi = 0, h = \mathrm{id}$ is the identity; therefore, as long as $h$ is fixed close to the identity, it is a $C^1$-diffeomorphism from a neighborhood of $0 \in C^0_0(\mathbb{T})$ to another neighborhood of $0 \in C^0_0(\mathbb{T})$.

Let $\psi_0 = (D\mathrm{Log}DF \circ h)Dh$. We have

$$D\psi_0 = (D^2\mathrm{Log}DF \circ h)(Dh)^2 + (D\mathrm{Log}DF \circ h)D^2 h,$$
$$D\Delta\psi_0 = \frac{1}{2}(D\mathrm{Log}DF \circ h)^2 (Dh)^2$$
$$+ (D\mathrm{Log}DF \circ h)(Dh)D\mathrm{Log}Dh - c(\psi_0, h),$$
$$D(\psi_0 - \Delta\psi_0) = (SF \circ h)(Dh)^2 + c(\psi_0, h),$$

and therefore, $\psi_0 - \Delta\psi_0 = P^*(\Phi(F, h))$.

On the other hand, let $\psi_1 = D\mathrm{Log}Dh \circ R_\omega - D\mathrm{Log}Dh$. We have

$$D\Delta\psi_1 = \frac{1}{2}[(D\mathrm{Log}Dh \circ R_\omega)^2 - (D\mathrm{Log}Dh)^2];$$

hence, $\psi_1 - \Delta\psi_1 = N_1 h \circ R_\omega - N_1 h$.

Equation (B.1) means that $\psi_0 - \Delta\psi_0 = \psi_1 - \Delta\psi_1$. We conclude that $\psi_0 = \psi_1$, i.e., equation (B.2) holds.

## Appendix C. Roth-type translation surfaces

Let $(M, \Sigma, \zeta)$ be a translation surface with no vertical connection, $I$ an open bounded horizontal segment in good position, $T = T_I$ the i.e.m. on $I$ that is the return map of the vertical flow.

Let $\mathcal{A}$ be the alphabet used to describe the combinatorics of $T$, $\pi$ the combinatorial data of $T$, $\mathcal{D}$ the Rauzy diagram having $\pi$ as a vertex. Let $\gamma(T)$ be the rotation number of $T$ (cf. Section 2.4). This is an infinite path in $\mathcal{D}$ starting from $\pi$. As in Section 3.3, write $\gamma(T)$ as an infinite concatenation

$$\gamma(T) = \gamma(1) * \cdots * \gamma(n) * \cdots$$

of finite complete paths of minimal length, and for $n > 0$, define,

$$Z(n) := B_{\gamma(n)}, \quad B(n) := B_{\gamma(1)*\cdots*\gamma(n)} = Z(n) \cdots Z(1).$$

For $n \geqslant 0$, let $T^{(n)}$ be the i.e.m. obtained from $T$ by the Rauzy-Veech steps corresponding to $\gamma(1) * \cdots * \gamma(n)$; $T^{(n)}$ is the return map of $T$ (or of the vertical flow) on some interval $I^{(n)} \subset I$ having the same left endpoint as $I = I^{(0)}$.

We first deal with condition (a) in the definition of a Roth-type i.e.m. (cf. Section 3.3).

Proposition C.1. *The following conditions are equivalent*:

(1) *Condition* (*a*) *of Section* 3.3 *is satisfied by* $T$: *for all* $\tau > 0$, $||Z(n+1)|| = \mathcal{O}(||B(n)||^{\tau})$.

(2) *For all* $\tau > 0$, *we have* $\max_{\mathcal{A}} |I_\alpha^{t,(n)}| = \mathcal{O}(\min_{\mathcal{A}} |I_\alpha^{t,(n)}|^{1-\tau})$.

(3) *For all* $\tau > 0$, *there exists* $C = C(\tau) > 0$ *such that for all* $1 \leqslant i, j \leqslant d-1$, *all* $x \in I$, *and all* $N > 0$, *we have*

$$\min_{0 \leqslant \ell < N} |T^{\ell}(u_i^b) - u_j^t| \geqslant C^{-1} N^{-1-\tau}$$

*and*

$$\min_{0 \leqslant \ell < N} |T^{\ell}(u_i^b) - x| \leqslant C N^{-1+\tau}, \ \min_{0 \leqslant \ell < N} |T^{-\ell}(u_i^t) - x| \leqslant C N^{-1+\tau}.$$

(4) *For all* $\tau > 0$, *there exists* $C = C(\tau) > 0$ *such that for any vertical separatrix segment* $S$ (*ingoing or outgoing*) *with an endpoint in* $\Sigma$ *of length* $|S| \geqslant 1$, *and all* $P \in M$, *there is a horizontal segment of length* $\leqslant C|S|^{-1+\tau}$ *from* $P$ *to* $S$, *but there is no horizontal segment of length* $\leqslant C|S|^{-1-\tau}$ *from a point of* $\Sigma$ *to* $S$.

*Proof.* We will show successively that (1) is equivalent to (2), that (3) is equivalent to (4), that (1)–(2) implies (3) and that (3) implies (2).

(1) $\Leftrightarrow$ (2). Recall from the proposition in [MMY05, p. 835] that one always has $\max_{\mathcal{A}} |I_\alpha^{t,(n)}| \geqslant ||B(n)||^{-1}|I| \geqslant \min_{\mathcal{A}} |I_\alpha^{t,(n)}|$ and that (1) is equivalent to

$$\max_{\mathcal{A}} |I_\alpha^{t,(n)}| = \mathcal{O}(||B(n)||^\tau \min_{\mathcal{A}} |I_\alpha^{t,(n)}|), \quad \forall \tau > 0.$$

The equivalence of this last relation with (2) is clear.

(3) $\Leftrightarrow$ (4). Represent $(M, \Sigma, \zeta)$ as a collection of rectangles whose top sides are the $I_\alpha^b$ and whose bottom sides are the $I_\alpha^t$. The $u_i^t$, $1 \leqslant i \leqslant d-1$, are the last intersection points of $I$ with the $d-1$ ingoing separatrices, while the $u_i^b$, $1 \leqslant i \leqslant d-1$, are the first intersection points of $I$ with the $d-1$ outgoing separatrices. As the return times to $I$ (the height of the rectangles) are bounded from above and bounded away from 0, the length of a (long enough) vertical segment and the cardinality of its intersection with $I$ are comparable. This makes clear the equivalence of (3) and (4).

(1) + (2) $\Rightarrow$ (3). We start with a result of independent interest. Recall that the return time $r_\alpha(n)$ of $I_\alpha^{t,(n)}$ in $I^{(n)}$ is given by $r_\alpha(n) = \sum_\beta B_{\alpha,\beta}(n)$.

LEMMA C.2. *Assume that property* (1) *holds. Then for all* $\tau > 0$, *there exists* $C = C(\tau) > 0$ *such that the entrance times* $r_i^b(n)$ *of* $u_i^b$ *under* $T$ *in* $I^{(n)}$ *and the entrance times* $r_i^t(n)$ *of* $u_i^t$ *under* $T^{-1}$ *in* $I^{(n)}$ *satisfy, for all* $1 \leqslant i \leqslant d-1$,

$$r_i^t(n) \geqslant C^{-1}||B(n)||^{1-\tau}, \quad r_i^b(n) \geqslant C^{-1}||B(n)||^{1-\tau}.$$

*Proof of lemma.* Recall ([MMY05, p. 833] and [Yoc10, Prop. 7.12]) that the product of $2d-3$ consecutive matrices $Z(n)$ have only positive coefficients. Then from the formula for the $r_\alpha(n)$ and property (1), it follows that for all $\tau > 0$,

$$\left(\min_{\mathcal{A}} r_\alpha(n))^{-1} = \mathcal{O}(||B(n)||^{-1+\tau}\right).$$

Let $1 \leqslant i \leqslant d-1$, and let $\alpha^* \in \mathcal{A}$ be the letter such that $u_i^b$ is the left endpoint of $I_{\alpha^*}^b$. Observe that $\alpha^* \neq {}_b\alpha$. We have the following dichotomy:

- Either all arrows of $\gamma(n+1)$ with loser $\alpha^*$ are of bottom type. Then we have $r_i^b(n+1) = r_i^b(n)$.
- Or $\gamma(n+1)$ contains one arrow of top type with loser $\alpha^*$. Then we have
$$r_i^b(n+1) \geqslant r_i^b(n) + \min_{\mathcal{A}} r_\alpha(n).$$

But the first case cannot happen more than $d+1$ consecutive times: each time ${}_b\alpha$ is a winner (necessarily of an arrow of top type), $\pi_b(\alpha^*)$ goes up by 1; once $\pi_b(\alpha^*) = d$, the next arrow with winner $\alpha^*$ is of bottom type; and any sequence of arrows of bottom type with winner $\alpha^*$ is followed by an arrow of top type with loser $\alpha^*$.

In this way we get the estimate for $r_i^b(n)$. The proof for $r_i^t(n)$ is similar. $\square$

We now assume that (1) and (2) hold . We prove the first inequality in (3). Let $N > 0$. Let $n$ be the smallest integer such that $N < r_i^b(n)$. From the lemma, we have $N \geqslant r_i^b(n-1) \geqslant C^{-1}||B(n-1)||^{1-\tau}$, which gives also using (1) that $N \geqslant C_1^{-1}||B(n)||^{1-2\tau}$. On the other hand, with this choice of $n$, we have that

$$\begin{aligned}\min_{0\leqslant \ell < N} |T^\ell(u_i^b) - u_j^t| &\geqslant \min_{\mathcal{A}} |I_\alpha^{t,(n)}| \\ &\geqslant C'^{-1}||B(n)||^{-1-\tau} \\ &\geqslant C'^{-1}_1 N^{-\frac{1+\tau}{1-2\tau}}.\end{aligned}$$

As $\tau > 0$ is arbitrary, this indeed proves the first inequality of (3).

We now prove the second part of property (3), regarding the forward orbit of $u_i^b$. (The proof for the backward orbit of $u_i^t$ is similar.) Let $\alpha^* \in \mathcal{A}, \alpha^* \neq {}_b\alpha$ be the letter such that $u_i^b$ is the left endpoint of $I_{\alpha^*}^b$. Let $n$ be the largest integer such that $N \geqslant 2B(n)$. We can assume that $n > 3d+4$, and from property (1), we have

$$||B(n)||^{-1} = \mathcal{O}(N^{-1+\tau}).$$

By an argument given in the proof of the lemma, there exists in the path $\gamma(n-d) * \cdots * \gamma(n)$ an arrow of top type with loser $\alpha^*$. This corresponds to a forward iterate $T^m(u_i^b)$ with $0 \leqslant m \leqslant ||B(n)||$, which belongs to $I^{(n-d-1)}$ but is *not* one of the endpoints of the $I_\alpha^{t,(n-d-1)}, \alpha \in \mathcal{A}$. Let $\beta^* \in \mathcal{A}$ such that $T^m(u_i^b) \in I_{\beta^*}^{t,(n-d-1)}$.

Consider the orbit segment

$$T^\ell(u_i^b),\ m \leqslant \ell < m + r_{\beta^*}(n-d-1).$$

Observe that $m + r_{\beta^*}(n-d-1) \leqslant N$.

One has a partition mod.0 of $I$ by the intervals

$$T^k(I_\alpha^{t,(n-3d-4)}), \quad \alpha \in \mathcal{A},\ 0 \leqslant k < r_\alpha(n-3d-4).$$

By [MMY05, p. 833] and [Yoc10, Prop. 7.12], every interval $T^k(I_\alpha^{t,(n-3d-4)})$ contains at least an interval $T^{k'}(I_{\beta^*}^{t,(n-d-1)})$ with $0 \leqslant k' < r_{\beta*}(n-d-1)$, and this last interval contains $T^{m+k'}(u_i^b)$. Choosing $k, \alpha$ such that $x$ belongs to the closure of $T^k(I_\alpha^{t,(n-3d-4)})$, we have

$$|x - T^{m+k'}(u_i^b)| \leqslant |I_\alpha^{t,(n-d-1)}|.$$

But for all $\tau > 0$, from property (2) we have

$$|I_\alpha^{t,(n-d-1)}| = \mathcal{O}(||B(n-d-1)||^{-1+\tau}).$$

Using once again property (1) and the definition of $N$, we have

$$|I_\alpha^{t,(n-d-1)}| = \mathcal{O}(N^{-1+\tau})$$

for all $\tau > 0$, which gives the required inequality.

(3) $\Rightarrow$ (2). Assume that property (3) is satisfied. Let $n$ be an integer and let $\alpha \in \mathcal{A}$. First assume that $\alpha \neq {}_b\alpha, \alpha_b$. Then the length of $I_\alpha^{b,(n)}$ is given for some $1 \leqslant i < j < d$ by

$$|I_\alpha^{b,(n)}| = |T^{r_i^b(n)}(u_i^b) - T^{r_j^b(n)}(u_j^b)|.$$

Assume, for instance, that $r_j^b(n) \leqslant r_i^b(n)$, and write $r := r_i^b(n) - r_j^b(n)$. After $r_j^b(n)$ backward iterations, we get $|I_\alpha^{b,(n)}| = |u_j^b - T^r(u_i^b)|$. This already gives a bound from below for $I_\alpha^{b,(n)}$ when $r \leqslant 1$; otherwise, iterating backwards once (if $\alpha \neq {}_t\alpha$) or twice (if $\alpha = {}_t\alpha$), we get $|I_\alpha^{t,(n)}| = |u_{j'}^t - T^{r-a}(u_j^b)|$ for some $j'$ and some $a \in \{1, 2\}$. As the entry times $r_j^b(n), r_i^b(n)$ are bounded above by the return times $r_\alpha(n), \alpha \in \mathcal{A}$, which are themselves bounded by $||B(n)||$, we get from the first inequality of (3) that

$$|I_\alpha^{b,(n)}|^{-1} = \mathcal{O}(||B(n)||^{1+\tau}), \quad \forall \tau > 0.$$

The cases $\alpha = {}_b\alpha$ and $\alpha = \alpha_b$ involve the endpoints of $I^{(n)}$ and require a slightly different argument, which we omit, but lead to the same estimate.

We now turn to a bound from above for $|I^{(n)}|$. Let $\alpha \in \mathcal{A}$ be the letter such that $r_\alpha(n)$ is the largest return time in $I^{(n)}$. We have that $r_\alpha(n) = ||B(n)||$ (choosing as norm the greatest column sum). Assume first that $\alpha \neq {}_b\alpha, {}_t\alpha$. There exists $1 \leqslant i, j \leqslant d-1$ such that $u_i^t$ is the left endpoint of $T^{r_i^t(n)}(I_\alpha^{t,(n)})$, $u_j^b$ is the left endpoint of $T^{-r_j^b(n)}(I_\alpha^{b,(n)})$, and $r_\alpha(n) = r_i^t(n) + r_j^b(n) + 1$. Assume, for instance, that $r_j^b(n) \geqslant r_i^t(n)$; hence, $r_j^b(n) \geqslant \frac{1}{3}||B(n)||$. For $0 \leqslant m < r_j^b(n)$, we have $T^m(u_j^b) \notin I^{(n)}$ by definition of the entrance time. In the second part of property (3), choosing $N = r_j^b(n)$ and choosing $x$ as the middle point in $I^{(n)}$ gives

$$|I^{(n)}| = \mathcal{O}(||B(n)||^{-1+\tau}).$$

The cases $\alpha = {}_b\alpha$, $\alpha = {}_t\alpha$ involve the left endpoint $u_0$ of $I$ and require a minor modification of the argument, but lead to the same estimate.

These two bounds on the $|I_\alpha^{b,(n)}|$ clearly imply property (2). □

We now can prove what was announced in Section 8.3.

Corollary C.3. *Assume that* $(M, \Sigma, \zeta)$ *is a translation surface of* (*restricted*) *Roth type. Then* $T_I$ *is an* i.e.m. *of* (*restricted*) *Roth type.*

*Proof.* We have to check that $T_I$ satisfies conditions (a), (b), and (c) of Section 3.3, and also (d) in the restricted case. By assumption, there exists an open bounded horizontal segment $I^\flat$ in good position such that the return map $T^\flat$ of the vertical flow to $I^\flat$ is an i.e.m. of (restricted) Roth type. Therefore, property (1) in the proposition is satisfied by $T^\flat$. Then, property (4) is satisfied by $(M, \Sigma, \zeta)$. Applying the proposition a second time, we conclude that property (1) is satisfied by $T_I$. This is condition (a) in Section 3.3. For conditions (b), (c) (and (d) in the restricted case), one has only to observe that *once* (a) *is satisfied*, they can be formulated directly in terms of the *continuous time* extended Kontsevich-Zorich cocycle over the Teichmüller flow in moduli space (without reference to the horizontal segment $I$). The main point is that the continuous times $t_n$ corresponding to the integers $n$ in $B(n)$ satisfy

$$t_{n+1} = \mathcal{O}(t_n^{1+\tau})$$

for all $\tau > 0$, so they are "dense enough" to imply the same conditions for all times $t$. We leave the details to the reader. □

## References


[Arn65] V. I. Arnold, Small denominators I: On the mappings of the circumference onto itself, in *Eleven Papers on Number Theory, Algebra and Functions of a Complex Variable*, *Trans. Amer. Math. Soc.* **46**, Amer. Math. Soc., Providence, RI, 1965, pp. 213–284. Zbl 0152.41905.

[AB07] A. Avila and A. Bufetov, Exponential decay of correlations for the Rauzy-Veech-Zorich induction map, in *Partially Hyperbolic Dynamics, Laminations, and Teichmüller Flow*, *Fields Inst. Commun.* **51**, Amer. Math. Soc., Providence, RI, 2007, pp. 203–211. MR 2388696. Zbl 1149.37004.

[AF07] A. Avila and G. Forni, Weak mixing for interval exchange transformations and translation flows, *Ann. of Math.* **165** (2007), 637–664. MR 2299743. Zbl 1136.37003. http://dx.doi.org/10.4007/annals.2007.165.637.

[AGY06] A. Avila, S. Gouëzel, and J.-C. Yoccoz, Exponential mixing for the Teichmüller flow, *Publ. Math. Inst. Hautes Études Sci.* (2006), 143–211. MR 2264836. Zbl 05117096. http://dx.doi.org/10.1007/s10240-006-0001-5.

[BHM10] X. Bressaud, P. Hubert, and A. Maass, Persistence of wandering intervals in self-similar affine interval exchange transformations, *Ergodic Theory Dynam. Systems* **30** (2010), 665–686. MR 2643707. Zbl 1200.37002. http://dx.doi.org/10.1017/S0143385709000418.

[Buf06] A. I. Bufetov, Decay of correlations for the Rauzy-Veech-Zorich induction map on the space of interval exchange transformations and the central limit theorem for the Teichmüller flow on the moduli space of abelian differentials, *J. Amer. Math. Soc.* **19** (2006), 579–623. MR 2220100. Zbl 1100.37002. http://dx.doi.org/10.1090/S0894-0347-06-00528-5.

[CG97] R. Camelier and C. Gutierrez, Affine interval exchange transformations with wandering intervals, *Ergodic Theory Dynam. Systems* **17** (1997), 1315–1338. MR 1488320. Zbl 0895.58019. http://dx.doi.org/10.1017/S0143385797097666.

[dlLG] R. de la Llave and C. Gutierrez, Absolute continuity of conjugacies among certain non-linear interval exchange transformations, manuscript, undated, circa 1999, communicated by Pascal Hubert.

[For97] G. Forni, Solutions of the cohomological equation for area-preserving flows on compact surfaces of higher genus, *Ann. of Math.* **146** (1997), 295–344. MR 1477760. Zbl 0893.58037. http://dx.doi.org/10.2307/2952464.

[For02] ———, Deviation of ergodic averages for area-preserving flows on surfaces of higher genus, *Ann. of Math.* **155** (2002), 1–103. MR 1888794. Zbl 1034.37003. http://dx.doi.org/10.2307/3062150.

[For] ———, Sobolev regularity of solutions of the cohomological equation. arXiv 0707.0940.

[GH55] W. H. Gottschalk and G. A. Hedlund, *Topological Dynamics*, *Amer. Math. Soc. Colloq. Publ.* **36**, Amer. Math. Soc., Providence, R. I., 1955. MR 0074810. Zbl 0067.15204.

[Her79] M. R. Herman, Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations, *Inst. Hautes Études Sci. Publ. Math.* **49** (1979), 5–233. MR 0538680. Zbl 0448.58019. http://dx.doi.org/10.1007/BF02684798.

[Her85] ———, Simple proofs of local conjugacy theorems for diffeomorphisms of the circle with almost every rotation number, *Bol. Soc. Brasil. Mat.* **16** (1985), 45–83. MR 0819805. Zbl 0651.58008. http://dx.doi.org/10.1007/BF02584836.

[Kea75] M. Keane, Interval exchange transformations, *Math. Z.* **141** (1975), 25–31. MR 0357739. Zbl 0278.28010. http://dx.doi.org/10.1007/BF01236981.

[Kea77] ———, Non-ergodic interval exchange transformations, *Israel J. Math.* **26** (1977), 188–196. MR 0435353. Zbl 0351.28012. http://dx.doi.org/10.1007/BF03007668.

[KR80] M. Keane and G. Rauzy, Stricte ergodicité des échanges d'intervalle, *Math. Z.* **174** (1980), 203–212. MR 0593819. Zbl 0479.28012. http://dx.doi.org/10.1007/BF01161409.

[KN76] H. B. Keynes and D. Newton, A "minimal", non-uniquely ergodic interval exchange transformation, *Math. Z.* **148** (1976), 101–105. MR 0409766. Zbl 0308.28014. http://dx.doi.org/10.1007/BF01214699.

[MMY05] S. Marmi, P. Moussa, and J.-C. Yoccoz, The cohomological equation for Roth-type interval exchange maps, *J. Amer. Math. Soc.* **18** (2005), 823–872. MR 2163864. Zbl 1112.37002. http://dx.doi.org/10.1090/S0894-0347-05-00490-X.

[MMY10] ———, Affine interval exchange maps with a wandering interval, *Proc. Lond. Math. Soc.* **100** (2010), 639–669. MR 2640286. Zbl 1196.37041. http://dx.doi.org/10.1112/plms/pdp037.

[Mas82] H. Masur, Interval exchange transformations and measured foliations, *Ann. of Math.* **115** (1982), 169–200. MR 0644018. Zbl 0497.28012. http://dx.doi.org/10.2307/1971341.

[Rau79] G. Rauzy, Échanges d'intervalles et transformations induites, *Acta Arith.* **34** (1979), 315–328. MR 0543205. Zbl 0414.28018. Available at http://pldml.icm.edu.pl/mathbwn/element/bwmeta1.element.bwnjournal-article-aav34i4p315bwm?q=c6c5218b-ddb2-46ff-952c-bd541af0e2f9$1&qt=IN_PAGE.

[Vee78] W. A. Veech, Interval exchange transformations, *J. Analyse Math.* **33** (1978), 222–272. MR 0516048. Zbl 0455.28006. http://dx.doi.org/10.1007/BF02790174.

[Vee82] ———, Gauss measures for transformations on the space of interval exchange maps, *Ann. of Math.* **115** (1982), 201–242. MR 0644019. Zbl 0486.28014. http://dx.doi.org/10.2307/1971391.

[Vee84a] ———, The metric theory of interval exchange transformations. I. Generic spectral properties, *Amer. J. Math.* **106** (1984), 1331–1359. MR 0765582. Zbl 0631.28006. http://dx.doi.org/10.2307/2374396.

[Vee84b] ———, The metric theory of interval exchange transformations. II. Approximation by primitive interval exchanges, *Amer. J. Math.* **106** (1984), 1361–1387. MR 0765583. Zbl 0631.28007. http://dx.doi.org/10.2307/2374397.

[Yoc84] J.-C. Yoccoz, Conjugaison différentiable des difféomorphismes du cercle dont le nombre de rotation vérifie une condition diophantienne, *Ann. Sci. École Norm. Sup.* **17** (1984), 333–359. MR 0777374. Zbl 0595.57027. Available at http://www.numdam.org/item?id=ASENS_1984_4_17_3_333_0.

[Yoc] J.-C. Yoccoz, Cours 2005 : Échange d'intervalles. Available at http://www.college-de-france.fr/site/jean-christophe-yoccoz/cours_et_seminaires_anterieurs.htm.

[Yoc06] ———, Continued fraction algorithms for interval exchange maps: an introduction, in *Frontiers in Number Theory, Physics, and Geometry. I*, Springer-Verlag, New York, 2006, pp. 401–435. MR 2261103. Zbl 1127.28011. http://dx.doi.org/10.1007/978-3-540-31347-2_12.

[Yoc10] ———, Interval exchange maps and translation surfaces, in *Homogeneous Flows, Moduli Spaces and Arithmetic*, *Clay Math. Proc.* **10**, Amer. Math. Soc., Providence, RI, 2010, pp. 1–69. MR 2648692. Zbl 05907408.

[Zor96] A. Zorich, Finite Gauss measure on the space of interval exchange transformations. Lyapunov exponents, *Ann. Inst. Fourier* (*Grenoble*) **46** (1996), 325–370. MR 1393518. Zbl 0853.28007. Available at http://www.numdam.org/item?id=AIF_1996__46_2_325_0.

[Zor97] ———, Deviation for interval exchange transformations, *Ergodic Theory Dynam. Systems* **17** (1997), 1477–1499. MR 1488330. Zbl 0958.37002. http://dx.doi.org/10.1017/S0143385797086215.

[Zor99] A. Zorich, How do the leaves of a closed 1-form wind around a surface?, in *Pseudoperiodic Topology*, *Amer. Math. Soc. Transl.* **197**, Amer. Math. Soc., Providence, RI, 1999, pp. 135–178. MR 1733872. Zbl 0976.37012.

[Zor06] ———, Flat surfaces, in *Frontiers in Number Theory, Physics, and Geometry. I*, Springer-Verlag, New York, 2006, pp. 437–583. MR 2261104. Zbl 1129.32012. http://dx.doi.org/10.1007/3-540-31347-8_13.

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