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Nonlinear wave equations as limits of convex minimization problems: proof of a conjecture by De Giorgi

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Abstract

We prove a conjecture by De Giorgi, which states that global weak solutions of nonlinear wave equations such as $\Box w + |w|^{p-2}w = 0$ can be obtained as limits of functions that minimize suitable functionals of the calculus of variations. These functionals, which are integrals in space-time of a convex Lagrangian, contain an exponential weight with a parameter ε, and the initial data of the wave equation serve as boundary conditions. As ε tends to zero, the minimizers v_ε converge, up to subsequences, to a solution of the nonlinear wave equation. There is no restriction on the nonlinearity exponent, and the method is easily extended to more general equations.

1. Introduction

In [1] Ennio De Giorgi formulated the following conjecture.

Conjecture. *Let $k > 1$ be an integer number. For $\varepsilon > 0$, let $v_\varepsilon(t,x)$ denote the minimizer of the convex functional*

$$F_\varepsilon(v) = \int_0^\infty \int_{\mathbb{R}^n} e^{-t/\varepsilon} \left\{ |v''(t,x)|^2 + \frac{1}{\varepsilon^2}|\nabla v(t,x)|^2 + \frac{1}{\varepsilon^2}|v(t,x)|^{2k} \right\} dxdt \tag{1}$$

subject to the boundary conditions

$$v(0,x) = \alpha(x), \quad v'(0,x) = \beta(x), \quad x \in \mathbb{R}^n, \tag{2}$$

where $\alpha, \beta \in C_0^\infty(\mathbb{R}^n)$ are given functions. Then, for almost every $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n$, the limit

$$w(t,x) = \lim_{\varepsilon \downarrow 0} v_\varepsilon(t,x)$$

exists and the function $w(t,x)$ solves in $\mathbb{R}^+ \times \mathbb{R}^n$ the nonlinear wave equation

$$w'' = \Delta w - k w^{2k-1} \tag{3}$$

with initial conditions

$$w(0,x)=\alpha(x),\quad w'(0,x)=\beta(x),\quad x\in\mathbb{R}^n. \tag{4}$$

This conjecture appeared in the *Duke Mathematical Journal* in Italian ([1]). An English version can be found in De Giorgi's selected papers [2].

The relevance of the conjecture lies in the possibility of casting a completely new bridge between hard *evolution* problems such as (3) and more easily tractable *convex minimization* problems. Louis Nirenberg comments in [5] that the "conjecture $\{\dots\}$ suggests a very interesting approach for solving the initial value problem for the wave equation with a nonlinear term involving a power of the function via a minimization problem $\{\dots\}$." Indeed, the variational approaches to the wave equation $w''=\Delta w$ and its nonlinear variants that can be found in the literature (see for example [6], [8], [9], [10] and references therein) are essentially based on the interpretation of $w''=\Delta w$ as the Euler equation of the functional

$$\iint |w'|^2\,dxdt-\iint |\nabla w|^2\,dxdt, \tag{5}$$

with possible lower order terms like $|w|^p$ to include the desired nonlinearity. These functionals are obviously neither convex nor bounded from below — even finiteness could be an issue for supercritical p — and one has to consider *critical points* (using the apparatus of Critical Point Theory) rather than *absolute minimizers*, which usually do not even exist. Unfortunately, functionals containing terms as (5) behave rather badly for the application of Critical Point Theory, and only partial results are obtainable in this way. On the contrary, the functionals appearing in (1) are *nonnegative* and *strictly convex*. As a consequence, existence and uniqueness of an absolute minimizer (in the natural function space) are easy to establish, regardless of the magnitude of the exponent $2k$ in (1).

A further point of interest (already pointed out by De Giorgi in [1]) is that since convexity and appropriate coercivity are just about the only two features characterizing the functional (1), and these are rather stable under perturbations, similar results are likely to hold for broader classes of problems. One could try, for instance, to replace the square of the gradient with some other power, or consider more general nonlinearities, or even, as suggested in [1], to substitute the "spatial" terms in (1) with generic functionals of the Calculus of Variations. One could then obtain similar results for quasilinear evolution equations involving the p-laplacian, or (nonlinear) Klein–Gordon equations, and so on. Some of these generalizations, those that require refinements of the present work, will be the object of further papers.

Here we confine ourselves to the following result, which essentially gives an affirmative answer to De Giorgi's conjecture.

THEOREM 1.1. *For $p \geq 2$ and $\varepsilon > 0$, let $v_\varepsilon(t,x)$ denote the unique minimizer of the strictly convex functional*

$$F_\varepsilon(v) = \int_0^\infty \int_{\mathbb{R}^n} e^{-t/\varepsilon} \left\{ |v''(t,x)|^2 + \frac{1}{\varepsilon^2} |\nabla v(t,x)|^2 + \frac{1}{\varepsilon^2} |v(t,x)|^p \right\} dxdt \tag{6}$$

under the boundary conditions (2), *where α and β are given functions such that*

$$\alpha, \beta \in H^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n). \tag{7}$$

Then

(a) Estimates. *There exists a constant C (which depends only on α, β, p and n) such that, for every $\varepsilon \in (0,1)$,*

$$\int_0^T \int_{\mathbb{R}^n} \left(|\nabla v_\varepsilon|^2 + |v_\varepsilon|^p \right) dxdt \leq CT \quad \forall T \geq \varepsilon, \tag{8}$$

$$\int_{\mathbb{R}^n} |v_\varepsilon'(t,x)|^2 \, dx \leq C, \quad \int_{\mathbb{R}^n} |v_\varepsilon(t,x)|^2 \, dx \leq C(1+t^2) \quad \forall t \geq 0 \tag{9}$$

and, for every function $h \in H^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$,

$$\left| \int_{\mathbb{R}^n} v_\varepsilon''(t,x) h(x) \, dx \right| \leq C \left(\|h\|_{L^p(\mathbb{R}^n)} + \|\nabla h\|_{L^2(\mathbb{R}^n)} \right) \quad \text{for a.e. } t > 0. \tag{10}$$

(b) Convergence. *Every sequence v_{ε_i} (with $\varepsilon_i \downarrow 0$) admits a subsequence that is convergent, in the strong topology of $L^q((0,T) \times A)$ for every $T > 0$ and every bounded open set $A \subset \mathbb{R}^n$ (with arbitrary $q \in [2,p)$ if $p > 2$, and with $q = 2$ if $p = 2$), almost everywhere on $\mathbb{R}^+ \times \mathbb{R}^n$, and in the weak topology of $H^1((0,T) \times \mathbb{R}^n)$ for every $T > 0$, to a function w such that*

$$w \in L^\infty(\mathbb{R}^+; L^p(\mathbb{R}^n)), \quad \nabla w \in L^\infty(\mathbb{R}^+; L^2(\mathbb{R}^n)), \tag{11}$$

$$w' \in L^\infty(\mathbb{R}^+; L^2(\mathbb{R}^n)), \quad w \in L^\infty((0,T); H^1(\mathbb{R}^n)) \quad \forall T > 0, \tag{12}$$

which solves in $\mathbb{R}^+ \times \mathbb{R}^n$ the nonlinear wave equation

$$w'' = \Delta w - \frac{p}{2} |w|^{p-2} w \tag{13}$$

with initial conditions as in (4).

(c) Energy inequality. *Letting*

$$\mathcal{E}(t) = \int_{\mathbb{R}^n} \left(|w'(t,x)|^2 + |\nabla w(t,x)|^2 + |w(t,x)|^p \right) dx, \tag{14}$$

the function $w(t,x)$ satisfies the energy inequality

$$\mathcal{E}(t) \leq \mathcal{E}(0) = \int_{\mathbb{R}^n} \left(|\beta(x)|^2 + |\nabla \alpha(x)|^2 + |\alpha(x)|^p \right) dx \quad \text{for a.e. } t > 0. \tag{15}$$

Some comments are in order.

De Giorgi's conjecture concerns the nonlinearity v^{2k}, with k integer. In Theorem 1.1 we work with $|v|^p$, and we drop the assumption that p be integer. Our assumptions on the initial data α, β are much weaker than in De Giorgi's conjecture (see the end of Remark 2.1 for more details). However, we obtain the main statement (convergence of the minimizers v_ε to a w that solves the wave equation) only up to extracting subsequences. This restriction seems hard to eliminate, because there is no bound on the magnitude of p: when the exponent is supercritical for the wave equation (13), the uniqueness of w solving (13), (4) is not guaranteed and there are several open question concerning this issue (see [6], [8], [9], [10]). On the other hand, we do not exclude that the conjecture is true as it is stated (i.e., convergence of v_ε to w without passing to a subsequence), yet we believe that a proof of this fact would require much additional work. (In particular, one should exploit the regularity of the initial data α, β as stated in the original conjecture, to get further *a priori* estimates on v_ε and obtain stronger compactness.) However, in all cases where uniqueness for the wave equation is known (e.g., $n = 2$ and arbitrary p, or $n > 2$ and $p \leq (2n-2)/(n-2)$), one can avoid passing to a subsequence.

Theorem 1.1 leads, as a corollary, to a *global* existence result for the wave equation (13) for arbitrarily large p and under very reasonable assumptions on the initial data. Note that the solution w obtained in this way also satisfies the energy inequality (15). (These kinds of solutions are called "of energy class;" see the paper of Struwe [9] and the references therein.) This global existence result, as such, is not new (see, e.g., [8]). Nevertheless, we would like to stress the novelty of this variational approach to existence results of this kind, which may provide new insight on Cauchy problems for the wave equation. In particular, if one could prove (as we believe) that the v_ε converge to w *without passing to subsequences*, then one could always single out a unique solution (the "variational solution") for Cauchy problems as (13), even when uniqueness of a weak solution is not guaranteed.

Our result is stated with $\mathbb{R}^n$ as the spatial environment, as in the original conjecture. However, our proof still works, more generally, if one replaces $\mathbb{R}^n$ by a generic open set Ω with appropriate boundary conditions. Spatial periodicity can also be treated. These variants would lead to some additional technicalities and, since De Giorgi's conjecture is stated in $\mathbb{R}^n$, we concentrate here only on the case $\Omega = \mathbb{R}^n$.

The presence of the exponential weight $\exp(-t/\varepsilon)$ in (1) makes it hard to obtain estimates on v_ε that are uniform in ε. However, an intuitive justification of the conjecture is easy to obtain. Indeed, note that the Euler equation in classical form reads

$$\left(\exp(-t/\varepsilon)v_\varepsilon''\right)'' = \varepsilon^{-2}\exp(-t/\varepsilon)\Delta v_\varepsilon - \varepsilon^{-2}\frac{p}{2}\exp(-t/\varepsilon)|v_\varepsilon|^{p-2}v_\varepsilon$$

and after an elementary computation,

$$\varepsilon^2 v_\varepsilon'''' - 2\varepsilon v_\varepsilon''' + v_\varepsilon'' = \Delta v_\varepsilon - \frac{p}{2}|v_\varepsilon|^{p-2}v_\varepsilon.$$

Formally, on dropping the first two terms, one gets the desidered wave equation.

Our plan to prove Theorem 1.1 is the following. First, we perform a time scaling letting $u_\varepsilon(t,x) := v_\varepsilon(\varepsilon t, x)$; this leads to an equivalent minimization problem, with a weight $\exp(-t)$ instead of $\exp(-t/\varepsilon)$. Next we obtain suitable estimates on u_ε of integral kind but, as we will see, they can be localized in time, up to scale $O(1)$. Getting back to the original v_ε, the integral estimates can be localized in time up to scale $O(\varepsilon)$: they provide weak convergence to w (up to subsequences) essentially in $H^1_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^n)$. However, the limit function w inherits the possibility of localizing the integral estimates in time, up to scale $O(\delta)$ for every δ, which eventually leads to the L^∞ estimates in (11), (12) via an argument involving Lebesgue points. Finally, the energy inequality (15) requires further refinements of the L^∞ estimates and additional technical arguments.

To our knowledge the present work is the first to give a proof of the conjecture (up to subsequences). The only paper devoted to this topic we know of is [7], which deals with a simplified version on bounded intervals. The result only establishes the convergence (up to subsequences) to a solution w of the wave equation, not the fact that the initial condition $w'(0,x) = \beta(x)$ is satisfied.

Remark on notation. Throughout the paper, the symbol ∇ (resp. Δ) denotes the gradient (resp. the Laplacian) in the spatial variable x, whereas a prime as in v', v'' etc. denotes partial differentiation with respect to the time variable t.

Moreover, in order to simplify the notation, through the rest of the paper symbols such as $\int v\,dx$ will always denote spatial integrals extended to the whole of $\mathbb{R}^n$, and short forms such as L^2, H^1 etc. will be used to denote $L^2(\mathbb{R}^n)$, $H^1(\mathbb{R}^n)$ etc., the underlying space $\mathbb{R}^n$ being understood.

2. **Time scaling and preliminary estimates**

In order to study the functional F_ε defined in (6), it is convenient to perform a time scaling and introduce, for $\varepsilon > 0$, the new functional

$$J_\varepsilon(u) = \int_0^\infty \int e^{-t}\left\{|u''(t,x)|^2 + \varepsilon^2|\nabla u(t,x)|^2 + \varepsilon^2|u(t,x)|^p\right\} dxdt. \tag{16}$$

The functionals F_ε and J_ε are equivalent in the sense that, whenever u and v are related by the change of variable $u(t,x) = v(\varepsilon t, x)$, we have $F_\varepsilon(v) = \varepsilon^{-3} J_\varepsilon(u)$. In particular, given α, β as in (7), the minimization of $F_\varepsilon(v)$ under the initial conditions (2) is equivalent to the minimization of $J_\varepsilon(u)$ under the initial conditions

$$u(0,x) = \alpha(x), \quad u'(0,x) = \varepsilon\beta(x). \tag{17}$$

To provide a precise functional setting for J_ε (and hence, indirectly, also for F_ε), we define the function space $\mathcal{U}$ as the set of (equivalence classes of) functions $u \in L^1_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}^n)$ such that

$$\nabla u \in L^1_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}^n), \quad u'' \in L^1_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}^n)$$

and such that the quantity

$$\|u\|_{\mathcal{U}} := \Big(\int_0^\infty \!\!\int e^{-t}\big\{|u''(t,x)|^2 + |\nabla u(t,x)|^2\big\}\, dxdt\Big)^{1/2} + \Big(\int_0^\infty \!\!\int e^{-t}|u(t,x)|^p\, dxdt\Big)^{1/p}$$

is finite. Note that $\mathcal{U}$, endowed with the norm $\|\cdot\|_{\mathcal{U}}$, is a Banach space that provides the natural setting for the functional J_ε.

Remark 2.1. Fix a bounded open set $A \subset \mathbb{R}^n$ and a number $T > 0$. As $p \geq 2$, it is immediate to check that $\mathcal{U} \hookrightarrow L^2((0,T)\times A)$ via the usual restriction operator which, combined with the fact that $\nabla u \in L^2((0,T)\times \mathbb{R}^n)$ for all $u \in \mathcal{U}$, gives

$$\mathcal{U} \hookrightarrow L^2((0,T); H^1(A)).$$

Moreover, since every $u \in \mathcal{U}$ satisfies $u'' \in L^2((0,T)\times A)$ and $u \in L^2((0,T)\times A)$, one infers by interpolation that also $u' \in L^2((0,T)\times A)$ and hence

$$\mathcal{U} \hookrightarrow H^2((0,T); L^2(A)).$$

In particular, for each $u \in \mathcal{U}$, the traces $u(0,x)$ and $u'(0,x)$ are well defined as elements of $L^2(A)$ and, from the arbitrariness of $A \subset \mathbb{R}^n$, also as elements of L^2_{loc}. In fact, somewhat sharper embeddings occur, namely $u(0,\cdot) \in H^{3/4}$ and $u'(0,\cdot) \in H^{1/4}$ (see [4, Th. 3.1]); this shows that the assumption (7) on β cannot be weakened to the more familiar request that $\beta \in L^2$.

In light of the previous remark, given α, β as in (7), one can check that the set

$$\mathcal{U}^\varepsilon_{\alpha,\beta} = \{u \in \mathcal{U} \mid u(0,x) = \alpha(x), \quad u'(0,x) = \varepsilon\beta(x)\}$$

is a closed and convex subset of $\mathcal{U}$. As $\mathcal{U}^\varepsilon_{\alpha,\beta} \neq \emptyset$ (consider for instance $u(t,x) = \alpha(x) + \varepsilon t\beta(x)$), it is easy to see that J_ε admits a unique minimizer in $\mathcal{U}^\varepsilon_{\alpha,\beta}$.

In this and in the next two sections, ε will not vary and, in order to avoid cumbersome notation, we will let

$$u(t,x) := u_\varepsilon(t,x), \quad \text{the minimizer of } J_\varepsilon \text{ in } \mathcal{U}^\varepsilon_{\alpha,\beta}. \tag{18}$$

Similarly, for several other quantities or functions that are defined in terms of u_ε, the dependence on ε will not be made explicit in the notation.

In particular, we define for almost every $t > 0$ the locally integrable function

$$L(t) = L_\varepsilon(t) := \int \Big(|u''(t,x)|^2 + \varepsilon^2 |\nabla u(t,x)|^2 + \varepsilon^2 |u(t,x)|^p \Big)\, dx, \tag{19}$$

which is a sort of "density" for $J_\varepsilon(u)$ at time t. Similarly, we define

$$H(t) = H_\varepsilon(t) := \int_t^\infty e^{-s} L(s)\, ds, \quad t \geq 0. \tag{20}$$

Note that H is continuous, nonnegative and nonincreasing, and $H(0) = J_\varepsilon(u)$. In particular, we see that $e^{-t}L(t) \in L^1(\mathbb{R}^+)$ and $H \in W^{1,1}((0,T))$ for every $T > 0$, with

$$H'(t) = -e^{-t} L(t) \quad \text{for a.e. } t > 0. \tag{21}$$

Finally, it is immediate to check that

$$\lim_{t \to \infty} H(t) = 0. \tag{22}$$

We also define the auxiliary functions

$$D(t) = D_\varepsilon(t) := \int |u''(t,x)|^2\, dx \quad \text{for a.e. } t > 0 \tag{23}$$

and

$$I(t) = I_\varepsilon(t) := \frac{1}{2} \int |u'(t,x)|^2\, dx \quad \forall t \geq 0, \tag{24}$$

which makes sense for *every* t due to Remark (2.1). Note that due to Lemma 2.4 below, $I \in W^{1,1}(0,T)$ for all $T > 0$ and

$$I'(t) = \int u'(t,x) u''(t,x)\, dx \quad \text{for a.e. } t > 0. \tag{25}$$

As we have mentioned, throughout this section ε is fixed. Later we shall be interested in letting $\varepsilon \downarrow 0$, so that from now on we always assume, without loss of generality, that $0 < \varepsilon < 1$, which will allow us to consider some constants as independent of ε.

LEMMA 2.2 (Level estimate). *There exists a constant $C = C(\alpha, \beta)$ such that the minimizer u defined in* (18) *satisfies*

$$J_\varepsilon(u) \leq \varepsilon^2 \int \Big(|\nabla \alpha(x)|^2 + |\alpha(x)|^p \Big)\, dx + C\varepsilon^3 \tag{26}$$

and, in particular,

$$J_\varepsilon(u) \leq C\varepsilon^2. \tag{27}$$

Throughout, $C(\alpha, \beta)$ will denote constants that depend only on α and β. (The dependence on the exponent p and the spatial dimension n is understood.)

Proof. As already observed, the function $w(t,x) = \alpha(x) + t\varepsilon\beta(x)$ belongs to $\mathcal{U}^{\varepsilon}_{\alpha,\beta}$, so that it is an admissible competitor for the minimizer u. As $w'' \equiv 0$, using the minimality of u, one finds by an elementary computation

$$\begin{aligned} J_\varepsilon(u) \le J_\varepsilon(w) &= \varepsilon^2 \int_0^\infty \!\! \int e^{-t} \big\{ |\nabla w(t,x)|^2 + |w(t,x)|^p \big\} \, dxdt \\ &\le \varepsilon^2 \int \big(|\nabla \alpha(x)|^2 + |\alpha(x)|^p \big) \, dx + C\varepsilon^3, \end{aligned}$$

and (26) is proved. □

The following simple lemma will be used to establish some summability properties of u and its time derivatives.

Lemma 2.3. *Let $w : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}$ be a function such that, for every open set $A \subset \mathbb{R}^n$ of finite measure and for every $T > 0$, $w \in L^2((0,T) \times A)$ and $w' \in L^2((0,T) \times A)$. Then*

$$\int_0^\infty \!\! \int e^{-t} |w(t,x)|^2 \, dxdt \le 2 \int |w(0,x)|^2 \, dx + 4 \int_0^\infty \!\! \int e^{-t} |w'(t,x)|^2 \, dxdt. \tag{28}$$

Note that we do not claim that any integral appearing in (28) is finite.

Proof. Given $T > 0$, for almost every $x \in \mathbb{R}^n$, the function $h(t) = w(t,x)$ belongs to $W^{1,2}(0,T)$. Integrating by parts, we have

$$\begin{aligned} \int_0^T e^{-t} |h(t)|^2 \, dt &= -e^{-t} |h(t)|^2 \Big|_0^T + 2 \int_0^T e^{-t} h(t) h'(t) \, dt \\ &\le |h(0)|^2 + 2 \left(\int_0^T e^{-t} |h(t)|^2 \, dt \right)^{1/2} \left(\int_0^T e^{-t} |h'(t)|^2 \, dt \right)^{1/2}, \end{aligned}$$

and using $2\sqrt{ab} \le a/2 + 2b$ to split the last product, we find that

$$\int_0^T e^{-t} |h(t)|^2 \, dt \le 2|h(0)|^2 + 4 \int_0^T e^{-t} |h'(t)|^2 \, dt. \tag{29}$$

Now, recalling that $h(t) = w(t,x)$, by integrating this inequality over $\mathbb{R}^n$ and letting $T \to \infty$ one proves the claim. □

Lemma 2.4. *There exists a constant $C = C(\alpha, \beta)$ such that the minimizer u defined in* (18) *satisfies*

$$\int_0^\infty \!\! \int e^{-t} |u''|^2 \, dxdt \le C\varepsilon^2, \tag{30}$$

$$\int_0^\infty \!\! \int e^{-t} |u'|^2 \, dxdt \le C\varepsilon^2, \tag{31}$$

and

$$\int_0^\infty \!\! \int e^{-t} |u|^2 \, dxdt \le C. \tag{32}$$

Proof. Note that (30) follows immediately from (27). Then, in light of Remark (2.1), we can apply Lemma 2.3 with $w(t,x) = u'(t,x)$, obtaining

$$\int_0^\infty \int e^{-t} |u'|^2 \, dxdt \le 2\varepsilon^2 \int |\beta(x)|^2 \, dx + C\varepsilon^2$$

(recall that $u'(0,x) = \varepsilon\beta(x)$), and (31) is established. Finally, (32) follows from Lemma 2.3 with $w = u$, combined with (31). $\square$

We point out for further reference that (30) and (31) can be rewritten in terms of the above defined functions $D(t)$ and $I(t)$, as

$$\int_0^\infty e^{-t} D(t) \, dt \le C\varepsilon^2, \tag{33}$$

$$\int_0^\infty e^{-t} I(t) \, dt \le C\varepsilon^2. \tag{34}$$

LEMMA 2.5. *The function $e^{-t} I(t)$ belongs to $W^{1,1}(\mathbb{R}^+)$. In particular,*

$$\lim_{T \to \infty} e^{-T} I(T) = 0. \tag{35}$$

Proof. We already know from (34) that $e^{-t} I(t)$ is in $L^1(\mathbb{R}^+)$. By (25) and the preceding discussion, $I \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^+)$ and for almost every t,

$$(e^{-t} I(t))' = -e^{-t} I(t) + e^{-t} I'(t),$$

so that we only have to check that the last term is in $L^1(\mathbb{R}^+)$. But by the Cauchy–Schwarz inequality, we have

$$\begin{aligned}
\int_0^\infty e^{-t} \left| I'(t) \right| \, dt \le & \int_0^\infty \int e^{-t} \left| u'(t,x) \right| \cdot \left| u''(t,x) \right| \, dxdt \\
\le & \left(\int_0^\infty \int e^{-t} |u'(t,x)|^2 \, dxdt \right)^{1/2} \left(\int_0^\infty \int e^{-t} |u''(t,x)|^2 \, dxdt \right)^{1/2} \\
\le & C\varepsilon^2
\end{aligned}$$

by (30) and (31). $\square$

3. The function $E(t)$ and its derivative

Recalling (20), (24) and (25), we define, for almost every $t \ge 0$, the function

$$\begin{aligned}
E(t) = E_\varepsilon(t) :=& \, I(t) - I'(t) + \frac{e^t}{2} H(t) \\
=& \, \frac{1}{2} \int |u'(t,x)|^2 \, dx - \int u'(t,x) u''(t,x) \, dx + \frac{e^t}{2} \int_t^\infty e^{-s} L(s) \, ds,
\end{aligned} \tag{36}$$

which will play a crucial role in proving estimates. (As before, here ε is fixed and the dependence on ε is omitted.)

Proposition 3.1. *The function $E(t)$ satisfies*

$$E' = -2D \quad \text{in } \mathcal{D}'(\mathbb{R}^+). \tag{37}$$

In particular, the function $E(t)$ is nonincreasing, belongs to $W^{1,1}((0,T))$ for every $T > 0$, and therefore it admits a continuous representative.

Remark 3.2. We point out that relation (37) can be obtained formally, though not in a straightforward way, by testing the weak form of the Euler equation, $J'_\varepsilon(u)\phi = 0$, with $\phi(t,x) = \eta(t)u'(t,x)$, the function η being in $C_0^\infty(\mathbb{R}^+)$. However this would produce terms such as $\eta u^{p-1}u'$ that may fail to be integrable when p is large. In the following proof this difficulty is overcome performing first variations using competitors of the kind $u(\varphi(t),x)$.

Proof. The last part of the statement follows from (33), which entails that $D \in L^1((0,T))$ for every $T > 0$. Therefore we only prove (37). Given an arbitrary $\eta \in C_0^\infty(\mathbb{R}^+)$, consider the primitive function

$$g(t) = \int_0^t \eta(s)\, ds. \tag{38}$$

It is clear that $g \in C^\infty(\mathbb{R}^+)$ and satisfies

$$g(t) \equiv 0 \quad \text{for } t \text{ close to zero.} \tag{39}$$

For every $\delta \in \mathbb{R}$ with $|\delta|$ small enough, the function

$$\varphi(t) = \varphi(t,\delta) = t - \delta g(t) \tag{40}$$

is a diffeomorphism of $\mathbb{R}^+$ of class C^∞. Now, for small δ, we construct the competitor

$$U(t,x) = u(\varphi(t),x)$$

(the dependence on δ, which is fixed, is omitted to simplify the notation), which satisfies the initial conditions

$$U(0,x) = \alpha(x), \quad U'(0,x) = \varepsilon\beta(x)$$

because $\varphi(t) = t$ for small t due to (39). We have

$$U'(t,x) = u'(\varphi(t),x)\varphi'(t), \quad U''(t,x) = u''(\varphi(t),x)|\varphi'(t)|^2 + u'(\varphi(t),x)\varphi''(t)$$

and hence

$$J_\varepsilon(U) = \int_0^\infty \int e^{-t}\Big\{\big(u''(\varphi(t),x)|\varphi'(t)|^2 + u'(\varphi(t),x)\varphi''(t)\big)^2 + \varepsilon^2|\nabla u(\varphi(t),x)|^2 + \varepsilon^2|u(\varphi(t),x)|^p\Big\}\, dxdt.$$

Now let $\psi = \varphi^{-1}$ be the inverse of φ (again, the dependence on δ is omitted); changing variable $t = \psi(s)$ in the integral, we have

$$J_\varepsilon(U) = \int_0^\infty \int \psi'(s) e^{-\psi(s)} \Big\{ \big(u''(s,x) |\varphi'(\psi(s))|^2 + u'(s,x)\varphi''(\psi(s)) \big)^2 + \varepsilon^2 |\nabla u(s,x)|^2 + \varepsilon^2 |u(s,x)|^p \Big\} \, dx ds. \tag{41}$$

Observe that, from (40), $s = \varphi(\psi(s)) = \psi(s) - \delta g(\psi(s))$, that is

$$\psi(s) = s + \delta g(\psi(s)). \tag{42}$$

In particular, we have $\psi(s) \geq s - \delta \|g\|_\infty$ and hence $e^{-\psi(s)} \leq e^{\delta \|g\|_\infty} e^{-s}$ which, together with (30), (31) and the finiteness of $\|\varphi'\|_\infty$ and $\|\varphi''\|_\infty$, shows that $J_\varepsilon(U)$ is finite and hence $U \in \mathcal{U}^\varepsilon_{\alpha,\beta}$.

Now, as $U(t,x)$ reduces to $u(t,x)$ when $\delta = 0$, the minimality of u entails that

$$\frac{d}{d\delta} J_\varepsilon(U) \Big|_{\delta=0} = 0. \tag{43}$$

In order to compute this derivative, we differentiate under the integral sign in (41). (Reasoning as above for the finiteness of $J_\varepsilon(U)$, it is easy to prove that this is possible.) First observe that, differentiating (42) with respect to δ, we have

$$\frac{\partial}{\partial\delta}\psi(s) = g(\psi(s)) + \delta g'(\psi(s)) \frac{\partial}{\partial\delta}\psi(s)$$

and hence

$$\frac{\partial}{\partial\delta}\psi(s)\Big|_{\delta=0} = g(s)$$

because $\psi(s) = s$ when $\delta = 0$. Similarly, differentiation with respect to s yields

$$\psi'(s) = 1 + \delta g'(\psi(s))\psi'(s),$$

and further differentiation with respect to δ gives

$$\frac{\partial}{\partial\delta}\psi'(s) = g'(\psi(s))\psi'(s) + \delta \left(g''(\psi(s))\psi'(s) \frac{\partial}{\partial\delta}\psi(s) + g'(\psi(s)) \frac{\partial}{\partial\delta}\psi'(s) \right);$$

hence, in particular,

$$\frac{\partial}{\partial\delta}\psi'(s)\Big|_{\delta=0} = g'(s).$$

As a consequence, we have that

$$\frac{\partial}{\partial\delta}\big(\psi'(s) e^{-\psi(s)}\big)\Big|_{\delta=0} = g'(s)e^{-s} - g(s)e^{-s}. \tag{44}$$

Furthermore, we have

$$\frac{\partial}{\partial\delta}|\varphi'(\psi(s))|^2 = 2\varphi'(\psi(s)) \frac{\partial}{\partial\delta}\varphi'(\psi(s))$$

and hence

$$\frac{\partial}{\partial\delta}|\varphi'(\psi(s))|^2\Big|_{\delta=0} = -2g'(s).$$

On the other hand, $\varphi''(\psi(s)) = -\delta g''(\psi(s))$, so that

$$\frac{\partial}{\partial\delta}\varphi''(\psi(s))\Big|_{\delta=0} = -g''(s).$$

Denoting by $\{\ldots\}$ the function within braces under the integral sign in (41), there holds

$$\{\ldots\}\Big|_{\delta=0} = |u''(s,x)|^2 + \varepsilon^2|\nabla u(s,x)|^2 + \varepsilon^2|u(s,x)|^p.$$

Moreover,

$$\frac{\partial}{\partial\delta}\{\ldots\}\Big|_{\delta=0} = 2(u''(s,x))(-2u''(s,x)g'(s) - u'(s,x)g''(s)).$$

Combining these facts, we obtain that

$$\begin{aligned}&\frac{\partial}{\partial\delta}\Big(\psi'(s)e^{-\psi(s)}\{\ldots\}\Big)\Big|_{\delta=0}\\ &\quad= e^{-s}\big(g'(s) - g(s)\big)\left\{|u''(s,x)|^2 + \varepsilon^2|\nabla u(s,x)|^2 + \varepsilon^2|u(s,x)|^p\right\}\\ &\qquad - e^{-s}\big(4|u''(s,x)|^2 g'(s) + 2u'(s,x)u''(s,x)g''(s)\big).\end{aligned}$$

Putting these things together and recalling (19), we see that (43) reduces to

$$\int_0^\infty e^{-s}\big(g'(s) - g(s)\big)L(s)\,ds - \int_0^\infty\!\!\int e^{-s}\big(4|u''|^2 g'(s) + 2u''u'g''(s)\big)\,dxds = 0$$

which, using the notation introduced in (23), (24) and (25), can be written in the equivalent form

$$\int_0^\infty e^{-s}\big(g'(s) - g(s)\big)L(s)\,ds - \int_0^\infty e^{-s}\big(4D(s)g'(s) + 2I'(s)g''(s)\big)\,ds = 0.$$

Recalling (21) we can integrate by parts the term involving $e^{-s}g(s)L(s)$, i.e.,

$$\begin{aligned}-\int_0^\infty e^{-s}g(s)L(s)\,dsdx &= g(s)H(s)\Big|_0^\infty - \int_0^\infty H(s)g'(s)\,ds\\ &= -\int_0^\infty H(s)g'(s)\,ds.\end{aligned}$$

(Note that the boundary terms vanish due to (39), (22) and the boundedness of g.) Plugging this into the previous identity, we have that

$$\int_0^\infty \big(e^{-s}L(s) - H(s)\big)g'(s)\,ds - \int_0^\infty e^{-s}\big\{4D(s)g'(s) + 2I'(s)g''(s)\big\}\,ds = 0$$

which, recalling (38) and rearranging terms, can be written as

$$\int_0^\infty \big(L(s) - e^s H(s) - 4D(s)\big)e^{-s}\eta(s)\,ds - 2\int_0^\infty I'(s)e^{-s}\eta'(s)\,ds = 0.$$

Now it is convenient to regard the product $e^{-s}\eta(s)$ as a single test function, that is, to rewrite the previous identity as

$$\int_0^\infty \big(L(s) - e^s H(s) - 4D(s) - 2I'(s)\big)(e^{-s}\eta(s))\,ds - 2\int_0^\infty I'(s)(e^{-s}\eta(s))'\,ds = 0.$$

But taking into account the fact that $\eta(s)$ (and hence also $e^{-s}\eta(s)$) is an arbitrary test function in $C_0^\infty(\mathbb{R}^+)$, the last identity means that

$$\frac{d}{dt}\big(-I'(t)\big) = \frac{1}{2}L(t) - \frac{1}{2}e^t H(t) - 2D(t) - I'(t) \quad \text{in } \mathcal{D}'(\mathbb{R}^+). \tag{45}$$

Now, since due to (21) there holds $\big(e^t H(t)\big)' = e^t H(t) - L(t)$, a direct computation of E' using (45) immediately yields (37). □

From now on, we will always identify $E(t)$ with its continuous representative.

Lemma 3.3. *The function $E(t)$ is nonnegative. More precisely,*

$$I(t) + \frac{e^t}{2}\int_t^\infty H(s)\,ds \le E(t) \quad \forall t \ge 0. \tag{46}$$

Proof. From Lemma 2.5, for almost every $s > 0$ we have that

$$-(e^{-s}I(s))' + \frac{1}{2}H(s) = e^{-s}E(s),$$

and hence, for any two numbers $T > t \ge 0$, integration over (t, T) yields

$$\begin{aligned} e^{-t}I(t) - e^{-T}I(T) + \frac{1}{2}\int_t^T H(s)\,ds &= \int_t^T e^{-s}E(s)\,ds \\ &\le E(t)\int_t^T e^{-s}\,ds = E(t)\big(e^{-t} - e^{-T}\big) \end{aligned}$$

since $E(s)$ is nonincreasing. Letting $T \to \infty$ and using (35) one obtains (46). □

Lemma 3.4 (Estimate for $E(0)$). *There exists a constant $C = C(\alpha, \beta)$ such that*

$$E(0) \le C\varepsilon^2, \tag{47}$$

and hence

$$0 \le E(t) \le C\varepsilon^2 \quad \forall t \ge 0. \tag{48}$$

In (47), of course, by $E(0)$ we mean the value at $t = 0$ of the continuous representative of $E(t)$.

Proof. From (33) and (34), we see that

$$\int_0^1 I(t)\,dt + \int_0^1 D(t)\,dt \le C\varepsilon^2 \tag{49}$$

and, since from Cauchy–Schwarz $|I'(t)| \le \sqrt{2I(t)D(t)}$, we also have

$$\int_0^1 |I'(t)|\,dt \le C\varepsilon^2.$$

Moreover, from (20) and (27),

$$H(t) \leq H(0) = J_\varepsilon(u) \leq C\varepsilon^2.$$

As a consequence of these inequalitites, integrating (36) we see that

$$\int_0^1 E(t)\, dt \leq C\varepsilon^2. \tag{50}$$

On the other hand, we have from (37) and $D(t) \geq 0$ that

$$E(0) = E(t) + 2\int_0^t D(s)\, ds \leq E(t) + 2\int_0^1 D(s)\, ds \quad \text{for a.e. } t \in (0,1).$$

Integrating this inequality over $(0,1)$, we find that

$$E(0) \leq \int_0^1 E(t)\, dt + 2\int_0^1 D(s)\, ds,$$

and (47) follows from (49) and (50).

Finally, (48) is obvious since $E(t)$ is nonincreasing and nonnegative. □

Remark 3.5. From the last two lemmata, we also note that

$$2\int_0^\infty D(t)\, dt = -\int_0^\infty E'(t)\, dt = -E(\infty) + E(0) \leq E(0) \leq C\varepsilon^2;$$

i.e.,

$$\int_0^\infty \int |u''(t,x)|^2\, dxdt \leq C\varepsilon^2,$$

a stronger version of (30).

4. **Estimates for the minimizers**

Now we are ready to prove some estimates for the minimizers u_ε.

THEOREM 4.1. *There exists a constant* $C = C(\alpha, \beta)$ *such that*

$$\int_t^{t+1} \int \left(|\nabla u_\varepsilon(s,x)|^2 + |u_\varepsilon(s,x)|^p\right) dxds \leq C \quad \forall t \geq 0, \tag{51}$$

and

$$2I(t) = \int |u_\varepsilon'(t,x)|^2\, dx \leq C\varepsilon^2 \quad \forall t \geq 0, \tag{52}$$

where u_ε *is the minimizer defined in* (18).

In the last statement, we have made explicit the dependence of u on ε, in view of letting $\varepsilon \downarrow 0$.

Proof. Estimate (52) follows immediately from (46) and (48).

Concerning (51), note that for every $t \geq 0$,

$$\begin{aligned}(53)\qquad &\varepsilon^2 \int_t^{t+1} \int \left(|\nabla u_\varepsilon(s,x)|^2 + |u_\varepsilon(s,x)|^p\right) dxds \\ &\leq \varepsilon^2 e^{t+1} \int_t^{t+1} e^{-s} \int \left(|\nabla u_\varepsilon(s,x)|^2 + |u_\varepsilon(s,x)|^p\right) dxds \\ &\leq e^{t+1} \int_t^{t+1} e^{-s} L(s)\, ds \leq e^{t+1} \int_t^{\infty} e^{-s} L(s)\, ds = e^{t+1} H(t),\end{aligned}$$

so that it suffices to estimate the right-hand side. Since $H(t)$ is nonincreasing, for $t \geq 0$, we have from (46) and (48)

$$\frac{e^t}{2} H(t+1) \leq \frac{e^t}{2} \int_t^{t+1} H(s)\, ds \leq \frac{e^t}{2} \int_t^{\infty} H(s)\, ds \leq E(t) \leq C\varepsilon^2,$$

which can be rewritten as

$$e^t H(t) \leq C\varepsilon^2 \quad \forall t \geq 1.$$

On the other hand, if $t \in [0,1]$, we still have

$$e^t H(t) \leq eH(0) = eJ_\varepsilon(u) \leq C\varepsilon^2$$

due to the level estimate (27). In conclusion, we obtain that

$$e^t H(t) \leq C\varepsilon^2 \quad \forall t \geq 0.$$

Inserting this into (53) and dividing by ε^2 gives the required estimate (51). □

THEOREM 4.2. *There exists $C = C(\alpha, \beta)$ such that, for every function h in $H^1 \cap L^p$,*

$$(54)\qquad \left| \int u_\varepsilon''(t,x) h(x)\, dx \right| \leq C\varepsilon^2 \big(\|h\|_{L^p} + \|\nabla h\|_{L^2}\big) \quad \textit{for a.e. } t > 0,$$

where u_ε is the minimizer defined in (18).

Proof. A standard argument using Gâteaux derivatives shows that the minimizer u_ε satisfies the Euler equation

$$(55)\qquad \int_0^{\infty} \int e^{-t} \Big(u_\varepsilon'' \eta'' + \varepsilon^2 \nabla u_\varepsilon \nabla \eta + \varepsilon^2 \frac{p}{2} |u_\varepsilon|^{p-2} u_\varepsilon \eta\Big)\, dxdt = 0$$

for every $\eta \in \mathcal{U}$ with null initial conditions $\eta(0,x) = 0$, $\eta'(0,x) = 0$.

Now consider the function $\varphi \in C^{1,1}(\mathbb{R})$ defined as

$$\varphi(t) = \begin{cases} 0 & \text{if} \quad t \leq 0, \\ t^2 & \text{if} \quad t \in (0,1), \\ 2t-1 & \text{if} \quad t \geq 1 \end{cases}$$

and, for $T > 0$ and $\delta > 0$ (we will keep T fixed and let $\delta \downarrow 0$), set

$$\varphi_\delta(t) = \delta \varphi((t-T)/\delta).$$

Given $h \in H^1 \cap L^p$, we choose $\eta(t,x) = \varphi_\delta(t)h(x)$ in the Euler equation (55). As $\varphi_\delta''(t) = 2\delta^{-1}\chi_{(T,T+\delta)}(t)$, this amounts to

$$\frac{2}{\delta}\int_T^{T+\delta}\int e^{-t}u_\varepsilon''(t,x)h(x)\,dxdt$$
$$= -\varepsilon^2\int_0^\infty\int e^{-t}\varphi_\delta(t)\Big\{\nabla u_\varepsilon(t,x)\nabla h(x) + \frac{p}{2}|u_\varepsilon(t,x)|^{p-2}u_\varepsilon(t,x)h(x)\Big\}\,dxdt.$$

Since $|\varphi_\delta(t)| \leq 2(t-T)^+$ and $\varphi_\delta(t) \to 2(t-T)^+$ as $\delta \downarrow 0$, by dominated convergence we easily conclude that the identity

$$2e^{-T}\int u_\varepsilon''(T,x)h(x)\,dx$$
$$= -2\varepsilon^2\int_0^\infty e^{-t}(t-T)^+\int\Big\{\nabla u_\varepsilon(t,x)\nabla h(x) + \frac{p}{2}|u_\varepsilon(t,x)|^{p-2}u_\varepsilon(t,x)h(x)\Big\}\,dxdt$$

is satisfied for almost every $T > 0$.

Therefore, for such T's,

$$\left|\int u_\varepsilon''(T,x)h(x)\,dx\right|$$
$$= \varepsilon^2\left|\int_T^\infty e^{-(t-T)}(t-T)\int\Big\{\nabla u_\varepsilon(t,x)\nabla h(x)+\frac{p}{2}|u_\varepsilon(t,x)|^{p-2}u_\varepsilon(t,x)h(x)\Big\}\,dxdt\right|$$
$$\leq \varepsilon^2\sum_{k=0}^\infty e^{-k}(k+1)\int_{T+k}^{T+k+1}\int\Big\{|\nabla u_\varepsilon(t,x)\nabla h(x)|+\frac{p}{2}|u_\varepsilon(t,x)|^{p-1}|h(x)|\Big\}\,dxdt.$$

But from Cauchy–Schwarz,

$$\int_{T+k}^{T+k+1}\int\Big|\nabla u_\varepsilon(t,x)\nabla h(x)\Big|\,dxdt$$
$$\leq \left(\int_{T+k}^{T+k+1}\int\Big|\nabla u_\varepsilon(t,x)\Big|^2\,dxdt\right)^{1/2}\left(\int_{T+k}^{T+k+1}\int\Big|\nabla h(x)\Big|^2\,dxdt\right)^{1/2}$$
$$= \left(\int_{T+k}^{T+k+1}\int\Big|\nabla u_\varepsilon(t,x)\Big|^2\,dxdt\right)^{1/2}\|\nabla h\|_{L^2} \leq C\|\nabla h\|_{L^2},$$

where in the last inequality we have used (51) with $t = T+k$. In the same way, using Hölder inequality and (51), we have

$$\int_{T+k}^{T+k+1}\int|u_\varepsilon(t,x)|^{p-1}|h(x)|\,dxdt$$
$$\leq \left(\int_{T+k}^{T+k+1}\int\Big|u_\varepsilon(t,x)\Big|^p\,dxdt\right)^{(p-1)/p}\|h\|_{L^p} \leq C\|h\|_{L^p}.$$

Putting these things together, for almost every $T > 0$ we obtain that

$$\left|\int u_\varepsilon''(T,x)h(x)\,dx\right| \leq \varepsilon^2 C\left(\|\nabla h\|_{L^2} + \|h\|_{L^p}\right)\sum_{k=0}^\infty e^{-k}(k+1)$$

and the claim follows. $\square$

The estimates thus far obtained on u are essentially based on Lemma 3.4. In order to prove the energy inequality (15), we need the following strong refinement of Lemma 3.4 and the subsequent technical lemma.

PROPOSITION 4.3 (Sharp estimate for $E(0)$). *There exists a constant $C = C(\alpha, \beta)$ such that*

$$E(0) \le \frac{\varepsilon^2}{2} \int \left(|\nabla \alpha(x)|^2 + |\alpha(x)|^p + |\beta(x)|^2 \right) dx + C\varepsilon^3. \tag{56}$$

Proof. We see from (36) that $E(t)$ is the sum of three terms. The first term $I(t)$ is continuous thanks to Lemma 2.5. In particular, from (17),

$$I(0) = \frac{1}{2} \int |u'(0,x)|^2 \, dx = \frac{\varepsilon^2}{2} \int |\beta(x)|^2 \, dx. \tag{57}$$

For the third term in (36), the situation is similar: the function $e^t H(t)$ is continuous and, since $H(0) = J_\varepsilon(u_\varepsilon)$, from (26) we obtain that

$$\frac{1}{2} e^t H(t) \Big|_{t=0} \le \frac{\varepsilon^2}{2} \int \left(|\nabla \alpha(x)|^2 + |\alpha(x)|^p \right) dx + C\varepsilon^3. \tag{58}$$

In remains to estimate the middle term in (36), namely $-I'(t)$, at $t = 0$. Note that, even though $I'(t)$ admits a continuous representative (because $E(t)$ does, or, more directly, by (45)), it is not possible to let $t = 0$ in (25), because (25) makes sense only for almost every $t \ge 0$. (Note that $u''(t,x)$ may have no trace at $t = 0$.) Therefore, to estimate the continuous representative of $I'(t)$ at $t = 0$, we estimate its integral averages close to zero.

For fixed $\delta > 0$, denoting by $\langle \cdot, \cdot \rangle$ the scalar product in L^2 and letting for simplicity $u'(t) = u'(t, \cdot)$ etc., we may write

$$I'(t) = \langle u''(t), u'(0) \rangle + \langle u''(t), u'(t) - u'(0) \rangle \quad \text{for a.e. } t \in (0, \delta). \tag{59}$$

But since $u'(0) = \varepsilon\beta$, from (54) with $h = \beta$ we have that

$$|\langle u''(t), u'(0) \rangle| = \varepsilon |\langle u''(t), \beta \rangle| \le C\varepsilon^3 \quad \text{for a.e. } t \in (0, \delta)$$

and hence, averaging (59) over $(0, \delta)$, we see that

$$\left| \frac{1}{\delta} \int_0^\delta I'(t) \, dt \right| \le C\varepsilon^3 + \frac{1}{\delta} \int_0^\delta |\langle u''(t), u'(t) - u'(0) \rangle| \, dt. \tag{60}$$

On the other hand, $u'(t)$ belongs to $H^1((0, \delta); L^2)$, so that $u'(t)$ belongs to $C^{1/2}((0, \delta); L^2)$ and, in particular,

$$\|u'(t) - u(0)\|_{L^2} \le \sqrt{t} \sqrt{\int_0^\delta D(s) \, ds} \quad \text{for every } t \in (0, \delta)$$

(recall (23)). Therefore from Cauchy–Schwarz, we have

$$\begin{aligned}|\langle u''(t), u'(t)-u'(0)\rangle| &\le \|u''(t)\|_{L^2}\sqrt{t}\sqrt{\int_0^\delta D(s)\,ds}\\ &= \sqrt{D(t)t}\sqrt{\int_0^\delta D(s)\,ds} \quad \text{for a.e. } t\in(0,\delta).\end{aligned}$$

Integrating on $(0,\delta)$ and using again Cauchy–Schwarz, we obtain

$$\begin{aligned}\int_0^\delta |\langle u''(t), u'(t)-u'(0)\rangle|\,dt &\le \sqrt{\int_0^\delta D(s)\,ds}\int_0^\delta \sqrt{D(t)t}\,dt\\ &\le \left(\int_0^\delta D(s)\,ds\right)\sqrt{\int_0^\delta t\,dt} \le \delta\int_0^\delta D(s)\,ds.\end{aligned}$$

Plugging this estimate into (60), we find that

$$\left|\frac{1}{\delta}\int_0^\delta I'(t)\,dt\right| \le C\varepsilon^3 + \int_0^\delta D(s)\,ds$$

and, as $\delta>0$ is arbitrary, letting $\delta\downarrow 0$ we find the estimate $|I'(0)|\le C\varepsilon^3$ for the continuous representative of $I'(t)$. Now (56) follows easily from (36), on combining (57), (58) and the last inequality. □

LEMMA 4.4. *Let $l(t)$, $m(t)$ be nonnegative functions in $L^1_{\rm loc}$ such that*

$$e^t\int_t^\infty\int_s^\infty e^{-z}l(z)\,dz\,ds \le m(t) \quad \text{for a.e. } t>0. \tag{61}$$

Then, for every pair of numbers $a>0$ and $\theta\in(0,1)$,

$$\left(\int_0^{\theta a} se^{-s}\,ds\right)\int_{T+\theta a}^{T+a} l(t)\,dt \le \int_T^{T+a} m(t)\,dt \quad \forall T\ge 0. \tag{62}$$

Proof. We have

$$\begin{aligned}e^t\int_t^\infty\int_s^\infty e^{-z}l(z)\,dz\,ds &= \int_t^\infty\int_0^\infty \chi_{(s,\infty)}(z)e^{t-z}l(z)\,dz\,ds\\ = \int_0^\infty e^{t-z}l(z)\left(\int_t^\infty \chi_{(s,\infty)}(z)\,ds\right)dz &= \int_0^\infty (z-t)^+e^{-(z-t)}l(z)\,dz;\end{aligned}$$

hence (61) can be rewritten as

$$m(t)\ge\int_0^\infty (z-t)^+e^{-(z-t)}l(z)\,dz \quad \text{for a.e. } t>0.$$

Now, for arbitrary $T \geq 0$, $a > 0$ and $\theta \in (0,1)$, integrating the last inequality on the interval $(T, T+a)$, we have

$$\begin{aligned}\int_T^{T+a} m(t)\,dt &\geq \int_0^\infty l(z)\left(\int_T^{T+a}(z-t)^+ e^{-(z-t)}\,dt\right)dz\\ &= \int_T^\infty l(z)\left(\int_T^{(T+a)\wedge z}(z-t)e^{-(z-t)}\,dt\right)dz\\ &\geq \int_{T+\theta a}^{T+a} l(z)\left(\int_T^{(T+a)\wedge z}(z-t)e^{-(z-t)}\,dt\right)dz\\ &= \int_{T+\theta a}^{T+a} l(z)\left(\int_T^{z}(z-t)e^{-(z-t)}\,dt\right)dz.\end{aligned}$$

Finally, changing variable $s = z - t$ in the inner integral, we see that

$$\begin{aligned}\int_T^{T+a} m(t)\,dt &\geq \int_{T+\theta a}^{T+a} l(z)\left(\int_0^{z-T} se^{-s}\,ds\right)dz\\ &\geq \int_{T+\theta a}^{T+a} l(z)\left(\int_0^{\theta a} se^{-s}\,ds\right)dz,\end{aligned}$$

and (62) follows. □

5. **Proof of Theorem 1.1**

Now we are in a position to prove Theorem 1.1. As discussed at the beginning of Section 2, the minimizers $v_\varepsilon(t,x)$ of F_ε mentioned in Theorem 1.1 are related to the minimizers $u_\varepsilon(t,x)$ of the functional J_ε (defined in (16), with the initial conditions (17)) by the change of variable

$$u_\varepsilon(t,x) = v_\varepsilon(\varepsilon t, x), \quad t \geq 0, \quad x \in \mathbb{R}^n \tag{63}$$

and, in particular, the v_ε satisfy the initial conditions

$$v_\varepsilon(0,x) = \alpha(x), \quad v'_\varepsilon(0,x) = \beta(x). \tag{64}$$

Clearly, all estimates concerning u_ε can be transferred to v_ε by scaling. Thus, keeping the same notation as in the statement of Theorem 1.1, we split its proof into several steps.

1. *A priori estimates.* Here we prove (8), (9) and (10).

Concerning (8), we rely on Theorem 4.1. Using (63), we can write (51) in terms of v_ε and, changing variable $\sigma = \varepsilon s$, we obtain

$$\int_{\varepsilon t}^{\varepsilon t+\varepsilon}\int \left(|\nabla v_\varepsilon(\sigma,x)|^2 + |v_\varepsilon(\sigma,x)|^p\right)dxd\sigma \leq C\varepsilon \quad \forall t \geq 0.$$

Since t is arbitrary, we can rename εt as τ, arriving at

$$\int_{\tau}^{\tau+\varepsilon}\int \left(|\nabla v_\varepsilon(\sigma,x)|^2 + |v_\varepsilon(\sigma,x)|^p\right)dxd\sigma \leq C\varepsilon \quad \forall \tau \geq 0. \tag{65}$$

Now fix T and ε with $T \geq \varepsilon$. Given $t \geq 0$, the interval $[t, t+T]$ is covered by $\lceil T/\varepsilon \rceil$ adjacent subintervals, each of length ε. On each of these subintervals $[\tau, \tau+\varepsilon]$ we can use (65) with the proper τ and then, summing the resulting inequalitites, we find that

$$\int_t^{t+T}\!\!\int \left(|\nabla v_\varepsilon(\sigma,x)|^2 + |v_\varepsilon(\sigma,x)|^p\right) dx d\sigma \leq C\varepsilon\lceil T/\varepsilon\rceil \leq 2CT \quad \forall t \geq 0, \tag{66}$$

which yields (8) as a particular case when $t = 0$.

Writing (52) with t/ε in place of t, and expressing the resulting inequality in terms of v'_ε using (63), proves the first part of (9), whereas the second part follows from the first and the elementary inequality

$$\int |v_\varepsilon(t,x)|^2\, dx \leq 2\int |v_\varepsilon(0,x)|^2\, dx + 2t\int_0^t\!\!\int |v'_\varepsilon(s,x)|^2\, dxds,$$

recalling that $v_\varepsilon(0,x) = \alpha(x)$ and $\alpha \in L^2$.

Finally, (10) follows immediately from Theorem 4.2, on combining (54) and (63).

2. *Passage to the limit.* Let Q^T denote the cylinder $(0,T)\times\mathbb{R}^n$. From (8) and (9) we see that, for every $T > 0$, there exists $C_T > 0$ such that

$$\|v_\varepsilon\|_{H^1(Q^T)} + \|v_\varepsilon\|_{L^p(Q^T)} \leq C_T.$$

Now take a sequence A_j of smooth, bounded open sets such that $\mathbb{R}^n = \bigcup A_j$, and let $A_j^T = (0,T)\times A_j$. Note that $H^1(Q^T)$ is compactly embedded in $L^2(A_j^T)$ (via the usual restriction operator) for each j. Moreover, when $p > 2$, we have that the v_ε are equibounded in $L^q(A_j^T)$ for every $q \in [2,p]$ and every j. As a consequence, by a standard diagonal argument, passing to a subsequence (not relabeled), we can assume that for every $T > 0$ and every j,

$$v_\varepsilon \rightharpoonup w \quad \text{in} \quad H^1(Q^T), \tag{67}$$

$$v_\varepsilon \rightharpoonup w \quad \text{in} \quad L^p(Q^T), \tag{68}$$

$$v_\varepsilon \to w \quad \text{in} \quad L^q(A_j^T) \quad \text{for every } q \in [2,p) \text{ (for } q = 2, \text{ if } p = 2) \tag{69}$$

(note the strong covergence in (69)) for a suitable function w which belongs to $H^1(Q^T)\cap L^p(Q^T)$ for all $T > 0$.

Note that v_ε, being the minimizer of F_ε under the initial conditions (2), satisfies the Euler equation

$$\int_0^\infty\!\!\int e^{-t/\varepsilon}\left(\varepsilon^2 v''_\varepsilon\eta'' + \nabla v_\varepsilon\nabla\eta + \frac{p}{2}|v_\varepsilon|^{p-2}v_\varepsilon\eta\right) dxdt = 0 \qquad \forall\eta\in C_0^\infty(\mathbb{R}^+\times\mathbb{R}^n),$$

which corresponds to (55) via (63). Integrating by parts, we may write this Euler equation in the equivalent form

$$-\int_0^\infty\!\!\int \varepsilon^2 v'_\varepsilon\left(e^{-t/\varepsilon}\eta''\right)' dxdt + \int_0^\infty\!\!\int e^{-t/\varepsilon}\left(\nabla v_\varepsilon\nabla\eta + \frac{p}{2}|v_\varepsilon|^{p-2}v_\varepsilon\eta\right) dxdt = 0.$$

Given an arbitrary $\varphi \in C_0^\infty(\mathbb{R}^+ \times \mathbb{R}^n)$, we can choose $\eta(t,x) = e^{t/\varepsilon}\varphi(t,x)$ in the last equation. A straightforward computation shows that

$$-\int_0^\infty\!\!\int v_\varepsilon'\big(\varepsilon^2\varphi'' + 2\varepsilon\varphi' + \varphi\big)'\,dxdt + \int_0^\infty\!\!\int \Big(\nabla v_\varepsilon \nabla\varphi + \frac{p}{2}|v_\varepsilon|^{p-2}v_\varepsilon\varphi\Big)\,dxdt = 0.$$

By (67) and (69), we can pass to the limit as $\varepsilon \downarrow 0$ and obtain

$$-\int_0^\infty\!\!\int w'\varphi'\,dxdt + \int_0^\infty\!\!\int \Big(\nabla w\nabla\varphi + \frac{p}{2}|w|^{p-2}w\varphi\Big)\,dxdt = 0,$$

namely that w is a weak solution, in $\mathbb{R}^+ \times \mathbb{R}^n$, of the nonlinear wave equation (13), as claimed in Theorem 1.1.

3. *The L^∞ bounds.* Here we prove (11) and (12). Letting $\varepsilon \downarrow 0$ in (66), by lower semicontinuity we get

$$\int_t^{t+T}\!\!\int \Big(|\nabla w|^2 + |w|^p\Big)\,dxds \le CT \quad \forall t \ge 0.$$

Dividing by T and letting $T \downarrow 0$, one obtains (11).

Finally, (12) follows immediately from (9) and the second part of (11).

4. *Passage to the limit in the initial conditions.* We have to show that w satisfies (4). The first condition, meant for instance in L^2, follows from (67) and the first condition in (64).

Passing to the limit in $v_\varepsilon'(0,\cdot)$ is more technical and can be done essentially as in [3, §1.4] but, for the sake of completeness, we give full details.

Let X denote the Banach space $H^1 \cap L^p$, normed with $\|\ \|_{H^1} + \|\ \|_{L^p}$. Given $\varphi \in L^2$, we can regard φ as an element of X' letting

$$\langle \varphi, h\rangle = \int \varphi h\,dx \quad \forall h \in X,$$

via the natural embedding $L^2 \hookrightarrow X'$. In this sense, (10) provides a bound for v_ε'' in $L^\infty(\mathbb{R}^+; X')$, which is uniform in ε. Similarly, the first part of (9) gives a uniform bound for v_ε' in $L^\infty(\mathbb{R}^+; X')$.

Thus, passing to a subsequence, we may assume that $v_\varepsilon' \rightharpoonup \psi_1$ and $v_\varepsilon'' \rightharpoonup \psi_2$ weakly-* in $L^\infty(\mathbb{R}^+; X')$ for suitable $\psi_1, \psi_2 \in L^\infty(\mathbb{R}^+; X')$, and (69) reveals that, *a fortiori*, $\psi_1 = w'$ and $\psi_2 = w''$. In particular, $w' \in W^{1,\infty}(\mathbb{R}^+; X')$ and $w'(0)$ is well defined as an element of X'.

Now, given $h \in X$ and $a \in C_0^\infty(\mathbb{R})$, from the second condition in (64) we have, for fixed ε,

$$\int_0^\infty a(t)\int v_\varepsilon'' h\,dxdt = -\int_0^\infty a'(t)\int v_\varepsilon' h\,dxdt - a(0)\int \beta(x)h(x)\,dx.$$

Letting $\varepsilon \downarrow 0$, since $a(t)h(x)$ belongs to $L^1(\mathbb{R}^+; X)$, we obtain that

$$\int_0^\infty a(t)\langle w''(t), h\rangle_{X',X}\,dt = -\int_0^\infty a'(t)\langle w'(t), h\rangle_{X',X}\,dt - a(0)\int \beta(x)h(x)\,dx$$

which, from the arbitrariness of $a \in C_0^\infty(\mathbb{R})$ and $h \in X$, proves that $w'(0) = \beta$, this equality being meant in X'. (Note that $\beta \in L^2 \hookrightarrow X'$.)

5. *The energy inequality.* Here we prove (15).

Inserting the estimate $E(t) \le E(0)$ into (46) and rearranging terms, we find that

$$e^t \int_t^\infty H(s)\,ds \le 2E(0) - 2I(t) \quad \forall t \ge 0. \tag{70}$$

Now, letting

$$l(t) = \int \left(|\nabla u(t,x)|^2 + |u(t,x)|^p \right) dx, \tag{71}$$

we have from (20) and (19) that

$$\frac{1}{\varepsilon^2} H(s) \ge \int_s^\infty e^{-z} l(z)\,dz, \quad s \ge 0,$$

and hence, dividing (70) by ε^2, we see that we can apply Lemma 4.4 with $l(z)$ defined as above and $m(t) = 2\frac{E(0)-I(t)}{\varepsilon^2}$. As a result of the lemma, rearranging terms we obtain the inequality

$$\left(\int_0^{\theta a} s e^{-s}\,ds \right) \int_{T+\theta a}^{T+a} l(t)\,dt + \frac{2}{\varepsilon^2} \int_T^{T+a} I(t)\,dt \le a \frac{2E(0)}{\varepsilon^2} \quad \forall T > 0$$

for arbitrary $a > 0$ and $\theta \in (0,1)$. Keeping a and θ fixed for the moment, we use this inequality in the weakened form

$$Y(\theta a) \int_{T+\theta a}^{T+a} l(t)\,dt + \frac{2}{\varepsilon^2} \int_{T+\theta a}^{T+a} I(t)\,dt \le a \frac{2E(0)}{\varepsilon^2} \quad \forall T > 0, \tag{72}$$

where we have set for simplicity

$$Y(z) = \int_0^z s e^{-s}\,ds. \tag{73}$$

Now, using (71) and (24), it is convenient to write (72) explicitly in terms of the function $u = u_\varepsilon$, thus obtaining

$$Y(\theta a) \int_{T+\theta a}^{T+a} \left(\int \left(|\nabla u_\varepsilon(t,x)|^2 + |u_\varepsilon(t,x)|^p \right) dx \right) dt \\ + \frac{1}{\varepsilon^2} \int_{T+\theta a}^{T+a} \left(\int |u_\varepsilon'(t,x)|^2\,dx \right) dt \le a \frac{2E_\varepsilon(0)}{\varepsilon^2} \quad \forall T > 0.$$

Substituting (63) into this estimate, after the change of variable $s = \varepsilon t$ in the time integrals we find that

$$Y(\theta a) \int_{T+\varepsilon\theta a}^{T+\varepsilon a} \left(\int \left(|\nabla v_\varepsilon(s,x)|^2 + |v_\varepsilon(s,x)|^p \right) dx \right) ds \\ + \int_{T+\varepsilon\theta a}^{T+\varepsilon a} \left(\int |v_\varepsilon'(s,x)|^2\,dx \right) ds \le a \frac{2E_\varepsilon(0)}{\varepsilon} \quad \forall T > 0.$$

(We have written T instead of εT since T is arbitrary.) Now, for fixed $\delta > 0$, any interval $(t, t+\delta)$, with $t \geq \varepsilon\theta a$, can be covered by $\lceil \delta/(1-\theta)\varepsilon a \rceil$ consecutive intervals of the kind $(T + \varepsilon\theta a, T + \varepsilon a)$ for suitable values of T; summing the corresponding inequalities as above, we find

$$Y(\theta a)\int_t^{t+\delta}\left(\int\left(|\nabla v_\varepsilon(s,x)|^2 + |v_\varepsilon(s,x)|^p\right)dx\right)ds \\ + \int_t^{t+\delta}\left(\int |v_\varepsilon'(s,x)|^2\,dx\right)ds \leq a\frac{2E_\varepsilon(0)}{\varepsilon}\left\lceil \frac{\delta}{(1-\theta)\varepsilon a}\right\rceil \quad \forall t \geq \varepsilon\theta a.$$

Keeping a, θ, δ and t fixed, we let $\varepsilon \downarrow 0$ in the previous inequality. Using (67) and (68), by lower semicontinuity we obtain that

(74)
$$Y(\theta a)\int_t^{t+\delta}\left(\int\left(|\nabla w(s,x)|^2 + |w(s,x)|^p\right)dx\right)ds + \int_t^{t+\delta}\left(\int |w'(s,x)|^2\,dx\right)ds \\ \leq a\limsup_{\varepsilon\downarrow 0}\left(\frac{2E_\varepsilon(0)}{\varepsilon}\left\lceil \frac{\delta}{(1-\theta)\varepsilon a}\right\rceil\right) \quad \forall t > 0.$$

To estimate the limsup, observe that

$$\lim_{\varepsilon\downarrow 0}\left(\varepsilon\left\lceil \frac{\delta}{(1-\theta)\varepsilon a}\right\rceil\right) = \frac{\delta}{(1-\theta)a}$$

and hence, from Proposition 4.3,

$$\limsup_{\varepsilon\downarrow 0}\left(\frac{2E_\varepsilon(0)}{\varepsilon}\left\lceil \frac{\delta}{(1-\theta)\varepsilon a}\right\rceil\right) \leq \frac{\delta}{(1-\theta)a}\int\left(|\nabla\alpha(x)|^2 + |\alpha(x)|^p + |\beta(x)|^2\right)dx$$

which, plugged into (74), yields

$$Y(\theta a)\int_t^{t+\delta}\left(\int\left(|\nabla w(s,x)|^2 + |w(s,x)|^p\right)dx\right)ds \\ + \int_t^{t+\delta}\left(\int |w'(s,x)|^2\,dx\right)ds \leq \frac{\delta}{1-\theta}\mathcal{E}(0) \quad \forall t > 0,$$

where $\mathcal{E}(0)$ is defined according to (15). Now, dividing by δ and choosing, for instance $\theta = \delta$ and $a = \delta^{-2}$, letting $\delta \downarrow 0$ and recalling (73), we find that

$$\int\left(|\nabla w(t,x)|^2 + |w(t,x)|^p\right)dx + \int |w'(t,x)|^2\,dx \leq \mathcal{E}(0) \quad \text{for a.e. } t > 0,$$

and the validity of (15) is established.

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