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# Linear Shafarevich conjecture

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## Abstract

In this paper we settle affirmatively Shafarevich's uniformization conjecture for varieties with linear fundamental groups. We prove the strongest to date uniformization result — the universal covering space of a complex projective manifold with a linear fundamental group is holomorphically convex. The proof is based on both known and newly developed techniques in non-abelian Hodge theory.

## Contents



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## Introduction

A complex analytic space $S$ is *holomorphically convex* if there is a proper holomorphic morphism $\pi : S \to T$ with $\pi_* \mathcal{O}_S = \mathcal{O}_T$ such that $T$ is a Stein space. $T$ is then called the *Cartan-Remmert reduction* of $S$.

The so-called Shafarevich conjecture of holomorphic convexity predicts that the universal covering space $\widetilde{X^{\mathrm{univ}}}$ of a complex compact projective manifold $X$ should be holomorphically convex. This is trivial if the fundamental group is finite. The Shafarevich conjecture is a corollary of the Riemann uniformization theorem in dimension 1.

*History of the problem.* The study of the Shafarevich conjecture for smooth projective surfaces was initiated in the mid 80's by Gurjar and Shastri [GS85] and Napier [Nap90]. In the mid 90's, new ideas introduced by J. Kollár and independently by F. Campana revolutionized the subject. The outcome was what is still the best result available with no assumption on the fundamental group, namely the construction of the Shafarevich map (aka $\Gamma$-reduction) [Cam94], [Kol93], [Kol95]. At the same time, Corlette and Simpson [Cor88], [Cor93], [Sim88], [Sim92], [Sim94] were developing non-abelian Hodge theory. A bit later a $p$-adic version of non-abelian Hodge theory in degree 1 was developed by Gromov and Schoen [GS92].

The idea that non-abelian Hodge theory can be used to prove the Shafarevich conjecture was introduced in 1994 by the second author. He proved the Shafarevich conjecture for nilpotent fundamental groups [Kat97]. At about the same time the second and the fourth author proved the Shafarevich conjecture for smooth projective surfaces with the fundamental group admitting faithful Zariski dense representation, in a reductive complex algebraic group [KR98].

The first author then found a way to extend this non-abelian Hodge theoretic argument to higher dimension and showed that the Shafarevich conjecture holds for any smooth projective variety with the fundamental group having a faithful representation, Zariski dense in a reductive complex algebraic group; see [Eys04]. Several influential contributions to these and closely related topics were also made by Lasell and the fourth author [LR96], Mok [Mok92] and Zuo [Zuo94].

The present article studies the conjecture in the case when $\pi_1(X, x)$ has a finite-dimensional complex linear representation with infinite monodromy group. It combines and develops further some known techniques in non-abelian Hodge theory. In particular, we prove the conjecture for projective manifolds $X$ whose fundamental group admits a faithful representation in $\mathrm{GL}_n(\mathbb{C})$. We explain the structure of the argument in more detail next.

*Main results and strategy of proof.* In what follows, $X$ will denote a connected projective algebraic complex manifold, $x \in X$ a point, $\overline{\mathbb{Q}} \subset \ell \subset \mathbb{C}$ a field of definition for $X$, and $Z$ a connected projective algebraic variety.

Suppose $X$ is a projective manifold whose fundamental group admits a faithful representation in $\mathrm{GL}_n(\mathbb{C})$. To show that the universal cover of $X$ is holomorphically convex we follow the reduction scheme proposed by Campana [Cam94] and Kollár [Kol93], [Kol95]. This reduction scheme breaks the problem into two parts:

(i) Construct a Shafarevich morphism $\mathrm{sh} : X \to \mathrm{Sh}(X)$ for $X$. That is - construct a morphism sh with connected fibers to a normal projective variety $\mathrm{Sh}(X)$ which contracts all subvarieties in $X$ whose fundamental group maps to a finite subgroup in $\pi_1(X)$.

(ii) Under (i) there is a natural Deligne-Mumford analytic stack structure on $\mathrm{Sh}(X)$ whose inertia group at a point is the image of the fundamental group of the corresponding fiber in $\pi_1(X)$. Show that the universal cover of this stack is a Stein space by constructing a plurisubharmonic exhaustion function.

Each part has its own difficulties, but the main challenge lies in the first part, namely in constructing and controlling the geometric behavior of the Shafarevich morphism.

Our general strategy for dealing with this difficulty has two main steps. First we use the given faithful linear representation to construct certain complex variations of mixed Hodge structures ($\mathbb{C}$-VMHS). Then we utilize the associated period mappings to construct a Shafarevich morphism. This is quite similar to the way period maps for complex variations of pure Hodge structures ($\mathbb{C}$-VHS) were used in [Eys04]. Once the Shafarevich morphism is constructed, holomorphic convexity is much simpler to obtain here.

In a nutshell, there are three main ideas that make our construction of the Shafarevich morphism possible:

- Utilize the general point in an absolutely constructible subvariety in the representation space of $\pi_1(X)$ to construct a large well behaved representation $\varrho$ of $\pi_1(X)$ into the group of points of $\mathrm{GL}_n$ over a non-archimedean local field.

- Use non-abelian Hodge theory to produce a proper pluriharmonic map, equivariant under the reductive part of $\varrho$, to a nonpositively curved space on which $\pi_1(X)$ acts by isometries.
- Use secondary period maps to extend this pluriharmonic map to a fully $\varrho$-equivariant proper map to a higher Albanese fibration over the nonpositively curved space. Check that the $\pi_1(X)$-quotient of this map is the Shafarevich morphism. It turns out that the crucial point in this construction of the Shafarevich morphism is a rather subtle rationality lemma, Theorem 4.4, whose proof relies on mixed Hodge theory.

Next we state our main theorem precisely and explain in detail how the above ideas are implemented in the proof.

THEOREM 1. *Let $G$ be a reductive algebraic group defined over $\mathbb{Q}$. Let $M = M_B(X, G)$ be the character scheme of $\pi_1(X, x)$ with values in $G$.*

(a) *Let $\widetilde{H_M^\infty} \subset \pi_1(X, x)$ be the intersection of the kernels of all representations $\pi_1(X, x) \to G(A)$, where $A$ is an arbitrary $\mathbb{C}$-algebra of finite type. Then, the associated Galois covering space of $X$,*

$$\widetilde{X_M^\infty} = \widetilde{X^{\mathrm{univ}}}/\widetilde{H_M^\infty},$$

*is holomorphically convex.*

(b) *There exists a natural nondecreasing family*

$$\widetilde{H_M^1} \subseteq \widetilde{H_M^2} \subseteq \cdots \subseteq \widetilde{H_M^k} \subseteq \cdots \subseteq \widetilde{H_M^\infty} \trianglelefteq \pi_1(X, x)$$

*of normal subgroups in $\pi_1(X, x)$. For a given $k \geq 1$, the group $\widetilde{H_M^k}$ corresponds to representations $\pi_1(X, x) \to G(A)$, with $A$ an Artin local algebra, and such that the Zariski closure of their monodromy group has $k$-step unipotent radical. For every $\widetilde{H_M^k}$, the associated cover*

$$\widetilde{X_M^k} = \widetilde{X^{\mathrm{univ}}}/\widetilde{H_M^k}$$

*is holomorphically convex.*

See Section 3.1 for the precise definition of $\widetilde{H_M^k}$. If $G = \mathrm{GL}_1$, this theorem is a restatement of [Kat97]. Actually, the theorem is likely to hold when we replace $M_B(X, G)$ by an arbitrary absolutely closed set $M$ defined over $\mathbb{Q}$ [Sim93]. The case $k = 0$ was established in [Eys04].

To put this theorem in perspective, let us recall succinctly the definition of Shafarevich morphisms. Assume $\Xi$ is a compact Kähler manifold and $H \subset \pi_1(\Xi)$ is a normal subgroup. Whether $\widetilde{\Xi^{\mathrm{univ}}}/H$ is holomorphically convex or not, an important result of [Cam94], [Kol93] states that a Shafarevich map (aka $\Gamma$-reduction) always exists, namely that there is an almost-holomorphic meromorphic fibration $\mathrm{sh}_H^w : X \to \mathrm{Sh}_H^w(X)$ whose very general fiber $Z$ is a maximal connected analytic subspace such that the image of $\pi_1(Z) \to \pi_1(X)/H$ is

finite. Here, $\mathrm{Sh}^w_H(X)$ is only defined up to bimeromorphic equivalence and is called a (relative) Shafarevich variety for $(X, H)$ and $\mathrm{sh}^w_H$ a (relative) Shafarevich map.

On the other hand, if $\widetilde{\Xi^{\mathrm{univ}}}/H$ is holomorphically convex, we can select a preferred model of $\mathrm{Sh}^w_H(X)$ that we will call the Shafarevich variety too, in order to avoid the terminology "Shafarevich variety in the strong sense." Consider, indeed, the Cartan-Remmert reduction $\pi : \widetilde{\Xi^{\mathrm{univ}}}/H \to T(\widetilde{\Xi^{\mathrm{univ}}}/H)$ . The quotient group $\Gamma := \pi_1(\Xi)/H$ acts properly discontinuously on $T(\widetilde{\Xi^{\mathrm{univ}}}/H)$ and $\pi$ is equivariant.

*Definition* 1. The Shafarevich variety of $(\Xi, H)$ is the normal compact complex analytic variety defined as the quotient $\mathrm{Sh}_H(\Xi) := T(\widetilde{\Xi^{\mathrm{univ}}}/H)/\Gamma$. The Shafarevich morphism is the morphism $s_H : \Xi \to S_H(\Xi)$ obtained by quotienting $\pi$.

Coming back to the discussion of Theorem 1, we want to stress an important feature of the construction already observed in [Kat97]. Let $\pi : \widetilde{X^k_M} \to \widetilde{S^k_M}(X)$ be the Cartan-Remmert reduction of $\widetilde{X^k_M}$. Then, the resulting Shafarevich morphism $\mathrm{sh}^k_M : X \to \mathrm{Sh}^k_M(X)$ is independent of $k \in \mathbb{N}^* \cup \infty$, although in general $\mathrm{sh}^1_M$and $\mathrm{sh}^0_M$ may not coincide. In the proof, we first construct the Cartan-Remmert reduction for $\widetilde{X^1_M}$ and establish its main properties, which is tantamount to constructing the Shafarevich morphism $\mathrm{sh}^1_M$, and the generalization to $k \geq 2$ is made easier by this basic observation. This also allows one to see that, for every subgroup $H \subset \pi_1(X, x)$ such that $\widetilde{H^\infty_M} \subset H \subset \widetilde{H^1_M}$, the covering space $\widetilde{X^{\mathrm{univ}}}/H$ is holomorphically convex as well.

The paper is organized as follows. Section 1 introduces Absolute Constructible Sets and recalls results from [Eys04]. Section 2 introduces a $\mathbb{C}$-VMHS constructed in [ES11] which serves as a main ingredient of the proof. Section 3 contains the proof of an important strictness statement. Section 4 contains a rationality lemma and the reduction to a finite number of local systems. Section 5 contains the construction of the Shafarevich morphism and the proof of the main theorem.

Given present-day technology, it seems difficult to go significantly further in the direction of proving the Shafarevich conjecture. Perhaps, the generalization to the Kähler case or understanding sufficient conditions for holomorphic convexity of the universal covering space of a singular projective variety might produce interesting developments. Several interesting observations have been made in cases of nonresidually finite fundamental groups. Bogomolov and the second author suggest [BK98] that the Shafarevich conjecture might fail in the case of nonresidually finite fundamental groups. From another point of view, papers by Bogomolov and de Oliveira [BdO05], [BdO06] suggest that

a big part of universal coverings of smooth projective varieties might still be holomorphically convex.

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## 1. **Absolute constructible sets**

1.1. *Basic facts.* Let $G$ be an algebraic reductive group defined over $\overline{\mathbb{Q}}$. The *representation scheme* of $\pi_1(X,x)$ is an affine $\overline{\mathbb{Q}}$-algebraic scheme described by its functor of points:

$$R(\pi_1(X,x),G)(\mathrm{Spec}(A)) := \mathrm{Hom}(\pi_1(X,x),G(A))$$

for any $\overline{\mathbb{Q}}$ algebra $A$. The *character scheme* of $\pi_1(X,x)$ with values in $G$ is the affine scheme

$$M_B(X,G) = R(\pi_1(X,x),G)//G.$$

Let $\bar{k}$ be an algebraically closed field of characteristic zero. Then the set of closed points $M_B(X,G)(\bar{k})$ is the set of $G(\bar{k})$-conjugacy classes of *reductive* representations of $\pi_1(X,x)$ with values in $G(\bar{k})$; see [LM85].

Character schemes of fundamental groups of complex projective manifolds are rather special. In [Sim94], two additional quasi-projective schemes over $\ell$ are constructed: $M_{\mathrm{DR}}(X,G)$ and $M_{\mathrm{Dol}}(X,G)$. The $\mathbb{C}$-points of $M_{\mathrm{DR}}(X,G)$ are in bijection with the equivalence classes of flat $G$-connections with reductive monodromy, and the $\mathbb{C}$-points of $M_{\mathrm{Dol}}(X,G)$ are in bijection with the isomorphism classes of polystable $G$-Higgs $G$-bundles with vanishing first and second Chern class. Whereas the notion of a polystable Higgs bundle depends on the choice of a polarization on $X$ the moduli space, $M_{\mathrm{Dol}}(X,G)$ does not; i.e., all moduli spaces one constructs for the different polarizations are naturally isomorphic [Sim94]. This is analogous to the classical statement that the usual Hodge decomposition on the de Rham cohomology is purely complex analytic, i.e., independent of a choice of a Kähler metric.[1] $M_{\mathrm{Dol}}(X,G)$ is acted upon algebraically by the multiplicative group $\mathbb{C}^*$. Furthermore, there is a biholomorphic map

$$RH : M_B(X,G)(\mathbb{C}) \to M_{\mathrm{DR}}(X,G)(\mathbb{C})$$

[1]The harmonic representative of a cohomology class depends in general on the Kähler metric. A helpful remark in the present context is that the harmonic representative of a degree 1 cohomology class actually does not depend on the Kähler metric.

and a real analytic homeomorphism

$$KH : M_B(X,G)(\mathbb{C}) \to M_{\mathrm{Dol}}(X,G)(\mathbb{C}).$$

$RH$ and $KH$ are also independent of the choice of a Kähler metric. When $l = \overline{\mathbb{Q}}$, one defines an *absolute constructible subset* of $M_B(X,G)(\mathbb{C})$ to be a subset $M$ such that

- $M$ is the set of complex points of a $\overline{\mathbb{Q}}$-constructible subset of $M_B(X,G)$,
- $RH(M)$ is the set of complex points of a $\overline{\mathbb{Q}}$-constructible subset of $M_{\mathrm{DR}}(X,G)$,
- $KH(M)$ is a $\mathbb{C}^*$-invariant set of complex points of a $\overline{\mathbb{Q}}$-constructible subset of $M_{\mathrm{Dol}}(X,G)$.

There is a rich theory describing the structure of absolutely constructible subsets in $M_B(X,G)$. Here we briefly summarize only those properties of absolutely constructible sets that we will need later. Full proofs and details can be found in [Sim93].

- The full moduli space $M_B(X,G)$ of representations of $\pi_1(X,x)$ in $G$ defined in [Sim94] is absolutely constructible and quasi-compact (acqc).
- The closure (in the classical topology) of an acqc subset is also acqc.
- Whenever $\rho$ is an isolated point in $M_B(X,G)$, $\{\rho\}$ is acqc.
- Absolute constructibility is invariant under standard geometric constructions. For instance, for any morphism $f : Y \to X$ of smooth connected projective varieties, the property of a subset being absolutely constructible is preserved when taking images and preimages via $f^* : M_B(X,G) \to M_B(Y,G)$. Similarly, for any homomorphism $\mu : G \to G'$ of reductive groups, taking images and preimages under $\mu_* : M_B(X,G) \to M_B(X,G')$ preserves absolute constructibility.
- Given a dominant morphism $f : Y \to X$ and $i \in \mathbb{N}$, the set $M_f^i(X, \mathrm{GL}_n)$ of local systems $V$ on $Y$ such that $R^i f_* V$ is a local system is ac. Also, taking images and inverse images under $R^i f_* : M_f^i(X, \mathrm{GL}_n) \to M_B(Y, \mathrm{GL}_{n'})$ preserves acqc sets.
- The complex points of a closed acqc set $M$ are stable under the $\mathbb{C}^*$ action defined by [Sim88] in terms of Higgs bundles. By [Sim88] the fixed point set $M^{\mathrm{VHS}} := M^{\mathbb{C}^*}$ consists of representations underlying polarizable complex variations of Hodge structure ($\mathbb{C}$-VHS, for short). Furthermore, $M$ is then the smallest closed acqc set in $M_B(X,G)$ containing $M^{\mathrm{VHS}}$.

1.2. *Reductive Shafarevich conjecture.* After complete results were obtained for surfaces in [KR98], the Shafarevich conjecture on holomorphic convexity for reductive linear coverings of arbitrary projective algebraic manifolds over $\mathbb{C}$ was settled affirmatively in [Eys04].

THEOREM 1.1. *Let $M \subset M_B(X,G)$ be an absolute constructible set of conjugacy classes of linear reductive representations of $\pi_1(X,x)$ in some reductive algebraic group $G$ over $\overline{\mathbb{Q}}$.*

*Define a normal subgroup $H_M \subset \pi_1(X,x)$ by*

$$H_M = \bigcap_{\rho \in M(\overline{\mathbb{Q}})} \ker(\rho).$$

*The Galois covering space $\widetilde{X_M} = \widetilde{X^{\mathrm{univ}}}/H_M$ is holomorphically convex.*

Without loss of generality we may assume in this theorem that $M$ is a closed absolutely constructible set since $\widetilde{X_M} = \widetilde{XM}$.

Let $\Gamma_M$ be the quotient group defined by

$$\Gamma_M = \pi_1(X,x)/\bigcap_{\rho \in M(\overline{\mathbb{Q}})} \ker(\rho).$$

$\Gamma_M$ is the Galois group of $\widetilde{X_M}$ over $X$ and acts in a proper discontinuous fashion on the Cartan-Remmert reduction $\widetilde{S_M}(X)$ of $\widetilde{X_M}$, which is a normal complex space. The quotient space

$$\mathrm{Sh}_M(X) = \widetilde{S_M}(X)/\Gamma_M$$

is then a normal projective variety, and the quotient morphism $\mathrm{sh}_M : X \to \mathrm{Sh}_M(X)$ is the Shafarevich morphism attached to $M$ in accordance to Definition 1. This morphism is a fibration, i.e., is surjective with connected fibers. Its fibers $Z$ are connected, have the property that $\pi_1(Z) \to \Gamma_M$ has finite image and are maximal with respect to these properties.

COROLLARY 1.2. *If $\pi_1(X,x)$ is almost reductive (i.e., has a Zariski dense representation with finite kernel in a reductive algebraic group over $\mathbb{C}$), then the Shafarevich conjecture holds for $X$.*

## 2. $\mathbb{C}$-VMHS attached to an absolute closed set

We will first review some of the results in [Hai98] and [ES11] that enable one to construct various $\mathbb{C}$-VMHS on $X$ out of $M$.

The results in [Hai98] are important, general and abstract since they deal with general compactifiable Kähler spaces. The results in [ES11] deal with the less general situation of a compact Kähler manifold but are more explicit and give some useful properties that will be exploited in Proposition 3.6. Furthermore, [ES11] suffices for most of our results and for all of those from Sections 3.3, 4 and 5, hence for the main theorem. On the other hand, the results in [Hai98, §§1–12] are needed for the optimal form of Proposition 3.6 and were one of our main sources of inspiration.

2.1. $\mathbb{C}$-VMHS, *definition*, *basic properties*. The notion of polarized $\mathbb{C}$-VHS was introduced in [Sim88] as a straightforward variant of [Gri70]. We will use another equivalent definition.

*Definition* 2.1. A $\mathbb{C}$-VHS (polarized complex variation of Hodge structures) on $X$ of weight $w \in \mathbb{Z}$ is a 5-tuple $(X, \mathbb{V}, \mathcal{F}^\bullet, \overline{\mathcal{G}}^\bullet, S)$, where

(1) $\mathbb{V}$ is a local system of finite-dimensional $\mathbb{C}$-vector spaces;
(2) $S$ is a nondegenerate flat sesquilinear pairing on $\mathbb{V}$;
(3) $\mathcal{F}^\bullet = (\mathcal{F}^p)_{p\in\mathbb{Z}}$ is a biregular decreasing filtration of $\mathbb{V}\otimes_{\mathbb{C}}\mathcal{O}_X$ by locally free coherent analytic sheaves such that $d'\mathcal{F}^p \subset \mathcal{F}^{p-1}\otimes\Omega^1_X$;
(4) $\overline{\mathcal{G}}^\bullet = (\overline{\mathcal{G}}^q)_{q\in\mathbb{Z}}$ is a biregular decreasing filtration of $\mathbb{V}\otimes_{\mathbb{C}}\mathcal{O}_{\bar X}$ by locally free coherent antianalytic sheaves such that $d''\overline{\mathcal{G}}^p \subset \overline{\mathcal{G}}^{p-1}\otimes\Omega^1_{\bar X}$;
(5) for every point $x \in X$, the fiber at $x(\mathbb{V}_x, \mathcal{F}^\bullet_x, \overline{\mathcal{G}}^\bullet_x)$ is a $\mathbb{C}$-MHS polarized by $S_x$.

The conjugate $\mathbb{C}$-VHS is the $\mathbb{C}$-VHS obtained on $\overline{\mathbb{V}}$ setting $\mathcal{F}^\bullet_{\overline{\mathbb{V}}} = \overline{\mathcal{G}}^\bullet$, etc. The local system $\mathbb{V}\oplus\overline{\mathbb{V}}$ carries a real polarized variation of Hodge structures.

Recall that a real reductive algebraic group $E$ is said to be of Hodge type if there is a morphism $h : U(1) \to \mathrm{Aut}(E)$ such that $h(-1)$ is a Cartan involution of $E$; see [Sim92, p. 46]. By definition, $h$ is a Hodge structure on $E$. Connected groups of Hodge type are precisely those admitting a compact Cartan subgroup. A Hodge representation of $E$ is a finite-dimensional complex representation $\alpha : E(\mathbb{R}) \to \mathrm{GL}(\mathbb{V}_{\mathbb{C}})$ such that $h$ fixes $\ker(\alpha)$. In this case, $\mathbb{V}_{\mathbb{C}}$ inherits a pure polarized Hodge structures of weight zero. The adjoint representation of a Hodge group is Hodge. Thus the Lie algebra $\mathfrak{E}$ of $E$ has a natural real Hodge structure of weight 0 compatible with the Lie bracket. The Lie algebra action $\mathfrak{E}_{\mathbb{C}}\otimes\mathbb{V}_{\mathbb{C}} \to \mathbb{V}_{\mathbb{C}}$ respects the Hodge structures.

The real Zariski closure $E_\rho$ of the monodromy group of a representation $\rho : \pi_1(X, x) \to G(\mathbb{C})$ underlying a $\mathbb{C}$-VHS is a group of Hodge type. We have $E_\rho \subset R_{\mathbb{C}|\mathbb{R}}G_{\mathbb{C}}$, where $R_{\mathbb{C}|\mathbb{R}}$ is the Weil restriction of scalars functor. Every Hodge representation $\alpha$ of $E$ gives rise to

$$\alpha\circ\rho : \pi_1(X, x) \to \mathrm{GL}(\mathbb{V}_\alpha),$$

a representation that underlies a $\mathbb{C}$-VHS [Sim92, Lemma 5.5].

The notion of $\mathbb{C}$-VMHS (or graded-polarized variation of $\mathbb{C}$-mixed Hodge structures) used in [ES11] is a straightforward generalization of that given in [SZ85], [Usu83].

*Definition* 2.2. A $\mathbb{C}$-VMHS on $X$ is a 6-tuple $(X, \mathbb{V}, \mathbb{W}_\bullet, \mathcal{F}^\bullet, \overline{\mathcal{G}}^\bullet, (S_k)_{k\in\mathbb{Z}})$, where

(1) $\mathbb{V}$ is a local system of finite-dimensional $\mathbb{C}$-vector spaces;
(2) $\mathbb{W}_\bullet = (\mathbb{W}_k)_{k\in\mathbb{Z}}$ is a decreasing filtration of $\mathbb{V}$ by local subsystems;

(3) $\mathcal{F}^\bullet = (\mathcal{F}^p)_{p\in\mathbb{Z}}$ is a biregular decreasing filtration of $\mathbb{V}\otimes_{\mathbb{C}}\mathcal{O}_X$ by locally free coherent analytic sheaves such that $d'\mathcal{F}^p \subset \mathcal{F}^{p-1}\otimes\Omega^1_X$;
(4) $\overline{\mathcal{G}}^\bullet = (\overline{\mathcal{G}}^q)_{q\in\mathbb{Z}}$ is a biregular decreasing filtration of $\mathbb{V}\otimes_{\mathbb{C}}\mathcal{O}_{\bar{X}}$ by locally free coherent antianalytic sheaves such that $d''\overline{\mathcal{G}}^p \subset \overline{\mathcal{G}}^{p-1}\otimes\Omega^1_{\bar{X}}$;
(5) for all $x\in X$, the stalk $(\mathbb{V}_x, \mathbb{W}_{\bullet,x}, \mathcal{F}^\bullet_x, \overline{\mathcal{G}}^\bullet_x)$ is a $\mathbb{C}$-MHS;
(6) $S_k$ is a flat sesquilinear nondegenerate pairing on $\mathrm{Gr}^{\mathbb{W}}_k\mathbb{V}$;
(7) $(X, \mathrm{Gr}^{\mathbb{W}}_k\mathbb{V}, \mathrm{Gr}^{\mathbb{W}\otimes_{\mathbb{C}}\mathcal{O}_X}_k\mathcal{F}^\bullet, \mathrm{Gr}^{\mathbb{W}\otimes_{\mathbb{C}}\mathcal{O}_{\bar{X}}}_k\overline{\mathcal{G}}^\bullet, S_k)$ is a $\mathbb{C}$-VHS.

We use the following terminology in the sequel.

*Definition* 2.3. A homomorphism of groups $\rho:\Gamma\to\Gamma'$ will be called trivial if $\rho(\Gamma)=\{e\}$. A VMHS will be called trivial (or constant) if its monodromy representation is trivial.

2.2. *mixed Hodge theory for the relative completion.* In [Hai98, Th. 13.10], certain $\mathbb{R}$-VMHS are attached to an $\mathbb{R}$-VHS on the compact Kähler manifold. In this section we review the results of [Hai98] relevant to our discussion and complement them with some explicit examples. We will omit the proofs of the statements that are not essential to our present goals, but we will describe in greater detail the examples we need.

2.2.1. *Hain's theorems.* Let us first review [Hai98, §§1–12]. Another reference where this material (and much more) has been nicely rewritten in a more general form is [Pri07, §6]. Let $E^\rho$ be a real reductive group of Hodge type. Let $\rho:\pi_1(X,x)\to E^\rho(\mathbb{R})$ be a Zariski dense representation underlying a VHS, and let

$$1\to\mathcal{U}^\rho_x\to\mathcal{G}^\rho_x\to E^\rho\to 1,\quad a:\pi_1(X,x)\to\mathcal{G}^\rho_x(\mathbb{R})$$

be its relative completion [Hai98].

$\mathcal{G}^\rho_x$ is a proalgebraic group over $\mathbb{R}$ and $\mathcal{U}^\rho_x$ is its prounipotent radical.

Let $k\geq 1$ be an integer, $\mathcal{U}^\rho_{x,k}$ be the $k$-th term of the lower central series of $\mathcal{U}^\rho_x$ and $\mathcal{G}^\rho_{x,k}$ be $\mathcal{G}^\rho_x/\mathcal{U}^\rho_{k_x}$.

The commutative Hopf algebra $\mathbb{R}[\mathcal{G}^\rho_x]$ of the regular functions on $\mathcal{G}^\rho_x$ carries a compatible $\mathbb{R}$-MHS with nonnegative weights. The increasing weight filtration is described by the formula

$$W_k\mathbb{R}[\mathcal{G}^\rho_x]=\mathbb{R}[\mathcal{G}^\rho_{x,k}],$$

where $\mathbb{R}[\mathcal{G}^\rho_{x,k}]$ is identified with its image in $\mathbb{R}[\mathcal{G}^\rho_x]$. Although these MHS are not necessarily finite dimensional, they are always filtered direct limits of finite-dimensional ones.

Let $\mathbb{M}_x=(M_x,W_\bullet,F^\bullet)$ be a finite-dimensional complex mixed Hodge structure, and consider $\alpha:\mathcal{G}^\rho_x(\mathbb{R})\to\mathrm{GL}(M_x)$ a representation of $\mathcal{G}^\rho_x$. We will say $\alpha$ is a mixed Hodge representation if and only if $\alpha$ is the representation

arising from the real points of a rational representation of $\mathcal{G}_x^\rho$ in $M_x$ and the coaction

$$\alpha^* : M_x \to M_x \otimes \mathbb{C}[\mathcal{G}_x^\rho]$$

respects the natural MHS.

The main result of [Hai98, §13] can now be stated as follows.

PROPOSITION 2.4. *Let $\alpha$ be a mixed Hodge representation. The representation $\alpha \circ a : \pi_1(X,x) \to \mathrm{GL}(M_x)$ underlies a* $\mathbb{C}$-VMHS. *Moreover, any* $\mathbb{C}$-VMHS $\mathbb{M}$ *whose graded constituents* $\mathrm{Gr}_W^k\mathbb{M}$ *are* VHS *such that their monodromy representations $\pi_1(X,x) \to \mathrm{GL}(\mathrm{Gr}_W^k\mathbb{M}_x)$ factor through $\rho$ is of this type. A similar statement holds for* $\mathbb{R}$-VMHS.

The recent preprint [Ara10] gives among other things an alternative approach to this material.

2.2.2. *Example.* In [Hai98] Hain describes the steps $\mathbb{W}_k\mathbb{M}$ of the weight filtration in Proposition 2.4 through iterated integrals. This, however, is somewhat technical and goes beyond the scope of the present paper. Instead of discussing the general construction, we will spell out the definition of the $\mathbb{C}$-VMHS underlying some very specific $\mathbb{C}$-mixed Hodge representations of $\mathcal{G}_{x,1}^\rho$ which will play a prominent role in our considerations.

Let $(X, \mathbb{V}, \mathcal{F}^\bullet, \overline{\mathcal{G}}^\bullet, S)$ be a $\mathbb{C}$-VHS that will be assumed with no loss of generality of weight 0. We will write $\mathbb{V}$ for short, since this will not cause any confusion.

Let $\mathcal{E}^\bullet(X,\mathbb{V})$ be the de Rham complex of $\mathbb{V}$. This de Rham complex inherits a Hodge filtration from $\mathcal{F}$ and the Hodge filtration on $\mathcal{E}^\bullet(X)$ and an anti-Hodge filtration from $\mathcal{G}$ and the anti-Hodge filtration on $\mathcal{E}^\bullet(X)$. The resulting two filtrations on its cohomology groups define on $H^p(X,\mathbb{V})$ a $\mathbb{C}$-Hodge structure of weight $p$. Furthermore, once we fix a Kähler form on $X$, there is a subspace $\mathcal{H}^p(X,\mathbb{V}) \subset \mathcal{E}^p(X,\mathbb{V})$ consisting of harmonic forms in a suitable sense such that the composite map $[\bullet] : \mathcal{H}^p(X,\mathbb{V}) \subset \mathcal{Z}^p(X,\mathbb{V}) \to H^p(X,\mathbb{V})$ is an isomorphism. This is standard and can be found in, e.g., [Zuc79] for $\mathbb{V}$ a $\mathbb{R}$-VHS. The general $\mathbb{C}$-VHS case follows in exactly the same way.

*Remark* 2.5. When $p = 1$, the space of harmonic forms is actually independent of the Kähler metric and of the polarization $S$. Furthermore, if $Y \to X$ is a morphism, then $f^*\mathcal{H}^1(X,\mathbb{V}) \subset \mathcal{H}^1(Y,\mathbb{V})$. Indeed $\mathcal{H}^1(X,\mathbb{V}) = \ker(D') \cap \ker(D'') \cap \mathcal{E}^1(X,\mathbb{V})$.

Consider $\alpha \in \mathcal{H}^1(X,\mathbb{V})$ such that $[\alpha]$ is of pure Hodge type $(P,Q)$. Then, for all $y \in X$, $\alpha(p) \in \mathbb{V}_y^{P-1,Q} \otimes \Omega^{1,0} \oplus \mathbb{V}_y^{P,Q-1} \otimes \Omega^{0,1}$.

Let $(\alpha_i)_{i\in I}$ be a $\mathbb{C}$-basis of $\mathcal{H}^1(X,\mathbb{V})$ such that each $[\alpha_i]$ is of pure Hodge type. Let $([\alpha_i]^*)_{i\in I}$ be the dual basis of the dual vector space $H^1(X,\mathbb{V})^*$, and

define

$$\Omega \in \mathcal{E}^1(X, \mathbb{V} \otimes H^1(X, \mathbb{V})^*)$$

by the formula

$$\Omega = \sum_i \alpha_i \otimes [\alpha_i]^*.$$

Note that $\Omega$ does not depend on the chosen basis. Now we define a new connection on the vector bundle underlying the local system $\mathbb{M}_0 = \mathbb{C} \oplus \mathbb{V}^* \otimes H^1(X, \mathbb{V})$ on $X$ by setting

$$d_{\mathbb{M}} = \begin{pmatrix} d_{\mathbb{C}} & \Omega \\ 0 & d_{\mathbb{V}^*} \otimes \mathrm{Id}_{H^1(X,\mathbb{V})} \end{pmatrix}.$$

The duality pairings $H^1(X, \mathbb{V}) \otimes H^1(X, \mathbb{V})^* \to \mathbb{C}$ and $\mathbb{V} \otimes \mathbb{V}^* \to \mathbb{C}$ are tacitly used in this formula. Since $d_{\mathbb{V}}\alpha_i = 0$, the connection follows that $d_{\mathbb{M}}$ is a flat connection and this gives rise to a local system $\mathbb{M}$. Furthermore, the connection $d_{\mathbb{M}}$ respects the 2-step filtration $W^0\mathbb{M} = \mathbb{C}$ $W^1\mathbb{M} = \mathbb{M}$; hence, $\mathbb{M}$ is a filtered local system whose graded parts are $\mathrm{Gr}^0_W\mathbb{M} = \mathbb{C}$ and $\mathrm{Gr}^1_W\mathbb{M} = \mathbb{V}^* \otimes H^1(X, \mathbb{V})$.

We now define a Hodge filtration $\mathcal{F}^\bullet$ of the smooth vector bundle underlying $\mathbb{M}$ by the formula valid for every $p \in X$:

$$\mathcal{F}^k_p = \begin{cases} \mathcal{F}^k_{\mathbb{V}^*_p \otimes H^1(X,\mathbb{V})} \subset \mathbb{V}^*_p \otimes H^1(X, \mathbb{V}) & \text{if } \ k > 0, \\ \mathbb{C}_p \oplus \mathcal{F}^k_{\mathbb{V}^*_p \otimes H^1(X,\mathbb{V})} \subset \mathbb{C}_p \oplus \mathbb{V}^*_p \otimes H^1(X, \mathbb{V}) & \text{if } \ k \leq 0. \end{cases}$$

Similarly, one defines an anti-Hodge filtration on $\mathbb{M}$, which we denote by $\overline{\mathcal{G}}^\bullet$, by the formula valid for every $p \in X$:

$$\overline{\mathcal{G}}^k_p = \begin{cases} \overline{\mathcal{G}}^k_{\mathbb{V}^*_p \otimes H^1(X,\mathbb{V})} \subset \mathbb{V}^*_p \otimes H^1(X, \mathbb{V}) & \text{if } \ k > 0, \\ \mathbb{C}_p \oplus \overline{\mathcal{G}}^k_{\mathbb{V}^*_p \otimes H^1(X,\mathbb{V})} \subset \mathbb{C}_p \oplus \mathbb{V}^*_p \otimes H^1(X, \mathbb{V}) & \text{if } \ k \leq 0. \end{cases}$$

It defines on each stalk $\mathbb{M}_p$ a $\mathbb{C}$-MHS such that $\mathrm{Gr}^0_W\mathbb{M}_p$ is the trivial Hodge structure on $\mathbb{C}$ and $\mathrm{Gr}^1_W\mathbb{M}_p$ is the given Hodge Structure on $\mathbb{V}^*_p \otimes H^1(X, \mathbb{V})$.

LEMMA 2.6. *$\mathcal{F}^k$ is a holomorphic subbundle of the holomorphic vector bundle $\mathcal{M}$ underlying $\mathbb{M}$ and satisfies Griffiths' transversality.*

*Proof.* First observe that the $\bar\partial$ operator of $\mathcal{M}$ is given by $d^{0,1}_{\mathbb{M}}$. Consider the original flat connection

$$d = \begin{pmatrix} d_{\mathbb{C}} & 0 \\ 0 & d_{\mathbb{V}^*} \otimes \mathrm{Id}_{H^1(X,\mathbb{V})} \end{pmatrix}.$$

Obviously

$$d^{0,1}_{\mathbb{M}} = d^{0,1} + \begin{pmatrix} 0 & \Omega^{0,1} \\ 0 & 0 \end{pmatrix}.$$

Since $d^{0,1}$ preserves $\mathcal{F}^k$, it follows that $\mathcal{F}_k$ is an holomorphic subbundle of $\mathcal{M}$ if and only if

$$\begin{pmatrix} 0 & \Omega^{0,1} \\ 0 & 0 \end{pmatrix} \mathcal{F}^k \subset \Omega^{0,1} \otimes \mathcal{F}^k,$$

where $\Omega^{0,1} \in \mathcal{E}^{0,1}(X, \mathbb{V}) \otimes H^1(X, \mathbb{V})^*$ is the $(0,1)$-component of $\Omega$. This condition is equivalent to $\Omega^{0,1} \cdot \mathcal{F}^k_{\mathbb{V}^* \otimes H^1(X,\mathbb{V})} = 0$ if $k > 0$.

We are thus reduced to checking that for every $\alpha \in \mathcal{H}^1(X, \mathbb{V})$ such that $[\alpha]$ is of pure Hodge type $(P, Q)$ and $[\beta]^\vee \in (H^1(X, \mathbb{V})^*)^{-P,-Q}$,

$$\alpha^{0,1} \otimes [\beta]^\vee \cdot \mathcal{F}^k_{\mathbb{V}^* \otimes H^1(X,\mathbb{V})} = 0 \text{ if } k > 0.$$

It is enough to check that $[\alpha] \otimes [\beta]^\vee \cdot H^{k,1-k}_{\mathbb{V}^* \otimes H^1(X,\mathbb{V})} = 0$, or further decomposing in Hodge type that

$$\alpha^{0,1} \otimes [\beta]^\vee \cdot h^{-P'+k,-Q'-k+1} \otimes [\beta]^{P',Q'} = O,$$

where $h^{-P'+k,-Q'-k+1} \in (\mathbb{V}^*)^{-P'+k,-Q'-k+1}$ and $[\beta]^{P',Q'} \in H^1(X, \mathbb{V})^{P',Q'}$. The only nontrivial case is when $P' = P, Q' = Q$, and this reduces to showing that $\mathbb{V}^{P,Q-1} \otimes \Omega^{0,1}.(\mathbb{V}^*)^{-P+k,-Q-k+1} = 0$, which is the case since $k > 0$.

Griffiths' transversality is the statement that $d^{1,0}_{\mathbb{M}} \mathcal{F}^k \subset \mathcal{F}^{k-1} \otimes \Omega^{1,0}$ and follows from the same argument. □

Antiholomorphicity and Griffiths' anti-transversality for $\overline{\mathcal{G}^\bullet}$ can be proved by the same method. Hence we have defined on $\mathbb{M}$ a graded polarizable VMHS with weights $0, 1$, the polarizations being the natural ones. In [HZ87], the case of $\mathbb{V} = \mathbb{C}_X$ is treated. In that case, the VMHS is actually defined over $\mathbb{Z}$.

*Definition* 2.7. $\mathbb{M} = \mathbb{M}(\mathbb{V}) := (X, \mathbb{M}, \mathbb{W}_\bullet, \mathcal{F}^\bullet_M, \overline{\mathcal{G}}^\bullet_M, (S_k)_{k=0,1})$ is the 1-step $\mathbb{C}$-VMHS attached to $\mathbb{V}$.

2.3. *mixed Hodge theory for the deformation functor.* In this paragraph, we review the construction of [ES11]. The "new" aspects of this construction actually grew out of the previous example. The older aspects, on the other hand, were part of Goldman-Millson's theory of deformations for representations of Kähler groups [GM88].

In this paragraph, we fix $N \in \mathbb{N}$ and assume that $G = \mathrm{GL}_N$ and $M = M_B(X, \mathrm{GL}_N)$. Let $\rho : \pi_1(X, x) \to \mathrm{GL}_N(\mathbb{C})$ be the monodromy representation of a $\mathbb{C}$-VHS. Let $\hat{\mathcal{O}}_\rho$ be the complete local ring of $[\rho] \in R(\pi_1(X, x), \mathrm{GL}_N)(\mathbb{C})$. Let

$$\mathsf{obs}_2 = [-;-] : S^2 H^1(X, \mathrm{End}(\mathbb{V}_\rho)) \to H^2(X, \mathrm{End}(\mathbb{V}_\rho))$$

be the Goldman-Millson obstruction to deforming $\rho$. Define $I_2, (I_n)_{n\geq 2}, (\Pi_n)_{n\geq 0}$, as follows:

$$\begin{aligned}
\Pi_0 &= \mathbb{C},\\
\Pi_1 &= H^1(X, \mathrm{End}(\mathbb{V}_\rho))^*,\\
I_2 &= \mathrm{Im}({}^t\mathsf{obs}_2) \subset S^2 H^1(X, \mathrm{End}(\mathbb{V}_\rho))^*,\\
I_n &= I_2 S^{n-2} H^1(X, \mathrm{End}(\mathbb{V}_\rho))^*,\\
\Pi_n &= S^n H^1(X, \mathrm{End}(\mathbb{V}_\rho))^*/I_n.
\end{aligned}$$

Then the complete local $\mathbb{C}$-algebra

$$(\hat{\mathcal{O}}_T, \mathfrak{m}) := \Big(\sum_{n\geq 0} \Pi_n, \sum_{n\geq 1} \Pi_n\Big)$$

is the function algebra of a formal scheme $T$, which is the germ at 0 of the quadratic cone

$$\mathsf{obs}_2^{-1}(0) \subset H^1(X, \mathrm{End}(\mathbb{V}_\rho)).$$

We endow $\hat{\mathcal{O}}_T$ with a split mixed Hodge structure with nonpositive weights, whose weight filtration is given by the formula $W_k\hat{\mathcal{O}}_T = \mathfrak{m}^{-k}$ for $k \leq 0$, arising from the identifications

$$\hat{\mathcal{O}}_T = \sum_{n\geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1} = \sum_{n\in\mathbb{N}} \Pi_n,$$

$\Pi_n$ being endowed with its natural $\mathbb{C}$-Hodge structure of weight $-n$. This mixed Hodge structure is infinite dimensional but can be described as the limit of the resulting finite-dimensional MHS on $\hat{\mathcal{O}}_T/\mathfrak{m}^n$.

In [GM88], an isomorphism between $Spf(\hat{\mathcal{O}}_\rho)$ and $T \times A$ is constructed, where $A$ is the germ at zero of a finite-dimensional vector space. In [ES11], this construction is revisited. A slight reinterpretation of the Goldman-Millson theory is that one can realize the formal local scheme $T$ as a hull of the deformation functor for $\rho$. Actually, there are *three* preferred such realizations $\mathcal{GM}^c, \mathcal{GM}', \mathcal{GM}''$, which are given by three canonical representations:

$$\begin{aligned}
\rho_T^{\mathrm{GM}} &:\pi_1(X, x) \to \mathrm{GL}_N(\widehat{\mathcal{O}}_T),\\
\rho_T^{\mathrm{GM}'} &:\pi_1(X, x) \to \mathrm{GL}_N(\widehat{\mathcal{O}}_T),\\
\rho_T^{\mathrm{GM}''} &:\pi_1(X, x) \to \mathrm{GL}_N(\widehat{\mathcal{O}}_T).
\end{aligned}$$

These three representations are conjugate up to an isomorphism of $T$.

We can now summarize the results developed by [ES11] in the form we shall need.

*Definition* 2.8. Let $\{\eta_i\} \subset E^\bullet(X, \mathrm{End}(\mathbb{V}_\rho))$ be a basis of the subspace $\mathcal{H}^1(X, \mathrm{End}(\mathbb{V}_\rho))$ of harmonic twisted one forms, each $\eta_i$ being of pure Hodge type $(P_i, Q_i)$ for the Deligne-Zucker $\mathbb{C}$-mixed Hodge Complex $E^\bullet(X, \mathrm{End}(\mathbb{V}_\rho))$. Denote the dual basis by $\{\eta_i^*\}$.

The $\mathrm{End}(\mathbb{V}_\rho)\otimes\Pi_1$-valued one-form $\alpha_1^v$ is defined by the formula

$$\alpha_1^v = \sum_{i=1}^{b} \eta_i \otimes \{\eta_i\}^*.$$

PROPOSITION 2.9. *For $k \geq 2$, we can construct a unique $D''$-exact form $\alpha_k^v \in E^1(X, \mathrm{End}(\mathbb{V}_\rho)) \otimes \Pi_k$ such that the following relation holds*:

$$D'\alpha_k^v + \alpha_{k-1}^v\alpha_1^v + \alpha_{k-2}^v\alpha_2^v + \cdots + \alpha_1^v\alpha_{k-1}^v = 0.$$

PROPOSITION 2.10. *Let $A_v = \sum \alpha_k^v$ acting on the vector bundle underlying the filtered local system $(\mathbb{V}_\rho \otimes \hat{\mathcal{O}}_T, W_k(\mathbb{V}_\rho \otimes \hat{\mathcal{O}}_T) = \mathbb{V}_\rho \otimes \mathfrak{m}^{k-\mathrm{wght}(\mathbb{V}_\rho)})$, whose connection will be denoted by $D$. Then, $D + A^v$ respects this weight filtration and satisfies Griffiths' transversality for the Hodge filtration $\mathcal{F}^\bullet$ defined by*

$$\mathcal{F}^p = \bigoplus_{k=-n}^{0} \mathcal{F}^p(\mathbb{V}_\rho \otimes \Pi_{-k}).$$

*We can construct an anti-Hodge filtration so that the resulting structure is a graded polarizable* $\mathbb{C}$-VMHS *whose monodromy representation is $\rho_T^{\mathrm{GM}''}$.*

A detailed proof of this proposition is given in [ES11]. The essential part is the construction of the anti-Hodge filtration, which is similar in spirit but somewhat subtler than the construction given in Example 2.2.2.

*Definition* 2.11. The $\mathbb{C}$-VMHS obtained by reduction mod $\mathfrak{m}^n$ corresponds to

$$\rho_{T,n} := (\rho_T^{c\mathrm{GM}^v} \mod \mathfrak{m}^n) : \pi_1(X, x) \to \mathrm{GL}_N(\widehat{\mathcal{O}}_T/\mathfrak{m}^n).$$

It will be called the $n$-th deformation of $\mathbb{V}_\rho$ and will be denoted by $\mathbb{D}_n(\mathbb{V}_\rho)$.

By construction, $D + A^v$ is an $\hat{\mathcal{O}}_T$-linear connection. As a consequence of the methods in [ES11, pp. 18–23], we also have

PROPOSITION 2.12. *There is an* MHS *on $\hat{\mathcal{O}}_T$ whose weight filtration is given by the powers of the maximal ideal and such that the natural map $\hat{\mathcal{O}}_T \to \mathrm{End}_{\mathbb{C}}(\mathbb{D}_n(\mathbb{V}_\rho))$ respects the natural* MHS.

This MHS is *not* the split MHS constructed above. This split MHS is just the weight graded counterpart of the true object. These MHS and VMHS are not uniquely defined when the deformation functor of $\rho$ is not prorepresentable. This phenomenon does not occur when the representation is irreducible.

*Remark* 2.13. The restriction $G = \mathrm{GL}_n$ in the above considerations was introduced only for convenience. It is not essential. In [ES11], similar statements are proven for arbitrary reductive groups $G$.

## 3. Subgroups of $\pi_1(X,x)$ attached to $M$

Let $G$ be a reductive algebraic group defined over $\overline{\mathbb{Q}}$. Suppose, as before, that $M \subset M_B(X,G)$ is an absolute closed subset.

3.1. *Definitions.*

*Definition* 3.1. Let $M^{\mathrm{VHS}}$ be the subset of $M(\mathbb{C})$ consisting of the conjugacy classes of $\mathbb{C}$-VHS that is $M^{\mathrm{VHS}} := KH^{-1}(M_{\mathrm{Dol}}(X,G)^{\mathbb{C}^*}(\mathbb{C}))$.

We choose a set $M^*$ of reductive representations $\rho : \pi_1(X,x) \to G(\mathbb{C})$ mapping onto $M^{\mathrm{VHS}}$ under the natural map $R(\pi_1(X,x),G) \to M_B(X,G)$. To be more precise, we define $M^*$ to be the union of the closed $G$-orbits on $R(\pi_1(X,x),G)$, or equivalently the set of reductive complex representations whose equivalence class lie in $M^{\mathrm{VHS}}$. Similarly, we define $M'$ to be the union of the closed $G$-orbits on $R(\pi_1(X,x),G)$ whose equivalence class lie in $M$. To each $\rho \in M^*$ we attach $E^\rho$, the real Zariski closure of its monodromy group, and the other constructions reviewed in paragraph 2.2.1.

*Definition* 3.2. The Tannakian categories $\mathcal{T}_M^{\mathrm{VHS}}$ and $\mathcal{T}_M$ are defined as follows:

- $\mathcal{T}_M^{\mathrm{VHS}}$ is the full Tannakian subcategory of the category of local systems on $X$ generated by the elements of $M^{\mathrm{VHS}}$.
- $\mathcal{T}_M$ is the full Tannakian subcategory of the category of local systems on $X$ generated by the elements of $M$.

Every object in $\mathcal{T}_M^{\mathrm{VHS}}$ is isomorphic to an object that is a subquotient of $\alpha_1(\rho_1)\otimes\cdots\otimes\alpha_s(\rho_s)$, where $\rho_1,\dots,\rho_s$ are elements of $M^*$ and $\alpha_i$ is a complex, linear finite-dimensional representation of $E^{\rho_i}(\mathbb{R})$. Let $M^{**}$ be the set of all such subquotients. The objects of $\mathcal{T}_M^{\mathrm{VHS}}$ underly polarizable $\mathbb{C}$-VHS.

Let $T_M^{\mathrm{VMHS}}$ be the thick Tannakian subcategory of ($\mathbb{C}$-VMHS) whose graded constituents are objects of $\mathcal{T}_M^{\mathrm{VHS}}$. The full subcategory of $\mathcal{T}_M^{\mathrm{VMHS}}$ with a weight filtration of length at most $k+1$ will be denoted by $T_M^{\mathrm{VMHS}}(k)$.

*Example* 3.3. For every $\rho \in M^{**}$, $\alpha$ as above and $\sigma = \alpha\circ\rho$, $\mathbb{D}_k(\mathbb{V}_\sigma)$ is an object of $T_M^{\mathrm{VMHS}}(k)$.

*Definition* 3.4. Given $X$, $G$, and $M \subset M_B(X,G)$ as above, and $k \in \mathbb{N}$, we define the following natural quotients of $\pi_1(X,x)$:

- $\Gamma_M^\infty$ is the quotient of $\pi_1(X,x)$ by the intersection $H_M^\infty$ of the kernels of the objects of $\mathcal{T}_M^{\mathrm{VMHS}}$ and of the objects of $M$.
- $\widetilde{\Gamma_M^\infty}$ is the quotient of $\pi_1(X,x)$ by the intersection $\widetilde{H_M^\infty}$ of the kernels of the monodromy representation of $\mathbb{D}_n(\mathbb{V}_\sigma)$, $\sigma \in M^{**}$, $n \in \mathbb{N}$, and of the objects of $M$.

- $\Gamma_M^k$ is the quotient of $\pi_1(X,x)$ by the intersection $H_M^k$ of the kernels of the objects of $\mathcal{T}_M^{\mathrm{VMHS}}(k)$ and of the objects of $M$.
- $\widetilde{\Gamma_M^k}$ is the quotient of $\pi_1(X,x)$ by the intersection $\widetilde{H_M^k}$ of the kernels of the monodromy representation of $\mathbb{D}_k(\mathbb{V}_\sigma)$, $\sigma \in M^{**}$, and of the objects of $M$.

It is likely that the canonical quotient morphism $\Gamma_M^k \to \widetilde{\Gamma_M^k}$ is an isomorphism, but we do not have a proof of this fact yet. We will thus have to work with the above slightly clumsy notation.

Note that we have the inclusions

$$\Gamma_M^\infty = \bigcap_{k\in\mathbb{N}} \Gamma_M^k \subset \Gamma_M^{k+1} \subset \Gamma_M^k \subset \Gamma_M^0 = \Gamma_M,$$
$$\widetilde{\Gamma_M^\infty} = \bigcap_{k\in\mathbb{N}} \widetilde{\Gamma_M^k} \subset \widetilde{\Gamma_M^{k+1}} \subset \widetilde{\Gamma_M^1} \subset \widetilde{\Gamma_M^0} = \widetilde{\Gamma_M}.$$

It should be noted that since $H_M^k$ (respectively $\widetilde{H_M^k}$) is normal, the various base point changing isomorphisms $\pi_X(X,x') \to \pi_1(X,x)$ respect $H_M^k$ (respectively $\widetilde{H_M^k}$). Hence, dropping the base point dependence in the notation $H_M^k$ (respectively $\widetilde{H_M^k}$) is harmless.

For future reference, we state the following lemma, whose proof is tautological.

LEMMA 3.5. *$H_M^k$ is the intersection of $\Gamma_M$ and the kernels of*

$$a_k^\rho : \pi_1(X,x) \to \mathcal{G}_{x,k}^\rho(\mathbb{R}).$$

3.2. *Strictness.* Let $z \in Z$ be a base point in the connected projective variety $Z$.

PROPOSITION 3.6. *For every $f : (Z,z) \to (X,x)$ such that $\pi_1(Z,z) \to \Gamma_M$ is trivial, the following are equivalent*:

(1) *For every $\mathbb{V}$ in $\mathcal{T}_M^{\mathrm{VHS}}$, $H^1(X,\mathbb{V}) \to H^1(Z,\mathbb{V})$ is trivial*;
(2) *$\pi_1(Z,z) \to \Gamma_M^1$ is trivial*;
(3) *$\pi_1(Z,z) \to \widetilde{\Gamma_M^1}$ is trivial*;
(4) *For every $\mathbb{V}$ in $\mathcal{T}_M^{\mathrm{VHS}}$, for every $\widehat{Z}_i \to Z$ a resolution of singularities of an irreducible component, the VMHS $M(\mathbb{V})_{\widehat{Z}_i}$ is trivial*;
(5) *For every $\sigma \in M^{**}$ and $k \in \mathbb{N}$, for every $\widehat{Z}_i \to Z$ a resolution of singularities of an irreducible component, the* VMHS $\mathbb{D}_k(\mathbb{V}_\sigma)_{\widehat{Z}_i}$ *is trivial*;
(6) *$\pi_1(Z,z) \to \widetilde{\Gamma_M^\infty}$ is trivial*;
(7) *$\pi_1(Z,z) \to \Gamma_M^\infty$ is trivial.*

*Proof.* $(1 \iff 2)$. Fix $\rho \in M^{**}$. Denote by $E^\rho$ the real Zariski closure of $\rho(\pi_1(X,x))$. By hypothesis $\rho(\pi_1(Z,z)) = \{e\}$ and thanks to [Hai98, §11],[2] we have a diagram

$$\begin{array}{ccc} \pi_1(Z,z) & \xrightarrow{a_Z} & \hat{\pi}_1^{\mathrm{DR}}(Z,z) \\ \downarrow & & \downarrow \\ \pi_1(X,x) & \xrightarrow{a_X} & \mathcal{U}_x^\rho \quad \subset \quad \mathcal{G}_x^\rho, \end{array}$$

where $\pi_1(Z,z) \overset{a_Z}{\mapsto} \hat{\pi}_1^{\mathrm{DR}}(Z,z) = \mathcal{U}^e(Z,z) = \mathcal{G}^e(Z,z)$ is the Malcev completion of $\pi_1(Z,z)$, i.e., its relative completion with respect to the trivial representation.

Let $\{V_\alpha\}_\alpha$ be a set of representatives of all isomorphism classes of complex irreducible left $E^\rho$-modules.

The prounipotent group morphism $f_* : \hat{\pi}_1^{\mathrm{DR}}(Z,z) \to \mathcal{U}_x^\rho$ gives rise to a morphism of proalgebraic complex vector groups (= limits of finite-dimensional complex vector spaces viewed as algebraic groups)

$$H_1(\hat{\pi}_1^{\mathrm{DR}}(Z,z))(\mathbb{C}) \to H_1(\mathcal{U}_x^\rho)(\mathbb{C}),$$

where $H_1(\mathcal{U}) = \mathcal{U}/\mathcal{U}'$ is the abelianization. One has identifications (see [Hai98, p. 73])

$$\begin{aligned} H_1(\hat{\pi}_1^{\mathrm{DR}}(Z,z))(\mathbb{C}) &= H_1(Z,\mathbb{C}), \\ H_1(\mathcal{U}_x^\rho)(\mathbb{C}) &= \prod_\alpha H_1(X,\mathbb{V}_\alpha) \otimes V_\alpha^*, \end{aligned}$$

where $\mathbb{V}_\alpha$ is the local system attached to $\rho$ and $V_\alpha$. The map is the transpose of the map

$$\bigoplus_\alpha H^1(X,\mathbb{V}_\alpha^*) \otimes V_\alpha \to H^1(X,\mathbb{C})$$

given on each factor by the composition

$$H^1(X,\mathbb{V}_\alpha^*) \otimes V_\alpha \overset{i_Z^* \otimes \mathrm{id}_{V_\alpha}}{\to} H^1(Z,\mathbb{V}_\alpha^*) \otimes V_\alpha = H^1(Z,\mathbb{C}) \otimes V_\alpha^* \otimes V_\alpha \overset{\mathrm{id}\otimes \mathrm{tr}}{\to} H^1(Z,\mathbb{C}).$$

For the middle equality in this formula, we used that $\mathbb{V}_{\alpha|Z}$ is the trivial local system, which follows from the assumption that $\pi_1(Z,z) \to \Gamma_M$ is trivial. Hence Condition 1 is equivalent to $H_1(\hat{\pi}_1^{\mathrm{DR}}(Z,z)(\mathbb{C})) \to H_1(\mathcal{U}_x^\rho)(\mathbb{C})$ being zero, which in turn is equivalent to Condition 2.

$(2 \implies 3)$ Condition 3 is obviously implied by Condition 2.

$(3 \implies 1)$ If 3 holds, $\mathbb{D}_1(\mathbb{V}_\sigma)|_Z$ is a trivial local system. But, by construction, this local system is a deformation of a trivial local system by a one-step

[2] *Stricto sensu*, in order to apply [Hai98, §11], we need that $Z$ be a smooth connected manifold. We can replace $Z$ by a neighborhood $U$ of it in some embedding in $\mathbf{P}^N(\mathbb{C})$ such that $Z \to U$ is an homotopy equivalence and apply [Hai98, §11] to $U$.

nilpotent matrix of closed one forms written in the following block form:

$$\begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}.$$

Hence, in the same basis, its monodromy on any $\gamma \in \pi_1(Z, z)$ is given by

$$\begin{pmatrix} 1 & \int_\gamma A \\ 0 & 1 \end{pmatrix}.$$

Hence the triviality of $\mathbb{D}_1(\mathbb{V}_\sigma)|_Z$ implies that $\int_\gamma A = 0$, or that the cohomology class of $A$ is zero. But by construction, the cohomology class of $A$ is zero if and only if Condition 1 holds.

$(1 \Longrightarrow 4)$ The cohomology class of the form $\alpha_1$ vanishes after restriction to $Z$ and so vanishes after pullback to $\widehat{Z}_i$. We denote by $f_i$ the composition of $f$ with the map $\widehat{Z}_i \to Z$. But $\alpha_1 \in \ker(D') \cap \ker(D'')$. Hence $f_i^*\alpha_1 \in \ker(D')_{i,1} \cap \ker(D'')_{i,1}$, where $D'_{i,1}, D''_{i,1}$ are the usual $D', D''$ acting on $E^1(\widehat{Z}_i, \mathrm{End}(f_i^*\mathbb{V}_\rho))$. As mentioned before, Hodge theory implies that

$$\ker(D')_{i,1} \cap \ker(D'')_{i,1} = \mathcal{H}^1(\widehat{Z}_i, \mathrm{End}(f_i^*\mathbb{V}_\rho)).$$

Hence $f_i^*\alpha_1$ is the harmonic representative of its class. From this it follows that $f_i^*\alpha_i = 0$. This implies that $f_i^*\mathbb{M}$ is the trivial deformation of $f_i^*\mathbb{M}_0$ and Condition 4 follows.

$(4 \Longrightarrow 1)$ The method we used to prove $(3 \Longrightarrow 1)$ works to yield that $H^1(X, \mathbb{V}) \to H^1(\hat{Z}_i, \mathbb{V})$ is zero. But this implies by the argument we used to show $(1 \Longrightarrow 4)$ that $f_i^*\alpha_1 = 0$. This in turn implies that $i_A^*\alpha_1 = 0$ if $f(Z) = \coprod A$ is a smooth stratification. Hence the holonomy of $M(\mathbb{V})_Z$ is trivial. Applying once again the method for $(3 \Longrightarrow 1)$ completes the argument.

$(1 \Longrightarrow 5)$ Continuing the same line of reasoning as in proving $(1 \Longrightarrow 4)$ and using the fact that the $(\alpha_k^v)$ constructed in Proposition 2.9 are uniquely determined, it follows that $(f_i^*\alpha_k^v)$ is the family of twisted forms one gets from applying the construction of Proposition 2.9 starting with $f_i^*\alpha_1 = 0$. Hence $f_i^*\alpha_k^v = 0$ and $f_i^*A^v = 0$. Condition 5 then follows.

$(1 \Longrightarrow 6)$ Continuing this line of reasoning, the argument made in $(4 \Longrightarrow 1)$ implies that the restriction of $\mathbb{D}_k(\mathbb{V}_\sigma)$ to $Z$ has trivial monodromy, which is equivalent to Condition 6.

$(6 \Longrightarrow 2)$ is trivial.

$(6 \Longrightarrow 5)$ comes from the fact that Condition 6 implies that the restriction of $\mathbb{D}_k(\mathbb{V}_\sigma)$ to $Z$ has trivial monodromy, Condition 5 follows *a fortiori*.

$(7 \Longrightarrow 3)$ is trivial.

$(1 \Longrightarrow 7)$ The proof is an easy adaptation of the argument of [Kat97, §2]. We nevertheless feel it is necessary to give some details.

The Lie algebras $L(Z, z) = \mathrm{Lie}(\hat{\pi}_1^{\mathrm{DR}}(Z, z))$ and $\mathfrak{U}_x^\rho = \mathrm{Lie}(\mathcal{U}_x^\rho)$ are nilpotent and so come equipped with a decreasing filtration given by their lower central

series. The map $i_Z$ gives rise to a Lie algebra morphism $(i_Z)_* : L(Z,z) \to \mathfrak{U}_x^\rho$. It is enough to show that $(i_Z)_* = 0$.

By relabeling, we can convert the lower central series into an increasing filtration $B^\bullet L(Z,z)$ and $B^\bullet \mathfrak{U}_x^\rho$ with indices $\leq -1$. For both Lie algebras, $\mathrm{Gr}_B^{-1}(\_) = H_1(\_)$ and $\mathrm{Gr}_B^{-1}(L(Z,z))$ generate the graded Lie algebra $\mathrm{Gr}_B^\bullet(L(Z,z))$. Hence Condition 3 implies that $\mathrm{Gr}_B^\bullet(i_Z)_* : \mathrm{Gr}_B^\bullet L(Z,z) \to \mathrm{Gr}_B^\bullet \mathfrak{U}_x^\rho$ is zero.

First consider the case where $Z$ is smooth. Then, by [Hai87a], [Hai98], both $L(Z,z)$ and $\mathfrak{U}_x^\rho$ carry a functorial mixed Hodge structure whose weight filtration is $B^\bullet$. Hence, since the map $(i_Z)_*$ respects the mixed Hodge structures, it is strict for the weight filtration and $\mathrm{Gr}_B^\bullet(i_Z)_* = 0 \Rightarrow (i_Z)_* = 0$.

Next we consider the case where $H_1(Z)$ is pure of weight one. We recall (see [Hai87a]) that $\mathbb{R}[\widehat{\pi}_1^{\mathrm{DR}}(Z,z)] = H^0(\overline{\mathsf{B}}(\mathbb{R}, E^\bullet(Z), \mathbb{R}))$, where $\overline{\mathsf{B}}$ is the reduced bar construction and $E^\bullet(Z)$ is a multiplicative mixed Hodge complex computing $H^\bullet(Z)$ endowed with a base point at $z$. $\overline{\mathsf{B}}(\mathbb{R}, E^\bullet(Z), \mathbb{R})$ carries an increasing filtration $\mathfrak{B}_\bullet$, the bar filtration. It follows from [Hai87a] that $\overline{\mathsf{B}}(\mathbb{R}, E^\bullet(Z), \mathbb{R})$ endowed with the bar filtration is a filtered mixed Hodge complex so that the bar filtration on $\mathbb{R}[\widehat{\pi}_1^{\mathrm{DR}}(Z,z)]$ is a filtration by MHS. The Eilenberg-Moore spectral sequence, which is the spectral sequence associated to the bar filtration, is a spectral sequence in the category of MHS and, since $H^1(Z)$ is pure of weight one, $E_1^{s,-s} = H^1(Z)^{\otimes s}$ is pure of weight $s$. Hence $\mathrm{Gr}_{\mathfrak{B}}^k \mathbb{R}[\widehat{\pi}_1^{\mathrm{DR}}(Z,z)]$ is pure of weight $k$. Since the bar filtration is a refinement of the weight filtration, it follows that the bar filtration and the weight filtration coincide. Combining this with the preceding argument, one easily finishes the proof of the case where $H_1(Z)$ is pure of weight one.

Finally, note that by passing to a hyperplane section we may assume that $Z$ is a curve, which without loss of generality can be taken to be seminormal, and the argument of [Kat97, pp. 340–341] applies verbatim. One concludes using Lemma 2.4, p. 342 in [Kat97]. □

*Remark* 3.7. If we skip Conditions 2 and 7 of the previous proposition, we obtain a strictness statement which can be proved without relying on [Hai98]. The equivalence of Conditions 2 and 7 with the other ones is not used in the sequel of this article.

*Remark* 3.8. As far as the equivalence of Condition 7 with the other ones is concerned, we believe that one can adapt the explicit argument made for $(1 \implies 6)$ using the more sophisticated iterated integrals of [Hai98]. Except perhaps for Condition 7, which depends on $X$ being projective, the proposition is valid in the compact Kähler case.

*Remark* 3.9. A generalization to the Kähler case of the main result in [Kat97] with an alternative proof was given in the unpublished thesis [Ler99]

(see also [Cla]) as a byproduct of her exegesis of [Hai87a] and [Hai87b]. The core of her argument could be reformulated in such a way that it becomes equivalent to the special case of the present one where $G = \{e\}$ is the trivial group.

3.3. *Reduction to using* VMHS.

PROPOSITION 3.10. *Let $n$ be a nonnegative integer. Let $H_n$ be the intersection of the kernels of all linear representations $\pi_1(X) \to \mathrm{GL}_n(A)$, $A$ being an arbitrary $\mathbb{C}$-algebra. Let $M = M(X, \mathrm{GL}_n)$. Then $H_n = \widetilde{H_M^\infty}$.*

*Proof.* The inclusion $H_n \subset \widetilde{H_M^\infty}$ is obvious. Now let $\gamma \in \widetilde{H_M^\infty}$. Then $\gamma$ defines a matrix valued regular function $F$ on $R(\pi_1(X,x), \mathrm{GL}_n)$ (i.e., $F \in \mathrm{Mat}_{n\times n}(\mathbb{C}[R(\pi_1(X,x), \mathrm{GL}_n)])$), which reduces to the constant function with value $I_n$ on $T_\rho \subset R(\pi_1(X,x), \mathrm{GL}_n)$ for every element $\rho \in M^{\mathrm{VHS}}$. The Goldman-Millson theory implies that the tautological representation $\pi_1(X,x) \to \mathrm{GL}_n(\widehat{\mathcal{O}_\rho})$ is conjugate to the pull back by $cGM$ of $\rho_T^{cGM}$. Hence $F$ induces the trivial matrix valued function when reduced to $Spf(\widehat{\mathcal{O}_\rho})$. Hence $F$ induces the constant matrix valued function with value $I_n$ on some complex analytic neighborhood of $M^{\mathrm{VHS}}$.

Let $\widetilde{\rho}$ be a reductive complex representation whose conjugacy class lies on $M - M^{\mathrm{VHS}}$. Then, by [Sim88], $\widetilde{\rho}$ correspond to a polystable Higgs bundle $(\mathcal{E}, \theta)$. For $t \in \mathbb{C}^*$, let $\tilde{\rho}(t)$ corresponds to $(\mathcal{E}, t.\theta)$. By applying the Goldman-Millson construction to each $\tilde{\rho}(t)$, we get a real analytic family of flat connections $(D_t)_{t\in\mathbb{C}^*}$ on the smooth vector bundle underlying $\mathcal{E} \otimes \mathcal{O}_{\tilde{\rho}(t)}$ (see for instance [Pri10, pp. 21]) such that the image $F_t$ of the matrix function $F$ in the complete local ring at $\tilde{\rho}(t)$ satisfies $F_t = \mathrm{hol}(D_t) \in \mathrm{Mat}_{n\times n}(\mathcal{O}_{\tilde{\rho}(t)})$. Since $F_t = I_n$ for small $t$, then $F_1 = I_n$. Hence $F$ maps to $I_n$ in $\mathrm{Mat}_{n\times n}(\mathbb{C}[R(\pi_1(X,x), \mathrm{GL}_n)]_{\tilde{\rho}})$. Hence $F = I_n$ in a complex analytic neighborhood of the set of reductive representations.

Given a nonreductive representation $\rho^{\mathrm{arb}}$, we may find a sequence $(\rho_m)_{m\in\mathbb{N}}$ of conjugate representations converging to a reductive one. Since $\rho_m(\gamma) = \mathrm{Id}_n$ for $m \gg 0$, we have that $\rho^{\mathrm{arb}}(\gamma) = \mathrm{Id}_n$. One concludes that $F = I_n$ or, in other words, that $\gamma$ lies in the kernel of every representation $\pi_1(X) \to \mathrm{GL}_n(A)$, for an arbitrary $\mathbb{C}$-algebra $A$. In particular, $\gamma \in H_n$. □

COROLLARY 3.11. *Assume that $\pi_1(X,x)$ has a faithful representation in $\mathrm{GL}_n(\mathbb{C})$. Then $\widetilde{H_M^\infty} = \{1\}$.*

## 4. **Rationality lemma**

4.1. *Some pure Hodge substructures attached to an absolute closed set $M$ and a fiber of* $\mathrm{sh}_M$. Let $f : Z \to X$ be a morphism and $M \subset M_B(X,G)$ be an absolute closed subset. For $\mathbb{V}$ an object of $\mathcal{T}_M$, we denote by $\mathrm{tr} : \mathbb{V} \otimes \mathbb{V}^* \to \mathbb{C}$

the natural contraction. Consider the subspace $P_{\mathbb{V}}(Z/X) \subset H^1(Z,\mathbb{C})$ defined by

$$P_{\mathbb{V}}(Z/X) := \mathrm{Im}\left[f^*H^1(X,\mathbb{V}) \otimes H^0(Z,\mathbb{V}^*) \xrightarrow{\cup} H^1(Z,\mathbb{V}\otimes\mathbb{V}^*) \xrightarrow{\mathrm{tr}} H^1(Z,\mathbb{C})\right].$$

In this formula, we denoted by $\mathbb{V}$ the local system on $Z$ defined as $f^*\mathbb{V}$. Obviously, no confusion can arise from this slight abuse of notation.

*Definition* 4.1. We also define $P_M(Z/X), \overline{P}_M(Z/X) \subset H^1(Z,\mathbb{C})$ as follows:

- $P_M(Z/X) \subset H^1(Z,\mathbb{C})$ is the subspace of $H^1(Z,\mathbb{C})$ spanned by the $P_{\mathbb{V}}(Z/X)$, when $\mathbb{V}$ runs over all objects in $\mathcal{T}_M^{\mathrm{VHS}}$.
- $\overline{P}_M(Z/X) \subset H^1(Z,\mathbb{C})$ is the subspace of $H^1(Z,\mathbb{C})$ spanned by the $P_{\mathbb{V}}(Z/X)$, when $\mathbb{V}$ runs over all objects in $\mathcal{T}_M$.

$H^1(Z,\mathbb{C})$ is defined over $\mathbb{Z}$ since it is the complexification of $H^1_{\mathrm{sing}}(Z,\mathbb{Z})$. This Betti integral structure is the one we will tacitly use.

LEMMA 4.2. *$P_M(Z/X)$ is a pure $\mathbb{C}$-Hodge substructure of weight one of the $\mathbb{C}$-MHS underlying Deligne's* MHS *on $H^1(Z,\mathbb{C})$.*

*Proof.* Since each $\mathbb{V}$ is a $\mathbb{C}$-VHS of weight zero, and $X$ is smooth, it follows that $H^1(X,\mathbb{V})$ is a pure $\mathbb{C}$-Hodge structure of weight one. Also by [Del71] the mixed Hodge structures on the cohomology of varieties with coefficients in variations of Hodge structures are functorial, and hence $P_{\mathbb{V}}(Z/X)$ is a $\mathbb{C}$-Hodge substructure of $H^1(Z,\mathbb{C})$. Finally, by strictness [Del71] the span $P_M(Z/X)$ of the $P_{\mathbb{V}}(Z/X)$'s will also be pure and of weight one. □

LEMMA 4.3. *If $G$ is defined over $\mathbb{Q}$ and if the absolutely closed subset $M \subset M_B(X,G)$ is defined over $\mathbb{Q}$, then $\overline{P}_M(Z/X)$ is defined over $\mathbb{Q}$.*

Assume now that $f(Z)$ is contained in a fiber of the reductive Shafarevich morphism for $M$ or that equivalently a finite étale cover of $Z$ lifts to a compact analytic subspace of $\widetilde{X_M}$. Then after a finite étale cover, we may assume that $f_*\pi_1(Z,z) \subset H_M$, i.e., that every object $\rho$ in $\mathcal{T}_M$ satisfies $\rho(\pi_1(Z,z)) = \{e\}$. The rationality lemma is the following statement.

THEOREM 4.4. *Assume $G$ is defined over $\mathbb{Q}$ and $M = M_B(X,G)$. Assume that $f_*\pi_1(Z,z) \subset H_M$. If $\pi_1(Z) \to \Gamma_M$ is trivial, then $P_M(Z/X) = \overline{P}_M(Z/X)$.*

COROLLARY 4.5. *If $G$ and $M = M_B(X,G)$ are defined over $\mathbb{Q}$, then $P_M(Z/X)$ is also defined over $\mathbb{Q}$.*

The rest of this section will be devoted to the proof of Theorem 4.4. We will also assume $\dim M > 0$ since the result is obvious for an absolute closed

subset consisting of isolated points. The proof will be done in several steps that reduce the general statement to special situations.

*Remark* 4.6. It seems likely that Theorem 4.4 holds true for arbitrary absolute closed subsets defined over $\mathbb{Q}$. One basically needs to adapt [ES11] to this situation.

4.2. *Reduction to the smooth case.* First we reduce to the case when $Z$ is smooth. We need the following lemma.

LEMMA 4.7. *$\overline{P}_M(Z/X)$ is a pure weight one substructure of Deligne's* MHS *on $H^1(Z)$.*

*Proof.* Let $\mathbb{V}$ be an object of $\mathcal{T}_M$. By [Sim97, Th. 4.1], the space $H^1(X,\mathbb{V})$ carries a pure twistor structure of weight one. Furthermore, by [Sim97, Th. 5.2], the space $H^1(Z,\mathbb{V})$ carries a canonical mixed twistor structure and $f^*H^1(X,\mathbb{V}) \subset H^1(Z,\mathbb{V})$ is a twistor substructure. By functoriality, $P_{\mathbb{V}}(Z/X) \subset H^1(Z,\mathbb{C})$ will be a pure weight-one twistor substructure, and hence the span

$$\overline{P}_M(Z/X) = \sum_{\mathbb{V}} P_{\mathbb{V}}(Z/X) \subset H^1(Z,\mathbb{C})$$

is a pure weight-one twistor substructure of the mixed Hodge structure $H^1(Z,\mathbb{C})$. However the Dolbeault realization of $\overline{P}_M(Z/X)$ is clearly preserved by $\mathbb{C}^*$ since, by assumption, $\mathbb{C}^*$ leaves $M^{\mathrm{Dol}}$ invariant. Therefore $\overline{P}_M(Z/X)$ is a sub-Hodge structure. □

In order to prove Theorem 4.4, since $P_M(Z/X) \subset \overline{P}_M(Z/X)$ is pure of weight one, it is enough to prove that $\mathrm{Gr}_1^W P_M(Z/X) = \mathrm{Gr}_1^W \overline{P}_M(Z/X)$. Hence, without a loss of generality, we can assume that $Z$ is smooth.

4.3. *Reduction to a finite number of local systems.*

LEMMA 4.8. *There is a finite set $S$ of objects of $\mathcal{T}_M^{\mathrm{VHS}}$ such that whenever a morphism $Z \to X$ has the property* $\mathrm{im}\,[\pi_1(Z) \to \Gamma_M] = 0$, *it follows that*

$$P_M(Z/X) = \sum_{\mathbb{V}\in S} P_{\mathbb{V}}(Z/X).$$

*Similarly, there is a finite set $\overline{S}$ of objects of $\mathcal{T}_M$, so that $\overline{P}_M(Z/X) = \sum_{\mathbb{V}\in\overline{S}} P_{\mathbb{V}}(Z/X)$. Furthermore, the set $\overline{S}$ can be chosen so that for any Higgs bundle $(E,\theta)$ corresponding to a $\mathbb{V} \in \overline{S}$, the* $\mathbb{C}$-VHS *associated to* $\lim_{t\to 0}(E,t.\theta)$ *belongs to $S$.*

*Proof.* Consider $(S_\alpha)_\alpha$ a stratification of $\mathrm{Sh}_M(X)$ by locally closed smooth algebraic subsets such that $s_\alpha := \mathrm{sh}_M|_{(\mathrm{sh}_M)^{-1}(S_\alpha)} : (\mathrm{sh}_M)^{-1}(S_\alpha) \to S_\alpha$ is a topological fibration. Fix $p_\alpha \in S_\alpha$. Let $Z_\alpha = s_\alpha^{-1}(p_\alpha)$, let $Z_{\alpha,o}$ be a connected

component and let $Z'_{\alpha,o} \to Z_{\alpha,o}$ be the topological covering space defined by $Z'_{\alpha,o} = \widetilde{Z^{\mathrm{univ}}_{\alpha,o}}/\ker(\pi_1(Z_{\alpha,o}) \to \Gamma_M)$.

Since $H^1(Z_{\alpha,o}, \mathbb{C})$ is finite dimensional, it follows that a finite set $S$ exists with the required properties for $Z = Z_{\alpha,o}$. Since the cohomology classes coming from $X$ are flat under the Gauss-Manin connection, this statement holds true for all fibers of $s_\alpha$. Since every $f : Z \to X$ with the required properties factors through one of the $Z_{\alpha,o}$'s, the lemma follows. □

4.4. *Hodge theoretical argument.* From now on, we really need to assume that $M = M_B(X, G)$ and that $G$ is defined over $\mathbb{Q}$.

Let $A$ be a noetherian $\mathbb{C}$-algebra and $\rho_A : \pi_1(X, x) \to \mathrm{GL}_N(A)$ be a representation. Let $\mathbb{V}_A$ be the local system of free $A$-modules attached to $\rho_A$ and $\mathbb{V}_A^\vee = \mathrm{Hom}_A(\mathbb{V}_A, A)$ be the local system associated to ${}^t\rho^{-1}$. We define

$$P(A) = \mathrm{Im}\left[H^1(X, \mathbb{V}_A) \otimes_A H^0(Z, \mathbb{V}_A^\vee) \to H^1(Z, \mathbb{C}) \otimes_{\mathbb{C}} A\right].$$

$P(A)$ is an $A$-submodule of the free $A$-module $H^1(Z, \mathbb{C}) \otimes_{\mathbb{C}} A$.

Let $\sigma$ in $M^{**}$ be a nonisolated point. In Section 2.3, we recalled the construction and basic properties of $T_\sigma \subset R(\pi_1(X, x), G)$ a formal local subscheme which gives rise to a hull of the deformation functor of $\sigma$. In follows from [GM90] that this formal subscheme is actually the formal neighborhood of $\sigma$ in an analytic germ $T^{\mathrm{an}}_\sigma \subset R(\pi_1(X, x), G)$. If we decompose the reduced germ of $T^{\mathrm{an}}_\sigma$ into the union $T^{\mathrm{an,red}}_\sigma = \cup_i T^{\mathrm{an,i}}$ of its analytic irreducible components, then we will denote by $T^i$ the formal neighborhood of $\sigma$ in $T^{\mathrm{an,i}}$. The irreducible components of an analytic germ being in one-to-one correspondence with the irreducible components of the associated formal germ, it follows that $T^{\mathrm{red}}_\sigma = \cup_i T^i$ is still the irreducible decomposition of the reduced formal local subscheme underlying $T_\sigma$. Note that $T^i$ is an integral formal subscheme of $T_\sigma$ and so its ideal $\mathfrak{P}^i$ is a minimal prime of $\widehat{\mathcal{O}}_{T_\sigma}$.

LEMMA 4.9. *The weight and Hodge filtrations on $\widehat{\mathcal{O}}_{T_\sigma}$ induce on $\mathfrak{P}^i \subset \widehat{\mathcal{O}}_{T_\sigma}$ a sub-*MHS *structure.*

*Proof.* First observe that the minimal associated primes of the graded ring $\mathrm{Gr}^{\mathfrak{m}^\bullet}\widehat{\mathcal{O}}_{T_\sigma}$ are graded ideals and also split sub-MHS of the split MHS on $\mathrm{Gr}^{\mathfrak{m}^\bullet}\widehat{\mathcal{O}}_{T_\sigma}$ since the $\mathrm{Res}_{\mathbb{C}|\mathbb{R}}\mathbb{C}^*$-action defining the Hodge decomposition is compatible with the ring structure.

By construction, there is a ring isomorphism $\widehat{\mathcal{O}}_{T_\sigma} \to \mathrm{Gr}^{\mathfrak{m}^\bullet}\widehat{\mathcal{O}}_{T_\sigma}$. This ring isomorphism takes minimal associated primes to minimal associated primes. Hence, $\mathrm{Gr}^{\mathfrak{m}^\bullet}\mathfrak{P}^i$ is a sub-Hodge structure of $\mathrm{Gr}^{\mathfrak{m}^\bullet}\widehat{\mathcal{O}}_{T_\sigma}$.

There is no canonical choice for this isomorphism, but it can be chosen in such a way that it respects the weight and Hodge filtrations — but not the three filtrations. This implies that the trace of the Hodge filtration of $\mathrm{Gr}^{\mathfrak{m}^\bullet}\widehat{\mathcal{O}}_{T_\sigma}$

on $\mathrm{Gr}^{\mathfrak{m}^\bullet}\mathfrak{P}^i$ is the filtration induced by the trace of the Hodge filtration of $\widehat{\mathcal{O}}_{T_\sigma}$ on $\mathfrak{P}^i$. The anti Hodge filtration satisfies a similar statement. These two facts imply that $\mathfrak{P}^i \subset \widehat{\mathcal{O}}_{T_\sigma}$ is a sub-MHS structure. □

Hence the complete local algebra $\widehat{O}_{T^i}$ carries a $\mathbb{C}$-MHS and $\rho_{\widehat{O}_{T^i}} : \pi_1(X,x) \to G(\widehat{O}_{T^i})$ is the monodromy of the local system $\mathbb{D}(\mathbb{V}_\sigma) \otimes_{\widehat{O}_{T_\sigma}} \widehat{O}_{T^i}$. Thanks to Lemmas 2.12 and 4.9, this local system underlies a $\mathbb{C}$-VMHS whose weight filtration corresponds to the powers of the maximal ideal in $\widehat{O}_{T^i}$.

By construction, the tautological representation

$$\rho_{\mathcal{O}_{T^{\mathrm{an,i}}}} : \pi_1(X,x) \to G(\mathcal{O}_{T^{\mathrm{an,i}}})$$

is a holomorphic family of representations parametrized by a reduced germ of complex space.

If there is a proper closed analytic subset $Z^i \subset T^{\mathrm{an,i}}$ such that for all $p \in T^{\mathrm{an,i}} - Z^i$, the representation $\rho_{\mathcal{O}_{T^{\mathrm{an,i}}}}(p)$ is a reductive representation, then the inclusion $f_*\pi_1(Z,z) \subset H_M$ implies that the restriction of $\rho_{\mathcal{O}_{T^{\mathrm{an,i}}}}(p)$ to $\pi_1(Z,z)$ is trivial for $p \notin Z^i$. Hence the restrictions of $\rho_{O_{T^{\mathrm{an,i}}}}$ and $\rho_{\widehat{O}_{T^i}}$ to $\pi_1(Z,z)$ are trivial as well.

If not, then for each irreducible component $M' \subset M$ containing $\sigma$, take a component $R'$ via $\sigma$ of the preimage $\pi^{-1}(M') \in R(\pi_1(X,x),G)$ that dominates $M'$. Let $(R')^{red} \subset R'$ be its maximal reduced subscheme. Consider the semisimplification of the representation attached to the generic point of the subscheme $(R')^{red} \subset R(\pi_1(X,x),G)$. It is conjugate to a Zariski dense representation with values in some $G' \subset G$, where $G'$ is reductive over $\overline{\mathbb{Q}}$. But $\mathrm{Im}(M_B(X,G') \to M_B(X,G))$ is a closed acqc set and so $M' \subset \mathrm{Im}(M_B(X,G') \to M_B(X,G))$. Thus, without a loss of generality, we may replace $G$ by $G'$ and also replace $T^{\mathrm{an}}_\sigma$ by an analytic Goldman-Millson slice through $\sigma$ in $R(\pi_1(X,x),G')$. With this new definition, the restriction of $\rho_{\widehat{O}_{T^i}}$ to $\pi_1(Z,z)$ is trivial too, and the corresponding local system on $Z$ is the constant local system $\mathbb{V}_\sigma \otimes_{\mathbb{C}} \widehat{O}_{T^i}$.

In particular, we have a canonical isomorphism of VMHS $\mathbb{V}^\vee_{\widehat{O}^i_T/\mathfrak{m}^k}|_Z \simeq \mathbb{V}^\vee_\sigma|_Z \otimes_{\mathbb{C}} \widehat{O}^i_T/\mathfrak{m}^k$. It now follows that, for all $k \in \mathbb{N}$,

$$P(\widehat{O}_{T^i}/\mathfrak{m}^k) = \mathrm{Im}(H^1(X, \mathbb{V}_{\widehat{O}^i_T/\mathfrak{m}^k}) \otimes_{\mathbb{C}} H^0(Z, \mathbb{V}^\vee_\sigma) \overset{H_{\widehat{O}^i_T/\mathfrak{m}^k}}{\to} H^1(Z,\mathbb{C}) \otimes_{\mathbb{C}} \widehat{O}_{T^i}/\mathfrak{m}^k).$$

$H_k := H_{\widehat{O}_{T^i}/\mathfrak{m}^k}$ preserves the natural mixed Hodge structures.

Proposition 4.10. $P_k := P(\widehat{O}_{T^i}/\mathfrak{m}^k) \subset P_1 \otimes \widehat{O}_{T^i}/\mathfrak{m}^k \subset P_M(Z/X) \otimes \widehat{O}_{T^i}/\mathfrak{m}^k$.

*Proof.* If $k = 1$, this is trivial; by construction, $P_1 \subset P_M(Z/X)$. We now argue by induction and assume that the result holds for $k' < k$.

The representation $\rho_k = \rho_{\widehat{O}^i_T/\mathfrak{m}^k}$ underlies a variation of complex mixed Hodge structure $\mathbb{M}_k$ on $X$. The weight filtration is given by the powers of $\mathfrak{m}$. Since $\rho_k$ is trivial on $\pi_1(Z)$, then its restriction to $Z$ is the trivial VMHS $\mathbb{H}\otimes_{\mathbb{C}}\widehat{O}^i_T/\mathfrak{m}^k$, where $\mathbb{H}$ is some Hodge structure of weight zero (with a possibly nontrivial Hodge vector) and, on $\widehat{O}_{T^i}/\mathfrak{m}^k$, the weight filtration is described by the powers of $\mathfrak{m}$:

$$P_k = \mathrm{Im}\left[H^1(X,\mathbb{M}_k\otimes_{\mathbb{C}}\mathbb{H})\xrightarrow{H_k} H^1(Z,\mathbb{C})\otimes_{\mathbb{C}}\widehat{\mathcal{O}}_{T^i}/\mathfrak{m}^k\right].$$

The weights of $\mathbb{M}_k$ are $0,\dots,-k+1$. Consider the following diagram of $MHS$, in which the rows are exact:

$$\begin{array}{ccccc} H^1(X,W_{-k+1}\mathbb{M}_k\otimes_{\mathbb{C}}\mathbb{H}) & \longrightarrow & H^1(X,\mathbb{M}_k\otimes_{\mathbb{C}}\mathbb{H}) & \longrightarrow & H^1(X,\mathbb{M}_{k-1}\otimes_{\mathbb{C}}\mathbb{H}) \\ \downarrow & & \downarrow & & \downarrow \\ H^1(Z)\otimes\mathfrak{m}^{k-1}/\mathfrak{m}^k & \longrightarrow & H^1(Z)\otimes\widehat{\mathcal{O}}_{T^i}/\mathfrak{m}^k & \longrightarrow & H^1(Z)\otimes\widehat{\mathcal{O}}_{T^i}/\mathfrak{m}^{k-1}. \end{array}$$

Remember we assume $Z$ to be smooth. The weights of the MHS in the first row are $2-m$, in the second $2-m,\dots,1$, in the third one $3-m,\dots,1$. Hence the second line is just the canonical exact sequence

$$W_{2-k}\left[H^1(Z)\otimes\widehat{\mathcal{O}}_{T^i}/\mathfrak{m}^k\right]\longrightarrow H^1(Z)\otimes\widehat{\mathcal{O}}_{T^i}/\mathfrak{m}^k\longrightarrow \mathrm{Gr}^W_{3-k}\left[H^1(Z)\otimes\widehat{\mathcal{O}}_{T^i}/\mathfrak{m}^k\right].$$

The main observation is now that, by strictness, we have

$$W_{2-k}P_k = \mathrm{Im}\left[H^1\left(X,W_{-k+1}\left(\mathbb{M}_k\otimes_{\mathbb{C}}\mathbb{H}\right)\right)\longrightarrow H^1(Z)\otimes\widehat{\mathcal{O}}_{T^i}/\mathfrak{m}^k\right].$$

From this it follows that $W_{2-k}P_k\in P_1\otimes\mathfrak{m}^{k-1}/\mathfrak{m}^k$. By induction, $P_{k-1}\subset H^1(Z)\otimes\widehat{O}_{T^i}/\mathfrak{m}^{k-1}\subset P_1\otimes\widehat{O}_{T^i}/\mathfrak{m}^{k-1}$. But $P_k$ (respectively $P_{k-1}$) is the image of the map in the third column (resp. the second). It follows that $P_k\subset P_1\otimes\widehat{O_{T^i}}/\mathfrak{m}^k$. □

4.5. *Proof of Theorem* 4.4 *if* $M = M_B(X,G)$. It follows from Proposition 4.10 that

$$P(\mathcal{O}_{T^{\mathrm{an,i}}})\subset P_M(Z/X)\otimes\mathcal{O}_{T^{\mathrm{an,i}}}.$$

It follows that for all $p$ in the complex analytic germ $T^{\mathrm{an,i}}$, we have

$$P_{\mathbb{V}_{\rho(p)}}(Z/X)\subset P_M(Z/X).$$

Since there is a complex analytic neighborhood $U$ of $\sigma$ in $M$ such that every point of $U$ has a (reductive) representative in $T^{\mathrm{an,i}}$, it follows that for every $\mathbb{V}\in U$, we have $P_{\mathbb{V}}(Z/X)\subset P_M(Z/X)$.

Now let $\overline{S}$ be the finite set from Lemma 4.8. Suppose $\mathbb{V}\in\overline{S}$ with an associated Higgs bundle $(E,\theta)$, and let $(\mathbb{V}_t)_{t\in\mathbb{C}^*}$ be the local systems corresponding to the Higgs bundle $(E,t\theta)$. For a small enough $t$, we have

$$P_{\mathbb{V}_t}(Z/X)\subset P_M(Z/X).$$

Fix $t$ small enough and nonzero. It follows that $\dim(\sum_{\mathbb{V}\in\bar{S}} P_{\mathbb{V}_t}(Z/X)) \leq \dim P_M(Z/X)$. Consider

$$P^{\mathrm{Dol}}_{\mathbb{V}_t} = \mathrm{Im}\left[ H^1_{\mathrm{Dol}}(X,\mathbb{V}_t)\otimes H^0_{\mathrm{Dol}}(Z,\mathbb{V}_t^*) \xrightarrow{\mathrm{Id}\otimes\mathrm{tr}} H^1_{\mathrm{Dol}}(Z)\right].$$

Using Simpson's Dolbeault isomorphism, we have

$$\dim\left(\sum_{\mathbb{V}\in\bar{S}} P^{\mathrm{Dol}}_{\mathbb{V}_t}(Z/X)\right) \leq \dim P_M(Z/X).$$

Recall that there is a natural isomorphism $s(t) : H^\bullet_{\mathrm{Dol}}(-,\mathbb{V}) \to H^\bullet_{\mathrm{Dol}}(-,\mathbb{V}_t)$. Let $(E,\theta)$ be a polystable Higgs bundle representing $\mathbb{V}$. Then $H^\bullet_{\mathrm{Dol}}(X,\mathbb{V}) := \mathbb{H}^\bullet(X,(E\otimes\Omega^\bullet_X,\theta))$. We can construct a quasi-isomorphism $(E\otimes\Omega^\bullet_X,\theta) \to (E\otimes\Omega^\bullet_X, t.\theta)$ by the formula

$$\begin{array}{ccccccc}
E & \xrightarrow{\theta} & E\otimes\Omega^1_X & \xrightarrow{\theta\wedge} & E\otimes\Omega^2_X & \xrightarrow{\theta\wedge} & \cdots \\
\downarrow & & \downarrow t & & \downarrow t^2 & & \\
E & \xrightarrow{t\theta} & E\otimes\Omega^1_X & \xrightarrow{t\theta\wedge} & E\otimes\Omega^1_X & \xrightarrow{t\theta\wedge} & \cdots .
\end{array}$$

Since $(E,t\theta)$ is the polystable Higgs bundle representing $\mathbb{V}_t$, this quasi-isomorphism indeed defines an isomorphism $s(t) : H^\bullet_{\mathrm{Dol}}(-,\mathbb{V}) \to H^\bullet_{\mathrm{Dol}}(-,\mathbb{V}_t)$.

In case $(E,\theta)$ is kept fixed by $\mathbb{C}^*$, which means that there is an isomorphism $\psi(t) : (E,\theta) \to (E,t\theta)$, $a(t) = \psi(t)^{-1}\circ s(t)$ is an automorphism of $H^1_{\mathrm{Dol}}(X,\mathbb{V})$ which comes from an action of $\mathbb{C}^*$. Here $(E,\theta)$ is not kept fixed by $\mathbb{C}^*$ but its restriction to $Z$ is. This gives a diagram

$$\begin{array}{ccc}
H^1_{\mathrm{Dol}}(X,\mathbb{V}) & \longrightarrow & H^1_{\mathrm{Dol}}(Z,\mathbb{V}) \\
\downarrow s(t) & & \downarrow a(t) \\
H^1_{\mathrm{Dol}}(X,\mathbb{V}_t) & \longrightarrow & H^1_{\mathrm{Dol}}(Z,\mathbb{V}).
\end{array}$$

By functoriality and the definition of $P^{\mathrm{Dol}}_{\mathbb{V}_t}(Z/X)$, we get a commutative diagram

$$\begin{array}{ccc}
P^{\mathrm{Dol}}_{\mathbb{V}}(Z/X) & \longrightarrow & H^1_{\mathrm{Dol}}(Z) \\
\downarrow \tilde{s}(t) & & \downarrow a(t) \\
P^{\mathrm{Dol}}_{\mathbb{V}_t}(Z/X) & \longrightarrow & H^1_{\mathrm{Dol}}(Z),
\end{array}$$

and $\tilde{s}(t)$ is an isomorphism.

Hence $\dim(\sum_{\mathbb{V}\in\bar{S}} P^{\mathrm{Dol}}_{\mathbb{V}}(Z/X)) \leq \dim P_M(Z/X)$. Since, by Simpson's Dolbeault isomorphism, the left-hand side is $\dim \overline{P}_M(Z/X)$ the theorem is proved.

## 5. Construction of the Shafarevich morphism

5.1. *Preliminary considerations.*

5.1.1. *Pure weight-one rational subspaces of $H^1(Z)$.* Let $Z$ be a complex projective variety. The possibly nonzero Hodge numbers of Deligne's mixed Hodge structure [Del71] on the first cohomology group $H^1(Z,\mathbb{Z})$ of the connected projective variety $Z$ are $h^{0,0}, h^{0,1}, h^{1,0}$.

In particular, we have an extension of $\mathbb{Q}$-MHS of a pure weight-one Hodge structure by a pure weight-zero Hodge structure:

$$0 \to W_0(H^1(Z,\mathbb{Q})) \to H^1(Z,\mathbb{Q}) \to \mathrm{Gr}_1^W(H^1(Z,\mathbb{Q})) \to 0. \tag{1}$$

Let $Z^{\mathrm{sn}} \to Z$ be the seminormalization of $Z$ (see [Kol96, Ch. I, Def. 7.2.1, p. 84 and the original references therein]). $H^1(Z) \to H^1(Z^{\mathrm{sn}})$ is an isomorphism of MHS since $Z^{\mathrm{sn}}(\mathbb{C}) \to Z(\mathbb{C})$ is a homeomorphism [Kol96, I.(7.2.1.1)].

Let $\mathcal{A}$ be a pure weight-one $\mathbb{Q}$-HS. There is an abelian variety (well defined up to isogeny) such that $H^1(A,\mathbb{Q}) = \mathcal{A}$.

Lemma 5.1. *Let $\widetilde{\phi} : \mathcal{A} \to H^1(Z,\mathbb{Q})$ be a morphism of* MHS. *Then there exist a rational number $r \neq 0$ and a morphism $\psi : Z^{\mathrm{sn}} \to A$ such that $r\widetilde{\phi} = H^1(\psi)$.*

We may as well assume $Z$ is seminormal. Assume moreover that $Z$ is a curve. Consider more generally $\phi : \mathcal{A} \to \mathrm{Gr}_1^W H^1(Z)$ a morphism $\mathbb{Q}$-HS of pure weight one.

Pulling back the extension (1) by the morphism $\phi$, we obtain an extension of $\mathbb{Q}$-MHS

$$0 \to W_0(H^1(Z,\mathbb{Q})) \to \mathcal{A}' \to \mathcal{A} \to 0. \tag{2}$$

*Proof.* Let $\nu : Z^\nu \to Z$ be the normalization of $Z$. Thanks to [Del71, lemme 10.3.1], the extension (1) is isomorphic to

$$0 \to W_0 \to H^1(Z) \xrightarrow{\nu^*} H^1(Z^\nu) \to 0.$$

Let $\gamma : [0,1] \to Z$ be a loop that is based at a singular point, meets the singular locus of $Z$ at finitely many points and is smooth outside these points. The preimage of $\gamma$ in $Z^\nu$ $\gamma$ is a finite union $\gamma_1, \ldots, \gamma_n$ of paths possibly lying in several connected components of $Z^\nu$. This defines a linear form $\int_\gamma : \omega \mapsto \sum_i \int_{\gamma_i} \omega$ on $H^0(Z^\nu, \Omega^1)$ and, upon composition with $\phi$, a linear form $\phi^* \int_\gamma$ on $\mathcal{A}^{1,0}$.

It follows from [Car87, Th. (1.13)] — see also the enlightening Example (1.17) — that (2) is split if and only if for every $\gamma$ as above, $\phi^* \int_\gamma$ is a rational multiple of a period of $\mathcal{A}$, i.e., lies in the image of $H_1(A,\mathbb{Q})$.

The datum $\widetilde{\phi}$ gives actually such a splitting, and the Abel-Jacobi construction gives a continuous mapping $Z \to A$ with the required property which is holomorphic when pulled back to $Z^\nu$. Since $Z$ is seminormal, this continuous mapping actually underlies a morphism.

The general case readily follows from the curve case. Assume first $Z$ is irreducible. Let $\lambda : Z^\nu \to Z$ be the normalisation of $Z$. Then we can construct a morphism $\psi^\nu : Z^\nu \to A$ and an integer $d$ such that $dH^1(\lambda) \circ \tilde{\phi} = H^1(\psi)$. This morphism is locally constant on the fibers of $\lambda : Z^\nu \to Z$. On the other hand, we can always find a connected curve $C$ passing through each connected component of a given positive-dimensional fiber $F$ of $\lambda$. Consider $C^{\mathrm{sn}} \to C$ the seminormalization of $C$. This is a homeomorphism that identifies $H^1(C)$ and $H^1(C^{\mathrm{sn}})$ with their respective mixed Hodge structures. The morphism $\psi^\nu|_C : C^{\mathrm{sn}} \to A$ is isogenous to the one predicted by Lemma 5.1, applied to $C$ and the resulting $\tilde{\phi}_C : \mathcal{A} \to H^1(C)$. Let $C'$ be the image of $C$ in $Z$. Since $\tilde{\phi}_C$ factors through $H^1(C')$, it follows that $\psi^\nu|_C$ is constant on the finite fiber of $C^{\mathrm{sn}} \to C'$. Hence $\psi^\nu$ assume the same value on all connected components of $F$. Hence it descends to a morphism $\psi : Z \to A$ since $Z$ is seminormal.

In general, if $Z$ has $m$ irreducible components, there are $m-1$ constants of integration to take care of, and a connected curve in $Z$ meeting every connected component of the smoooth locus will do the bookkeeping. □

5.1.2. *Period mappings for* $\mathbb{C}$-VMHS. $\mathbb{R}$-MHS have period domains and $\mathbb{R}$-VMHS period mappings generalizing those constructed by Griffiths for $\mathbb{R}$-VHS ([Usu83]; see also [Car87]).[3]

Recall that $X$ is a complex projective manifold and let $(X, \mathbb{V}, \mathcal{F}^\bullet, S)$ be a $\mathbb{R}$-VHS of weight zero. Let $M$ be the real Zariski closure of its monodromy group computed at some base point $x \subset X$. Let $U \subset M$ be the isotropy group of the Hodge filtration on $\mathbb{V}_x$. Then the period domain of $\mathbb{V}$ is the complex manifold $D(\mathbb{V}) := M/U$. It is endowed with a certain horizontal distribution, which can be described in terms of the Hodge structure on the Lie algebra $\mathfrak{m}$ of $M$. It is actually the actually the period domain attached to the Hodge semisimple group $M^{\mathrm{ad}}$. Furthermore, $D(\mathbb{V})$ is a moduli space of Hodge structures on $M$; see [GS69] for more details.

Let $(X, \mathbb{V}, \mathbb{W}_\bullet, \mathcal{F}^\bullet, (S_k)_{k\in\mathbb{Z}})$ be a $\mathbb{R}$-VMHS. Again we have a period domain $MD(\mathbb{V})$ for this variation and a holomorphic fibration of period domains $\psi : MD(\mathbb{V}) \to \prod_k D(\mathrm{Gr}_k^{\mathbb{W}}\mathbb{V})$ which is compatible to the horizontal distributions.

The domain $MD(\mathbb{V})$ is a homogeneous space of the form $H/U'$, where $H$ is the subgroup of $W_0\mathrm{GL}(\mathbb{V}_x)$ mapping to $\prod_k M(S_k)$ under the natural surjection $W_0\mathrm{GL}(\mathbb{V}_x) \to \mathrm{GL}(\mathrm{Gr}_0^{\mathbb{W}}\mathbb{V}_x)$.

[3]Actually, $\mathbb{C}$-VMHS also have period mappings of their own, but since this would not give additional information, we will stick to the usual conventions used in the literature.

Accordingly, there is an equivariant holomorphic horizontal period mapping $\phi_{\mathbb{V}} : \widetilde{X^{\mathrm{univ}}} \to MD(\mathbb{V})$ with a commutative diagram

$$\begin{array}{ccc} \widetilde{X^{\mathrm{univ}}} & \xrightarrow{\phi_{\mathbb{V}}} & MD(\mathbb{V}) \\ & \searrow & \downarrow \psi \\ & & \prod_k D(\mathrm{Gr}_k^{\mathbb{W}}\mathbb{V}). \end{array}$$

Let $\mathbb{M}$ be a $\mathbb{R}$-VMHS of weights $-1, 0$ and $MD$ be its period domain. Let $D$ be product of the the period domains corresponding to the graded parts of $\mathbb{M}$. The map $MD \to D$ is then an affine bundle.

The following lemma can be extracted from [Car87, p. 200].

LEMMA 5.2. *$MD \to D$ is a holomorphic vector bundle. The fiber $V(H_{-1}, H_0)$ of $MD \to D$ at $(H_{-1}, H_0)$ is canonically isomorphic to $\mathrm{Hom}(H_0, H_{-1})_{\mathbb{C}}/F^0$ where $\mathrm{Hom}(H_0, H_{-1})$ is endowed of its natural Hodge structure of weight $-1$.*

Consider $f : Z \to X$ a morphism such that $f^*\mathrm{Gr}_i\mathbb{M}$ is a VHS with trivial monodromy. Let $P_{\mathbb{M}} = \sum_i P_{f^*\mathrm{Gr}_i\mathbb{M}}(Z/X) \subset H^1(Z)$. Then we have the following lemma.

LEMMA 5.3. *There is a commutative diagram*

$$\begin{array}{ccc} \widetilde{Z^{\mathrm{univ}}} & \longrightarrow & (P^{1,0}_{\mathbb{M}})^* \\ & \searrow & \downarrow g_{\mathbb{M}} \\ & & V(H_{-1}, H_0), \end{array}$$

*where $g_{\mathbb{M}}$ is linear and injective and, when $Z$ is smooth, the horizontal map is given by integration of closed holomorphic forms.*

*Proof.* The proof is straightforward and is left to the reader. The case when $\mathbb{V} = \mathbb{C}$ is standard, and the general case follows by the same reasoning. □

5.2. *Proof of Theorem* 1.

5.2.1. *Notation.* In what follows, $M = M_B(X, G)$, where $G$ is a reductive group defined over $\mathbb{Q}$.

LEMMA 5.4. *There exists an object $\mathbb{M}_1$ of $\mathcal{T}_M^{\mathrm{MVHS}}(1)$ such that for every $f : Z \to X$ for which $\pi_1(Z) \to \Gamma_M$ is trivial, we have that $P_{\mathbb{M}_1} = P_M(Z/X)$ and that $g_{\mathbb{M}_1}$ is injective.*

*Proof.* Take

$$\mathbb{M}_1 := \sum_{\sigma\in S} \left(\mathbb{D}_1(\mathbb{V}_\sigma) + \overline{\mathbb{D}_1(\mathbb{V}_\sigma)}\right),$$

where $S \subset \mathcal{T}_M^{\mathrm{VHS}}$ is the finite set constructed in Lemma 4.8, and $\mathbb{D}_1(\mathbb{V}_\sigma)$ is the $\mathbb{C}$-VMHS from Definition 2.11. □

Let $\widetilde{X_M^k}$ be the covering space of $X$ defined as $\widetilde{X^{\mathrm{univ}}}/\widetilde{H_M^k}$. This covering is Galois with Galois group $\Gamma_M^k$.

Consider the local systems that belong to the finite set $S$ in $\mathcal{T}_M^{\mathrm{VHS}}$ from Lemma 4.8. Without loss of generality, we may assume that they underly real VHS of weight zero. Every $\rho$ in $S$ underlies a Zariski dense representation $\pi_1(X) \to G_\rho$, where $G_\rho$ is a real Lie group of Hodge type. Let $\rho_S : \pi_1(X) \to G_S = \prod_{\rho\in S} G_\rho$ be the direct sum representation.

5.2.2. *Construction of the Shafarevich morphism in case $k = 1$.* In this paragraph, we assume that $k = 1$. Choose a finite-dimensional real representation as in Lemma 5.4 of $\mathcal{G}_S^1(\mathbb{R})$ such that the associated local system $\mathbb{W}(1)$ underlies a graded polarizable real variation of mixed Hodge structure with the finite weight filtration $0 = \mathbb{W}_{-2} \subset \mathbb{W}_{-1} \subset \mathbb{W}_0 = \mathbb{W}(1)$.

Associated with $\mathbb{W}(1)$, we have a holomorphic Griffiths' transversal period mapping $q_S^1 : \widetilde{X^{\mathrm{univ}}} \to \mathcal{D}_S^1$, where $\mathcal{D}_S^1$ is is the corresponding period domain for MHS. The period domain $\mathcal{D}_S^1$ has a holomorphic fibration $\pi : \mathcal{D}_S^1 \to \mathcal{D}_S$, which makes it an affine fibration over the period domain $\mathcal{D}_S$. The composition $\pi \circ q_S^1$ is the period mapping for the associated graded object of $\mathcal{T}_M^{\mathrm{VHS}}$.

The map $q_S^1$ factors through a holomorphic horizontal map $Q_M^1 : \widetilde{X_M^1} \to \mathcal{D}_S^1$. Consider the holomorphic map $q_S : \widetilde{X_M^1} \xrightarrow{Q_M^1 \times \mathrm{Sh}_M} \mathcal{D}_S^1 \times \mathrm{Sh}_M(X)$.

Lemma 5.5. *Every connected component of a fiber of $q_S$ is compact.*

*Proof.* Such a component $\Phi$ is contained in the lift of some fiber $Z$ of $X \to \mathrm{Sh}_M(X)$. Replacing $Z$ by an étale cover, we may assume $\rho(\pi_1(Z)) = \{e\}$ whenever the conjugacy class of the reductive representation $\rho$ is in $M$. Hence $\Phi$ is a connected component of a fiber of the map $q'$ defined as $q_S$ restricted to $\widetilde{Z_M^1} = \widetilde{Z^{\mathrm{univ}}}/\ker(\pi_1(Z) \to \widetilde{\Gamma_M^1})$.

Now $\pi \circ q'$ is the constant map and $\Phi$ is a connected component of a fiber of an holomorphic map $\psi : \widetilde{Z_M^1} \to V$, where $V$ is a complex vector space that is a fiber of $\pi$.

Apply Lemma 5.1 to $X = Z$ and $\mathcal{A} = P_M(Z/X)$. The rationality hypothesis is fulfilled thanks to Theorem 4.4. We find a map to an abelian variety $Z^{\mathrm{sn}} \to A$, and using the universal covering space $P_M^{1,0}(Z/X)^* \to A$ of $A$ we can lift this map to a proper holomorphic map $\psi' : \widetilde{Z_M^1}^{\mathrm{sn}} \to P_M^{1,0}(Z/X)^*$.

Our claim follows from the fact that we have a commutative diagram

$$\begin{array}{ccc} \widetilde{Z^1_M}^{\mathrm{sn}} & \xrightarrow{\psi} & P^{1,0}_M(Z/X)^* \\ \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle i} \\ \widetilde{Z^1_M} & \xrightarrow{\psi} & V, \end{array}$$

where $s$ is the seminormalization and $i$ is an injective linear map. □

Next, recall the following classical result.

LEMMA 5.6 ([Car79, vol. 2, pp. 797–811]). *Let $X, S$ be two complex spaces and $f : X \to S$ a morphism. Assume a connected component $F$ of a fiber of $f$ is compact. Then, $F$ has a neighborhood $V$ such that $g(V)$ is a local analytic subvariety of $S$ and $V \to g(V)$ is proper.*

*Furthermore, assume any connected component of a fiber of $f$ is compact and $X$ and $S$ are normal. Then, the set $\bar{S}$ of connected components of a fiber of $f$ can be endowed with a structure of normal complex space such that the quotient mapping $e : X \to \bar{S}$ is holomorphic, proper, with connected fibers.*

Using this lemma, we construct a surjective proper holomorphic mapping with connected fibers to a normal complex space $r^1_M : \widetilde{X^1_M} \to \widetilde{S^1_M}(X)$ such that its fibers are precisely the connected components of the fibers of $q$. Since $q$ is $\Gamma^1_M$-equivariant, it follows that $r^1_M$ is $\Gamma^1_M$-equivariant too. Note that $\Gamma^1_M$ acts on $\widetilde{S}^1_M(X)$ in a proper discontinuous fashion and hence has at most finite stabilizers.

LEMMA 5.7. *The fibers of $r^1_M$ are precisely the maximal compact connected analytic subvarieties of $\widetilde{X^1_M}$.*

*Proof.* It is enough to show that whenever $Z$ is a connected compact analytic subvariety of $\widetilde{X^1_M}$, $r^1_M$ is constant. Fix such a $Z$.

The map $f : Z \to X$ has the property that the group homomorphism $\pi_1(Z) \to \Gamma^1_M$ induced by $\pi_1(f)$ has finite image. Let $Z'$ be a connected étale cover of $Z$ such that $\pi_1(Z') \to \Gamma^1_M$ is trivial. By abuse of notation, we write $f : Z' \to X$ for the resulting map. Then, for every representation $\rho$ in $M$, $f^*\rho$ is trivial and for every object $\mathbb{V}$ of $\mathcal{T}^{\mathrm{VHS}}_M$, the restriction map $H^1(X, \mathbb{V}) \to H^1(Z', \mathbb{V})$ is zero. This implies, through the proof of Lemma 5.5 that $q$ is constant on $Z'$ and thus $r^1_M$. □

*Remark.* In fact, it can be shown that $\widetilde{S^1_M}(X)/\Gamma^1_M$ is a normal algebraic variety. This follows from recent work of G. Pearlstein but is not used in the the main theorem, and so we will not discuss it here.

5.2.3. *Stein property in the case $k = 1$.*

PROPOSITION 5.8. *$\widetilde{X^1_M}$ is holomorphically convex and $r^1_M$ is its Cartan-Remmert factorization.*

*Proof.* Consider the natural period mapping $\widetilde{S_M}(X) \to \mathcal{D}_S$ and the affine bundle $V_S(X) = \widetilde{S_M}(X) \times_{\mathcal{D}_S} \mathcal{D}^1_S \to \widetilde{S_M}(X)$. The previous consideration implies that $\widetilde{S^1_M}(X) \to V_S(X)$ is proper and finite to one, hence finite.

An affine bundle over a Stein space is Stein [Ser53, p. 68]. In particular $V_S(X)$ is Stein. Hence $\widetilde{S^1_M}(X)$ is Stein. □

5.2.4. *General case.*

THEOREM 5.9. *Let $\widetilde{X} = \widetilde{X^{\mathrm{univ}}}/\Gamma$ be a Galois covering space of $X$ with $\widetilde{H^\infty_M} \subset \Gamma \subset \widetilde{H^1_M}$. Then $\widetilde{X}$ is holomorphically convex.*

*Proof.* Consider the map $q : \widetilde{X} \to \widetilde{S^1_M}(X)$. We claim that every connected component $\Phi$ of a fiber of $q$ is compact. Indeed, $\Phi$ has to be a connected lift of a projective variety $Z \subset X$ which is mapped to a point in $S^1_M(X)$. Replacing $Z$ by an étale cover, we may assume $\pi_1(Z) \to \Gamma^1_M$ is trivial; hence, $\mathrm{Im}(\pi_1(Z) \to \pi_1(X)) \subset \Gamma$ by Proposition 3.6. This implies that $\Phi$ is compact.

In particular, we may construct its Stein factorization $\widetilde{X} \to \widetilde{S}$, and it follows from the previous argument that $p : \widetilde{S} \to \widetilde{S^1_M}(X)$ has the following property.

LEMMA 5.10. *Every point $x \in \widetilde{S^1_M}(X)$ has a neighborhood $U$ such that $p^{-1}(U)$ is the disjoint union of open sets $V$, and $p|_V$ is a quotient map by a finite group $G$.*

This certainly implies that $\widetilde{S}$ is Stein. In fact, the finite group in question is $\ker(\pi_1((r^1_M)^{-1}(x)) \to \Gamma/\Gamma^1_M)$ and injects into the real points of a prounipotent proalgebraic group. It is thus a trivial group; hence, $\widetilde{S} \to \widetilde{S^1_M}(X)$ is a topological covering map. □

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