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Addendum to: Subelliptic $\mathrm{Spin}_{\mathbb{C}}$ Dirac operators, III

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Abstract

We prove the relative index conjecture, which in turn implies that the set of embeddable deformations of a strictly pseudoconvex CR-structure on a compact 3-manifold is closed in the $\mathcal{C}^\infty$-topology.

1. Proof of the Relative Index Conjecture

Let Y denote an oriented, compact, 3-dimensional manifold, with $H \subset TY$ a plane field, defining a contact structure. A strictly pseudoconvex CR-structure on Y is defined by a complex structure on the fibers of H, which we can represent as the bundle of $-i$-eigenspaces, denoted $T_b^{0,1}Y$. The CR-structure, in turn, defines a differential operator,

$$\bar{\partial}_b f = df \restriction_{T_b^{0,1}Y} . \tag{1}$$

The space of CR-functions on Y is the null-space of $\bar{\partial}_b$. A Szegő projector is an L^2-orthogonal projection onto the L^2-closure of the $\ker \bar{\partial}_b$, defined by the choice of a smooth, nondegenerate density on Y. None of our results depend upon the choice of this density.

A CR-structure is embeddable, or fillable if the $\ker \bar{\partial}_b$ contains sufficiently many functions to embed Y into $\mathbb{C}^N$ for some N. This is equivalent to the requirement that the CR-manifold $(Y, T_b^{0,1}Y)$ arises as the boundary of a compact normal Stein space; see pp. 4 and 5 of [2].

Recall that the deformations of a reference CR-structure, ${}^0T_b^{0,1}Y$, on (Y, H) are parametrized by

$$\mathrm{Def}(Y, H, \mathcal{S}_0) = \{\Phi \in \mathcal{C}^\infty(Y; \mathrm{Hom}({}^0T_b^{0,1}Y, {}^0T_b^{1,0}Y)) : \|\Phi\|_{L^\infty} < 1\} \tag{2}$$

via the prescription

$${}^\Phi T_{b,y}^{0,1}Y = \{\overline{Z}_y + \Phi_y(\overline{Z}_y) : \overline{Z}_y \in {}^0T_{b,y}^{0,1}Y\}. \tag{3}$$

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Here and in the sequel we often use the Szegő projector (instead of Φ) to label a CR-structure. From now on we assume that the reference CR-structure, with Szegő projector $\mathcal{S}_0$, is fillable.

Let $\mathcal{E} \subset \operatorname{Def}(Y, H, \mathcal{S}_0)$ be the subset consisting of the fillable deformations. In Theorem A of [2], [3] we showed that if $\mathcal{S}_0$ is the Szegő projector defined by the (fillable) reference CR-structure and $\mathcal{S}_1$ that defined by a deformation, then the deformed structure is fillable if and only if the restriction

$$\mathcal{S}_1 : \operatorname{Im} \mathcal{S}_0 \longrightarrow \operatorname{Im} \mathcal{S}_1 \tag{4}$$

is a Fredholm operator. Let R-Ind$(\mathcal{S}_0, \mathcal{S}_1)$ denote its Fredholm index, which we call the *relative index*. For each $m \in \mathbb{N} \cup \{0\}$ and any $\delta > 0$, let

$$\mathfrak{S}_m^\delta = \Big\{ \mathcal{S}_1 \in \operatorname{Def}(Y, H, \mathcal{S}_0) : \ -\infty < \text{R-Ind}(\mathcal{S}_0, \mathcal{S}_1) \leq m \\ \text{and } \|\Phi\|_{L^\infty}^2 \leq \frac{1}{2} - \delta \Big\}. \tag{5}$$

Proposition 10.1 in [2] shows that there is an integer k_0, so that if a sequence $\langle \Phi_n \rangle \subset \mathfrak{S}_m^\delta$ converges to Φ in the $\mathcal{C}^{k_0}$-norm, then the structure defined by Φ is fillable.

In this addendum to [5], we show how the formula for the relative index between the Szegő projectors $\mathcal{S}_0, \mathcal{S}_1$, defined by two fillable CR-structures on a contact 3-manifold (Y, H), gives a proof of the Relative Index Conjecture.

THEOREM 1. *Let (Y, H) be a compact 3-dimensional co-oriented, contact manifold, and let $\mathcal{S}_0$ be the Szegő projector defined by an fillable CR-structure on Y, with underlying plane field H. There is a nonnegative integer M such that for the Szegő projector $\mathcal{S}_1$ defined by any fillable deformation of the reference structure, with underlying plane field, H, we have the upper bound*

$$\text{R-Ind}(\mathcal{S}_0, \mathcal{S}_1) \leq M. \tag{6}$$

Combining (6) with Proposition 10.1 of [2] we prove

COROLLARY 1. *Under the hypotheses of Theorem 1, the set of fillable deformations of the CR-structure on Y is closed in the $\mathcal{C}^\infty$-topology.*

Proof of the corollary. Suppose that $\langle \Phi_n \rangle$ is a sequence of fillable deformations in $\mathcal{E} \subset \operatorname{Def}(Y, H, \mathcal{S}_0)$ converging to $\Phi \in \operatorname{Def}(Y, H, \mathcal{S}_0)$, in the $\mathcal{C}^\infty$-topology. Recall that, by definition, $\|\Phi\|_{L^\infty} < 1$.

Let Ψ_1 and Ψ_2 be deformations of the reference structure, with local representations

$$\Psi_j = \psi_j Z \otimes \bar{\omega}, \tag{7}$$

where Z locally spans ${}^0T_b^{1,0}Y$ and $\bar{\omega}$ is the $(0,1)$-form dual to $\overline{Z}$; see page 12 in [2]. The analogous local coordinate representation of Ψ_2 as a deformation of Ψ_1 is given by

$$\psi_{21} = \frac{\psi_2 - \psi_1}{1 - \overline{\psi_1}\psi_2}; \tag{8}$$

see equation (5.5) in [2]. We can represent Φ as a deformation of any of the structures in the sequence. From equation (8) it is clear that there is an integer N so that, as deformations of Φ_N, a tail of the sequence and its limit lie in the L^∞-ball in $\mathrm{Def}(Y, H, \mathcal{S}_N)$, centered at 0, of radius $\frac{1}{4}$. Theorem 1 shows that there is an M so that

$$\text{R-Ind}(\mathcal{S}_N, \mathcal{S}_n) \leq M \text{ for all } n \in \mathbb{N}. \tag{9}$$

Proposition 10.1 from [2] then implies that the limiting structure Φ is also fillable, completing the proof of the corollary. □

Before proving Theorem 1 we recall the formula for the relative index, which is Theorem 13 in [5]. This formula involves topological and analytic invariants, which we now define, of the complex manifolds that fill the pair of CR-structures. Let X be a 4-dimensional manifold with boundary, and let $\widehat{H}^2(X)$ denote the image of $H^2(X, bX)$ in $H^2(X)$ under the natural map. The signature of the nondegenerate quadratic form on $\widehat{H}^2(X)$, defined by

$$([\alpha], [\beta]) \mapsto \int_X \alpha \wedge \beta, \tag{10}$$

is denoted $\mathrm{sig}[X]$, and $\chi[X]$ is the topological Euler characteristic

$$\chi[X] = \sum_{j=0}^{4} b_j(X)(-1)^j, \text{ where } b_j(X) = \dim H_j(X; \mathbb{Q}). \tag{11}$$

The final element needed for the proof of Theorem 1 is the relative index formula itself.

THEOREM 2. *Let (Y, H) be a compact 3-dimensional co-oriented, contact manifold, and let $\mathcal{S}_0, \mathcal{S}_1$ be Szegő projectors for fillable CR-structures with underlying plane field H. Suppose that (X_0, J_0), (X_1, J_1) are strictly pseudoconvex complex manifolds with boundary $(Y, H, \mathcal{S}_0)$, $(Y, H, \mathcal{S}_1)$, respectively. Then*

$$\text{R-Ind}(\mathcal{S}_0, \mathcal{S}_1) = \dim H^{0,1}(X_0, J_0) - \dim H^{0,1}(X_1, J_1) + \frac{\mathrm{sig}[X_0] - \mathrm{sig}[X_1] + \chi[X_0] - \chi[X_1]}{4}. \tag{12}$$

If $(Y, T_b^{0,1}Y)$ is fillable, then the normal Stein space, X, that it bounds is unique. By the definition of a normal singularity, the algebra of CR-functions on $(Y, T_b^{0,1}Y)$ is isomorphic to the algebra of holomorphic functions on X. If $\widehat{X}$ is obtained from X by resolving the singularities, then the algebras of holomorphic functions on X and $\widehat{X}$ are isomorphic, and therefore the Szegő projector defined by this CR-structure is the projection onto the boundary values of holomorphic functions on X, or any resolution of X.

Proof of Theorem 1. Recall that $\mathcal{S}_0, \mathcal{S}_1$ are Szegő projectors defined by fillable CR-structures on (Y, H). We let X_0 and X_1 denote complex manifolds with strictly pseudoconvex boundaries, obtained as the minimal resolutions of the normal Stein spaces bounded by $(Y, \mathcal{S}_0)$ and $(Y, \mathcal{S}_1)$ respectively. In Theorem 2′ of [1], Bogomolov and De Oliveira prove that there are small perturbations of the complex structures on X_0 and X_1 making them into Stein manifolds. Hence it follows that X_0 and X_1, with the deformed complex structures, have strictly plurisubharmonic exhaustion functions. Therefore both X_0 and X_1 have the homotopy type of 2-dimensional CW-complexes. This implies that the Betti numbers $b_3(X_i)$ and $b_4(X_i)$ are zero.

The long exact sequence of the pair (X_i, bX_i) in homology, reads, in part

$$\cdots \longrightarrow H_1(bX_i) \longrightarrow H_1(X_i) \longrightarrow H_1(X_i, bX_i) \longrightarrow \cdots . \tag{13}$$

Poincaré-Lefschetz duality states that $H_1(X_i, bX_i) \simeq H^3(X_i)$, for $i = 0, 1$. As X_0 and X_1 have the homotopy type of 2-complexes, and the singular cohomology groups are homotopy invariant, it follows that $H^3(X_i) = 0$, and therefore, as $bX_i = Y$,

$$\dim H_1(X_i) \leq \dim H_1(Y), \text{ for } i = 0, 1; \tag{14}$$

see also page 328 in [9]. Poincaré-Lefschetz duality implies the isomorphism $H^2(X_i, bX_i) \simeq H_2(X_i)$. If $b_2^+(X_i)$ $(b_2^-(X_i))$ is the dimension of the maximal subspace on which the pairing in (10) is positive definite (negative definite), and $b_2^0(X_i)$ is the dimension of the null-space of the map $H^2(X_i, bX_i) \to H^2(X_i)$, then we see that

$$\dim H_2(X_i) = b_2(X_i) = b_2^+(X_i) + b_2^-(X_i) + b_2^0(X_i) = \dim H^2(X_i, bX_i) \text{ and } \operatorname{sig}[X_i] = b_2^+(X_i) - b_2^-(X_i). \tag{15}$$

Taking advantage of these facts we can rewrite the formula in (12) as

$$\text{R-Ind}(\mathcal{S}_0, \mathcal{S}_1) = C_0 - C_1, \tag{16}$$

where C_i denotes the contribution of the terms from X_i:

$$C_i = \dim H^{0,1}(X_i, J_i) + \frac{2b_2^+(X_i) + b_2^0(X_i) - b_1(X_i)}{4}. \tag{17}$$

From equations (16) and (17), and the fact that $b_1(X_1) \leq b_1(Y)$, we conclude that

$$\text{R-Ind}(\mathcal{S}_0, \mathcal{S}_1) \leq C_0 + \frac{b_1(Y)}{4}. \tag{18}$$

This completes the proof of the theorem. □

1.1. *A new proof of Lempert's stability.* It is a consequence of Theorem D in [2] that $\text{R-Ind}(\mathcal{S}_0, \mathcal{S}_1) \geq 0$ for sufficiently small deformations. If $Y = S^3$ and $X_0 \subset \mathbb{C}^2$ is diffeomorphic to the 4-ball, then (14) shows that $b_1(X_0) = b_1(X_1) = 0$ and $C_0 = 0$ in (16). The relative index formula takes the very simple form:

$$\text{(19)} \qquad \text{R-Ind}(\mathcal{S}_0, \mathcal{S}_1) = -\left[\dim H^{0,1}(X_1, J_1) + \frac{2b_2^+(X_1) + b_2^0(X_1)}{4}\right].$$

The nonnegativity of $\text{R-Ind}(\mathcal{S}_0, \mathcal{S}_1)$ for small deformations and (19) show, in the present circumstance that for small deformations, the relative index $\text{R-Ind}(\mathcal{S}_0, \mathcal{S}_1)$ must vanish. When this is so, then a small extension of the results in Section 5 of [2] shows that for any nonnegative integer k, there is an integer l_k and a constant M_k so that the $\mathcal{C}^k$-operator norm of the difference, $\|\mathcal{S}_0 - \mathcal{S}_1\|_{\mathcal{C}^k}$, is bounded by $M_k\|\Phi\|_{\mathcal{C}^{l_k}}$. Here Φ is the deformation tensor for the CR-structure defining $\mathcal{S}_1$ as a deformation of that defining $\mathcal{S}_0$.

The coordinate functions $z_1 \restriction_{bX_0}, z_2 \restriction_{bX_0}$ define a CR-embedding of $(Y, \mathcal{S}_0)$ into $\mathbb{C}^2$. By definition of the Szegő projector, the functions

$$\text{(20)} \qquad \varphi_i = \mathcal{S}_1[z_i \restriction_{bX_0}], \text{ for } i = 1, 2,$$

are CR-functions relative to the deformed structure. If $\|\mathcal{S}_0 - \mathcal{S}_1\|_{\mathcal{C}^1}$ is sufficiently small, then $y \mapsto (\varphi_1(y), \varphi_2(y))$ defines a CR-embedding of $(Y, {}^{\Phi}T^{0,1}Y)$ into $\mathbb{C}^2$, which is a $\mathcal{C}^1$-small deformation of bX_0. This completes the proof of the following proposition.

PROPOSITION 1. *Suppose that X_0 is an embedding of the standard 4-ball into $\mathbb{C}^2$ with a smooth strictly pseudoconvex boundary diffeomorphic to S^3. There is an $\varepsilon > 0$ and an l so that any embeddable deformation of the induced CR-structure on bX_0 with deformation tensor Φ, satisfying $\|\Phi\|_{\mathcal{C}^l} < \varepsilon$, arises as a small deformation of bX_0 in $\mathbb{C}^2$.*

This gives a new proof of a generalization of Lempert's first stability theorem, Theorem 4.5 in [6]. Lempert's original result assumes that X_0 is a strictly linearly convex domain. He uses the existence of "inner and outer S^1-actions" to verify that the deformed structure can be embedded as a small perturbation of the reference structure. In particular, Lempert's argument makes extensive usage of a "pseudoconcave cap" to compactify the deformed Stein space. This type of compactification is not needed for our analysis, but our results also say nothing about the existence of inner S^1-actions.

Suppose that X_0 is strictly linearly convex. As noted above, if the deformation tensor is sufficiently small in the $\mathcal{C}^{l_2}$-norm, then the $\mathcal{C}^2$-operator norm of the difference $\|\mathcal{S}_0 - \mathcal{S}_1\|_{\mathcal{C}^2}$ will also be small. From this it follows, as in [6], that the deformed structure has an embedding that is also strictly linearly convex. In a subsequent paper, [7], Lempert removed the hypothesis of strict linear convexity and extended his stability result to the boundaries of smoothly bounded, strictly pseudoconvex domains in $\mathbb{C}^2$.

1.2. *Remarks on the Ozbagci-Stipsicz Conjecture.* As noted above, $\operatorname{sig}[X_1] + b_2(X_1) = 2b_2^+(X_1) + b_2^0(X_1)$. A global bound on $|\,\text{R-Ind}(\mathcal{S}_0, \mathcal{S}_1)|$, among all Szegő projectors $\mathcal{S}_1$ defined by elements of $\mathcal{E}$, is therefore equivalent to an upper bound for the quantity

$$b_2^+(X_1) + b_2^0(X_1) + \dim H^{0,1}(X_1),$$

among all Stein spaces, X_1 filling (Y, H). The existence of an upper bound on $b_2^+(X_1) + b_2^0(X_1)$ was conjectured by Ozbagci and Stipsicz, and proved in some special cases; see [9].

The fact, noted above, that $\text{R-Ind}(\mathcal{S}_0, \mathcal{S}_1) \geq 0$, for sufficiently small deformations and (16) show that for such deformations,

$$\begin{aligned} \dim H^{0,1}(X_1) + \frac{2b_2^+(X_1) + b_2^0(X_1)}{4} \\ \leq \dim H^{0,1}(X_0) + \frac{2b_2^+(X_0) + b_2^0(X_0) + b_1(Y) - b_1(X_0)}{4}. \end{aligned} \tag{21}$$

On page 328 of [9], Stipsicz proves the existence of a constant $K_{(Y,H)}$ (which may be positive or negative) so that for any Stein filling of (Y, H), we have the estimate

$$b_2^-(X_1) \leq 5b_2^+(X_1) + 2 - K_{(Y,H)} + 2b_1(Y). \tag{22}$$

These estimates, along with (15) and (21), prove a "germ" form of the Ozbagci–Stipsicz conjecture.

Proposition 2. *With (Y, H) as above, let $\mathcal{S}_0$ be a fillable reference* CR-*structure. Among sufficiently small, fillable deformations of this* CR-*structure the set of numbers*

$$\{b_1(X_1), \operatorname{sig}(X_1), \chi(X_1)\}$$

is finite. Here X_1 ranges over the minimal resolutions of the normal Stein spaces bounded by the deformed structures $(Y, H, \mathcal{S}_1)$.

The notion of smallness here depends on the size of the gap at 0 in the spectrum of the $\Box_b$-operator of the reference CR-structure. This can vary quite dramatically from fillable structure to fillable structure, which is why we call this a germ form of the Ozbagci–Stipsicz conjecture.

1.3. *Open problems and a possible strategy*: Our results suggest a strategy for proving a lower bound on $\text{R-Ind}(\mathcal{S}_0, \mathcal{S}_1)$, among deformations Φ with $\|\Phi\|_{L^\infty} < 1 - \varepsilon$, for an $\varepsilon > 0$. Suppose that no such bound exists, one could then choose a sequence $\langle \Phi_n \rangle \subset \mathcal{E}$ for which $\text{R-Ind}(\mathcal{S}_0, \mathcal{S}_n)$ tends to $-\infty$. A contradiction would follow immediately if we could show that $\langle\Phi_n\rangle$ is bounded in the $\mathcal{C}^{k_0+1}$-norm.

While such an *a priori* bound seems unlikely for the original sequence, it would suffice to replace the sequence $\langle\Phi_n\rangle$ with a "wiggle-equivalent" sequence. Let M_n denote a projective surface containing $(Y, {}^{\Phi_n}T_b^{0,1}Y)$ as a separating hypersurface; see Theorem 8.1 in [8]. An equivalent sequence with better regularity might be obtained by wiggling the hypersurfaces defined by $(Y, {}^{\Phi_n}T_b^{0,1}Y)$ within M_n, perhaps using some sort of heat-flow. After composing the resultant deformations with suitable contact transformations, we might be able to obtain a sequence $\langle\Phi_n'\rangle$ with $\text{R-Ind}(\mathcal{S}_0, \mathcal{S}_n') = \text{R-Ind}(\mathcal{S}_0, \mathcal{S}_n)$ that does satisfy an *a priori* $\mathcal{C}^{k_0+1}$-bound. Such an argument would seem to require an improved understanding of the metric geometry of $\mathrm{Def}(Y, H, \mathcal{S}_0)$, as well as the relationship of an abstract deformation to the local extrinsic geometry of Y as a hypersurface in M_n.

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