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Căldăraru's conjecture and Tsygan's formality

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Abstract

In this paper we complete the proof of Căldăraru's conjecture on the compatibility between the module structures on differential forms over poly-vector fields and on Hochschild homology over Hochschild cohomology. In fact we show that twisting with the square root of the Todd class gives an isomorphism of precalculi between these pairs of objects.

Our methods use formal geometry to globalize the local formality quasi-isomorphisms introduced by Kontsevich and Shoikhet. (The existence of the latter was conjectured by Tsygan.) We also rely on the fact — recently proved by the first two authors — that Shoikhet's quasi-isomorphism is compatible with cap products after twisting with a Maurer-Cartan element.

Contents

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1. **Introduction and statement of the main results**

Throughout k is a ground field of characteristic 0. In this introduction $(X, \mathcal{O})$ is a ringed site[1] such that $\mathcal{O}$ is a sheaf of commutative k-algebras. In addition, we fix a Lie algebroid $\mathcal{L}$ over $(X, \mathcal{O})$.

[1]We work over sites instead of spaces to cover some additional cases that are important for algebraic geometry (like algebraic spaces and Deligne–Mumford stacks). Readers not interested in such generality may assume that $(X, \mathcal{O})$ is just a ringed space.

Roughly speaking a Lie algebroid is a sheaf of $\mathcal{O}$-modules that is also a sheaf of Lie algebras that acts on $\mathcal{O}$ by derivations. See Section 3.1. Standard examples of Lie algebroids are the tangent bundle on a smooth manifold and the holomorphic tangent bundle on a complex manifold. Readers not familiar with Lie algebroids are advised to think of $\mathcal{L}$ as a tangent bundle (holomorphic or not) for the rest of this introduction. Concepts like "connection" take their familiar meaning in this context. In fact, our main reason for working in the setting of Lie algebroids is that these allow us to treat the algebraic, holomorphic and C^∞-cases in a uniform way.

1.1. *The Atiyah and Todd class of a Lie algebroid.* From now on we make the additional assumption that the Lie algebroid $\mathcal{L}$ is locally free of rank d as an $\mathcal{O}$-module.

The Atiyah class $A(\mathcal{L}) \in \operatorname{Ext}^1(\mathcal{L}, \mathcal{L}^* \otimes \mathcal{L}) = \mathrm{H}^1(X, \mathcal{L}^* \otimes \operatorname{End}_{\mathcal{O}}(\mathcal{L}))$ of $\mathcal{L}$ may, for example, be defined as the obstruction against the existence of a global $\mathcal{L}$-connection on $\mathcal{L}$. See Section 6 for more details.

The i-th scalar Atiyah class $a_i(\mathcal{L})$ of $\mathcal{L}$ is defined as

$$a_i(\mathcal{L}) = \operatorname{tr}\Big(\bigwedge^i A(\mathcal{L})\Big) \in \mathrm{H}^i\Big(X, \bigwedge^i \mathcal{L}^*\Big),$$

where $\bigwedge^i$ is the map

$$\bigwedge^i : (\mathcal{L}^* \otimes \operatorname{End}(\mathcal{L}))^{\otimes i} \to \bigwedge^i \mathcal{L}^* \otimes \operatorname{End}(\mathcal{L})$$

given by composition on $\operatorname{End}(\mathcal{L})^{\otimes i}$ and the exterior product on $(\mathcal{L}^*)^{\otimes i}$ and where tr is the usual trace on $\operatorname{End}(\mathcal{L})$, extended linearly to a map $\bigwedge^i \mathcal{L}^* \otimes \operatorname{End}(\mathcal{L}) \to \bigwedge^i \mathcal{L}^*$.

The Todd class $\operatorname{td}(\mathcal{L})$ of $\mathcal{L}$ is derived from the Atiyah class $A(\mathcal{L})$ by the following familiar formula:

$$\operatorname{td}(\mathcal{L}) = \det\left(\frac{A(\mathcal{L})}{1 - \exp\left(-A(\mathcal{L})\right)}\right) \in \bigoplus_{i \geq 0} \mathrm{H}^i\Big(X, \bigwedge^i \mathcal{L}^*\Big), \tag{1.1}$$

where the function

$$q(x) = \frac{x}{1 - \exp(-x)} \tag{1.2}$$

is extended to $\bigwedge \mathcal{L}^* \otimes_{\mathcal{O}} \operatorname{End}(\mathcal{L})$ via its formal Taylor expansion. In this way the Todd class $\operatorname{td}(\mathcal{L})$ of $\mathcal{L}$ can be expressed in terms of the scalar Atiyah classes of $\mathcal{L}$.

1.2. *Gerstenhaber algebras and precalculi.* By definition a Gerstenhaber algebra is a graded vector space equipped with a Lie bracket $[-,-]$ of degree

0 and a commutative, associative cup product $\cup$ of degree 1[2] such that the Leibniz rule is satisfied

$$[a, b \cup c] = [a, b] \cup c + (-1)^{|a|(|b|+1)} b \cup [a, c].$$

If A is a Gerstenhaber algebra, then a precalculus [13] over A is a quadruple $(A, M, \imath, \mathrm{L})$ where M is a graded vector space and $\imath : A \otimes M \to M$ and $\mathrm{L} : A \otimes M \to M$ are linear maps of degree 1 and 0 respectively such that $\imath$ makes M into an $(A[-1], \cup)$-module and L makes M into an $(A, [-,-])$-Lie module and such that the following compatibilities hold for $a, b \in A$

$$\imath_a \mathrm{L}_b - (-1)^{(|a|+1)|b|} \mathrm{L}_b \imath_a = \imath_{[a,b]}, \tag{1.3}$$

$$\mathrm{L}_a \imath_b + (-1)^{|a|+1} \imath_a \mathrm{L}_b = \mathrm{L}_{a \cup b}. \tag{1.4}$$

A precalculus is *not* the same as a Gerstenhaber module. The second equation in the previous display is not correct for a Gerstenhaber module.

Below $\imath$ will be referred to as "contraction" and L as the "Lie derivative." Furthermore, we will often write $a \cap m$ for $\imath_a(m)$ and as such refer to it as the "cap product."

1.3. *Poly-vector fields, poly-differential operators, differential forms and Hochschild chains in the Lie algebroid framework.* For a Lie algebroid $\mathcal{L}$ the sheaves of $\mathcal{L}$-poly-vector fields and $\mathcal{L}$-differential forms are defined as

$$T^{\mathcal{L}}_{\mathrm{poly}}(X) = \bigoplus_{n \geq -1} \bigwedge^{n+1} \mathcal{L}, \qquad \Omega^{\mathcal{L}}(X) = \bigoplus_{n \leq 0} \bigwedge^{-n} \mathcal{L}^*,$$

where the wedge products are taken over $\mathcal{O}_X$.

The sheaf $T^{\mathcal{L}}_{\mathrm{poly}}(X)$ becomes a sheaf of Gerstenhaber algebras when endowed with the trivial differential, the Lie algebroid version of the Schouten–Nijenhuis Lie bracket and the exterior product. Our grading convention is such that the Lie bracket and wedge product are of degree 0 and 1 respectively.

We equip $\Omega^{\mathcal{L}}(X)$ with the trivial(!) differential,[3] and also with the contraction operator and Lie derivative with respect to $\mathcal{L}$-poly-vector fields. In this way the pair $(T^{\mathcal{L}}_{\mathrm{poly}}(X), \Omega^{\mathcal{L}}(X))$ becomes a sheaf of precalculi. In our conventions the contraction operator and Lie derivative have degrees 1 and 0 respectively.

The Lie algebroid generalization of the sheaf of $\mathcal{L}$-poly-differential operators is denoted by $D^{\mathcal{L}}_{\mathrm{poly}}(X)$ [27], [3]. It is the tensor algebra over $\mathcal{O}$ of the universal enveloping algebra of $\mathcal{L}$ (see §3.3 below).

[2]Note that our grading conventions are shifted with respect to the usual ones.

[3]The de Rham differential d_L on $\Omega^{\mathcal{L}}(X)$ is not part of the precalculus structure. In the operadic setting of [13], d_L appears as a unary operation and not as a differential.

The sheaf $D^{\mathcal{L}}_{\mathrm{poly}}(X)$ has properties similar to the standard sheaf of poly-differential operators on X (see, e.g., [16]). In particular it is a differential graded Lie algebra (shortly, from now on, a DG-Lie algebra) and also a Gerstenhaber algebra *up to homotopy*. For the definition of the differential, the Lie bracket (of degree 0) and the cup product (of degree 1), see Section 3.3.

The sheaf of $\mathcal{L}$-Hochschild chains $C^{\mathcal{L}}_{\mathrm{poly}}(X)$ may be defined as the $\mathcal{O}$-dual of $D^{\mathcal{L}}_{\mathrm{poly}}(X)$ (although we use a slightly different approach). Furthermore, there is a differential b_H as well as actions $\cap$, L of $D^{\mathcal{L}}_{\mathrm{poly}}(X)$ on $C^{\mathcal{L}}_{\mathrm{poly}}(X)$ that make the pair $(D^{\mathcal{L}}_{\mathrm{poly}}(X), C^{\mathcal{L}}_{\mathrm{poly}}(X))$ into a precalculus up to homotopy. We refer to Section 3.4 for more detail.

Finally, we recall that there is a Hochschild–Kostant–Rosenberg (HKR for short) quasi-isomorphism from $T^{\mathcal{L}}_{\mathrm{poly}}(X)$ to $D^{\mathcal{L}}_{\mathrm{poly}}(X)$; dually, there is a HKR quasi-isomorphism from $C^{\mathcal{L}}_{\mathrm{poly}}(X)$ to $\Omega^{\mathcal{L}}(X)$. As in the classical case where $\mathcal{L}$ is the tangent bundle neither of these HKR quasi-isomorphisms is compatible with the Gerstenhaber and precalculus structures up to homotopy.

1.4. *Main results.* Now we consider the derived category $D(X)$ of sheaves of k-vector spaces over X. When equipped with the derived tensor product this becomes a symmetric monoidal category. Furthermore, viewed as objects in $D(X)$, both $T^{\mathcal{L}}_{\mathrm{poly}}(X)$ and $D^{\mathcal{L}}_{\mathrm{poly}}(X)$ are honest Gerstenhaber algebras and their combination with $\Omega^{\mathcal{L}}(X)$ and $C^{\mathcal{L}}_{\mathrm{poly}}(X)$ yields precalculi.

Our first main result relates the Todd class of a Lie algebroid (as discussed in §1.1) to the failure of the HKR isomorphisms to preserve these precalculi structures.

THEOREM 1.1. *Let $\mathcal{L}$ be a locally free Lie algebroid of rank d over the ringed site $(X, \mathcal{O}_X)$. Then we have the following commutative diagram of precalculi in the category $D(X)$:*

$$\begin{tikzcd} T^{\mathcal{L}}_{\mathrm{poly}}(X) \arrow[r, "\mathrm{HKR}\circ\iota_{\sqrt{\mathrm{td}(\mathcal{L})}}"] \arrow[d, squiggly] & D^{\mathcal{L}}_{\mathrm{poly}}(X) \arrow[d, squiggly] \\ \Omega^{\mathcal{L}}(X) & C^{\mathcal{L}}_{\mathrm{poly}}(X), \arrow[l, "(\sqrt{\mathrm{td}(\mathcal{L})}\wedge -)\circ\mathrm{HKR}"] \end{tikzcd} \tag{1.5}$$

where the vertical arrows indicate actions *and the horizontal arrows are isomorphisms. Here $\wedge$ denotes the left multiplication in $\Omega^{\mathcal{L}}(X)$ and ι denotes the contraction action of $\Omega^{\mathcal{L}}(X)$ on $T^{\mathcal{L}}(X)$.*[4]

[4]Note that normally we view $\Omega^{\mathcal{L}}(X)$ as a module over $T^{\mathcal{L}}(X)$. In the definition of the horizontal arrows in the diagram (1.5) the opposite actions appear for reasons that are mysterious to the authors.

The convention that wavy arrows indicate actions will be used throughout this article.

The following corollary will be applied to Căldăraru's conjecture below.

Corollary 1.2. *There is a commutative diagram of precalculi*:

$$\begin{tikzcd} \bigoplus_{m,n\geq 0} \mathrm{H}^m(X, \bigwedge^n \mathcal{L}) \arrow[r, "\mathrm{HKR}\circ\iota_{\sqrt{\mathrm{td}(\mathcal{L})}}"] \arrow[d, rightsquigarrow] & \mathbb{H}^\bullet(X, D^{\mathcal{L}}_{\mathrm{poly}}(X)) \arrow[d, rightsquigarrow] \\ \bigoplus_{m,n\geq 0} \mathrm{H}^m(X, \bigwedge^n \mathcal{L}^*) & \mathbb{H}^\bullet(X, C^{\mathcal{L}}_{\mathrm{poly}}(X)), \arrow[l, "(\sqrt{\mathrm{td}(\mathcal{L})}\wedge -)\circ\mathrm{HKR}"'] \end{tikzcd} \tag{1.6}$$

with $\mathbb{H}^\bullet(X,-)$ *denoting the hypercohomology functor.*

Proof. This follows by applying the functor $\mathbb{H}^\bullet(X,-)$ to the commutative diagram (1.5). □

If we consider only the Lie brackets and the Lie algebra actions, then the horizontal isomorphisms in the commutative diagram (1.5) are obtained from the horizontal arrows in diagram (1.7), which is part of our second main result.

Theorem 1.3. *Assume that* $\mathbb{R}\subset k$. *Let* $\mathcal{L}$ *be a locally free Lie algebroid of rank* d *over the ringed site* $(X,\mathcal{O})$. *There exist sheaves of differential graded Lie algebras* $(\mathfrak{g}_i^{\mathcal{L}}, \mathrm{d}_i, [\ ,\ ]_i)$ *and sheaves of* DG-*Lie modules* $(\mathfrak{m}_i^{\mathcal{L}}, \mathrm{b}_i, \mathrm{L}_i)$ *over them as well as* L_∞-*quasi-isomorphisms* $\mathfrak{U}_{\mathcal{L}}$ *from* $\mathfrak{g}_1^{\mathcal{L}}$ *to* $\mathfrak{g}_2^{\mathcal{L}}$ *and* $\mathfrak{S}_{\mathcal{L}}$ *from* $\mathfrak{m}_2^{\mathcal{L}}$ *to* $\mathfrak{m}_1^{\mathcal{L}}$, *which fit into the following commutative diagram*:

$$\begin{tikzcd} T^{\mathcal{L}}_{\mathrm{poly}}(X) \arrow[r, hook] \arrow[d, rightsquigarrow, "\mathrm{L}"'] & \mathfrak{g}_1^{\mathcal{L}} \arrow[r, "\mathfrak{U}_{\mathcal{L}}"] \arrow[d, rightsquigarrow, "\mathrm{L}_1"'] & \mathfrak{g}_2^{\mathcal{L}} \arrow[d, rightsquigarrow, "\mathrm{L}_2"'] & D_{\mathrm{poly}}(X) \arrow[l, hook'] \arrow[d, rightsquigarrow, "\mathrm{L}"'] \\ \Omega^{\mathcal{L}}(X) \arrow[r, hook] & \mathfrak{m}_1^{\mathcal{L}} & \mathfrak{m}_2^{\mathcal{L}} \arrow[l, "\mathfrak{S}_{\mathcal{L}}"'] & C^{\mathcal{L}}_{\mathrm{poly}}(X), \arrow[l, hook'] \end{tikzcd} \tag{1.7}$$

where the hooked arrows are strict (*i.e.*, DG-*Lie*) *quasi-isomorphisms.*

1.4.1. *Comments on the results and the proofs.* The proofs of Theorems 1.1 and 1.3 depend on the simultaneous globalization of a number of local formality results due to Kontsevich [16] (see also [17]), Tsygan [25], Shoikhet [20] and the first two authors [8], [6], [7]. This globalization is performed by a functorial version of formal geometry [4] (see also [28]).

The proof of Theorem 1.1, roughly speaking, involves the construction of a morphism up to homotopy between the precalculus structures up to homotopy on $(T^{\mathcal{L}}_{\mathrm{poly}}(X),\ \Omega^{\mathcal{L}}(X))$ and $(D^{\mathcal{L}}_{\mathrm{poly}}(X), C^{\mathcal{L}}_{\mathrm{poly}}(X))$. In this paper we do not construct a full "precalculus$_\infty$"-quasi-isomorphism between these structures. In the case that $\mathcal{L}$ is a tangent bundle this was done in [13] using operadic methods; actually, in loc. cit. the authors work in the "calculus$_\infty$" setting, encoding also the de Rham differential, which is not part of the precalculus structure as observed before. On the other hand, in contrast to loc. cit., the results we prove are explicit and this fact is essential to recover Căldăraru's conjecture as formulated in [10] (see Theorem 1.4 below).

We are able to obtain such explicit results by starting with the local quasi-isomorphisms of Kontsevich and Shoikhet that are given by explicit formulae (in contrast to, say, Tamarkin's local G_∞-quasi-isomorphism [23]). While these are *a priori* only L_∞-quasi-isomorphisms they are nonetheless compatible with products up to homotopy [16], [7] in a strong explicit sense, and this turns out to be enough for our purposes.

As the local quasi-isomorphisms of Kontsevich and Shoikhet are defined over $\mathbb{R}$ (see §5.4) we have to assume $\mathbb{R} \subset k$ in the statement of Theorem 1.3. However enough coefficients are rational (and computable), which in turn allows us to prove Theorem 1.1 over an arbitrary field of characteristic zero. This idea was already used in [4]. See Section 7.3.1. For Theorem 1.3 we could likely have started with a Tamarkin-style local quasi-isomorphism [23] defined over $\mathbb{Q}$, but since the coefficients of such a local quasi-isomorphism are not explicit, the result would not be immediately applicable to Theorem 1.1.

The existence of the upper horizontal isomorphism in (1.5) was proved independently in [12], [4], while its explicit form was computed in [4]. The existence of the lower horizontal isomorphism was shown in [13]. As observed above, our approach via Kontsevich's and Shoikhet's local formality formulae allows us to compute it explicitly.

1.5. *Căldăraru's conjecture.* Assume now that X is a smooth algebraic or complex variety. Căldăraru's conjecture (stated originally in the algebraic case) asserts the existence of various compatibilities between the Hochschild (co)homology and tangent (co)homology of X (see below). For the full statement we refer to [10]. The results in this paper complete the proof of Căldăraru's conjecture.

We now explain this in more detail. The Hochschild (co)homology [22] of X is defined as

$$\mathrm{HH}^n(X) = \mathrm{Ext}^n_{\mathcal{O}_{X\times X}}(\mathcal{O}_\Delta, \mathcal{O}_\Delta) \qquad (n \geq 0),$$
$$\mathrm{HH}_n(X) = \mathrm{Tor}_{-n}^{\mathcal{O}_{X\times X}}(\mathcal{O}_\Delta, \mathcal{O}_\Delta) \qquad (n \leq 0),$$

where $\Delta \subset X\times X$ is the diagonal. From these definitions it is clear that $\mathrm{HH}^\bullet(X)$ has a canonical algebra structure (by the Yoneda product) and $\mathrm{HH}_\bullet(X)$ is a module over it.

Furthermore, if we put $\mathcal{L} = T_X$ then it is proved in [9] (and partially in [30]) that there are isomorphisms of algebras and modules

$$\begin{tikzcd}
\mathrm{HH}^\bullet(X) \arrow[r] \arrow[d, rightsquigarrow] & \mathbb{H}^\bullet(X, D^{\mathcal{L}}_{\mathrm{poly}}(X)) \arrow[d, rightsquigarrow] \\
\mathrm{HH}_\bullet(X) \arrow[r] & \mathbb{H}^\bullet(X, C^{\mathcal{L}}_{\mathrm{poly}}(X)),
\end{tikzcd}$$

where on the right-hand side we consider only the part of the precalculus given by the cup and cap product.

We define the tangent (co)homology of X by

$$\mathrm{HT}^\bullet(X) = \bigoplus \mathrm{H}^\bullet(X, \bigwedge^{\bullet} T_X), \ \mathrm{H}\Omega_\bullet(X) = \bigoplus \mathrm{H}^\bullet(X, \Omega_X^{-\bullet}),$$

where now $\Omega_X^\bullet$ denotes the graded sheaf of differential forms on X.

The commutative diagram (1.6) then yields the following

Theorem 1.4 ("Căldăraru's conjecture"). *For a smooth algebraic or complex variety X over k there is a commutative diagram of k-algebras and modules*

$$\begin{tikzcd}
\mathrm{HT}^\bullet(X) \arrow[r, "\mathrm{HKR}\circ\iota_{\sqrt{\mathrm{td}(X)}}"] \arrow[d, rightsquigarrow] & \mathrm{HH}^\bullet(X) \arrow[d, rightsquigarrow] \\
\mathrm{H}\Omega_\bullet(X) & \mathrm{HH}_\bullet(X), \arrow[l, "(\sqrt{\mathrm{td}(X)}\wedge -)\circ\mathrm{HKR}"]
\end{tikzcd} \tag{1.8}$$

where $\mathrm{td}(X)$ *is the Todd class for* $\mathcal{L} = T_X$.

Theorem 1.4 completes the proof of the parts of Căldăraru's conjecture [10] that do not depend on X being proper. The cohomological part (the upper row in the above diagram) was already proved in [4] and is also an unpublished result of Kontsevich.

In the proper case there is an additional assertion in Căldăraru's conjecture that involves the natural bilinear form on $\mathrm{HH}_\bullet(X)$. We do not consider this assertion in the present paper as it has already been proved by Markarian [18] and Ramadoss [19]. If we combine Theorem 1.8 with the results of Markarian and Ramadoss, we obtain a full proof of Căldăraru's conjecture. Let us also mention that in the compact Calabi-Yau case Căldăraru's conjecture was proved in [15].

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2. Notation and conventions

As stated already we always work over a ground field k of characteristic 0; unadorned tensor products are over k.

Most objects we consider are equipped with a topology that will be explicitly specified when needed. However if an object is introduced without a specific topology, or if the topology is not clear from the context, then it is assumed to be equipped with the discrete topology.

Many objects we will encounter are $\mathbb{Z}$-graded. Koszul's sign rule is always assumed in this context. For a double or higher complex we apply the sign rule with respect to total degree.

3. Some recollections on Lie algebroids and related topics

3.1. *Generalities on Lie algebroids.* In this section R is a commutative k-algebra.

Definition 3.1. A *Lie algebroid* L over R is a Lie algebra over k that is in addition an R-module and is endowed with an anchor map $\rho : L \to \mathrm{Der}_k(R)$ satisfying the compatibility

$$[l_1, rl_2] = \rho(l_1)(r)l_2 + r[l_1, l_2],\ r \in R,\ l_i \in L,\ i = 1, 2. \tag{3.1}$$

The basic example of a Lie algebroid over R is $L = \mathrm{Der}_k(R)$ with the identity anchor map and the commutator Lie bracket.

If L is a Lie algebroid, then $R \oplus L$ is a Lie algebra with Lie bracket $[(r, l), (r', l')] = (\rho(l)(r') - \rho(l')(r), [l, l'])$. We define the *universal enveloping algebra* $\mathrm{U}_R(L)$ of L to be the quotient of the augmentation ideal of the enveloping algebra associated to the Lie algebra $R \oplus L$ by the relation $r \otimes l = rl$ ($r \in R$, $l \in R \oplus L$).

For the sake of simplicity, below we will usually omit the anchor map ρ from the notation, unless it is necessary for the sake of clarity.

The universal enveloping algebra of a Lie algebroid satisfies a universal property similar to that of an ordinary enveloping algebra. This implies, for example, that the anchor map ρ uniquely extends to an algebra morphism from $\mathrm{U}_R(L)$ to $\mathrm{End}_k(R)$, or equivalently: it yields a left $\mathrm{U}_R(L)$-module structure on R.

For reasons that will become clear later we assume that our Lie algebroids are free of rank d over R.

3.1.1. *L-poly-vector fields and L-differential forms over R.* To a Lie algebroid L over R we associate

$$T^L_{\mathrm{poly}}(R) = \bigoplus_{n \geq -1} \wedge_R^{n+1} L, \tag{3.2}$$

$$\Omega^L(R) = \bigoplus_{n \leq 0} \wedge_R^{-n} L^*,\ L^* = \mathrm{Hom}_R(L, R). \tag{3.3}$$

We refer to (3.2) and (3.3) as the spaces of *L-poly-vector fields* and *L-forms* on R.

As an exterior algebra $T^L_{\mathrm{poly}}(R)$ has a wedge product that we denote by $\cup$ ("the cup product"). The extension of the Lie bracket on L to a bi-derivation on $T^L_{\mathrm{poly}}(R)$ defines a Lie bracket that is called the Schouten–Nijenhuis bracket and is denoted by $[-,-]$. Note that with our grading conventions the cup product has degree 1 and the Lie bracket has degree 0. The cup product and the Lie bracket make $T^L_{\mathrm{poly}}(R)$ into a (shifted) Gerstenhaber algebra with trivial differential.

On the other hand, $\Omega^L(R)$ is obviously a graded algebra with respect to the wedge product. In addition there is an analogue d_L of the de Rham differential on $\Omega^L(R)$, which is given on generators by

$$\mathrm{d}_L(r)(l) = l(r),$$
$$\mathrm{d}_L(l^*)(l_1,l_2) = l_1(l^*(l_2)) - l_2(l^*(l_1)) - l^*([l_1,l_2])$$

for $r \in R$, $l, l_i \in L$, $i = 1, 2$, $l^* \in L^*$ and is extended uniquely by Leibniz's rule.

The natural contraction operation of L-forms on R with respect to L-poly-vector fields is denoted by $\cap$ (the "cap product"). The Lie derivative L of L-forms on R with respect to L-poly-vector fields is specified in the usual way via Cartan's homotopy formula as the commutator of d_L and the contraction. The pair

$$((T^L_{\mathrm{poly}}(R), [-,-], \cup), (\Omega^L(R), \cap, \mathrm{L}))$$

forms a precalculus (see §1.2).

3.1.2. *L-connections.* As usual L is a Lie algebroid over R.

Definition 3.2. Let M be an R-module M. An *L-connection* on M is a k-linear map ∇ from M to $L^* \otimes_R M$, which satisfies Leibniz's rule

$$\nabla(rm) = \mathrm{d}_L(r) \otimes m + r\nabla m, \ r \in R, \ m \in M. \tag{3.4}$$

The L-connection ∇ is said to be *flat* if $\nabla^2 = 0$. Equivalently, the assignment $l \mapsto \nabla_l$, where ∇_l denotes the action of ∇ followed by contraction with respect to l, defines a Lie algebra morphism from L to $\mathrm{End}_k(M)$.

If we let $l \in L$ act as ∇_l, then a flat L-connection on M extends to a left $\mathrm{U}_R(L)$-module structure on M.

Furthermore, a flat L-connection ∇ on M can be extended to a differential (denoted by the same symbol) on the graded R-module $\Omega^L(R) \otimes_R M$ via Leibniz's rule

$$\nabla(\omega \otimes_R m) = \mathrm{d}_L\omega \otimes_R m + (-1)^{|\omega|}\omega \wedge \nabla m, \ \omega \in \Omega^L(R), \ m \in M.$$

3.1.3. *L-differential operators over R.* In this section we define the algebra of poly-differential operators of a Lie algebroid and we list some of its properties. We give some explicit formulae along the lines of [7].

As in the case of ordinary Lie algebras, $\mathrm{U}_R(L)$ (see §3.1) may be naturally filtered by giving R filtered degree 0 and L filtered degree 1. In particular,

$$F_0\mathrm{U}_R(L) = R,\ F_1\mathrm{U}_R(L) = R \oplus L.$$

We view $\mathrm{U}_R(L)$ as an R-*central* bimodule via the natural embedding of R into $\mathrm{U}_R(L)$. Explicitly, if we denote this embedding by i, then[5]

$$rD \overset{!}{=} Dr \overset{\text{def}}{=} i(r)D,\ r \in R, D \in \mathrm{U}_R(L). \tag{3.5}$$

Moreover $\mathrm{U}_R(L)$ is an R-coalgebra [27]; i.e., $\mathrm{U}_R(L)$ possesses an R-linear coproduct $\Delta : \mathrm{U}_R(L) \to \mathrm{U}_R(L) \otimes_R \mathrm{U}_R(L)$ and an R-linear counit, satisfying the usual axioms. The comultiplication actually takes values in

$$\begin{aligned}&(\mathrm{U}_R(L) \otimes_R \mathrm{U}_R(L))' \\ &\quad= \left\{\sum_j D_j \otimes E_j \in U_R(L) \otimes_R \mathrm{U}_R(L) \mid \forall r \in R : \sum_j D_j i(r) \otimes E_j = \sum_j D_j \otimes E_j i(r)\right\},\end{aligned}$$

which is an R-algebra even though $\mathrm{U}_R(L) \otimes_R U_R(L)$ is not.

The comultiplication Δ and counit ε are given by similar formulae as in the Lie algebra case:

$$\begin{aligned}\Delta(r) &= r \otimes_R 1 = 1 \otimes_R r && r \in R,\\ \Delta(l) &= l \otimes_R 1 + 1 \otimes_R l && l \in L,\\ \Delta(DE) &= D_{(1)}E_{(1)} \otimes D_{(2)}E_{(2)} && D, E \in \mathrm{U}_R(L),\\ \varepsilon(D) &= D(1).\end{aligned} \tag{3.6}$$

In the third formula we have used Sweedler's convention. The expression on the right-hand side is well defined because it is the product inside the algebra $(\mathrm{U}_R(L) \otimes_R \mathrm{U}_R(L))'$. In the fourth formula we have used the natural action of $\mathrm{U}_R(L)$ on R (see §3.1).

The algebra (better: in the terminology of [27], [3] "the Hopf algebroid") $\mathrm{U}_R(L)$ may be thought of as an algebra of L-differential operators on R. In the case $L = \mathrm{Der}_k(R)$ and R smooth over k then $\mathrm{U}_R(L)$ coincides with the algebra of differential operators on R.

[5] Note that there is, at first sight, a more natural right R-module structure on $U_R(L)$ given by the formula $Dr = Di(r)$. This alternative right module structure will not be used in this paper.

3.1.4. *L-jets.* Let $(\mathrm{U}_R(L))_{\leq n}$ be the elements of degree $\leq n$ with respect to the canonical filtration on $\mathrm{U}_R(L)$ introduced in Section 3.1.3. The L-n-jets are defined as

$$J^n L = \mathrm{Hom}_R(\mathrm{U}_R(L)_{\leq n}, R).$$

(This is unambiguous, as the left and right R-modules structures on $\mathrm{U}_R(L)$ are the same; see (3.5).) We also put

$$JL = \mathrm{Hom}_R(\mathrm{U}_R(L), R) = \operatorname{proj}\lim_n J^n L \qquad (\text{as } \mathrm{U}_R(L) = \operatorname{inj}\lim_n (\mathrm{U}_R(L))_{\leq n}). \tag{3.7}$$

JL has a natural commutative algebra structure obtained from the comultiplication on $\mathrm{U}_R(L)$. Thus, for $\phi_1, \phi_2 \in JL$, $D \in \mathrm{U}_R(L)$, we have

$$(\phi_1\phi_2)(D) = \phi_1(D_{(1)})\phi_2(D_{(2)})\,,$$

and the unit in JL is given by the counit on $\mathrm{U}_R(L)$.

In addition JL has two commuting left $\mathrm{U}_R(L)$-module structures which we now elucidate. First of all there are two distinct monomorphisms of k-algebras

$$\begin{aligned}\alpha_1 :& R \to JL : r \mapsto (D \mapsto r\varepsilon(D))\,,\\ \alpha_2 :& R \to JL : r \mapsto (D \mapsto D(r))\,.\end{aligned}$$

It will be convenient to write $R_i = \alpha_i(R)$ and to view JL as an $R_1 - R_2$-bimodule.

There are also two distinct commuting actions by derivations of L on JL. Let $l \in L$, $\phi \in JL$, $D \in \mathrm{U}_R(L)$.

$$\begin{aligned}{}^1\nabla_l(\phi)(D) &= l(\phi(D)) - \phi(lD),\\ {}^2\nabla_l(\phi)(D) &= \phi(Dl).\end{aligned}$$

Again it will be convenient to write L_i for L acting by ${}^i\nabla$. Then ${}^i\nabla$ defines a flat L_i-connection on JL, considered as an R_i-module. The connection ${}^1\nabla$ is the well-known *Grothendieck connection.* It follows that JL is a $\mathrm{U}_R(L)_1 - \mathrm{U}_R(L)_2$-bimodule (with both $\mathrm{U}_R(L)_1$ and $\mathrm{U}_R(L)_2$ acting on the left).

The $\mathrm{U}_R(L)_2$ action on JL takes the very simple form

$$(D \cdot \phi)(E) = \phi(ED)$$

(for $D, E \in \mathrm{U}_R(L)_2$, $\phi \in JL$).

Define $\varepsilon : JL \to R$ by $\varepsilon(\phi) = \phi(1)$, and put $J^c L = \ker \varepsilon$. Then JL is complete for the J^cL-adic topology and the filtration on JL induced by (3.7) coincides with the J^cL-adic filtration. If we filter JL with the J^cL-adic filtration, then we obtain

$$\operatorname{gr} JL = \mathrm{S}_R L^* \tag{3.8}$$

and the R_1 and R_2-action on the right-hand side of this equation coincide. (Here and below the letter S stands for "symmetric algebra.")

The induced actions on $\operatorname{gr} JL = \mathrm{S}_R L^*$ of $l \in L$, considered as an element of L_1 and L_2, are given by the contractions i_{-l} and i_l, respectively.

In case R is the coordinate ring of a smooth affine algebraic variety and $L = \mathrm{Der}_k(R)$ then we may identify JL with the completion $R\widehat{\otimes}R$ of $R \otimes R$ at the kernel of the multiplication map $R \otimes R \to R$. The two actions of R on JL are respectively $R\widehat{\otimes}1$ and $1\widehat{\otimes}R$.

Similarly, a derivation on R can be extended to $R\widehat{\otimes}R$ in two ways by letting it act respectively on the first and second factor. Since derivations are continuous they act on adic completions and hence in particular on JL. This provides the two actions of L on JL.

In the sequel we will view the action labelled by "1" as the default action; i.e., we will usually not write the 1 explicitly.

3.2. *Relative poly-vector fields, poly-differential operators.* We need relative poly-differential operators and poly-vector fields. So assume that $A \to B$ is a morphism of commutative k-algebras. Then

$$T_{\mathrm{poly},A}(B) = \bigoplus_{n \geq -1} T^n_{\mathrm{poly},A}(B),$$

$$D_{\mathrm{poly},A}(B) = \bigoplus_{n \geq -1} D^n_{\mathrm{poly},A}(B),$$

where $T^n_{\mathrm{poly},A}(B) = \bigwedge_B^{n+1} \mathrm{Der}_A(B)$. Similarly, $D^n_{\mathrm{poly},A}(B) \subseteq \mathrm{Hom}_A(B^{\otimes_A (n+1)}, B)$ consists of those A-linear maps from $B^{\otimes_A (n+1)}$ to B that are A-linear differential operators on B in each argument.

It is easy to see that $T_{\mathrm{poly},A}(B)$ is a Gerstenhaber algebra when equipped with the Schouten bracket and the exterior product. Similarly, $D_{\mathrm{poly},A}(B)$ is a graded subspace of the relative Hochschild complex $C^\bullet_A(B)$, and since differential operators are closed under composition one easily sees that it is in fact a sub-B_∞-algebra; see Appendix A for more details on B_∞-algebras.

If A and B are DG-algebras, then we equip $T_{\mathrm{poly},A}(B)$, $D_{\mathrm{poly},A}(B)$ with the total differentials $[\mathrm{d}_B, -]$ and $[\mathrm{d}_B, -] + \mathrm{d}_H$, where d_H denotes the Hochschild differential. Similar results now apply.

3.3. *The sheaf of L-poly-differential operators.*

Definition 3.3. For a Lie algebroid L over R, we define the graded vector space $D^L_{\mathrm{poly}}(R)$ of *L-poly-differential operators on R* as the tensor algebra over R of $\mathrm{U}_R(L)$ with shifted degree, i.e.,

$$D^L_{\mathrm{poly}}(R) = \bigoplus_{n \geq -1} \mathrm{U}_R(L)^{\otimes_R (n+1)}.$$

The action of $\mathrm{U}_R(L)$ on R extends to a map

$$D^{L,n}_{\mathrm{poly}}(R) \to \mathrm{Hom}_k(R^{\otimes n+1}, R) \tag{3.9}$$

defined by

$$(D_1 \otimes \cdots \otimes D_{n+1})(r_1 \otimes \cdots \otimes r_{n+1}) \mapsto D_1(r_1) \cdots D_{n+1}(r_{n+1}),$$

whose image lies in the space $D_{\mathrm{poly}}(R)$ of poly-differential operators on R.

$D^L_{\mathrm{poly}}(R)$ is a B_∞-algebra. In particular it is a DG-Lie algebra and furthermore it is a Gerstenhaber algebra up to homotopy. In Appendix A we give the formulae for the full B_∞-structure. Here we content ourselves by reminding the reader of the basic operations.

The *Gerstenhaber bracket* on $D^L_{\mathrm{poly}}(R)$ is defined by

$$[D_1, D_2] = D_1\{D_2\} - (-1)^{|D_1||D_2|} D_2\{D_1\},\ D_i \in D^L_{\mathrm{poly}}(R),\ i = 1, 2, \tag{3.10}$$

where

$$D_1\{D_2\} = \sum_{i=0}^{|D_1|} (-1)^{i|D_2|} (\mathrm{id}^{\otimes i} \otimes \Delta^{|D_2|} \otimes \mathrm{id}^{\otimes |D_1|-i})(D_1) \cdot (1^{\otimes i} \otimes D_2 \otimes 1^{\otimes |D_1|-i})\,.$$

It is a Lie bracket of degree 0. The special element $\mu = 1 \otimes_R 1 \in D^{L,1}_{\mathrm{poly}}(R) = \mathrm{U}_R(L) \otimes_R \mathrm{U}_R(L)$ satisfies $[\mu, \mu] = 0$. The *Hochschild differential* is defined as the operator $\mathrm{d_H} = [\mu, -]$.

The *cup product* on $D^L_{\mathrm{poly}}(R)$ is defined by

$$D_1 \cup D_2 = (-1)^{(|D_1|-1)(|D_2|-1)} D_1 \otimes_R D_2. \tag{3.11}$$

(See also Appendix A for an explicit derivation of the previous formula.)

One may now show that these operations make the 4-tuple

$$(D^L_{\mathrm{poly}}(R), \mathrm{d_H}, [\ ,\ ], \cup)$$

into a Gerstenhaber algebra up to homotopy (see Lemma A.1). Indeed if R is smooth over k and $L = \mathrm{Der}_k(R)$ is the tangent bundle, then the operations we have defined are the same as those one obtains from the identification $D^L_{\mathrm{poly}}(R) = D_{\mathrm{poly}}(R)$ where we view the right-hand side as a sub-B_∞-algebra of the Hochschild complex $C^\bullet(R)$ of R (cf. §3.2).

It is in fact, as we explain now, not necessary to verify that we have defined a homotopy Gerstenhaber structure on $D^L_{\mathrm{poly}}(R)$. Indeed the results can be obtained directly from the known results for the Hochschild complex (see [14], [26]). Similarly, it is not necessary to write explicit formulae for $[-,-]$ and $\cup$ (or for the whole B_∞-structure for that matter). This point of view will be useful when we consider Hochschild chains, as in that case the formulae become more complicated.

The L_2-action on JL commutes with the R_1-action (see §3.1.4) so we obtain a ring homomorphism

$$\mathrm{U}_{R_2}(L_2) \to D_{R_1}(JL) : D \mapsto (\theta \mapsto D(\theta))$$

and hence a map

$$D^{L_2}_{\mathrm{poly}}(R_2) \to D_{\mathrm{poly},R_1}(JL) \tag{3.12}$$

of Gerstenhaber algebras up to homotopy. The right-hand side has an R_1-connection given by $[{}^1\nabla, -]$, and it follows from [4, Prop. 4.2.4, Lemma 4.3.4] that the left-hand side of (3.12) is given by the horizontal sections for this connection.

Now as discussed in Section 3.2, we know that $D_{\mathrm{poly},R_1}(JL)$ is a B_∞-algebra and it is an easy verification that the braces and the differential, which make up the B_∞-structure, are horizontal for $[{}^1\nabla, -]$. Hence the B_∞-structure on $D_{\mathrm{poly},R_1}(JL)$ descends to $D^L_{\mathrm{poly}}(R)$, and one verifies that its basic operations are indeed given by the formulae we gave earlier.

3.4. *The Hochschild complex of L-chains over R.* We start with the following definition.

Definition 3.4. For a Lie algebroid L over R, the graded R-module

$$C^L_{\mathrm{poly},p}(R) = \begin{cases} JL^{\widehat{\otimes}_R -p}, & p < 0, \\ R, & p = 0, \end{cases} \tag{3.13}$$

is called the space of *Hochschild L-chains over R*.

Our aim in this section will be to show that the pair

$$(D^L_{\mathrm{poly}}(R), C^L_{\mathrm{poly}}(R))$$

is a precalculus up to homotopy. We will do this without relying on explicit formulae (as they are quite complicated). Instead we will reduce to a relative version of [7] that discusses Hochschild (co)homology. Explicit formulae are given in Appendix B.

Let us first remind the reader that if A is a k-algebra, then the pair $(C^\bullet(A), C_\bullet(A))$ consisting of the spaces of Hochschild cochains and chains is a precalculus up to homotopy. For $C^\bullet(A)$ this is just the (shifted) homotopy Gerstenhaber structure that we have already mentioned in Section 3.3 and that was introduced in [14], [26].

The full precalculus structure up to homotopy on $(C^\bullet(A), C_\bullet(A))$ is a more intricate object. A complete treatment in a very general setting was given in [7]. It is shown that the precalculus structure can be obtained from two interacting B_∞-module structures on $C_\bullet(A)$. These B_∞-module structures are obtained from brace-type operations. For more operadic approaches see [13].

Although we do not really use them, for the benefit of the reader we state the well-known formulae for the contraction, the Lie derivative and the differential. If $P \in C^{m-1}(A) = \mathrm{Hom}(A^{\otimes m}, A)$ and $(a_0|\cdots|a_t) \in C_{-t}(A) =$

$A^{\otimes t+1}$, then we have

$$\imath_P(a_0|\cdots|a_t) = (a_0 P(a_1,\ldots,a_m)|a_{m+1}|\cdots|a_t),$$

$$\mathrm{L}_P(a_0|\ldots|a_t) = \sum_{i=0}^{t-m+1}(-1)^{(m-1)i}(a_0|\cdots|a_{i-1}|P(a_i,\ldots,a_{i+m-1})|a_{i+m}|\cdots|a_t)$$
$$+\sum_{l=t-m+2}^{t+1}(-1)^{lt}(P(a_l,\ldots,a_t,a_0,\ldots,a_{m-t+l-2})|a_{m-t+l-1}|\cdots|a_{l-1}).$$

The differential b_H is defined as L_μ, where μ is the multiplication, considered as an element of $\mathrm{Hom}(A^{\otimes 2}, A)$.

To construct the precalculus structure up to homotopy on $(D^L_{\mathrm{poly}}(R), C^L_{\mathrm{poly}}(R))$ we proceed as in Section 3.3. We first define an object that is larger than $C^L_{\mathrm{poly}}(R)$.

Definition 3.5. The space of *L-poly-jets over R* is the *completed* space of relative Hochschild chains $\widehat{C}_{R_1,\bullet}(JL)$. Explicitly,

$$\widehat{C}_{R_1,\bullet}(JL) = \bigoplus_{p\leq 0} JL^{\widehat{\otimes}_{R_1}-p-1}. \tag{3.14}$$

The Grothendieck connection ${}^1\nabla$ on JL (see §3.1.4) yields a connection on $\widehat{C}_{R_1,\bullet}(JL)$ by Leibniz's rule, which we also refer to as the Grothendieck connection. The following result was proved in [5].

Proposition 3.6. *For a Lie algebroid L over a commutative ring R as above, there is an isomorphism of graded vector spaces*

$$\widehat{C}_{R_1,\bullet}(JL)^{{}^1\nabla} \to C^L_{\mathrm{poly}}(R) \tag{3.15}$$

that sends

$$\phi_1\otimes\phi_2\otimes\cdots\otimes\phi_p \in \widehat{C}_{R_1,-p}(JL)^{{}^1\nabla}$$

to

$$\varepsilon(\phi_1)\phi_2\otimes\cdots\otimes\phi_p \in C^L_{\mathrm{poly},1-p}(R).$$

Proof. The arguments of the proof of [5, Prop. 1.11] can be repeated almost verbatim. □

The formulae from [7] for the Hochschild complexes now yield that

$$(D_{\mathrm{poly},R_1}(JL), \widehat{C}_{R_1,\bullet}(JL)) \subset (C^{\mathrm{cont},\bullet}_{R_1}(JL), \widehat{C}_{R_1,\bullet}(JL))$$

is a precalculus up to homotopy. Furthermore, one verifies that the formulae in [7] are compatible with the Grothendieck connection ${}^1\nabla$. Hence the precalculus descends to one on

$$(D_{\mathrm{poly},R_1}(JL)^{{}^1\nabla}, \widehat{C}_{R_1,\bullet}(JL)^{{}^1\nabla}) = (D^L_{\mathrm{poly}}(R), C^L_{\mathrm{poly}}(R)), \tag{3.16}$$

where we use (3.12) as well as Proposition 3.6.

It remains to check that this construction coincides with the standard one for a smooth commutative algebra. Namely, if R/k is smooth and L is the tangent bundle, then we have

$$D^L_{\mathrm{poly}}(R) = D_{\mathrm{poly}}(R).$$

We also have $JL = R\widehat{\otimes}R$ (see §3.1.4) and in this way we obtain an isomorphism

$$\begin{aligned}(3.17)\quad & C^L_{\mathrm{poly},-p}(R) = (R\widehat{\otimes}R)^{\widehat{\otimes}_{R_1}p} \\ & \quad \to R^{\widehat{\otimes}p+1} : (r_1\widehat{\otimes}s_1)\widehat{\otimes}\cdots\widehat{\otimes}(r_p\widehat{\otimes}s_p) \mapsto (r_1\cdots r_p)\widehat{\otimes}s_1\widehat{\otimes}\cdots\widehat{\otimes}s_p\end{aligned}$$

that yields an isomorphism of graded vector spaces

$$C^L_{\mathrm{poly}}(R) = \widehat{C}_\bullet(R).$$

Thus, we have an isomorphism of pairs of graded vector spaces

$$(D^L_{\mathrm{poly}}(R), C^L_{\mathrm{poly}}(R)) = (D_{\mathrm{poly}}(R), \widehat{C}_\bullet(R)). \tag{3.18}$$

The right-hand side is a precalculus up to homotopy (as it is basically a pair of spaces of Hochschild chains/cochains).

Lemma 3.7. *The precalculus up to homotopy on the right-hand side of* (3.18) *is the same as the one we have constructed on the left-hand side.*

Proof. Note that going from the pair (k, R) to (R, JL) is a base extension by R (since $JL = R\widehat{\otimes}R$). Since the formulae in [7] are clearly compatible with base extension we have that the precalculus structure on

$$(D_{\mathrm{poly},R}(JL), \widehat{C}_{R,\bullet}(JL)) = (R\widehat{\otimes}D_{\mathrm{poly}}(R), R\widehat{\otimes}\widehat{C}_\bullet(R)) \tag{3.19}$$

is obtained by base extension from the one on

$$(D_{\mathrm{poly}}(R), \widehat{C}_\bullet(R)).$$

Furthermore, one checks that the Grothendieck connections on $D_{\mathrm{poly},R}(JL)$ and $\widehat{C}_{R,\bullet}(JL)$ under the isomorphism (3.19) act by the standard Grothendieck connection on the copy of R appearing on the left of $\widehat{\otimes}$ and trivially on $D_{\mathrm{poly}}(R)$, $\widehat{C}_\bullet(R)$. Hence its invariants are precisely $D_{\mathrm{poly}}(R)$, $\widehat{C}_\bullet(R)$. This finishes the proof. □

3.5. *The Hochschild–Kostant–Rosenberg Theorem.* We recall the Lie algebroid version of the famous cohomological Hochschild–Kostant–Rosenberg (shortly, HKR) quasi-isomorphism; for a proof, we refer to [3].

Theorem 3.8. *We consider a Lie algebroid L over R in the sense of Definition 3.1, which is assumed to be free of rank d over R.*

Then the map

$$\mathrm{HKR}(l_1 \wedge \cdots \wedge l_p) = (-1)^{\frac{p(p-1)}{2}} \frac{1}{p!} \sum_{\sigma \in \mathfrak{S}_p} (-1)^{\sigma} l_{\sigma(1)} \otimes_R \cdots \otimes_R l_{\sigma(p)} \tag{3.20}$$

defines a quasi-isomorphism of complexes from $(T^L_{\mathrm{poly}}(R), 0)$ *to* $(D^L_{\mathrm{poly}}(R), \mathrm{d_H})$.

There is a dual version of Theorem 3.8, which will also be needed.

THEOREM 3.9. *The quasi-isomorphism* (3.20) *induces the quasi-isomorphism*

$$\mathrm{HKR}(a) = a \circ \mathrm{HKR}$$

of complexes from $(C^L_{\mathrm{poly}}(R), \mathrm{b_H})$ *to* $(\Omega^L(R), 0)$.

4. Fedosov resolutions in the Lie algebroid framework

4.1. *Introduction.* The aim of this section is to discuss Fedosov resolutions [11] in the Lie algebroid framework. These are needed to formulate and prove the globalization result, which in turn leads to the main results.

To help the reader understand our algebraic setup (which was inspired by [28]) we give some motivation for the definitions in the subsequent sections. For the sake of exposition, we assume in this introduction that X is some kind of d-dimensional smooth space and $\mathcal{L}$ is an appropriate version of the tangent bundle of X.

One of the applications of formal geometry is the globalization of local *coordinate dependent* constructions. For example using the Darboux Lemma it is trivial to quantize a symplectic manifold locally but such local quantizations are coordinate dependent and they do not globalize easily. The same is true for the local formality morphisms (see §5.4 below for more details) that we use in this paper.

The idea is then to replace X by a much larger infinite dimensional space $X^{\mathrm{coord}} \to X$ that parametrizes formal local coordinate systems on X. For example if X is an algebraic variety, then the fiber at $x \in X$ in X^{coord} is given by the k-algebra isomorphisms $\widehat{\mathcal{O}}_{X,x} \to k[[t_1, \ldots, t_d]]$. An equivalent way of saying this is that X^{coord} universally trivializes the jet bundle $(\widehat{\mathcal{O}}_{X,x})_{x \in X}$ over X.

Local constructions can be tautologically globalized to X^{coord}, and this should be followed by some type of descent for X^{coord}/X. A general procedure to do this is to resolve $\mathcal{O}_X$ by a de Rham-type complex over $\mathcal{O}_{X^{\mathrm{coord}}}$, but this does not really work as the fibers of $X^{\mathrm{coord}} \to X$ are not contractible.

However in the aforementioned examples the local constructions are all compatible with *linear* coordinate changes. So if we define $X^{\mathrm{aff}} = X^{\mathrm{coord}} / \mathrm{Gl}_d$, then the constructions descend to X^{aff}, and as the fibers of X^{aff}/X are contractible we can descend further to X.

In this paper we work over a general locally free Lie algebroid $\mathcal{L}$ rather than T_X. In this setting we define the analogue of X^{coord} as the space which universally trivializes the space of jet bundles for $\mathcal{L}$ (see §3.1.4).

4.2. *Setup.* As a general principle we work on the presheaf level in this paper, performing sheafification only as the very last step of the constructions. This means that throughout we may replace all spaces by rings and locally free sheaves may be treated as free modules.

As before we consider a Lie algebroid L over a ring R in the sense of Definition 3.1; i.e., L is free of rank d over R.

First we discuss Fedosov resolutions of L-poly-vector fields and L-poly-differential operators as Gerstenhaber algebras up to homotopy, referring to [4] for details. Finally, we discuss Fedosov resolutions of $\Omega^L(R)$ (see (3.3)) and $C^L_{\mathrm{poly}}(R)$ (see (3.13)) that are compatible with the precalculus structure up to homotopy.

4.3. *The (affine) coordinate space of a Lie algebroid.* For a Lie algebroid L over R as above, its *coordinate space* $R^{\mathrm{coord},L}$ was introduced and discussed in detail in [2], [4], to which we refer for a more extensive treatment.

As explained in Section 4.1, the main property of $R^{\mathrm{coord},L}$ is the existence of an isomorphism of $R^{\mathrm{coord},L}$-algebras

$$t : R^{\mathrm{coord},L}\widehat{\otimes}_{R_1} JL \to R^{\mathrm{coord},L}[\![x_1,\dots,x_d]\!] = R^{\mathrm{coord},L}\widehat{\otimes} F,\ \ F = k[\![x_1,\dots,x_d]\!], \tag{4.1}$$

and $R^{\mathrm{coord},L}$ is universal with respect to this property; that is, if there is an R-algebra W, such that there is a W-linear isomorphism $W\widehat{\otimes}_{R_1} JL \to W[\![x_1,\dots,x_d]\!]$, then there exists a unique morphism $R^{\mathrm{coord},L} \to W$.

In particular, we note that in contrast to JL, the ring $R^{\mathrm{coord},L}$ is *not* an adic topological ring; it is equipped with the discrete topology (like R).

Example 4.1. Assume $R = k[x_1,\dots,x_d]$ and $L = \mathrm{Der}_k(R)$. As explained in [2, §6.1.5], [28], we have

$$R^{\mathrm{coord},L} = R[y_{i,\alpha} :\ i = 1,\dots,d,\ \alpha \in \mathbb{N}^d \setminus \{0\}]_{\det(y_{i,e_j})},$$

where e_j is the j-th standard basis vector in $\mathbb{Z}^d$ and the subscript $\det(y_{i,e_j})$ refers to the localization at the indicated element. As in this case $X = \operatorname{Spec} R$ has global coordinates $x_1,\dots,x_n$, the coordinate ring of the jet bundle JL is equal to $R[\![y_1,\dots,y_d]\!]$, where y_i is a local version of the global coordinate x_i. The morphism t is the "universal Taylor expansion" morphism

$$t(y_i) = \sum_\alpha y_{i,\alpha} t^\alpha.$$

As a consequence of the universal property of $R^{\mathrm{coord},L}$, $R^{\mathrm{coord},L}$ admits an action of $\mathrm{GL}_d(k)$, such that the following identity holds true on $R^{\mathrm{coord},L}\widehat{\otimes} F$

for $A \in \mathrm{GL}_d(k)$:

$$(A^{-1}\widehat{\otimes}A)|_{JL} = \mathrm{Id}_{JL},$$

where JL is considered as a subalgebra of $R^{\mathrm{coord},L}\widehat{\otimes}F$ through (4.1).

By means of $R^{\mathrm{coord},L}$, we consider the graded algebra

$$C^{\mathrm{coord},L} = \Omega_{R^{\mathrm{coord},L}} \otimes_{\Omega_{R_1}} \Omega^{L_1}(R_1).$$

It has the structure of a DG-algebra with differential $\mathrm{d}_{C^{\mathrm{coord},L}} = \mathrm{d}_{\Omega_{R^{\mathrm{coord},L}}} \otimes_{\Omega_{R_1}} 1 + 1 \otimes_{\Omega_{R_1}} \mathrm{d}_{L_1}$ and inherits from $R^{\mathrm{coord},L}$ a rational $\mathrm{GL}_d(k)$-action.

The universal isomorphism (4.1) extends to an isomorphism

$$t : C^{\mathrm{coord},L}\widehat{\otimes}_{R_1} JL \to C^{\mathrm{coord},L}[\![x_1, \ldots, x_d]\!], \tag{4.2}$$

where we used the respective obvious identifications

$$\begin{aligned} C^{\mathrm{coord},L}\widehat{\otimes}_{R^{\mathrm{coord},L}} \left(R^{\mathrm{coord},L}\widehat{\otimes}_{R_1} JL\right) &\cong \Omega_{R^{\mathrm{coord},L}}\widehat{\otimes}_{\Omega_{R_1}} \left(\Omega^{L_1}(R_1) \otimes_{R_1} JL\right) \\ &\cong C^{\mathrm{coord},L}\widehat{\otimes}_{R_1} JL, \\ C^{\mathrm{coord},L}\widehat{\otimes}_{R^{\mathrm{coord},L}} \left(R^{\mathrm{coord},L}\widehat{\otimes}F\right) &\cong C^{\mathrm{coord},L}[\![x_1, \ldots, x_d]\!]. \end{aligned}$$

We endow the graded algebra on the left-hand, resp. right-hand, side of (4.2) with the following natural differential:

$$^1\nabla^{\mathrm{coord}} = \mathrm{d}_{\Omega_{R^{\mathrm{coord},L}}}\widehat{\otimes}_{\Omega_{R_1}} 1 + 1\widehat{\otimes}_{\Omega_{R_1}} {}^1\nabla, \text{ resp.} \tag{4.3}$$

$$\mathrm{d} = \mathrm{d}_{C^{\mathrm{coord},L}}\widehat{\otimes}1, \tag{4.4}$$

where $^1\nabla$ was introduced in Section 3.4. Both (4.3) and (4.4) are, by construction, flat $C^{\mathrm{coord},L}$-connections on the respective spaces, and the obvious inclusions from $C^{\mathrm{coord},L}$ into $C^{\mathrm{coord},L}\widehat{\otimes}_{R_1} JL$ and $C^{\mathrm{coord},L}[\![x_1, \ldots, x_d]\!]$ are morphisms of DG-algebras.

The main property of the connections (4.3) and (4.4) lies in the existence of a *canonical Maurer–Cartan element in* $C^{\mathrm{coord},L}$; namely, according to [2, §1.6] and [4, §5.2], there exists a unique element ω of $C^{\mathrm{coord},L}\widehat{\otimes}\mathrm{Der}(F)$ of degree 1, satisfying

$$t \circ {}^1\nabla^{\mathrm{coord}} \circ t^{-1} - \mathrm{d} = \omega,$$

where the expression on the left-hand side is naturally viewed as a $C^{\mathrm{coord},L}$-linear derivation of F. Furthermore, ω satisfies the Maurer–Cartan equation in the DG-Lie algebra $C^{\mathrm{coord},L}\widehat{\otimes}\mathrm{Der}(F)$, i.e.,

$$\mathrm{d}\omega + \frac{1}{2}[\omega, \omega] = 0,$$

(which implies that $\mathrm{d} + [\omega, \bullet]$ is a flat connection on $C^{\mathrm{coord},L}[\![x_1, \ldots, x_d]\!]$) and the verticality condition

$$\iota_v\omega = 1 \otimes v, \;\; v \in \mathfrak{gl}_d(k). \tag{4.5}$$

(Here, ι_v on the left-hand side denotes the contraction operation on $C^{\mathrm{coord},L}$ with respect to v, coming from the infinitesimal action of $\mathfrak{gl}_d(k)$ on $R^{\mathrm{coord},L}$; v on the right-hand side denotes the linear vector field associated to v, acting on F.)

Finally, we consider the *affine coordinate space* $R^{\mathrm{aff},L}$ of a Lie algebroid L over R; it is simply the $\mathrm{GL}_d(k)$-invariant ring

$$R^{\mathrm{aff},L} = \left(R^{\mathrm{coord},L}\right)^{\mathrm{GL}_d(k)}.$$

It is an R-algebra in an obvious way, and enjoys a universal property similar to the one satisfied by $R^{\mathrm{coord},L}$, for which we refer to [4, §5.4].

Example 4.2. Continuing Example 4.1, assume $R = k[x_1, \ldots, x_d]$ and $L = \mathrm{Der}_k(R)$. We now have

$$R^{\mathrm{aff},L} = R[y_{i,\alpha} :\ i = 1, \ldots, d,\ |\alpha| \geq 2],$$

where $|\bullet|$ denotes the norm of a multiindex in $\mathbb{N}^d$. We observe that $R^{\mathrm{aff},L}$ is an (infinite) polynomial ring, while $R^{\mathrm{coord},L}$ is not, due to the localization.

Similarly, we have the DG-algebra $C^{\mathrm{aff},L} = \Omega_{R^{\mathrm{aff},L}} \otimes_{\Omega_{R_1}} \Omega^{L_1}(R_1)$, with differential $\mathrm{d}_{C^{\mathrm{aff},L}} = \mathrm{d}_{\Omega_{R^{\mathrm{aff},L}}} \otimes_{\Omega_{R_1}} 1 + 1 \otimes_{\Omega_{R_1}} \mathrm{d}_{L_1}$. We may further consider the graded algebra

$$C^{\mathrm{aff},L} \widehat{\otimes}_{R^{\mathrm{aff},L}} \left(R^{\mathrm{aff},L} \widehat{\otimes}_{R_1} JL\right) \cong \Omega_{R^{\mathrm{aff},L}} \widehat{\otimes}_{\Omega_{R_1}} \left(\Omega^{L_1}(R_1) \otimes_{R_1} JL\right) \cong C^{\mathrm{aff},L} \widehat{\otimes}_{R_1} JL,$$

endowed with the natural differential

$${}^1\nabla^{\mathrm{aff}} = \mathrm{d}_{\Omega_{R^{\mathrm{aff},L}}} \widehat{\otimes}_{\Omega_{R_1}} 1 + 1 \widehat{\otimes}_{\Omega_{R_1}} {}^1\nabla,$$

making the natural inclusion $C^{\mathrm{aff},L} \hookrightarrow C^{\mathrm{aff},L} \widehat{\otimes}_{R_1} JL$ into a morphism of DG-algebras. Obviously, ${}^1\nabla^{\mathrm{coord}}$ descends by its very construction to $C^{\mathrm{aff},L} \widehat{\otimes}_{R_1} JL$ and identifies with ${}^1\nabla^{\mathrm{aff}}$.

LEMMA 4.3. *$R^{\mathrm{aff},L}$ is of the form $S \otimes R$ where S is an (infinitely generated) polynomial ring.*

Proof. See [4, §5.3]. □

Note that this depends on our standing assumption that L is free and furthermore the decomposition $R^{\mathrm{aff},L} = S \otimes R$ is not canonical.

4.4. *Fedosov resolutions of L-poly-vector fields and L-poly-differential operators on R.* In this section, we recall briefly the main results of [4, §4.3], to which we refer for more details. We consider relative poly-differential operators and poly-vector fields (see §3.2) in the following situation: $(A, \mathrm{d}_A) = (C^{\mathrm{aff},L}, \mathrm{d}_{C^{\mathrm{aff},L}})$ and $(B, \mathrm{d}_B) = (C^{\mathrm{aff},L} \widehat{\otimes}_{R_1} JL, {}^1\nabla^{\mathrm{aff}})$.

THEOREM 4.4. *For a Lie algebroid L over R as above, there exist quasi-isomorphisms of Gerstenhaber algebras up to homotopy*:

$$\begin{aligned}(T_{\mathrm{poly}}^{L}(R),0,[\ ,\ ],\cup) &= (T_{\mathrm{poly}}^{L_2}(R_2),0,[\ ,\ ],\cup)\\ &\hookrightarrow \left(T_{\mathrm{poly},C^{\mathrm{aff},L}}(C^{\mathrm{aff},L}\widehat{\otimes}_{R_1}JL),{}^1\nabla^{\mathrm{aff}},[\ ,\ ],\cup\right),\end{aligned}\tag{4.6}$$

$$\begin{aligned}(D_{\mathrm{poly}}^{L}(R),\mathrm{d_H},[\ ,\ ],\cup) &= (D_{\mathrm{poly}}^{L_2}(R_2),\mathrm{d_H},[\ ,\ ],\cup)\\ &\hookrightarrow \left(D_{\mathrm{poly},C^{\mathrm{aff},L}}(C^{\mathrm{aff},L}\widehat{\otimes}_{R_1}JL),{}^1\nabla^{\mathrm{aff}}+\mathrm{d_H},[\ ,\ ],\cup\right).\end{aligned}\tag{4.7}$$

Proof. We refer to [4] for details. For example the map (4.7) is derived by suitable base extension from (3.12). For the fact that the maps are quasi-isomorphisms, we refer to [4, Prop. 7.3.1]. □

4.5. *The Fedosov resolution of L-forms on R.* We consider the precalculus $(\Omega^L(R),0,\mathrm{L},\cap)$ of L-forms over the Gerstenhaber algebra $(T_{\mathrm{poly}}^L(R),0,[\ ,\ ],\cup)$, described in Section 3.1. We now describe a resolution of $(\Omega^L(R),0,\mathrm{L},\cap)$ compatible with the Fedosov resolution $\left(T_{\mathrm{poly},C^{\mathrm{aff},L}}(C^{\mathrm{aff},L}\widehat{\otimes}_{R_1}JL),{}^1\nabla^{\mathrm{aff}},[\ ,\ ],\cup\right)$ from Theorem 4.4.

THEOREM 4.5. *For a Lie algebroid L over R as above, there exists a quasi-isomorphism of precalculi as in the following commutative diagram:*[6]

$$\begin{tikzcd}(T_{\mathrm{poly}}^{L}(R),0,[\ ,\ ],\cup)=(T_{\mathrm{poly}}^{L_2}(R_2),0,[\ ,\ ],\cup) \arrow[r, hook] \arrow[d, squiggly] & \left(T_{\mathrm{poly},C^{\mathrm{aff},L}}(C^{\mathrm{aff},L}\widehat{\otimes}_{R_1}JL),{}^1\nabla^{\mathrm{aff}},[\ ,\ ],\cup\right) \arrow[d, squiggly]\\ (\Omega^L(R),0,\mathrm{L},\cap)=(\Omega^{L_2}(R_2),0,\mathrm{L},\cap) \arrow[r, hook] & \left(\Omega_{C^{\mathrm{aff},L}\widehat{\otimes}_{R_1}JL/C^{\mathrm{aff},L}},{}^1\nabla^{\mathrm{aff}},\mathrm{L},\cap\right),\end{tikzcd}$$

the vertical arrows denoting the contraction and Lie derivative.

Proof. We refer to [4, §4.3.3]. We observe that the construction of the quasi-isomorphism uses a dualization of the construction of the quasi-isomorphism (4.6) and that contraction operations and differentials are preserved by the above quasi-isomorphism, whence all algebraic structures are preserved. □

4.6. *The Fedosov resolution of L-chains on R.* We consider the DG-algebra $(C^{\mathrm{aff},L}\widehat{\otimes}_{R_1}JL,{}^1\nabla^{\mathrm{aff}})$, and to it we associate the $C^{\mathrm{aff},L}$-relative Hochschild chain complex, i.e.,

$$\begin{aligned}\widehat{C}_{C^{\mathrm{aff},L},\bullet}(C^{\mathrm{aff},L}\widehat{\otimes}_{R_1}JL) &= \bigoplus_{p\le 0}\left(C^{\mathrm{aff},L}\widehat{\otimes}_{R_1}JL\right)^{\widehat{\otimes}_{C^{\mathrm{aff},L}}(-p+1)}\\ &\cong \bigoplus_{p\le 0}\left(C^{\mathrm{aff},L}\widehat{\otimes}_{R_1}JL^{\widehat{\otimes}_{R_1}(-p+1)}\right)=C^{\mathrm{aff},L}\widehat{\otimes}_{R_1}\widehat{C}_{R_1,\bullet}(JL).\end{aligned}$$

[6] Ω_A, for a topological k-algebra A, denotes the continuous de Rham complex. A similar convention holds for an extension of topological algebras B/A.

Further, we have the identification

$$C^{\mathrm{aff},L}\widehat{\otimes}_{R_1}\widehat{C}_{R_1,\bullet}(JL) \cong \Omega_{R^{\mathrm{aff},L}}\widehat{\otimes}_{\Omega_{R_1}}\left(\Omega^{L_1}(R_1)\widehat{\otimes}_{R_1}\widehat{C}_{R_1,\bullet}(JL)\right),$$

and one checks that the differentials coming from the Grothendieck connection on each side are the same. That is,

$${}^1\nabla^{\mathrm{aff}} = \mathrm{d}_{\Omega_{R^{\mathrm{aff},L}}}\widehat{\otimes}\, 1 + 1\,\widehat{\otimes}\, {}^1\nabla.$$

PROPOSITION 4.6. *For a Lie algebroid L over R as above, the cohomology of $\left(\widehat{C}_{C^{\mathrm{aff},L},\bullet}(C^{\mathrm{aff},L}\widehat{\otimes}_{R_1}JL), {}^1\nabla^{\mathrm{aff}}\right)$ is concentrated in degree 0, where*

$$\mathrm{H}^0\left(\widehat{C}_{C^{\mathrm{aff},L},\bullet}(C^{\mathrm{aff},L}\widehat{\otimes}_{R_1}JL), {}^1\nabla^{\mathrm{aff}}\right) \cong C^L_{\mathrm{poly}}(R).$$

Proof. Taking the inverse of (3.15) we obtain a morphism

$$C^L_{\mathrm{poly}}(R) \cong \widehat{C}_{R_1,\bullet}(JL)^{{}^1\nabla} \hookrightarrow \widehat{C}_{R_1,\bullet}(JL)$$

that extends to a morphism

$$(C^L_{\mathrm{poly}}(R),0) \to (\Omega_{R^{\mathrm{aff},L}}\widehat{\otimes}_{\Omega_{R_1}}\left(\Omega^{L_1}(R_1)\widehat{\otimes}_{R_1}\widehat{C}_{R_1,\bullet}(JL)\right), \mathrm{d}_{\Omega_{R^{\mathrm{aff},L}}}\otimes 1 + 1\otimes {}^1\nabla). \tag{4.8}$$

We will show that it is a quasi-isomorphism. To this end we make use of the identification $R^{\mathrm{aff},L} = S\otimes R$ given in Lemma 4.3. The right-hand side of the extended morphism becomes

$$(\Omega_S\,\widehat{\otimes}\,\left(\Omega^{L_1}(R_1)\widehat{\otimes}_{R_1}\widehat{C}_{R_1,\bullet}(JL)\right), \mathrm{d}_S\otimes 1 + 1\otimes {}^1\nabla).$$

Using a filtration argument together with a suitable version of Poincaré's Lemma for S, the previous complex is quasi-isomorphic to

$$(\Omega^{L_1}(R_1)\otimes_{R_1}\widehat{C}_{R_1,\bullet}(JL), {}^1\nabla).$$

It remains to show that for each $p\le 0$,

$$(\Omega^{L_1}(R_1)\otimes_{R_1}\widehat{C}_{R_1,p}(JL), {}^1\nabla)$$

has cohomology in degree 0. Filtering this complex with respect to the J-adic filtration and taking the associated graded complex one verifies that one obtains

$$(\Omega^L(R)\otimes_{R_1}\mathrm{S}(L^*)^{\otimes -p-1}, \mathrm{d}),$$

where the differential d is obtained from the action of L on $\mathrm{S}(L^*)^{\otimes p+1}$ by contraction. Using again a suitable version of Poincaré's Lemma one finds that the resulting complex is indeed exact in degrees <0. □

THEOREM 4.7. *For a Lie algebroid L over R as above, there is a quasi-isomorphism of precalculi up to homotopy as in the following commutative* *diagram*:

$$\begin{tikzcd}
(D^L_{\mathrm{poly}}(R), \mathrm{d_H}, [\ ,\ ], \cup) \arrow[r, hook] \arrow[d, squiggly] & \left(D_{\mathrm{poly},C^{\mathrm{aff},L}}(C^{\mathrm{aff},L}\widehat{\otimes}_{R_1} JL), {}^1\nabla^{\mathrm{aff}} + \mathrm{d_H}, [\ ,\ ], \cup\right) \arrow[d, squiggly] \\
(C^L_{\mathrm{poly}}(R), \mathrm{b_H}, \mathrm{L}, \cap) \arrow[r, hook] & \left(\widehat{C}_{C^{\mathrm{aff},L},\bullet}(C^{\mathrm{aff},L}\widehat{\otimes}_{R_1} JL), {}^1\nabla^{\mathrm{aff}} + \mathrm{b_H}, \mathrm{L}, \cap\right),
\end{tikzcd}$$

the vertical arrows denoting the contraction and Lie derivative.

5. Globalization of Tsygan's formality in the Lie algebroid framework

The present section is devoted to the proof of Theorem 1.3. We first briefly review some basic facts on L_∞-algebras, L_∞-modules and related morphisms. This is discussed in [4, §6] for L_∞-morphisms. Here we add a discussion on the descent procedure for L_∞-modules over L_∞-algebras and related morphisms.

Then we add a short *excursus* on Kontsevich's and Shoikhet's formality theorems. We focus on the main properties of both formality morphisms, without delving into the technical details of their respective constructions.

Finally, we give the main lines, along which the globalization of Tsygan's formality can be proved. The proof is a combination of the properties of Kontsevich's and Shoikhet's L_∞-morphisms with the Fedosov resolutions from Section 4.

5.1. *Descent for L_∞-algebras and L_∞-modules.* We discuss a series of descent scenarios for L_∞-algebras, L_∞-modules and related morphisms, which are modelled after the formalism for descent of differential forms in differential geometry. The verification of the results in this section are along the same lines as [2, §§7.6, 7.7]. To clearly separate all the various cases we have numbered them.

(1) To start it is convenient to work over an arbitrary DG operad $\mathcal{O}$ with underlying graded operad $\widetilde{\mathcal{O}}$. (Thus, we forget the differential on $\mathcal{O}$.) Assume that $\mathfrak{g}$ is an algebra over $\mathcal{O}$ and consider a set of $\widetilde{\mathcal{O}}$-derivations $(\iota_v)_{v\in\mathfrak{s}}$ of degree -1 on $\mathfrak{g}$. ($\mathfrak{s}$ is an index set, without any additional structure.) Put $\mathrm{L}_v = \mathrm{d}_\mathfrak{g}\iota_v + \iota_v\mathrm{d}_\mathfrak{g}$. This is a derivation of $\mathfrak{g}$ of degree 0 that commutes with $\mathrm{d}_\mathfrak{g}$. Put

$$\mathfrak{g}^\mathfrak{s} = \{w \in \mathfrak{g} \mid \forall v \in \mathfrak{s} : \iota_v w = \mathrm{L}_v w = 0\}. \tag{5.1}$$

It is easy to see that $\mathfrak{g}^\mathfrak{s}$ is an algebra over $\mathcal{O}$ as well. Informally we will call such a set of derivations $(\iota_v)_{v\in\mathfrak{s}}$ an $\mathfrak{s}$-action.

(2) Assume that M is a $\mathfrak{g}$-module, and assume that $\mathfrak{s}$ also acts on M, in a way compatible with the action of $\mathfrak{s}$ on $\mathfrak{g}$; i.e., a general element v of $\mathfrak{s}$ determines an operator ι_v on M, such that Leibniz's rule holds true for the

operations $\widetilde{\mathcal{O}}(n)\otimes(\mathfrak{g}^{\otimes n-1}\otimes M)\to M$. Again, we set $\mathrm{L}_v=\mathrm{d}_M\iota_v+\iota_v\mathrm{d}_M$, which is a derivation of degree 0 on M compatible with the derivations L_v on $\mathfrak{g}$, d_M being the differential on M.

(3) The above constructions apply, in particular, if $\mathfrak{g}$ is an L_∞-algebra. Assume that it has Taylor coefficients Q_n, $n\geq 1$. Then L_v is defined by means of $\mathrm{d}_{\mathfrak{g}}=Q_1$, and the derivation property of ι_v reads as

$$(5.2)\quad \iota_v\left(Q_n(x_1,\dots,x_n)\right) = \sum_{i=1}^n(-1)^{\sum_{j=1}^{i-1}|x_j|+i}Q_n(x_1,\dots,\iota_v x_i,\dots,x_n),\ x_j\in\mathfrak{g},\ j=1,\dots,n.$$

Under these conditions the L_∞-structure descends to $\mathfrak{g}^{\mathfrak{s}}$.

(4) Similarly, if M is an L_∞-module over $\mathfrak{g}$ defined by Taylor coefficients R_n, then the compatibility condition is

$$(5.3)\quad \begin{aligned}\iota_v\left(R_n(x_1,\dots,x_n;m)\right) &= \sum_{i=1}^n(-1)^{\sum_{j=1}^{i-1}|x_j|+i}R_n(x_1,\dots,\iota_v x_i,\dots,x_n;m)\\ &\quad+(-1)^{\sum_{i=1}^{n}|x_i|+n-1}R_n(x_1,\dots,x_n;\iota_v m),\ m\in M,\ x_j\in\mathfrak{g},\ j=1,\dots,n.\end{aligned}$$

If this holds true, then $M^{\mathfrak{s}}$ becomes an L_∞-module over $\mathfrak{g}^{\mathfrak{s}}$.

(5) We also need descent for L_∞-morphisms. This does not immediately fall under the operadic framework given in (1), (2) but it is easy enough to give explicit formulae like (5.2), (5.3). Thus, assume $\psi:\mathfrak{g}\to\mathfrak{h}$ is an L_∞-morphism between L_∞-algebras with $\mathfrak{s}$-action. Under the following compatibility condition

$$(5.4)\qquad \iota_v(\psi_n(x_1,\dots,x_n))=\sum_{i=1}^n(-1)^{\sum_{j=1}^{i-1}|x_j|+(i-1)}\psi_n(x_1,\dots,\iota_v x_i,\dots,x_n)$$

for $x_j\in\mathfrak{g},\ j=1,\dots,n,\ n\geq 1$, ψ descends to an L_∞-morphism $\psi^{\mathfrak{s}}:\mathfrak{g}^{\mathfrak{s}}\to\mathfrak{h}^{\mathfrak{s}}$.

(6) Let $\psi:\mathfrak{g}\to\mathfrak{h}$ be a morphism between L_∞-algebras with $\mathfrak{s}$-action such that the descent condition (5.4) holds, and let N be an L_∞-module over $\mathfrak{h}$ equipped with a compatible $\mathfrak{s}$-action. Let N_ψ be the pullback of N along ψ. Then the $\mathfrak{s}$-action on N_ψ is compatible with the $\mathfrak{s}$-action on $\mathfrak{g}$.

(7) Now assume that $\mathfrak{g}$ is an L_∞-algebra and M, N are L_∞-modules over $\mathfrak{g}$. Assume that all objects are equipped with an $\mathfrak{s}$-action and that the descent conditions are satisfied.

Assume that $\varphi : M \to N$ is an L_∞-module morphism. Then the condition for φ to descend to an L_∞-morphism $M^{\mathfrak{s}} \to N^{\mathfrak{s}}$ is
(5.5)

$$\iota_v\left(\varphi_n(x_1,\ldots,x_n;m)\right) = \sum_{i=1}^{n}(-1)^{\sum_{j=1}^{i-1}|x_j|+(i-1)}\varphi_n(x_1,\ldots,\iota_v x_i,\ldots,x_n;m) + (-1)^{\sum_{i=1}^{n}|x_i|+n}\varphi_n(x_1,\ldots,x_n;\iota_v m)$$

for $m \in M,\ x_j \in \mathfrak{g},\ j = 1,\ldots,n,\ n \geq 1$.

5.2. *Twisting of L_∞-algebras and L_∞-modules.* We refer to [11, §2], for a very detailed exposition of L_∞-algebras, L_∞-modules and the associated twisting procedures. See also [29].

Convention. We will work with infinite sums. We assume throughout that the occurring sums are convergent and that standard series manipulations are allowed. This will be the case in our applications.

If $(\mathfrak{g}, Q)$ is an L_∞-algebra, then the Maurer-Cartan equation is defined as

$$\sum_{j=1}^{\infty}\frac{1}{j!}Q_n(\underbrace{\omega,\cdots,\omega}_{j}) = 0, \tag{5.6}$$

and a solution $\omega \in \mathfrak{g}_1$ is called a Maurer–Cartan element (MC element for short). Below we will only use DG-Lie algebras and in this case (5.6) reduces to the finite sum

$$d\omega + \frac{1}{2}[\omega,\omega] = 0.$$

An MC element defines a new "twisted" DG-Lie structure on $\mathfrak{g}$ (denoted by $\mathfrak{g}_\omega$) with Taylor coefficients

$$Q_{\omega,n}(x_1,\ldots,x_n) = \sum_j \frac{1}{j!}Q_{n+j}(\underbrace{\omega,\ldots,\omega}_{j},x_1,\ldots,x_n), \quad n \geq 1.$$

If $\mathfrak{g}$ is actually a DG-Lie algebra, then twisting keeps the bracket but changes the differential to

$$\mathrm{d}_\omega = \mathrm{d}_{\mathfrak{g}} + [\omega,-].$$

If $\mathfrak{h}$ is another L_∞ algebra, ψ is an L_∞-morphism from $\mathfrak{g}$ to $\mathfrak{h}$ and ω is an MC element in $\mathfrak{g}$, then

$$\psi(\omega) = \sum_{n\geq 1}\frac{1}{n!}\psi_n(\underbrace{\omega,\ldots,\omega}_{n}) \tag{5.7}$$

is an MC element in $\mathfrak{h}$.

We may also twist ψ with respect to ω, so as to get an L_∞-morphism ψ_ω from $\mathfrak{g}_\omega$ to $\mathfrak{g}_{\psi(\omega)}$, where

$$\psi_{\omega,n}(x_1,\ldots,x_n)=\sum_{j\geq 0}\frac{1}{j!}\psi_{n+j}(\underbrace{\omega,\ldots,\omega}_{j},x_1,\ldots,x_n),\ n\geq 1.$$

If M is an L_∞-module over a DG-Lie algebra with Taylor coefficients R_n and $\omega\in\mathfrak{g}_1$ is an MC element, then we may define a twisted L_∞ structure on M_ω over $\mathfrak{g}_\omega$ by the formula

$$R_{\omega,n}(x_1,\ldots,x_n;m)=\sum_{j\geq 0}\frac{1}{j!}R_{n+j}(\underbrace{\omega,\ldots,\omega}_{j},x_1,\ldots,x_n;m),\quad n\geq 0.$$

If $\mathfrak{g}$ is a DG-Lie algebra and M is a DG-Lie module over $\mathfrak{g}$, then twisting keeps the $\mathfrak{g}$-action on M but changes the differential on M to

$$\mathrm{d}_\omega=d+\omega\bullet .$$

Twisting of modules is compatible with pullback. More precisely if $\psi:\mathfrak{g}\to\mathfrak{h}$ is an L_∞-morphism, N is an L_∞-module over $\mathfrak{h}$ and $\omega\in\mathfrak{g}_1$ is an MC element, then we have

$$(N_{\psi(\omega)})_{\psi_\omega}=(N_\psi)_\omega. \tag{5.8}$$

If $\varphi:M\to N$ is an L_∞-morphism of DG-Lie modules over the DG-Lie algebra $\mathfrak{g}$ and ω is an MC element in $\mathfrak{g}_1$, then we obtain a twisted L_∞-morphism $\varphi_\omega:M_\omega\to N_\omega$, which is defined by

$$\varphi_{\omega,n}(x_1,\ldots,x_n;m)=\sum_{j\geq 0}\frac{1}{j!}\varphi_{n+j}(\underbrace{\omega,\ldots,\omega}_{j},x_1,\ldots,x_n;m),\ n\geq 1. \tag{5.9}$$

5.3. *Compatibility of twisting and descent.* Assume now that $\mathfrak{g}$ is a DG-Lie algebra equipped with an $\mathfrak{s}$-action and that $\omega\in\mathfrak{g}_1$ is an MC element. Then $\mathfrak{s}$ still acts on $\mathfrak{g}_\omega$, where we forget here about the differential: in fact, the concept of an $\mathfrak{s}$-action only refers to the underlying Lie algebra structure on $\mathfrak{g}$. However $\mathfrak{g}^{\mathfrak{s}}$ and $\mathfrak{g}^{\mathfrak{s}}_\omega$ will be different (as the Lie derivative L_v for $v\in\mathfrak{s}$ will be different).

If (M,R) is an L_∞-module over $\mathfrak{g}$ that is also equipped with a compatible $\mathfrak{s}$-action, then the $\mathfrak{s}$-actions on $\mathfrak{g}_\omega$ and M_ω are compatible provided the following condition holds:

$$R_n(\iota_v\omega,x_2,\ldots,x_n;m)=0, x_i\in\mathfrak{g},\ i=2,\ldots,n,\ n\geq 2,\ m\in M. \tag{5.10}$$

This condition is automatic if M is a DG-Lie module.

If $\psi:\mathfrak{g}\to\mathfrak{h}$ is an L_∞-morphism of DG-Lie algebras equipped with an $\mathfrak{s}$-action and the descent condition (5.4) is satisfied for ψ, then an easy computation (see, e.g., [2, §7.7]) shows that the same descent condition will be satisfied for ψ_ω if the following condition holds:

$$\psi_n(\iota_v\omega, x_2, \dots, x_n) = 0,\ x_i \in \mathfrak{g},\ i = 2, \dots, n\ s \in \mathfrak{s},\ n \geq 2. \tag{5.11}$$

Furthermore, if in this setting N is an L_∞-module over $\mathfrak{h}$ with compatible $\mathfrak{s}$-action such that the compatibility condition (5.10) holds, then the corresponding condition will hold for N_ψ.

Similarly, if we have an L_∞-morphism $\varphi : M \to N$ between DG-Lie modules over a DG-Lie algebra $\mathfrak{g}$ such that $\mathfrak{g}$, M, N are equipped with compatible $\mathfrak{s}$-actions in such a way that the descent condition (5.5) holds for φ then the same descent condition will be satisfied for φ_ω if the following condition holds:

$$\varphi_n(\iota_v\omega, x_2, \dots, x_n; m) = 0,\ m \in M,\ x_i \in \mathfrak{g},\ i = 2, \dots, n\ n \geq 2,\ s \in \mathfrak{s}.$$

5.4. *Kontsevich's and Shoikhet's formality theorems.* In this brief section, we quote (without proofs) Kontsevich's and Shoikhet's formality theorems, along with the relevant properties, which we will need later in the proof of globalization results.

We consider the algebra $F = k[\![x_1, \dots, x_d]\!]$ of formal power series in d variables over a field k containing $\mathbb{R}$.

To F, we associate the DG-Lie algebras $(T_{\mathrm{poly}}(F), 0, [\ ,\ ])$, resp. $(D_{\mathrm{poly}}(F), \mathrm{d_H}, [\ ,\ ])$, of formal poly-vector fields, resp. formal poly-differential operators, on F; further, we consider the DG-Lie modules $(\Omega_F, 0, \mathrm{L})$, resp. $(\widehat{C}_\bullet(F), \mathrm{b_H}, \mathrm{L})$, over $(T_{\mathrm{poly}}(F), 0, [\ ,\ ])$, resp. $(D_{\mathrm{poly}}(F), \mathrm{d_H}, [\ ,\ ])$, where Ω_F denotes the continuous de Rham complex of F with de Rham differential d, and $\widehat{C}_\bullet(F)$ is the continuous Hochschild chain complex of F.

The following is Kontsevich's celebrated "formality" result.

THEOREM 5.1 ([16]). *There is an L_∞-quasi-isomorphism*

$$\mathcal{U} : (T_{\mathrm{poly}}(F), 0, [\ ,\ ]) \to (D_{\mathrm{poly}}(F), \mathrm{d_H}, [\ ,\ ]),$$

enjoying the following properties:

(i) *The first Taylor coefficient of $\mathcal{U}$ coincides with the Hochschild–Kostant–Rosenberg quasi-isomorphism (of DG-vector spaces)*

$$\mathrm{HKR}(\partial_{i_1} \wedge \cdots \wedge \partial_{i_p}) = (-1)^{\frac{p(p-1)}{2}} \frac{1}{p!} \sum_{\sigma \in \mathfrak{S}_p} (-1)^\sigma \partial_{i_{\sigma(1)}} \otimes \cdots \otimes \partial_{i_{\sigma(p)}}$$

from $(T_{\mathrm{poly}}(F), 0)$ to $(D_{\mathrm{poly}}(F), \mathrm{d_H})$.

(ii) *If $n \geq 2$, and γ_i, $i = 1, \dots, n$, are elements of $T^0_{\mathrm{poly}}(F)$, then*

$$\mathcal{U}_n(\gamma_1, \dots, \gamma_n) = 0.$$

(iii) *If $n \geq 2$, γ_1 is a linear vector field on F (i.e., an element of $\mathfrak{gl}_d$), γ_i, $i = 2, \dots, n$ are general elements of $T_{\mathrm{poly}}(F)$, then*

$$\mathcal{U}_n(\gamma_1, \gamma_2, \dots, \gamma_n) = 0.$$

By composing the action L of $D_{\mathrm{poly}}(F)$ on $\widehat{C}_\bullet(F)$ with the L_∞-quasi-isomorphism $\mathcal{U}$ from Theorem 5.1, $\widehat{C}_\bullet(F)$ inherits an L_∞-module structure over the DG-Lie algebra $(T_{\mathrm{poly}}(F), 0, [\ ,\ ])$.

The first part of the following theorem was a conjecture by Tsygan [25], which was proved by Shoikhet in [20]. The second part was proved in [11].

THEOREM 5.2. *There is an L_∞-quasi-isomorphism*

$$\mathcal{S} : (\widehat{C}_\bullet(F), \mathrm{b_H}, \mathrm{L} \circ \mathcal{U}) \to (\Omega_F, 0, \mathrm{L})$$

of L_∞-modules over the DG-*Lie algebra* $(T_{\mathrm{poly}}(F), 0, [\ ,\ ])$, *enjoying the following properties*:

(i) *The 0-th Taylor coefficient of $\mathcal{S}$ coincides with the Hochschild–Kostant–Rosenberg quasi-isomorphism*

$$\mathrm{HKR}((a_0|\cdots|a_p)) = \frac{1}{p!} a_0 \mathrm{d}a_1 \cdots \mathrm{d}a_p$$

from the DG-*vector space* $(\widehat{C}_\bullet(F), \mathrm{b_H})$ *to the* DG-*vector space* $(\Omega_F, 0)$.

(ii) *If $n \geq 1$, γ_1 is a linear vector field on F, γ_i, $i = 2, \ldots, n$ are general elements of $T_{\mathrm{poly}}(F)$ and c is a general element of $\widehat{C}_\bullet(F)$, then*

$$\mathcal{S}_n(\gamma_1, \ldots, \gamma_n; c) = 0.$$

5.5. *Formality theorem in the ring case.* This section is devoted to the proof of a Tsygan-like formality theorem in the case of a Lie algebroid L over a k-algebra R, such that L is free over R of rank d. The proof combines Shoikhet's Formality Theorem 5.2 with the Fedosov resolutions from Section 4.

THEOREM 5.3. *Assume $\mathbb{R} \subset k$. For a Lie algebroid L over R as above, there exist* DG-*Lie algebras* $(\mathfrak{g}_i^L, \mathrm{d}_i, [\ ,\ ]_i)$, DG-*Lie modules* $(\mathfrak{m}_i^L, \mathrm{b}_i, \mathrm{L}_i)$ *over* $\mathfrak{g}_i$, $i = 1, 2$, *and L_∞-quasi-isomorphisms $\mathfrak{U}_L$ from $\mathfrak{g}_1^L$ to $\mathfrak{g}_2^L$ and $\mathfrak{S}_L$ from $\mathfrak{m}_2^L$ to $\mathfrak{m}_1^L$ that fit into the following commutative diagram*:

$$\begin{tikzcd} T^L_{\mathrm{poly}}(R) \arrow[r, hook] \arrow[d, squiggly, "\mathrm{L}"'] & \mathfrak{g}_1^L \arrow[r, "\mathfrak{U}_L"] \arrow[d, squiggly, "\mathrm{L}_1"'] & \mathfrak{g}_2^L \arrow[d, squiggly, "\mathrm{L}_2"'] & D^L_{\mathrm{poly}}(R) \arrow[l, hook'] \arrow[d, squiggly, "\mathrm{L}"'] \\ \Omega^L(R) \arrow[r, hook] & \mathfrak{m}_1^L & \mathfrak{m}_2^L \arrow[l, "\mathfrak{S}_L"'] & C^L_{\mathrm{poly}}(R) \arrow[l, hook'] \end{tikzcd} \tag{5.12}$$

such that the induced maps

$$T^L_{\mathrm{poly}}(R) \to \mathrm{H}^\bullet(D^L_{\mathrm{poly}}(R), \mathrm{d_H}),\ \mathrm{H}^\bullet(C^L_{\mathrm{poly}}(R), \mathrm{b_H}) \to \Omega^L(R)$$

on (co)homology coincide with the respective HKR-*quasi-isomorphisms. The morphisms indicated by hooked arrows are actual quasi-isomorphisms of* DG-*Lie algebras and* DG-*Lie modules respectively.*

Proof. The first step in the proof of Theorem 5.3 may be borrowed from [4, §7.3]. Namely, we consider the following graded vector spaces:

$$C^{\mathrm{coord},L}\widehat{\otimes}T_{\mathrm{poly}}(F)\cong T_{\mathrm{poly},C^{\mathrm{coord},L}}(C^{\mathrm{coord},L}\widehat{\otimes}F),$$
$$C^{\mathrm{coord},L}\widehat{\otimes}D_{\mathrm{poly}}(F)\cong D_{\mathrm{poly},C^{\mathrm{coord},L}}(C^{\mathrm{coord},L}\widehat{\otimes}F),$$
$$C^{\mathrm{coord},L}\widehat{\otimes}\Omega_F\cong \Omega_{C^{\mathrm{coord},L}\widehat{\otimes}F/C^{\mathrm{coord},L}},$$
$$C^{\mathrm{coord},L}\widehat{\otimes}\widehat{C}_\bullet(F)\cong \widehat{C}_{C^{\mathrm{coord},L},\bullet}(C^{\mathrm{coord},L}\widehat{\otimes}F),$$

where $\widehat{C}_{C^{\mathrm{coord},L},\bullet}(C^{\mathrm{coord},L}\widehat{\otimes}F)$ denotes the $C^{\mathrm{coord},L}$-relative Hochschild chain complex of the DG-algebra $C^{\mathrm{coord},L}\widehat{\otimes}F$. (Recall that the DG-algebra $C^{\mathrm{coord},L}$ was introduced in Section 4.3.)

The Maurer–Cartan form on $C^{\mathrm{coord},L}\widehat{\otimes}F$ introduced in Section 4.3 defines a twisted differential $\mathrm{d}_\omega=\mathrm{d}+\omega$ on the listed graded vector spaces and as explained in Section 5.2 d_ω is compatible with the respective DG-Lie algebra and DG-Lie module structures.

Thus, formal geometry provides us with the following DG-Lie algebras and respective DG-Lie modules:

$$\begin{array}{cc}
\left(T_{\mathrm{poly},C^{\mathrm{coord},L}}(C^{\mathrm{coord},L}\widehat{\otimes}F),\mathrm{d}_\omega,[\ ,\ ]\right), & \left(D_{\mathrm{poly},C^{\mathrm{coord},L}}(C^{\mathrm{coord},L}\widehat{\otimes}F),\mathrm{d}_\omega+\mathrm{d}_{\mathrm{H}},[\ ,\ ]\right)\\
\rightsquigarrow & \rightsquigarrow\\
\left(\Omega_{C^{\mathrm{coord},L}\widehat{\otimes}F/C^{\mathrm{coord},L}},\mathrm{d}_\omega,\mathrm{L}\right) & \left(\widehat{C}_{C^{\mathrm{coord},L}}(C^{\mathrm{coord},L}\widehat{\otimes}F),\mathrm{d}_\omega+\mathrm{b}_{\mathrm{H}},\mathrm{L}\right).
\end{array}$$

We repeat that, viewing all DG-Lie algebra and DG-Lie module structures above as L_∞-structures, the differential d_ω is the twist of the standard structures with respect to the MC element ω of

$$C^{\mathrm{coord},L}\widehat{\otimes}\mathrm{Der}(F)=T^0_{\mathrm{poly},C^{\mathrm{coord},L}}(C^{\mathrm{coord},L}\widehat{\otimes}F).$$

The L_∞-quasi-isomorphism $\mathcal{U}$ of Theorem 5.1 extends $C^{\mathrm{coord},L}$-linearly to an L_∞-quasi-isomorphism

$$\mathcal{U}_L:\left(T_{\mathrm{poly},C^{\mathrm{coord},L}}(C^{\mathrm{coord},L}\widehat{\otimes}F),\mathrm{d},[\ ,\ ]\right)\\ \to\left(D_{\mathrm{poly},C^{\mathrm{coord},L}}(C^{\mathrm{coord},L}\widehat{\otimes}F),\mathrm{d}+\mathrm{d}_{\mathrm{H}},[\ ,\ ]\right).$$

The composition of the DG-Lie action L of $D_{\mathrm{poly},C^{\mathrm{coord},L}}(C^{\mathrm{coord},L}\widehat{\otimes}F)$ on $\widehat{C}_{C^{\mathrm{coord},L},\bullet}(C^{\mathrm{coord},L}\widehat{\otimes}F)$ with the L_∞-quasi-isomorphism $\mathcal{U}_L$ endows the latter graded vector space with a structure of L_∞-module over the DG-Lie algebra $T_{\mathrm{poly},C^{\mathrm{coord},L}}(C^{\mathrm{coord},L}\widehat{\otimes}F)$, which is obtained by $C^{\mathrm{coord},L}$-base extension of the corresponding L_∞-module structure of $\widehat{C}_\bullet(F)$ over $T_{\mathrm{poly}}(F)$.

Accordingly, the L_∞-quasi-isomorphism $\mathcal{S}$ of Theorem 5.2 extends to an L_∞-quasi-isomorphism of L_∞-modules

$$\mathcal{S}_L:\left(\widehat{C}_{C^{\mathrm{coord},L},\bullet}(C^{\mathrm{coord},L}\widehat{\otimes}F),\mathrm{d}+\mathrm{b}_{\mathrm{H}},\mathrm{L}\circ\mathcal{U}_L\right)\to\left(\Omega_{C^{\mathrm{coord},L}\widehat{\otimes}F/C^{\mathrm{coord},L}},\mathrm{d},L\right),$$

both viewed as L_∞-modules over $T_{\mathrm{poly},C^{\mathrm{coord},L}}(C^{\mathrm{coord},L}\widehat{\otimes}F)$.

As outlined in Section 5.2 we may apply the twisting procedures for L_∞-algebras, L_∞-modules and L_∞-morphisms to the present case, where the MC element is the Maurer–Cartan form ω. Thus, we get an L_∞-morphism $\mathcal{U}_{L,\omega}$

$$\mathcal{U}_{L,\omega} : \left(T_{\mathrm{poly},C^{\mathrm{coord},L}}(C^{\mathrm{coord},L}\widehat{\otimes} F), \mathrm{d}_\omega, [\ ,\ ]\right) \\ \to \left(D_{\mathrm{poly},C^{\mathrm{coord},L}}(C^{\mathrm{coord},L}\widehat{\otimes} F), \mathrm{d}_\omega + \mathrm{d}_{\mathrm{H}}, [\ ,\ ]\right),$$

where here and below we used Property (ii) of Theorem 5.1, which yields that the MC element $\mathcal{U}(\omega)$ equals ω. The L_∞-morphism $U_{L,\omega}$ yields an L_∞-module structure on $\widehat{C}_{C^{\mathrm{coord},L},\bullet}(C^{\mathrm{coord},L}\widehat{\otimes} F)$ over the ω-twisted DG-Lie algebra $T_{\mathrm{poly},C^{\mathrm{coord},L}}(C^{\mathrm{coord},L}\widehat{\otimes} F)$.

Translating (5.8) to the present case, we have

$$\left(\widehat{C}_{C^{\mathrm{coord},L},\bullet}(C^{\mathrm{coord},L}\widehat{\otimes} F), \mathrm{d}_\omega + \mathrm{b}_{\mathrm{H}}, \mathrm{L}\circ\mathcal{U}_{L,\omega}\right) \\ = \left(\widehat{C}_{C^{\mathrm{coord},L},\bullet}(C^{\mathrm{coord},L}\widehat{\otimes} F), \mathrm{d} + \mathrm{b}_{\mathrm{H}}, \mathrm{L}\circ\mathcal{U}_{L}\right)_\omega$$

from which we get an L_∞-quasi-isomorphism

$$\mathcal{S}_{L,\omega} : \left(\widehat{C}_{C^{\mathrm{coord},L},\bullet}(C^{\mathrm{coord},L}\widehat{\otimes} F), \mathrm{d}_\omega + \mathrm{b}_{\mathrm{H}}, \mathrm{L}\circ\mathcal{U}_{L,\omega}\right) \\ \to \left(\Omega_{C^{\mathrm{coord},L}\widehat{\otimes} F/C^{\mathrm{coord},L}}, \mathrm{d}_\omega, \mathrm{L}\right)$$

of L_∞-modules.

Using the isomorphism (4.2) we obtain isomorphisms of DG-Lie algebras and respective DG-Lie modules

$$\left(T_{\mathrm{poly},C^{\mathrm{coord},L}}(C^{\mathrm{coord},L}\widehat{\otimes} F), \mathrm{d}_\omega, [\ ,\ ]\right) \\ \cong \left(T_{\mathrm{poly},C^{\mathrm{coord},L}}(C^{\mathrm{coord},L}\widehat{\otimes}_{R_1} JL), {}^1\nabla^{\mathrm{coord}}, [\ ,\ ]\right),$$

$$\left(D_{\mathrm{poly},C^{\mathrm{coord},L}}(C^{\mathrm{coord},L}\widehat{\otimes} F), \mathrm{d}_\omega + \mathrm{d}_{\mathrm{H}}, [\ ,\ ]\right) \\ \cong \left(D_{\mathrm{poly},C^{\mathrm{coord},L}}(C^{\mathrm{coord},L}\widehat{\otimes}_{R_1} JL), {}^1\nabla^{\mathrm{coord}} + \mathrm{d}_{\mathrm{H}}, [\ ,\ ]\right),$$

$$\left(\Omega_{C^{\mathrm{coord},L}\widehat{\otimes} F/C^{\mathrm{coord},L}}, \mathrm{d}_\omega, \mathrm{L}\right) \\ \cong \left(\Omega_{C^{\mathrm{coord},L}\widehat{\otimes}_{R_1} JL/C^{\mathrm{coord},L}}, {}^1\nabla^{\mathrm{coord}}, \mathrm{L}\right),$$

$$\left(\widehat{C}_{C^{\mathrm{coord},L},\bullet}(C^{\mathrm{coord},L}\widehat{\otimes} F), \mathrm{d}_\omega + \mathrm{b}_{\mathrm{H}}, \mathrm{L}\right) \\ \cong \left(\widehat{C}_{C^{\mathrm{coord},L},\bullet}(C^{\mathrm{coord},L}\widehat{\otimes}_{R_1} JL), {}^1\nabla^{\mathrm{coord}} + \mathrm{b}_{\mathrm{H}}, \mathrm{L}\right),$$

an L_∞-morphism

$$\mathcal{U}_L^{\mathrm{coord}} : \left(T_{\mathrm{poly},C^{\mathrm{coord},L}}(C^{\mathrm{coord},L}\widehat{\otimes}_{R_1} JL), {}^1\nabla^{\mathrm{coord}}, [\ ,\ ]\right) \\ \to \left(D_{\mathrm{poly},C^{\mathrm{coord},L}}(C^{\mathrm{coord},L}\widehat{\otimes}_{R_1} JL), {}^1\nabla^{\mathrm{coord}} + \mathrm{d}_{\mathrm{H}}, [\ ,\ ]\right),$$

which yields an L_∞-module structure on $\widehat{C}_{C^{\mathrm{coord},L},\bullet}(C^{\mathrm{coord},L}\widehat{\otimes}_{R_1} JL)$ over

$$T_{\mathrm{poly},C^{\mathrm{coord},L}}(C^{\mathrm{coord},L}\widehat{\otimes}_{R_1} JL),$$

and finally an L_∞-morphism

$$\mathcal{S}_L^{\mathrm{coord}} : \left(\widehat{C}_{C^{\mathrm{coord},L},\bullet}(C^{\mathrm{coord},L}\widehat{\otimes}_{R_1} JL), {}^1\nabla^{\mathrm{coord}} + \mathrm{b_H}, \mathrm{L}\circ \mathcal{U}_L^{\mathrm{coord}}\right) \to \left(\Omega_{C^{\mathrm{coord},L}\widehat{\otimes}F/C^{\mathrm{coord},L}}, {}^1\nabla^{\mathrm{coord}}, \mathrm{L}\right).$$

We recall from Section 4.3 that there is a rational action of $\mathrm{GL}_d(k)$ on $C^{\mathrm{coord},L}$ extending in a natural way to a (topological) rational action on all DG-Lie algebras and DG-Lie modules above. The previous actions determine infinitesimally actions of $\mathfrak{gl}_d(k)$ on all DG-Lie algebras and DG-Lie modules considered so far in the sense of Section 5.2.

The L_∞-morphism $\mathcal{U}_{L,\omega}$ descends with respect to the action of the set $\mathfrak{s} = \mathfrak{gl}_d(k)$ (using the notation of §5.2), because the descent condition (5.11) is satisfied as a consequence of Property (iii) of Theorem 5.1 and of the verticality property (4.5) of ω.

Similarly, Property (ii) of Theorem 5.2, together with the verticality property of ω, implies that $\mathcal{S}_{L,\omega}$ descends with respect to the action of $\mathfrak{gl}_d(k)$ (see §5.3). Summarizing all arguments so far, and because of the compatibility of the $\mathrm{GL}_d(k)$-action with the isomorphism (4.2), we get L_∞-morphisms

$$(U_L^{\mathrm{coord}})^{\mathfrak{gl}_d(k)} : \left(T_{\mathrm{poly},C^{\mathrm{coord},L}}(C^{\mathrm{coord},L}\widehat{\otimes}_{R_1} JL), {}^1\nabla^{\mathrm{coord}}, [\ ,\ ]\right)^{\mathfrak{gl}_d(k)} \to \left(D_{\mathrm{poly},C^{\mathrm{coord},L}}(C^{\mathrm{coord},L}\widehat{\otimes}_{R_1} JL), {}^1\nabla^{\mathrm{coord}} + \mathrm{d_H}, [\ ,\ ]\right)^{\mathfrak{gl}_d(k)}$$

and

$$(S_L^{\mathrm{coord}})^{\mathfrak{gl}_d(k)} : \left(\widehat{C}_{C^{\mathrm{coord},L},\bullet}(C^{\mathrm{coord},L}\widehat{\otimes}_{R_1} JL), {}^1\nabla^{\mathrm{coord}} + \mathrm{b_H}, \mathrm{L}\circ \mathcal{U}_L^{\mathrm{coord}}\right)^{\mathfrak{gl}_d(k)} \to \left(\Omega_{C^{\mathrm{coord},L}\widehat{\otimes}F/C^{\mathrm{coord},L}}, {}^1\nabla^{\mathrm{coord}}, \mathrm{L}\right)^{\mathfrak{gl}_d(k)}.$$

Repeating almost verbatim the arguments at the end of [4, §7.3.3], there are obvious isomorphisms of DG-Lie algebras and DG-Lie modules

$$\left(T_{\mathrm{poly},C^{\mathrm{aff},L}}(C^{\mathrm{aff},L}\widehat{\otimes}_{R_1} JL), {}^1\nabla^{\mathrm{aff}}, [\ ,\ ]\right) \cong \left(T_{\mathrm{poly},C^{\mathrm{coord},L}}(C^{\mathrm{coord},L}\widehat{\otimes}_{R_1} JL), {}^1\nabla^{\mathrm{coord}}, [\ ,\ ]\right)^{\mathfrak{gl}_d(k)},$$

$$\left(D_{\mathrm{poly},C^{\mathrm{aff},L}}(C^{\mathrm{aff},L}\widehat{\otimes}_{R_1} JL), {}^1\nabla^{\mathrm{aff}} + \mathrm{d_H}, [\ ,\ ]\right) \cong \left(D_{\mathrm{poly},C^{\mathrm{coord},L}}(C^{\mathrm{coord},L}\widehat{\otimes}_{R_1} JL), {}^1\nabla^{\mathrm{coord}} + \mathrm{d_H}, [\ ,\ ]\right)^{\mathfrak{gl}_d(k)},$$

$$\left(\Omega_{C^{\mathrm{aff},L}\widehat{\otimes}F/C^{\mathrm{aff},L}}, {}^1\nabla^{\mathrm{aff}}, \mathrm{L}\right) \cong \left(\Omega_{C^{\mathrm{coord},L}\widehat{\otimes}F/C^{\mathrm{coord},L}}, {}^1\nabla^{\mathrm{coord}}, \mathrm{L}\right)^{\mathfrak{gl}_d(k)},$$

$$\left(\widehat{C}_{C^{\mathrm{aff},L},\bullet}(C^{\mathrm{aff},L}\widehat{\otimes}_{R_1} JL), {}^1\nabla^{\mathrm{aff}} + \mathrm{b_H}, \mathrm{L}\right) \cong \left(\widehat{C}_{C^{\mathrm{coord},L},\bullet}(C^{\mathrm{coord},L}\widehat{\otimes}_{R_1} JL), {}^1\nabla^{\mathrm{coord}} + \mathrm{b_H}, \mathrm{L}\right)^{\mathfrak{gl}_d(k)}.$$

We now set

$$\mathfrak{g}_1^L = T_{\mathrm{poly},C^{\mathrm{aff},L}}(C^{\mathrm{aff},L}\widehat{\otimes}_{R_1} JL), \qquad \mathfrak{g}_2^L = D_{\mathrm{poly},C^{\mathrm{aff},L}}(C^{\mathrm{aff},L}\widehat{\otimes}_{R_1} JL),$$
$$\mathfrak{m}_1^L = \Omega_{C^{\mathrm{aff},L}\widehat{\otimes}_{R_1} JL/C^{\mathrm{aff},L}}, \qquad \mathfrak{m}_2^L = \widehat{C}_{C^{\mathrm{aff},L},\bullet}(C^{\mathrm{aff},L}\widehat{\otimes}_{R_1} JL),$$

and $\mathfrak{U}_L = \mathcal{U}_L^{\mathrm{aff}}$, $\mathfrak{S}_L = \mathcal{S}_L^{\mathrm{aff}}$. Combining all the results so far, we get the commutative diagram (5.12), and to prove the claim, it remains to show that $\mathfrak{U}_L$ and $\mathfrak{S}_L$ are L_∞-quasi-isomorphisms.

The proof of the fact that $\mathfrak{U}_L$ is a quasi-isomorphism can be found in [4, §7.3.4]. The proof of the fact that $\mathfrak{S}_L$ is a quasi-isomorphism is dual. We will now sketch it.

The L_∞-morphism $\mathfrak{S}_L$ is obtained from $\mathcal{S}_{L,\omega}$ using the isomorphism (4.2) and by (5.9) the Taylor components of $\mathcal{S}_{L,\omega}$ are given by

$$\mathcal{S}_{L,\omega,n}(\gamma_1,\dots,\gamma_n;c) = \sum_{m\geq 0}\frac{1}{m!}\mathcal{S}_{L,n+m}(\underbrace{\omega,\dots,\omega}_{m},\gamma_1,\dots,\gamma_n;c),$$
$$\gamma_i \in T_{\mathrm{poly},C^{\mathrm{coord},L}}(C^{\mathrm{coord},L}\widehat{\otimes}F),\ c\in \widehat{C}_{C^{\mathrm{coord},L},\bullet}(C^{\mathrm{coord},L}\widehat{\otimes}F).$$

$T_{\mathrm{poly},C^{\mathrm{coord},L}}(C^{\mathrm{coord},L}\widehat{\otimes}F)$, $D_{\mathrm{poly},C^{\mathrm{coord},L}}(C^{\mathrm{coord},L}\widehat{\otimes}F)$, $\Omega_{C^{\mathrm{coord},L}\widehat{\otimes}F/C^{\mathrm{coord},L}}$ and $\widehat{C}_{C^{\mathrm{coord},L},\bullet}(C^{\mathrm{coord},L}\widehat{\otimes}F)$ are bi-graded complexes. The first degree is the natural degree coming from $C^{\mathrm{coord},L}$, while the second degree is associated to poly-vector degree, (shifted) Hochschild degree, (negative) form degree and (negative) Hochschild degree respectively.

The component $\mathcal{S}_{L,\omega,0}$ can be written into a sum

$$\mathcal{S}_{L,\omega,0}(c) = \sum_{n\geq 0}\frac{1}{n!}\mathcal{S}_n(\underbrace{\omega,\dots,\omega}_{n};c).$$

The grading property of the L_∞-quasi-isomorphism $\mathcal{S}$ of Theorem 5.2 implies that the component $\mathcal{S}_{L,\omega,0}^n$ of $\mathcal{S}_{L,\omega,0}$ indexed by n has bi-degree $(n,-n)$.

Dualizing [4, Lemma 7.3.2], and using Property (i) of Theorem 5.2, we get the following commutative diagram of graded vector spaces:

(5.13)
$$\begin{tikzcd} C^L_{\mathrm{poly}}(R) \arrow[r, hook] \arrow[d, "\mathrm{HKR}"'] & C^{\mathrm{coord},L}\widehat{\otimes}\widehat{C}_\bullet(F) \arrow[d, "\mathcal{S}^0_{L,\omega,0}"] \\ \Omega^L(R) \arrow[r, hook] & C^{\mathrm{coord},L}\widehat{\otimes}\Omega_F, \end{tikzcd}$$

where the morphism HKR on the left vertical arrow was defined in Theorem 3.9.

The twisting procedure and the descent procedure by the isomorphism (4.2) produce the commutative diagram

$$\begin{tikzcd} C^L_{\mathrm{poly}}(R) \arrow[r, hook] \arrow[d, "\mathrm{HKR}"'] & \widehat{C}_{C^{\mathrm{aff},L},\bullet}(C^{\mathrm{aff},L}\widehat{\otimes}_{R_1}JL) \arrow[d, "\mathfrak{S}^0_{L,0}"] \\ \Omega^L(R) \arrow[r, hook] & \Omega_{C^{\mathrm{aff},L}\widehat{\otimes}_{R_1}JL/C^{\mathrm{aff},L}} \end{tikzcd}$$

out of the commutative diagram (5.13); the above bi-gradings naturally translate into bi-gradings on $\Omega_{C^{\mathrm{aff},L}\widehat{\otimes}F/C^{\mathrm{aff},L}}$ and $\widehat{C}_{C^{\mathrm{aff},L},\bullet}(C^{\mathrm{aff},L}\widehat{\otimes}F)$. The component $\mathfrak{S}_{L,0}$ is a sum of terms $\mathfrak{S}^n_{L,0}$, $n \geq 0$, of bi-degree $(n,-n)$.

We now prove that the morphisms $\mathfrak{S}_{L,0}$ and $\mathfrak{S}^0_{L,0}$ coincide at the level of cohomology. For this, we consider on the double complexes $\Omega_{C^{\mathrm{aff},L}\widehat{\otimes}F/C^{\mathrm{aff},L}}$ and $\widehat{C}_{C^{\mathrm{aff},L},\bullet}(C^{\mathrm{aff},L}\widehat{\otimes}F)$ the filtration with respect to the second degree. Then, the corresponding spectral sequences degenerate at their first terms, because of the results of Sections 4.5 and 4.6, and the resulting complexes consist of single columns $(\Omega^L(R),0)$ and $(C^L_{\mathrm{poly}}(R),\mathrm{b_H})$. Thus, the respective second terms of the spectral sequences coincide with $\Omega^L(R)$ and with $\mathrm{H}^\bullet(C^L_{\mathrm{poly}}(R),\mathrm{b_H})$. Since both spectral sequences degenerate at their first term (i.e., the cohomology with respect to the first degree is concentrated in degree 0), $\mathfrak{S}_{L,0}$ and $\mathfrak{S}^0_{L,0}$ obviously coincide at the level of cohomology, and this ends the proof. □

5.6. *Functoriality property of Theorem 5.3.* We consider two Lie algebroids (L,R), (M,S) as above.

Definition 5.4. An *algebraic morphism* from (L,R) to (M,S) consists of a pair (ℓ,λ), where (i) λ is a k-algebra morphism from R to S and (ii) ℓ is a Lie algebra morphism from L to M, enjoying the following compatibility properties with respect to the corresponding anchor maps:

$$\lambda(l(r)) = \ell(l)(\lambda(r)),\ \ell(rl) = \lambda(r)\ell(l),\ r\in R,\ l\in L.$$

The universal property of the universal enveloping algebra of a Lie algebroid yields, for any algebraic morphism $\varphi=(\ell,\lambda)$ from (L,R) to (M,S), a Hopf algebroid morphism $\varphi_D : \mathrm{U}_R(L)\to\mathrm{U}_S(M)$. Thus, (ℓ,λ) defines a morphism φ_D of B_∞-algebras from $D^L_{\mathrm{poly}}(R)$ to $D^M_{\mathrm{poly}}(S)$. In particular, it restricts to a morphism of Gerstenhaber algebras up to homotopy.

Further, the algebraic morphism φ defines a morphism $\varphi_T : T^L_{\mathrm{poly}}(R)\to T^M_{\mathrm{poly}}(S)$ by extending (via the S-linear wedge product) the assignment

$$\varphi_T : S\otimes_R L\to M : s\otimes_R l\mapsto s\ell(l).$$

Since (ℓ,λ) preserves the anchor map and Lie bracket, we have a morphism of Gerstenhaber algebras from $T^L_{\mathrm{poly}}(R)$ to $T^M_{\mathrm{poly}}(S)$.

Proposition 5.5. *We assume (L,R), (M,S) to be Lie algebroids over R and S respectively and $\varphi=(\ell,\lambda)$ to be an algebraic morphism between them as in Definition 5.4. We further assume that the morphism*

$$\varphi_T : S\otimes_R L \to M : s\otimes_R l \mapsto s\ell(l)$$

is an isomorphism of S-modules.

The morphism (ℓ,λ) determines a morphism of DG-*algebras*

$$\left(\Omega^L(R),\mathrm{d}_L\right) \overset{\varphi_\Omega}{\to} \left(\Omega^M(S),\mathrm{d}_M\right)$$

that satisfies

$$\varphi_\Omega(\gamma\cap\omega)=\varphi_T(\gamma)\cap\varphi_\Omega(\omega),\ \gamma\in T_{\mathrm{poly}}^L(R),\ \omega\in\Omega^L(R), \tag{5.14}$$

and a morphism of algebras

$$\varphi_J : JL\to JM$$

that satisfies

$$\lambda(\alpha(E))=\varphi_J(\alpha)(\varphi_D(E)),\ \alpha\in JL,\ E\in \mathrm{U}_R(L), \tag{5.15}$$

$$\varphi_J\left({}^1\nabla_l\alpha\right)={}^1\nabla_{\ell(l)}\varphi_J(\alpha),\ \alpha\in JL,\ l\in L, \tag{5.16}$$

$$\varphi_J({}^2\nabla_l\alpha)={}^2\nabla_{\ell(l)}\varphi_J(\alpha),\ \alpha\in JL,\ l\in L \tag{5.17}$$

and that commutes with the algebra monomorphisms α_i, $i=1,2$ (see §3.1.4).

Proof. Since φ_T is an isomorphism of S-modules, we define φ_Ω on L-differential forms on R via

$$\varphi_\Omega(r)=\lambda(r),\qquad \varphi_\Omega(l^*)(s\ell(l))=s\lambda(l^*(l)),\ r\in R,\ s\in S,\ l\in L,\ l^*\in L^*,$$

and we extend it to $\Omega^R(L)$ by R-linearity and by multiplicativity with respect to the wedge product.

To prove that φ_Ω intertwines d_L and d_M, it suffices to verify the claim on R and L^*. In the first case, we have

$$\begin{aligned}\varphi_\Omega(\mathrm{d}_L(r))(s\ell(l))&=s\lambda(\mathrm{d}_L(r)(l))=s\lambda(l(r))\\&=s\ell(l)(\lambda(r))=s\ell(l)(\varphi_\Omega(r))=\mathrm{d}_M(\varphi_\Omega(r))(s\ell(l))\end{aligned}$$

for a general element r of R, s of S and l of L, while in the second case we have

$$\begin{aligned}&\varphi_\Omega(\mathrm{d}_L l^*)(s_1\ell(l_1),s_2\ell(l_2))\\&\quad=s_1s_2\lambda(\mathrm{d}_L l^*(l_1,l_2))\\&\quad=s_1s_2\lambda(l_1(l^*(l_2)))-s_1s_2\lambda(l_2(l^*(l_1)))-s_1s_2\lambda(l^*([l_1,l_2]))\\&\quad=s_1\ell(l_1)(s_2)\lambda(l^*(l_2))+s_1s_2\ell(l_1)(\lambda(l^*(l_2)))\\&\qquad-s_2\ell(l_2)(s_1)\lambda(l^*(l_1))-s_1s_2\ell(l_2)(\lambda(l^*(l_1)))\\&\qquad-s_1\ell(l_1)(s_2)\lambda(l^*(l_2))+s_2\ell(l_2)(s_1)\lambda(l^*(l_1))-s_1s_2\lambda(l^*([l_1,l_2]))\end{aligned}$$

$$\begin{aligned}
&= (s_1\ell(l_1))(\varphi_\Omega(l^*)(s_2\ell(l_2))) - (s_2\ell(l_2))(\varphi_\Omega(l^*)(s_1\ell(l_1))) \\
&\quad - \varphi_\Omega(l^*)(s_1\ell(l_1)(s_2)\ell(l_2) - s_2\ell(l_2)(s_1)\ell(l_1) + s_1s_2\ell([l_1,l_2])) \\
&= \mathrm{d}_M(\varphi_\Omega(l^*))(s_1\ell(l_1), s_2\ell(l_2)).
\end{aligned}$$

By compatibility with wedge products, it suffices to prove (5.14) for γ in R or in L, and for a general ω. We check exemplarily the claim for $\gamma = l \in L$, i.e.,

$$\begin{aligned}
\varphi_\Omega(l \cap \omega)(s_1\ell(l_1), \dots, s_p\ell(l_p)) &= s_1 \cdots s_p \lambda((l \cap \omega)(l_1, \dots, l_p)) \\
&= s_1 \cdots s_p \lambda(\omega(l, l_1, \dots, l_p)) \\
&= \varphi_\Omega(\omega)(\ell(l), s_1\ell(l_1), \dots, s_p\ell(l_p)) \\
&= (\varphi_T(l) \cap \varphi_\Omega(\omega))(s_1\ell(l_1), \dots, s_p\ell(l_p)).
\end{aligned}$$

We now define the morphism φ_J on JL. For a general element α of JL, we set

$$\varphi_J(\alpha)(s) = s\lambda(\alpha(1)),\ \ \varphi_J(\alpha)(s\ell(l_1)\cdots\ell(l_p)) = s\lambda(\alpha(l_1\cdots l_p)),\ \ s \in S,\ \ l_i \in L.$$

It is sufficient to define φ_J on such elements of $\mathrm{U}_S(M)$ since, being φ_T an isomorphism of S-modules, a general element of $\mathrm{U}_S(M)$ is a sum of elements of the form

$$\begin{aligned}
(s_1\ell(l_1))\cdots(s_p\ell(l_p)) &= s_1\ell(l_1)s_2\ell(l_2)\cdots s_p\ell(l_p) \\
&= s_1(\ell(l_1)(s_2))\ell(l_2)\cdots s_p\ell(l_p) + s_1s_2\ell(l_1)\ell(l_2)\cdots s_p\ell(l_p) = \cdots,
\end{aligned}$$

where the product has to be understood in $\mathrm{U}_S(M)$.

Since φ_D is defined by extending λ and ℓ in a way compatible with the Lie algebroid structure of $\mathrm{U}_R(L)$, (5.15) follows immediately.

As for (5.16), it suffices to check the identity on R and on elements of $\mathrm{U}_S(M)$ of the form $s\ell(l_1)\cdots\ell(l_p)$. In the first case, for $s \in S$, $l \in L$, we have

$$\begin{aligned}
\left({}^1\nabla_{\ell(l)}\varphi_J(\alpha)\right)(s) &= \ell(l)(\varphi_J(\alpha)(s)) - \varphi_J(\alpha)(\ell(l)s) \\
&= \ell(l)(s\lambda(\alpha(1))) - \varphi_J(\alpha)(\ell(l)s) \\
&= \ell(l)(s)\lambda(\alpha(1)) + s\ell(l)\lambda(\alpha(1)) \\
&\quad - \varphi_J(\alpha)(\ell(l)(s)) - \varphi_J(\alpha)(s\ell(l)) \\
&= s\ell(l)\lambda(\alpha(1)) - s\lambda(\alpha(l)) \\
&= s\lambda(({}^1\nabla_l\alpha)(1)) \\
&= \varphi_J({}^1\nabla_l\alpha)(s).
\end{aligned}$$

As for the second case, for $\alpha \in JL,\ l, l_i \in L,\ i = 1, \dots, p,\ s \in S$, we have

$$\begin{aligned}
&\left({}^1\nabla_{\ell(l)}\varphi_J(\alpha)\right)(s\ell(l_1)\cdots\ell(l_p)) \\
&\qquad = \ell(l)\left(\varphi_J(\alpha)(s\ell(l_1)\cdots\ell(l_p))\right) - \varphi_J(\alpha)(\ell(l)s\ell(l_1)\cdots\ell(l_p)) \\
&\qquad = \ell(l)(s\lambda(\alpha(l_1\cdots l_p))) - \varphi_J(\alpha)(\ell(l)s\ell(l_1)\cdots\ell(l_p)) \\
&\qquad = \ell(l)(s)\lambda(\alpha(l_1\cdots l_p)) + s\ell(l)(\lambda(\alpha(l_1\cdots l_p))) \\
&\qquad\qquad - \varphi_J(\alpha)(\ell(l)(s)\ell(l_1)\cdots\ell(l_p)) - \varphi_J(\alpha)(s\ell(l)\ell(l_1)\cdots\ell(l_p)) \\
&\qquad = s\lambda(l(\alpha(l_1\cdots l_p))) - s\lambda(\alpha)(ll_1\cdots l_p) \\
&\qquad = \varphi_J({}^1\nabla_l\alpha)(s\ell(l_1)\cdots\ell(l_p)).
\end{aligned}$$

The identity (5.17) as well as the compatibility with α_i, $i = 1, 2$ are verified by similar computations. □

Assume now that $\varphi = (\ell, \lambda) : (L, R) \to (M, S)$ is as in the previous lemma and that $\varphi_T : S \otimes_R L \to M$ is an isomorphism. As always we assume that L (and hence M) is free of rank d. Looking at associated graded objects we see that the extended map

$$S_1 \otimes_{R_1} JL \to JM : s \otimes \alpha \mapsto s\varphi_J(\alpha) \tag{5.18}$$

is an isomorphism. Hence any R_1-linear differential operator on JL can be extended to an S_1-linear differential operator on JM. We use this to define a map

$$\varphi_D : D_{R_1}(JL) \to D_{S_1}(JM)$$

and a corresponding map of B_∞-algebras

$$\varphi_D : D_{\mathrm{poly},R_1}(JL) \to D_{\mathrm{poly},S_1}(JM) \tag{5.19}$$

such that the following diagram is commutative

$$\begin{tikzcd}
D^L_{\mathrm{poly}}(R) \arrow[r, "\varphi_D"] \arrow[d, hook] & D^M_{\mathrm{poly}}(S) \arrow[d, hook] \\
D_{\mathrm{poly},R_1}(JL) \arrow[r, "\varphi_D"'] & D_{\mathrm{poly},S_1}(JM),
\end{tikzcd} \tag{5.20}$$

where the vertical monomorphisms have been defined in (3.12).

An easy computation shows that φ_D in (3.12) commutes with the action of the Grothendieck connection $[{}^1\nabla, -]$. It follows by the discussion in Section 3.3 that if we take the invariants for $[{}^1\nabla, -]$ of the lower line in (5.20), we obtain the upper line.

We extend φ_J to a map of graded vector spaces

$$\varphi_C : \widehat{C}_{R,\bullet}(JL) \to \widehat{C}_{S,\bullet}(JM) : \alpha_1 \otimes \cdots \otimes \alpha_n \mapsto \varphi_J(\alpha_1) \otimes \cdots \otimes \varphi_J(\alpha_n),$$

which is again essentially just base extension over S/R. This map obviously commutes with the Grothendieck connection ${}^1\nabla$. We obtain a map of pairs of graded vector spaces

$$(\varphi_D, \varphi_C) : (D_{\mathrm{poly},R}(JL), \widehat{C}_{R,\bullet}(JL)) \to (D_{\mathrm{poly},S}(JM), \widehat{C}_{R,\bullet}(JL)),$$

and as this map is just base extension over S/R, it is compatible with all structures defined in [7] hence, in particular, with the DG-Lie algebra and DG-Lie module structures and also with the precalculi up to homotopy.

Taking invariants for ${}^1\nabla$ and using (3.16) we obtain a commutative diagram of precalculus structure up to homotopy:

$$\begin{tikzcd}
(D^L_{\mathrm{poly}}(R), \mathrm{d_H}, [\ ,\ ], \cup) \arrow[r, "\varphi_D"] \arrow[d, rightsquigarrow] & (D^M_{\mathrm{poly}}(S), \mathrm{d_H}, [\ ,\ ], \cup) \arrow[d, rightsquigarrow] \\
(C^L_{\mathrm{poly}}(R), \mathrm{b_H}, \mathrm{L}, \cap) \arrow[r, "\varphi_C"'] & (C^M_{\mathrm{poly}}(S), \mathrm{b_H}, \mathrm{L}, \cap).
\end{tikzcd}$$

One also obtains from Proposition 5.5 a commutative diagram of precalculi:

$$\begin{tikzcd}
(T^L_{\mathrm{poly}}(R), 0, [\ ,\ ], \cup) \arrow[r, "\varphi_T"] \arrow[d, rightsquigarrow] & (T^M_{\mathrm{poly}}(S), 0, [\ ,\ ], \cup) \arrow[d, rightsquigarrow] \\
(\Omega^L(R), 0, \mathrm{L}, \cap) \arrow[r, "\varphi_\Omega"'] & (\Omega^M(S), 0, \mathrm{L}, \cap).
\end{tikzcd}$$

Furthermore, from (5.18) and the universal property of coordinate spaces (see (4.1)) we obtain an R-algebra morphism from $R^{\mathrm{coord},L}$ to $S^{\mathrm{coord},M}$. It extends further to a morphism of DG-algebras from $C^{\mathrm{coord},L}$ to $C^{\mathrm{coord},M}$ thanks to (5.16) and the fact that φ_Ω is a morphism of DG-algebras from $\Omega^L(R)$ to $\Omega^M(S)$.

Finally, the algebraic morphism (ℓ, λ) induces precalculi morphisms (up to homotopy) between all corresponding Fedosov resolutions, since the monomorphism α_2 and the connection ${}^2\nabla$, which are needed in the construction of the Fedosov resolutions of Section 4 (we refer to [4] for more details thereabout), have been proved to be preserved by (ℓ, λ).

As a consequence of these arguments, we deduce the following theorem, which expresses the functoriality properties of the commutative diagram (5.12) of Theorem 5.3.

THEOREM 5.6. *For a general algebraic morphism $\varphi = (\ell, \lambda)$ from (L, R) to (M, S) as in Definition 5.4, which induces an isomorphism $S \otimes_R L \cong M$ of S-modules, and such that L is free of rank d, the L_∞-quasi-isomorphisms $\mathfrak{U}_L$, $\mathfrak{U}_M$, $\mathfrak{S}_L$ and $\mathfrak{S}_M$ of* DG-*Lie algebras and* DG-*Lie modules fit into the* *commutative diagram*

$$
\begin{tikzcd}
T_{\mathrm{poly}}^L(R) \arrow[r, hook] \arrow[d, "\varphi_T"'] & \mathfrak{g}_1^L \arrow[r, "\mathfrak{U}_L"] \arrow[d, "\varphi_T"'] & \mathfrak{g}_2^L \arrow[d, "\varphi_D"'] & D_{\mathrm{poly}}^L(R) \arrow[l, hook'] \arrow[d, "\varphi_D"'] \\
T_{\mathrm{poly}}^M(S) \arrow[r, hook] \arrow[d, squiggly, "\mathrm{L}"'] & \mathfrak{g}_1^M \arrow[r, "\mathfrak{U}_M"] \arrow[d, squiggly, "\mathrm{L}_1"'] & \mathfrak{g}_2^M \arrow[d, squiggly, "\mathrm{L}_2"'] & D_{\mathrm{poly}}^M(S) \arrow[l, hook'] \arrow[d, squiggly, "\mathrm{L}"'] \\
\Omega^M(S) \arrow[r, hook] & \mathfrak{m}_1^M & \mathfrak{m}_2^M \arrow[l, "\mathfrak{S}_M"'] & C_{\mathrm{poly}}^M(S) \arrow[l, hook'] \\
\Omega^L(R) \arrow[r, hook] \arrow[u, "\varphi_\Omega"] & \mathfrak{m}_1^L \arrow[u, "\varphi_\Omega"] & \mathfrak{m}_2^L \arrow[l, "\mathfrak{S}_L"'] \arrow[u, "\varphi_J"] & C_{\mathrm{poly}}^L(R), \arrow[l, hook'] \arrow[u, "\varphi_J"]
\end{tikzcd}
\tag{5.21}
$$

where we have borrowed notation from Proposition 5.5; *all such morphisms are compatible with respect to the composition of algebraic morphisms between Lie algebroids.*

Note that Theorem 5.6 makes no reference to the (homotopy) precalculus structures that we discussed above; we will need these below.

5.7. *Proof of Theorem* 1.3. We now collect the results of Sections 5.5 and 5.6 to give the proof of Theorem 1.3, via a well-suited gluing procedure.

We consider a ringed site $(X, \mathcal{O})$ and a sheaf of Lie algebroids $\mathcal{L}$ such that $\mathcal{L}$ is locally free of rank d over $\mathcal{O}$. We replace X by its full subcategory of objects U such that $\mathcal{L}(U)$ is free over $\mathcal{O}(U)$. This does not change the category of sheaves.

All sheaves of DG-Lie algebras and DG-Lie modules in the commutative diagram (1.7) are obtained by sheafifying the corresponding presheaves of DG-Lie algebras and DG-Lie modules; i.e.,

$$
U \to T_{\mathrm{poly}}^{\mathcal{L}(U)}(\mathcal{O}(U)), \qquad U \to D_{\mathrm{poly}}^{\mathcal{L}(U)}(\mathcal{O}(U)),
$$
$$
U \to \Omega^{\mathcal{L}(U)}(\mathcal{O}(U)), \qquad U \to C_{\mathrm{poly}}^{\mathcal{L}(U)}(\mathcal{O}(U)).
$$

Since $\mathcal{L}$ is locally free of order d over $\mathcal{O}$, for a morphism $V \to U$ in X, the corresponding restriction morphism $(\mathcal{O}(U), \mathcal{L}(U)) \to (\mathcal{O}(V), \mathcal{L}(V))$ yields an isomorphism

$$
\mathcal{O}(V) \otimes_{\mathcal{O}(U)} \mathcal{L}(U) \cong \mathcal{L}(V).
$$

Thus, any restriction morphism as above may be viewed as an algebraic morphism between Lie algebroids, satisfying the isomorphism property of Theorem 5.6.

If we then consider the DG-Lie algebras and DG-Lie modules

$$
U \to \mathfrak{g}_i^{\mathcal{L}(U)},\ U \to \mathfrak{m}_i^{\mathcal{L}(U)},\ i = 1, 2,
$$

Theorem 5.3 produces, for any U in X, L_∞-quasi-isomorphisms $\mathfrak{U}_{\mathcal{L}(U)}$ and $\mathfrak{S}_{\mathcal{L}(U)}$ that fit into a commutative diagram (5.12). By Theorem 5.6 these are actually morphisms of presheaves.

Sheafifying all presheaves and morphisms between presheaves concludes the proof.

6. The relationship between Atiyah classes and jet bundles

In the present section we review some technical results from [4, §8], to which we refer for more details. We need only the main notation and conventions for use in Section 7.

For a field k of characteristic 0, we consider a sheaf $\mathcal{L}$ of Lie algebroids over a ringed site $(X, \mathcal{O})$, which is locally free of rank d over $\mathcal{O}$.

We have a short exact sequence of $\mathcal{O}_1$-$\mathcal{O}_2$-bimodules

$$0 \to \mathcal{L}^* \to J^1\mathcal{L} \to \mathcal{O} \to 0,$$

where $\mathcal{O}_i$, $i = 1, 2$, denotes a copy of $\mathcal{O}$ embedded in $J\mathcal{L}$ via the monomorphism α_i and where $J^1\mathcal{L}$ was introduced in Section 3.1.4.

For a general $\mathcal{O}$-module $\mathcal{E}$, tensoring over $\mathcal{O}_2$ yields a short exact sequence

$$0 \longrightarrow \mathcal{L}^* \otimes_{\mathcal{O}} \mathcal{E} \longrightarrow J^1\mathcal{L} \otimes_{\mathcal{O}_2} \mathcal{E} \longrightarrow \mathcal{E} \longrightarrow 0,$$

which we will call the $\mathcal{L}$-Atiyah sequence. The $\mathcal{L}$-Atiyah class $A_{\mathcal{L}}(\mathcal{E})$ of $\mathcal{E}$ over $\mathcal{L}$ is the extension class of this sequence in $\mathrm{Ext}^1_{\mathcal{O}}(\mathcal{E}, \mathcal{L}^* \otimes_{\mathcal{O}} \mathcal{E})$. As explained in Section 1.1, if $\mathcal{E}$ is a vector bundle, the i-th scalar Atiyah class $a_{\mathcal{L},i}(\mathcal{E})$ of $\mathcal{E}$ is defined as

$$a_{\mathcal{L},i}(\mathcal{E}) = \mathrm{tr}\Big(\bigwedge^i A_{\mathcal{L}}(\mathcal{E})\Big) \in \mathrm{H}^i\Big(X, \bigwedge^i \mathcal{L}^*\Big). \tag{6.1}$$

Below we will only consider the case $\mathcal{E} = \mathcal{L}$. In that case we simplify the notation to

$$A(\mathcal{L}) = A_{\mathcal{L}}(\mathcal{L}), \qquad a_i(\mathcal{L}) = a_{i,\mathcal{L}}(\mathcal{L}).$$

Observe that the $a_i(\mathcal{L})$ are cohomology classes. We now outline how we may realize them as explicit cocycles.

By the very construction of $C^{\mathrm{coord},\mathcal{L}}$ and $C^{\mathrm{aff},\mathcal{L}}$, there are natural morphisms of DG-algebras

(6.2)
$$\Omega^{\mathcal{L}_2}(X) \xhookrightarrow{\theta} C^{\mathrm{aff},\mathcal{L}} \widehat{\otimes}_{\mathcal{O}_1} \Omega_{J\mathcal{L}/\mathcal{O}_1} \hookrightarrow C^{\mathrm{coord},\mathcal{L}} \widehat{\otimes}_{\mathcal{O}_1} \Omega_{J\mathcal{L}/\mathcal{O}_1} \cong C^{\mathrm{coord},\mathcal{L}} \widehat{\otimes} \Omega_F\ .$$

The differentials on the first three DG-algebras are the natural ones (see §4.3). The differential on the fourth DG-algebra is $\mathrm{d} + \mathrm{L}_\omega$ for a certain MC element $\omega \in \Gamma\left(X, C^{\mathrm{coord},\mathcal{L}} \widehat{\otimes}\, \mathrm{Der}(F)\right)$ and d the natural differential. See again Section 4.3.

The MC element ω can be expressed as

$$\omega = \eta_\alpha \omega_{\alpha,i} \partial_{x_i}, \; i = 1, \dots, d,$$

where η_α is in $C^{\mathrm{coord},\mathcal{L}}$ and has degree 1, $\omega_{\alpha,i}$ belongs to F and $\partial_{x_i} = \partial/\partial x_i$.

If we define Ξ to be the matrix with entries

$$\Xi_{ij} = \eta_\alpha \mathrm{d}_F(\partial_{x_j} \omega_{\alpha,i}) \in \Gamma\left(X, C^{\mathrm{coord},\mathcal{L}} \widehat{\otimes} \Omega_F\right), \tag{6.3}$$

where d_F is the de Rham differential on Ω_F, then on the nose, we have

$$\mathrm{Tr}(\Xi^n) \in \Gamma\left(X, C^{\mathrm{coord},\mathcal{L}} \widehat{\otimes} \Omega_F\right).$$

Furthermore, it is true that

$$(\mathrm{d} + \mathrm{L}_\omega)(\mathrm{Tr}(\Xi^n)) = 0.$$

It is shown in [4, §8] that $\mathrm{Tr}(\Xi^n)$ is actually the image of a (necessarily unique) element in $\Gamma\left(X, C^{\mathrm{aff},\mathcal{L}} \widehat{\otimes}_{\mathcal{O}_1} \Omega_{J\mathcal{L}/\mathcal{O}_1}\right)$. Abusing notation somewhat we will still write this element as $\mathrm{Tr}(\Xi^n)$. It is still a cocycle and in this way represents an element of

$$\overline{\mathrm{Tr}(\Xi^n)} \in \Gamma(X, \mathrm{H}^{2n}(C^{\mathrm{aff},\mathcal{L}} \widehat{\otimes}_{\mathcal{O}_1} \Omega_{J\mathcal{L}/\mathcal{O}_1}))$$

that maps naturally to the hypercohomology

$$\mathbb{H}^{2n}(X, C^{\mathrm{aff},\mathcal{L}} \widehat{\otimes}_{\mathcal{O}_1} \Omega_{J\mathcal{L}/\mathcal{O}_1}).$$

Further, we observe that the injection $\Omega^{\mathcal{L}}(X) \xrightarrow{\theta} C^{\mathrm{aff},\mathcal{L}} \widehat{\otimes}_{\mathcal{O}_1} \Omega_{J\mathcal{L}/\mathcal{O}_1}$ of DG-algebras is a quasi-isomorphism, as discussed in Section 4.5. Thus, θ induces an isomorphism

$$\bigoplus_{m,n} \mathrm{H}^m(X, \wedge^n \mathcal{L}^*) = \mathbb{H}^\bullet(X, \Omega^{\mathcal{L}}(X)) \xrightarrow{\mathbb{H}(\theta)} \mathbb{H}^\bullet(X, C^{\mathrm{aff},\mathcal{L}} \widehat{\otimes}_{\mathcal{O}_1} \Omega_{J\mathcal{L}/\mathcal{O}_1}).$$

The following identity is [4, eq. (8.8)]

$$a_n(\mathcal{L}) = \mathbb{H}(\theta)^{-1}\left(\overline{\mathrm{Tr}(\Xi^n)}\right), \; n \geq 1, \tag{6.4}$$

which indeed expresses $a_n(\mathcal{L})$ in terms of the explicit cocycle $\mathrm{Tr}(\Xi^n)$.

7. Proof of Theorem 1.1

The aim of this Section is to prove Theorem 1.1, which implies Căldăraru's conjecture (Theorem 1.4) as was outlined in the introduction.

For this purpose, we first remind the reader of the main result of [7] about compatibility between cap products. We then prove a ring-theoretical globalized version of this result (compare to the proof of Theorem 5.3). By functoriality (see §5.6), we obtain the sheaf-theoretical globalization. Finally, using results of [7], we compute explicitly the isomorphism appearing in the compatibility between cap products, which we identify with the action of the homological HKR-quasi-isomorphism followed by left multiplication by the square root of the (modified) Todd class.

7.1. *A* memento *of compatibility between cup and cap products.* In this section, we present a *memento* of the main results of [7], [4] concerning compatibility between cup and cap products respectively.

First of all, as before, F is the algebra of formal power series in d variables over the field k that is assumed to contain $\mathbb{R}$ for now. We recall the existence of (homotopy) Gerstenhaber algebra structures on $T_{\mathrm{poly}}(F)$ and $D_{\mathrm{poly}}(F)$, which together with Ω_F and $\widehat{C}_\bullet(F)$ yield (homotopy) precalculi [7].

We recall also the L_∞-quasi-isomorphisms $\mathcal{U}$ introduced in Theorem 5.1 and $\mathcal{S}$ introduced in Theorem 5.2. We denote by $\mathcal{U}_n$, $n \geq 1$, resp. $\mathcal{S}_n$, $n \geq 0$, the n-th Taylor component of $\mathcal{U}$, resp. $\mathcal{S}$.

We further consider a commutative DG-algebra $(\mathfrak{m}, \mathrm{d}_\mathfrak{m})$. The precalculus structures on $(T_{\mathrm{poly}}(F), \Omega_F)$ and $(D_{\mathrm{poly}}(F), \widehat{C}_\bullet(F))$ can be extended by $\mathfrak{m}$-linearity to precalculi

$$(T^\mathfrak{m}_{\mathrm{poly}}(F), \Omega^\mathfrak{m}_F) = (T_{\mathrm{poly}}(F)\widehat{\otimes}\mathfrak{m}, \Omega_F\widehat{\otimes}\mathfrak{m})$$

and

$$(D^\mathfrak{m}_{\mathrm{poly}}(F), \widehat{C}^\mathfrak{m}_\bullet(F)) = (D_{\mathrm{poly},\mathfrak{m}}(F\widehat{\otimes}\mathfrak{m}), \widehat{C}_{\bullet,\mathfrak{m}}(F\widehat{\otimes}\mathfrak{m}))).$$

Convention. Below we will work with potentially infinite series with coefficients in $\mathfrak{m}$. We make the standard assumption that we are in a setting where all these series converge and standard series manipulations are allowed. In our actual application all series will be finite for degree reasons.

An MC element γ of $T^\mathfrak{m}_{\mathrm{poly}}(F)$ can be written as a sum

$$\gamma = \gamma_{-1} + \gamma_0 + \gamma_1 + \gamma_2 + \cdots,$$

where γ_i is an element of $T^\mathfrak{m}_{\mathrm{poly}}(F)$ of poly-vector degree i, $i \geq -1$, which satisfies the Maurer–Cartan equation

$$\mathrm{d}_\mathfrak{m}\gamma + \frac{1}{2}[\gamma, \gamma] = 0.$$

We denote by $\mathcal{U}(\gamma)$ the image of an MC element γ as above with respect to $\mathcal{U}$ (see (5.7)). This is again an MC element. Further, we set

$$\mathcal{U}_{\gamma,1}(\gamma_1) = \sum_{n\geq 0} \frac{1}{n!}\mathcal{U}_{n+1}(\underbrace{\gamma, \ldots, \gamma}_{n}, \gamma_1),\ \ \gamma_1 \in T^\mathfrak{m}_{\mathrm{poly}}(F),$$

$$\mathcal{S}_{\gamma,0}(c) = \sum_{n\geq 0} \frac{1}{n!}\mathcal{S}_n(\underbrace{\gamma, \ldots, \gamma}_{n}; c),\ \ c \in \widehat{C}^\mathfrak{m}_\bullet(F).$$

Since $\mathcal{U}$ and $\mathcal{S}$ are L_∞-quasi-isomorphisms, $\mathcal{U}_{\gamma,1}$ and $\mathcal{S}_{\gamma,0}$ are both quasi-isomorphisms of DG-vector spaces.

THEOREM 7.1. *For a general commutative* DG-*algebra* $(\mathfrak{m}, \mathrm{d}_\mathfrak{m})$ *as above and for a general MC element* γ *of* $T_{\mathrm{poly}}^\mathfrak{m}(F)$, $\mathcal{U}_{\gamma,1}$ *and* $\mathcal{S}_{\gamma,0}$ *descend to quasi-isomorphisms of (homotopy) precalculi, fitting into the commutative diagram*

$$\begin{tikzcd} \left(T_{\mathrm{poly}}^\mathfrak{m}(F), \mathrm{d}_\mathfrak{m} + [\gamma, \bullet], [\ ,\ ], \cup\right) \arrow[r, "\mathcal{U}_{\gamma,1}"] \arrow[d, squiggly] & \left(D_{\mathrm{poly}}^\mathfrak{m}(F), \mathrm{d}_\mathfrak{m} + \mathrm{d}_\mathrm{H} + [\mathcal{U}(\gamma), \bullet], [\ ,\ ], \cup\right) \arrow[d, squiggly] \\ (\Omega_F^\mathfrak{m}, \mathrm{d}_\mathfrak{m} + \mathrm{L}_\gamma, \mathrm{L}, \cap) & \left(\widehat{C}_\bullet^\mathfrak{m}(F), \mathrm{d}_\mathfrak{m} + \mathrm{b}_\mathrm{H} + \mathrm{L}_{\mathcal{U}(\gamma)}, \mathrm{L}, \cap\right) \arrow[l, "\mathcal{S}_{\gamma,0}"'] \end{tikzcd}$$

in the sense that $\mathcal{U}_{\gamma,1}$ *and* $\mathcal{S}_{\gamma,0}$ *preserve Lie brackets, Lie actions, cup and cap products up to homotopy.*

Kontsevich [16] first stated and proved that $\mathcal{U}_{\gamma,1}$ defines a quasi-isomorphism of Gerstenhaber algebras up to homotopy from $T_{\mathrm{poly}}^\mathfrak{m}(F)$ to $D_{\mathrm{poly}}^\mathfrak{m}(F)$ in the sense specified above. We observe that the identity $\mathcal{U}_{\gamma,1}([\gamma_1, \gamma_2]) = [\mathcal{U}_{\gamma,1}(\gamma_1), \mathcal{U}_{\gamma_1}(\gamma_2)]$ at the level of cohomology, for γ_i in $T_{\mathrm{poly}}^\mathfrak{m}(F)$, $i = 1, 2$, holds true, because $\mathcal{U}$ is an L_∞-morphism. In particular, there is a homotopy operator describing the compatibility with Lie brackets, expressible in terms of the Taylor components of $\mathcal{U}$ twisted by the MC element γ. On the other hand, the identity $\mathcal{U}_{\gamma,1}(\gamma_1 \cup \gamma_2) = \mathcal{U}_{\gamma,1}(\gamma_1) \cup \mathcal{U}_{\gamma,1}(\gamma_2)$ at the level of cohomology comes from a more complicated identity up to homotopy. In this situation, the homotopy operator is not expressible in terms of the Taylor components of $\mathcal{U}$. For an explicit description of the homotopy operator, we refer to [17], [4], [7].

The actual formulation of Theorem 7.1 was first proposed by Shoikhet [20] as a conjecture in the particular case, where γ is a (formal) Poisson structure. This conjecture was first proved in [21] only in degree 0 and later in [6] for all degrees. A more general result was stated and proved in [7], to which we refer for more details. The identity $\mathcal{S}_{\gamma,0}(\mathrm{L}_{\mathcal{U}_{\gamma,1}}(\gamma_1)(c)) = \mathrm{L}_{\gamma_1}(\mathcal{S}_{\gamma,0}(c))$ at the level of cohomology, for γ_1 in $T_{\mathrm{poly}}^\mathfrak{m}(F)$, c in $\widehat{C}_\bullet^\mathfrak{m}(F)$, is a consequence of the fact that $\mathcal{S}_{L,\gamma}$ is an L_∞-morphism of L_∞-modules. (In particular, there is a homotopy formula involving the Taylor components of $\mathcal{U}$ and $\mathcal{S}$, twisted by γ.) The identity $\mathcal{S}_{\gamma,0}(\mathcal{U}_{\gamma,1}(\gamma_1) \cap c) = \gamma_1 \cap \mathcal{S}_{\gamma,0}(c)$ at the level of cohomology holds true in virtue of a homotopy formula, but the corresponding homotopy operator does not involve the Taylor components of $\mathcal{U}$ and $\mathcal{S}$: such an operator was explicitly described in [7].

In Section 7.1.1 we briefly review the construction of the homotopy operator for the compatibility between cap products.

7.1.1. *The homotopy formula for the compatibility between cap products.* For later computations, we write down the explicit homotopy operator for

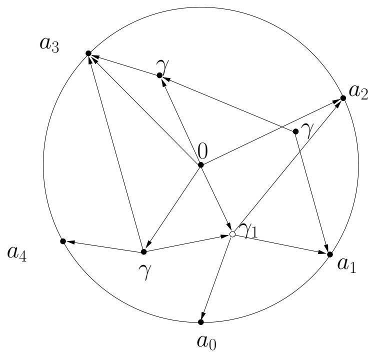

Figure 1. An S-admissible graph of type $(4, 5)$.

the compatibility between the $\cap$-actions. Namely, for an MC element γ as in Theorem 7.1, for γ_1 a general element of $T^{\mathfrak{m}}_{\mathrm{poly}}(F)$ and c a general element of $\widehat{C}^{\mathfrak{m}}_{\bullet}(F)$, we have the homotopy relation

(7.1)
$$\mathcal{S}_{\gamma,0}(\mathcal{U}_{\gamma,1}(\gamma_1)\cap c) - \gamma_1 \cap \mathcal{S}_{\gamma,0}(c) = (\mathrm{d}_{\mathfrak{m}} + \mathrm{L}_\gamma)\mathcal{H}^{\mathcal{S}}_\gamma(\gamma_1, c) + \mathcal{H}^{\mathcal{S}}_\gamma(\mathrm{d}_{\mathfrak{m}}\gamma_1 + [\gamma,\gamma_1], c) + (-1)^{|\gamma_1|}\mathcal{H}^{\mathcal{S}}_\gamma(\gamma_1, \mathrm{d}_{\mathfrak{m}}c + \mathrm{b}_{\mathrm{H}}c + \mathrm{L}_{\mathcal{U}(\gamma)}c),$$

where

$$\mathcal{H}^{\mathcal{S}}_\gamma(\gamma_1, c) = \sum_{n\geq 0}\frac{1}{n!}\sum_{\Gamma\in\mathcal{G}^S_{n+1,m+1}} \mathring{W}_{D,\Gamma}\mathcal{S}_\Gamma(\gamma_1, \underbrace{\gamma,\dots,\gamma}_{n}, c), \tag{7.2}$$

c being of Hochschild degree $-m$.

In (7.2), the second sum is over "S-admissible graphs" of type $(n+1, m+1)$. These are directed graphs with $n+2$ vertices of the first type and $m+1$ cyclically ordered vertices of the second type and with an orientation of the outgoing edges from vertices of the first type, and with a special vertex of the first type, labelled by 0. The vertices of the second type can be only endpoints of edges, and S-admissible graphs do not contain edges starting and ending at the same vertex; finally, the vertex 0 has only incoming edges.

To the vertex 1 of the first type of an S-admissible graph Γ is assigned the poly-vector field γ_1. The number of outgoing edges from 1 equals the poly-vector degree of γ_1 plus 1. To any other vertex of the first type, except 0, is assigned a copy of the MC element γ. To the i-th vertex of the second type is assigned the $i+1$-th component of the Hochschild chain c. Pictorially, here is an S-admissible graph of type $(4, 5)$, with corresponding coloring by poly-vector fields and Hochschild chains (see Figure 1).

The differential form $S_\Gamma(\gamma_1, \underbrace{\gamma,\dots,\gamma}_{n}, c)$ is defined explicitly in [20], [6], [7].

More important for our purposes is the integral weight $\mathring{W}_{D,\Gamma}$, for a general S-admissible graph of type $(n+1, m+1)$,

$$\mathring{W}_{D,\Gamma} = \int_{\mathring{\mathcal{Y}}^+_{n+1,m+1}} \omega_{D,\Gamma}. \tag{7.3}$$

First of all, $\mathring{\mathcal{Y}}^+_{n+1,m+1}$ denotes the codimension-1-submanifold (with corners) of the compactified configuration space $\mathcal{D}^+_{n+1,m+1}$ of $n+1$ points in the punctured unit disk $D^\times$ and $m+1$ cyclically oriented points in S^1, consisting of configurations of points, where the point labelled by 1 moves on a smooth curve from the origin to the first point $\overline{1}$ (with respect to the cyclic order) in S^1. Graphically,

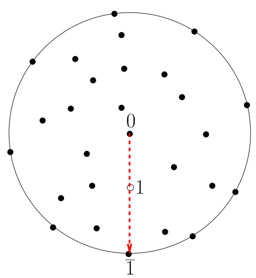

Figure 2. A general configuration of points in $\mathring{\mathcal{Y}}^+_{n+1,m+1}$.

In Figure 2, the dashed line represents the curve, along which the point 1 (labelled as "$\circ$") moves. The differential form $\omega_{D,\Gamma}$ associated to a graph in $\mathcal{G}^S_{n+1,m+1}$ is a product of smooth 1-forms on $\mathcal{D}^+_{n+1,m+1}$. The basic ingredient is a slight modification of the exterior derivative of Kontsevich's angle function; see [16], [7] for more details.

For the globalization procedure of the compatibility between cap products, we need the following technical lemma, which corresponds, in the present framework, to Theorem 5.2(ii).

Lemma 7.2. *If Γ is an S-admissible graph in $\mathcal{G}^S_{n+1,m+1}$, $n \geq 1$, and at least one of the poly-vector fields γ_i, $i \neq 1$, is linear on F, then*

$$\mathring{W}_{D,\Gamma} \mathcal{S}_\Gamma(\gamma_1, \gamma_2, \dots, \gamma_{n+1}, c) = 0.$$

Proof. The first point of the first type in $\mathring{\mathcal{Y}}^+_{n+1,m+1}$, by the very construction of $\mathring{\mathcal{Y}}^+_{n+1,m+1}$, moves from the origin 0 to the first point in S^1 with respect to the cyclic order. To the former point is associated the poly-vector field γ_1. Any other point associated to a vertex of the first type moves freely in the punctured unit disk $D^\times$.

Without loss of generality, we assume γ_2 to be an $\mathfrak{m}$-valued linear vector field. The valence (i.e., the number of outgoing edges) of the corresponding vertex of the first type is 1, while the linearity of γ_2 implies that there can be at most one incoming edge to the vertex corresponding to γ_2. This follows from the construction of the differential form $\mathcal{S}_\Gamma(\gamma_1, \gamma_2, \ldots, \gamma_{n+1}, c)$.

Thus, we may safely restrict to S-admissible graphs Γ, such that the vertex 2 has valence exactly 1 and with at most one incoming edge.

If the vertex labelled by 2 does not have incoming edges, the corresponding integral weight $\mathring{W}_{D,\Gamma}$ vanishes by dimensional reasons. In fact, we integrate a 1-form (corresponding to the only outgoing edge from 2) over a 2-dimensional submanifold (with corners) of $D^\times$.

If the vertex labelled by 2 has exactly one incoming and one outgoing edge, we may apply [7, Lemma 6.1] to yield the vanishing of the corresponding weight $\mathring{W}_{D,\Gamma}$. □

7.2. *The proof of Theorem* 1.1 *in the ring case.* We will first assume that the ground field contains $\mathbb{R}$. At the end of the section we will show how to get rid of this restriction.

We consider a Lie algebroid L over R, as in Definition 3.1, free of rank d over R. Then we set $(\mathfrak{m}, \mathrm{d}_\mathfrak{m}) = (C^{\mathrm{coord},L}, \mathrm{d})$, where $\mathrm{d} = \mathrm{d}_{\Omega_{R^{\mathrm{coord},L}}} \widehat{\otimes}_{\Omega_{R_1}} 1 + 1 \widehat{\otimes}_{\Omega_{R_1}} \mathrm{d}_{L_1}$ (see §4.3 for more details), and the Maurer–Cartan form ω is an $\mathfrak{m}$-valued vector field on F obeying

$$\mathrm{d}\omega + \frac{1}{2}[\omega, \omega] = 0.$$

By Theorem 5.1(ii) we have $\mathcal{U}(\omega) = \omega$. Furthermore, one checks that by degree reasons $\mathcal{U}_\omega$ and $\mathcal{S}_\omega$ yield finite sums when evaluated on specific elements. The same goes for the associated homotopies. So the results of Section 7.1 apply.

Combining the arguments of the proof of Theorem 5.3 with Theorem 7.1 we get the following commutative diagram of precalculus structures up to homotopy:

(7.4)

$$\begin{tikzcd}
\left(T_{\mathrm{poly},C^{\mathrm{aff},L}}(C^{\mathrm{aff},L} \widehat{\otimes}_{R_1} JL), {}^1\nabla^{\mathrm{aff}}, [\ ,\ ], \cup\right) \arrow[dr, "\mathfrak{U}_{L,1}"] \arrow[dd, squiggly] & \\
& \left(D_{\mathrm{poly},C^{\mathrm{aff},L}}(C^{\mathrm{aff},L} \widehat{\otimes}_{R_1} JL), {}^1\nabla^{\mathrm{aff}} + \mathrm{d}_{\mathrm{H}}, [\ ,\ ], \cup\right) \arrow[dd, squiggly] \\
\left(\Omega_{C^{\mathrm{aff},L} \widehat{\otimes}_{R_1} JL / C^{\mathrm{aff},L}}, {}^1\nabla^{\mathrm{aff}}, \mathrm{L}, \cap\right) & \\
& \left(\widehat{C}_{C^{\mathrm{aff},L}, \bullet}(C^{\mathrm{aff},L} \widehat{\otimes}_{R_1} JL), {}^1\nabla^{\mathrm{aff}} + \mathrm{b}_{\mathrm{H}}, \mathrm{L}, \cap\right) \arrow[ul, "\mathfrak{S}_{L,0}"']
\end{tikzcd}$$

The fact that $\mathfrak{U}_{L,1}$ preserves the respective Lie brackets up to homotopy is a consequence of the fact that $\mathfrak{U}_L$ is an L_∞-morphism; similarly, the fact that $\mathfrak{S}_{L,0}$ preserves the Lie module structure up to homotopy is a consequence of the fact that $\mathfrak{S}_L$ is an L_∞-morphism of L_∞-modules.

On the other hand, $\mathfrak{U}_{L,1}$ is compatible with respect to the products labelled by $\cup$ up to homotopy by the results of [4, §10.1].

As for the compatibility between the actions labelled by $\cap$ up to homotopy, we first observe that the homotopy formula (7.1) is well defined in the case $(\mathfrak{m}, \mathrm{d}_{\mathfrak{m}}) = (C^{\mathrm{coord},L}, \mathrm{d})$ and $\gamma = \omega$, with the same notation as above. By the same arguments as in the proof of Theorem 5.3 it remains to prove that the homotopy operator (7.2) descends to a homotopy operator

$$\mathfrak{H}_L^{\mathcal{S}} : T_{\mathrm{poly},C^{\mathrm{aff},L}}(C^{\mathrm{aff},L}\widehat{\otimes}_{R_1} JL) \otimes \widehat{C}_{C^{\mathrm{aff},L},\bullet}(C^{\mathrm{aff},L}\widehat{\otimes}_{R_1} JL) \to \Omega_{C^{\mathrm{aff},L}\widehat{\otimes}_{R_1} JL/C^{\mathrm{aff},L}}.$$

This holds true as a consequence of Lemma 7.2 together with the verticality property of the Maurer–Cartan form ω; see Section 4.3.

If we now couple the commutative diagram (7.4) with the results of Sections 4.4, 4.5 and 4.6, and using the same notation introduced at the end of the proof of Theorem 5.3, we get the following commutative diagram of precalculi up to homotopy

$$\begin{tikzcd} T^L_{\mathrm{poly}}(R) \arrow[r, hook] \arrow[d] & \mathfrak{g}_1^L \arrow[r, "\mathfrak{U}_{L,1}"] \arrow[d] & \mathfrak{g}_2^L \arrow[d] & D^L_{\mathrm{poly}}(R) \arrow[l, hook'] \arrow[d] \\ \Omega^L(R) \arrow[r, hook] & \mathfrak{m}_1^L & \mathfrak{m}_2^L \arrow[l, "\mathfrak{S}_{L,0}"'] & C^L_{\mathrm{poly}}(R). \arrow[l, hook'] \end{tikzcd} \tag{7.5}$$

The quasi-isomorphisms $\mathfrak{U}_{L,1}$ and $\mathfrak{S}_{L,0}$ are obtained from $\mathcal{U}_{L,\omega,1}$ and $\mathcal{S}_{L,\omega,0}$ respectively by means of the descent procedure. Since ω is an $\mathfrak{m}$-valued vector field in $T^{\mathfrak{m}}_{\mathrm{poly}}(F) = \mathfrak{g}_1^L$, for $\mathfrak{m} = C^{\mathrm{coord},L}$, we can use the results of [4, §10.1], and [7, §6], to evaluate explicitly $\mathcal{U}_{L,\omega,1}$ and $\mathcal{S}_{L,\omega,0}$; namely,

$$\mathcal{U}_{L,\omega,1} = \mathrm{HKR} \circ \iota_{j(\omega)}, \ \mathcal{S}_{L,\omega,0} = j(\omega) \wedge \mathrm{HKR}, \tag{7.6}$$

where

$$j(\omega) = \det \sqrt{\frac{\Xi}{\exp\left(\frac{\Xi}{2}\right) - \exp\left(-\frac{\Xi}{2}\right)}}, \tag{7.7}$$

with Ξ as defined in (6.3). To interpret (7.7) one should expand the right-hand side formally in terms of $\mathrm{Tr}(\Xi^n)$ and then substitute the expression for Ξ given in (6.3). This yields an element of $C^{\mathrm{coord},L}\widehat{\otimes}\Omega_F$ of degree $2n$. Thus $j(\omega)$ is a sum of elements in $C^{\mathrm{coord},L}\widehat{\otimes}\Omega_F$ of even total degree.

By the discussion in Section 6 the element $\mathrm{Tr}(\Xi^n) \in C^{\mathrm{coord},L}\widehat{\otimes}\Omega_F$ may be interpreted as an element in $C^{\mathrm{aff},L}\widehat{\otimes}_{\mathcal{O}_1}\Omega_{JL/\mathcal{O}_1}$ via the inclusions (6.2). Hence the same holds for $j(\omega)$. We keep the same notation for this reinterpreted version of $j(\omega)$.

We thus get the following formulae:

$$\mathfrak{U}_{L,1} = \mathrm{HKR} \circ \iota_{j(\omega)}, \ \mathfrak{S}_{L,0} = j(\omega) \wedge \mathrm{HKR}. \tag{7.8}$$

7.3. *Functoriality properties of the commutative diagram* (7.5). The computations in the proof of Proposition 5.5 imply the following theorem, expressing the functoriality properties of the commutative diagram (7.5).

THEOREM 7.3. *For a general algebraic morphism (ℓ, λ) from (L, R) to (M, S) as in Definition 5.4, which induces an isomorphism $S \otimes_R L \cong M$ of S-modules, and such that L is free of rank d over R, there exist quasi-isomorphisms $\mathfrak{U}_{L,1}$, $\mathfrak{U}_{M,1}$, $\mathfrak{S}_{L,0}$ and $\mathfrak{S}_{M,0}$, fitting into the commutative diagram of precalculi up to homotopy*

$$\begin{tikzcd}
T^L_{\mathrm{poly}}(R) \arrow[r, hook] \arrow[d, "\varphi_T"'] & \mathfrak{g}_1^L \arrow[rr, "\mathfrak{U}_{L,1}=\mathrm{HKR}\circ\iota_{j(\omega_L)}"] \arrow[d, "\varphi_T"'] & & \mathfrak{g}_2^L \arrow[d, "\varphi_D"'] & D^L_{\mathrm{poly}}(R) \arrow[l, hook'] \arrow[d, "\varphi_D"'] \\
T^M_{\mathrm{poly}}(S) \arrow[r, hook] \arrow[d, squiggly, "\mathrm{L}"'] & \mathfrak{g}_1^M \arrow[rr, "\mathfrak{U}_{M,1}=\mathrm{HKR}\circ\iota_{j(\omega_M)}"] \arrow[d, squiggly, "\mathrm{L}_1"'] & & \mathfrak{g}_2^M \arrow[d, squiggly, "\mathrm{L}_2"'] & D^M_{\mathrm{poly}}(S) \arrow[l, hook'] \arrow[d, squiggly, "\mathrm{L}"'] \\
\Omega^M(S) \arrow[r, hook] & \mathfrak{m}_1^M & & \mathfrak{m}_2^M \arrow[ll, "\mathfrak{S}_{M,0}=j(\omega_M)\wedge\mathrm{HKR}"'] & C^M_{\mathrm{poly}}(S) \arrow[l, hook'] \\
\Omega^L(R) \arrow[r, hook] \arrow[u, "\varphi_\Omega"] & \mathfrak{m}_1^L \arrow[u, "\varphi_\Omega"] & & \mathfrak{m}_2^L \arrow[ll, "\mathfrak{S}_{L,0}=j(\omega_L)\wedge\mathrm{HKR}"'] \arrow[u, "\varphi_J"] & C^L_{\mathrm{poly}}(R), \arrow[l, hook'] \arrow[u, "\varphi_J"]
\end{tikzcd} \tag{7.9}$$

where we borrow notation from Proposition 5.5, and where ω_L and ω_M, denote the Maurer–Cartan form on $C^{\mathrm{coord},L}$ and $C^{\mathrm{coord},M}$ respectively. The precalculus structures up to homotopy on $(\mathfrak{g}_i^*, \mathfrak{m}_i^*)$, $* = L, M$, $i = 1, 2$, are defined as in Section 5.5. Moreover the implied homotopies are in a similar way functorial for algebraic morphisms (ℓ, λ) from (L, R) to (M, S) satisfying $S \otimes_R L \cong M$.

Almost all important objects appearing in Theorem 7.3 have already appeared in Theorem 5.6, hence the functoriality properties extend to the present situation. The commutativity of the upper and lower squares involving $j(\omega)$ follows from the compatibility of the inclusions (6.2) with the base extension S/R. The functoriality properties of the implied homotopies are verified in the same way. See [4, Lemma 10.1.1] for results on $\mathfrak{U}_{*,1}$ and related homotopies. In virtue of Lemma 7.2, the homotopy expressing the compatibility of $\mathfrak{S}_{*,0}$ with cap products descends correctly on $C^{\mathrm{aff},*}$, and the functoriality properties of such a homotopy follow along the same lines of the functoriality properties in Theorem 5.6, as the homotopy under consideration is expressed in terms of scalar combinations of poly-differential operators associated with graphs of a certain type as the L_∞-quasi-isomorphisms of Kontsevich and Shoikhet.

7.3.1. *Arbitrary base fields.* We now briefly indicate how we may replace k by a general field of characteristic zero. Our arguments depend on the existence of a number of explicit homotopies. These homotopies are constructed as scalar linear combinations of poly-differential operators indexed by certain graphs, where the scalars depend only on the corresponding graphs. For the arguments to work the coefficients need to satisfy certain linear equations. These equations have a solution over $\mathbb{R}$ (given that over this field we have homotopies that work). Thus, they have a solution over any field of characteristic zero.

We will now be more specific. We refer to [4, §10.4] for what concerns Lie brackets and cup products; here we concentrate on the compatibility between cap products. We embed k in a field K containing $\mathbb{R}$. By virtue of [7, §6], $\mathcal{U}_{\gamma,1}$ and $\mathcal{S}_{\gamma,0}$ are defined over $\mathbb{Q}$ and thus k (while they are *a priori* defined over $\mathbb{R} \subset K$). Then observe that equation (7.1) is linear in the coefficients $\mathring{W}_{D,\Gamma}$ of $\mathcal{H}^{\mathcal{S}}_{\gamma}$. Since we already have a solution of these equations in $\mathbb{R} \subset K$, we get one in k by applying any projection $K \to k$.

7.4. *Proof of Theorem* 1.1 *in the global case.* Let $(X, \mathcal{O})$ be a ringed site and $\mathcal{L}$ be a locally free sheaf of Lie algebroids over $\mathcal{O}$ of rank d. We denote by $D(X)$ the derived category of sheaves of k-vector spaces over X. According to the results of Section 3, transported to the framework of sheaves of k-vector spaces, $(T^{\mathcal{L}}_{\mathrm{poly}}(X), \Omega^{\mathcal{L}}(X))$ and $(D^{\mathcal{L}}_{\mathrm{poly}}(X), C^{\mathcal{L}}_{\mathrm{poly}}(X))$ are precalculi up to homotopy. Therefore, viewed as objects of $D(X)$ they are genuine precalculi.

Additionally, the sheafification procedure can be applied to the commutative diagram (7.9), in virtue of the results of Section 7.3 (using the fact that the homotopies are functorial as well). If we further consider the resulting commutative diagram of sheaves of k-vector spaces in the derived category $D(X)$, then using (6.4) we get the commutative diagram of precalculi

$$\begin{tikzcd}
T^{\mathcal{L}}_{\mathrm{poly}}(X) \arrow[rr] \arrow[dd, "\mathrm{HKR}\circ\iota_{\widetilde{\mathrm{td}}(\mathcal{L})^{1/2}}"'] \arrow[dr, rightsquigarrow] & & \mathfrak{g}^{\mathcal{L}}_1 \arrow[dd, "\mathrm{HKR}\circ\iota_{j(\omega)}" near start] \arrow[dr, rightsquigarrow] & \\
& \Omega^{\mathcal{L}}(X) \arrow[rr] & & \mathfrak{m}^{\mathcal{L}}_1 \\
D^{\mathcal{L}}_{\mathrm{poly}}(X) \arrow[rr] \arrow[dr, rightsquigarrow] & & \mathfrak{g}^{\mathcal{L}}_2 \arrow[dr, rightsquigarrow] & \\
& C^{\mathcal{L}}_{\mathrm{poly}}(X) \arrow[rr] \arrow[uu, "\widetilde{\mathrm{td}}(\mathcal{L})^{1/2}\wedge\mathrm{HKR}" near end] & & \mathfrak{m}^{\mathcal{L}}_2, \arrow[uu, "j(\omega)\wedge\mathrm{HKR}"']
\end{tikzcd}$$

where all horizontal and vertical arrows represent isomorphisms in the derived category $D(X)$. Here $\widetilde{\mathrm{td}}(\mathcal{L})$ is the modified Todd class of $\mathcal{L}$ that is obtained by replacing the function $q(x)$ in the definition of the Todd class (see (1.2)) by

$$\widetilde{q}(x) = \frac{x}{e^{x/2} - e^{-x/2}}.$$

Hence at this point we have proved Theorem 1.1, provided that we replace the Todd class by the modified one. To obtain the result for the ordinary Todd class we follow the method of [4, §10.3]. We have

$$\begin{aligned}\widetilde{\mathrm{td}}(\mathcal{L}) &= \mathrm{td}(\mathcal{L}) \det(e^{-A(\mathcal{L})/2}) \\ &= \mathrm{td}(\mathcal{L}) e^{-\operatorname{Tr}(A(\mathcal{L}))/2} \\ &= \mathrm{td}(\mathcal{L}) e^{-a_1(\mathcal{L})/2}.\end{aligned}$$

In other words, it is sufficient to prove that $(\iota_{e^{-a_1(\mathcal{L})/4}}, e^{a_1(\mathcal{L})/4} \wedge -)$ defines an automorphism of the precalculus $(T^{\mathcal{L}}_{\mathrm{poly}}(X), \Omega^{\mathcal{L}}(X))$.

Via the inclusions (6.2) together with (6.4), we may as well prove that

$$(\iota_{e^{-\operatorname{Tr}(\Xi)/4}}, e^{\operatorname{Tr}(\Xi)/4} \wedge -)$$

defines an automorphism of the precalculus $(C^{\mathrm{coord}} \widehat{\otimes} T_{\mathrm{poly}}(F), C^{\mathrm{coord}} \widehat{\otimes} \Omega_F)$ or equivalently that $(\iota_{\operatorname{Tr}(\Xi)}, -\operatorname{Tr}(\Xi) \wedge -)$ act as derivations. The fact that $\iota_{\operatorname{Tr}(\Xi)}$ is a derivation with respect to the cup product and Lie bracket was checked in [4, §10.3]. So it remains to show compatibility with the cap product and Lie derivative.

As $\operatorname{Tr}(\Xi) = \sum_{i,\alpha} \eta_\alpha \mathrm{d}_F(\partial_i \omega^i_\alpha)$ we first derive some identities for $\iota_{\mathrm{d}_F b}$ and $\mathrm{d}_F b \wedge -$ with b in F.

First we claim

$$\mathrm{d}_F b \wedge (D \cap \sigma) = -\iota_{\mathrm{d}_F b}(D) \cap \sigma + (-1)^{|D|+1} D \cap (\mathrm{d}_F b \wedge \sigma) \tag{7.10}$$

for $b \in F$, $D \in T_{\mathrm{poly}}(F)$, $\sigma \in \Omega_F$. If $D = D_1 \cup D_2$ and (7.10) holds for D_1, D_2, then it holds for D as well. To see this, note that

$$\begin{aligned}&\mathrm{d}_F b \wedge ((D_1 \cup D_2) \cap \sigma) \\ &\qquad = \mathrm{d}_F b \wedge (D_1 \cap (D_2 \cap \sigma)) \\ &\qquad = -\iota_{\mathrm{d}_F b}(D_1) \cap (D_2 \cap \sigma) + (-1)^{|D_1|+1} D_1 \cap (\mathrm{d}_F b \wedge (D_2 \cap \sigma)) \\ &\qquad = -\iota_{\mathrm{d}_F b}(D_1) \cap (D_2 \cap \sigma) - (-1)^{|D_1|+1} D_1 \cap \iota_{\mathrm{d}_F b}(D_2) \cap \sigma \\ &\qquad\quad + (-1)^{|D_1|+|D_2|} D_1 \cap D_2 \cap (\mathrm{d}_F b \wedge \sigma) \\ &\qquad = -\iota_{\mathrm{d}_F b}(D_1 \cup D_2) \cap \sigma + (-1)^{|D_1 \cup D_2|+1} (D_1 \cup D_2) \cap (\mathrm{d}_F b \wedge \sigma).\end{aligned}$$

So we only have to consider the case where D is a function or a vector field. The case that D is a function is trivial, so assume that D is a vector field. In that case for the right-hand side of (7.10), we find

$$\begin{aligned} -\iota_{\mathrm{d}_F b}(D) \cap \sigma + (-1)^{|D|+1} D \cap (\mathrm{d}_F b \wedge \sigma) &= Db \cap \sigma - Db \wedge \sigma + \mathrm{d}_F b \wedge (D \cap \sigma) \\ &= \mathrm{d}_F b \wedge (D \cap \sigma), \end{aligned}$$

which is equal to the left-hand side of (7.10).

For the Lie derivative we use $\mathrm{L}_D = [\mathrm{d}_F, D \cap -]$. It is clear that d_F and $\mathrm{d}_F b \wedge -$ commute. Using (7.10) we then compute

$$\begin{aligned} \mathrm{d}_F b \wedge \mathrm{L}_D \sigma &= \mathrm{d}_F b \wedge (\mathrm{d}_F (D \cap \sigma) - (-1)^{|D|+1} D \cap \mathrm{d}_F \sigma) \\ &= -\mathrm{d}_F (\mathrm{d}_F b \wedge (D \cap \sigma)) + (-1)^{|D|} \mathrm{d}_F b \wedge (D \cap \mathrm{d}_F \sigma) \\ &= \mathrm{d}_F (\iota_{\mathrm{d}_F b}(D) \cap \sigma) + (-1)^{|D|} \mathrm{d}_F (D \cap (\mathrm{d}_F b \wedge \sigma)) \\ &\qquad + (-1)^{|D|+1} \iota_{\mathrm{d}_F b}(D) \cap d\sigma - D \cap (\mathrm{d}_F b \wedge \mathrm{d}_F \sigma) \\ &= \mathrm{L}_{\iota_{\mathrm{d}_F b} D}(\sigma) + (-1)^{|D|} \mathrm{L}_D (\mathrm{d}_F b \wedge \sigma). \end{aligned}$$

If η is an odd element in C^{coord}, then $\iota_{\eta \mathrm{d}_F b} D = \eta \iota_{\mathrm{d}_F b} D$ and $\mathrm{L}_{\iota_{\eta \mathrm{d}_F b} D}(\sigma) = \mathrm{L}_{\eta \iota_{\mathrm{d}_F b} D}(\sigma) = -\eta \mathrm{L}_{\iota_{\mathrm{d}_F b}}(\sigma)$. Using this we find

$$\mathrm{Tr}(\Xi) \wedge (D \cap \sigma) = -\iota_{\mathrm{Tr}(\Xi)}(D) \cap \sigma + D \cap (\mathrm{Tr}(\Xi) \wedge \sigma)$$

and

$$\mathrm{Tr}(\Xi) \wedge \mathrm{L}_D \sigma = -\mathrm{L}_{\iota_{\mathrm{Tr}(\Xi)} D}(\sigma) + \mathrm{L}_D (\mathrm{Tr}(\Xi) \wedge \sigma).$$

We conclude that $(\imath_{\mathrm{Tr}(\Xi)}, -\mathrm{Tr}(\Xi) \wedge -)$ does indeed define a derivation of precalculi.

Appendix A. Explicit formulae for the B_∞-structure on poly-differential operators

In this appendix and the next one we develop the precalculus structure on L-chains over L-cochains up to homotopy. The results in these appendices are provided for background and are not essential for the results in the body of the paper.

The graded vector space $V = D^L_{\mathrm{poly}}(R)$ is naturally a B_∞-algebra. This means that the cofree coassociative coalgebra (with counit) $\mathrm{T}(V)$ is canonically equipped with the structure of a DG bialgebra. The notion of B_∞-algebra was introduced in [1]. However, we make use here mainly of the B_∞-algebra structure given by braces [26], [14], to which we refer for more details; see also [7, §§1, 2].

The corresponding associative product m on $\mathrm{T}(V)$ is uniquely determined by its Taylor components $m_{p,q} : \mathrm{T}^p(V) \otimes \mathrm{T}^q(V) \to V$. We have $m_{p,q} = 0$ if $p \neq 1$ and

$$\begin{aligned}(\mathrm{A}.1)\quad \mathrm{m}_{1,q}(D \otimes (D_1 \otimes \cdots \otimes D_q)) &= D\{D_1, \ldots, D_q\} \\ &= \sum_{1 \le i_1 \le \cdots \le i_q \le |D| + \sum_{b=1}^{q-1}|D_b|+1} (-1)^{\sum_{k=1}^{q}|D_k|(i_k-1)} \\ &\Big(1^{\otimes(i_1-1)} \otimes \Delta^{|D_1|} \otimes 1^{\otimes(i_2-i_1-|D_1|-1)} \otimes \Delta^{|D_2|} \otimes \cdots \otimes 1^{\otimes(i_q-i_{q-1}-|D_{q-1}|-1)} \\ &\otimes \Delta^{|D_q|} \otimes 1^{\otimes(|D|+\sum_{b=1}^{q-1}|D_b|-i_q)}\Big)(D) \\ &\Big(1^{\otimes(i_1-1)} \otimes D_1 \otimes 1^{\otimes(i_2-i_1-|D_1|-1)} \otimes D_2 \otimes \cdots \otimes 1^{\otimes(i_q-i_{q-1}-|D_{q-1}|-1)} \\ &\otimes D_q \otimes 1^{\otimes(|D|+\sum_{b=1}^{q-1}|D_b|-i_q)}\Big)\end{aligned}$$

for elements D, D_i, $i = 1, \ldots, q$, of $D_{\mathrm{poly}}^L(R)$, where $|-|$ denotes the (shifted) degree of elements of $D_{\mathrm{poly}}^L(R)$. Accordingly, we have $|D\{D_1, \ldots, D_q\}| = |D| + \sum_{a=1}^{q}|D_a|$, and thus all brace operations are of degree 0. In the sum (A.1), we have $1 \le i_1$, $i_k + |D_k| + 1 \le i_{k+1}$, $k = 1, \ldots, q-1$, $i_q + |D_q| \le |D| + \sum_{a=1}^{q}|D_a| + 1$. The sign conventions are taken from [7]. The brace operations (A.1) satisfy an infinite family of quadratic identities (see, e.g., [7]), which are equivalent to the associativity of the product m.

We define the cup product by means of the brace operations (see also [26], [7]) via the assignment

$$D_1 \cup D_2 = (-1)^{|D_1|+1}\mu\{D_1, D_2\}, \ D_i \in D_{\mathrm{poly}}^L(R), \ i = 1, 2. \tag{A.2}$$

It is obvious that the cup product has (shifted) degree 1. An easy verification using Formula (A.1) shows that the previous definition of cup product coincides with the one given in formula (3.11).

We now have the following compatibilities.

Lemma A.1. *The degree* 0 *operation* (3.10) *and the degree* 1 *operation* (A.2) *satisfy the following properties*:

$$[D_1, D_2] = -(-1)^{|D_1||D_2|}[D_2, D_1], \tag{A.3}$$

$$[D_1, [D_2, D_3]] = [[D_1, D_2], D_3] + (-1)^{|D_1||D_2|}[D_2, [D_1, D_3]], \tag{A.4}$$

$$\begin{aligned}D_1 \cup D_2 &= (-1)^{(|D_1|-1)(|D_2|-1)} D_2 \cup D_1 \\ &\pm \left(\mathrm{d_H}(D_1\{D_2\}) - (\mathrm{d_H}D_1)\{D_2\} - (-1)^{|D_1|} D_1\{\mathrm{d_H}D_2\}\right),\end{aligned} \tag{A.5}$$

$$D_1 \cup (D_2 \cup D_3) = (D_1 \cup D_2) \cup D_3, \tag{A.6}$$

and

(A.7)
$$[D_1, D_2 \cup D_3] = [D_1, D_2] \cup D_3 + (-1)^{|D_1|(|D_2|-1)} D_2 \cup [D_1, D_3] \\ + (-1)^{|D_1|}\Big(\mathrm{d_H}(D_1\{D_2, D_3\}) - (\mathrm{d_H} D_1)\{D_2, D_3\} - (-1)^{|D_1|} D_1\{\mathrm{d_H} D_2, D_3\} \\ - (-1)^{|D_1|+|D_2|} D_1\{D_2, \mathrm{d_H} D_3\}\Big)$$

for general elements D_i *of* $D^L_{\mathrm{poly}}(R)$, $i = 1, 2, 3$ *and where* $\mathrm{d_H} = [\mu, \bullet]$, $\mu = 1 \otimes_R 1$.

Appendix B. The precalculus structure on L-chains

We need results from [7], [24] about algebraic structures on Hochschild (co)chains, which have to be adapted to the Lie algebroid framework.

According to [24] and [7], there are two distinct, noncompatible, left B_∞-module structures on the Hochschild chain complex of A, viewed as a B_∞-algebra with respect to the brace operations (A.1). Equivalently, we view the two left B_∞-module structures on the Hochschild chain complex as the data of two left actions $\mathrm{m}_{L,i}$, $i = 1, 2$, on the left comodule cofreely cogenerated by the Hochschild chain complex of A over the coalgebra cofreely cogenerated by the Hochschild cochain complex of A.

These results can be applied to the present situation with due changes: $\widehat{C}_{R,\bullet}(JL)$ has two left B_∞-module structures over the B_∞-algebra $D^L_{\mathrm{poly}}(R)$.

We borrow the main notation and sign conventions from [7]. We denote by $\mathrm{m}_{L,i}$, $i = 1, 2$ the two left B_∞-module structures on $\widehat{C}_{R,\bullet}(JL)$. They are uniquely determined by their Taylor components

(B.1)
$$\Big(\mathrm{m}^{1,q,r}_{L,1}(P \otimes (Q_1 \otimes \cdots \otimes Q_q) \otimes a \otimes (R_1 \otimes \cdots \otimes R_r))\Big)(D) \\ = \sum_{l=-|a|-\sum_{b=1}^q |Q_b|-q+1 \bmod (-|a|+1)}^{-|a|-|P|-\sum_{b=1}^q |Q_b|+r+1 \bmod (-|a|+1)} \sum_{\substack{l \le j_1 \le \cdots \le j_q \le -|a| \\ 1 \le k_1 \le \cdots \le k_r \le |a|+|P|+l}} (-1)^{l(-|a|-l+1)+\sum_{b=1}^q |Q_b|(j_b-l)+\sum_{c=1}^r |R_c|(k_c-l-1)} \\ \sigma^{(-|a|-l+1)}(a)\Big((\Delta^{(|P|+\sum_{b=1}^q |Q_b|+\sum_{c=1}^r |R_c|)} \otimes 1^{\otimes(-|a|-|P|-\sum_{b=1}^q |Q_b|-\sum_{c=1}^r |R_c|)})(D) \\ \Big(1^{(j_1-l)} \otimes \Delta^{|Q_1|} \otimes \cdots \otimes 1^{\otimes(j_q-j_{q-1}-|Q_{q-1}|-1)} \otimes \Delta^{|Q_q|} \otimes 1^{\otimes(-|a|-j_q-|Q_q|+k_1)} \\ \otimes \Delta^{|R_1|} \otimes \cdots \otimes 1^{\otimes(k_r-k_{r-1}-|R_{r-1}|-1)} \otimes \Delta^{|R_r|} \\ \otimes 1^{\otimes(|a|+|P|+\sum_{b=1}^q |Q_b|+\sum_{c=1}^{r-1} |R_c|+l-k_r-1)} \otimes 1^{\otimes(-|a|-|P|-\sum_{b=1}^q |Q_b|-\sum_{c=1}^r |R_c|)}\Big)(P) \\ \Big(1^{(j_1-l)} \otimes Q_1 \otimes \cdots \otimes 1^{\otimes(j_q-j_{q-1}-|Q_{q-1}|-1)} \otimes Q_q \otimes 1^{\otimes(-|a|-j_q-|Q_q|+k_1)} \otimes R_1 \otimes \cdots \\ \otimes 1^{\otimes(k_r-k_{r-1}-|R_{r-1}|-1)} \otimes R_r \otimes 1^{\otimes(|a|+|P|+\sum_{b=1}^q |Q_b|+\sum_{c=1}^{r-1} |R_c|+l-k_r-1)} \\ \otimes 1^{\otimes(-|a|-|P|-\sum_{b=1}^q |Q_b|-\sum_{c=1}^r |R_c|)}\Big)\Big),$$

where σ is the operator on $\widehat{C}_{R,\bullet}(JL)$ defined via

$$\sigma(a)(D_0\otimes\cdots D_{-|a|})=a(D_1\otimes\cdots D_{-|a|}\otimes D_0),\ D_i\in \mathrm{U}_R(L),\ i=0,\dots,-|a|,$$

which obviously satisfies $\sigma^{(-|a|+1)}=\mathrm{id}$, and the indices in the summation satisfy $l\le j_1$, $j_i+|Q_i|+1\le j_{i+1}$, $i=1,\dots,q-1$, $j_q+|Q_q|\le -|a|$, $k_i+|R_i|+1\le k_{i+1}$, $i=1,\dots,r-1$, $k_r+|R_r|\le |a|+|P|+\sum_{b=1}^q|Q_b|+\sum_{j=1}^c|R_c|+l-1$, and

$$\begin{aligned}
&\left(\mathrm{m}_{L,2}^{0,0,r}(a\otimes(R_1\otimes\cdots\otimes R_r))\right)(D)\\
&\quad=\sum_{1\le i_1\le\cdots\le i_p\le -|a|}(-1)^{\sum_{c=1}^r|R_c|(i_c-1)}\\
&\qquad a\Big(\Big(1^{\otimes i_1}\otimes\Delta^{|R_1|}\otimes 1^{\otimes(i_2-i_1-|R_1|-1)}\otimes\Delta^{|R_2|}\otimes\cdots\\
&\qquad\otimes 1^{\otimes(i_r-i_{r-1}-|R_{r-1}|-1)}\otimes\Delta^{|R_r|}\otimes 1^{\otimes(|D|+\sum_{c=1}^{r-1}|R_c|-i_r)}\Big)(D)\\
&\qquad\Big(1^{\otimes i_1}\otimes R_1\otimes 1^{\otimes(i_2-i_1-|R_1|-1)}\otimes R_2\otimes\cdots\\
&\qquad\otimes 1^{\otimes(i_r-i_{r-1}-|R_{r-1}|-1)}\otimes R_r\otimes 1^{\otimes(|D|+\sum_{c=1}^{r-1}|R_c|-i_r)}\Big)\Big),
\end{aligned}\tag{B.2}$$

where the summation is over indices $i_1,\dots,i_r$ such that $1\le i_1$, $i_k+|D_k|+1\le i_{k+1}$, $k=1,\dots,p-1$, $i_p+|D_p|\le -|a|$. We observe that the components of m_L, resp. m_L, are nontrivial only if $p\le 1$, with no restrictions on q,r, resp. only if $q=r=0$, with no restrictions on p.

It is not difficult but quite tedious to verify that both (B.1) and (B.2) have degree 0 and satisfy an infinite family of quadratic relations involving braces.

The Taylor components of $\mathrm{m}_{L,i}$, $i=1,2$, permit to define a pairing of degree 0 between $D_{\mathrm{poly}}^L(R)$ and $\widehat{C}_{R,\bullet}(JL)$ via

$$\mathrm{L}_D a=\mathrm{m}_{L,1}^{1,0,0}(D\otimes a)+(-1)^{|D|}\mathrm{m}_{L,2}^{0,0,1}(a\otimes D),\ D\in D_{\mathrm{poly}}^L(R),\ a\in\widehat{C}_{R,\bullet}(JL).\tag{B.3}$$

Similarly, we consider two distinct pairings between $D_{\mathrm{poly}}^L(R)$ and $\widehat{C}_{R,\bullet}(JL)$: for μ as above,

$$D\cap a=(-1)^{|D|}\mathrm{m}_{L,1}^{1,1,0}(\mu\otimes D\otimes a),\tag{B.4}$$

$$a\cap D=(-1)^{|a|}\mathrm{m}_{L,1}^{1,0,1}(\mu\otimes a\otimes D),\ D\in \mathrm{D}_{\mathrm{poly}}^L(R),\ a\in\widehat{C}_{R,\bullet}(JL).\tag{B.5}$$

It follows from their very definition that both (B.4) and (B.5) have degree 1.

LEMMA B.1. *The pairing* (B.3) *of degree* 0 *and the pairings* (B.4), (B.5) *of degree* 1 *satisfy the following properties*:

$$\mathrm{L}_{[D_1,D_2]}a = \mathrm{L}_{D_1}(\mathrm{L}_{D_2}a) - (-1)^{|D_1||D_2|}\mathrm{L}_{D_2}(\mathrm{L}_{D_1}a), \tag{B.6}$$

$$\begin{aligned} D \cap a &= (-1)^{(|D|-1)(|a|-1)} a \cap D \\ &\quad \pm \left(\mathrm{b_H}(\mathrm{m}_{L,1}^{1,0,0}(D \otimes a) - \mathrm{m}_{L,1}^{1,0,0}(\mathrm{d_H}D \otimes a) - (-1)^{|D|}\mathrm{m}_{L,1}^{1,0,0}(D \otimes \mathrm{b_H}a)\right), \end{aligned} \tag{B.7}$$

$$D_1 \cap (D_2 \cap a) = (D_1 \cup D_2) \cap a, \tag{B.8}$$

$$(a \cap D_1) \cap D_2 = a \cap (D_1 \cup D_2), \tag{B.9}$$

(B.10)

$$\begin{aligned} \mathrm{L}_{D_1}(D_2 \cap a) &= [D_1, D_2] \cap a + (-1)^{|D_1|(|D_2|-1)} D_2 \cap \mathrm{L}_{D_1}a \\ &\quad + (-1)^{|D_1|}\left(\mathrm{b_H}(\mathrm{m}_{L,1}^{1,1,0}(D_1 \otimes D_2 \otimes a))\right. \\ &\qquad -\mathrm{m}_{L,1}^{1,1,0}(\mathrm{d_H}D_1 \otimes D_2 \otimes a) - (-1)^{|D_1|}\mathrm{m}_{L,1}^{1,1,0}(D_1 \otimes \mathrm{d_H}D_2 \otimes a) \\ &\qquad \left. -(-1)^{|D_1|+|D_2|}\mathrm{m}_{L,1}^{1,1,0}(D_1 \otimes D_2 \otimes \mathrm{b_H}a)\right), \\ \mathrm{L}_{D_1}(a \cap D_2) &= \mathrm{L}_{D_1}a \cap D_2 + (-1)^{|D_1|(|a|-1)} a \cap [D_1, D_2] \\ &\quad + (-1)^{|D_1|}\left(\mathrm{b_H}(\mathrm{m}_{L,1}^{1,0,1}(D_1 \otimes a \otimes D_2))\right. \\ &\qquad -\mathrm{m}_{L,1}^{1,0,1}(\mathrm{d_H}D_1 \otimes a \otimes D_2) - (-1)^{|D_1|}\mathrm{m}_{L,1}^{1,0,1}(D_1 \otimes \mathrm{b_H}a \otimes D_2) \\ &\qquad \left. -(-1)^{|D_1|+|a|}\mathrm{m}_{L,1}^{1,0,1}(D_1 \otimes a \otimes \mathrm{d_H}D_2)\right), \end{aligned}$$

and finally

(B.11)

$$\begin{aligned} &\mathrm{L}_{D_1 \cup D_2}a + (-1)^{(|D_1|-1)(|D_2|-1)}\mathrm{L}_{D_2 \cup D_1}a \\ &= \left(D_1 \cap \mathrm{L}_{D_2}a + (-1)^{(|D_1|-1)(|D_2|+|a|-1)}\mathrm{L}_{D_2}a \cap D_1\right) \\ &\quad + (-1)^{|a|(|D_2|-1)}\left(\mathrm{L}_{D_1}a \cap D_2 + (-1)^{(|D_1|+|a|-1)(|D_2|-1)} D_2 \cap \mathrm{L}_{D_1}a\right) \\ &\quad + (-1)^{(|D_2|-1)}\left([D_1, D_2] \cap a + (-1)^{(|a|-1)(|D_1|+|D_2|-1)} a \cap [D_1, D_2]\right) \\ &\quad + (-1)^{|D_2|}\mathrm{b_H}(\mathrm{m}_{L,2}^{0,0,2}(a \otimes R_1 \otimes R_2)) - (-1)^{|D_1|}\mathrm{m}_{L,2}^{0,0,2}(\mathrm{b_H}a \otimes D_1 \otimes D_2) \\ &\quad + (-1)^{|D_2|}\mathrm{m}_{L,2}^{0,0,2}(a \otimes \mathrm{d_H}D_1 \otimes D_2) + (-1)^{|D_1|+|D_2|}\mathrm{m}_{L,2}^{0,0,2}(a \otimes D_1 \otimes \mathrm{d_H}D_2) \\ &\quad + (-1)^{|D_1|}\mathrm{b_H}(\mathrm{m}_{L,2}^{0,0,2}(a \otimes D_2 \otimes D_1)) - (-1)^{|D_2|}\mathrm{m}_{L,2}^{0,0,2}(\mathrm{b_H}a \otimes D_2 \otimes D_1) \\ &\quad + (-1)^{|D_1|}\mathrm{m}_{L,2}^{0,0,2}(a \otimes \mathrm{d_H}D_2 \otimes D_1) + (-1)^{|D_1|+|D_2|}\mathrm{m}_{L,2}^{0,0,2}(a \otimes D_2 \otimes \mathrm{d_H}D_1) \end{aligned}$$

for a general element a *of* $\widehat{C}_{R,\bullet}(JL)$ *and general elements* D, D_i, $i = 1, 2$, *of* $D_{\mathrm{poly}}^L(R)$, *and where* $\mathrm{b_H} = \mathrm{L}_\mu$, *for* μ *as before.*

As for Lemma A.1, the proof essentially makes use of the brace identities, of the fact that $\mathrm{m}_{L,i}$, $i = 1, 2$, is a left action with respect to the brace operations, and of the fact that $\mathrm{m}_{L,1}$ and $\mathrm{m}_{L,2}$ satisfy a weak compatibility, as explained in more details in [7].

Both actions $\mathrm{m}_{L,1}$ and $\mathrm{m}_{L,2}$ are compatible with the Grothendieck connection; i.e.,

$$\begin{aligned}&{}^1\nabla_l\Big(\mathrm{m}_{L,1}^{1,q,r}(D\otimes Q_1\otimes\cdots\otimes a\otimes R_1\otimes\cdots)\Big)\\&\qquad\qquad=\mathrm{m}_{L,1}^{1,q,r}(D\otimes Q_1\otimes\cdots\otimes{}^1\nabla_l a\otimes R_1\otimes\cdots),\ q,r\geq 0,\\&{}^1\nabla_l\Big(\mathrm{m}_{L,2}^{0,0,r}(D_1\otimes\cdots\otimes a)\Big)=\mathrm{m}_{L,2}^{p,0,0}(D_1\otimes\cdots\otimes{}^1\nabla_l a),\ p\geq 0\end{aligned}$$

for D, D_i ($i=1,\ldots,p$), Q_j ($j=1,\ldots,q$), R_k ($k=1,\ldots,r$) elements of $D_{\mathrm{poly}}^L(R)$, and a of $\widehat{C}_{R,\bullet}(JL)$. Both identities follow from the fact that ${}^1\nabla$ commutes with the operator σ and from the fact that $\mathrm{U}_R(L)$ is a Hopf algebroid; in particular, the comultiplication is an algebra morphism.

Then in virtue of Lemma B.1, the pairings (B.3), (B.4) and (B.5) are compatible with the Grothendieck connection implying, in particular, that the Hochschild differential is also compatible therewith. By the very same arguments, formulae (B.7), (B.8), (B.9), (B.10), (B.11) and (B.11) are compatible with the Grothendieck connection, whence $(\mathrm{Ker}({}^1\nabla)\cap\widehat{C}_{R,\bullet}(JL),\mathrm{b_H},\mathrm{L},\cap)$, where $\cap$ denotes here both (B.4) and (B.5), inherits a structure of precalculus up to homotopy over the Gerstenhaber algebra $(D_{\mathrm{poly}}^L(R),\mathrm{d_H},[\ ,\ ],\cup)$ up to homotopy.

For the sake of completeness, we write down explicit formulae for the Hochschild differential $\mathrm{b_H}$ on the complex of Hochschild L-chains on R and for the pairing (B.5) between $D_{\mathrm{poly}}^L(R)$ and $C_{\mathrm{poly}}^L(R)$; in [9] we deduce the same formulae in the framework of homological algebra and derived functors. Explicitly,

$$\begin{aligned}&\mathrm{b_H}(a)=a\circ\mathrm{d_H},\\&a\cap D=(-1)^{|a|}a(D\otimes_R\bullet),\ a\in C_{\mathrm{poly}}^L(R),\ D\in D_{\mathrm{poly}}^L(R).\end{aligned}$$

We observe that (B.6) implies that $\mathrm{b_H}$, the Hochschild differential on L-chains, is compatible with respect to (B.3), and that (B.10) and (B.11), in the special case $D_1=\mu$, imply that $\mathrm{b_H}$ satisfies Leibniz's rule with respect to (B.4) and (B.5) respectively.

Thus, combining these arguments with Proposition 3.6, we have the following important

THEOREM B.2. *For a Lie algebroid L over the ring R as above, the twist of* (B.3), (B.4), (B.5) *and of the Hochschild differential* $\mathrm{b_H}$ *with respect to the isomorphism* (3.15) *endow $C_{\mathrm{poly}}^L(R)$ with a structure of precalculus up to homotopy over the Gerstenhaber algebra $(D_{\mathrm{poly}}^L(R),\mathrm{d_H},[\ ,\ ],\cup)$ up to homotopy.*

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