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The dimension and structure of the space of harmonic 2-spheres in the m-sphere

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Abstract

We prove the conjecture, posed in 1993 by Bolton and Woodward, that the dimension of the space of harmonic maps from the 2-sphere to the $2n$-sphere is $2d + n^2$. We also give an explicit algebraic method to construct all harmonic maps from the 2-sphere to the m-sphere.

1. Introduction

A *harmonic map* is a map $\varphi : M \to N$ between Riemannian manifolds that extremizes the energy functional $\int_D |d\varphi|^2 d\,\mathrm{vol}$ over compact domains D in M; this functional generalizes the Dirichlet integral. Examples of harmonic maps include harmonic functions, geodesics and minimal surfaces. Harmonic maps have been used to prove important results in geometry, including rigidity results (see, for example, [16]).

In this paper we study the space of harmonic maps from S^2 to S^m. Since a harmonic map from a 2-sphere is automatically weakly conformal (see, for example, [26]), a map from a 2-sphere is harmonic *if and only if* it is a minimal branched immersion [12].

Following the twistor lift approach initiated by Calabi in [6], the moduli space of harmonic 2-spheres in S^m was studied, among other sources, in [1], [2], [10], [11], [17], [20], [21], [23], [24], [25]. It is known that the space of *linearly full* (i.e., not lying in a proper sub-sphere) harmonic maps from S^2 to S^{2n} of *degree* d is isomorphic to two copies of the space $\mathrm{SO}(2n+1, \mathbb{C})$ when $d = n(n+1)/2$, and it is empty if $d < n(n+1)/2$. Apart from these remarkable results, not much is known for arbitrary d and n. The dimension was only known when $n = 2$ [20], [23], [24], [25] and $n = 3$ [9].

In [3], Bolton and Woodward conjectured, using heuristic arguments, that the dimension of the space of linearly full harmonic 2-spheres of degree d in S^{2n} is $2d + n^2$. In this paper we give a proof of this conjecture. To this end, we first find a *completely explicit* algebraic method to construct *any* harmonic

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map from S^2 to S^{2n}. The idea is to generalize the methods used in [2], [4], [5], [20], [23], [24], [25], where harmonic maps from S^2 to S^4 are constructed, via the twistor lift approach, using holomorphic maps from S^2 to $\mathbb{CP}^3$ satisfying a differential system; this is possible because the twistor space of S^4 is biholomorphic to $\mathbb{CP}^3$. Hence, a completely explicit algebraic construction is obtained, allowing the study of the dimension and structure of the space of harmonic maps from S^2 to S^4.

In the case $n > 2$, the twistor space of the $2n$-sphere is certainly *not* biholomorphic to any complex projective space. Nevertheless, these two spaces are birationally equivalent, and it turns out that this is sufficient in order to do a *local* study. In fact, the moduli space of harmonic maps from S^2 to S^{2n} of a given degree d is *locally isomorphic* to a space of holomorphic maps from S^2 to $\mathbb{CP}^{n(n+1)/2}$ of degree d and satisfying a particular differential system, namely equation (7). This differential system, in a different form, also appears in [5], [13].

The next step is to study the space of solutions of this differential system. As a space of maps from the 2-sphere to complex projective space, we can restrict our work to *tuples of polynomials*, as in [20]. The naïve approach—namely to use the standard basis for polynomials and convert the differential system into a large set of quadratic equations on the coefficients of the polynomials—does not work because the system is *too big*. However, a different kind of basis for the space of polynomials leads to a description of the set of solutions essentially as a *determinantal variety* on a set of parameters that determine the polynomials, or alternatively, as the set of *integral elements* of an exterior differential system in the space of parameters.

With this description, and using elementary intersection theory, we find that $2d+n^2$ is a lower bound for the dimension of the space of harmonic maps from S^2 to S^{2n}.

To prove that $2d + n^2$ is also an upper bound, we introduce the concept of *extendable* harmonic map. A similar concept also appears in [18] (as 'maps with extra eigenvalues') and [19] (as 'collapses of maps'). A harmonic map from S^2 to S^{2n} is extendable if, after embedding S^{2n} into $S^{2(n+1)}$ geodesically, the map can be obtained as a suitable deformation through linearly full harmonic maps whose codomain is $S^{2(n+1)}$. These deformations provide a local projection from the set of harmonic maps into $S^{2(n+1)}$ to the set of harmonic maps into S^{2n}. This projection is used to produce an inductive procedure to show that $2d+n^2$ is an upper bound of the dimension of the space of harmonic maps from S^2 to S^{2n}.

It is worth noting that the proof implies that the dimension of the set of linearly full harmonic maps from S^2 to S^{2n} is *pure*, i.e., all irreducible components have the same dimension.

The paper is organized as follows. In Section 2 we review Calabi's twistor construction. Section 3 describes how to translate the problem into the study of maps into complex projective space. To prove that this translation is good enough for our purposes is highly technical, so for expository reasons we postpone it to Section 7. In Section 4 we give an explicit algebraic recipe to construct *all* harmonic maps from S^2 to spheres. Finally, in Section 5 we use this construction to prove Bolton and Woodward's conjecture for the linearly full case, and in Section 6 we consider the nonlinearly full case.

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2. **Preliminaries**

Recall ([7], for example) that a map $\varphi : S^2 \to S^m \subset \mathbb{R}^{m+1}$ is *harmonic* if $\Delta^{S^2}\varphi = \lambda\varphi$ for some function $\lambda : S^2 \to \mathbb{R}$. Such a map is called *linearly full* if its image does not lie in a proper geodesic sub-sphere of S^m. Using the topology of S^2, Calabi showed in [6] that for $\varphi : S^2 \to S^m$ to be linearly full, m must be an even number $2n$.

We do a quick review of the twistor construction in [6]. The twistor space of the $2n$ sphere, denoted $\mathcal{Z}_n$ (or $\mathcal{Z}_n(\mathbb{C}^{2n+1})$), is the complex manifold of n-dimensional linear subspaces of $\mathbb{C}^{2n+1}$ that are *isotropic* with respect to the complex bilinear product

$$(\boldsymbol{z}, \boldsymbol{w}) = ((z_1, \ldots, z_{2n+1}), (w_1, \ldots, w_{2n+1})) = \sum_{k=1}^{2n+1} z_k w_k. \tag{1}$$

In other words, $\mathcal{Z}_n$ is the submanifold of the Grassmannian of n-planes in $\mathbb{C}^{2n+1}$ given by

$$\mathcal{Z}_n = \{P \in \mathrm{Gr}(n, \mathbb{C}^{2n+1}) : (v, w) = 0 \ \text{ for all } v, w \in P\}.$$

We will use $\langle \boldsymbol{z}, \boldsymbol{w} \rangle := (\boldsymbol{z}, \overline{\boldsymbol{w}})$ to denote the Hermitian product in $\mathbb{C}^{2n+1}$, and the word 'perpendicular' will always mean perpendicular with respect to this Hermitian product.

The manifold $\mathcal{Z}_n$ is a complex submanifold of $\mathrm{Gr}(n, \mathbb{C}^{2n+1})$, so we can restrict the Plücker embedding $\mathrm{Pl} : \mathrm{Gr}(n, \mathbb{C}^{2n+1}) \to \mathbb{P}\Lambda^n\mathbb{C}^{2n+1}$ to $\mathcal{Z}_n$. This restriction (also denoted by Pl) has degree 2 [1], [21]. The tangent plane of $\mathcal{Z}_n$ at a point P is the subspace of the tangent plane to the Grassmannian at P—which can be described as $T_P\mathrm{Gr}(n, \mathbb{C}^{2n+1}) = \mathrm{Hom}(P, P^\perp)$—given by [21]

$$T_P\mathcal{Z}_n = \{L \in \mathrm{Hom}(P, P^\perp) : (Lu, u) = 0 \ \ \forall u \in P\}.$$

There is a projection $\pi : \mathcal{Z}_n \to S^{2n}$ defined as follows: For $P \in \mathcal{Z}_n$, $\pi(P)$ is the unique real unit vector in $\mathbb{C}^{2n+1}$ such that $\{\pi(P), P_1, \dots, P_n, \bar{P}_1, \dots, \bar{P}_n\}$ is a positively oriented basis of $\mathbb{C}^{2n+1}$, where $\{P_1, \dots, P_n\}$ is a basis of P. Note that in [10], π is denoted by π_+, and π_- is used to denote the map $-\pi$ (i.e., π composed with the antipodal map).

Given a linearly full harmonic map $\varphi : S^2 \to S^{2n}$ and isothermic coordinates $(z, \bar{z})$ in S^2, Calabi defined $\psi : S^2 \to \mathcal{Z}_n$ by

$$\psi := \operatorname{Span}\left\{\frac{\partial \varphi}{\partial \bar{z}}, \dots, \frac{\partial^n \varphi}{\partial \bar{z}^n}\right\}$$

and proved that ψ is well defined and is holomorphic and horizontal (i.e., perpendicular to the fibers of π). In other words, $\partial\psi/\partial z$ belongs to the subspace [21]

$$H_P \mathcal{Z}_n = \{L \in T_P \mathcal{Z}_n : L(P) \perp \bar{P}\}. \tag{2}$$

In addition, ψ satisfies $\pi \circ \psi = \varphi$ or $-\varphi$ and it is linearly full, in the sense that the image of ψ is not contained in any submanifold of the form

$$\mathcal{Z}_n^F := \{W \oplus F \in \mathcal{Z}_n : W \in \mathcal{Z}_r((F \oplus \bar{F})^\perp)\} \cong \mathcal{Z}_r, \tag{3}$$

where F is an $(n-r)$-dimensional subspace of $\mathbb{C}^{2n+1}$, with $r < n$ (see [10]).

Conversely, if $\psi : S^2 \to \mathcal{Z}_n$ is holomorphic, horizontal and linearly full, then $\pi \circ \psi$ and $-\pi \circ \psi$ are harmonic and linearly full. Therefore, we have a 2-to-1 correspondence between linearly full harmonic maps from S^2 to S^{2n} and linearly full holomorphic horizontal maps from S^2 to $\mathcal{Z}_n$.

Since $H_2(\mathcal{Z}_n, \mathbb{Z}) = \mathbb{Z}$ [1], [21], the homology class induced by ψ is a positive multiple $d = \deg(\psi)$ of a generator of $H_2(\mathcal{Z}_n, \mathbb{Z})$. The number d is called the *twistor degree* of φ; since the Plücker embedding has degree 2, we have that the degree of the curve $\mathrm{Pl} \circ \psi$ in $\mathbb{P}\Lambda^n\mathbb{C}^{2n+1}$ is twice the twistor degree of φ, and so the number d can also be characterized as $\mathrm{Area}(\varphi(S^2))/4\pi$ [21].

Hence both harmonic and holomorphic and horizontal maps are graded by the degree. Let

$$\begin{aligned}
\mathrm{Harm}_d^f(S^2, S^{2n}) &= \{\text{Linearly full harmonic maps from } S^2 \text{ to } S^{2n} \text{ of area } 4\pi d\},\\
\mathrm{HH}_d^f(S^2, \mathcal{Z}_n) &= \{\text{Horizontal, holomorphic, full maps}\\
&\qquad \text{from } S^2 \text{ to } \mathcal{Z}_n \text{ of degree } d\},
\end{aligned}$$

and let

$$\begin{aligned}
\mathrm{Harm}_d^{f,+}(S^2, S^{2n}) &= \{\pi \circ \psi : \psi \in \mathrm{HH}_d^f(S^2, \mathcal{Z}_n)\},\\
\mathrm{Harm}_d^{f,-}(S^2, S^{2n}) &= \{-\pi \circ \psi : \psi \in \mathrm{HH}_d^f(S^2, \mathcal{Z}_n)\}.
\end{aligned}$$

The results discussed above then imply

THEOREM 1 ([6], [1], [10]).

$$\mathrm{Harm}_d^f(S^2, S^{2n}) = \mathrm{Harm}_d^{f,+}(S^2, S^{2n}) \sqcup \mathrm{Harm}_d^{f,-}(S^2, S^{2n}).$$

For most of the remainder of this paper we will study the properties of $\mathrm{HH}_d^f(S^2, \mathcal{Z}_n)$. First we translate the condition of horizontality into a differential system in projective space. This will be done in the following section.

3. **Translation of the problem**

In [8], a birational map from $\mathbb{CP}^{N_n}$ (where $N_n := n(n+1)/2$) to $\mathcal{Z}_n$ was constructed. Composition with the inverse of this map translates the problem of finding holomorphic and horizontal maps into $\mathcal{Z}_n$ to solving a system of differential equations on $\mathbb{CP}^{N_n}$. We review here some of the main results, referring to [8] for some of the proofs.

Let $\beta = \{E_0, E_1, \dots, E_n, \overline{E}_1, \dots, \overline{E}_n\}$ be a basis of $\mathbb{C}^{2n+1}$ such that

$$(E_0, E_0) = 1, \quad (E_0, E_i) = (E_0, \overline{E}_i) = (E_i, E_j) = (\overline{E}_i, \overline{E}_j) = 0, \quad (E_i, \overline{E}_j) = \delta_{ij}; \tag{4}$$

i.e., β is a unitary basis where $2n$ of the vectors are pairwise conjugate. In this paper, these bases will be called *isotropic* bases.

Let E be the isotropic n-plane spanned by the vectors E_i, $1 \le i \le n$, and let U_E be the open subset of $\mathcal{Z}_n$ consisting of planes whose orthogonal projection over E is onto. Then every $P \in U_E$ can be written as the graph of a map from E to $E^\perp$. Namely, P can be written as the span of n vectors of the form

$$\alpha_i E_0 + E_i + \sum_{k=1}^{n} c_{ik} \overline{E}_k, \quad 1 \le i \le n,$$

where α_i, c_{ik} are complex numbers. Since P is isotropic, we have

$$\alpha_i \alpha_j + c_{ij} + c_{ji} = 0, \ 1 \le i, j \le n,$$

which implies that $c_{ij} = -(\alpha_i \alpha_j + \tau_{ij})/2$ for some $\tau_{ij} \in \mathbb{C}$ satisfying $\tau_{ij} = -\tau_{ji}$.

This defines a bijective, holomorphic map from an affine open subset of $\mathbb{CP}^{N_n}$ onto U_E, which can be extended to a birational map $b_\beta : \mathbb{CP}^{N_n} \to \mathcal{Z}_n$. We use the subscript β to emphasize the dependency on the basis β chosen. Explicitly, using homogeneous coordinates in $\mathbb{CP}^{N_n}$, $b_\beta : \mathbb{CP}^{N_n} \to \mathcal{Z}_n$ is the birational map that takes

$$[s : \alpha_1 : \cdots : \alpha_n : \tau_{12} : \cdots : \tau_{1n} : \tau_{23} : \cdots : \tau_{n-1,n}]$$

to the n-plane generated by the vectors

$$\frac{\alpha_\ell}{s} E_0 + E_\ell - \sum_{k=1}^{n} \left(\frac{\alpha_\ell \alpha_k}{2s^2} + \frac{\tau_{\ell k}}{2s} \right) \overline{E}_k, \quad 1 \le \ell \le n$$

where, by definition, $\tau_{ji} = -\tau_{ij}$ for $1 \leq i \leq j \leq n$. Using matrix notation in the basis β, b_β is therefore the n-plane spanned by the rows of the matrix

(5)
$$\left(\begin{array}{c|cccc|cccc} \frac{\alpha_1}{s} & 1 & 0 & \cdots & 0 & -\frac{\alpha_1^2}{2s^2} & -\frac{\alpha_1\alpha_2}{2s^2} - \frac{\tau_{12}}{2s} & \cdots & -\frac{\alpha_1\alpha_n}{2s^2} - \frac{\tau_{1n}}{2s} \\ \frac{\alpha_2}{s} & 0 & 1 & \cdots & 0 & -\frac{\alpha_2\alpha_1}{2s^2} + \frac{\tau_{12}}{2s} & -\frac{\alpha_2^2}{2s^2} & \cdots & -\frac{\alpha_2\alpha_n}{2s^2} - \frac{\tau_{2n}}{2s} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\alpha_n}{s} & 0 & 0 & \cdots & 1 & -\frac{\alpha_n\alpha_1}{2s^2} + \frac{\tau_{1n}}{2s} & -\frac{\alpha_n\alpha_2}{2s^2} + \frac{\tau_{2n}}{2s} & \cdots & -\frac{\alpha_n^2}{2s^2} \end{array}\right),$$

or in shorter notation,

$$(\alpha/s,\ I_n,\ -(\alpha\,{}^t\alpha + sT)/2s^2), \tag{6}$$

where the superscript ^{t} on the left denotes the transpose, $\alpha = {}^t(\alpha_1, \dots, \alpha_n)$, and T is the skew-symmetric matrix whose ij-entry is τ_{ij}, $1 \leq i, j \leq n$.

This map is just a common way to parametrize $\mathcal{Z}_n$ (see, e.g., [15, p. 235]). An alternative way to see this map appears in [13]: $\mathcal{Z}_n$ is the quotient of $\mathrm{SO}(2n+1, \mathbb{C})$ by the isotropy subgroup $(G_c)_0$ at a point of $\mathcal{Z}_n$. If $(\mathfrak{g}_c)_0$ is the Lie subalgebra of this subgroup, the vector space $\mathrm{so}(2n+1, \mathbb{C})$ can be written as the direct sum of $(\mathfrak{g}_c)_0$ and a nilpotent subalgebra $\mathfrak{n}$ parametrized by the complex quantities α_i, τ_{jk}, with the property that $\xi^3 = 0$ for all $\xi \in \mathfrak{n}$. The map shown above is just the equivalence class in $\mathrm{SO}(2n+1, \mathbb{C})/(G_c)_0 \simeq \mathcal{Z}_n$ of the exponential map restricted to $\mathfrak{n}$.

Given $\psi \in \mathrm{HH}_d^f(S^2, \mathcal{Z}_n)$, we would like to define $\tilde{\psi} := b_\beta^{-1} \circ \psi$, thus translating the problem from maps to $\mathcal{Z}_n$ into maps to $\mathbb{CP}^{N_n}$, as in the following diagram:

$$\begin{tikzcd} & \mathbb{CP}^{N_n} \arrow[d, dashed, "b_\beta"] \\ & \mathcal{Z}_n \arrow[d, "\pi"] \\ S^2 \arrow[uur, "\tilde{\psi}"] \arrow[ur, "\psi"] \arrow[r, "\varphi"] & S^{2n}. \end{tikzcd}$$

Figure 1. Lifts of harmonic maps.

There is, however, an initial problem: b_β^{-1} is only birational, so it is not defined in the whole of $\mathcal{Z}_n$, so it may not be defined in the image of ψ at all. Due to the fact that ψ is linearly full, it turns out that this never happens.

Lemma 1. *If $\psi \in \mathrm{HH}_d^f(S^2, \mathcal{Z}_n)$ and β is an isotropic basis, then the image of ψ is contained in the image of b_β except for finitely many points.*

Proof. See [8, Lemma 2.4]. $\square$

Thus, given $\psi \in \mathrm{HH}_d^f(S^2, \mathcal{Z}_n)$, the birational map $b_\beta^{-1} \circ \psi$ is well defined except at finitely many points, and therefore it can be completed to give a map $\tilde{\psi} : S^2 \to \mathbb{CP}^{N_n}$. It is clear that $\tilde{\psi}$ is holomorphic. The condition of horizontality for ψ translates into a system of differential equations on the functions s, α_i, τ_{jk}. To find this system, let us first recall how to describe $\psi'(z)$ as an element of $\mathrm{Hom}(P, P^\perp)$, where $P = \psi(z)$ and the dashes denote derivatives with respect to the variable z: if c is a curve in $\mathbb{C}^{2n+1}$ such that $c(z) \in \psi(z)$, then $\psi'(z)$ takes the vector $c(z) \in P$ to the perpendicular projection of $c'(z)$ into $P^\perp$.

In our case, ψ can be described as the span of the rows of the matrix $(\alpha/s,\ I_n,\ -(\alpha\, {}^t\!\alpha + sT)/2s^2)$. To simplify the notation, let $\gamma = \alpha/2s$ and $R = T/2s$. Then we can write the curve $c(z)$ as

$$ {}^t\!x \left(2\gamma,\ I_n,\ -(2\gamma\, {}^t\!\gamma + R)\right) $$

where, for our purposes, $x \in \mathbb{C}^n$ can be taken as constant. The linear map $\psi'(z)$ takes this vector to the projection of

$$ {}^t\!x\ \left(2\gamma',\ \mathrm{O}_n,\ -(2\gamma'\, {}^t\!\gamma + 2\gamma\, {}^t\!\gamma' + R')\right) $$

on $P^\perp$, where O_n denotes the zero $n \times n$ matrix. Since $\psi'(z)$ is in the horizontal subspace $H_P\mathcal{Z}_n$ of $T_P\mathcal{Z}_n$ defined by expression (2), we must have

$$ \mathrm{proj}_{P^\perp}\Big({}^t\!x\ \left(2\gamma',\ \mathrm{O}_n,\ -(2\gamma'\, {}^t\!\gamma + 2\gamma\, {}^t\!\gamma' + R')\right)\Big) \perp \bar{P}, $$

where the symbol '$\perp$' means perpendicular with respect to the Hermitian product $\langle \boldsymbol{z}, \boldsymbol{w} \rangle := (\boldsymbol{z}, \overline{\boldsymbol{w}})$ for $\boldsymbol{z}, \boldsymbol{w} \in \mathbb{C}^{2n+1}$, with $(\ ,\ )$ defined by formula (1). This implies that for all $v \in P$,

$$ \Big(\mathrm{proj}_{P^\perp}\Big({}^t\!x\ \left(2\gamma',\ \mathrm{O}_n,\ -(2\gamma'\, {}^t\!\gamma + 2\gamma\, {}^t\!\gamma' + R')\right)\Big),\ \ v\Big) = 0. $$

Since P is isotropic, the projection operator in the last equation is irrelevant. On the other hand, v can be written as ${}^t\!y\ (2\gamma,\ I_n,\ -(2\gamma\, {}^t\!\gamma + R))$ for some $y \in \mathbb{C}^n$, so the condition of horizontality of ψ is equivalent to the condition

$$ \Big({}^t\!x\ \left(2\gamma',\ \mathrm{O}_n,\ -(2\gamma'\, {}^t\!\gamma + 2\gamma\, {}^t\!\gamma' + R')\right),\ {}^t\!y\ \left(2\gamma,\ I_n,\ -(2\gamma\, {}^t\!\gamma + R)\right)\Big) = 0 $$

for all $x, y \in \mathbb{C}^n$. Computing this product—remember that each row vector on the left-hand side of the last expression is written in terms of an isotropic basis β as in (4)—we obtain ${}^t\!x\,(2\gamma'\, {}^t\!\gamma - 2\gamma\, {}^t\!\gamma' - R')\, y = 0$ for all $x, y \in \mathbb{C}^n$. This implies

$$ 2\gamma'\, {}^t\!\gamma - 2\gamma\, {}^t\!\gamma' = R', $$

or in terms of the original functions,

$$ \alpha'\, {}^t\!\alpha - \alpha\, {}^t\!\alpha' = sT' - s'T. $$

Thus we have the following

Proposition 1. *Let* $\psi \in \mathrm{HH}_d^f(S^2, \mathcal{Z}_n)$. *Then* $\tilde{\psi} := \beta^{-1} \circ \psi = [s : \alpha_1 : \cdots : \alpha_n : \tau_{12} : \cdots : \tau_{n-1,n}]$ *satisfies*

$$ {}^t\alpha' \alpha - {}^t\alpha \alpha' = sT' - s'T, $$

or in components

$$ \alpha_i' \alpha_j - \alpha_i \alpha_j' = s\tau_{ij}' - s'\tau_{ij}. \tag{7} $$

In addition, linear fullness of ψ *translates into the condition*

$$ W\left(\left(\frac{\alpha_1}{s}\right)', \ldots, \left(\frac{\alpha_n}{s}\right)'\right) \not\equiv 0, \tag{8} $$

where W *denotes the Wronskian.*

Conversely, if $\tilde{\psi} : S^2 \to \mathbb{CP}^{N_n}$ *is holomorphic and satisfies conditions* (7) *and* (8), *then* $\psi := b_\beta \circ \tilde{\psi}$ *is linearly full, holomorphic and horizontal.*

Proof. For the Wronskian condition, see [8, Prop. 2.2 and Th. 2.3]. Note that for notational convenience, the τ_{ij} used in this paper differ from the ones defined in [8] by a factor of 2 in order to get rid of the annoying factor of 2 appearing in equation 2.17 of [8]. $\square$

Definition 1. Let

$$ \begin{aligned} \mathrm{PD}_d^f(S^2, \mathbb{CP}^{N_n}) = \Big\{ & [s : \alpha_1 : \cdots : \alpha_n : \tau_{12} : \cdots : \tau_{n-1,n}] \ \text{ algebraic maps} \\ & \text{of degree } d \text{ satisfying } \alpha_i'\alpha_j - \alpha_i\alpha_j' = s\tau_{ij}' - s'\tau_{ij}, \\ & \text{and } \left(\frac{\alpha_i}{s}\right)',\ 1 \le i \le n, \ \text{independent} \Big\}. \end{aligned} $$

Then we have well-defined maps

$$ \begin{aligned} \mathcal{B}_\beta : \mathrm{PD}_d^f(S^2, \mathbb{CP}^{N_n}) &\longrightarrow \bigcup_{k=0}^{\infty} \mathrm{HH}_k^f(S^2, \mathcal{Z}_n) \\ \tilde{\psi} &\longrightarrow \psi = b_\beta \circ \tilde{\psi} \end{aligned} $$

and

$$ \begin{aligned} \mathcal{C}_\beta : \mathrm{HH}_d^f(S^2, \mathcal{Z}_n) &\longrightarrow \bigcup_{k=0}^{\infty} \mathrm{PD}_k^f(S^2, \mathbb{CP}^{N_n}) \\ \psi &\longrightarrow \tilde{\psi} = b_\beta^{-1} \circ \psi. \end{aligned} $$

These maps are algebraic, and clearly $\mathcal{B}_\beta \circ \mathcal{C}_\beta$ and $\mathcal{C}_\beta \circ \mathcal{B}_\beta$ are equal to the identity in their respective domains.

We would like these maps to preserve the degree d, but this is not always the case (see [8]). However, it turns out that a slightly weaker result holds. First note that all the functions s, α_i and τ_{jk}, $1 \le i, j, k \le n$ can be considered as coprime polynomials in one complex variable z of maximum degree d. For reasons that will become clear in Section 4, we define the following subvariety of $\mathrm{PD}_d^f(S^2, \mathbb{CP}^{N_n})$.

Definition 2. Let

$$\mathrm{PD}_{d,0}^f(S^2, \mathbb{CP}^{N_n}) = \Big\{[s : \alpha_1 : \cdots : \alpha_n : \tau_{12} : \cdots : \tau_{n-1,n}] \in \mathrm{PD}_d^f(S^2, \mathbb{CP}^{N_n}) \\ \text{with } s = \prod_{\ell=1}^{d}(z - z_\ell),\ z_\ell \in \mathbb{C} \ \text{ distinct, and } \alpha_1(z_\ell) \neq 0,\ \forall \ell\Big\}.$$

Then $\mathrm{PD}_{d,0}^f(S^2, \mathbb{CP}^{N_n})$ is open in $\mathrm{PD}_d^f(S^2, \mathbb{CP}^{N_n})$, and the following important result holds.

THEOREM 2. *Given $\psi \in \mathrm{HH}_d^f(S^2, \mathcal{Z}_n)$, there exists a basis β and an open set $\mathcal{U}_\beta \ni \psi$ such that $\mathcal{B}_\beta : \mathrm{PD}_{d,0}^f(S^2, \mathbb{CP}^{N_n}) \to \mathcal{U}_\beta$ is an algebraic isomorphism.*

The proof of this theorem is complicated and technical, so we will postpone it to Section 7. Now we continue translating the problem of describing harmonic maps from S^2 to S^{2n} into a suitable 'parameter space' via a simple algebraic construction.

4. **Explicit algebraic construction**

In this section we analyze the system of equations (7) given by

$$\alpha_i'\alpha_j - \alpha_i\alpha_j' = s\tau_{ij}' - s'\tau_{ij}, \quad 1 \leq i, j \leq n$$

and the condition (8) given by

$$W\left(\left(\frac{\alpha_1}{s}\right)', \ldots, \left(\frac{\alpha_n}{s}\right)'\right) \not\equiv 0,$$

where the functions s, α_i, τ_{jk} are polynomials in one complex variable z of maximum degree d and without common factors.

This analysis will lead to an explicit algebraic construction of *any* linearly full harmonic map from S^2 to S^{2n}. The approach is quite simple: solve system (7) for the polynomials τ_{ij} and find a smaller more compact condition on the remaining polynomials.

System (7) is equivalent to the conditions

$$\alpha_i'\alpha_j - \alpha_i\alpha_j' = s^2\left(\frac{\tau_{ij}}{s}\right)', \quad 1 \leq i < j \leq n, \tag{9}$$

which is equivalent to

$$\frac{\alpha_i'\alpha_j - \alpha_i\alpha_j'}{s^2} \ \text{ has no residues} \tag{10}$$

and

$$\tau_{ij} = s\int \frac{\alpha_i'\alpha_j - \alpha_i\alpha_j'}{s^2}\,dz \ \text{ is a polynomial of degree} \leq d. \tag{11}$$

It is not easy to translate condition (10) into a simple formula unless we assume something about the zeros of s. This is actually the motivation for the definition of $\mathrm{PD}^f_{d,0}(S^2, \mathbb{CP}^{N_n})$. From this point on *we will assume that s has d distinct complex zeros* located at $\{z_1, \ldots, z_d\}$. This requirement allows us to find a simple formula for the residues of $(\alpha_j \alpha_i' - \alpha_i \alpha_j')/s^2$, as follows.

Lemma 2. *The function $(\alpha_i'\alpha_j - \alpha_i\alpha_j')/s^2$ has residues only at z_ℓ, $1 \leq \ell \leq d$, and*

$$\operatorname*{res}_{z=z_\ell} \frac{\alpha_i'\alpha_j - \alpha_i\alpha_j'}{s^2} = \frac{1}{s'} \left(\frac{\alpha_i'\alpha_j - \alpha_i\alpha_j'}{s'} \right)'_{|z=z_\ell}.$$

Proof. It is clear that the residues of $(\alpha_i'\alpha_j - \alpha_i\alpha_j')/s^2$ are only at the z_ℓ. To find the value of the residue at z_ℓ, use the formula

$$\operatorname*{res}_{z=z_\ell} \frac{\alpha_i'\alpha_j - \alpha_i\alpha_j'}{s^2} = \lim_{z\to z_\ell} \left((z-z_\ell)^2 \, \frac{\alpha_i'\alpha_j - \alpha_i\alpha_j'}{s^2} \right)'.$$

The right-hand side gives

$$\begin{aligned} \lim_{z\to z_\ell} \left((z-z_\ell)^2 \, \frac{\alpha_i'\alpha_j - \alpha_i\alpha_j'}{s^2} \right)' &= \lim_{z\to z_\ell} \left(s' \, \frac{(z-z_\ell)^2}{s^2} \, \frac{\alpha_i'\alpha_j - \alpha_i\alpha_j'}{s'} \right)' \\ &= \lim_{z\to z_\ell} \left(s' \, \frac{(z-z_\ell)^2}{s^2} \right)' \left(\frac{\alpha_i'\alpha_j - \alpha_i\alpha_j'}{s'} \right)_{|z=z_\ell} + \frac{1}{s'} \left(\frac{\alpha_i'\alpha_j - \alpha_i\alpha_j'}{s'} \right)'_{|z=z_\ell}. \end{aligned}$$

It is elementary to show that the first term of the last expression is 0. This proves the lemma. □

The following lemma translates system (7) into a completely algebraic condition.

Lemma 3. *The polynomials s, α_i, τ_{jk} are solutions of the system* (7) *if and only if s divides $W(s, \alpha_i, \alpha_j)$, and τ_{ij} is given by formula* (11), $1 \leq i, j \leq n$.

Proof. Suppose that $\alpha_i'\alpha_j - \alpha_i\alpha_j' = s\tau_{ij}' - s'\tau_{ij}$. Differentiating, we obtain

$$\alpha_i''\alpha_j - \alpha_i\alpha_j'' = s\tau_{ij}'' - s''\tau_{ij}.$$

This implies

$$\begin{aligned} s''(\alpha_i\alpha_j' - \alpha_i'\alpha_j) - s'(\alpha_i\alpha_j'' - \alpha_i''\alpha_j) &= s''(s'\tau_{ij} - s\tau_{ij}') - s'(s''\tau_{ij} - s\tau_{ij}'') \\ &= s(s'\tau_{ij}'' - s''\tau_{ij}'). \end{aligned}$$

Thus,

$$\begin{aligned} W(s, \alpha_i, \alpha_j) &= s(\alpha_i'\alpha_j'' - \alpha_i''\alpha_j') - s'(\alpha_i\alpha_j'' - \alpha_i''\alpha_j) + s''(\alpha_i\alpha_j' - \alpha_i'\alpha_j) \\ &= s(\alpha_i'\alpha_j'' - \alpha_i''\alpha_j' + s'\tau_{ij}'' - s''\tau_{ij}'). \end{aligned}$$

This proves the 'only if' part. Note that it holds in general, i.e., without the restriction that s has simple zeros.

Now we prove the 'if' part. If $s|W(s,\alpha_i,\alpha_j)$, then $s|(s''(\alpha_i'\alpha_j-\alpha_i\alpha_j')-s'(\alpha_i''\alpha_j-\alpha_i\alpha_j''))$. Since

$$s''(\alpha_i'\alpha_j-\alpha_i\alpha_j')-s'(\alpha_i''\alpha_j-\alpha_i\alpha_j'')=-(s')^3\frac{1}{s'}\left(\frac{\alpha_i'\alpha_j-\alpha_i\alpha_j'}{s'}\right)'$$

and since $s'(z_\ell)\neq 0$, $1\le\ell\le d$, we have that the function

$$\frac{1}{s'}\left(\frac{\alpha_i'\alpha_j-\alpha_i\alpha_j'}{s'}\right)'$$

must vanish at the points z_ℓ, $1\le\ell\le d$. Thus, by Lemma 2, $(\alpha_i'\alpha_j-\alpha_i\alpha_j')/s^2$ has no residues at the zeros of s, so it cannot have residues at all. Hence,

$$\int\frac{\alpha_i'\alpha_j-\alpha_i\alpha_j'}{s^2}$$

is a rational function whose poles are simple and at the zeros of s. Furthermore, an elementary calculation shows that the functions

$$\tau_{ij}=t_{ij0}s+s\int\frac{\alpha_i'\alpha_j-\alpha_i\alpha_j'}{s^2},$$

where t_{ij0} are arbitrary integration constants, are polynomials of degree less than or equal to d. □

Now we analyze the condition $s\,|W(s,\alpha_i,\alpha_j)$ further. This condition is satisfied if and only if $W(s,\alpha_i,\alpha_j)=0$ at the points z_ℓ, $1\le\ell\le d$, which happens if and only if the three vectors

$$^t(s(z_\ell),\alpha_1(z_\ell),\dots,\alpha_n(z_\ell)),\quad {}^t(s'(z_\ell),\alpha_1'(z_\ell),\dots,\alpha_n'(z_\ell)),$$
$$^t(s''(z_\ell),\alpha_1''(z_\ell),\dots,\alpha_n''(z_\ell))$$

are linearly dependent for each ℓ. Or equivalently, if and only if for each ℓ, $1\le\ell\le d$, there exist complex numbers p_ℓ,q_ℓ,r_ℓ not all equal to 0 such that

$$p_\ell s(z_\ell)+r_\ell s'(z_\ell)+q_\ell s''(z_\ell)=0, \tag{12}$$

$$p_\ell\alpha(z_\ell)+r_\ell\alpha'(z_\ell)+q_\ell\alpha''(z_\ell)=0, \tag{13}$$

where $\alpha:={}^t(\alpha_1,\alpha_2,\dots,\alpha_n)\in(\mathbb{C}[z]_d)^n$, as before.

Since $s(z_\ell)=0$ and $s'(z_\ell)\neq 0$, equation (12) is equivalent to $r_\ell=-q_\ell s''(z_\ell)/s'(z_\ell)$. Renaming the constants, solving equations (12) and (13) is therefore equivalent to solving

$$p_\ell\alpha(z_\ell)-q_\ell(s''(z_\ell)\alpha'(z_\ell)-s'(z_\ell)\alpha''(z_\ell))=0,\quad 1\le\ell\le d. \tag{14}$$

Since s is monic with distinct roots at $z_1,\dots,z_d\in\mathbb{C}$, the polynomials $\{s,s/(z-z_1),\dots,s/(z-z_d)\}$ form a basis of $\mathbb{C}[z]_d$, and we can write

$$s=\prod_{\ell=1}^{d}(z-z_\ell),\quad \alpha_i=a_{i0}s+\sum_{\ell=1}^{d}a_{i\ell}\frac{s}{z-z_\ell},\quad \tau_{jk}=t_{jk0}s+\sum_{\ell=1}^{d}t_{jk\ell}\frac{s}{z-z_\ell}, \tag{15}$$

for some complex numbers $a_{i\ell}$, $t_{jk\ell}$, with $1\le i,j,k\le n$ and $0\le\ell\le d$.

The idea now is to introduce expressions (15) into equation (14). But first we translate condition (8) in terms of the quantities z_ℓ, $a_{i\ell}$, $t_{jk\ell}$.

LEMMA 4. *Suppose that $d \geq n$. Then*

$$W\left(\left(\frac{\alpha_1}{s}\right)', \dots, \left(\frac{\alpha_n}{s}\right)'\right) \not\equiv 0 \iff \operatorname{Rank}\begin{pmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ a_{12} & a_{22} & \dots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1d} & a_{2d} & \dots & a_{nd} \end{pmatrix} = n.$$

Proof. Since all the functions involved are rational, the vanishing of the Wronskian implies linear dependence [22]. Notice that $\alpha_i(z_\ell) = a_{i\ell}\, s'(z_\ell)$. Thus

$$\begin{aligned}
&W\left(\left(\frac{\alpha_1}{s}\right)', \dots, \left(\frac{\alpha_n}{s}\right)'\right) \equiv 0 \\
&\iff \sum_{i=1}^n b_i \left(\frac{\alpha_i}{s}\right)' = 0 \ \text{ for some } b_1, \dots, b_n \in \mathbb{C} \text{ not all } 0 \\
&\iff \sum_{i=1}^n b_i \alpha_i = cs, \quad b_1, \dots, b_n \in \mathbb{C} \text{ not all } 0, \text{for some } c \in \mathbb{C} \\
&\iff \sum_{i=1}^n b_i \alpha_i(z_\ell) = 0, \quad 1 \leq \ell \leq d, \quad b_1, \dots, b_n \in \mathbb{C} \text{ not all } 0 \\
&\iff \sum_{i=1}^n b_i a_{i\ell} s'(z_\ell) = 0, \quad 1 \leq \ell \leq d, \quad b_1, \dots, b_n \in \mathbb{C} \text{ not all } 0 \\
&\iff \operatorname{Rank}\begin{pmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ a_{12} & a_{22} & \dots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1d} & a_{2d} & \dots & a_{nd} \end{pmatrix} < n.
\end{aligned}$$
□

The following proposition provides an explicit way to construct maps in $\mathrm{PD}^f_{d,0}(S^2, \mathbb{CP}^{N_n})$ in terms of the quantities $s_\ell, a_{i\ell}, t_{jk\ell}$.

PROPOSITION 2. *Let $\tilde{\psi} : S^2 \to \mathbb{CP}^{N_n}$ be a holomorphic curve of degree d, and write*

$$\tilde{\psi} = [s : \alpha_1 : \dots : \alpha_n : \tau_{12} : \dots : \tau_{n-1,n}],$$

where s, α_i, τ_{jk} are coprime polynomials in $z \in \mathbb{C}$. Suppose that s has only simple roots at $z_1, \dots, z_d \in \mathbb{C}$, and write

$$s = \prod_{\ell=1}^d (z - z_\ell), \quad \alpha_i = a_{i0} s + \sum_{\ell=1}^d a_{i\ell} \frac{s}{z - z_\ell}, \quad \tau_{jk} = t_{jk0} s + \sum_{\ell=1}^d t_{jk\ell} \frac{s}{z - z_\ell}.$$

Then $\tilde{\psi} \in \mathrm{PD}^f_{d,0}(S^2, \mathbb{CP}^{N_n})$ if and only if the following conditions hold.

(i) $$a_{i\ell}\sum_{u\neq\ell}\frac{a_{ju}}{(z_\ell-z_u)^2}-a_{j\ell}\sum_{u\neq\ell}\frac{a_{iu}}{(z_\ell-z_u)^2}=0,\quad 1\le i,j\le n.$$

(ii) $$t_{jk\ell}=a_{k0}a_{j\ell}-a_{j0}a_{k\ell}+\sum_{r\neq\ell}\frac{a_{j\ell}a_{kr}-a_{k\ell}a_{jr}}{z_\ell-z_r},\quad 1\le j,k\le n.$$

(iii) $a_{1\ell}\neq 0,\ 1\le\ell\le d.$

(iv) $$\operatorname{Rank}\begin{pmatrix} a_{11} & a_{21} & \dots & a_{n1}\\ a_{12} & a_{22} & \dots & a_{n2}\\ \vdots & \vdots & \ddots & \vdots\\ a_{1d} & a_{2d} & \dots & a_{nd}\end{pmatrix}=n.$$

Proof. In view of Lemma 4 and the discussion leading to equations (12) and (13), $\tilde\psi=[s:\alpha_1:\cdots:\tau_{12}:\cdots]\in \mathrm{PD}^f_{d,0}(S^2,\mathbb{CP}^{N_n})$ if and only if (iii) and (iv) hold, τ_{ij} is given by equation (11), and for each ℓ, $1\le\ell\le d$, there are complex numbers p_ℓ,q_ℓ not both zero such that equation (14) holds, i.e.,

$$p_\ell\alpha(z_\ell)-q_\ell(s''(z_\ell)\alpha'(z_\ell)-s'(z_\ell)\alpha''(z_\ell))=0,\quad 1\le\ell\le d.$$

Note that if $q_\ell=0$ for some ℓ, then $\alpha_1(z_\ell)=0$, which is false by assumption.

Introduce the expressions (15) into equations (11) and (12). Long and straightforward computations then show that equation (11) is equivalent to (ii) and that, for $1\le j\le n$,

$$s''(z_\ell)\alpha_j'(z_\ell)-s'(z_\ell)\alpha_j''(z_\ell)=2\sum_{u\neq\ell}a_{ju}\frac{(s'(z_\ell))^2}{(z_\ell-z_u)^2}+a_{j\ell}\left(\frac{(s''(z_\ell))^2}{2}-\frac{s'(z_\ell)s'''(z_\ell)}{3}\right).$$

Hence equation (14) is equivalent to

(16) $$p_\ell a_{j\ell}s'(z_\ell)-q_\ell\left(2\sum_{u\neq\ell}a_{ju}\frac{(s'(z_\ell))^2}{(z_\ell-z_u)^2}+a_{j\ell}\left(\frac{(s''(z_\ell))^2}{2}-\frac{s'(z_\ell)s'''(z_\ell)}{3}\right)\right)=0.$$

Now we simplify this expression. Since q_ℓ and $s'(z_\ell)$ are nonzero for $1\le\ell\le d$, we can divide equation (16) by $-2q_\ell(s'(z_\ell))^2$ to obtain

$$\lambda_\ell a_{j\ell}+\sum_{u\neq\ell}\frac{a_{ju}}{(z_\ell-z_u)^2}=0,\quad 1\le j\le n,\ 1\le\ell\le d,$$

where λ_ℓ, $1\le\ell\le d$, are suitable constants.

Up to this point every step is reversible, so it only remains to prove that the last equation is equivalent to (i). Thus let $1\le i,j\le n$. Then we have

$$a_{i\ell}\sum_{u\neq\ell}\frac{a_{ju}}{(z_\ell-z_u)^2}=-\lambda_\ell a_{i\ell}a_{j\ell}=a_{j\ell}\sum_{u\neq\ell}\frac{a_{iu}}{(z_\ell-z_u)^2}$$

as desired. Conversely, suppose that

$$a_{i\ell}\sum_{u\neq\ell}\frac{a_{ju}}{(z_\ell-z_u)^2}=a_{j\ell}\sum_{u\neq\ell}\frac{a_{iu}}{(z_\ell-z_u)^2},\quad 1\le i,j\le n.$$

Since $a_{1\ell} \neq 0$ for all ℓ, defining

$$\lambda_\ell = -\frac{1}{a_{1\ell}} \sum_{u\neq\ell} \frac{a_{1u}}{(z_\ell - z_u)^2}$$

we have

$$\lambda_\ell a_{j\ell} + \sum_{u\neq\ell} \frac{a_{ju}}{(z_\ell - z_u)^2} = 0, \quad 1 \leq j \leq n,\ 1 \leq \ell \leq d. \qquad \square$$

Remark. Equation (i) of Proposition 2 can be written, in matrix notation, as

$$(17)\quad \begin{pmatrix} \lambda_1 & \frac{1}{(z_1-z_2)^2} & \cdots & \frac{1}{(z_1-z_d)^2} \\ \frac{1}{(z_2-z_1)^2} & \lambda_2 & \cdots & \frac{1}{(z_2-z_d)^2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{(z_d-z_1)^2} & \frac{1}{(z_d-z_2)^2} & \cdots & \lambda_d \end{pmatrix} \begin{pmatrix} a_{11} & a_{21} & \ldots & a_{n1} \\ a_{12} & a_{22} & \ldots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1d} & a_{2d} & \ldots & a_{nd} \end{pmatrix} = 0.$$

Note that the definition of λ_ℓ is implicit in the previous formula. This essentially describes equation (i) of Proposition 2 as a determinantal variety. We will use this description when we calculate the dimension of $\mathrm{PD}^f_{d,0}(S^2, \mathbb{CP}^{N_n})$.

An alternative and interesting way to think of equation (i) of Proposition 2 is the following. Let $\mathcal{Z} = \{(z_1, \ldots, z_d) \in \mathbb{C}^d : z_i \neq z_j \text{ if } i \neq j\}$. Consider the exterior differential system in $\mathcal{Z}$ generated algebraically by the forms

$$\omega_\ell := \sum_{u\neq\ell} \frac{1}{(z_u - z_\ell)^2}\, dz_u \wedge dz_\ell, \quad 1 \leq \ell \leq d.$$

Then the columns of the matrix $(a_{i\ell})_{i\ell}$ form a basis for the *integral elements* of this exterior differential system. It may be interesting to explore this point of view in order to understand more deeply the structure of $\mathrm{Harm}^f_d(S^2, S^{2n})$.

Note that the construction above provides the following 'recipe': to construct *every* linearly full harmonic map from S^2 to S^{2n} of a given degree d,

1) Find a meromorphic function $g : S^2 \to S^2$ bounded at ∞ (g corresponds to α_1/s above) with only simple poles at $z_1, z_2, \ldots, z_d \in \mathbb{C}$ and with residue $a_{1\ell} \neq 0$ at z_ℓ such that

$$\dim\ker \begin{pmatrix} \lambda_1 & \frac{1}{(z_1-z_2)^2} & \cdots & \frac{1}{(z_1-z_d)^2} \\ \frac{1}{(z_2-z_1)^2} & \lambda_2 & \cdots & \frac{1}{(z_2-z_d)^2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{(z_d-z_1)^2} & \frac{1}{(z_d-z_2)^2} & \cdots & \lambda_d \end{pmatrix} \geq n,$$

where $\lambda_\ell = \sum_{u\neq\ell} \frac{-a_{1u}}{a_{1\ell}} (z_u - z_\ell)^2$.

2) Find vectors ${}^t(a_{i1},\dots,a_{id})\in\mathbb{C}^d$, $2\le i\le n$, in the kernel of the matrix above such that the set $\{{}^t(a_{i1},\dots,a_{id}),\ 1\le i\le n\}$ is linearly independent.
3) Choose arbitrary complex numbers a_{i0}, $1\le i\le n$, and t_{ij0}, $1\le i<j\le n$, and write s, α_i, τ_{jk} as in expression (15), with the $t_{jk\ell}$ given by expression (ii) of Proposition 2.
4) Choose an isotropic basis β, and let $\psi: S^2\to\mathcal{Z}_n$ be given by the span of the rows of the matrix (5).
5) Let $\varphi=\pi\circ\psi$. Then π is harmonic, linearly full and has degree d.

Using this recipe one can construct *completely explicit* examples of harmonic maps. Note that the main difficulty is to find the function g of item 1). In fact, once this function is found, the rest of the procedure is essentially linear. Exact (i.e., not approximate) examples of such functions g can be found for given values of d, n and the distinct complex numbers $z_1,\dots,z_d$. In fact, for $n=2$, $d=3$ it is possible to find a formula that gives all such functions g: if the (3×3 in this case) matrix in condition 1) above has nullity 2, then it is easy to first find the λ_ℓ and then a vector ${}^t(a_{11},\dots,a_{1d})$ with nonzero entries in the kernel of that matrix. Then let $g(z)=\sum_{\ell=1}^d a_{1\ell}/(z-z_\ell)$. This gives a family of functions depending on two nonzero complex parameters c_1,c_2:

$$g(z)=c_1\left(\frac{(z_2-z_3)^2}{z-z_3}-\frac{(z_2-z_1)^2}{z-z_1}\right)+c_2\left(\frac{(z_3-z_2)^2}{z-z_2}-\frac{(z_3-z_1)^2}{z-z_1}\right).$$

For $d=4$, $n=2$, one can obtain a similar, yet more complicated, formula. For higher values of n and d (of course with $d\ge n(n+1)/2$), given distinct numbers $z_1,\dots,z_d$ one strategy is to solve equation (17) (which is quadratic in the λ_ℓ and $a_{j\ell}$) by giving arbitrary values to some of the variables—so some equations become linear—and solving for the others. With this procedure and the help of a computer one can find examples, for instance, when $n=3$, $d\ge 8$ and when $n=4$, $d\ge 12$. The formulas, however, generally involve very large numbers.

We do not know the meaning of the condition on the meromorphic function g in 1) above. It is interesting that much of the information about the space of harmonic maps from S^2 to S^{2n} is encoded in this function. The function g is, in the terminology of the next section, *extendable* in the sense that from it, using the process above, one can generate harmonic maps into higher dimensional spheres.

Now we define the parameter space that we will analyze in the next section.

Definition 3. Let $\mathrm{PSS}_d^n\subset\mathbb{C}^{d+(d+1)n+(d+1)n(n-1)/2}$ be the quasi-affine variety given, in the coordinates

$$(z_1,\cdots z_d,a_{10},\cdots a_{1d},a_{20},\cdots,\cdots,a_{nd},t_{120},\cdots,t_{12d},t_{130},\cdots,\cdots t_{n-1,n,d}),$$

by the conditions

- $z_1, \ldots, z_d$ distinct;
- $a_{1,\ell} \neq 0$ for $1 \le \ell \le d$;
- a_{i0} and t_{jk0} arbitrary for all i, j, k;
- $\begin{pmatrix} \lambda_1 & \frac{1}{(z_1-z_2)^2} & \cdots & \frac{1}{(z_1-z_d)^2} \\ \frac{1}{(z_2-z_1)^2} & \lambda_2 & \cdots & \frac{1}{(z_2-z_d)^2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{(z_d-z_1)^2} & \frac{1}{(z_d-z_2)^2} & \cdots & \lambda_d \end{pmatrix} \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1d} & a_{2d} & \cdots & a_{nd} \end{pmatrix} = 0,\ \lambda_i \in \mathbb{C};$
- $\text{Rank} \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1d} & a_{2d} & \cdots & a_{nd} \end{pmatrix} = n;$
- $t_{jk\ell} = a_{k0}a_{j\ell} - a_{j0}a_{k\ell} + \sum_{r \neq \ell} \dfrac{a_{j\ell}a_{kr} - a_{k\ell}a_{jr}}{z_\ell - z_r}.$

For convenience, we will use the short notation $(\boldsymbol{z}, \boldsymbol{a}, \boldsymbol{t})$ to denote an element of PSS_d^n.

Although PSS_d^n and $\text{PD}_{d,0}^f(S^2, \mathbb{CP}^{N_n})$ are not algebraically equivalent, there is an algebraic map from PSS_d^n to $\text{PD}_{d,0}^f(S^2, \mathbb{CP}^{N_n})$. Thus we have

Theorem 3. *The space* PSS_d^n *has the same dimension as* $\text{PD}_{d,0}^f(S^2, \mathbb{CP}^{N_n})$.

Proof. Proposition 2 implies that the algebraic map

$$\begin{aligned} \text{PSS}_d^n &\longrightarrow \text{PD}_{d,0}^f(S^2, \mathbb{CP}^{N_n}) \\ (\boldsymbol{z}, \boldsymbol{a}, \boldsymbol{t}) &\longrightarrow [s : \alpha_1 : \cdots \alpha_n : \tau_{12} : \cdots : \tau_{n-1,n}], \end{aligned} \tag{18}$$

where $s = \prod_{\ell=1}^{d} (z - z_\ell), \quad \alpha_i = a_{i0}s + \sum_{\ell=1}^{d} a_{i\ell} \dfrac{s}{z - z_\ell}, \quad \tau_{jk} = t_{jk0}s + \sum_{\ell=1}^{d} t_{jk\ell} \dfrac{s}{z - z_\ell},$

is onto and finite-to-one (the inverse image of any point in $\text{PD}_{d,0}^f(S^2, \mathbb{CP}^{N_n})$ is given by all the possible permutations of the 'ℓ' index in expression (18)). Therefore, the two spaces have the same dimension. □

In view of this theorem, it only remains to find the dimension of PSS_d^n in order to prove Bolton and Woodward's conjecture. This is done in the next section.

5. Study of solutions in parameter space

We will first find that the dimension of PSS_d^n is at least $2d + n^2$. The methods used are elementary intersection theory. For convenience, we will use the notation

$$\Sigma_{\boldsymbol{z},\boldsymbol{\lambda}} := \begin{pmatrix} \lambda_1 & \frac{1}{(z_1-z_2)^2} & \cdots & \frac{1}{(z_1-z_d)^2} \\ \frac{1}{(z_2-z_1)^2} & \lambda_2 & \cdots & \frac{1}{(z_2-z_d)^2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{(z_d-z_1)^2} & \frac{1}{(z_d-z_2)^2} & \cdots & \lambda_d \end{pmatrix}.$$

PROPOSITION 3. *The dimension of* PSS_d^n *is at least* $2d+n^2$.

Proof. Let $\mathcal{Z} = \{\boldsymbol{z} = (z_1,\ldots,z_d) \in \mathbb{C}^d : z_i \neq z_j \text{ if } i \neq j\}$, and consider the maps

$$\mathrm{PSS}_d^n \xrightarrow{\nu_1} \mathcal{Z} \times \mathbb{C}^d \xrightarrow{\nu_2} \mathrm{Sym}_d(\mathbb{C})$$

given by

$$\nu_1(\boldsymbol{z},\boldsymbol{a},\boldsymbol{t}) = \left(\boldsymbol{z}, \frac{1}{a_{11}} \sum_{u\neq 1} \frac{-a_{1u}}{(z_u-z_1)^2}, \ldots, \frac{1}{a_{1d}} \sum_{u\neq d} \frac{-a_{1u}}{(z_u-z_d)^2}\right),\ \nu_2(\boldsymbol{z},\boldsymbol{\lambda}) = \Sigma_{\boldsymbol{z},\boldsymbol{\lambda}}.$$

The image of $\nu_2 \circ \nu_1$ is the open subset of the quasi-affine variety

$$\left\{\Sigma_{\boldsymbol{z},\boldsymbol{\lambda}} : (\boldsymbol{z},\boldsymbol{\lambda}) \in \mathcal{Z} \times \mathbb{C}^d\right\} \cap \{M \in \mathrm{Sym}_d(\mathbb{C}) \text{ with nullity at least } n\}$$

consisting of matrices that have an element in the kernel whose components are all different from 0. (Note that this is needed so that $(a_{11},\ldots,a_{1d})$ has this property.) The dimension of the set of matrices of the form $\Sigma_{\boldsymbol{z},\boldsymbol{\lambda}}$ is $2d-1$, and the codimension of the set of symmetric matrices with nullity at least n is $n(n+1)/2$. The set PSS_d^n is not empty, so $\nu_2 \circ \nu_1(\mathrm{PSS}_d^n)$ is not empty, so

$$\dim(\nu_2 \circ \nu_1(\mathrm{PSS}_d^n)) \geq 2d - 1 - \frac{n(n+1)}{2}.$$

The fiber of the map ν_2 over $\Sigma_{\boldsymbol{z},\boldsymbol{\lambda}}$ has dimension 1 everywhere (namely $\nu_2^{-1}(\Sigma_{\boldsymbol{z},\boldsymbol{\lambda}}) = \{(z_1+u,\ldots,z_d+u,\lambda_1,\ldots,\lambda_d) : u \in \mathbb{C}\}$), so

$$\dim(\nu_1(\mathrm{PSS}_d^n) \geq 2d - \frac{n(n+1)}{2}.$$

The fiber of the map ν_1 over any point $(\boldsymbol{z},\boldsymbol{\lambda}) \in \mathcal{Z}\times\mathbb{C}^d$ consists of all tuples $(\boldsymbol{z},\boldsymbol{a},\boldsymbol{t})$ such that the n vectors ${}^t(a_{i1},\ldots,a_{id})$, $1 \leq i \leq n$, span $\ker(\Sigma_{\boldsymbol{z},\boldsymbol{\lambda}})$, a_{i0}, $1 \leq i \leq n$ and t_{ij0}, $1 \leq i < l \leq n$ are arbitrary complex numbers, and the rest of the $t_{ij\ell}$ are given by expression (ii) of Proposition 2. This set is isomorphic to an open subset of $\mathbb{C}^{n^2} \times \mathbb{C}^n \times \mathbb{C}^{n(n-1)/2}$, so it has dimension $n^2+n+n(n-1)/2$. Therefore,

$$\dim(\mathrm{PSS}_d^n) \geq 2d - \frac{n(n+1)}{2} + n^2 + n + n(n-1)/2 = 2d + n^2. \qquad \square$$

The opposite inequality, namely that $\dim(\mathrm{Harm}_d(S^2,S^{2n})) \leq 2d+n^2$, appears at the end of [18]. Before this came to our knowledge, a proof of this fact was found, so we include it here for the sake of completeness. It seems also that this fact was essentially known by Bolton and Woodward, at least for some particular cases.

The methods we use are actually very similar to those in [18] (namely doing induction on n), and the concept of 'k-extendable maps' defined below turns out to be a particular case of the concept of 'maps with k pairs of extra eigenfunctions' used in [18]. It may be, in fact, that these two definitions are equivalent.

Definition 4. An element $(\boldsymbol{z},\boldsymbol{a},\boldsymbol{t})\in \mathrm{PSS}_d^n$ is *'k-extendable'* if $\dim(\ker(\Sigma_{\boldsymbol{z},\boldsymbol{\lambda}})) = n+k$. The set of k-extendable elements in PSS_d^n will be denoted by $\mathcal{E}_{n,d}^{(\geq k)}$.

The term 'k-extendable' comes from the fact that an element of PSS_d^n can be *extended* to an element of PSS_d^{n+k}. A different way to say this is the following. Consider the projections

$$\begin{array}{ccc} (\boldsymbol{z},a_{10},\cdots,a_{n+1,d},\dots,t_{12d},\dots,t_{n,n+1,d}) & \in & \mathrm{PSS}_d^{n+1} \\ \downarrow p_n & & \\ (\boldsymbol{z},a_{10},\cdots,a_{n,d},\dots,t_{12d},\dots,t_{n-1,n,d}) & \in & \mathrm{PSS}_d^{n} \end{array}$$

given by deletion of all the components $a_{n+1\ell}$ and $t_{j,n+1,\ell}$, $1\leq \ell\leq d$, $1\leq j\leq n$. Then $\mathcal{E}_{n,d}^{(\geq k)} = p_{n+k}\circ\cdots\circ p_n(\mathrm{PSS}_d^{n+k})$.

The corresponding objects in $\mathrm{PD}_{d,0}^f(S^2,\mathbb{CP}^{N_n})$ are as follows. The projections p_n correspond to

$$\begin{array}{cc} [s:\alpha_1:\cdots:\alpha_n:\alpha_{n+1}:\tau_{12}:\cdots:\tau_{1,n+1}:\tau_{23}:\cdots:\tau_{n,n+1}] & \in \mathrm{PD}_{d,0}^f(S^2,\mathbb{CP}^{N_{n+1}}) \\ \downarrow & \\ [s:\alpha_1:\cdots:\alpha_n:\tau_{12}:\cdots:\tau_{1,n}:\tau_{23}:\cdots:\tau_{n-1,n}] & \in \mathrm{PD}_{d,0}^f(S^2,\mathbb{CP}^{N_n}) \end{array}$$

given by deletion of all the components that have $n+1$ as subscript. It is well defined since α_1 is not zero at the zeros of s, and s has degree d. Then an element in $\mathrm{PD}_{d,0}^f(S^2,\mathbb{CP}^{N_n})$ is k-extendable if it is in the image of k consecutive projections.

The projection $p_n : \mathrm{PSS}_d^{n+1} \to \mathcal{E}_{n,d}^{(\geq 1)} \subseteq \mathrm{PSS}_d^n$ is onto. Therefore, by computing the dimension of the fibers of p_n and the dimension of $\mathcal{E}_{n,d}^{(\geq 1)}$ we will be able to find a relation between the dimensions of PSS_d^{n+1} and PSS_d^n.

LEMMA 5. *The variety $\mathcal{E}_{n,d}^{(\geq 1)}$ has codimension at least* 1 *in* PSS_d^n.

Proof. Suppose not. Then $\mathcal{E}_{n,d}^{(\geq 1)}$ would contain some open subset of PSS_d^n, and then there would be a point $\phi\in\mathcal{E}_{n,d}^{(\geq 1)}$ such that every continuous curve ϕ^u in PSS_d^n with $\phi^0=\phi$ would be contained in $\mathcal{E}_{n,d}^{(\geq 1)}$ for u in a neighborhood of 0.

We will prove that this is not the case. Let $\phi = (\boldsymbol{z},\boldsymbol{a},\boldsymbol{t})\in\mathcal{E}_{n,d}^{(\geq 1)}$. Then the corresponding matrix $\Sigma_{\boldsymbol{z},\boldsymbol{\lambda}}$ has nullity greater than or equal to $n+1$. Let $q := d-n$ and let ℓ be the rank of $\Sigma_{\boldsymbol{z},\boldsymbol{\lambda}}$, so $\ell < q$. By reordering the indexes we can assume that the first ℓ columns of $\Sigma_{\boldsymbol{z},\boldsymbol{\lambda}}$ are linearly independent. Consider the matrix

$$\Sigma^u_{\boldsymbol{z},\boldsymbol{\lambda}} = \Sigma_{\boldsymbol{z},\boldsymbol{\lambda}} + \left(\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & u\, I_{q-l} & 0 \\ \hline 0 & 0 & 0 \end{array}\right)$$

obtained by adding $u \in \mathbb{C}$ to the diagonal entries of $\Sigma_{\boldsymbol{z},\boldsymbol{\lambda}}$ from $\ell+1$ to q. The nullity of this matrix is at least $d-q=n$. For $1 \leq i,j,k \leq n$, $1 \leq \ell \leq d$, let $(a^u_{i\ell})_{i\ell}$ be a continuous family of matrices of rank n whose columns are in the kernel of $\Sigma^u_{\boldsymbol{z},\boldsymbol{\lambda}}$ and such that $a^0_{i\ell} = a_{i\ell}$, let $a^u_{i0} = a_{i0}$, let $t^u_{ij\ell}$ be given by the formula in Proposition 2(ii) and let $t^u_{jk0} = t_{jk0}$. Then the curve $\phi^u = (z_\ell, a^u_{i\ell}, t^u_{jk}) \in \mathrm{PSS}^n_d$ constructed this way satisfies $\phi^0 = \phi$, and for $u \neq 0$, it is not hard to see that the matrix $\Sigma^u_{\boldsymbol{z},\boldsymbol{\lambda}}$ has nullity n. Therefore, $\phi^u \notin \mathcal{E}^{(\geq 1)}_{n,d}$ for $u \neq 0$. This proves the lemma. □

PROPOSITION 4. *The dimension of* PSS^n_d *is less than or equal to* $2d+n^2$.

Proof. We proceed by induction. The case $n=1$ is straightforward since $\mathrm{PD}^f_{d,0}(S^2,\mathbb{CP}^1)$ is an open subset of the set of meromorphic functions of degree d from S^2 to $\mathbb{CP}^1$, and therefore it has dimension $2d+1$. The cases $n=2$ and $n=3$ were proved in [20], [23], [24], [25] and [9] respectively.

Suppose that the dimension of PSS^n_d is less than or equal to $2d+n^2$. Consider the projection

$$p_n : \mathrm{PSS}^{n+1}_d \to \mathcal{E}^{(\geq 1)}_{n,d}.$$

Note that $p_n(\mathcal{E}^{(\geq k)}_{n+1,d}) = \mathcal{E}^{(\geq k+1)}_{n,d}$. Thus p_n restricts to

$$\mathrm{PSS}^{n+1}_d \setminus \mathcal{E}^{(\geq 1)}_{n+1,d} \to \mathcal{E}^{(\geq 1)}_{n,d} \setminus \mathcal{E}^{(\geq 2)}_{n,d}.$$

This restriction is onto, so we can find the dimension of PSS^{n+1}_d by adding the dimension of $\mathcal{E}^{(\geq 1)}_{n,d} \setminus \mathcal{E}^{(\geq 2)}_{n,d}$ and the dimension of the fiber.

Let $\phi = (\boldsymbol{z},\boldsymbol{a},\boldsymbol{t}) \in \mathcal{E}^{(\geq 1)}_{n,d} \setminus \mathcal{E}^{(\geq 2)}_{n,d} \subset \mathrm{PSS}^n_d$. Then $p_n^{-1}(\phi)$ consists of those $(z_\ell, a_{i\ell}, t_{jk\ell}) \in \mathrm{PSS}^{n+1}_d$, where $1 \leq i,j,k \leq (n+1)$ and $0 \leq \ell \leq d$, obtained by inserting $a_{n+1,\ell}$, $t_{j,n+1,\ell}$, $0 \leq \ell \leq d$, $1 \leq j \leq n$, in the appropriate slots in the original $(\boldsymbol{z},\boldsymbol{a},\boldsymbol{t})$, where

- ${}^t(a_{n+1,1}, a_{n+1,2}, \ldots, a_{n+1,d}) \in \ker(\Sigma_{\boldsymbol{z},\boldsymbol{\lambda}})$.
- $t_{j,n+1,\ell}$, $1 \leq j \leq n$, $1 \leq \ell \leq d$ are given by Proposition 2(ii).
- $a_{n+1,0}$ and $t_{j,n+1,0}$, $1 \leq j \leq n$, are arbitrary complex numbers.

Therefore, $p_n^{-1}(\phi)$ is isomorphic to an open subset of $\ker(\Sigma_{\boldsymbol{z},\boldsymbol{\lambda}})$, which has dimension $n+1$ since $\phi \in \mathcal{E}^{(\geq 1)}_{n,d} \setminus \mathcal{E}^{(\geq 2)}_{n,d}$, times $\mathbb{C} \times \mathbb{C}^n$, and therefore $p_n^{-1}(\phi)$ has dimension $n+1+1+n = 2n+2$.

On the other hand, Lemma 5 implies that the dimension of $p_n(\mathrm{PSS}^{n+1}_d \setminus \mathcal{E}^{(\geq 1)}_{n+1,d}) = \mathcal{E}^{(\geq 1)}_{n,d} \setminus \mathcal{E}^{(\geq 2)}_{n,d}$ is less than or equal to $\dim(\mathrm{PSS}^n_d) - 1$ and that PSS^{n+1}_d and $\mathrm{PSS}^{n+1}_d \setminus \mathcal{E}^{(\geq 1)}_{n+1,d}$ have the same dimension. Putting it all together, we obtain

$$\begin{aligned}\dim(\mathrm{PSS}_d^{n+1}) &= \dim(\mathrm{PSS}_d^{n+1} \setminus \mathcal{E}_{n+1,d}^{(\geq 1)}) \\ &\leq \dim(\text{Image of } p_n) + \dim(\text{Fiber of } p_n) \\ &\leq (\dim(\mathrm{PSS}_d^n) - 1) + 2n + 2 \\ &\leq 2d + n^2 + 2n + 1 \\ &= 2d + (n+1)^2. \qquad \square\end{aligned}$$

THEOREM 4. *The space* $\mathrm{Harm}_d^f(S^2, S^{2n})$ *has pure dimension* $2d + n^2$.

Proof. In each irreducible component, Theorem 3 and Propositions 3 and 4 show that $\mathrm{PD}_{d,0}^f(S^2, \mathbb{CP}^{N_n})$ has pure dimension $2d+n^2$. Then Theorem 2 shows that every element of $\mathrm{HH}_d^f(S^2, \mathcal{Z}_n)$ is contained in an open subset of dimension $2d + n^2$, so $\mathrm{HH}_d^f(S^2, \mathcal{Z}_n)$ has pure dimension $2d + n^2$. Finally, use Theorem 1 to identify $\mathrm{Harm}_d^{f,+}(S^2, S^{2n})$ and $\mathrm{Harm}_d^{f,-}(S^2, S^{2n})$ with $\mathrm{HH}_d^f(S^2, \mathcal{Z}_n)$. $\square$

6. The nonlinearly full case

In order to complete the study of the dimension of the moduli space of harmonic maps from S^2 to S^m, we need to consider the set $\mathrm{Harm}_d(S^2, S^m)$ of *all* (full and nonfull) harmonic maps from S^2 to S^m. Evidently,

$$\mathrm{Harm}_d(S^2, S^m) = \mathrm{Harm}_d^f(S^2, S^m) \sqcup \mathrm{Harm}_d^{nf}(S^2, S^m),$$

where $\mathrm{Harm}_d^{nf}(S^2, S^m)$ denotes the set of nonfull maps. Note that when m is odd, the set $\mathrm{Harm}_d^f(S^2, S^m)$ is empty [6]. So it is clear that the properties of these sets are quite different depending on the parity of m.

The variety of all holomorphic and horizontal maps from S^2 to $\mathcal{Z}_n$ will be denoted $\mathrm{HH}_d(S^2, \mathcal{Z}_n)$, and $\mathrm{HH}_d^{nf}(S^2, \mathcal{Z}_n)$ will denote the subvariety of nonfull maps.

It is convenient to split the space of nonfull harmonic maps into pieces corresponding to the dimension of the sphere where the image of the map lies, as follows.

Definition 5. For $\ell \leq m$, let

$$\mathrm{Harm}_d^{(\leq \ell)}(S^2, S^m) = \Big\{\varphi \in \mathrm{Harm}_d(S^2, S^m) : \varphi(S^2) \subseteq S^m \cap V, \\ \text{for some } V \in \mathrm{Gr}(\ell+1, \mathbb{R}^{m+1})\Big\},$$

and let

$$\mathrm{Harm}_d^{(2k)}(S^2, S^m) = \mathrm{Harm}_d^{(\leq 2k)}(S^2, S^m) \setminus \mathrm{Harm}_d^{(\leq 2(k-1))}(S^2, S^m).$$

The corresponding objects in $\mathrm{HH}_d^{nf}(S^2, \mathcal{Z}_n)$ are

$$\mathrm{HH}_d^{(\leq k)}(S^2, \mathcal{Z}_n) = \big\{\psi \in \mathrm{HH}_d^{nf}(S^2, \mathcal{Z}_n) : \psi(S^2) \subset \mathcal{Z}_n^F \\ \text{for some } F \in \mathrm{Gr}(n-k, \mathbb{C}^{2n+1})\big\}$$

and

$$\mathrm{HH}_d^{(k)}(S^2, \mathcal{Z}_n) = \mathrm{HH}_d^{(\leq k)}(S^2, \mathcal{Z}_n) \setminus \mathrm{HH}_d^{(\leq k-1)}(S^2, \mathcal{Z}_n),$$

where, for $F \in \mathrm{Gr}(n-r, \mathbb{C}^{2n+1})$, $\mathcal{Z}_n^F := \{W \oplus F \in \mathcal{Z}_n : W \in \mathcal{Z}_r((F \oplus \bar{F})^\perp)\}$.

These definitions also appear in [10]. Note that the set $\mathrm{Harm}_d^{(2k)}(S^2, S^m)$ consists of maps that are linearly full in some $2k$-dimensional geodesic subsphere of S^m.

The map $\pi : \mathcal{Z}_n \to S^{2n}$ induces surjective maps

$$\Pi^\pm : \mathrm{HH}_d(S^2, \mathcal{Z}_n) \to \mathrm{Harm}_d^{f,\pm}(S^2, S^{2n}) \sqcup \mathrm{Harm}_d^{nf}(S^2, S^{2n})$$

defined by $\Pi^\pm(\psi) = \pm\pi \circ \psi$. Note that $\Pi^\pm$ map the variety $\mathrm{HH}_d^{(k)}(S^2, \mathcal{Z}_n)$ onto $\mathrm{Harm}_d^{(2k)}(S^2, S^{2n})$ for $k < n$ and take $\mathrm{HH}_d^{(n)}(S^2, \mathcal{Z}_n) \equiv \mathrm{HH}_d^{f}(S^2, \mathcal{Z}_n)$ bijectively onto $\mathrm{Harm}_d^{f,\pm}(S^2, S^{2n})$ (Theorem 1).

Directly from the definition, and using the fact that $\mathrm{Harm}_d^{(\leq 2k+1)}(S^2, S^m) = \mathrm{Harm}_d^{(\leq 2k)}(S^2, S^m)$ [6], we have [10]

$$\mathrm{Harm}_d(S^2, S^{2n}) = \mathrm{Harm}_d^{f,+}(S^2, S^{2n}) \sqcup \mathrm{Harm}_d^{f,-}(S^2, S^{2n}) \\ \sqcup \mathrm{Harm}_d^{(2(n-1))}(S^2, S^{2n}) \sqcup \mathrm{Harm}_d^{(2(n-2))}(S^2, S^{2n}) \sqcup \cdots \sqcup \mathrm{Harm}_d^{(2)}(S^2, S^{2n}).$$

and

$$\mathrm{Harm}_d(S^2, S^{2n+1}) = \mathrm{Harm}_d^{(2n)}(S^2, S^{2n+1}) \\ \sqcup \mathrm{Harm}_d^{(2(n-1))}(S^2, S^{2n+1}) \sqcup \cdots \sqcup \mathrm{Harm}_d^{(2)}(S^2, S^{2n+1}).$$

Now we address what is meant by the *dimension* of these sets. For the linearly full case this was clear: we implicitly assumed that $\mathrm{Harm}_d^{f,\pm}(S^2, S^{2n})$ had the structure induced by the *bijective* maps $\Pi^\pm$. In the nonlinearly full case, if $k < n-1$, the maps $\Pi^\pm : \mathrm{HH}_d^{(k)}(S^2, \mathcal{Z}_n) \to \mathrm{Harm}_d^{(2k)}(S^2, S^{2n})$ are onto but not bijective; furthermore, when m is odd, we need to discuss the structure of $\mathrm{Harm}_d^{(2k)}(S^2, S^m)$ before being able to study its dimension.

A natural topological structure for the sets $\mathrm{Harm}_d(S^2, S^m)$ and $\mathrm{HH}_d(S^2, \mathcal{Z}_n)$ is the compact-open topology. On the other hand, since $\mathcal{Z}_n$ is a subvariety of degree 2 [1] of $\mathbb{P}(\Lambda^n \mathbb{C}^{2n+1}) \simeq \mathbb{CP}^{M_n}$ (where $M_n := \binom{2n+1}{n} - 1$) via the Plücker embedding, the space $\mathrm{HH}_d(S^2, \mathcal{Z}_n)$ is a subvariety of the space of holomorphic maps of degree $2d$ from S^2 to $\mathbb{CP}^{M_n}$, which is regarded as the projectivization of the set of $(M_n + 1)$-tuples of coprime polynomials in z with maximum degree exactly $2d$. Therefore, $\mathrm{HH}_d(S^2, \mathcal{Z}_n)$ is a quasi-projective subvariety of $\mathbb{P}(\mathbb{C}[z]_{2d}^{M_n+1})$ and, in particular, each irreducible component is a topological manifold, maybe with singularities. (In fact this topology coincides with the compact-open topology.) It is shown in [10] that the maps $\Pi^\pm : \mathrm{HH}_d(S^2, \mathcal{Z}_n) \to \mathrm{Harm}_d(S^2, S^{2n})$ are continuous and closed. Further, $\Pi^\pm : \mathrm{HH}_d^{(k)}(S^2, \mathcal{Z}_n) \to \mathrm{Harm}_d^{(2k)}(S^2, S^{2n})$ is a fiber bundle with fiber $\mathrm{SO}(2(n-k), \mathbb{R})/U(n-k)$. For any m, we also have the fiber bundle

$$\begin{aligned}\rho_k : \mathrm{Harm}_d^{(2k)}(S^2, S^m) &\to \mathrm{Gr}(2k+1, \mathbb{R}^{m+1})\\ \varphi &\to (2k+1)\text{-subspace where } \varphi(S^2) \text{ lies},\end{aligned}$$

with fiber $\mathrm{Harm}_d^{f,+}(S^2, S^{2k})$. Hence the sets $\mathrm{Harm}_d^{(2k)}(S^2, S^m)$ are finite dimensional topological manifolds away from possible singular points. This is all the structure we need in order to calculate their dimension.

In addition, the space $\mathrm{HH}_d(S^2, \mathcal{Z}_n)$ has an analytic structure as a subvariety of $\mathbb{P}(\mathbb{C}[z]_{2d}^{M_n+1})$, and so does $\mathrm{Harm}_d(S^2, S^m)$ as a subvariety of the manifold $C^\infty(S^2, S^m)$. With these structures, the maps $\Pi^\pm$ are real analytic submersions, and the bundles described above are real analytic, of course away from the (possible) singular locus. (See [19] for the $n = 2$ case; for $n > 2$ it is similar.)

Further, for $k \geq n-1$, the complex structure of $\mathrm{HH}_d^{(k)}(S^2, \mathcal{Z}_n)$ can be transferred to $\mathrm{Harm}_d^{(2k)}(S^2, S^{2n})$ via the diffeomorphisms $\Pi^\pm$. However, for $k < n-1$, we do not know if the spaces $\mathrm{Harm}_d^{(2k)}(S^2, S^{2n})$ admit a complex structure; we use complex instead of real dimension in the theorem below just to have a more compact statement.

Now we find the dimension of $\mathrm{Harm}_d^{(2k)}(S^2, S^m)$ using the fiber bundle given by ρ_k explained above.

Theorem 5. *When $m = 2n$ is even,*

$$\dim_{\mathbb{C}}(\mathrm{Harm}_d^{(2k)}(S^2, S^{2n})) = 2d + n^2 - (n-k)(n-k-1).$$

In particular,

$$\dim_{\mathbb{C}}(\mathrm{Harm}_d(S^2, S^{2n})) = 2d + n^2.$$

When $m = 2n+1$ is odd,

$$\dim_{\mathbb{R}}(\mathrm{Harm}_d^{(2k)}(S^2, S^{2n+1})) = 2[2d + n^2 - (n-k)(n-k-1)] + 2k + 1.$$

In particular,

$$\dim_{\mathbb{R}}(\mathrm{Harm}_d(S^2, S^{2n+1})) = 2(2d + n^2) + 2n + 1.$$

Proof. The fiber of the surjective map

$$\rho_k : \mathrm{Harm}_d(S^2, S^{2k}) \to \mathrm{Gr}(2k+1, \mathbb{R}^{m+1})$$

is $\mathrm{Harm}_d^{f,+}(S^2, S^{2k}))$ at every point. Therefore

$$\begin{aligned}\dim_{\mathbb{R}}(\mathrm{Harm}_d^{(2k)}(S^2, S^m)) &= \dim_{\mathbb{R}}(\mathrm{Harm}_d^{f,+}(S^2, S^{2k})) + \dim_{\mathbb{R}}(\mathrm{Gr}(2k+1, \mathbb{R}^{m+1})) \\ &= 2(2d + k^2) + (2k+1)(m-2k),\end{aligned}$$

so for m even,

$$\dim_{\mathbb{R}}(\mathrm{Harm}_d^{(2k)}(S^2, S^{2n})) = 2[2d + n^2 - (n-k)(n-k-1)],$$

and therefore

$$\dim_{\mathbb{C}}(\mathrm{Harm}_d^{(2k)}(S^2, S^{2n})) = 2d + n^2 - (n-k)(n-k-1),$$

and for m odd,

$$\dim_{\mathbb{R}}(\mathrm{Harm}_d^{(2k)}(S^2, S^{2n+1})) = 2[2d + n^2 - (n-k)(n-k-1)] + 2k + 1. \quad \square$$

It is an interesting fact that $\dim_{\mathbb{C}}(\mathrm{Harm}_d^{nf}(S^2, S^{2n}))$ is $2d + n^2$, i.e., equal to $\dim_{\mathbb{C}}(\mathrm{Harm}_d^{f}(S^2, S^{2n}))$.

7. Proof of Theorem 2

We fix the following notation.

- If S is a subset of a vector space, we will use $\langle S \rangle$ to denote the span of S.
- For an isotropic basis $\beta = \{E_0, E_1, \dots, E_n, \bar{E}_1, \dots, \bar{E}_n\}$ of $\mathbb{C}^{2n+1}$, the subspaces $\langle E_1, \dots, E_n \rangle$ and $\langle \bar{E}_1, \dots, \bar{E}_n \rangle$ will be denoted by E_β and $\bar{E}_\beta$, respectively.
- Given two subspaces V_1, V_2 of $\mathbb{C}^N$, we will denote ('ISO' refers to 'isotropic')
$$\mathrm{Hom}_{\mathrm{ISO}}(V_1, V_2) := \{L \in \mathrm{Hom}(V_1, V_2) : (L(v_1), v_1) = 0 \ \ \forall v_1 \in V_1\}.$$
- Let $\psi \in \mathrm{HH}_d^f(S^2, \mathcal{Z}_n)$, and let β be an isotropic basis of $\mathbb{C}^{2n+1}$. The components of $\mathcal{C}_\beta(\psi)$ corresponding to s and α_1 in homogeneous coordinates (i.e., the first and second components) will be denoted by $(\mathcal{C}_\beta(\psi))_s$ and $(\mathcal{C}_\beta(\psi))_{\alpha_1}$, respectively. In other words, writing $\mathcal{C}_\beta(\psi) = [s : \alpha_1 : \cdots : \alpha_n : \tau_{12} : \cdots : \tau_{n-1,n}]$, define
$$(\mathcal{C}_\beta(\psi))_s := s \quad \text{and} \quad (\mathcal{C}_\beta(\psi))_{\alpha_1} := \alpha_1.$$

The proof of Theorem 2 goes as follows:

(i) Show that $\mathcal{B}_\beta(\mathrm{PD}_{d,0}^f(S^2, \mathbb{CP}^{N_n})) \subset \mathrm{HH}_d^f(S^2, \mathcal{Z}_n)$. This is done in Proposition 5.

(ii) Show that $\mathcal{B}_\beta(\mathrm{PD}_{d,0}^f(S^2, \mathbb{CP}^{N_n}))$ is open in $\mathrm{HH}_d^f(S^2, \mathcal{Z}_n)$ and that every $\psi \in \mathrm{HH}_d^f(S^2, \mathcal{Z}_n)$ is in the image of $\mathcal{B}_\beta$ for some basis β. This is done in Lemma 9, after preparing it with several lemmas.

Since $\mathcal{B}_\beta$ is algebraic and has algebraic inverse $\mathcal{C}_\beta$, this will prove the theorem.

The following proposition asserts that $\mathcal{B}_\beta(\tilde{\psi})$ has the same degree as $\tilde{\psi}$ when $\tilde{\psi} \in \mathrm{PD}_{d,0}^f(S^2, \mathbb{CP}^{N_n})$. Its proof is unexpectedly simple.

Proposition 5. *Let β be an isotropic basis of $\mathbb{C}^{N_n}$.*
(i) *If $\tilde{\psi} \in \mathrm{PD}_d^f(S^2, \mathbb{CP}^{N_n})$, then the degree of $\mathcal{B}_\beta(\tilde{\psi})$ is at least d.*
(ii) $\mathcal{B}_\beta(\mathrm{PD}_{d,0}^f(S^2, \mathbb{CP}^{N_n})) \subset \mathrm{HH}_d^f(S^2, \mathcal{Z}_n)$.

Proof. Let $\tilde{\psi} = [s : \alpha_1 : \cdots : \alpha_n : \tau_{12} : \cdots : \tau_{n-1,n}] \in \mathrm{PD}_d^f(S^2, \mathbb{CP}^{N_n})$, and let $\psi = \mathcal{B}(\tilde{\psi}) = b_\beta \circ \tilde{\psi}$. We need to prove that the degree of ψ is d, or equivalently, composing with the Plücker embedding, that the degree of $\mathrm{Pl} \circ \psi$ is $2d$.

The map $\mathrm{Pl} \circ \psi = \mathrm{Pl} \circ b_\beta \circ \tilde{\psi} : S^2 \to \mathbb{P}\Lambda^n \mathbb{C}^{2n+1}$ is given, in a suitable basis of $\Lambda^n \mathbb{C}^{2n+1}$, by the projectivization of the vector whose components are the $n \times n$ minors of the $n \times (2n+1)$ matrix

$$(\alpha/s,\ I_n,\ -(\alpha\, {}^t\!\alpha + sT)/2s^2)$$

(notation as in expressions (5) and (6)). To study the degree of $\mathrm{Pl} \circ \psi$, we will multiply the components of the vector formed by the n by n minors of this matrix by a suitable homogenizing factor h so as to obtain a vector whose components are polynomials without common factors. Since all the minors of the matrix above are homogeneous rational functions of degree 0, and since one of them is 1, the degree of $\mathrm{Pl} \circ \psi$ must be equal to the degree of the homogenizing factor h.

Note that the expressions α_i^2/s^2, $1 \le i \le n$, and τ_{jk}^2/s^2, $1 \le j,k \le n$, appear as minors of the matrix $(\alpha/s,\ I_n,\ -(\alpha\, {}^t\!\alpha + sT)/2s^2)$. Since s, α_i and τ_{jk} cannot vanish simultaneously, h must be a multiple of s^2. This implies that the degree of $\mathcal{B}_\beta(\tilde{\psi})$ is at least d, proving (i).

If $\psi \in \mathrm{PD}_{d,0}^f(S^2, \mathbb{CP}^{N_n})$, we can use the assumption that s has simple zeros $z_1, z_2, \ldots, z_d$ and $\alpha_1(z_m) \ne 0$, $1 \le m \le d$. Since the denominators of all the minors divide a power of s, the homogenizing factor h can only have roots at the zeros $z_1, z_2, \ldots, z_d$ of s. Performing row operations on the matrix $(\alpha/s, I_n, -(\alpha\, {}^t\!\alpha + sT)/2s^2)$ does not alter the map $\mathrm{Pl} \circ \psi$. So to row i, $2 \le i \le n$, let us add the first row times the factor

$$\frac{-\alpha_1 \alpha_i + s\tau_{1i}}{\alpha_1^2}$$

so as to obtain zeros in the $n+2$ column, rows 2 to n.

Then, for $2 \le i \le n$, the $(i,1)$-entry of the new matrix becomes τ_{1i}/α_1, the $(i,2)$-entry becomes

$$\frac{-\alpha_1 \alpha_i + s\tau_{1i}}{\alpha_1^2},$$

the $(i, n+2)$-entry becomes 0 and the (i,j)-entry, $n+3 \le j \le 2n+1$, becomes

$$\frac{\alpha_1 \tau_{ji} + \alpha_i \tau_{1j} + \alpha_j \tau_{i1}}{2s\alpha_1} - \frac{\tau_{1i}\tau_{1j}}{2\alpha_1^2}.$$

Here is where the magic appears: since $\tilde{\psi} \in \mathrm{PD}^f_{d,0}(S^2, \mathbb{CP}^{N_n})$, the equations

$$\begin{cases} \alpha_1'\alpha_j - \alpha_1\alpha_j' &= s\tau_{1j}' - s'\tau_{1j}, \\ \alpha_j'\alpha_i - \alpha_j\alpha_i' &= s\tau_{ji}' - s'\tau_{ji}, \\ \alpha_i'\alpha_1 - \alpha_i\alpha_1' &= s\tau_{i1}' - s'\tau_{i1} \end{cases}$$

must hold for $1 < i, j < n$. (When $i = j$, define $\tau_{ij} = 0$.) Multiplying the first equation by α_i, the second by α_1, the third by α_j, and adding, the left-hand side cancels, and we obtain the equation

$$s'(\alpha_1\tau_{ji} + \alpha_i\tau_{1j} + \alpha_j\tau_{i1}) = s(\alpha_1\tau_{ji}' + \alpha_i\tau_{1j}' + \alpha_j\tau_{i1}').$$

Therefore, the matrix obtained after doing these row operations and multiplying the first row by $2s^2$ is given by

(19)
$$\left(\begin{array}{c|c:ccc|c:c} 2s\alpha_1 & 2s^2 & 0 & \cdots & 0 & -\alpha_1^2 & (-\alpha_1\alpha_j - s\tau_{1j})_{1j} \\ \hdashline \left(\frac{\tau_{1i}}{\alpha_1}\right)_{i1} & \left(\frac{-\alpha_1\alpha_i + s\tau_{1i}}{\alpha_1^2}\right)_{i2} & & I_{n-1} & & 0 & \left(\frac{\alpha_1\tau_{ji}' + \alpha_i\tau_{1j}' + \alpha_j\tau_{i1}'}{2s'\alpha_1} - \frac{\tau_{1i}\tau_{1j}}{2\alpha_1^2}\right)_{ij} \end{array}\right).$$

As before, multiply the components of the vector formed by the $n \times n$ minors of this matrix by a suitable homogenizing factor $\hat{h}$ to obtain a vector whose components are polynomials without common factors. The same argument as before shows that $\hat{h}$ can only vanish at the zeros of s' and α_1, which by hypothesis are different from those of s.

But then the component corresponding to E_β (i.e., the minor of columns 2 to $n+1$) is, on the one hand equal to the homogenizing factor h (which has zeros only at the zeros of s), and on the other hand equal to $2s^2\hat{h}$. Since this last expression only vanishes to order 2 at the zeros of s, the homogenizing factor h must be a constant multiple of s^2. Since s has degree d, h has degree $2d$, and we conclude that the degree of $\mathrm{Pl} \circ \psi$ is $2d$, which implies that the degree of $\psi = \mathcal{B}_\beta(\tilde{\psi})$ is d. $\square$

Now we must prove two things: first, that $\mathcal{B}_\beta(\mathrm{PD}^f_{d,0}(S^2, \mathbb{CP}^{N_n}))$ is open in $\mathrm{HH}^f_d(S^2, \mathcal{Z}_n)$, and second, that every $\psi \in \mathrm{HH}^f_d(S^2, \mathcal{Z}_n)$ is in the image of $\mathcal{B}_\beta(\mathrm{PD}^f_{d,0}(S^2, \mathbb{CP}^{N_n}))$ for some β. What we will actually do is, given $\psi \in \mathrm{HH}^f_d(S^2, \mathcal{Z}_n)$, describe an open subset of $\mathrm{HH}^f_d(S^2, \mathcal{Z}_n)$ containing ψ (essentially the set of those $\widehat{\psi}$ so that $\dim(\widehat{\psi}(z) \cap \bar{E}_\beta) \leq 1\ \forall z$) and show that it equals $\mathcal{B}_\beta(\mathrm{PD}^f_{d,0}(S^2, \mathbb{CP}^{N_n}))$.

First we need some tools. Note that the 'trouble' for $\tilde{\psi} = \mathcal{B}_\beta(\psi)$ only happens at the zeros of its first component s. These points correspond, for the map ψ, to incidence with the n-plane $\bar{E}_\beta$. This motivates the following definition.

Definition 6. For $F \in \mathcal{Z}_n$, let

$$\begin{aligned} I_F^{(\geq k)} &= \{P \in \mathcal{Z}_n : \dim(F \cap P) \geq k\}, \\ I_F^{(k)} &= \{P \in \mathcal{Z}_n : \dim(F \cap P) = k\}. \end{aligned}$$

The relationship between the vanishing of s and the variety $I_{\bar{E}_\beta}^{(\geq 1)}$ is clarified in the following lemma.

Lemma 6.

(i) *Let* $\psi \in \mathrm{HH}_d^f(S^2, \mathcal{Z}_n)$, *and let* $\beta = \{E_0, E_1, \dots, E_n, \bar{E}_1, \dots, \bar{E}_n\}$ *be an isotropic basis of* $\mathbb{C}^{N_n}$. *Write* $\mathcal{C}_\beta(\psi) = [s : \alpha_1 : \cdots : \alpha_n : \tau_{12} : \cdots : \tau_{n-1,n}]$. *Then*

$$\psi(z) \in I_{\bar{E}_\beta}^{(\geq 1)} \implies s(z) = 0.$$

(ii) *Let* $\tilde{\psi} = [s : \alpha_1 : \cdots : \alpha_n : \tau_{12} : \cdots : \tau_{n-1,n}] \in \mathrm{PD}_d^f(S^2, \mathbb{CP}^{N_n})$. *Then, if* s *has only simple zeros* (*in particular if* $\tilde{\psi} \in \mathrm{PD}_{d,0}^f(S^2, \mathbb{CP}^{N_n})$),

$$s(z) = 0 \implies (\mathcal{B}_\beta(\tilde{\psi}))(z) \in I_{\bar{E}_\beta}^{(1)}.$$

Proof. (i) If $s(z) \neq 0$, then $\psi(z)$ is the span of the vectors

$$\frac{\alpha_\ell}{s} E_0 + E_\ell - \sum_{k=1}^{n} \left(\frac{\alpha_\ell \alpha_k}{2s^2} + \frac{\tau_{\ell k}}{2s} \right) \overline{E}_k, \quad 1 \leq \ell \leq n,$$

with s, α_i and τ_{jk} evaluated at z. This implies $\psi(z) \cap \bar{E}_\beta = \{0\}$, and therefore $\psi(z) \notin I_{\bar{E}_\beta}^{(\geq 1)}$.

(ii) As in the proof of Proposition 5, if $s(z) = 0$ and $\alpha_1(z) \neq 0$, then $\mathcal{B}_\beta(\tilde{\psi})$ can be expressed as the subspace spanned by the rows of the matrix (19):

$$\left(\begin{array}{c|c:c|c:c} 0 & 0 & 0 \ \cdots \ 0 & -\alpha_1^2 & (-\alpha_1 \alpha_j)_{1j} \\ \hdashline \left(\frac{\tau_{1i}}{\alpha_1}\right)_{i1} & \left(\frac{-\alpha_i}{\alpha_1}\right)_{i2} & I_{n-1} & 0 & \left(\frac{\alpha_1 \tau'_{ji} + \alpha_i \tau'_{1j} + \alpha_j \tau'_{i1}}{2s'\alpha_1} - \frac{\tau_{1i}\tau_{1j}}{2\alpha_1^2} \right)_{ij} \end{array} \right),$$

where all the functions involved are evaluated at z. (Note that since s has only simple roots, $s'(z)$ does not vanish.) This implies that $(\mathcal{B}_\beta(\tilde{\psi}))(z) \cap \bar{E}_\beta = \langle -\alpha_1(\alpha_1 \bar{E}_1 + \cdots + \alpha_n \bar{E}_n) \rangle$, and therefore $(\mathcal{B}_\beta(\tilde{\psi}))(z) \in I_{\bar{E}_\beta}^{(1)}$.

If $s(z) = 0$ and $\alpha_1(z) = 0$, note that since $\tilde{\psi}$ satisfies equation (7), and since s has only simple roots, not all the α_i, $1 \leq i \leq n$, can vanish simultaneously at any of the zeros of s, for otherwise equation (7) would imply that the τ_{ij} also vanish at that point, which is impossible. Therefore, we can proceed as in the proof of Proposition 5 to obtain an expression similar to (19) using α_i instead of α_1 and then use the same argument as in the last paragraph. $\square$

Now the idea is the following. The variety $I^{(\geq 1)}_{\bar{E}_\beta}$ generates the codimension 2 homology of $\mathcal{Z}_n$. Therefore, a map $\psi \in \mathrm{HH}^f_d(S^2, \mathcal{Z}_n)$ will intersect $I^{(\geq 1)}_{\bar{E}_\beta}$ at exactly d points, counted with multiplicity. If we are able to choose a basis β so that ψ intersects $I^{(\geq 1)}_{\bar{E}_\beta}$ transversely and only in $I^{(1)}_{\bar{E}_\beta}$, it will do so at d distinct points, and therefore s will have d distinct roots; with some additional light conditions on β, this will imply that $\mathcal{C}_\beta(\psi)$ lies in $\mathrm{PD}^f_{d,0}(S^2, \mathbb{CP}^{N_n})$.

It is for this reason that we now study some properties of the sets $I^{(\geq k)}_{\bar{E}_\beta}$. The following is well known.

LEMMA 7. *Let $F \in \mathcal{Z}_n$. The set $I^{(\geq k)}_F$ is a subvariety of $\mathcal{Z}_n$ of codimension $k(k+1)/2$ and $I^{(k)}_F$ is an open subvariety of $I^{(\geq k)}_F$.*

Proof. Let $\{F_1, F_2, \dots, F_n\}$ be a basis of F. Then $I^{(\geq k)}_F$ can be expressed as

$$I^{(\geq k)}_F = \{P \in \mathcal{Z}_n : \mathrm{Pl}(P) \wedge F_{i_1} \wedge F_{i_2} \wedge \cdots \wedge F_{i_{n-k+1}} = 0,\ 1 \leq i_1, \dots, i_{n-k+1} \leq n\}.$$

Thus $I^{(\geq k)}_F$ is a projective variety. This description also shows that, for all k, $I^{(\geq k+1)}_F$ is a proper closed subvariety of $I^{(\geq k)}_F$. Hence, since $I^{(\geq k)}_F = I^{(k)}_F \cup I^{(\geq k+1)}_F$, $I^{(k)}_F$ must be an open subvariety of $I^{(\geq k)}_F$.

To compute the dimension of $I^{(\geq k)}_F$, consider the incidence correspondence

$$\Psi_k = \{(\Gamma, P) : \Gamma \subset P\} \subset \mathrm{Gr}(k, F) \times \mathcal{Z}_n.$$

The projection into the first factor is onto with the fiber over $\Gamma \in \mathrm{Gr}(k, F)$ being the set of pairs $(\Gamma, \Gamma \oplus W)$ where W is an isotropic $(n-k)$-plane lying in $(\Gamma \oplus \bar{\Gamma})^\perp$. This set is isomorphic to $\mathcal{Z}_{n-k}$ and hence irreducible of dimension $(n-k)(n-k+1)/2$. Hence Ψ_k is irreducible of dimension

$$k(n-k) + (n-k)(n-k+1)/2 = n(n+1)/2 - k(k+1)/2$$

(see for example [14, Lecture 11]). Since the variety $I^{(\geq k)}_F$ is the image of Ψ_k under the second projection, which is one-to-one except in the closed subset $I^{(\geq k+1)}_F \subset I^{(\geq k)}_F$, we have that $I^{(\geq k)}_F$ is also irreducible with codimension $k(k+1)/2$. □

The following lemma gives a criterion for transversality. It is also proved that $I^{(\geq 1)}_F$ generates the codimension 2 homology of $\mathcal{Z}_n$.

LEMMA 8. *Let $F \in \mathcal{Z}_n$ and let $P \in I^{(1)}_F$. Write $P = \langle P_1 \rangle \oplus P'$, where $P_1 \in P \cap F$ and $P' \perp P_1$. Let $0 \neq P_0 \in (P \oplus \bar{P})^\perp$ with $P_0 = \bar{P}_0$. Then*

(i) $P^\perp = \langle P_0 \rangle \oplus \langle \bar{P}_1 \rangle \oplus (P + F) \cap P^\perp$.

(ii) $T_P\mathcal{Z}_n = \mathrm{Hom}(\langle P_1 \rangle, \langle P_0 \rangle) \oplus \mathrm{Hom}(P', \langle P_0 \rangle) \oplus \mathrm{Hom}_{\mathrm{ISO}}(P, \langle \bar{P}_1 \rangle \oplus (P+F) \cap P^\perp)$. (Recall that $\mathrm{Hom}_{\mathrm{ISO}}(V_1, V_2)$ was defined at the beginning of Section 7.)

(iii) $T_P I_F^{(1)} = \mathrm{Hom}(P', \langle P_0\rangle) \oplus \mathrm{Hom}_{\mathrm{ISO}}(P,\ \langle \bar{P}_1\rangle \oplus (P+F)\cap P^\perp)$. *Therefore,* $I_F^{(1)}$ *is regular.*

(iv) (*Criterion for transversality*). *Let* $\gamma : S^2 \to \mathcal{Z}_n$ *be a holomorphic curve such that* $P = \gamma(z) \in I_F^{(1)}$ *for some* $z \in S^2$. *Let* $0 \neq P_1 \in P \cap F$, *and suppose that* $\gamma'(z)(P_1) \in (P \oplus \bar{P})^\perp$. *Then* γ *intersects* $I_F^{(1)}$ *transversely at* P *if and only if* $\gamma'(z)(P_1) \neq 0$.

(v) *The variety* $I_F^{(\geq 1)}$ *intersects a generator of* $H_2(\mathcal{Z}_n, \mathbb{Z}) = \mathbb{Z}$ [21] *transversely in a single point. Therefore, a curve in* $\mathcal{Z}_n$ *that intersects* $I_F^{(\geq 1)}$ *transversely and only at regular points of* $I_F^{(\geq 1)}$ *has degree equal to the number of points of intersection.*

Proof. (i) It is clear that $P^\perp \supseteq \langle P_0\rangle + \langle \bar{P}_1\rangle + (P+F)\cap P^\perp$. In addition, since $P+F$ has dimension $2n-1$, $(P+F)\cap P^\perp$ must have dimension at least $n-1$, so we only have to prove linear independence.

Suppose that $a_0P_0 + a_1\bar{P}_1 + v_P + v_F = 0$, where $a_0, a_1 \in \mathbb{C}$, $v_P \in P$, $v_F \in F$, and $v_P + v_F \in (P+F)\cap P^\perp$. Since $(P_1, P_0) = (P_1, v_P) = (P_1, v_F) = 0$, we must have $0 = (P_1, a_0P_0 + a_1\bar{P}_1 + v_P + v_F) = a_1\,(P_1, \bar{P}_1)$, so $a_1 = 0$.

Therefore, $a_0P_0 + v_P = -v_F$, so since $(P_0, v_P) = (v_P, v_P) = (v_F, v_F) = 0$, we must have $a_0^2\,(P_0, P_0) = (a_0P_0 + v_P, a_0P_0 + v_P) = (v_F, v_F) = 0$, so $a_0 = 0$ and therefore $v_P + v_F = 0$.

(ii) Let $L \in \mathrm{Hom}(P, P^\perp)$. Then L can be written as $L = L_0 + L_1$, where $L_0 \in \mathrm{Hom}(P, \langle P_0\rangle)$ and $L_1 \in \mathrm{Hom}(P, \langle \bar{P}_1\rangle \oplus (P+F)\cap P^\perp)$.

The map $L_0 + L_1$ is in $T_P\mathcal{Z}_n$ if and only if, for all $u \in P$, $(L_0(u) + L_1(u), u) = 0$. Since $(L_0(u), u) = 0$ for all $u \in P$, $(L_0(u) + L_1(u), u) = 0$ if and only if $(L_1(u), u) = 0$, i.e., if and only if $L_0 \in \mathrm{Hom}(P, \langle P_0\rangle) = \mathrm{Hom}(\langle P_1\rangle, \langle P_0\rangle) \oplus \mathrm{Hom}(P', \langle P_0\rangle)$, $L_1 \in \mathrm{Hom}_{\mathrm{ISO}}(P, \langle \bar{P}_1\rangle \oplus (P+F)\cap P^\perp)$.

(iii) The geometric idea is the following. Given a curve in $I_F^{(\geq 1)}$, any curve in $\mathbb{C}^{2n+1}$ tracing the intersection between the given curve and F must have derivative contained in F, which essentially implies the claim.

More algebraically, the variety $I_F^{(\geq 1)} \subset \mathcal{Z}_n$ is the zero locus of $f(Q) = \mathrm{Pl}(Q) \wedge \mathrm{Pl}(F)$. Thus $T_P I_F^{(1)}$ consists of those $L \in T_P\mathcal{Z}_n$ such that $df_P(L) = 0$. We need to prove that $L(P_1) \in (F+P)\cap P^\perp$.

So let $L \in T_P\mathcal{Z}_n$ and let $\{P_2, \ldots, P_n\}$ be a basis of P'. Take curves $c_i(t)$ in $\mathbb{C}^{2n+1}$ such that $c_i(0) = P_i$ and $c_i'(0) = L(P_i) \in P^\perp$, $1 \leq i \leq n$. Then, since $P_1 \in F$,

$$\begin{aligned} df_P(L) &= \frac{d}{dt}_{|t=0} c_1(t) \wedge \cdots \wedge c_n(t) \wedge \mathrm{Pl}(F) \\ &= c_1'(0) \wedge P_2 \wedge \cdots \wedge P_n \wedge \mathrm{Pl}(F) \\ &= L(P_1) \wedge P_2 \wedge \cdots \wedge P_n \wedge \mathrm{Pl}(F) \end{aligned}$$

which is 0 if and only if $L(P_1) \in (F+P)\cap P^\perp$. This proves the claim.

(iv) Since $T_P I_F^{(1)}$ has codimension 1 in $T_P \mathcal{Z}_n$, proving transversality is equivalent to showing that $0 \neq \gamma'(z) \notin T_P I_F^{(1)}$. Using the hypotheses on γ and part (iii), $\gamma'(z) \in T_P I_F^{(1)}$ if and only if $\gamma'(z)(P_1) \in ((P+F) \cap P^{\perp}) \cap (P \oplus \bar{P})^{\perp} = \{0\}$.

(v) The curve $G : \mathbb{CP}^1 \to \mathcal{Z}_n$ given by

$$G([w_0 : w_1]) = \left\langle w_0 w_1 P_0 + w_0^2 P_1 - \frac{w_1^2}{2} \bar{P}_1 \right\rangle \oplus \mathrm{Pl}(P')$$

is a generator of $H_2(\mathcal{Z}_n, \mathbb{Z})$ [21]. This curve intersects $I_F^{(\geq 1)}$ only at P, i.e., when $w_0 = 1$, $w_1 = 0$.

Let $g(z) \in G([1 : z])$ be the curve $g(z) = zP_0 + P_1 - z^2 \bar{P}_1$. Then $g(0) = P_1$ and $g'(0) = P_0 \neq 0$, which implies that $0 \neq G'([1:0])(P_1) = P_0 \in (P \oplus \bar{P})^{\perp}$, and therefore, by (iv), the curve G intersects $T_P I_F^{(1)}$ transversely at a single point. □

Suppose that $\psi \in \mathrm{HH}_d^f(S^2, \mathcal{Z}_n)$. In view of the previous lemma, our goal is now to find a basis β such that ψ intersects $\bar{E}_\beta = \langle \bar{E}_1, \ldots, \bar{E}_n \rangle$ transversely at d points and with some additional properties. This is what we do in the following lemma.

LEMMA 9. *Given $\psi \in \mathrm{HH}_d^f(S^2, \mathcal{Z}_n)$ there exists a basis β such that the image of ψ intersects $I_{\bar{E}_\beta}^{(\geq 1)}$ only at $I_{\bar{E}_\beta}^{(1)}$ and transversely, such that $(\mathcal{C}_\beta(\psi))_{\alpha_1}$ is not* 0 *at the points of intersection, and such that $(\mathcal{C}_\beta(\psi))_s$ is not* 0 *at the point $\infty \in S^2$.*

Proof. The idea is to show that the set of elements $\bar{E} \in \mathcal{Z}_n$ that are 'bad' (meaning that ψ intersects $I_{\bar{E}}^{(\geq 2)}$, or intersects $I_{\bar{E}}^{(\geq 1)}$ nontransversely, etc.) is a proper closed subset of $\mathcal{Z}_n$ so its complement is nonempty.

To this end, consider the following subvarieties of $\mathcal{Z}_n$:

$$\begin{aligned} \mathcal{A}_1 = \Big\{ \bar{E} \in \mathcal{Z}_n : \psi(z) \in I_{\bar{E}}^{(\geq 1)} \ \text{ for some } z \in S^2 \\ \text{and } \psi'(z)(P_1) = 0 \quad \text{for some } 0 \neq P_1 \in \psi(z) \cap \bar{E} \Big\}, \\ \mathcal{A}_2 = \{ \bar{E} \in \mathcal{Z}_n : \psi \text{ intersects } I_{\bar{E}}^{(\geq 2)} \}. \end{aligned}$$

To study $\mathcal{A}_1$, consider the incidence correspondence

$$\Psi_1 = \{ (z, P_1, \bar{E}) \in S^2 \times (\mathbb{C}^{2n+1} \setminus \{0\}) \times \mathcal{Z}_n : P_1 \in \bar{E} \cap \psi(z) \ \text{ and } \ \psi'(z)(P_1) = 0 \}.$$

Then $\mathcal{A}_1 = \pi_3(\Psi_1)$, where π_3 denotes projection on the third factor. We calculate the dimension of $\pi_3(\Psi_1)$ as follows. Consider the incidence correspondence

$$\widehat{\Psi}_1 = \{ (z, P_1) \in S^2 \times (\mathbb{C}^{2n+1} \setminus \{0\}) : P_1 \in \psi(z) \ \text{ and } \ \psi'(z)(P_1) = 0 \}.$$

Projection in $\widehat{\Psi}_1$ into the first factor is clearly onto S^2, and the fiber over $z \in S^2$ is the set of pairs (z, P_1) such that $P_1 \neq 0$ is in the kernel of $\psi'(z) : \psi(z) \to (\psi(z))^{\perp}$. Since $\psi \in \mathrm{HH}_d^f(S^2, \mathcal{Z}_n)$, $\psi'(z)$ is horizontal so the image of $\psi'(z)$ lies in $(\psi(z) \oplus \overline{\psi(z)})^{\perp}$. Hence the kernel of $\psi'(z)$ has dimension $n - 1$

except at the finite set of points where ψ is singular, where it has dimension n. This implies that $\widehat{\Psi}_1$ has dimension n.

Now look at the projection $\pi_{12} : \Psi_1 \to \widehat{\Psi}_1$ defined by $\pi_{12}(z, P_1, \bar{E}) = (z, P_1)$. This projection is onto, and its fiber over a point $(z, P_1) \in \widehat{\Psi}_1$ is given by $\{(z, P_1, \bar{E}) : \bar{E} \ni P_1\}$. Given $0 \neq P_1 \in \mathbb{C}^{2n+1}$ isotropic, we have

$$\{\bar{E} \in \mathcal{Z}_n : \bar{E} \ni P_1\} = \{P_1 \oplus \bar{E}' : \bar{E}' \in \mathcal{Z}((P_1 \oplus \bar{P}_1)^{\perp})\},$$

which is isomorphic to $\mathcal{Z}_{n-1}$ and therefore has dimension $n(n-1)/2$. This implies that Ψ_1 has dimension $n + n(n-1)/2 = n(n+1)/2$.

Finally consider the projection $\pi_3 : \Psi_1 \to \mathcal{Z}_n$ on the third factor. The fiber of this projection over $\bar{E} \in \mathcal{Z}_n$ consists of the triples $(z, P_1, \bar{E})$ such that $\psi(z) \in I_{\bar{E}}^{(\geq 1)}$, and $0 \neq P_1 \in \bar{E} \cap \ker(\psi'(z))$. The set of $z \in S^2$ satisfying the first condition is finite (because ψ is linearly full) and, for each z in this set, the set of P_1 satisfying the second condition has dimension at least 1. This implies that $\mathcal{A}_1 = \pi_3(\Psi_1)$ is a subvariety of $\mathcal{Z}_n$ with codimension at least 1.

To study $\mathcal{A}_2$, use the incidence correspondence given by

$$\Psi_2 = \{(z, \bar{E}) \in S^2 \times \mathcal{Z}_n : \psi(z) \in I_{\bar{E}}^{(\geq 2)}\}.$$

Note that $\mathcal{A}_2 = \pi_2(\Psi_2)$, where π_2 denotes the projection on the second factor. Projection over the first factor in Ψ_2 is onto, and the fiber over $z \in S^2$ is the set of pairs $(z, \bar{E})$ such that $\bar{E} \in I_{\psi(z)}^{(\geq 2)}$, which has codimension 3 in $\mathcal{Z}_n$. Hence $\dim(\Psi_2) = \dim(\mathcal{Z}_n) - 2$, which implies that $\pi_2(\Psi_2) = \mathcal{A}_2$ must be a subvariety of $\mathcal{Z}_n$ of codimension at least two.

Therefore, we have that $\mathcal{A}_1$ and $\mathcal{A}_2$ are proper subvarieties of $\mathcal{Z}_n$. If β is any basis such that $\bar{E}_\beta \in \mathcal{Z}_n$ is outside of these two sets, then ψ does not intersect $I_{\bar{E}_\beta}^{(\geq 2)}$, ψ intersects $I_{\bar{E}_\beta}^{(\geq 1)}$ only in $I_{\bar{E}_\beta}^{(1)}$, and by Lemma 8(iv), ψ intersects $I_{\bar{E}_\beta}^{(1)}$ transversely.

Now we have to deal with the remaining constraints. Let

$$\mathcal{A}_3 = I_{\psi(\infty)}^{(1)} \cap (\mathcal{Z}_n \setminus (\mathcal{A}_1 \cup \mathcal{A}_2)).$$

The quasi-algebraic variety $\mathcal{A}_3$ has codimension 1, and if β is any basis such that $\bar{E}_\beta \notin \mathcal{A}_3$, then Lemma 8(v), together with Lemma 6(i), implies that $(\mathcal{C}_\beta(\psi))_s$ has only d simple zeros, located at the points z where $\psi(z) \in I_{\bar{E}_\beta}^{(1)}$. Since $\psi \in \mathcal{A}_3$, $\psi(\infty) \notin I_{\bar{E}_\beta}^{(1)}$, Lemma 6(ii) implies that $(\mathcal{C}_\beta(\psi))_s$ is not 0 at infinity.

Finally, let $\beta = \{E_0, E_1, \ldots, E_n, \bar{E}_1, \ldots, \bar{E}_n\}$ be a basis such that $\bar{E}_\beta \in \mathcal{Z}_n$ is outside of $\mathcal{A}_3$. For convenience, write $(\mathcal{C}_\beta(\psi)) = [s : \alpha_1 : \cdots : \alpha_n : \tau_{12} : \cdots : \tau_{n-1,n}]$. Since s has only simple zeros, equation (7) implies that at least one of the α_i, $1 \leq i \leq n$, must be nonzero at the zeros of s. Therefore, we can find a matrix $(a_{ij})_{ij} \in U(n)$ such that $a_{11}\alpha_1 + \cdots + a_{1n}\alpha_n$ is not zero where s

vanishes. Then the basis

$$\beta' = \left\{E_0, \sum_{i=1}^{n} a_{1i}E_i, \dots, \sum_{i=1}^{n} a_{ni}E_i, \sum_{i=1}^{n} \bar{a}_{1i}\bar{E}_i, \dots, \sum_{i=1}^{n} \bar{a}_{ni}\bar{E}_i\right\}$$

is isotropic and satisfies $\bar{E}_{\beta'} = \bar{E}_\beta \notin \mathcal{A}_3$ and $(\mathcal{C}_{\beta'}(\psi))_{\alpha_1} = a_{11}\alpha_1 + \cdots + a_{1n}\alpha_n$, which by hypothesis does not vanish at the zeros of s. □

We restate Theorem 2 here and complete its proof.

Theorem 2. *Given* $\psi \in \mathrm{HH}^f_d(S^2, \mathcal{Z}_n)$, *there exists a basis* β *and an open set* $\mathcal{U}_\beta \subseteq \mathrm{HH}^f_d(S^2, \mathcal{Z}_n)$ *containing* ψ *such that*

$$\mathcal{B}_\beta : \mathrm{PD}^f_{d,0}(S^2, \mathbb{CP}^{N_n}) \to \mathcal{U}_\beta$$

is an algebraic isomorphism.

Proof. Let β be a basis with the properties of Lemma 9. Then consider the set

$$\mathcal{U}_\beta = \{\widehat{\psi} \in \mathrm{HH}^f_d(S^2, \mathcal{Z}_n) : \mathrm{Im}(\widehat{\psi}) \cap I^{(\geq 2)}_{\bar{E}_\beta} = \emptyset, \ \ \widehat{\psi} \text{ intersects } I^{(1)}_{\bar{E}_\beta} \text{ transversely,}$$
$$(\mathcal{C}_\beta(\widehat{\psi}))_s(\infty) \neq 0, \text{ and } (\mathcal{C}_\beta(\widehat{\psi}))_{\alpha_1}(z) \neq 0 \text{ at the points of intersection}\}.$$

The set $\mathcal{U}_\beta \subseteq \mathrm{HH}^f_d(S^2, \mathcal{Z}_n)$ is defined by open conditions and it is nonempty since $\psi \in \mathcal{U}_\beta$.

So let $\widehat{\psi} \in \mathcal{U}_\beta$, and write $\mathcal{C}_\beta(\widehat{\psi}) = [s : \alpha_1 : \cdots : \alpha_n : \tau_{12} : \cdots : \tau_{n-1,n}]$. Since $\widehat{\psi}$ intersects $I^{(\geq 1)}_{\bar{E}_\beta}$ transversely and only in $I^{(1)}_{\bar{E}_\beta}$, which by Lemma 8(iii) is regular, Lemma 8(v) implies that $\widehat{\psi}$ intersects $I^{(1)}_{\bar{E}_\beta}$ only at d distinct points $\{\widehat{\psi}(z_1), \dots, \widehat{\psi}(z_d)\}$.

Now Lemma 6(ii) implies that $s(z_\ell) = 0$, $1 \leq \ell \leq d$. Since $s(\infty) \neq 0$, $z_\ell \in \mathbb{C}$ for $1 \leq \ell \leq d$. Also, since $\widehat{\psi} \in \mathcal{U}_\beta$, $\alpha_1(z_i) \neq 0$, and therefore the degree of $\mathcal{C}_\beta(\widehat{\psi})$ is at least d. But on the other hand, Proposition 5(i) asserts that this degree cannot exceed d. Therefore, $\mathcal{C}_\beta(\hat{\psi})$ has degree d, s has d distinct roots $z_m \in \mathbb{C}$ and $\alpha_1(z_\ell) \neq 0$ for $1 \leq \ell \leq d$. This implies that $\mathcal{C}_\beta(\mathcal{U}_\beta) \subseteq \mathrm{PD}^f_{d,0}(S^2, \mathbb{CP}^{N_n})$.

On the other hand, from Proposition 5(ii) and Lemma 6(ii) it follows that $\mathcal{B}_\beta(\mathrm{PD}^f_{d,0}(S^2, \mathbb{CP}^{N_n})) \subseteq \mathcal{U}_\beta$. Since $\mathcal{C}_\beta$ is, by definition, the inverse of $\mathcal{B}_\beta$ and they are both algebraic, the proof is complete. □

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