

Bounds for the multiplicities of cohomological automorphic forms on GL_2

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Abstract

We prove a power saving for the dimension of the space of cohomological automorphic forms of fixed level and growing weight on GL_2 over any number field that is not totally real. Our proof involves the theory of p -adically completed cohomology developed by Calegari and Emerton and a bound for the growth of coinvariants in certain finitely generated noncommutative Iwasawa modules.

1. Introduction

Let F be a number field of degree n , with r_1 real places and r_2 complex places, and with ring of adèles \mathbb{A} and finite adèles \mathbb{A}_f . Let $F_\infty = F \otimes_{\mathbb{Q}} \mathbb{R}$, so that $\mathrm{GL}_2(F_\infty) = \mathrm{GL}_2(\mathbb{R})^{r_1} \times \mathrm{GL}_2(\mathbb{C})^{r_2}$, and let Z_∞ be the centre of $\mathrm{GL}_2(F_\infty)$. Let K_f be a compact open subgroup of the finite adèle group $\mathrm{GL}_2(\mathbb{A}_f)$, and define $X = \mathrm{GL}_2(F) \backslash \mathrm{GL}_2(\mathbb{A}) / K_f Z_\infty$. If $\mathbf{d} = (d_1, \dots, d_{r_1+r_2})$ is an $(r_1 + r_2)$ -tuple of positive even integers, we shall let $S_{\mathbf{d}}(K_f)$ denote the space of cusp forms on X which are of cohomological type with weight \mathbf{d} , and define $\Delta(\mathbf{d})$ to be

$$\Delta(\mathbf{d}) = \prod_{i \leq r_1} d_i \times \prod_{i > r_1} d_i^2.$$

In this paper, we shall investigate the dimension of $S_{\mathbf{d}}(K_f)$ as \mathbf{d} varies with K_f held fixed. When F is totally real, Shimizu [15] has proven that

$$\dim S_{\mathbf{d}}(K_f) \sim C \Delta(\mathbf{d})$$

for some constant C independent of \mathbf{d} , while if F is not totally real, it may be proven using the trace formula that

$$(1) \quad \dim S_{\mathbf{d}}(K_f) = o(\Delta(\mathbf{d}))$$

(see for instance [9]). The purpose of this paper is to strengthen (1) by a power in the case where some entries of \mathbf{d} are held fixed while the rest grow

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uniformly. To be precise, if I is a subset of $[1, \dots, r_1 + r_2]$ and $\mathbf{n} = (n_i | i \in I)$ is an $|I|$ -tuple of positive even integers, we define $\mathcal{D}(\mathbf{n})$ to be the set of weights \mathbf{d} such that $d_i = n_i$ for $i \in I$. We then prove

THEOREM 1. *If F is not totally real, then for any fixed K_f , I , and \mathbf{n} , we have*

$$(2) \quad \dim S_{\mathbf{d}}(K_f) \ll_{\varepsilon} (\min_{i \notin I} d_i)^{-1/3+\varepsilon} \Delta(\mathbf{d})$$

for all $\mathbf{d} \in \mathcal{D}(\mathbf{n})$.

We may restate this more simply in the case of parallel weight as follows.

COROLLARY 2. *If F is not totally real, then for any fixed K_f and $\mathbf{d} = (d, \dots, d)$ parallel, we have*

$$\dim S_{\mathbf{d}}(K_f) \ll_{\varepsilon} d^{n-1/3+\varepsilon}.$$

Note that Theorem 1 strengthens (1) by a power if we restrict to $\mathbf{d} \in \mathcal{D}(\mathbf{n})$ such that

$$c \leq \ln d_i / \ln d_j \leq C, \quad i, j \notin I$$

for some $C, c > 0$. It is interesting to compare our theorem with results of Finis, Grunewald, and Tirao [9] in the case when F is imaginary quadratic. They prove the bounds

$$(3) \quad d \ll \dim S_{\mathbf{d}}(K_f) \ll \frac{d^2}{\ln d}, \quad \mathbf{d} = (d)$$

using base change and the trace formula respectively, where the upper bound is valid for any K_f and the lower bound for any K_f contained in the product of the standard maximal compact subgroups of the p -adic groups $\mathrm{GL}_2(F_p)$. For imaginary quadratic F , Theorem 1 reads

$$\dim S_{\mathbf{d}}(K_f) \ll_{\varepsilon} d^{5/3+\varepsilon},$$

and (3) demonstrates that the actual growth rate of $\dim S_{\mathbf{d}}(K_f)$ is a smaller power of d (which is probably d , as the experimental data of [9] shows). When F is contained in a solvable extension of its maximal totally real subfield F_0 and $\mathbf{d} = (d, \dots, d)$ is parallel, Rajan [14] has also used base change to show that

$$\dim S_{\mathbf{d}}(K_f) \gg d^{|F_0:\mathbb{Q}|}$$

after shrinking K_f if necessary. (Note the distinction between this result and that of [9], which shows that this lower bound holds for K_f maximal.)

Automorphic forms in $\dim S_{\mathbf{d}}(K_f)$ are tempered but not in the discrete series, and bounds for the multiplicities of such forms which improve over the trivial bound by a power are quite rare. Indeed, the best known bounds for tempered multiplicities that may be proven using purely analytic methods such as the trace formula only strengthen the trivial bound by a power of log,

and to obtain more than this it seems necessary to exploit some additional number theoretic or cohomological properties of the automorphic forms. To our knowledge, there are only two other families of automorphic forms for which bounds of this kind are known. The first of these is $S_1(q)$, the space of classical holomorphic forms of weight 1, level q , and character the Legendre symbol $(\frac{\cdot}{q})$, for which bounds were proven by Duke [7] and Michel and Venkatesh [13] using the restrictions placed on the Fourier coefficients of such forms by the theorem of Deligne and Serre.

The second is the collection of automorphic forms of cohomological type appearing in a ‘ p -adic congruence tower,’ studied by Calegari and Emerton in [4]. Here they prove a bound for the multiplicity of cohomological forms of fixed weight and full level Np^k with $k \rightarrow \infty$ on any reductive group G , provided the form makes a contribution to cohomology outside of the degree in which the discrete series of G (if any) appears. One of the interesting features of the proof of Theorem 1 is that it draws heavily on the methods used by Calegari and Emerton, in spite of the differences between the families of automorphic forms the two results deal with.

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2. Notation and outline of proof

Let us first give a rough outline of the proof of Theorem 1, which we shall expand on in the remainder of the section. Let Y be the locally symmetric space attached to X . Because the forms in $S_{\mathbf{d}}$ are of cohomological type, bounding their multiplicity is equivalent to bounding the cohomology of certain complex local systems $W_{\mathbf{d}}$ on Y , and because Y was arithmetic we are able to replace $W_{\mathbf{d}}$ with analogous systems $V_{\mathbf{d}}$ over \mathbb{Q}_p . By choosing a lattice in $V_{\mathbf{d}}$ and reducing mod p it will suffice to bound the \mathbb{F}_p homology of a family of congruence covers of Y . The family of covers that appears is sufficiently similar to the kind studied by Calegari and Emerton that we may apply their theory of p -adically completed homology, which converts the statement about \mathbb{F}_p growth that we require into one about the coinvariants of noncommutative Iwasawa

modules. This is Proposition 4 below, whose proof will be discussed separately in Section 3 and which should be regarded as the key ingredient in Theorem 1.

2.1. *The Eichler-Shimura isomorphism.* X may be written as a disjoint union

$$X = \coprod_{i=1}^N \Gamma_i \backslash \mathrm{SL}_2(F_\infty),$$

where Γ_i are lattices of the form $\mathrm{SL}_2(F) \cap K_i$ for compact open subgroups K_i of $\mathrm{SL}_2(\mathbb{A}_f)$, and by shrinking K_f if necessary we may assume that Γ_i are torsion free. We define

$$\begin{aligned} Y &= \coprod_{i=1}^N \Gamma_i \backslash \mathrm{SL}_2(F_\infty) / K_\infty \\ &= \coprod_{i=1}^N Y_i \end{aligned}$$

to be the associated locally symmetric spaces, where $K_\infty \subset \mathrm{SL}_2(F_\infty)$ is the standard maximal compact subgroup. We define $W_{\mathbf{d}}$ to be the representation of $\mathrm{SL}_2(F_\infty)$ obtained by taking the tensor product of the representation Sym^{d_i-2} of $\mathrm{SL}_2(F_{v_i})$ when v_i is a real place and the representation $\mathrm{Sym}^{d_i/2-1} \otimes \overline{\mathrm{Sym}}^{d_i/2-1}$ of $\mathrm{SL}_2(F_{v_i})$ when v_i is complex. We also use $W_{\mathbf{d}}$ to denote the local system on Y obtained by restricting $W_{\mathbf{d}}$ to each of the groups Γ_i . It should be noted that while one may consider more general local systems by allowing representations of the form $\mathrm{Sym}^a \otimes \overline{\mathrm{Sym}}^b$ with $a \neq b$ at complex places, the cuspidal cohomology with coefficients in such systems is trivial by a theorem of Borel-Wallach ([3, Th. 6.7, VII, p. 226]). This theorem is stated only in the case of compact quotient, but the proof (via vanishing of (\mathfrak{g}, K) -cohomology) applies more generally to the cuspidal cohomology in the noncompact case, as this may also be computed via (\mathfrak{g}, K) -cohomology.

Let $H^i(Y, W_{\mathbf{d}})$ be the cohomology groups of the local system $W_{\mathbf{d}}$, and let $H_c^i(Y, W_{\mathbf{d}})$ be the subspace of classes whose restriction to some neighbourhood of the cusps is trivial. It follows from the Eichler-Shimura isomorphism (see [10, §3] or [1, Th.3.5] and [2, Cor. 5.5]) that if $\dim W_{\mathbf{d}} > 1$, then

$$(4) \quad \dim H_c^{r_1+r_2}(Y, W_{\mathbf{d}}) = 2^{r_1} \dim S_{\mathbf{d}}(K_f).$$

Using the duality between H_c^i and H_i , we see that Theorem 1 would be implied by the following proposition.

PROPOSITION 3. *Let $Y = \mathrm{SL}_2(F) \backslash \mathrm{SL}_2(\mathbb{A}) / K_f K_\infty$ for some compact open $K_f \subset \mathrm{SL}_2(\mathbb{A}_f)$, and let I and \mathbf{n} be as in Theorem 1. For any fixed K_f, I , and \mathbf{n} , we have*

$$(5) \quad \dim H_i(Y, W_{\mathbf{d}}) \ll_\varepsilon \left(\min_{i \notin I} d_i \right)^{-1/3+\varepsilon} \Delta(\mathbf{d})$$

for all i and all $\mathbf{d} \in \mathcal{D}(\mathbf{n})$.

Remark. While the notational convention we have introduced for $W_{\mathbf{d}}$ allows us to state the Eichler-Shimura isomorphism (4) simply, it will be more convenient in the remainder of the paper to adopt the following convention: let $\{\sigma_1, \dots, \sigma_n\}$ be the complex embeddings of F , and let \mathbf{d} be an n -tuple of nonnegative integers indexed by the σ_i for which $d_i = d_j$ when σ_i and σ_j are complex conjugates. We then let $W_{\mathbf{d}}$ be the representation of $\mathrm{SL}_2(F_{\infty})$ obtained by forming the tensor product of the representations Sym^{d_i} of $\mathrm{SL}_2(F_{v_i})$ when σ_i corresponds to a real place v_i and $\mathrm{Sym}^{d_i} \otimes \overline{\mathrm{Sym}}^{d_i}$ of $\mathrm{SL}_2(F_{v_i})$ when σ_i is either of the two embeddings corresponding to a complex place v_i .

2.2. *Completed homology.* Define the compact p -adic analytic subgroups $G, G(p^k), H(p^k)$ and $T(p^k)$ of $\mathrm{SL}_2(\mathbb{Z}_p)$ by

$$G(p^k) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a - 1, b, c, d - 1 \in p^k \mathbb{Z}_p \right\}, \quad G = G(p),$$

$$H(p^k) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| b \in p^k \mathbb{Z}_p \right\} \cap G, \quad \text{and} \quad T(p^k) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| b, c \in p^k \mathbb{Z}_p \right\} \cap G.$$

Moreover, if $t \geq 1$ is an integer and $\mathbf{k} = (k_1, \dots, k_t)$ is a t -tuple of positive integers, define

$$(6) \quad \mathcal{G} = \prod_{i=1}^t G_i, \quad \mathcal{G}_{\mathbf{k}} = \prod_{i=1}^t G_i(p^{k_i}), \quad \mathcal{H}_{\mathbf{k}} = \prod_{i=1}^t H_i(p^{k_i}) \quad \text{and} \quad \mathcal{T}_{\mathbf{k}} = \prod_{i=1}^t T_i(p^{k_i}),$$

where $G_i \simeq G$ for all i , etc. In Section 5 we shall prove that if we choose $t = n$, there exists an algebraic representation $V_{\mathbf{d}}$ of \mathcal{G} over \mathbb{Q}_p and an injection $\Gamma \rightarrow \mathcal{G}$ such that if we also let $V_{\mathbf{d}}$ denote the local system on Y obtained by restricting $V_{\mathbf{d}}$ to Γ , we have

$$(7) \quad \dim_{\mathbb{C}} H_i(Y, W_{\mathbf{d}}) = \dim_{\mathbb{Q}_p} H_i(Y, V_{\mathbf{d}}).$$

The advantage of this p -adic reformulation is that we may study the right-hand side of (7) using the completed homology modules defined by Emerton in [8] and studied by Calegari and Emerton in [4], [5]. These are finitely generated modules over noncommutative Iwasawa algebras $\Lambda_{\mathbb{Q}_p}$ and Λ , defined to be

$$\Lambda_{\mathbb{Q}_p} = \Lambda_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p, \quad \Lambda_{\mathbb{Z}_p} = \varprojlim_{\mathbf{k}} \mathbb{Z}_p[\mathcal{G}/\mathcal{G}_{\mathbf{k}}],$$

$$\Lambda = \varprojlim_{\mathbf{k}} \mathbb{F}_p[\mathcal{G}/\mathcal{G}_{\mathbf{k}}],$$

where the projections are given by the trace maps $\mathbb{Z}_p[\mathcal{G}/\mathcal{G}_{\mathbf{k}'}] \rightarrow \mathbb{Z}_p[\mathcal{G}/\mathcal{G}_{\mathbf{k}}]$ for $\mathbf{k}' \geq \mathbf{k}$. We shall describe the structure of these algebras and their associated modules in more depth in Section 3. In Section 5 we shall define finitely generated $\Lambda_{\mathbb{Q}_p}$ modules $\widetilde{H}_j(\mathcal{V}_{\mathbf{d}})$, which may be used to calculate the right-hand

side of (7) via a spectral sequence

$$E_2^{i,j} = H_i(\mathcal{G}_{\mathcal{P}}, \widetilde{H}_j(\mathcal{V}_{\mathbf{d}}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) \implies H_{i+j}(Y, V_{\mathbf{d}}).$$

This spectral sequence allows us to reduce our problem to one of bounding the dimension of $H_{i,\text{con}}(\mathcal{G}, M \otimes V_{\mathbf{d}})$, where M is a fixed torsion $\Lambda_{\mathbb{Q}_p}$ module and \mathbf{d} varies. (Note that hereafter we shall always assume homology groups of \mathcal{G} to be computed with continuous chains, and drop the subscript ‘con.’) We shall do this by choosing lattices $L \subset M$ and $\mathcal{V}_{\mathbf{d}} \subset V_{\mathbf{d}}$, so that $L \otimes \mathcal{V}_{\mathbf{d}}$ is a lattice in $M \otimes V_{\mathbf{d}}$, and applying the bound

$$(8) \quad \dim_{\mathbb{Q}_p} H_i(\mathcal{G}, M \otimes V_{\mathbf{d}}) \leq \dim_{\mathbb{F}_p} H_i(\mathcal{G}, L \otimes (\mathcal{V}_{\mathbf{d}}/p)).$$

In Section 4 we shall prove that $\mathcal{V}_{\mathbf{d}}$ may be chosen to have the property that $\mathcal{V}_{\mathbf{d}}/p$ is a submodule of $\mathbb{F}_p[\mathcal{G}/\mathcal{H}_{\mathbf{k}}]$ for any \mathbf{k} satisfying $p^{k_i-1} > d_i$ for all i . We will also provide a characterization of the submodules of $\mathbb{F}_p[\mathcal{G}/\mathcal{H}_{\mathbf{k}}]$ that allows us to apply Shapiro’s lemma to the right-hand side of (8) and transfers the problem to one of bounding $H_i(\mathcal{H}_{\mathbf{k}}, M')$ for M' a fixed torsion Λ module and \mathbf{k} varying. Finally, by using the commensurator of Γ to replace $\mathcal{H}_{\mathbf{k}}$ with $\mathcal{T}_{\mathbf{k}}$, it suffices to bound $H_i(\mathcal{T}_{\mathbf{k}}, M')$. The trivial bound for the dimension of this space is

$$(9) \quad \dim H_i(\mathcal{T}_{\mathbf{k}}, M') \ll |\mathcal{G} : \mathcal{T}_{\mathbf{k}}|,$$

and it turns out that the reductions we have made are tight in the sense that we may recover the trivial bound for $S_{\mathbf{d}}(K_f)$ from (9). The problem then becomes one of making a power improvement in (9), and this is solved by the following proposition, which lies at the heart of our proof. (We define the rank r of a Λ module in Section 3; it suffices here to know that $r = 0$ when M is torsion.)

PROPOSITION 4. *For any $t \geq 1$, let \mathcal{G} and $\mathcal{T}_{\mathbf{k}}$ be as in (6) and let M be a finitely generated Λ module of rank r . Then*

$$(10) \quad \dim M_{\mathcal{T}_{\mathbf{k}}} = (r + O(\eta^{\kappa}))|\mathcal{G} : \mathcal{T}_{\mathbf{k}}|,$$

$$(11) \quad \dim H_i(\mathcal{T}_{\mathbf{k}}, M) \ll \eta^{\kappa} |\mathcal{G} : \mathcal{T}_{\mathbf{k}}|, \quad i \geq 1,$$

for all t -tuples \mathbf{k} , where $\kappa = \min(k_i)$ and $\eta = 10p^{-2/3}$.

Remark. We have attempted to extend this proof to higher rank groups, but have so far been prevented from doing so by the fact that the subgroup \mathcal{H} for which we can realize $\mathcal{V}_{\mathbf{d}}$ as a subrepresentation of $\mathbb{F}_p[\mathcal{G}/\mathcal{H}]$, and the subgroups \mathcal{T} for which we are able to prove analogues of Proposition 4, are not conjugate to one another under the noncompact p -adic group containing them.

Structure of Paper. Proposition 4 is proven in Section 3, and Section 4 contains results on the structure of $\mathbb{F}_p[\mathcal{G}/\mathcal{H}_{\mathbf{k}}]$ including how to choose the lattice $\mathcal{V}_{\mathbf{d}} \subset V_{\mathbf{d}}$ such that $\mathcal{V}_{\mathbf{d}}/p \subset \mathbb{F}_p[\mathcal{G}/\mathcal{H}_{\mathbf{k}}]$. We apply the theory of p -adically

completed cohomology in Section 5 and combine these ingredients in Section 6 to conclude the proof.

3. Coinvariants of Λ -modules

This section contains the proof of Proposition 4. Once we have proven the case $i = 0$ and $r = 0$ the others will follow by the same arguments used by Harris [11] and Calegari and Emerton [4], and so we restate this case as a separate proposition.

PROPOSITION 5. *If $t \geq 1$, \mathcal{G} and $\mathcal{T}_{\mathbf{k}}$ are as in (6), and M is any torsion Λ module, we have*

$$(12) \quad \dim M_{\mathcal{T}_{\mathbf{k}}} \ll \eta^\kappa |\mathcal{G} : \mathcal{T}_{\mathbf{k}}|,$$

for all \mathbf{k} , where $\kappa = \min(k_i)$ and $\eta = 10p^{-2/3}$.

We shall prove Proposition 5 by induction on t , the number of factors of \mathcal{G} . Proposition 7 forms the base case of the induction, and the inductive step is carried out in Section 3.1.

We begin the proof by describing the structure of the algebras Λ and $\Lambda_{\mathbb{Q}_p}$ in more detail. (Although we will not work with $\Lambda_{\mathbb{Q}_p}$ in this section, it is convenient to present its structure theory together with that of Λ .) Λ is a noncommutative Noetherian integral domain, and so its field of fractions \mathcal{L} is a division ring that is flat over Λ on both sides (and likewise for $\Lambda_{\mathbb{Q}_p}$ and its field of fractions $\mathcal{L}_{\mathbb{Q}_p}$). If M is a finitely generated Λ (resp. $\Lambda_{\mathbb{Q}_p}$) module, then $M \otimes_{\Lambda} \mathcal{L}$ (resp. $M \otimes_{\Lambda_{\mathbb{Q}_p}} \mathcal{L}_{\mathbb{Q}_p}$) is a finite dimensional \mathcal{L} (resp. $\mathcal{L}_{\mathbb{Q}_p}$) vector space, and we define the rank of M to be the dimension of this vector space. We see that rank is additive in short exact sequences by the flatness of \mathcal{L} over Λ and that M has rank 0 if and only if it is torsion.

The basic result on the growth of coinvariants in a finitely generated module M for Λ or $\Lambda_{\mathbb{Q}_p}$ is due to Harris [11]. To state it in the case under consideration, let \mathcal{G}_n denote the group $\mathcal{G}_{\mathbf{k}}$ with \mathbf{k} chosen to be (n, \dots, n) . We then have

THEOREM 6. *Let M be a finitely generated module for either $\Lambda_{\mathbb{Q}_p}$ or Λ of rank r . We then have*

$$(13) \quad \dim M_{\mathcal{G}_n} = r|\mathcal{G} : \mathcal{G}_n| + O(p^{(3t-1)n}).$$

The significance of the exponent $3t - 1$ is that it is one less than the dimension of \mathcal{G} , so that $|\mathcal{G} : \mathcal{G}_n| = cp^{3tn}$ for some c and hence the error term in Theorem 6 is smaller than the main term by a power. When M is torsion and $\mathbf{k} = (k, \dots, k)$ is parallel, (12) and (13) become

$$\dim M_{\mathcal{T}_{\mathbf{k}}} \ll 10^k |\mathcal{G} : \mathcal{T}_{\mathbf{k}}|^{1-1/3t} \quad \text{and} \quad \dim M_{\mathcal{G}_n} \ll |\mathcal{G} : \mathcal{G}_n|^{1-1/3t},$$

so that Proposition 5 extends the Λ -module case of Theorem 6 to the family of subgroups $\mathcal{T}_{\mathbf{k}}$ (up to the factor of 10^k , which becomes negligible when p is

large). The key difference between these two results is that the groups \mathcal{T}_k are not shrinking uniformly to the identity.

We shall break the proof of Proposition 5 into two parts, the first of which establishes the case $t = 1$ and the second of which applies induction in t . It is possible that the method we have used in the case $t = 1$ could be extended to provide the entire proof, but we believe that the inductive argument is simpler as it avoids having to keep track of large numbers of incidence relations. The case $t = 1$ is proven in Proposition 7 below, and the inductive step is carried out in Section 3.1.

PROPOSITION 7. *Let M be a $\Lambda = \mathbb{F}_p[[G]]$ module, and suppose that for some $C, k > 0$, M satisfies the bound*

$$(14) \quad \dim M_{G(p^l)} \leq Cp^{2l}$$

for all $l \leq k$. Then we have

$$\dim M_{T(p^k)} \leq C\eta^{k-2}p^{2k}.$$

Proof. We assume that $p \neq 2$, as $\eta > 1$ in that case. We shall apply an inductive argument to a family of subgroups that interpolates between $G(p^k)$ and $T(p^k)$. For $0 \leq j \leq l - 1$, let $T(l, j)$ be the group

$$T(l, j) = \left\{ \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} \mid a \equiv 1(p^{l-j}) \right\} + G(p^l),$$

so that for fixed l , $T(l, j)$ forms a family growing from $T(l, 0) = G(p^l)$ to $T(l, l - 1) = T(p^l)$. Let (l, j) be given, and assume that we have the bound

$$(15) \quad \dim M_{T(l', j')} \leq C\eta^{j'-1}p^{2l'}$$

for all pairs (l', j') that are smaller than (l, j) in the lexicographic ordering. We shall then deduce this bound for the pair (l, j) . Note that (15) follows from the assumption (14) when $j = 0, 1$, and so we may assume that $j \geq 2$.

We shall establish (15) by applying inclusion-exclusion counting to the groups lying between $T(l - 1, j - 1)$ and $T(l, j - 1)$. It may be checked that $V = T(l - 1, j - 1)/T(l, j - 1)$ is Abelian and isomorphic to the vector space \mathbb{F}_p^3 . We define the plane $U \subset V$ to be the image of $G(p^{l-1})$ in V and the line μ to be the image of $T(l, j)$. It will be more convenient in what follows to work with invariants rather than coinvariants, and so we define $A = M_{T(l, j-1)}^* \cdot V$ acts on A , and if $L \subseteq V$ is any subset, we let A^L denote the subspace of A fixed by all elements of L . The problem is now to deduce the bound

$$\dim A^\mu = \dim M_{T(l, j)} \leq C\eta^{j-1}p^{2l}$$

from the inductive hypotheses

$$(16) \quad \dim A = \dim M_{T(l, j-1)} \leq C\eta^{j-2}p^{2l}, \quad \dim A^V = \dim M_{T(l-1, j-1)} \leq C\eta^{j-2}p^{2l-2}.$$

We shall do this using the following two lemmas.

LEMMA 8. *If $\ell \not\subset U$ is a line, there is $g \in G(p)$ whose action on $G(p)$ by conjugation descends to V and such that $g\ell g^{-1} = \mu$.*

Proof. Define the elements $\alpha(l, j)$, N and \bar{N} of $G(p)$ by

$$\alpha(l, j) = \begin{pmatrix} 1 + p^{l-j} & 0 \\ 0 & (1 + p^{l-j})^{-1} \end{pmatrix}, \quad N = \begin{pmatrix} 1 & p^{j-1} \\ 0 & 1 \end{pmatrix}, \quad \bar{N} = \begin{pmatrix} 1 & 0 \\ p^{j-1} & 1 \end{pmatrix}.$$

We shall show that we may take g in the statement of the lemma to be of the form $N^a \bar{N}^b$. We have

$$T(l-1, j-1) = \langle \alpha(l-1, j-1) \rangle G(p^{l-1}), \quad T(l, j-1) = \langle \alpha(l, j-1) \rangle G(p^l).$$

It can be seen that N normalizes $G(p^l)$ and $G(p^{l-1})$, and a calculation gives (17)

$$\begin{aligned} N\alpha(l-1, j-1)N^{-1} &= \begin{pmatrix} 1 + p^{l-j} & p^{j-1}[(1 + p^{l-j})^{-1} - (1 + p^{l-j})] \\ 0 & (1 + p^{l-j})^{-1} \end{pmatrix} \in T(l-1, j-1), \\ N\alpha(l, j-1)N^{-1} &= \begin{pmatrix} 1 + p^{l-j+1} & p^{j-1}[(1 + p^{l-j+1})^{-1} - (1 + p^{l-j+1})] \\ 0 & (1 + p^{l-j+1})^{-1} \end{pmatrix} \in T(l, j-1), \end{aligned}$$

so that it also normalizes $T(l-1, j-1)$ and $T(l, j-1)$. It may be shown in the same way that \bar{N} normalizes these groups, so that N and \bar{N} act on V .

To calculate this action, we define N' and \bar{N}' to be

$$N' = \begin{pmatrix} 1 & p^{l-1} \\ 0 & 1 \end{pmatrix} \quad \bar{N}' = \begin{pmatrix} 1 & 0 \\ p^{l-1} & 1 \end{pmatrix}$$

and let $\{v_1, v_2, v_3\}$ be the basis of V consisting of the respective images of $\alpha(l-1, j-1)$, N' and \bar{N}' . We clearly have $Nv_2N^{-1} = v_2$, and it follows from equation (17) and our assumption that $p \neq 2$ that $Nv_1N^{-1} = v_1 - 2v_2$. A calculation shows that

$$N\bar{N}'N^{-1} = \begin{pmatrix} 1 + p^{l+j-2} & -p^{l+2j-3} \\ p^{l-1} & 1 - p^{l+j-2} \end{pmatrix} \in \bar{N}' + G(p^l),$$

so that $Nv_3N^{-1} = v_3$. (Note that we have used the assumption $j \geq 2$ here.) Therefore N acts on V by shearings that leave the plane U fixed and translate v_1 in the direction v_2 , and a similar calculation shows that \bar{N} acts by shearings in the direction v_3 . The group generated by these is transitive on lines not contained in U , which completes the proof. \square

LEMMA 9. *Let $P \subset V$ be a plane that is not equal to U , and let $\ell_1, \dots, \ell_k, \ell$ be distinct lines in P that do not lie in U . If we define $N = \sum_{i=1}^k A^{\ell_i}$, then we have*

$$\dim A^\ell + \dim N \leq \dim(A^\ell + N) + k \dim A^P.$$

Proof. Inclusion-exclusion counting gives

$$\dim A^\ell + \dim N = \dim(A^\ell + N) + \dim A^\ell \cap N,$$

so it suffices to show that $\dim A^\ell \cap N \leq k \dim A^P$. Choose $v \in P$ that does not lie in ℓ or U , and consider the action of the element $(\mathbf{0} - v)^k \in \mathbb{F}_p[V]$ on $z \in A^\ell \cap N$. For any ℓ_i , we can find $u_i \in \ell$ so that $u_i + v = v_i \in \ell_i$. We have $(\mathbf{0} - v)x = (\mathbf{0} - v_i)x$ for any $x \in A^\ell$, and because $(\mathbf{0} - v_i)A^{\ell_i} = 0$, this implies that

$$(\mathbf{0} - v)^k z = \prod_{i=1}^k (\mathbf{0} - v_i) z = 0.$$

It follows that $A^\ell \cap N \subset \ker(\mathbf{0} - v)^k$. $\mathbf{0} - v$ maps A^ℓ to itself, and the kernel is $A^v \cap A^\ell = A^P$. We therefore have

$$\dim \ker(\mathbf{0} - v)^k : A^\ell \rightarrow A^\ell \leq k \dim A^P,$$

which completes the proof. □

If $P \neq U$ is a plane and $\ell_1, \dots, \ell_k \subset P$ are distinct lines that do not lie in U , we may apply Lemma 9 repeatedly to show that

$$\sum_{i=1}^k \dim A^{\ell_i} \leq \dim \sum_{i=1}^k A^{\ell_i} + \frac{k(k-1)}{2} \dim A^P.$$

It follows from Lemma 8 that $\dim A^\ell = \dim A^\mu$ for all $\ell \not\subset U$, and applying this to the above equation gives

$$(18) \quad k \dim A^\mu \leq \dim A + \frac{k(k-1)}{2} \dim A^P$$

$$\frac{2}{k-1} \dim A^\mu - \frac{2}{k(k-1)} \dim A \leq \dim A^P.$$

If P_1, \dots, P_j are planes containing μ , we may apply the argument in the proof of Lemma 9 to the space V/μ to show that

$$\sum_{i=1}^j \dim A^{P_i} \leq \dim A^\mu + \frac{j(j-1)}{2} \dim A^V.$$

Applying (18) to each summand $\dim A^{P_i}$ and rearranging gives

$$\left(\frac{2j}{k-1} - 1 \right) \dim A^\mu \leq \frac{2j}{k(k-1)} \dim A + \frac{j(j-1)}{2} \dim A^V.$$

After combining this with the inductive hypotheses (16), we have

$$\left(\frac{2j}{k-1} - 1\right) \dim A^\mu \leq C\eta^{j-2}p^{2l} \left(\frac{2j}{k(k-1)} + \frac{j(j-1)}{2}p^{-2}\right).$$

Choosing $j = k = \lfloor p^{2/3} \rfloor$ (note that $k \geq 2$ as $p \geq 3$) gives

$$\dim A^\mu \leq C\eta^{j-2}p^{2l}(10p^{-2/3}),$$

which completes the proof. □

3.1. *Proof of Proposition 5: the inductive step.* We now assume that $t > 1$ and factorize the groups and rings under consideration as

$$\mathcal{G} = G_1 \times \mathcal{G}', \quad \mathcal{T}_k = T_1(p^{k_1}) \times \mathcal{T}'_k, \quad \text{and} \quad \Lambda = \Lambda_1 \otimes \Lambda'.$$

A natural approach to Proposition 5 would be to consider $M_{\mathcal{T}'_k}$ as a Λ_1 module and apply Proposition 7 to derive a bound for $M_{\mathcal{T}_k}$ from a bound for the coinvariant spaces $(M_{\mathcal{T}'_k})_{G_1(p^l)}$ for $l \leq k_1$. Indeed, it would suffice to know that

$$(19) \quad \dim M_{G_1(p^l) \times \mathcal{T}'_k} \ll p^{2l} |\mathcal{G}' : \mathcal{T}'_k|$$

uniformly in \mathbf{k} and l . However, $M_{\mathcal{T}'_k}$ need not be torsion for Λ_1 , and so (19) will not hold in general. We get around this difficulty by showing that the Λ_1 -module $M_{\mathcal{T}'_k}$ may be written as an extension of K by L , where K satisfies (19) and L has few generators. $K_{\mathcal{T}_k}$ may be thought of as the part of $M_{\mathcal{T}_k}$ that may be bounded using the action of Λ_1 , and $L_{\mathcal{T}_k}$ as the part that may be bounded using the action of Λ' .

We construct L by using the action of Λ_1 to define a filtration of M by Λ' modules, one of which must be torsion. To demonstrate this in a simple case, assume that M_{G_1} is torsion as a Λ' module. If we apply the inductive hypothesis that Proposition 5 holds for the lower dimensional group \mathcal{G}' , we obtain

$$\dim(M_{\mathcal{T}'_k})_{G_1} \ll \eta^k |\mathcal{G}' : \mathcal{T}'_k|.$$

We may lift a basis for $(M_{\mathcal{T}'_k})_{G_1}$ to a generating set for $M_{\mathcal{T}'_k}$ as a Λ_1 module, and (12) then follows trivially from this bound on the number of generators. In this case, we therefore see that we may take $K = 0$, $L = M_{\mathcal{T}'_k}$.

In the general case, first assume without loss of generality that M is a cyclic Λ module. To define the filtration we shall use, we need the following structure theorem for Λ , taken from [6].

THEOREM 10. *Let g_1, \dots, g_d be a topological generating set for \mathcal{G} . The completed group ring $\Lambda = \mathbb{F}_p[[\mathcal{G}]]$ is generated by $z_i = 1 - g_i$, and every element of it can be uniquely expressed as a sum over multi-indices α ,*

$$x = \sum_{\alpha} \lambda_{\alpha} z^{\alpha},$$

where $z^\alpha = \prod_{i=1}^d z_i^{\alpha_i}$ and all $\lambda_\alpha \in \mathbb{F}_p$. Moreover, all such sums are in Λ . The filtration by degree gives Λ the structure of a filtered ring whose associated graded ring is commutative, i.e., $z^\alpha z^\beta = z^{\alpha+\beta}$ up to terms of degree $> |\alpha| + |\beta|$.

When applied to Λ_1 , Theorem 10 says that Λ_1 is an almost commutative polynomial algebra in three variables, equal to the \mathbb{F}_p span of $z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3}$ for $\alpha \in (\mathbb{Z}_{\geq 0})^3$. Let us define a total ordering \succ on such triples α by requiring that $\alpha \succ \beta$ if $|\alpha| > |\beta|$ and ordering those triples α with a common value of $|\alpha|$ lexicographically. We denote the successor of α under this ordering by α^+ . We shall also define a partial ordering \geq on triples by declaring that $\alpha \geq \beta$ if this inequality holds entrywise.

Define I_α to be the subspace of Λ_1 spanned by all monomials z^β with $\beta \succeq \alpha$, which is a two-sided ideal by the almost commutativity of Λ_1 , and if N is any Λ_1 module, we define $N_\alpha = I_\alpha N$.

LEMMA 11. *If $\alpha \geq \beta$, multiplication by $z^{\alpha-\beta}$ induces a surjection*

$$z^{\alpha-\beta} : N_\beta/N_{\beta^+} \longrightarrow N_\alpha/N_{\alpha^+}.$$

Proof. This follows from the definition of N_α and the fact that multiplication by $z^{\alpha-\beta}$ induces an isomorphism $I_\beta/I_{\beta^+} \simeq I_\alpha/I_{\alpha^+}$. □

We note that the first quotient of the filtration $\{N_\alpha\}$ of N is equal to N_{G_1} . We shall consider the filtration $\{M_\alpha\}$ of M . If $v \in M$ is a generator, the successive quotients M_α/M_{α^+} of this filtration are cyclic Λ' modules generated by $z^\alpha v$. Write M as Λ/I for some left ideal $I \subset \Lambda$, and let α be the last tuple in the ordering \succ such that $I \subset I_\alpha \otimes \Lambda'$. Because M_α/M_{α^+} is a nontrivial quotient of $(I_\alpha/I_{\alpha^+}) \otimes \Lambda'$, which is free of rank one for Λ' , we see that M_α/M_{α^+} is a torsion Λ' -module. We then deduce that

$$(20) \quad \dim(M_\alpha/M_{\alpha^+})_{\mathcal{T}'_k} \ll \eta^\kappa |\mathcal{G}' : \mathcal{T}'_k|$$

by applying the inductive hypothesis to M_α/M_{α^+} as a Λ' module, and this will allow us to control the size of various filtered pieces of the Λ_1 module $N = M_{\mathcal{T}'_k}$. Because the projection $M \twoheadrightarrow N$ commutes with the action of Λ_1 , it induces projections $M_\alpha \twoheadrightarrow N_\alpha$ and hence a surjection

$$(M_\alpha/M_{\alpha^+})_{\mathcal{T}'_k} \twoheadrightarrow N_\alpha/N_{\alpha^+}.$$

It then follows from (20) that

$$(21) \quad \dim N_\alpha/N_{\alpha^+} \ll \eta^\kappa |\mathcal{G}' : \mathcal{T}'_k|.$$

By Lemma 11 and the subsequent comment, there is a surjection

$$(22) \quad z^\alpha : N_{G_1} \longrightarrow N_\alpha/N_{\alpha^+},$$

and we may choose a subspace $\bar{L} \subset N_{G_1}$ for which the restriction of (22) is an isomorphism. It follows from (21) that

$$\dim \bar{L} \ll \eta^\kappa |\mathcal{G}' : \mathcal{T}'_{\mathbf{k}}|.$$

Choose a basis for \bar{L} , lift its elements to N , and let $L \subset N$ be the submodule they generate under the action of Λ_1 . Because L has $\ll \eta^\kappa |\mathcal{G}' : \mathcal{T}'_{\mathbf{k}}|$ generators, we have

$$\begin{aligned} L_{T_1(p^k)} &\ll \eta^\kappa |G_1 : T_1(p^k)| \times |\mathcal{G}' : \mathcal{T}'_{\mathbf{k}}| \\ &= \eta^\kappa |\mathcal{G} : \mathcal{T}_{\mathbf{k}}|, \end{aligned}$$

so that it suffices to bound the $T_1(p^k)$ coinvariants in $K = N/L$.

LEMMA 12. $K_\beta/K_{\beta+} = 0$ for all $\beta \geq \alpha$, where we recall that \geq is the partial ordering introduced after Theorem 10.

Proof. We have

$$z^\alpha : L_{G_1} \rightarrow N_\alpha/N_{\alpha+}$$

by construction, so that $L_\alpha/L_{\alpha+} \rightarrow N_\alpha/N_{\alpha+}$. We therefore have $L_\alpha + N_{\alpha+} = N_\alpha = L_\alpha + N_\alpha$, so $N_{\alpha+} + L = N_\alpha + L$. However

$$K_\alpha = (N_\alpha + L)/L \quad \text{and} \quad K_{\alpha+} = (N_{\alpha+} + L)/L,$$

so that $K_\alpha/K_{\alpha+} = 0$ and hence $K_\beta/K_{\beta+} = 0$ for all $\beta \geq \alpha$ by Lemma 11. \square

Our last step is to use the triviality of $K_\beta/K_{\beta+}$ to bound $K_{G_1(p^l)}$, so that Lemma 7 may be applied. Define the two-sided ideal $I(p^l)$ of Λ_1 by

$$0 \rightarrow I(p^l) \rightarrow \mathbb{F}_p[[G_1]] \rightarrow \mathbb{F}_p[G_1/G_1(p^l)] \rightarrow 0.$$

$I(p^l)$ may also be described as the span of the monomials z^γ for

$$\gamma \not\leq (p^l - 1, p^l - 1, p^l - 1).$$

Lemma 11 provides us with maps

$$(23) \quad K_{G_1} \rightarrow K_\beta/K_{\beta+} = I_\beta K/I_{\beta+} K \rightarrow (I_\beta + I(p^l))K/(I_{\beta+} + I(p^l))K$$

for all β , so that

$$(24) \quad \begin{aligned} \dim(I_\beta + I(p^l))K/(I_{\beta+} + I(p^l))K &\leq \dim K_{G_1} \\ &\leq \dim N_{G_1} \\ &\leq |\mathcal{G}' : \mathcal{T}'_{\mathbf{k}}|. \end{aligned}$$

Lemma 12 and the second map in equation (23) also imply that $(I_\beta + I(p^l))K = (I_{\beta+} + I(p^l))K$ for all $\beta \geq \alpha$.

If S_l is the set of indices $\beta \leq (p^l - 1, p^l - 1, p^l - 1)$, we have

$$(25) \quad |\{\beta \in S_l \mid \alpha \not\leq \beta\}| \ll_\alpha p^{2l}.$$

Because the ideals $I_\beta + I(p^l)$ form an exhaustive filtration of $I(p^l)$, we have

$$\dim K_{G_1(p^l)} = \dim K/I(p^l)K = \sum_{\beta} \dim(I_\beta + I(p^l))K/(I_{\beta+} + I(p^l))K.$$

Suppose that $z^\beta \in I(p^l)$. Then $I_\beta + I(p^l) = I_{\beta+} + (z^\beta) + I(p^l) = I_{\beta+} + I(p^l)$, and the quotient above is zero. Hence it follows that

$$\dim K_{G_1(p^l)} = \dim K/I(p^l)K = \sum_{\beta \in S_l} \dim(I_\beta + I(p^l))K/(I_{\beta+} + I(p^l))K.$$

If $\beta \geq \alpha$, then the corresponding term is zero. Because there are at most p^{2l} nonzero terms by (25), and every such term has order at most $|\mathcal{G}' : \mathcal{T}_k|$, we conclude that

$$\dim K_{G_1(p^l)} \ll_{\alpha} p^{2l} |\mathcal{G}' : \mathcal{T}_k|.$$

The required bound for $\dim K_{T_1(p^k)}$ now follows from Proposition 7, which completes the proof of Proposition 5.

3.2. *Extension to higher homological degree.* We finish this section by deducing the remaining statements of Proposition 4 from Proposition 5, following Harris [11] and Calegari and Emerton [4]. Recall that we must prove that if M is a finitely generated Λ module of rank r , then

$$(26) \quad \dim M_{\mathcal{T}_k} = (r + O(\eta^k)) |\mathcal{G} : \mathcal{T}_k|,$$

$$(27) \quad \dim H_i(\mathcal{T}_k, M) \ll \eta^k |\mathcal{G} : \mathcal{T}_k|, \quad i \geq 1.$$

We begin with equation (26), which we have just proven in the case $r = 0$. In general, there is an exact sequence

$$0 \rightarrow T \rightarrow M \rightarrow M' \rightarrow 0$$

with T torsion and M' torsion free of rank r , and by applying the rank 0 result to T , we deduce

$$|\dim M_{\mathcal{T}_k} - \dim M'_{\mathcal{T}_k}| \ll \eta^k |\mathcal{G} : \mathcal{T}_k|,$$

so that we may assume M is torsion free. This implies the existence of morphisms $\Lambda^r \rightarrow M$ and $M \rightarrow \Lambda^r$ with torsion cokernels, from which (26) follows from the associated long exact sequences on homology and $\dim \Lambda^r_{\mathcal{T}_k} = r |\mathcal{G} : \mathcal{T}_k|$.

Turning to $i > 0$, because M is finitely generated, there exists a short exact sequence

$$0 \rightarrow N \rightarrow \Lambda^n \rightarrow M \rightarrow 0$$

of finitely generated Λ modules for some $n \geq 0$. Because Λ^n is acyclic, the associated long exact sequence in homology gives

$$(28) \quad 0 \rightarrow H_1(\mathcal{T}_k, M) \rightarrow N_{\mathcal{T}_k} \rightarrow \Lambda^n_{\mathcal{T}_k} \rightarrow M_{\mathcal{T}_k} \rightarrow 0,$$

$$(29) \quad H_i(\mathcal{T}_k, M) \simeq H_{i-1}(\mathcal{T}_k, N), \quad i \geq 2.$$

The lemma for $i = 1$ now follows from (28) and (26), taking into account the fact that rank is additive in short exact sequences. For higher i , we use

induction. Assume the result for all $i \leq m$ and all finitely generated modules, in particular for N . We then obtain it for M in degree $m+1$ from the isomorphism (29), which completes the proof.

4. **The structure of $\mathbb{F}_p[G/H(p^k)]$**

The goal of this section is to prove the following structure result, which will allow us to decompose a subrepresentation $L \subset \mathbb{F}_p[G/H(p^k)]$ into pieces to which Shapiro’s lemma can be applied. More precisely, we shall filter L with quotients isomorphic to $\mathbb{F}_p[G/H(p^l)]$ for $l \leq k$, in a manner analogous to the base p expansion of an integer.

PROPOSITION 13. $\mathbb{F}_p[G/H(p^k)]$ has a filtration $0 = F(0) \subset \dots \subset F(p^{k-1}) = \mathbb{F}_p[G/H(p^k)]$ such that for all $1 \leq l \leq k$ and all $0 \leq a < p^{k-l}$,

$$F((a + 1)p^{l-1})/F(ap^{l-1}) \simeq \mathbb{F}_p[G/H(p^l)].$$

Moreover, $F(i)$ is the unique subrepresentation of $\mathbb{F}_p[G/H(p^k)]$ of dimension i .

The filtration of L which we may construct from this is described below.

COROLLARY 14. Let $L \subset \mathbb{F}_p[G/H(p^k)]$ be a submodule of dimension d , and let the base p expansion of d be written

$$d = \sum_{i=1}^l p^{\alpha(i)},$$

where $\alpha(i)$ is a nonincreasing sequence of nonnegative integers (that is, we write the larger powers of p first). Then there exists a filtration $0 = L_0 \subset \dots \subset L_l = L$ of L by submodules L_i such that $L_i/L_{i-1} \simeq \mathbb{F}_p[G/H(p^{\alpha(i)+1})]$.

Proof. Let the partial sums of the expansion of d be

$$s(i) = \sum_{j=1}^i p^{\alpha(j)},$$

and let $L_i = F(s(i))$. We have $s(i) = s(i - 1) + p^{\alpha(i)}$ and $p^{\alpha(i)} | s(i - 1)$, so by the proposition, $L_i/L_{i-1} \simeq \mathbb{F}_p[G/H(p^{\alpha(i)+1})]$ as required. \square

We begin the proof of Proposition 13 by defining $\phi : G \rightarrow p\mathbb{Z}_p$ by

$$\phi : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{c}{a}.$$

It can be seen that ϕ intertwines the right action of G on itself with the action on $p\mathbb{Z}_p$ by fractional linear transformations given by

$$g : z \mapsto \frac{dz + c}{bz + a}.$$

Moreover, if we let $\bar{\phi}$ denote the composition of ϕ with reduction modulo $p^k\mathbb{Z}_p$, then it may be seen that $\bar{\phi}$ factors through the right action of $H(p^k)$. Therefore, ϕ defines an isomorphism between the actions of G on $G/H(p^k)$ and $p\mathbb{Z}_p/p^k$, and hence the representations $\mathbb{F}_p[G/H(p^k)]$ and $\mathbb{F}_p[p\mathbb{Z}_p/p^k]$. There is an important collection of subspaces of $\mathbb{F}_p[p\mathbb{Z}_p/p^k]$, which we shall denote by $F(i)$ for $0 \leq i \leq p^{k-1}$, and which may be defined as the space of all functions on $p\mathbb{Z}_p$ obtained by taking the reductions modulo p of polynomials $p : \mathbb{Q}_p \rightarrow \mathbb{Q}_p$ of degree at most $i - 1$ that take integer values on that set (and where we set $F(0) = 0$). It is a theorem of Lucas [12] that such functions are in fact constant on cosets of $p^k\mathbb{Z}_p$, and moreover, that a basis for $F(i)$ is given by the binomial coefficients $x \mapsto \binom{x/p}{t}$, $0 \leq t \leq i - 1$, so that $\dim F(i) = i$. The lower unipotent matrix

$$\bar{N} = \begin{pmatrix} 1 & 0 \\ -p & 1 \end{pmatrix}$$

acts on $p\mathbb{Z}_p$ by $z \mapsto z - p$ and so acts on the function $\binom{x/p}{t}$ by

$$(30) \quad \begin{aligned} \bar{N} \binom{x/p}{t} &= \binom{x/p + 1}{t} \\ &= \binom{x/p}{t} + \binom{x/p}{t - 1}. \end{aligned}$$

Equation (30) implies that the action of $\bar{N} - I$ on $\mathbb{F}_p[p\mathbb{Z}_p/p^k]$ with respect to the basis

$$\{x \mapsto \binom{x/p}{t} \mid 0 \leq t \leq p^{k-1} - 1\}$$

is given by a matrix with ones along the upper off-diagonal and zeros elsewhere, and it follows from this that $F(i)$ is the unique i -dimensional subspace of $\mathbb{F}_p[p\mathbb{Z}_p/p^k]$ that is stable under \bar{N} .

We may prove more about the spaces $F(i)$ after defining an object that will play a key role later in the proof. For $0 \leq d \leq p^{k-1} - 1$, let Sym^d be the standard d th symmetric power representation of G , realized on the space of functions $f : G \rightarrow \mathbb{Q}_p$ of the form

$$f : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto p(a, c)$$

for some homogeneous polynomial p of degree d .

Definition 15. $\mathcal{V}_d \subset \text{Sym}^d$ is the lattice of integrally valued functions.

With this in mind, we may prove

LEMMA 16. *The subspaces $F(i)$ are stable under G .*

Proof. For $f \in \mathcal{V}_d$, we have

$$\begin{aligned}
 (31) \quad f \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right] &= p(a, c) \\
 &= a^d p\left(1, \frac{c}{a}\right) \\
 &\equiv p\left(1, \frac{c}{a}\right) \pmod{p},
 \end{aligned}$$

so that when we transfer functions in \mathcal{V}_d to $p\mathbb{Z}_p$ via ϕ and reduce modulo p , we obtain exactly the space $F(d + 1)$. \mathcal{V}_d is clearly preserved by G , and because ϕ was an intertwiner, we see that $F(d + 1)$ is also. \square

The proof of Lemma 16 also allows us to see the following important property of \mathcal{V}_d .

LEMMA 17. \mathcal{V}_d/p occurs as a submodule of $\mathbb{F}_p[G/H(p^k)]$.

Proof. If $f \in \mathcal{V}_d$, it follows from equation (31) that the reduction of f modulo p is invariant from the right under $H(p^k)$. \square

In light of Lemma 16, we see that $F(i)$ are exactly the subspaces described in the filtration of Proposition 13. The key idea behind the main claim of Proposition 13 is a certain recursive characterization of the subspaces $F(i)$. If we define $F(a, l)$ to be the subspace of $\mathbb{F}_p[p^l\mathbb{Z}_p/p^k]$ obtained by reducing polynomials of degree at most $a - 1$ that are integral valued on $p^l\mathbb{Z}_p$ modulo p , we then see that $F(ap^{l-1})$ is exactly the subspace of $\mathbb{F}_p[p\mathbb{Z}_p/p^k]$ consisting of elements whose restrictions to the cosets $\mathbb{F}_p[z + p^l\mathbb{Z}_p/p^k]$ all lie in $F(a, l)$. Indeed, this follows from the observation that both spaces have the same dimension and are stable under the lower unipotent subgroup. We therefore have

$$\begin{aligned}
 (32) \quad F((a + 1)p^{l-1})/F(ap^{l-1}) &\simeq (F(a + 1, l)/F(a, l)) \otimes \mathbb{F}_p[p\mathbb{Z}_p/p^l] \\
 &\simeq \mathbb{F}_p[p\mathbb{Z}_p/p^l]
 \end{aligned}$$

as vector spaces, and so the proposition would follow from knowing that the identification (32) commutes with the action of G on both sides. To show this, let $g \in G$ be given. By restriction, g gives a map from $z + p^l\mathbb{Z}_p/p^k$ to $gz + p^l\mathbb{Z}_p/p^k$, and when we choose an identification of both of these sets with $p\mathbb{Z}_p/p^{k-l+1}$ in the natural way, we see that this map is equal to a fractional linear transformation $x \mapsto g'x$ for some $g' \in G$. Therefore, using our claim that the fractional linear action of G preserves $F(a, l)$, we see that the map

$$\mathbb{F}_p[z + p^l\mathbb{Z}_p/p^k] \longrightarrow \mathbb{F}_p[gz + p^l\mathbb{Z}_p/p^k]$$

preserves $F(a, l)$ and $F(a + 1, l)$, and because $F(a + 1, l)/F(a, l)$ is one dimensional, it acts trivially on the quotient. It follows that the isomorphism (32) commutes with G , which concludes the proof.

5. Completed homology

In this section, we shall construct the p -adic local system $V_{\mathbf{d}}$ with the property that

$$(33) \quad \dim_{\mathbb{C}} H_i(Y, W_{\mathbf{d}}) = \dim_{\mathbb{Q}_p} H_i(Y, V_{\mathbf{d}}).$$

We shall then apply the theory of p -adically completed cohomology developed by Calegari and Emerton [4], [5], [8] to convert the problem of bounding the right-hand side to one of bounding $H_i(\mathcal{G}, M \otimes V_{\mathbf{d}})$, where M is a fixed $\Lambda_{\mathbb{Q}_p}$ modules and \mathbf{d} varies. To begin, let p be a prime that is totally split in F . If $\{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$ are the primes of F above p , and $F_{\mathfrak{p}_i}$ is the completion of F at \mathfrak{p}_i , Γ has an embedding

$$\phi : \Gamma \longrightarrow \mathrm{SL}_2(F_p) := \prod_i \mathrm{SL}_2(F_{\mathfrak{p}_i}),$$

the closure of whose image is a compact open subgroup of the target p -adic group. If we let \mathcal{G} be as in (6) with $t = n$, then $\mathrm{SL}_2(F_p)$ contains \mathcal{G} as a compact open subgroup, and by passing to a finite index sublattice we may assume that $\phi(\Gamma) \subset \mathcal{G}$. We shall in fact assume that $\overline{\phi(\Gamma)} = \mathcal{G}$, which is not necessary for the proof but allows us to avoid cluttering the previous sections with excessive notation. In any case, it may always be arranged after first choosing p at which Γ has full level.

Recalling our convention that \mathbf{d} was an n -tuple of nonnegative integers, we define the representation $V_{\mathbf{d}}$ of \mathcal{G} to be the tensor products of the representations Sym^{d_i} of G_i , and we denote both the restriction of $V_{\mathbf{d}}$ to Γ under ϕ and the associated local system on Y in the same way. Note that this definition relies on a choice of bijection between the complex embeddings $\{\sigma_i\}$ and p -adic embeddings $\{\mathfrak{p}_i\}$ of F . The following lemma shows that (33) holds if this choice is made in a natural way.

LEMMA 18. *Let \overline{F} be the Galois closure of F . There exist a complex and p -adic place σ and \mathfrak{p} of \overline{F} , and a bijection between the set of all complex and p -adic places $\{\sigma_i\}$ and $\{\mathfrak{p}_i\}$ of F , such that for all $\mathbf{d} \in (\mathbb{Z}_{\geq 0})^n$, there exists a representation ρ of Γ over \overline{F} such that*

- $\rho \otimes_{\sigma} \mathbb{C} \simeq W_{\mathbf{d}}$,
- $\rho \otimes_{\mathfrak{p}} \mathbb{Q}_p \simeq V_{\mathbf{d}}$.

Proof. Let $G = \mathrm{Gal}(\overline{F}/\mathbb{Q})$ and $H = \mathrm{Stab}_G(F)$. If $\sigma = \sigma_0$ is a chosen complex embedding of \overline{F} and $g_i \in G/H$ is a fixed system of coset representatives, the set of all complex embeddings of F is equal to the restrictions of $\sigma_i = \sigma \circ g_i$. Likewise, if we choose a p -adic place \mathfrak{p} of \overline{F} , the restrictions of $\mathfrak{p}_i = \mathfrak{p} \circ g_i$ form a complete set of p -adic embeddings of F .

Let Sym^d denote the d th symmetric power representation of $\text{SL}_2(\overline{F})$. If we define the representation ρ of $\text{SL}_2(\overline{F})$ by

$$\rho \simeq \bigotimes_i \text{Sym}^{d_i} \circ g_i,$$

then on restriction to $\text{SL}_2(F)$ we have

$$\begin{aligned} \rho \otimes_{\sigma} \mathbb{C} &\simeq \bigotimes_i (\text{Sym}^{d_i} \circ g_i) \otimes_{\sigma} \mathbb{C} \\ &\simeq \bigotimes_i \text{Sym}^{d_i} \otimes_{\sigma_i} \mathbb{C} \\ &\simeq W_{\mathbf{d}}, \end{aligned}$$

and likewise for $V_{\mathbf{d}}$. The lemma follows on restriction to Γ . □

We now introduce the p -adic tools that we shall use to study the right-hand side of (33). Let $\mathcal{P} \subset \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$ be the set of places at which we are allowing the weight to vary, so that \mathcal{P} is the complement of I under the bijection of Lemma 18. Let

$$\mathcal{G}_{\mathcal{P}} = \prod_{\mathfrak{p}_i \in \mathcal{P}} G_i,$$

and factorize $V_{\mathbf{d}}$ as $V_{\mathbf{d},\mathcal{P}} \otimes V_{\mathbf{d}}^{\mathcal{P}}$. We shall choose a \mathcal{G} -stable lattice $\mathcal{V}_{\mathbf{d}} \subset V_{\mathbf{d}}$ using Lemma 16, by letting \mathbf{k} be the smallest t -tuple of integers ≥ 1 satisfying $p^{k_i-1} > d_i$ and $4|k_i - 1$, and choosing $\mathcal{V}_{d_i} \subset V_{d_i}$ such that $\mathcal{V}_{d_i}/p \subset \mathbb{F}_p[G_i/H_i(p^{k_i})]$ for all i . We let $\mathcal{V}_{\mathbf{d}} = \otimes \mathcal{V}_{d_i}$, which we factorize as $\mathcal{V}_{\mathbf{d},\mathcal{P}} \otimes V_{\mathbf{d}}^{\mathcal{P}}$. Let

$$\mathcal{G}_{\mathcal{P},r} = \prod_{\mathfrak{p}_i \in \mathcal{P}} G_i(p^r)$$

be the principal congruence subgroups of $\mathcal{G}_{\mathcal{P}}$, $\Gamma_r = \Gamma \cap \mathcal{G}_{\mathcal{P},r}$, and let Y_r be the corresponding covers of Y . Following Emerton, we define

$$\widetilde{H}_i(\mathcal{V}_{\mathbf{d}}) = \varprojlim_s \varprojlim_r H_i(Y_r, \mathcal{V}_{\mathbf{d}}/p^s)$$

to be the i th completed homology module of the tower $\{Y_r\}$ with coefficients in $\mathcal{V}_{\mathbf{d}}$. We shall use the following fact about these modules, taken from [4], [8]:

- (1) $\widetilde{H}_i(\mathcal{V}_{\mathbf{d}})$ is a p -adically complete and separated \mathbb{Z}_p module.
- (2) $\widetilde{H}_i(\mathcal{V}_{\mathbf{d}})$ has the structure of a finitely generated $\mathbb{Z}_p[[\mathcal{G}_{\mathcal{P}}]]$ module that extends the natural action of $\mathcal{G}_{\mathcal{P}}$ by conjugation.
- (3) Because $\text{SL}(2, \mathbb{C})$ does not admit discrete series, $\widetilde{H}_i(\mathcal{V}_{\mathbf{d}})$ is a torsion $\mathbb{Z}_p[[\mathcal{G}_{\mathcal{P}}]]$ module for all i .
- (4) $\widetilde{H}_i(\mathcal{V}_{\mathbf{d}})$ carries a natural action of $\text{SL}(2, F_{\mathfrak{p}_i})$ for those $\mathfrak{p}_i \in \mathcal{P}$ satisfying $d_i = 0$, which extends the action of G_i .
- (5) There is a spectral sequence

$$(34) \quad E_2^{i,j} = H_i(\mathcal{G}_{\mathcal{P}}, \widetilde{H}_j(\mathcal{V}_{\mathbf{d}}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) \implies H_{i+j}(Y, V_{\mathbf{d}}).$$

The spectral sequence (34) implies an upper bound

$$(35) \quad \dim H_q(Y, V_{\mathbf{d}}) \leq \sum_{i+j=q} \dim H_i(\mathcal{G}_{\mathcal{P}}, \tilde{H}_j(\mathcal{V}_{\mathbf{d}}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$$

for the classical homology group we are interested in, and after some simplifications this will reduce the problem of a power saving to a statement about torsion $\mathbb{Z}_p[[\mathcal{G}_{\mathcal{P}}]]$ modules with a compatible SL_2 action, which will follow from the results of Sections 3 and 4. Because we have defined the representation $\mathcal{V}_{\mathbf{d}}$ by pulling back, a representation of \mathcal{G} , $\mathcal{V}_{\mathbf{d},\mathcal{P}}/p^s$ is eventually trivial as a representation of Γ_r . We therefore have

$$\begin{aligned} \tilde{H}_j(\mathcal{V}_{\mathbf{d}}) &\simeq \varprojlim_s \varprojlim_r H_i(Y_r, \mathcal{V}_{\mathbf{d}}^{\mathcal{P}}/p^s) \otimes \mathcal{V}_{\mathbf{d},\mathcal{P}}/p^s \\ &\simeq \tilde{H}_j(\mathcal{V}_{\mathbf{d}}^{\mathcal{P}}) \otimes \mathcal{V}_{\mathbf{d},\mathcal{P}} \end{aligned}$$

as representations of $\mathcal{G}_{\mathcal{P}}$, and because we are fixing d_i for those $\mathfrak{p}_i \notin \mathcal{P}$, we shall simply write \tilde{H}_j for $\tilde{H}_j(\mathcal{V}_{\mathbf{d}}^{\mathcal{P}})$ and $\tilde{H}_{j,\mathbb{Q}_p}$ for $\tilde{H}_j(\mathcal{V}_{\mathbf{d}}^{\mathcal{P}}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Furthermore, as we do not need to give any further consideration to the primes not in \mathcal{P} , we shall ignore them from this point on and write \mathcal{G} for $\mathcal{G}_{\mathcal{P}}$, $V_{\mathbf{d}}$ for $\mathcal{V}_{\mathbf{d},\mathcal{P}}$, and assume that $\mathcal{P} = \{\mathfrak{p}_1, \dots, \mathfrak{p}_t\}$. The upper bound (35) may then be rewritten

$$(36) \quad \dim H_q(Y, V_{\mathbf{d}}) \leq \sum_{i+j=q} \dim H_i(\mathcal{G}, \tilde{H}_{j,\mathbb{Q}_p} \otimes V_{\mathbf{d}}).$$

6. Reduction modulo p

We now combine the results of the previous sections to prove an upper bound for the right-hand side of (36) by choosing a lattice inside $\tilde{H}_{j,\mathbb{Q}_p} \otimes V_{\mathbf{d}}$ that we then reduce modulo p . The lattice we take will be the tensor product of the image of \tilde{H}_j in $\tilde{H}_{j,\mathbb{Q}_p}$ and a lattice $\mathcal{V}_{\mathbf{d}} \subset V_{\mathbf{d}}$ that is the tensor product of the lattices \mathcal{V}_{d_i} defined in Definition 15. By Lemma 17, we know that $\mathcal{V}_{\mathbf{d}}/p$ is a submodule of $\mathbb{F}_p[\mathcal{G}/\mathcal{H}_{\mathbf{k}}]$ that we denote by L .

The image of \tilde{H}_j in $\tilde{H}_{j,\mathbb{Q}_p}$ is isomorphic to the p -torsion free quotient $\tilde{H}_{j,\mathrm{tf}}$ of \tilde{H}_j , and we denote the reduced lattice $\tilde{H}_{j,\mathrm{tf}}/p$ by M_j . By property (2), $\tilde{H}_{j,\mathbb{Q}_p}$ is a torsion $\Lambda_{\mathbb{Q}_p}$ module, which implies that M_j is a torsion Λ module. Moreover, $\tilde{H}_{j,\mathrm{tf}}$ is stable under $\mathrm{SL}(2, F_{\mathfrak{p}_i})$ for any \mathfrak{p}_i , so that M_j also carries an action of these groups. We may now reduce our chosen lattice modulo p to obtain

$$\dim_{\mathbb{Q}_p} H_i(\mathcal{G}, \tilde{H}_{j,\mathbb{Q}_p} \otimes V_{\mathbf{d}}) \leq \dim_{\mathbb{F}_p} H_i(\mathcal{G}, M_j \otimes L),$$

and the required bound on the right-hand side is obtained by combining the results of Sections 3 and 4 in the following lemma.

LEMMA 19. *Let M be a torsion Λ module with a compatible action of SL_2 , and let L be any subrepresentation of $\mathbb{F}_p[\mathcal{G}/\mathcal{H}_{\mathbf{k}}]$ that factorizes as $\otimes L_i$*

with $L_i \subset \mathbb{F}_p[G_i/H_i(p^k)]$. We then have

$$(37) \quad \dim H_i(\mathcal{G}, M \otimes L) \ll \alpha^\kappa |\mathcal{G} : \mathcal{H}_k|$$

for all i , where $\alpha = \eta^{1/2}$ and the implied constant depends only on M .

We shall prove Lemma 19 with the aid of the following lemma, which will allow us to make approximations to the subgroups of \mathcal{G} appearing there.

LEMMA 20. *Let G be a pro- p group, and let $H \leq G$ be a subgroup of index p . If M is a representation of G over \mathbb{F}_p , we have*

$$\dim H_i(H, M) \leq p \dim H_i(G, M)$$

for any $i \geq 0$.

Proof. By Shapiro’s lemma, $H_i(H, M) \simeq H_i(G, M \otimes \mathbb{F}_p[G/H])$. As all the composition factors of the G -module $\mathbb{F}_p[G/H]$ are isomorphic to \mathbb{F}_p , the lemma follows. \square

Proof of Lemma 19. When $L = \mathbb{F}_p[\mathcal{G}/\mathcal{H}_k]$, we apply Shapiro’s lemma and use a diagonal element of SL_2 to conjugate \mathcal{H}_k to a group $g\mathcal{H}_kg^{-1}$ with the following properties:

$$g\mathcal{H}_kg^{-1} \leq \mathcal{T}_{k'}, \quad |\mathcal{T}_{k'} : g\mathcal{H}_kg^{-1}| \leq p^{3t}, \quad |k'_i - k_i/2| \leq 2.$$

The inequality (37) then follows from Lemma 20 and Proposition 5.

For general L , we apply Proposition 13 to the factors L_i to obtain a filtration $L = F_0 \supset F_1 \supset \dots$ such that every quotient F_i/F_{i+1} is isomorphic to $\mathbb{F}_p[\mathcal{G}/\mathcal{H}_1]$ for some $1 \leq k$ and each isomorphism class of quotient occurs at most p^t times. By applying (37) in the case $L = \mathbb{F}_p[\mathcal{G}/\mathcal{H}_1]$ to the quotient modules, we obtain

$$\begin{aligned} \dim H_i(\mathcal{G}, M \otimes L) &\leq p^t \sum_{1 \leq k} \dim H_i(\mathcal{H}_1, M) \\ &\ll \sum_{1 \leq k} \alpha^{\min(l_i)} |\mathcal{G} : \mathcal{H}_1| \\ &\ll \sum_{i=1}^t \sum_{1 \leq k} \alpha^{l_i} |\mathcal{G} : \mathcal{H}_1| \\ &\ll \sum_{i=1}^t \alpha^{k_i} |\mathcal{G} : \mathcal{H}_k| \\ &\ll \alpha^\kappa |\mathcal{G} : \mathcal{H}_k|. \end{aligned} \quad \square$$

It remains to follow the bound we have proven back to one for the original cohomology group $H_i(Y, W_{\mathbf{d}})$. Lemma 19 gives

$$\dim H_i(Y, V_{\mathbf{d}}) \ll \alpha^\kappa |\mathcal{G} : \mathcal{H}_k|,$$

where \mathbf{k} is a $|\mathcal{P}|$ -tuple of integers satisfying $|k_i - \log_p d_i| \leq 1$ for $i \in \mathcal{P}$. We therefore have

$$\begin{aligned} \dim H_i(Y, V_{\mathbf{d}}) &\ll \alpha^\kappa \prod p^{k_i} \\ &\ll (\min_{i \in \mathcal{P}} d_i)^{-1/3 + (\ln 10)/(2 \ln p)} \prod_{i \in \mathcal{P}} d_i. \end{aligned}$$

If we express this in terms of the original notation for $W_{\mathbf{d}}$ with $\mathbf{d} \in (\mathbb{Z}_{\geq 0})^{r_1+r_2}$, and choose p to be sufficiently large, we see that this implies Proposition 3 and hence Theorem 1.

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