Generic mean curvature flow I; generic singularities

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Abstract

It has long been conjectured that starting at a generic smooth closed embedded surface in $\mathbf{R}^3$, the mean curvature flow remains smooth until it arrives at a singularity in a neighborhood of which the flow looks like concentric spheres or cylinders. That is, the only singularities of a generic flow are spherical or cylindrical. We will address this conjecture here and in a sequel. The higher dimensional case will be addressed elsewhere.

The key to showing this conjecture is to show that shrinking spheres, cylinders, and planes are the only stable self-shrinkers under the mean curvature flow. We prove this here in all dimensions. An easy consequence of this is that every singularity other than spheres and cylinders can be perturbed away.

0. Introduction

One of the most important problems in mean curvature flow (MCF) is to understand the possible singularities that the flow goes through. Singularities are unavoidable as the flow contracts any closed embedded hypersurface in Euclidean space eventually leading to extinction of the evolving hypersurfaces. Classically, mean curvature flow was only defined up to the first singular time, but a number of ways to define weak solutions have been developed over the last 30 years by Brakke, [Bra78], Evans-Spruck, [ES91], and Chen-Giga-Goto, [CGG91]; cf. Osher-Sethian, [OS88].

Through the combined work of Abresch-Langer, Calabi, Epstein, Gage, Grayson, and Hamilton, among others, singularities for the curve shortening flow in the plane, that is, mean curvature flow for curves, are well understood. In higher dimensions, a key starting point for singularity analysis is Huisken's montonicity formula, [Hui90]. This is because the monotonicity implies, according to Huisken, [Hui90], that the flow is asymptotically self-similar near a

given singularity and, thus, is modeled by self-shrinking solutions of the flow. Huisken's original proof applied to so-called type I singularities, but Ilmanen and White extended Huisken's formula to weak solutions and proved asymptotic self-similarity for all singularities of the flow with convergence now in the weak sense of geometric measure theory; see Lemma 8 in [Ilm95b].

Here and in papers to follow, we will show a long-standing conjecture (of Huisken in general and Angenent-Chopp-Ilmanen in $\mathbf{R}^3$, [AIC95]) classifying the singularities of mean curvature flow starting at a generic closed embedded surface. The current paper shows that, in all dimensions, the only singularities that cannot be perturbed away are cylinders and spheres. It also gives an application to generic mean curvature flow in $\mathbf{R}^3$ showing that generic mean curvature flow that disappears in a compact point does so in a round point. Well-known examples of Angenent and numerics of Angenent, Chopp, and Ilmanen discussed below show that there is virtually no hope of classifying the singularities for mean curvature flow starting at an arbitrary hypersurface in $\mathbf{R}^{n+1}$ for $n > 1$[1]; the thrust of Huisken's conjecture is that this can be done for generic initial hypersurfaces.

Under various assumptions, like convexity, mean convexity, and two-convexity, or for curves in the plane, the blow ups of singularities have been classified by various authors including Huisken, Gage-Hamilton, Grayson, Huisken-Sinestrari, and White. Unlike in our case, in all previous classifications of possible singularities for $n > 1$, assumptions, like mean convexity, were made that immediately guaranteed that none of the exotic singularities described above could occur. For instance, since mean convexity is preserved under the flow, if the initial surface is mean convex, then so are all blow ups and, since none of the exotic self-similar flows mentioned above are mean convex, they are immediately ruled out as singularities.

Most of the results of this paper apply to immersed hypersurfaces, but embeddedness is used in two places. The first is the construction of piecewise MCF, where the compactness theorem of [CMb] uses embeddedness in a crucial way. The second place is in the classification of mean convex self-shrinkers. Here embeddedness is used to rule out the immersed locally convex Abresch-Langer examples and products of these with Euclidean factors. That these can be ruled out is used in the classification of generic singularities, but these statements actually hold without embeddedness, as we will explain below, using instead a paper of Epstein and Weinstein, [EW87]. We are grateful to the referee for pointing this out.

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[1]Since time slices of self-shrinkers are minimal hypersurfaces for a conformally changed metric on $\mathbf{R}^{n+1}$ (see §3), some classification of self-shrinkers may be possible in $\mathbf{R}^3$ using the ideas of [CM06], [CM04a], [CM04b], [CM04c], [CM04d], [CMc].

A one-parameter family M_t of hypersurfaces in $\mathbf{R}^{n+1}$ flows by mean curvature if

$$(0.1) \qquad\qquad (\partial_t x)^\perp = \bar{H},$$

where $\bar{H} = -H\mathbf{n}$ is the mean curvature vector, $\mathbf{n}$ is the outward unit normal, $v^\perp$ is the normal part of a vector v, and the mean curvature H is given by

$$(0.2) \qquad\qquad H = \operatorname{div} \mathbf{n}.$$

With this convention, H is n/R on the n-sphere of radius R in $\mathbf{R}^{n+1}$ and H is k/R on the "cylinder" $\mathbf{S}^k \times \mathbf{R}^{n-k} \subset \mathbf{R}^{n+1}$ of radius R. Thus, in either case, the mean curvature vector "points inwards" and, hence, the flow contracts.

The simplest (nonstatic) mean curvature flow is given by the one-parameter family of shrinking spheres $M_t \subset \mathbf{R}^{n+1}$ centered at the origin and with radius $\sqrt{-2nt}$ for $t \leq 0$. This is a smooth flow except at the origin at time 0 when the flow becomes extinct. In [Hui84], Huisken showed that MCF starting at any smooth compact convex initial hypersurface in $\mathbf{R}^{n+1}$ remains smooth and convex until it becomes extinct at a point and if we rescale the flow about the point in space-time where it becomes extinct, then the rescalings converge to round spheres. Huisken-Sinestrari, [HS99a], [HS99b], and White, [Whi00], [Whi03], have proven a number of striking and important results about MCF of mean convex hypersurfaces and their singularities and Huisken-Sinestrari, [HS09], have developed a theory for MCF with surgery for two-convex hypersurfaces in $\mathbf{R}^{n+1}$ ($n \geq 3$) using their analysis of singularities (and their blow ups).

Huisken's proof that convex hypersurfaces become extinct in round points applied for $n \geq 2$, but the corresponding result for convex curves was proven by Gage and Hamilton in [GH86]. In fact, in [Gra87] (see also [Gra89b], [EG87], [Ham95], [Hui98]), Grayson showed that any simple closed smooth curve in $\mathbf{R}^2$ stays smooth under the curve shortening flow, eventually becomes convex, and thus will become extinct in a "round point." The situation is more complicated for surfaces where there are many other potential types of singularities that can arise. For instance, Grayson constructed a rotationally symmetric dumbbell in [Gra89a] where the neck pinches off before the two bells become extinct. For rescalings of the singularity at the neck, the resulting blow ups cannot be extinctions and, thus, are certainly not spheres. In fact, rescalings of the singularity converge to shrinking cylinders; we refer to White's survey [Whi02] for further discussion of this example.

The family of shrinking spheres of radius $\sqrt{-2nt}$ is self-similar in the sense that M_t is given by

$$(0.3) \qquad\qquad M_t = \sqrt{-t}\, M_{-1}.$$

A MCF M_t satisfying (0.3) is called a self-shrinker. Self-shrinkers play an important role in the study of mean curvature flow, not least because they describe all possible blow ups at a given singularity of a mean curvature flow. To explain this, we will need the notion of a tangent flow, cf. [Ilm95b], [Whi97], which generalizes the tangent cone construction from minimal surfaces. The basic idea is that we can rescale a MCF in space and time to obtain a new MCF thereby expanding a small neighborhood of the point that we want to focus on. Huisken's monotonicity gives uniform control over these rescalings and a standard compactness theorem then gives a subsequence converging to a limiting solution of the MCF. This limit is called a tangent flow. A tangent flow will achieve equality in Huisken's monotonicity formula and, thus, must be a self-shrinker by [Hui90], [Ilm95b].

The precise definition of a *tangent flow* at a point (x_0, t_0) in space-time of a MCF M_t is as follows. First translate M_t in space-time to move (x_0, t_0) to $(0, 0)$ and then take a sequence of parabolic dilations $(x, t) \to (c_j x, c_j^2 t)$ with $c_j \to \infty$ to get MCF's $M_t^j = c_j \left(M_{c_j^{-2} t + t_0} - x_0 \right)$. Using Huisken's monotonicity formula, [Hui90], and Brakke's compactness theorem, [Bra78], White [Whi97] and Ilmanen [Ilm95b] show that a subsequence of the M_t^j's converges weakly to a limiting flow $\mathcal{T}_t$ that we will call a *tangent flow* at (x_0, t_0). Moreover, another application of Huisken's monotonicity shows that $\mathcal{T}_t$ is a self-shrinker. It is not known whether $\mathcal{T}_t$ is unique; that is, whether different sequences of dilations might lead to different tangent flows.

In [Ilm95b], Ilmanen proved that in $\mathbf{R}^3$ tangent flows at the first singular time must be smooth, although he left open the possibility of multiplicity. However, he conjectured that the multiplicity must be one.

CONJECTURE 0.4 (Ilmanen [Ilm95b, p. 7]; cf. [Eck07]). *For a smooth one-parameter family of closed embedded surfaces in $\mathbf{R}^3$ flowing by mean curvature, every tangent flow at the first singular time has multiplicity one.*

If this conjecture holds, then it would follow from Brakke's regularity theorem that near a singularity the flow can be written as a graph of a function with small gradient over the tangent flow.

We will say that a MCF is smooth up to and including the first singular time if every tangent flow at the first singular time is smooth and has multiplicity one. Conjecturally, all embedded MCF in $\mathbf{R}^3$ are smooth up to the first singular time.

A self-similarly shrinking solution to MCF is completely determined by the $t = -1$ time-slice and, thus, we sometimes think of a self-similar flow as just that time-slice.

The simplest self-shrinkers are $\mathbf{R}^n$, the sphere of radius $\sqrt{-2nt}$, and more generally cylindrical products $\mathbf{S}^k \times \mathbf{R}^{n-k}$ (where the $\mathbf{S}^k$ has radius $\sqrt{-2kt}$).

All of these examples are mean convex (i.e., have $H \geq 0$) and, in fact, these are the only mean convex examples under mild assumptions; see [Hui90], [Hui93], [AL86], and Theorem 0.17.

Without the assumption on mean convexity, then there are expected to be many more examples of self-shrinkers in $\mathbf{R}^3$. In particular, Angenent, [Ang92], has constructed a self-shrinking torus ("shrinking donut") of revolution[2] and there is numerical evidence for a number of other examples (see Chopp, [Cho94], Angenent-Chopp-Ilmanen, [AIC95], Ilmanen, [Ilm95a], and Nguyen, [Ngu09], [Ngu10]). We will see in this paper that all self-shrinkers except the simplest are highly unstable and, thus, hard to find. We will use this instability to perturb them away in a generic flow.

Angenent used his self-shrinking torus to give a new proof that the dumbbell has a neck pinching singularity before the two bells become extinct. The idea is to make the neck of the dumbbell long and thin and the bells on either side large enough to contain two large round spheres. By the maximum principle, the interior of the MCF of the dumbbell will contain the shrinking spheres and, thus, cannot become extinct until after the spheres do. On the other hand, Angenent used the self-shrinking torus to encircle the neck of the dumbbell and, thus, conclude that the neck would pinch off before the spheres had shrunk to points; see also White, [Whi02], for a beautiful expository discussion of the dumbbell, tangent flows, and self-similar solutions.

While M_t will always be a one-parameter family of hypersurfaces flowing by mean curvature, we will use Σ to denote a single hypersurface and Σ_s to denote a one-parameter variation of Σ. Frequently, Σ will be the time $t = -1$ slice of a self-shrinking solution M_t of the mean curvature flow.

Given $x_0 \in \mathbf{R}^{n+1}$ and $t_0 > 0$, define the functional F_{x_0,t_0} (see [Hui90], [AIC95, eq. (6)], or [Ilm95a, p. 6]; cf. [Ilm94, 2.4]) by

$$(0.5) \qquad F_{x_0,t_0}(\Sigma) = (4\pi t_0)^{-\frac{n}{2}} \int_\Sigma \mathrm{e}^{\frac{-|x-x_0|^2}{4t_0}} \, d\mu.$$

The main point of these functionals is that Σ is a critical point of F_{x_0,t_0} precisely when it is the time $t = -t_0$ slice of a self-shrinking solution of the mean curvature flow that becomes extinct at $x = x_0$ and $t = 0$. The *entropy* $\lambda = \lambda(\Sigma)$ of Σ will be the supremum of the F_{x_0,t_0} functionals (cf. [MM09, Def. 1.9])

$$(0.6) \qquad \lambda = \sup_{x_0,t_0} F_{x_0,t_0}(\Sigma).$$

The key properties of the entropy λ are

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[2]In fact, for every $n \geq 2$, by rotating a curve in plane, Angenent constructed an embedded self-shrinker in $\mathbf{R}^{n+1}$ that is topologically $\mathbf{S}^1 \times \mathbf{S}^{n-1}$. The curve satisfies an ODE that can be interpreted as the geodesic equation for a singular metric.

- λ is nonnegative and invariant under dilations, rotations, or translations of Σ.
- $\lambda(M_t)$ is nonincreasing in t if the hypersurfaces M_t flow by mean curvature.
- The critical points of λ are self-shrinkers for the mean curvature flow.

These properties are the main *advantages* of the entropy functional over the F functionals. The main *disadvantage* of the entropy is that it need not depend smoothly on Σ. To deal with this, we will say that a self-shrinker is *entropy-stable* if it is a local minimum for the entropy functional.[3]

To illustrate our results, we will first specialize to the case where $n = 2$; that is, to mean curvature flow of surfaces in $\mathbf{R}^3$.

THEOREM 0.7. *Suppose that $\Sigma \subset \mathbf{R}^3$ is a smooth complete embedded self-shrinker without boundary and with polynomial volume growth.*

- *If Σ is not a sphere, a plane, or a cylinder, then there is a graph $\tilde{\Sigma}$ over Σ of a compactly supported function with arbitrarily small C^m norm (for any fixed m) so that $\lambda(\tilde{\Sigma}) < \lambda(\Sigma)$.*

In particular, Σ cannot arise as a tangent flow to the MCF starting from $\tilde{\Sigma}$.

Motivated by this theorem, we will next define an ad hoc notion of generic MCF that requires the least amount of technical set-up, yet should suffice for many applications. A piece-wise MCF is a finite collection of MCF's M_t^i on time intervals $[t_i, t_{i+1}]$ so that each $M_{t_{i+1}}^{i+1}$ is the graph over $M_{t_{i+1}}^i$ of a function u_{i+1}:

$$(0.8) \qquad \qquad \operatorname{Area}\left(M_{t_{i+1}}^{i+1}\right) = \operatorname{Area}\left(M_{t_{i+1}}^i\right),$$

$$(0.9) \qquad \qquad \lambda\left(M_{t_{i+1}}^{i+1}\right) \leq \lambda\left(M_{t_{i+1}}^i\right).$$

With this definition, area is nonincreasing in t even across the jumps.

For simplicity, we assume in Theorem 0.10 below that the MCF is smooth up to and including the first singular time. As mentioned, this would be the case for any MCF in $\mathbf{R}^3$ if the multiplicity one conjecture, Conjecture 0.4, holds.

The following theorem can be thought of as a generalization of the results of Gage-Hamilton [GH86], Grayson [Gra87], and Huisken [Hui84].

THEOREM 0.10. *For any closed embedded surface $\Sigma \subset \mathbf{R}^3$, there exists a piece-wise MCF M_t starting at Σ and defined up to time t_0 where the surfaces*

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[3]Here "local" means with respect to hypersurfaces that can be written as a graph over the given hypersurface of a function with small C^2 norm. In particular, we do not require the support to be compact.

become singular. Moreover, M_t can be chosen so that if

$$(0.11) \qquad\qquad \liminf_{t \to t_0} \ \frac{\mathrm{diam} M_t}{\sqrt{t_0 - t}} < \infty,$$

then M_t becomes extinct in a round point.

The meaning of (0.11) is that each time slice of a tangent flow (at time t_0) has uniformly bounded diameter after rescaling.

It is worth noting that the piece-wise MCF constructed in Theorem 0.10 can be made arbitrarily close in C^m norm to the smooth flow, with the number of break-points possibly going to infinity as the C^m distance goes to zero.

Theorem 0.10 will eventually be a corollary of our main theorem in [CMa] about generic MCF. However, it follows directly from our classification of entropy stable self-shrinkers together with compactness of the space of all self-shrinkers with a fixed bound on area and genus and serves to illustrate some of the central ideas about generic MCF. (The compactness of self-shrinkers with area and genus bound was proven in [CMb].) Here is why it follows directly from these two results (a detailed proof is given in §8 of this paper).

> Starting at the given surface, flow by mean curvature until the evolving surface is sufficiently close to a time slice in a tangent flow. If this time slice is not a sphere yet has diameter bounded by a fixed number, then by the classification of stable self-shrinkers we can find a small graph over it where the entropy has gone down by a fixed amount. Start the MCF at this new surface and flow until the evolving surface is sufficiently close to a time slice in the corresponding tangent flow. If this time slice is also not a sphere yet has a fixed bound for the diameter, we can continue the process of making a replacement and get again that the entropy has gone down a fixed amount. As the entropy is always positive, this process has to terminate and we get the theorem. Note that the entropy goes down by a definite amount after each replacement follows from the compactness theorem of [CMb].

One consequence of Theorem 0.10 is that if the initial surface is topologically not a sphere, then the piece-wise flow must develop a noncompact (after rescaling) singularity.

Note that even flows that become extinct at a point can develop noncompact singularities. For example, Altschuler-Angenent-Giga, [AAG95], constructed a mean convex initial surface in $\mathbf{R}^3$ whose MCF is smooth until it becomes extinct in a "noncompact" point. In fact, the tangent flow at the extinction point is a cylinder. This is in contrast to the case of curves where

Grayson, [Gra87], has shown that all tangent flows are compact; cf. [Ham95] and [Hui98].

0.1. *Higher dimensions.* As already mentioned, the main result of this paper is the classification of generic singularities. In higher dimensions, this is the following.

THEOREM 0.12. *Suppose that Σ is a smooth complete embedded self-shrinker without boundary and with polynomial volume growth.*

(1) *If Σ is not equal to $\mathbf{S}^k \times \mathbf{R}^{n-k}$, then there is a graph $\tilde{\Sigma}$ over Σ of a function with arbitrarily small C^m norm (for any fixed m) so that $\lambda(\tilde{\Sigma}) < \lambda(\Sigma)$.*
(2) *If Σ is not $\mathbf{S}^n$ and does not split off a line, then the function in (1) can be taken to have compact support.*

In particular, in either case, Σ cannot arise as a tangent flow to the MCF *starting from $\tilde{\Sigma}$.*

In our earlier theorem showing that in $\mathbf{R}^3$ entropy stable self-shrinkers are all standard, we assumed that the self-shrinker was smooth. This was a reasonable assumption in $\mathbf{R}^3$ for applications as (by the theorem of Ilmanen) any tangent flow in $\mathbf{R}^3$ is indeed smooth. However, examples of Velázquez, [Vel94], show that tangent flows need not be smooth in higher dimensions[4] and, instead, one has the following well-known conjecture (see page 8 of [Ilm95b]).

CONJECTURE 0.13. *Suppose that $M_0 \subset \mathbf{R}^{n+1}$ is a smooth closed embedded hypersurface. A time slice of any tangent flow of the* MCF *starting at M_0 has multiplicity one and the singular set is of dimension at most $n - 3$.[5]*

We will show that when $n \leq 6$, our results classifying entropy stable self-shrinkers in $\mathbf{R}^{n+1}$ hold even for self-similar shrinkers satisfying the smoothness of Conjecture 0.13. In fact, we show here the following stronger result.

THEOREM 0.14. *Theorem 0.12 holds when $n \leq 6$ and Σ is an oriented integral varifold that is smooth off of a singular set with locally finite $(n-2)$-dimensional Hausdorff measure.*

In the case where Σ is a nonsmooth minimal cone in Theorem 0.14, the perturbed surfaces will also be cones and the perturbation will be scale-invariant.

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[4]In [Vel94], Velázquez constructed smooth embedded hypersurfaces evolving by MCF in $\mathbf{R}^8$ whose tangent flow at the first singular time is the static Simons cone and is, in particular, not smooth.

[5]By the dimension reduction of [Whi97], the estimate on the singular set would follow from ruling out static planes, quasi-static planes, and various polyhedral cones as potential blow ups. The key for proving the conjecture is to rule out higher multiplicity static planes.

Moreover, the perturbation can be chosen with arbitrarily small scale-invariant C^m norm (for any fixed m).

0.2. *Outline of the proof.* The key results of this paper are the classification of entropy-stable self-shrinkers given in Theorem 0.7 and its higher dimensional generalization Theorem 0.12. The main technical tool for proving this is the concept of F-stability, which we will define below.

We will see that there are several characterizations of self-shrinkers. One of the most useful is that self-shrinkers are the critical points for the $F_{0,1}$ functional (see Proposition 3.6 below). Since this functional is the volume in a conformally changed metric, they are also minimal surfaces in the conformally changed metric. From either point of view, every self-shrinker Σ is unstable in the usual sense; i.e., there are always nearby hypersurfaces where the $F_{0,1}$ functional is strictly less (see Theorem 9.2 below). This is because translating a self-shrinker in space (or time) always lowers the functional. The stability that we are interested in, which we will call F-stability, mods out for these translations.

> A critical point Σ for F_{x_0,t_0} is F-*stable* if for every compactly supported variation[6] Σ_s with $\Sigma_0 = \Sigma$, there exist variations x_s of x_0 and t_s of t_0 that make $F'' = (F_{x_s,t_s}(\Sigma_s))'' \geq 0$ at $s = 0$.

We will show that entropy-stable self-shrinkers that do not split off a line must be F-stable.

THEOREM 0.15. *Suppose that $\Sigma \subset \mathbf{R}^{n+1}$ is a smooth complete self-shrinker with $\partial \Sigma = \emptyset$, with polynomial volume growth, and Σ does not split off a line isometrically. If Σ is F-unstable, then there is a compactly supported variation Σ_s with $\Sigma_0 = \Sigma$ so that $\lambda(\Sigma_s) < \lambda(\Sigma)$ for all $s \neq 0$.*

Thus, we are led to classifying F-stable self-shrinkers.

THEOREM 0.16. *If Σ is a smooth[7] complete embedded self-shrinker in $\mathbf{R}^{n+1}$ without boundary and with polynomial volume growth that is F-stable with respect to compactly supported variations, then it is either the round sphere or a hyperplane.*

The main steps in the proof of Theorem 0.16 are
- show that F-stability implies mean convexity (i.e., $H \geq 0$),
- classify the mean convex self-shrinkers (see Theorem 0.17 below).

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[6] A compactly supported variation Σ_s of Σ is a one-parameter family of graphs over Σ given by $\{x + s\,f(x)\mathbf{n}(x) | x \in \Sigma\}$, where $\mathbf{n}$ is the unit normal and f is a function on Σ with compact support.

[7] The theorem holds when $n \leq 6$ and Σ is an oriented integral varifold that is smooth off of a singular set with locally finite $(n-2)$-dimensional Hausdorff measure. See Corollary 12.2.

The connection between F-stability and mean convexity comes from that the mean curvature H turns out to be an eigenfunction for the second variation operator for the $F_{0,1}$ functional on a self-shrinker Σ; see Theorems 4.14 and 5.2. When Σ is closed (so that the spectral theory is easiest), it is then almost immediate to see that F-stability is equivalent to H being the lowest eigenfunction which, in turn, is equivalent to that H does not change sign (i.e., mean convexity). Although spectral theory of open manifolds is more complicated, we show the corresponding result in Section 9 to get that F-stability implies mean convexity.

The classification of mean convex self-shrinkers began with [Hui90], where Huisken showed that the only smooth *closed* self-shrinkers with nonnegative mean curvature in $\mathbf{R}^{n+1}$ (for $n > 1$) are round spheres (i.e., $\mathbf{S}^n$). When $n = 1$, Abresch and Langer, [AL86], had already shown that the circle is the only simple closed self-shrinking curve. In a second paper, [Hui93], Huisken dealt with the noncompact case. He showed in [Hui93] that the only smooth *open* embedded self-shrinkers in $\mathbf{R}^{n+1}$ with $H \geq 0$, polynomial volume growth, and $|A|$ bounded are isometric products of a round sphere and a linear subspace (i.e., $\mathbf{S}^k \times \mathbf{R}^{n-k} \subset \mathbf{R}^{n+1}$). We will show that Huisken's classification holds even without the $|A|$ bound, which will be crucial for our applications.

THEOREM 0.17. $\mathbf{S}^k \times \mathbf{R}^{n-k}$ *are the only smooth complete embedded self-shrinkers without boundary, with polynomial volume growth, and $H \geq 0$ in* $\mathbf{R}^{n+1}$.

The $\mathbf{S}^k$ factor in Theorem 0.17 is round and has radius $\sqrt{2k}$; we allow the possibilities of a hyperplane (i.e., $k = 0$) or a sphere ($n-k = 0$). Embeddedness is used in Theorem 0.17 to rule out the products of immersed Abresch-Langer shrinkers with Euclidean factors.

In [EW87], Epstein and Weinstein studied the local nonlinear stability of the rescaled curve shortening flow of locally convex curves in $\mathbf{R}^2$. They first classify the locally convex static solutions (i.e., locally convex self-shrinkers), obtaining a two-parameter family of solutions $w_{p,q}$ where $2\pi p$ is the total curvature and $2q$ is the number of "vertices" (or critical points of the curvature). Next, they prove the existence of stable and unstable manifolds[8] at each solution and compute the codimensions of these. As a consequence, they show that

(1) Every immersed solution except the round circle is locally dynamically unstable.

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[8]Recall that the stable manifold is the set of curves that flows into the static solution as t goes to infinity. The unstable manifold is the stable manifold for the flow with time reversed.

(2) The embedded round circle is a stable solution (recovering a local version of Gage-Hamilton).

(3) The round circle with multiplicity $p > 1$ is locally dynamically unstable.

In fact, it follows from the linear analysis in [EW87] that the immersed self-shrinking curves are F-unstable. As a result, the assumption of embeddedness is not needed in Theorem 0.16 and, thus, embeddedness is also not needed in Theorems 0.7 and 0.12 (or in Theorem 4.31 below).

0.3. *The proofs of the main theorems.* The classification of F-stable self-shrinkers, Theorem 0.16, will be divided into the (easier) compact case and the noncompact case. The compact case is stated in Theorem 4.30 and proven in Section 6. The noncompact case is stated in Theorem 4.31 and proven in Section 11.

The splitting theorem, Theorem 0.15, is proven in Section 7. Also in this section, using this and the classification of compact stable self-shrinkers from Theorem 4.30, we prove the compact case of the entropy-stability result in Corollary 7.56.

In Section 8, we combine the splitting theorem, the classification of compact stable self-shrinkers, and the compactness theorem of [CMb] to construct piece-wise MCF and prove Theorem 0.10.

In Section 10, we classify mean convex self-shrinkers and, thus, prove Theorem 0.17.

Section 11 contains the proofs of 4.31 and Theorem 0.16, thus allowing us to complete the proofs of Theorems 0.7 and 0.12.

Section 12 extends the main results to F-stationary varifolds with sufficiently small singular sets, including the proof of Theorem 0.14.

0.4. *Conventions and notation.* Using the definition (0.2), H is $2/R$ on the sphere of radius R in $\mathbf{R}^3$ and H is $1/R$ on the cylinder of radius R. If e_i is an orthonormal frame for Σ and $\mathbf{n}$ is a unit normal, the coefficients of the second fundamental form are defined to be

$$(0.18) \qquad\qquad a_{ij} = \langle \nabla_{e_i} e_j, \mathbf{n} \rangle.$$

In particular, we have

$$(0.19) \qquad\qquad \nabla_{e_i} \mathbf{n} = -a_{ij} e_j.$$

Since $\langle \nabla_{\mathbf{n}} \mathbf{n}, \mathbf{n} \rangle = 0$, we have that $H = \langle \nabla_{e_i} \mathbf{n}, e_i \rangle = -a_{ii}$, where by convention we are summing over repeated indices.

When L is a differential operator, we will say that u is an eigenfunction with eigenvalue μ if $Lu = -\mu u$ and u is not identically zero. The Laplacian Δ is defined to be the divergence of the gradient; thus, on $\mathbf{R}^n$, it is given by

$\Delta u = \sum_{i=1}^{n} u_{x_i x_i}$. With this convention, the eigenvalues of the Laplacian are nonnegative on any closed manifold.

The parabolic distance between two pairs of points (x_1, t_1) and (x_2, t_2) in space-time is denoted by $d_P((x_1, t_1), (x_2, t_2))$ and equal to

$$\max\left\{ |x_1 - x_2|, \sqrt{|t_1 - t_2|} \right\}.$$

Finally, there is a second, and related, way to study the asymptotic structure near a singularity. In this case, one considers a new flow which combines the mean curvature flow with a continuous rescaling process $\tilde{M}_s = \frac{1}{\sqrt{-t}} M_t$ where $s(t) = -\log(-t)$ is a reparameterization of time. The resulting flow is called the *rescaled mean curvature flow*[9] and a one-parameter family $\tilde{M}_s$ satisfies it if

$$(0.20) \qquad\qquad (\partial_s x)^\perp = -H\mathbf{n} + \frac{x^\perp}{2},$$

where x is the position vector. In this case, we are rescaling about $x = 0$ and $t = 0$, but translations in space and time give similar equations for rescaling about other points. It is not hard to see that the fixed points of the rescaled mean curvature flow are precisely the self-shrinkers. The result of Huisken, [Hui84], about extinction in round points, can be reformulated in terms rescaled MCF flow as stating that the rescaled MCF of a convex hypersurface exists for all time and converges asymptotically to the round sphere. Obviously, this implies that the tangent flows at the singularity are round spheres.

We are grateful to the referee for comments and suggestions and, in particular, for bringing to our attention the paper of Epstein and Weinstein, [EW87].

1. The F functional and Huisken's monotonicity formula

We will need to recall Huisken's monotonicity formula (see [Hui90], [Eck05], [Eck04]). To do this, first define the nonnegative function Φ on $\mathbf{R}^{n+1} \times (-\infty, 0)$ by

$$(1.1) \qquad\qquad \Phi(x, t) = [-4\pi t]^{-\frac{n}{2}} e^{\frac{|x|^2}{4t}}$$

and then set $\Phi_{(x_0, t_0)}(x, t) = \Phi(x - x_0, t - t_0)$. If M_t is a solution to the MCF and u is a C^2 function, then

$$(1.2) \quad \frac{d}{dt} \int_{M_t} u \Phi_{(x_0, t_0)} = -\int_{M_t} \left| H\mathbf{n} - \frac{(x - x_0)^\perp}{2(t_0 - t)} \right|^2 u \Phi_{(x_0, t_0)} + \int_{M_t} [u_t - \Delta u]\, \Phi.$$

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[9]Our definition of the rescaled flow differs slightly from Huisken's definition in [Hui90] since Huisken's rescaling fixes the time $t = -\frac{1}{2}$ slice rather than the $t = -1$.

When u is identically one, we get the monotonicity formula

$$(1.3) \qquad \frac{d}{dt} \int_{M_t} \Phi_{(x_0,t_0)} = - \int_{M_t} \left| H\mathbf{n} - \frac{(x - x_0)^\perp}{2(t_0 - t)} \right|^2 \Phi_{(x_0,t_0)}.$$

Huisken's density is the limit of $\int_{M_t} \Phi_{x_0,t_0}$ as $t \to t_0$. That is,

$$(1.4) \qquad \Theta_{x_0,t_0} = \lim_{t \to t_0} \int_{M_t} \Phi_{x_0,t_0}.$$

This limit exists by the monotonicity (1.3) and the density is nonnegative as the integrand Φ_{x_0,t_0} is nonnegative. Note also that it easily follows from the monotonicity that Huisken's density is upper semi-continuous in the following sense. Given x_0, t_0, and $\varepsilon > 0$, there exists $\delta > 0$ such that if $d_P((x,t),(x_0,t_0)) < \delta$, then

$$(1.5) \qquad \Theta_{x,t} < \Theta_{x_0,t_0} + \varepsilon.$$

The F functional and Huisken's weighted volume are related by the following identity:

$$(1.6) \qquad F_{x_0,t_0}(M_{-1}) = (4\pi t_0)^{-n/2} \int_{M_{-1}} e^{-\frac{|x-x_0|^2}{4t_0}} = \int_{M_{-1}} \Phi_{(x_0,t_0)}(x,0).$$

Remark 1.7. The density was defined so that it is one on a hyperplane. In [Sto94], A. Stone calculated Huisken's density, and thus the F functional, on spheres and cylinders. The density of $\mathbf{S}^2$ is $4/e \approx 1.47$, and the density of $\mathbf{S}^1 \times \mathbf{R}$ is $\sqrt{2\pi/e} \approx 1.52$.

We will use two simple properties of the F_{x_0,t_0} functional defined in (0.5). The first is that if we scale Σ by some $\alpha > 0$ about the origin, then the scaled hypersurface $\alpha \Sigma$ satisfies

$$(1.8) \qquad F_{0,\alpha^2 t_0}(\alpha \Sigma) = F_{0,t_0}(\Sigma).$$

Similarly, we get a corresponding equality for rescalings about an arbitrary $x_0 \in \mathbf{R}^{n+1}$. The second property that we will need is that if the one-parameter family of hypersurfaces M_t flows by mean curvature and $t > s$, then Huisken's monotonicity formula (1.3) gives

$$(1.9) \qquad F_{x_0,t_0}(M_t) \leq F_{x_0,t_0+(t-s)}(M_s).$$

Recall that we defined the entropy $\lambda = \lambda(\Sigma)$ of Σ in (0.6) as the supremum of the F_{x_0,t_0} functionals

$$(1.10) \qquad \lambda = \sup_{x_0,t_0} F_{x_0,t_0}(\Sigma).$$

We conclude that the entropy is monotone under the mean curvature flow.

LEMMA 1.11. *The entropy λ is nonincreasing in t if the hypersurfaces M_t flow by mean curvature or by the rescaled mean curvature flow.*

Proof. The first claim follows immediately from (1.9), while the second claim follows from (1.9) and (1.8). □

2. Self-shrinkers and self-shrinking MCF

A hypersurface is said to be a *self-shrinker* if it is the time $t = -1$ slice[10] of a self-shrinking MCF that disappears at $(0,0)$, i.e., of a MCF satisfying $M_t = \sqrt{-t}\, M_{-1}$. Thanks to the next lemma we will later use slight abuse of notation and identify a self-shrinker with the corresponding self-shrinking MCF, and sometimes we will also simply think of a self-shrinker as a hypersurface satisfying the following equation for the mean curvature and the normal:

$$(2.1) \qquad\qquad H = \frac{\langle x, \mathbf{n}\rangle}{2}.$$

LEMMA 2.2. *If a hypersurface Σ satisfies (2.1), then $M_t = \sqrt{-t}\,\Sigma$ flows by mean curvature and*

$$(2.3) \qquad\qquad H_{M_t} = -\frac{\langle x, \mathbf{n}_{M_t}\rangle}{2t}.$$

Conversely if a one-parameter family of hypersurfaces M_t flows by mean curvature, then M_t satisfies $M_t = \sqrt{-t}\, M_{-1}$ if and only if M_t satisfies (2.3).

Proof. If Σ is a hypersurface that satisfies that $H = \frac{\langle x, \mathbf{n}\rangle}{2}$, then we set $M_t = \sqrt{-t}\,\Sigma$ and $x(p,t) = \sqrt{-t}\,p$ for $p \in \Sigma$. It follows that $\mathbf{n}_{M_t}(x(p,t)) = \mathbf{n}_\Sigma(p)$, $H_{M_t}(x(p,t)) = \frac{H_\Sigma(p)}{\sqrt{-t}}$, and $\partial_t x = -\frac{p}{2\sqrt{-t}}$. Thus, $(\partial_t x)^\perp = -\frac{\langle p, \mathbf{n}\rangle}{2\sqrt{-t}} = -H_{M_t}(x(p,t))$. This proves that M_t flows by mean curvature and shows (2.3).

On the other hand, suppose that M_t is a one-parameter family of hypersurfaces flowing by mean curvature. A computation shows that

$$(2.4) \qquad\qquad (-t)^{\frac{3}{2}}\, \partial_t\left(\frac{x}{\sqrt{-t}}\right) = -t\,\partial_t x + \frac{x}{2}.$$

If $\frac{M_t}{\sqrt{-t}} = M_{-1}$, then

$$(2.5) \qquad 0 = (-t)^{\frac{3}{2}}\left\langle \partial_t\left(\frac{x}{\sqrt{-t}}\right), \mathbf{n}_{M_{-1}}\right\rangle = -t\langle\partial_t x, \mathbf{n}_{M_{-1}}\rangle + \frac{1}{2}\langle x, \mathbf{n}_{M_{-1}}\rangle.$$

Hence, since M_t flow by the mean curvature, it follows that

$$(2.6) \qquad\qquad H_{M_{-1}} = -\langle\partial_t x, \mathbf{n}_{M_{-1}}\rangle = \frac{\langle x, \mathbf{n}_{M_{-1}}\rangle}{2}.$$

The equation for H_{M_t}, for general t, follows by scaling.

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[10]In [Hui90], Huisken defines self-shrinkers to be the time $t = -\frac{1}{2}$ slice of a self-shrinking MCF; consequently, Huisken gets that $H = \langle x, \mathbf{n}\rangle$.

Finally, if M_t flows by mean curvature and satisfies (2.3), then, by the first part of the lemma, it follows that $N_t = \sqrt{-t}M_{-1}$ also flows by the mean curvature and has the same initial condition as M_t; thus $M_t = N_t$ for $t \geq -1$. $\quad\square$

COROLLARY 2.7. *If the hypersurfaces M_t flow by mean curvature, then M_t is a self-similar shrinking MCF if and only $\int_{M_t} \Phi$ is constant.*

Proof. This follows by combining Lemma 2.2 with (1.9). $\quad\square$

Another consequence of Lemma 2.2 is the following standard corollary.

COROLLARY 2.8. *If Σ is a self-shrinker and $H \equiv 0$, then Σ is a minimal cone. In particular, if Σ is also smooth and embedded, then it is a hyperplane through 0.*

Proof. Since $H \equiv 0$, $M_t = \Sigma$ is a static solution of the MCF. Hence, by Lemma 2.2, $\sqrt{-t}\,\Sigma = \Sigma$ for all $t < 0$ and, thus, Σ is a cone. The second claim follows since the only smooth cone through 0 is a hyperplane. $\quad\square$

2.1. *Volume bounds and a priori bounds for blow ups.* Let $M_t \subset \mathbf{R}^{n+1}$ be a MCF with initial hypersurface M_0 where M_0 is smooth closed and embedded.

LEMMA 2.9. *If M_0 and M_t are as above and $T > 0$ is given, then there exists a constant $V = V(\operatorname{Vol}(M_0), T) > 0$ so that for all $r > 0$, all $x_0 \in \mathbf{R}^{n+1}$, and all $t \geq T$,*

$$(2.10) \qquad \operatorname{Vol}(B_r(x_0) \cap M_t) \leq V r^n.$$

Proof. For any $t_0 > t$ to be chosen later, Huisken's monotonicity formula gives

$$(2.11) \qquad [4\pi(t_0 - t)]^{-\frac{n}{2}} \, \mathrm{e}^{-\frac{1}{4}} \, \operatorname{Vol}(B_{\sqrt{t_0-t}}(x_0) \cap M_t)$$

$$\leq [4\pi(t_0 - t)]^{-\frac{n}{2}} \int_{B_{\sqrt{t_0-t}}(x_0) \cap M_t} \mathrm{e}^{\frac{|x-x_0|^2}{4(t-t_0)}}$$

$$\leq \int_{M_t} \Phi_{x_0,t_0}(\cdot, t) \leq \int_{M_0} \Phi_{x_0,t_0}(\cdot, 0)$$

$$= [4\pi t_0]^{-\frac{n}{2}} \int_{M_0} \mathrm{e}^{-\frac{|x-x_0|^2}{4t_0}} \leq (4\pi T)^{-\frac{n}{2}} \, \operatorname{Vol}(M_0).$$

Hence,

$$(2.12) \qquad \operatorname{Vol}(B_{\sqrt{t_0-t}}(x_0) \cap M_t) \leq \mathrm{e}^{\frac{1}{4}} \left(\frac{(t_0 - t)}{T} \right)^{\frac{n}{2}} \operatorname{Vol}(M_0).$$

Setting $t_0 = t + r^2$ and $V = \mathrm{e}^{\frac{1}{4}} \operatorname{Vol}(M_0) T^{-\frac{n}{2}}$, the claim now easily follows. $\quad\square$

Given a MCF M_t as above, a *limit flow* (or a *blow up flow*), $\mathcal{B}_t$, is defined as follows (cf., for instance, [Whi00, pp. 675–676] and [Ilm94, Ch. 7]). Let (x_j, t_j)

be a sequence of points in space-time and c_j a sequence of positive numbers with $c_j \to \infty$. A *limit flow* is a Brakke limit of the sequence of rescaled flows $t \to c_j(M_{c_j^{-2}t+t_j} - x_j)$. Such limits exist by Brakke's compactness theorem, [Bra78], and the *a priori* volume bound in Lemma 2.9. Note that when all x_j are equal to a common point x_0 and the t_j's are equal to a common time t_0, then the limit flow is a tangent flow at (x_0, t_0) (translated to the origin in space-time).

COROLLARY 2.13. *If M_0 and M_t are as above, then there exists a constant $V > 0$ so that for any limit flow $\mathcal{B}_t$, all $r > 0$, $x_0 \in \mathbf{R}^{n+1}$, and all t,*

$$(2.14) \qquad \mathrm{Vol}(B_r(x_0) \cap \mathcal{B}_t) \leq Vr^n.$$

Proof. This follows immediately from the lemma after noting that, since the initial hypersurface M_0 is smooth, the original flow remains smooth for a short time and, thus, any blow up near (in time) the initial hypersurface is a static plane. Thus, to prove the corollary, we may assume that the blow up is for times $t_j \geq T$ for some $T > 0$; the corollary now follows from the lemma. $\square$

If Σ is a hypersurface in $\mathbf{R}^{n+1}$, then we say that Σ has polynomial volume growth if there are constants C and d so that for all $r \geq 1$,

$$(2.15) \qquad \mathrm{Vol}(B_r(0) \cap \Sigma) \leq Cr^d.$$

Note that, by the corollary, any time-slice of a blow up has polynomial volume growth.

3. A variational characterization of self-shrinkers

In Lemma 2.2, we saw that a hypersurface is a self-shrinker if and only if

- it satisfies the second order differential equation $H = \frac{\langle x, \mathbf{n} \rangle}{2}$.

In this section, we will see that there are two other equivalent characterizations for a hypersurface Σ to be a self-shrinker:

- it is a minimal hypersurface in the conformally changed metric $g_{ij} = \mathrm{e}^{-\frac{|x|^2}{2n}} \delta_{ij}$ on $\mathbf{R}^{n+1}$;
- it is a critical point for the F_{x_0,t_0} functionals with respect to variations in all three parameters, i.e., critical for variations of Σ, x_0, and t_0.

The equivalence of these is proven in Lemma 3.1 and Proposition 3.6, respectively.

Unless explicitly stated otherwise, the results of the rest of the paper hold for hypersurfaces in $\mathbf{R}^{n+1}$ for any n.

3.1. *The first variation of F_{x_0,t_0}.* We will say that the one-parameter family of smooth immersed hypersurfaces $\Sigma_s \subset \mathbf{R}^{n+1}$ is a variation of Σ if Σ_s is given by a one-parameter family of embeddings $X_s : \Sigma \to \mathbf{R}^{n+1}$ with X_0 equal to the identity. We will call the vector field $\left.\frac{\partial X_s}{\partial s}\right|_{s=0}$ the variation vector field.

LEMMA 3.1. *Let $\Sigma_s \subset \mathbf{R}^{n+1}$ be a variation of Σ with variation vector field $\Sigma_0' = f\mathbf{n}$. If x_s and t_s are variations of x_0 and t_0 with $x_0' = y$ and $t_0' = h$, then $\frac{\partial}{\partial s}\left(F_{x_s,t_s}(\Sigma_s) \right)$ is*

$$(3.2) \quad (4\pi\, t_0)^{-\frac{n}{2}} \int_\Sigma \left[f \left(H - \frac{\langle x - x_0, \mathbf{n} \rangle}{2t_0} \right) \right.$$
$$\left. + h \left(\frac{|x - x_0|^2}{4t_0^2} - \frac{n}{2t_0} \right) + \frac{\langle x - x_0, y \rangle}{2t_0} \right] e^{\frac{-|x-x_0|^2}{4t_0}}\, d\mu.$$

Proof. From the first variation formula (for area), we know that

$$(3.3) \quad\quad\quad\quad\quad\quad (d\mu)' = fH\, d\mu.$$

The s derivative of the weight $e^{-|x-x_s|^2/(4t_s)}$ will have three separate terms coming from the variation of the surface, the variation of x_s, and the variation of t_s. Using, respectively, that $\nabla|x - x_s|^2 = 2(x - x_s)$,

$$(3.4) \quad\quad \partial_{t_s} \log\left[(4\pi t_s)^{-n/2}\, e^{-\frac{|x-x_s|^2}{4t_s}} \right] = \frac{-n}{2t_s} + \frac{|x - x_s|^2}{4t_s^2},$$

and $\partial_{x_s}|x - x_s|^2 = 2(x_s - x)$, we get that the derivative of

$$\log\left[(4\pi\, t_s)^{-\frac{n}{2}}\, e^{-|x-x_s|^2/(4t_s)} \right]$$

at $s = 0$ is given by

$$(3.5) \quad\quad -\frac{f}{2t_0}\langle x - x_0, \mathbf{n} \rangle + h\left(\frac{|x - x_0|^2}{4t_0^2} - \frac{n}{2t_0} \right) + \frac{1}{2t_0}\langle x - x_0, y \rangle.$$

Combining this with (3.3) gives (3.2). $\square$

3.2. *Critical points of F_{x_0,t_0} are self-shrinkers.* We will say that Σ is a critical point for F_{x_0,t_0} if it is critical with respect to all normal variations in Σ and all variations in x_0 and t_0. Strictly speaking, it is the triplet (Σ, x_0, t_0) that is a critical point of F, but we will refer to Σ as a critical point of F_{x_0,t_0}. The next proposition shows that Σ is a critical point for F_{x_0,t_0} if and only if it is the time $-t_0$ slice of a self-shrinking solution of the mean curvature flow that becomes extinct at the point x_0 and time 0.

PROPOSITION 3.6. *Σ is a critical point for F_{x_0,t_0} if and only if $H = \frac{\langle x - x_0, \mathbf{n} \rangle}{2t_0}$.*

Proof. It suffices to show that if $H = \frac{\langle x-x_0, \mathbf{n}\rangle}{2t_0}$, then the last two terms in (3.2) vanish for every h and every y. Obviously it is enough to show this for $x_0 = 0$ and $t_0 = 1$, and this follows from Lemma 3.25 below (or more precisely (3.26) and (3.27) below). $\qquad \square$

3.3. *Identities at self-shrinkers.* In this subsection, $\Sigma \subset \mathbf{R}^{n+1}$ is a smooth embedded hypersurface; Δ, div, and ∇ are the (submanifold) Laplacian, divergence, and gradient, respectively, on Σ.

We will need below that the linear operator

$$(3.7) \qquad \mathcal{L}v = \Delta v - \frac{1}{2}\langle x, \nabla v\rangle = e^{\frac{|x|^2}{4}} \operatorname{div}\left(e^{\frac{-|x|^2}{4}} \nabla v\right)$$

is self-adjoint in a weighted L^2 space; that is, the following lemma.

LEMMA 3.8. *If $\Sigma \subset \mathbf{R}^{n+1}$ is a hypersurface, u is a C^1 function with compact support, and v is a C^2 function, then*

$$(3.9) \qquad \int_\Sigma u(\mathcal{L}v)e^{\frac{-|x|^2}{4}} = -\int_\Sigma \langle \nabla v, \nabla u\rangle e^{\frac{-|x|^2}{4}}.$$

Proof. The lemma follows immediately from Stokes' theorem and (3.7). $\qquad \square$

The next corollary is an easy extension of Lemma 3.8, used later to justify computations when Σ is not closed.

COROLLARY 3.10. *Suppose that $\Sigma \subset \mathbf{R}^{n+1}$ is a complete hypersurface without boundary. If u, v are C^2 functions with*

$$(3.11) \qquad \int_\Sigma \left(|u\nabla v| + |\nabla u|\, |\nabla v| + |u\mathcal{L}v|\right) e^{\frac{-|x|^2}{4}} < \infty,$$

then we get

$$(3.12) \qquad \int_\Sigma u(\mathcal{L}v)e^{\frac{-|x|^2}{4}} = -\int_\Sigma \langle \nabla v, \nabla u\rangle e^{\frac{-|x|^2}{4}}.$$

Proof. Within this proof, we will use square brackets $[\cdot]$ to denote weighted integrals

$$(3.13) \qquad [f] = \int_\Sigma f e^{\frac{-|x|^2}{4}}.$$

Given any ϕ that is C^1 with compact support, we can apply Lemma 3.8 to $\phi\, u$ and v to get

$$(3.14) \qquad [\phi\, u\mathcal{L}v] = -[\phi\langle \nabla v, \nabla u\rangle] - [u\langle \nabla v, \nabla \phi\rangle].$$

Next, we apply this with $\phi = \phi_j$, where ϕ_j is one on the intrinsic ball of radius j in Σ centered at a fixed point and ϕ_j cuts off linearly to zero between B_j and

B_{j+1}. Since $|\phi_j|$ and $|\nabla\phi_j|$ are bounded by one, $\phi_j \to 1$, and $|\nabla\phi_j| \to 0$, the dominated convergence theorem (which applies because of (3.11)) gives that

(3.15) $[\phi_j\, u\mathcal{L}v] \to [u\mathcal{L}v]\,,$

(3.16) $[\phi_j\langle\nabla v, \nabla u\rangle] \to [\langle\nabla v, \nabla u\rangle]\,,$

(3.17) $[u\langle\nabla v, \nabla\phi\rangle] \to 0.$

Substituting this into (3.14) gives the corollary.   □

We will use Corollary 3.10 in the proof of Lemma 3.25 below. To keep things short, we will say that a function u is "in the weighted $W^{2,2}$ space" if

(3.18) $\int_\Sigma \left(|u|^2 + |\nabla u|^2 + |\mathcal{L}u|^2\right) e^{\frac{-|x|^2}{4}} < \infty.$

The point is that if u and v are both in the weighted $W^{2,2}$ space, then (3.11) is satisfied and we can apply Corollary 3.10 to u and v. In particular, since the right side of (3.12) is symmetric in u and v, it follows that

(3.19) $\int_\Sigma u(\mathcal{L}v)\, e^{\frac{-|x|^2}{4}} = \int_\Sigma v(\mathcal{L}u) e^{\frac{-|x|^2}{4}}.$

The next lemma applies the self-adjoint operator $\mathcal{L}$ to natural geometric quantities on a self-shrinker; x_i is the i-th component of the position vector x, i.e., $x_i = \langle x, \partial_i\rangle$.

LEMMA 3.20. *If $\Sigma \subset \mathbf{R}^{n+1}$ is a hypersurface with $H = \frac{\langle x, \mathbf{n}\rangle}{2}$, then*

(3.21) $\mathcal{L}x_i = -\frac{1}{2}x_i,$

(3.22) $\mathcal{L}|x|^2 = 2n - |x|^2.$

Proof. Since $\Delta x = -H\mathbf{n}$ on any hypersurface and $H = \frac{\langle x, \mathbf{n}\rangle}{2}$ on a self-shrinker, we have

(3.23) $\Delta x_i = \langle -H\mathbf{n}, \partial_i\rangle = \frac{-1}{2}\langle x^\perp, \partial_i\rangle = \frac{-1}{2}\langle x, \partial_i\rangle + \frac{1}{2}\langle x, (\partial_i)^T\rangle,$

giving (3.21). Combining this with $\nabla|x|^2 = 2x^T$, where x^T is the tangential projection of x, gives

(3.24) $\Delta|x|^2 = 2\langle \Delta x, x\rangle + 2|\nabla x|^2 = -\left|x^\perp\right|^2 + 2n.$

The second claim (3.22) follows since $\frac{1}{2}\langle x, \nabla|x|^2\rangle = \langle x, x^T\rangle = |x^T|^2$.   □

In the rest of this section, we will always assume that Σ is smooth, complete, $\partial\Sigma = \emptyset$, Σ has polynomial volume growth, and $H = \frac{\langle x, \mathbf{n}\rangle}{2}$.

Equations (3.26) and (3.27) in the next lemma are used in the proof of Proposition 3.6 whereas (3.28), (3.29), and the corollary that follow the next

lemma are first used in the next section when we compute the second variation of the F functional at a critical point.

LEMMA 3.25. *If Σ is complete, $\partial\Sigma = \emptyset$, Σ has polynomial volume growth, and $H = \frac{\langle x, \mathbf{n}\rangle}{2}$, then*

$$(3.26) \qquad \int_\Sigma \left(|x|^2 - 2n\right) e^{\frac{-|x|^2}{4}} = 0,$$

$$(3.27) \qquad \int_\Sigma x\, e^{\frac{-|x|^2}{4}} = 0 = \int_\Sigma x\, |x|^2 e^{\frac{-|x|^2}{4}},$$

$$(3.28) \qquad \int_\Sigma \left(|x|^4 - 2n(2n+4) + 16H^2\right) e^{\frac{-|x|^2}{4}} = 0.$$

Finally, if $w \in \mathbf{R}^{n+1}$ is a constant vector, then

$$(3.29) \qquad \int_\Sigma \langle x, w\rangle^2 e^{\frac{-|x|^2}{4}} = 2\int_\Sigma \left|w^T\right|^2 e^{\frac{-|x|^2}{4}}.$$

Proof. In this proof, we will use square brackets $[\cdot]$ to denote weighted integrals

$$(3.30) \qquad [f] = \int_\Sigma f e^{\frac{-|x|^2}{4}}.$$

We will repeatedly use that the constant functions, the function x_i, the function $\langle x, w\rangle$, and the function $|x|^2$ are all in the weighted $W^{2,2}$ space and, thus, that we can apply (the self-adjointness) Corollary 3.10 to any pair of these functions.

To get the first claim (3.26), use $u = 1$ and $v = |x|^2$ in (3.19) to get

$$(3.31) \qquad 0 = \left[1\mathcal{L}|x|^2\right] = \left[2n - |x|^2\right],$$

where the last equality used that $\mathcal{L}|x|^2 = 2n - |x|^2$ by (3.22) in Lemma 3.20. The first equality in (3.27) follows similarly by taking $u = 1$ and $v = x_i$ and using that $\mathcal{L}x_i = -\frac{1}{2}x_i$ by (3.21).

To get the second equality in (3.27), argue similarly with $u = x_i$ and $v = |x|^2$ to get

$$(3.32) \qquad -\frac{1}{2}\left[x_i|x|^2\right] = \left[|x|^2\mathcal{L}x_i\right] = \left[x_i\mathcal{L}|x|^2\right] = \left[x_i(2n - |x|^2)\right] = -\left[x_i|x|^2\right],$$

where the last equality used that $[x_i] = 0$. Since the constants in front of the $\left[x_i|x|^2\right]$ terms differ, we get that $\left[x_i|x|^2\right] = 0$ as claimed.

To get (3.28), apply (3.12) with $u = v = |x|^2$ and (3.22) to get

$$(3.33) \qquad 2n\left[|x|^2\right] - \left[|x|^4\right] = \left[|x|^2\mathcal{L}|x|^2\right] = -\left[\left|\nabla|x|^2\right|^2\right] = -4\left[\left|x^T\right|^2\right].$$

This gives (3.28) since $\left|x^T\right|^2 = |x|^2 - \left|x^\perp\right|^2 = |x|^2 - 4H^2$ and $\left[|x|^2\right] = [2n]$. Finally, the last claim (3.29) follows from (3.12) with $u = v = \langle x, w\rangle$ since $\mathcal{L}\langle x, w\rangle = -\frac{1}{2}\langle x, w\rangle$ and $\nabla\langle x, w\rangle = w^T$. $\qquad\square$

COROLLARY 3.34. *If Σ is as in Lemma 3.25, then*

$$(3.35) \qquad \int_\Sigma \left[\left(\frac{|x|^2}{4} - \frac{n}{2} \right)^2 - \frac{n}{2} \right] e^{\frac{-|x|^2}{4}} = - \int_\Sigma H^2 \, e^{\frac{-|x|^2}{4}}.$$

Proof. As in the proof of Lemma 3.25, square brackets $[\cdot]$ will denote weighted integrals. Squaring things out gives

(3.36)

$$4 \left[\left(\frac{|x|^2}{4} - \frac{n}{2} \right)^2 - \frac{n}{2} \right] = \left[\frac{|x|^4}{4} - n |x|^2 + n^2 - 2n \right]$$

$$= \left[n(n+2) - 4H^2 - n(2n) + n^2 - 2n \right] = -4 \left[H^2 \right],$$

where the second equality used Lemma 3.25. $\qquad \square$

4. The second variation of F_{x_0,t_0} and the operator L_{x_0,t_0}

In this section, we will calculate the second variation formula of the F functional for simultaneous variations in all three parameters Σ, x_0, and t_0. The most important case is when Σ is a critical point where we will use our calculation to formulate a notion of stability.

As we saw in Section 3, critical points of the F functional are the same as critical points for F_{x_0,t_0}, where x_0 and t_0 are fixed and we vary the hypersurface alone. These are easily seen to characterize self-shrinkers and are equivalent to being a minimal hypersurface in $\mathbf{R}^{n+1}$ in a conformally changed metric. In Section 5 we will see that it turns out that as minimal hypersurfaces in the conformally changed metric, they are unstable in the usual sense of minimal surfaces (cf. [CM06, Ch. 1]). Equivalently, every critical point of the F functional is unstable if you fix x_0 and t_0 and vary Σ alone.

In spite of this, it is still possible to formulate a natural version of stability for the full F functionals. The point is that the apparent instability comes from translating the hypersurface in space and time, and our notion of stability will account for this.

4.1. *The general second variation formula.* In the next theorem, Σ_s will be a one-parameter family of hypersurfaces, where the s derivative $\partial_s \Sigma_s$ will be assumed to be normal to Σ_s for each s. Namely, we assume that $\partial_s \Sigma_s = f\mathbf{n}$, where f is a function and $\mathbf{n}$ is the unit normal to Σ_s (and we have suppressed the s dependence). This allows us to apply Lemma 3.1 for each s and then differentiate with respect to s.

THEOREM 4.1. *If Σ_s is a normal variation of Σ, x_s is a variation of x_0, and t_s is a variation of t_0 with*

$$(4.2) \qquad \partial_s \big|_{s=0} \Sigma_s = f\mathbf{n}, \; \partial_s \big|_{s=0} x_s = y, \; and \; \partial_s \big|_{s=0} t_s = h,$$

$$(4.3) \qquad \partial_s\big|_{s=0} f = f', \ \partial_{ss}\big|_{s=0} x_s = y', \ and \ \partial_{ss}\big|_{s=0} t_s = h',$$

then setting $F'' = \partial_{ss}\big|_{s=0} (F_{x_s,t_s}(\Sigma_s))$ *gives*

$$(4.4) \quad F'' = (4\pi t_0)^{-n/2} \int_\Sigma \left[-f L_{x_0,t_0} f + fh \frac{\langle x - x_0, \mathbf{n}\rangle}{t_0^2} - h^2 \frac{|x - x_0|^2 - nt_0}{2t_0^3} \right.$$

$$+ f \frac{\langle y, \mathbf{n}\rangle}{t_0} - \frac{|y|^2}{2t_0} - h \frac{\langle x - x_0, y\rangle}{t_0^2}$$

$$+ \left(f \left(H - \frac{\langle x - x_0, \mathbf{n}\rangle}{2t_0} \right) + h \left(\frac{|x - x_0|^2}{4t_0^2} - \frac{n}{2t_0} \right) + \langle \frac{x - x_0}{2t_0}, y\rangle \right)^2$$

$$+ f' \left(H - \frac{\langle x - x_0, \mathbf{n}\rangle}{2t_0} \right)$$

$$\left. + h' \left(\frac{|x - x_0|^2}{4t_0^2} - \frac{n}{2t_0} \right) + \frac{\langle x - x_0, y'\rangle}{2t_0} \right] e^{\frac{-|x-x_0|^2}{4t_0}} d\mu,$$

where $L_{x_0,t_0} = \Delta + |A|^2 - \frac{1}{2t_0}\langle x - x_0, \nabla(\cdot)\rangle + \frac{1}{2t_0}$ *is a (nonsymmetric) second order operator.*

    Proof. Within this proof, we will use square brackets $[\cdot]_{x_s,t_s}$ to denote weighted integrals, i.e.,

$$(4.5) \qquad\qquad [g]_{x_s,t_s} = (4\pi t_s)^{-\frac{n}{2}} \int g\, e^{\frac{-|x-x_s|^2}{4t_s}} d\mu.$$

Letting primes denote derivatives with respect to s at $s = 0$, differentiating (3.2) gives

(4.6)

$$F'' = \left[f \left(H - \frac{\langle x - x_s, \mathbf{n}\rangle}{2t_s} \right)' + h \left(\frac{|x - x_s|^2}{4t_s^2} - \frac{n}{2t_s} \right)' + \left\langle \left(\frac{x - x_s}{2t_s} \right)', y \right\rangle \right.$$

$$+ \left(f \left(H - \frac{\langle x - x_0, \mathbf{n}\rangle}{2t_0} \right) + h \left(\frac{|x - x_0|^2}{4t_0^2} - \frac{n}{2t_0} \right) + \langle \frac{x - x_0}{2t_0}, y\rangle \right)^2$$

$$\left. + f' \left(H - \frac{\langle x - x_0, \mathbf{n}\rangle}{2t_0} \right) + h' \left(\frac{|x - x_0|^2}{4t_0^2} - \frac{n}{2t_0} \right) + \frac{\langle x - x_0, y'\rangle}{2t_0} \right]_{x_0,t_0},$$

where we used (3.5) to compute the term on the second line. Standard calculations for the first variations of H and $\mathbf{n}$ (see, for instance, in [HP99, Th. 3.2] or [CMd]) give

$$(4.7) \qquad\qquad\qquad H' = -\Delta f - |A|^2 f,$$

$$(4.8) \qquad\qquad\qquad \mathbf{n}' = -\nabla f.$$

Since $x' = f\mathbf{n}$, we get

$$(4.9) \qquad\qquad \langle x - x_s, \mathbf{n}\rangle' = f - \langle y, \mathbf{n}\rangle - \langle x - x_0, \nabla f\rangle.$$

Hence, since $(t_s^{-1})' = -h \, t_0^{-2}$ and $(x_s)' = y$, we have

$$(4.10) \quad \left(H - \frac{\langle x - x_s, \mathbf{n} \rangle}{2t_s} \right)'$$

$$= -\Delta f - |A|^2 f - \frac{f - \langle y, \mathbf{n} \rangle - \langle x - x_0, \nabla f \rangle}{2t_0} + \frac{h}{2t_0^2} \langle x - x_0, \mathbf{n} \rangle$$

$$= -L_{x_0,t_0} f + \frac{h}{2t_0^2} \langle x - x_0, \mathbf{n} \rangle + \frac{\langle y, \mathbf{n} \rangle}{2t_0},$$

where the second equality used the definition of L_{x_0,t_0}. The second term is given by

$$(4.11)$$

$$\left(\frac{|x - x_s|^2}{4t_s^2} - \frac{n}{2t_s} \right)' = -h \frac{|x - x_0|^2}{2t_0^3} + \frac{\langle x - x_0, f\mathbf{n} \rangle}{2t_0^2} - \frac{\langle x - x_0, y \rangle}{2t_0^2} + \frac{n}{2t_0^2} h$$

$$= f \frac{\langle x - x_0, \mathbf{n} \rangle}{2t_0^2} - h \frac{|x - x_0|^2 - nt_0}{2t_0^3} - \frac{\langle x - x_0, y \rangle}{2t_0^2}.$$

For the third term, observe that

$$(4.12) \quad \left(\frac{x - x_s}{2t_s} \right)' = \frac{f\mathbf{n} - y - h(x - x_0)/t_0}{2t_0}.$$

Combining these gives the theorem. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

4.2. *The second variation at a critical point.* In this subsection, we specialize our calculations from the previous section to the case where Σ is a critical point.

In the next theorem and throughout the remainder of this paper, we let $L = L_{0,1}$ be the (nonsymmetric) second order operator

$$(4.13) \qquad L = \mathcal{L} + |A|^2 + \frac{1}{2} = \Delta + |A|^2 - \frac{1}{2}\langle x, \nabla(\cdot) \rangle + \frac{1}{2}.$$

THEOREM 4.14. *Suppose that Σ is complete, $\partial\Sigma = \emptyset$, Σ has polynomial volume growth, and Σ is a critical point for $F_{0,1}$. If Σ_s is a normal variation of Σ, x_s, t_s are variations with $x_0 = 0$ and $t_0 = 1$, and*

$$(4.15) \qquad \partial_s\big|_{s=0} \Sigma_s = f\mathbf{n}, \ \partial_s\big|_{s=0} x_s = y, \ \text{and} \ \partial_s\big|_{s=0} t_s = h,$$

then setting $F'' = \partial_{ss}\big|_{s=0} (F_{x_s,t_s}(\Sigma_s))$ gives

$$(4.16)$$

$$F'' = (4\pi)^{-n/2} \int_\Sigma \left(-fLf + 2fh \, H - h^2 H^2 + f\langle y, \mathbf{n} \rangle - \frac{\langle y, \mathbf{n} \rangle^2}{2} \right) e^{\frac{-|x|^2}{4}} \, d\mu.$$

Proof. Since Σ is a critical point for $F_{0,1}$, we have by (3.2) that

$$(4.17) \qquad\qquad\qquad\qquad H = \frac{\langle x, \mathbf{n} \rangle}{2}.$$

Within this proof, we will use square brackets $[\cdot]$ to denote weighted integrals:

$$(4.18) \qquad [g] = (4\pi)^{-\frac{n}{2}} \int_\Sigma g\, e^{\frac{-|x|^2}{4}}\, d\mu.$$

It follows from Lemma 3.25 that

$$(4.19) \qquad \left[\frac{|x|^2}{4} - \frac{n}{2} \right] = 0 \text{ and } [x] = \left[x\,|x|^2 \right] = 0.$$

Theorem 4.1 (with $x_0 = 0$ and $t_0 = 1$) gives

$$(4.20) \quad F'' = \left[-fLf + fh\langle x, \mathbf{n}\rangle - h^2\frac{|x|^2 - n}{2} + f\langle y, \mathbf{n}\rangle - \frac{|y|^2}{2} - h\langle x, y\rangle \right]$$
$$+ \left[\left(h\left(\frac{|x|^2}{4} - \frac{n}{2}\right) + \frac{1}{2}\langle x, y\rangle \right)^2 \right],$$

where we used (4.17) and (4.19) to see that a number of the terms from Theorem 4.1 vanish. Squaring out the last term and using (4.17) and (4.19) gives

$$(4.21) \quad F'' = \left[-fLf + 2fh\,H - \frac{n}{2}\,h^2 + f\langle y, \mathbf{n}\rangle - \frac{|y|^2}{2} \right]$$
$$+ \left[h^2\left(\frac{|x|^2}{4} - \frac{n}{2}\right)^2 + \frac{1}{4}\langle x, y\rangle^2 \right] + h\left[\left(\frac{|x|^2}{4} - \frac{n}{2}\right)\langle x, y\rangle \right].$$

Using the last two equalities in (4.19), we see that the last term in (4.21) vanishes and thus

$$(4.22)$$
$$F'' = \left[-fLf + 2fh\,H + h^2\left(\frac{|x|^2}{4} - \frac{n}{2}\right)^2 - \frac{n\,h^2}{2} + f\langle y, \mathbf{n}\rangle - \frac{|y|^2}{2} + \frac{1}{4}\langle x, y\rangle^2 \right]$$
$$= \left[-fLf + 2fh\,H - h^2 H^2 + f\langle y, \mathbf{n}\rangle - \frac{\left|y^\perp\right|^2}{2} \right],$$

where the second equality used Corollary 3.34 and (3.29). $\qquad\square$

4.3. *F-stable self-shrinkers.* We will say that a critical point Σ for $F_{0,1}$ is *F-stable* if, for every normal variation $f\mathbf{n}$, there exist variations of x_0 and t_0 that make $F'' \geq 0$.

Roughly speaking, Σ is stable when the only way to decrease the $F_{0,1}$ functional is to translate in space or time. It is easy to see that the sphere of radius $\sqrt{2n}$ is F-stable in $\mathbf{R}^{n+1}$.

LEMMA 4.23. *The n-sphere of radius $\sqrt{2n}$ in $\mathbf{R}^{n+1}$ is F-stable.*

Proof. Note that $x^T = 0$, A is $1/\sqrt{2n}$ times the metric, and $L = \Delta + 1$. Therefore, by Theorem 4.14, the lemma will follow from showing that given an arbitrary normal variation $f\mathbf{n}$, there exist $h \in \mathbf{R}$ and $y \in \mathbf{R}^{n+1}$ so that

$$(4.24) \qquad \int_{\mathbf{S}^n} \left[-f(\Delta f + f) + \sqrt{2n}\, fh - \frac{n}{2} h^2 + f\langle y, \mathbf{n}\rangle - \frac{\langle y, \mathbf{n}\rangle^2}{2} \right] \geq 0.$$

Recall that the eigenvalues of the Laplacian[11] on the n-sphere of radius one are given by $k^2 + (n-1)k$ for $k = 0, 1, \dots$ with 0 corresponding to the constant function and the first nonzero eigenvalue n corresponding to the restrictions of the linear functions in $\mathbf{R}^{n+1}$. It follows that the eigenvalues of Δ on the sphere of radius $\sqrt{2n}$ are given by

$$(4.25) \qquad\qquad\qquad \mu_k = \frac{k^2 + (n-1)k}{2n},$$

with $\mu_0 = 0$ corresponding to the constant functions and $\mu_1 = \frac{1}{2}$ corresponding to the linear functions. Let E be the space of $W^{1,2}$ functions that are orthogonal to constants and linear functions; equivalently, E is the span of all the eigenfunctions for μ_k for all $k \geq 2$. Therefore, we can choose $a \in \mathbf{R}$ and $z \in \mathbf{R}^{n+1}$ so that

$$(4.26) \qquad\qquad\qquad f_0 \equiv f - a - \langle z, \mathbf{n}\rangle \in E.$$

Using the orthogonality of the different eigenspaces, we get that

$$(4.27)$$
$$\int_{\mathbf{S}^n} -f(\Delta f + f) \geq (\mu_2 - 1)\int_{\mathbf{S}^n} f_0^2 + (\mu_1 - 1)\int_{\mathbf{S}^n} \langle z, \mathbf{n}\rangle^2 + (\mu_0 - 1)\int_{\mathbf{S}^n} a^2$$
$$= \frac{1}{n}\int_{\mathbf{S}^n} f_0^2 - \frac{1}{2}\int_{\mathbf{S}^n} \langle z, \mathbf{n}\rangle^2 - \int_{\mathbf{S}^n} a^2.$$

Again using the orthogonality of different eigenspaces, we get

$$(4.28) \qquad \int_{\mathbf{S}^n} \left[\sqrt{2n}\, fh + f\langle y, \mathbf{n}\rangle \right] = \int_{\mathbf{S}^n} \left[\sqrt{2n}\, ah + \langle z, \mathbf{n}\rangle\langle y, \mathbf{n}\rangle \right].$$

Combining (4.27) and (4.28), we get that the left-hand side of (4.24) is greater than or equal to

$$(4.29) \qquad \int_{\mathbf{S}^n} \left[\frac{f_0^2}{n} - \frac{1}{2}\langle z, \mathbf{n}\rangle^2 - a^2 + \sqrt{2n}\, ah - \frac{n}{2} h^2 + \langle z, \mathbf{n}\rangle\langle y, \mathbf{n}\rangle - \frac{\langle y, \mathbf{n}\rangle^2}{2} \right]$$
$$= \int_{\mathbf{S}^n} \left[\frac{f_0^2}{n} - \frac{1}{2}(\langle z, \mathbf{n}\rangle - \langle y, \mathbf{n}\rangle)^2 - \left(a - \frac{\sqrt{n}\, h}{\sqrt{2}}\right)^2 \right].$$

This can be made nonnegative by choosing $y = z$ and $h = \frac{\sqrt{2}\, a}{\sqrt{n}}$. $\qquad\qquad \square$

We will prove the following converse.

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[11]See, e.g., Chavel [Cha84, eq. (14), p. 35].

THEOREM 4.30. *The n-sphere with radius $\sqrt{2n}$ and center 0 is the only* (*smooth*) *closed F-stable hypersurface in $\mathbf{R}^{n+1}$ for any $n \geq 2$.*

In the noncompact case, we show

THEOREM 4.31. *The n-planes through 0 are the only complete* (*smooth*) *embedded noncompact F-stable hypersurfaces in $\mathbf{R}^{n+1}$ with no boundary and polynomial volume growth.*

We will prove Theorem 4.30 in Section 6 and then prove Theorem 4.31 in Section 11. Theorem 0.16 from the introduction follows from combining Theorems 4.30 and 4.31.

5. Eigenfunctions and eigenvalues of L

Throughout this section, $\Sigma \subset \mathbf{R}^{n+1}$ is a smooth hypersurface without boundary, $H = \frac{\langle x, \mathbf{n} \rangle}{2}$, and

$$(5.1) \qquad L = \Delta + |A|^2 + \frac{1}{2} - \frac{1}{2} \langle x, \nabla(\cdot) \rangle$$

is as in Theorem 4.14.

In this section we will show that H and the translations are eigenfunctions of L. We will see that the eigenvalue corresponding to H is -1, whereas the eigenvalue corresponding to the translations (in space) is $-\frac{1}{2}$. We will also see that in a weighted space these eigenfunctions are orthogonal. In fact, orthogonality is a direct consequence of that they are eigenfunctions corresponding to different eigenvalues and that in a weighted space the operator L is self-adjoint. Namely, we show the following theorem.

THEOREM 5.2. *The mean curvature H and the normal part $\langle v, \mathbf{n} \rangle$ of a constant vector field v are eigenfunctions of L with*

$$(5.3) \qquad LH = H \ \text{ and } \ L\langle v, \mathbf{n} \rangle = \frac{1}{2} \langle v, \mathbf{n} \rangle.$$

Moreover, if Σ is compact, then L is self-adjoint in the weighted space and

$$(5.4) \qquad -\int_{\Sigma} (u_1 L u_2) \mathrm{e}^{\frac{-|x|^2}{4}} = \int_{\Sigma} \left(\langle \nabla u_1, \nabla u_2 \rangle - |A|^2 u_1 u_2 - \frac{1}{2} u_1 u_2 \right) \mathrm{e}^{\frac{-|x|^2}{4}}.$$

Once we have this, then we can use a theorem of Huisken together with the usual characterization of the first eigenvalue to conclude that if Σ is a closed F-stable self-shrinker, then Σ must be a sphere.

5.1. *H and $\langle v, \mathbf{n} \rangle$ are eigenfunctions of L.*

LEMMA 5.5. *$LH = H$ and $L\langle v, \mathbf{n} \rangle = \frac{1}{2} \langle v, \mathbf{n} \rangle$ for any constant vector field v.*

Proof. Letting e_i be an orthonormal frame for Σ and differentiating $H = \frac{1}{2}\langle x, \mathbf{n}\rangle$ as in Theorem 4.1 of [Hui90] (which has a different normalization) gives

$$(5.6) \qquad 2\nabla_{e_i} H = \langle e_i, \mathbf{n}\rangle + \langle x, \nabla_{e_i}\mathbf{n}\rangle = -a_{ij}\langle x, e_j\rangle,$$

where the first equality used that $\nabla_e x = e$ for any vector e and the second equality used (0.19). Working at a point p and choosing the frame e_i so that $\nabla_{e_i}^T e_j(p) = 0$, differentiating again gives at p that

$$(5.7) \qquad 2\nabla_{e_k}\nabla_{e_i} H = -a_{ij,k}\langle x, e_j\rangle - a_{ij}\langle e_k, e_j\rangle - a_{ij}\langle x, \nabla_{e_k} e_j\rangle.$$

Using that $\nabla_{e_i}^T e_j = 0$ at p, the last term in (5.7) can be rewritten as

$$(5.8) \qquad - a_{ij}\langle x, \nabla_{e_k} e_j\rangle = -a_{ij}\langle x, a_{kj}\mathbf{n}\rangle = -2Ha_{ij}a_{kj}.$$

Taking the trace of (5.7), using Codazzi's equation (i.e., $a_{ij,i} = a_{ii,j} = -H_j$; see, e.g., (B.5) in [Sim83]), and substituting (5.8) gives at p that

$$(5.9) \qquad 2\Delta H = \langle x, \nabla H\rangle + H - 2|A|^2 H,$$

and thus,

$$(5.10) \qquad LH \equiv \Delta H + |A|^2 H + \frac{1}{2}H - \frac{1}{2}\langle x, \nabla H\rangle = H.$$

Since p is arbitrary and (5.10) is independent of the frame, we conclude that $LH = H$.

Fix a constant vector $v \in \mathbf{R}^{n+1}$, and set $f = \langle v, \mathbf{n}\rangle$. We have

$$(5.11) \qquad \nabla_{e_i} f = \langle v, \nabla_{e_i}\mathbf{n}\rangle = -a_{ij}\langle v, e_j\rangle.$$

Working again at a fixed point p and choosing the frame as above, differentiating again and using the Codazzi equation gives at p that

$$(5.12) \qquad \nabla_{e_k}\nabla_{e_i} f = -a_{ik,j}\langle v, e_j\rangle - a_{ij}\langle v, a_{jk}\mathbf{n}\rangle.$$

Taking the trace gives

$$(5.13) \qquad \Delta f = \langle v, \nabla H\rangle - |A|^2 f.$$

(Note that we have not used the self-shrinker equation for H in (5.13), but we will use it next.) From (5.6), we know that

$$(5.14) \qquad \langle v, \nabla H\rangle = -\frac{1}{2}\langle x, e_j\rangle a_{ij}\langle v, e_i\rangle = \frac{1}{2}\langle x, \nabla f\rangle,$$

where the last equality used (5.11). Substituting this back into (5.13), we finally get that $Lf = \Delta f + |A|^2 f + \frac{1}{2}f - \frac{1}{2}\langle x, \nabla f\rangle = \frac{1}{2}f$. $\qquad\square$

Proof of Theorem 5.2. The two claims in (5.3) are proven in Lemma 5.5. The last claim follows immediately from Lemma 3.8. $\qquad\square$

5.2. *The eigenvalues of L.* Recall that we will say that $\mu \in \mathbf{R}$ is an eigenvalue of L if there is a (not identically zero) function u with $Lu = -\mu u$. The operator L is not self-adjoint with respect to the L^2-inner product because of the first order term, but it is self-adjoint with respect to the weighted inner product by Theorem 5.2. In particular, standard spectral theory[12] gives the following corollary.

COROLLARY 5.15. *If $\Sigma \subset \mathbf{R}^{n+1}$ is a smooth closed hypersurface, then*

(1) *L has real eigenvalues $\mu_1 < \mu_2 \leq \ldots$ with $\mu_k \to \infty$.*
(2) *There is an orthonormal basis $\{u_k\}$ for the weighted L^2 space with $Lu_k = -\mu_k u_k$.*
(3) *The lowest eigenvalue μ_1 is characterized by*

$$(5.16) \qquad \mu_1 = \inf_f \ \frac{\int_\Sigma \left(|\nabla f|^2 - |A|^2 f^2 - \frac{1}{2}\, f^2 \right) \mathrm{e}^{-\frac{|x|^2}{4}}}{\int_\Sigma f^2 \mathrm{e}^{-\frac{|x|^2}{4}}},$$

where the infimum is taken over smooth functions f (that are not identically zero).
(4) *Any eigenfunction for μ_1 does not change sign and, consequently, if u is any weak solution of $Lu = -\mu_1 u$, then u is a constant multiple of u_1.*

Remark 5.17. There is a corresponding result when Σ is not closed. In this case, we choose a smooth open subset $\Omega \subset \Sigma$ with compact closure and restrict to functions with compact support in Ω. In this case, (1) and (2) are unchanged, the infimum in (3) is over smooth functions with compact support in Ω, and (4) is for Dirichlet eigenfunctions (i.e., ones that vanish on $\partial\Omega$).

6. **Spheres are the only closed F-stable self-shrinkers**

We now have all of the tools that we need to prove that spheres are the only smooth, embedded closed self-shrinkers that are F-stable (this is Theorem 4.30).

Proof of Theorem 4.30. Since Σ is closed, the mean curvature H cannot vanish identically. By Huisken's classification, [Hui90], it suffices to show that H does not change sign. (By the Harnack inequality, it is then positive everywhere.) We will argue by contradiction, so suppose that H changes sign.

In order to show that Σ is unstable, we must find a function f on Σ so that $F'' < 0$ no matter what values of h and y that we choose. Given a variation

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[12]These facts are proven in a similar setting (Dirichlet eigenvalues for divergence operators on smooth, bounded open sets in $\mathbf{R}^n$) in Theorems 1 and 2 on pages 335 and 336, respectively, in Evans, [Eva98]. The proof uses only linear elliptic theory and self-adjointness, and it carries over to this setting with standard modifications.

$f\mathbf{n}$ of Σ, a variation y of x_0, and a variation h of t_0, Theorem 4.14 gives

$$(6.1) \quad F'' = (4\pi)^{-n/2} \int_\Sigma \left[-fLf + 2fh\,H - h^2 H^2 + f\langle y, \mathbf{n}\rangle - \frac{\langle y, \mathbf{n}\rangle^2}{2} \right] e^{\frac{-|x|^2}{4}}.$$

We know from Theorem 5.2 that H is an eigenfunction for L with eigenvalue -1. However, since H changes sign, (4) in Theorem 5.15 implies that -1 is not the smallest eigenvalue for L. Thus, we get a positive function f with $Lf = \mu f$, where $\mu > 1$. Since L is self-adjoint in the weighted L^2 space by (5.4), we see that f must be orthogonal to the eigenspaces with different eigenvalues (this is (2) in Theorem 5.15). Therefore, by Theorem 5.2, f is orthogonal to H and the translations; i.e., for any $y \in \mathbf{R}^{n+1}$, we have

$$(6.2) \qquad\qquad 0 = \int_\Sigma fH\,e^{\frac{-|x|^2}{4}} = \int_\Sigma f\langle y, \mathbf{n}\rangle e^{\frac{-|x|^2}{4}}.$$

Substituting this into (6.1) gives

$$(6.3) \quad (4\pi)^{n/2}\,F'' = \int_\Sigma \left[-f\mu f - h^2 H^2 - \frac{\langle y, \mathbf{n}\rangle^2}{2} \right] e^{\frac{-|x|^2}{4}} \leq -\mu \int_\Sigma f^2 e^{\frac{-|x|^2}{4}}.$$

Since $\mu > 1$ is positive, it follows that $F'' < 0$ for every choice of h and y as desired. $\qquad\square$

7. Entropy

The entropy $\lambda(\Sigma)$ of a hypersurface Σ is defined to be

$$(7.1) \qquad\qquad \lambda(\Sigma) = \sup_{x_0 \in \mathbf{R}^{n+1}, t_0 > 0} F_{x_0, t_0}(\Sigma).$$

Even though the supremum is over noncompact space-time, we will see that this supremum is achieved on two important classes of hypersurfaces: compact hypersurfaces with $\lambda > 1$ and self-shrinkers with polynomial volume growth.

7.1. *The entropy is achieved for compact hypersurfaces with $\lambda > 1$.* We will need a few simple properties of the F functionals that we summarize in the following lemma.

LEMMA 7.2. *If $\Sigma \subset \mathbf{R}^{n+1}$ is a smooth complete embedded hypersurface without boundary and with polynomial volume growth, then*

(1) *$F_{x_0, t_0}(\Sigma)$ is a smooth function of x_0 and t_0 on $\mathbf{R}^{n+1} \times (0, \infty)$.*
(2) *Given any $t_0 > 0$ and any x_0, we have $\partial_{t_0} F_{x_0, t_0}(\Sigma) \geq -\frac{\lambda(\Sigma)}{4} \sup_\Sigma H^2$.*
(3) *For each x_0, $\lim_{t_0 \to 0} F_{x_0, t_0}(\Sigma)$ is 1 if $x_0 \in \Sigma$ and is 0 otherwise.*
(4) *If Σ is closed, then $\lambda(\Sigma) < \infty$.*

Proof. Property (1) follows immediately from differentiating under the integral sign and the polynomial volume growth.

Since the hypotheses and conclusion in (2) are invariant under translation, it suffices to do the case where $x_0 = 0$. The first variation formula (Lemma 3.1) gives

$$(7.3) \qquad \partial_{t_0} F_{0,t_0}(\Sigma) = (4\pi t_0)^{-\frac{n}{2}} \int_\Sigma \frac{|x|^2 - 2n t_0}{4 t_0^2} \, \mathrm{e}^{-\frac{|x|^2}{4 t_0}} .$$

Since $\Delta |x|^2 = 2n - 2H\langle x, \mathbf{n}\rangle$ and $\Delta \, \mathrm{e}^f = \mathrm{e}^f \left(\Delta f + |\nabla f|^2\right)$, we have

$$(7.4)$$

$$\mathrm{e}^{\frac{|x|^2}{4 t_0}} \Delta \, \mathrm{e}^{-\frac{|x|^2}{4 t_0}} = \frac{|x^T|^2}{4 t_0^2} - \frac{2n}{4 t_0} + \frac{H\langle x, \mathbf{n}\rangle}{2 t_0} = \frac{|x|^2 - 2n \, t_0}{4 t_0^2} - \frac{|x^\perp|^2}{4 t_0^2} + \frac{H\langle x, \mathbf{n}\rangle}{2 t_0}$$

$$\leq \frac{|x|^2 - 2n \, t_0}{4 \, t_0^2} + \frac{H^2}{4},$$

where the inequality used $2ab \leq a^2 + b^2$. Since Σ has polynomial volume growth and the vector field $\nabla \mathrm{e}^{-\frac{|x|^2}{4 t_0}}$ decays exponentially, Stokes' theorem gives

$$(7.5) \qquad \partial_{t_0} F_{0,t_0}(\Sigma) \geq -(4\pi t_0)^{-\frac{n}{2}} \int_\Sigma \frac{H^2}{4} \, \mathrm{e}^{-\frac{|x|^2}{4 t_0}}$$

$$\geq -\frac{1}{4} \, F_{x_0,t_0}(\Sigma) \, \sup_\Sigma H^2 \geq -\frac{\lambda(\Sigma)}{4} \, \sup_\Sigma H^2,$$

where the last inequality used that, by definition, $\lambda(\Sigma) \geq F_{x_0,t_0}(\Sigma)$.

Property (3) is a standard consequence of the fact that smooth hypersurfaces are approximated by a hyperplane on small scales (recall that the function $(4\pi t_0)^{-\frac{n}{2}} \mathrm{e}^{-\frac{|x-x_0|^2}{4 t_0}}$ is a heat kernel on a hyperplane through x_0 and has integral one on the hyperplane independent of t_0).

To see property (4), observe that since Σ is smooth and closed, there is a constant $V > 0$ so that for every x_0 and for every radius $R > 0$,

$$(7.6) \qquad \mathrm{Vol}(B_R(x_0) \cap \Sigma) \leq V R^n .$$

Property (4) follows easily from this. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

LEMMA 7.7. *If $\Sigma \subset \mathbf{R}^{n+1}$ is a smooth closed embedded hypersurface and $\lambda(\Sigma) > 1$, then there exist $x_0 \in \mathbf{R}^{n+1}$ and $t_0 > 0$ so that $\lambda = F_{x_0,t_0}(\Sigma)$.*

Proof. For each fixed $t_0 > 0$, it is easy to see that $\lim_{|x_0|\to\infty} F_{x_0,t_0}(\Sigma) = 0$ by the exponential decay of the weight function together with the compactness of Σ. In particular, for each fixed $t_0 > 0$, the maximum of $F_{x_0,t_0}(\Sigma)$ is achieved at some x_0. Moreover, the first variation formula (Lemma 3.1) shows that this maximum occurs when the weighted integral of $(x - x_0)$ vanishes, but this can only occur when x_0 lies in the convex hull of Σ. It remains to take the supremum of these maxima as we vary t_0; i.e., we must show that the F functionals are strictly less than $\lambda(\Sigma)$ when t_0 goes to either zero or infinity.

It is easy to see that $F_{x_0,t_0}(\Sigma)$ goes to zero (uniformly in x_0) as $t_0 \to \infty$ since

(7.8) $$F_{x_0,t_0}(\Sigma) \le (4\pi\, t_0)^{-\frac{n}{2}}\, \mathrm{Vol}(\Sigma).$$

Fix an $\varepsilon > 0$ with $1 + 3\varepsilon < \lambda(\Sigma)$. Given any x, property (3) in Lemma 7.2 gives a $t_x > 0$ so that $F_{x,t_0}(\Sigma) < 1 + \varepsilon$ for all $t_0 \le t_x$. Using the continuity (in x) of $F_{x,t_x}(\Sigma)$ and the compactness of Σ, we can choose a finite collection of points x_i, positive numbers $t_i < \frac{\varepsilon}{C_2}$ (with C_2 from (2) in Lemma 7.2), and radii $r_i > 0$ so that

- $\Sigma \subset \cup_i B_{r_i}(x_i)$;
- for every $x \in B_{2r_i}(x_i)$, we have $F_{x,t_i}(\Sigma) < 1 + \varepsilon$.

If we let $\bar{r}$ be the minimum of the r_i's and let $\bar{t}$ be the minimum of the t_i's, then property (2) in Lemma 7.2 gives, for any x_0 in the $\bar{r}$ tubular neighborhood of Σ and any $t_0 \le \bar{t}$, that

(7.9) $$F_{x_0,t_0}(\Sigma) \le F_{x_0,t_{i(x_0)}}(\Sigma) + C_2(t_{i(x_0)} - t_0) \le 1 + 2\varepsilon,$$

where $i(x_0)$ satisfies $x_0 \subset B_{2r_{i(x_0)}}(x_{i(x_0)})$. It follows that $F_{x_0,t_0}(\Sigma)$ is strictly less than λ for t_0 sufficiently small, and the lemma follows. $\qquad\square$

7.2. *The entropy is achieved for self-shrinkers with polynomial volume growth.* The next lemma shows that, for a self-shrinker Σ with polynomial volume growth, the function $(x_0, t_0) \to F_{x_0,t_0}(\Sigma)$ has a strict (global) maximum at $x_0 = 0, t_0 = 1$, unless Σ splits off a line isometrically. Most of the proof is concerned with handling the case where Σ is noncompact; when Σ is closed, the lemma is an easy consequence of Huisken's monotonicity for the associated self-similar flow (cf. [Whi97, §8]).

LEMMA 7.10. *Suppose that Σ is a smooth complete self-shrinker with $\partial\Sigma = \emptyset$, polynomial volume growth, and Σ does not split off a line isometrically. Given $\varepsilon > 0$, there exists $\delta > 0$ so that*

(7.11) $$\sup\{F_{x_0,t_0}(\Sigma)|\, |x_0| + |\log t_0| > \varepsilon\} < \lambda - \delta.$$

Proof. We will prove first that the function $(x_0, t_0) \to F_{x_0,t_0}(\Sigma)$ has a strict local maximum at $(0, 1)$. We do this by showing that $(0, 1)$ is a critical point (in fact, the only critical point) and then showing that the second derivative at $(0, 1)$ is strictly negative. The bulk of the proof is then devoted to showing that the function is decreasing along a family of paths that start at $(0, 1)$ and run off to infinity (or time goes to zero) and whose union is all of space-time.

Since Σ is fixed and has polynomial volume growth, we may think of $F_{x_0,t_0}(\Sigma)$ as a smooth function of x_0 and t_0. Since Σ is a self-shrinker, it follows from Proposition 3.6 that the $\mathbf{R}^{n+1} \times \mathbf{R}$-gradient of this function vanishes at $x_0 = 0$ and $t_0 = 1$.

The second variation formula (Theorem 4.14) with $\Sigma_s \equiv \Sigma$ gives that the second derivative of $F_{x_s,t_s}(\Sigma)$ at $s = 0$ along the path $x_s = sy, t_s = 1 + sh$ is

$$(7.12) \qquad -(4\pi)^{-n/2} \int_\Sigma \left(h^2 H^2 + \frac{\left| y^\perp \right|^2}{2} \right) \mathrm{e}^{\frac{-|x|^2}{4}} \, d\mu \,.$$

This expression is clearly nonpositive. Moreover, the second term vanishes only when y is everywhere tangent to Σ; this can happen only when $y = 0$ since Σ does not split off a line. Likewise, the first term can vanish only when $h = 0$ or when Σ is a minimal cone; the latter does not occur here since hyperplanes are the only smooth minimal cones. We conclude that $F_{x_0,t_0}(\Sigma)$ has a strict local maximum at $x_0 = 0$ and $t_0 = 1$. Thus, the lemma will follow from the following claim.

Claim. If we fix $y \in \mathbf{R}^{n+1}$ and $a \in \mathbf{R}$ and then define $g(s)$ by

$$(7.13) \qquad g(s) = F_{sy, 1 + a\, s^2}(\Sigma),$$

then $g'(s) \leq 0$ for all $s > 0$ with $1 + as^2 > 0$. (The second condition is automatic for $a \geq 0$.)

The rest of the proof will be devoted to proving the claim.

Preliminaries: *weighted spaces and self-adjoint operators.* We will use square brackets $[\cdot]_s$ to denote the weighted integral

$$[f]_s = (4\pi\, t_s)^{-\frac{n}{2}} \int_\Sigma f \mathrm{e}^{-\frac{|x - x_s|^2}{4\, t_s}} \, d\mu$$

with $x_s = sy$ and $t_s = 1 + a\, s^2$ and define the operator $\mathcal{L}_s$

$$(7.14) \qquad \mathcal{L}_s v = \Delta v - \left\langle \frac{x - x_s}{2t_s}, \nabla v \right\rangle = \mathrm{e}^{\frac{|x - x_s|^2}{4t_s}} \operatorname{div} \left(\mathrm{e}^{\frac{-|x - x_s|^2}{4t_s}} \nabla v \right).$$

A slight variation of the proof of Corollary 3.10 gives that

$$(7.15) \qquad -[u\mathcal{L}_s v]_s = [\langle \nabla u, \nabla v \rangle]_s$$

whenever u and v are C^2 functions with $[|u\nabla v| + |\nabla u|\,|\nabla v| + |u\mathcal{L}_s v|]_s < \infty$.

To apply this, we will compute $\mathcal{L}_s$ on $\langle x, y \rangle$ and $|x|^2$ (cf. Lemma 3.20, where we did this for $s = 0$). First, since $\nabla \langle x, y \rangle = y^T$ and $\Delta \langle x, y \rangle = -H \langle \mathbf{n}, y \rangle = -\frac{\langle x, \mathbf{n} \rangle}{2} \langle \mathbf{n}, y \rangle = -\frac{\langle x, y^\perp \rangle}{2}$, we get

$$(7.16) \qquad 2\mathcal{L}_s \langle x, y \rangle = -\langle x, y^\perp \rangle - \left\langle \frac{x - x_s}{t_s}, y^T \right\rangle.$$

Similarly, using $\nabla |x|^2 = 2x^T$ and $\Delta |x|^2 = 2n - \left| x^\perp \right|^2$ by (3.24) gives

$$(7.17) \qquad \mathcal{L}_s |x|^2 = 2n - \left| x^\perp \right|^2 - \left\langle \frac{x - x_s}{t_s}, x^T \right\rangle.$$

It follows from the polynomial volume growth and (7.16) that we can apply (7.15) with $u = 1$ and $v = \langle x, y \rangle$ to get

$$(7.18) \qquad \left[\langle (x - x_s)^T, y^T \rangle \right]_s = -t_s \left[\langle x^\perp, y^\perp \rangle \right]_s.$$

Similarly, the polynomial volume growth and (7.17) allow us to apply (7.15) with $u = 1$ and $v = |x|^2$ to get

$$(7.19) \qquad \left[\langle (x - x_s)^T, x^T \rangle \right]_s = t_s \left[2n - |x^\perp|^2 \right]_s.$$

The derivative of g. The first variation formula (Lemma 3.1) gives

$$(7.20) \qquad g'(s) = \left[t'_s \left(\frac{|x - x_s|^2}{4t_s^2} - \frac{n}{2t_s} \right) + \frac{\langle x - x_s, y \rangle}{2t_s} \right]_s.$$

It is useful to set $z = x - x_s$ so that $x = z + sy$. Using (7.18), we compute the weighted L^2 inner product of y^T and z^T:

$$(7.21) \qquad \left[\frac{\langle z^T, y^T \rangle}{t_s} \right]_s = - \left[\langle (z + sy)^\perp, y^\perp \rangle \right]_s = - \left[\langle z^\perp, y^\perp \rangle + s |y^\perp|^2 \right]_s.$$

Similarly, we can rewrite (7.19) as

$$(7.22) \qquad \left[\frac{|z^T|^2}{t_s} + s \frac{\langle z^T, y^T \rangle}{t_s} \right]_s = \left[2n - |z^\perp|^2 - 2s \langle z^\perp, y^\perp \rangle - s^2 |y^\perp|^2 \right]_s.$$

Using (7.21) to evaluate the $\langle z^T, y^T \rangle$ term on the left gives

$$(7.23) \qquad \left[\frac{|z^T|^2}{t_s} \right]_s = s \left[\langle z^\perp, y^\perp \rangle + s |y^\perp|^2 \right]_s$$

$$+ \left[2n - |z^\perp|^2 - 2s \langle z^\perp, y^\perp \rangle - s^2 |y^\perp|^2 \right]_s$$

$$= \left[2n - |z^\perp|^2 - s \langle y^\perp, z^\perp \rangle \right]_s.$$

Using (7.23), the first term in (7.20) can be written as

$$(7.24) \qquad t'_s \left[\frac{|z|^2}{4t_s^2} - \frac{n}{2t_s} \right]_s = \frac{t'_s}{4 t_s} \left[\frac{|z^T|^2}{t_s} + \frac{|z^\perp|^2}{t_s} - 2n \right]_s$$

$$= \frac{t'_s}{4 t_s} \left[\left(\frac{1}{t_s} - 1 \right) |z^\perp|^2 - s \langle y^\perp, z^\perp \rangle \right]_s.$$

Similarly, using (7.21), we can rewrite the second term in (7.20) as

$$(7.25) \qquad \left[\frac{\langle z, y \rangle}{2t_s} \right]_s = \frac{1}{2} \left[\frac{\langle z^T, y^T \rangle}{t_s} + \frac{\langle z^\perp, y^\perp \rangle}{t_s} \right]_s$$

$$= \frac{1}{2} \left[\left(\frac{1}{t_s} - 1 \right) \langle z^\perp, y^\perp \rangle - s |y^\perp|^2 \right]_s.$$

Since we set $t_s = 1 + as^2$, we have $t'_s = 2a\,s$ and

$$(7.26) \qquad -\frac{t'_s}{4t_s}\,s = \frac{1}{2}\left(\frac{1}{t_s} - 1\right) = \frac{-2a\,s^2}{4t_s},$$

so the cross-terms (i.e., the $\langle z^\perp, y^\perp \rangle$ terms) are equal in (7.24) and (7.25). This gives

$$(7.27) \qquad 4t_s\,g'(s) = t'_s\left(\frac{1}{t_s} - 1\right)\left[|z^\perp|^2\right]_s - 4\,a\,s^2\left[\langle z^\perp, y^\perp \rangle\right]_s - 2s\,t_s\left[\left|y^\perp\right|^2\right]_s$$

$$= -\frac{2s}{t_s}\left[a^2s^2|z^\perp|^2 + 2a\,s\,t_s\langle z^\perp, y^\perp \rangle + t_s^2\left|y^\perp\right|^2\right]_s$$

$$= -\frac{2s}{t_s}\left[\left|a\,s\,z^\perp + t_s y^\perp\right|^2\right]_s.$$

This is clearly nonpositive for all $a \in \mathbf{R}$, $y \in \mathbf{R}^{n+1}$, and $s > 0$ with $t_s > 0$. $\quad\square$

7.3. *The equivalence of F-stability and entropy-stability.* We can now prove that F-stability and entropy-stability are equivalent for self-shrinkers that do not split off a line isometrically. Within the proof, we will need the following elementary lemma.

LEMMA 7.28. *There is a constant C_n depending only on n so that if $\Sigma \subset \mathbf{R}^{n+1}$ is a complete hypersurface, then for any $x_0 \in \mathbf{R}^{n+1}$ and $t_0 > 0$,*

$$(7.29) \qquad (4\pi t_0)^{-\frac{n}{2}} \int_\Sigma |x|^2\,e^{-\frac{|x-x_0|^2}{4t_0}} \leq 2\lambda(\Sigma)\left(C_n\,t_0 + |x_0|^2\right).$$

Proof. We can assume that $\lambda(\Sigma) < \infty$ since there is otherwise nothing to prove. The translation and dilation $x \to y = \frac{x-x_0}{\sqrt{t_0}}$ takes Σ to $\tilde{\Sigma}$ with $d\mu_\Sigma = t_0^{\frac{n}{2}}\,d\mu_{\tilde{\Sigma}}$ so that

$$(7.30) \qquad (4\pi t_0)^{-\frac{n}{2}} \int_\Sigma |x|^2\,e^{-\frac{|x-x_0|^2}{4t_0}}\,d\mu_\Sigma$$

$$= (4\pi)^{-\frac{n}{2}} \int_{\tilde{\Sigma}} \left|\sqrt{t_0}\,y + x_0\right|^2\,e^{-\frac{|y|^2}{4}}\,d\mu_{\tilde{\Sigma}}$$

$$\leq 2|x_0|^2\lambda(\tilde{\Sigma}) + 2t_0\,(4\pi)^{-\frac{n}{2}} \int_{\tilde{\Sigma}} |y|^2\,e^{-\frac{|y|^2}{4}}\,d\mu_{\tilde{\Sigma}}.$$

To bound the second term by the entropy, note that

$$(7.31) \qquad \left(4\pi R^2\right)^{-\frac{n}{2}}\,e^{-\frac{1}{4}}\,\mathrm{Vol}(B_R \cap \tilde{\Sigma}) \leq F_{0,R^2}(\tilde{\Sigma}) \leq \lambda(\tilde{\Sigma}).$$

Using this volume bound and chopping the integral up into annuli $B_{j+1} \setminus B_j$ gives

$$(7.32) \quad (4\pi)^{-\frac{n}{2}} \int_{\tilde{\Sigma}} |y|^2\,e^{-\frac{|y|^2}{4}}\,d\mu_{\tilde{\Sigma}} \leq e^{\frac{1}{4}}\,\lambda(\tilde{\Sigma}) \sum_{j=0}^{\infty}\left(e^{-\frac{j^2}{4}}(j+1)^{n+2}\right) = C_n\,\lambda(\tilde{\Sigma}),$$

where C_n depends only on n. The lemma follows from these bounds and the invariance of the entropy under dilations and translations that gives $\lambda(\Sigma) = \lambda(\tilde{\Sigma})$. $\qquad\qquad\square$

Proof of Theorem 0.15. Assume that Σ is not F-stable and, thus, there is a one-parameter normal variation Σ_s for $s \in [-2\varepsilon, 2\varepsilon]$ with $\Sigma_0 = \Sigma$ so that

(V1) For each s, the variation vector field is given by a function f_{Σ_s} times the normal $\mathbf{n}_{\Sigma_s}$ where every f_{Σ_s} is supported in a fixed compact subset of $\mathbf{R}^{n+1}$.

(V2) For any variations x_s and t_s with $x_0 = 0$ and $t_0 = 1$, we get that

$$(7.33) \qquad\qquad \partial_{ss}\Big|_{s=0} F_{x_s, t_s}(\Sigma_s) < 0.$$

We will use this to prove that Σ is also entropy-unstable.

Setting up the proof. Define a function $G : \mathbf{R}^{n+1} \times \mathbf{R}^+ \times [-2\varepsilon, 2\varepsilon] \to \mathbf{R}^+$ by

$$(7.34) \qquad\qquad G(x_0, t_0, s) = F_{x_0, t_0}(\Sigma_s).$$

We will show that there exists some $\varepsilon_1 > 0$ so that if $s \neq 0$ and $|s| \leq \varepsilon_1$, then

$$(7.35) \qquad \lambda(\Sigma_s) \equiv \sup_{x_0, t_0} G(x_0, t_0, s) < G(0, 1, 0) = \lambda(\Sigma),$$

and this will give the theorem with $\tilde{\Sigma}$ equal to Σ_s for any $s \neq 0$ in $(-\varepsilon_1, \varepsilon_1)$; by taking $s > 0$ small enough, we can arrange that $\tilde{\Sigma}$ is as close as we like to $\Sigma_0 = \Sigma$.

The remainder of the proof is devoted to establishing (7.35). The key points will be

(1) G has a strict local maximum at $(0, 1, 0)$.
(2) The restriction of G to Σ_0, i.e., $G(x_0, t_0, 0)$, has a strict global maximum at $(0, 1)$.
(3) $|\partial_s G|$ is uniformly bounded on compact sets.
(4) $G(x_0, t_0, s)$ is strictly less than $G(0, 1, 0)$ whenever $|x_0|$ is sufficiently large.
(5) $G(x_0, t_0, s)$ is strictly less than $G(0, 1, 0)$ whenever $|\log t_0|$ is sufficiently large.

The proof of (7.35) *assuming* (1)–(5). We will divide into three separate regions depending on the size of $|x_0|^2 + (\log t_0)^2$.

First, it follows from steps (4) and (5) that there is some $R > 0$ so that (7.35) holds for every s whenever

$$(7.36) \qquad\qquad x_0^2 + (\log t_0)^2 > R^2.$$

Second, as long as s is small, step (1) implies that (7.35) holds when $x_0^2 + (\log t_0)^2$ is sufficiently small.

Finally, in the intermediate region where $x_0^2 + (\log t_0)^2$ is bounded from above and bounded uniformly away from zero, step (2) says that G is strictly less than $\lambda(\Sigma)$ at $s = 0$ and step (3) says that the s derivative of G is uniformly bounded. Hence, there exists some $\varepsilon_3 > 0$ so that $G(x_0, t_0, s)$ is strictly less than $\lambda(\Sigma)$ whenever (x_0, t_0) is in the intermediate region as long as $|s| \le \varepsilon_3$.

This completes the proof of (7.35) assuming (1)–(5).

The proof of (1)–(5).

Step (1). Since Σ is a self-shrinker, it follows from Proposition 3.6 that ∇G vanishes at $(0, 1, 0)$. Given $y \in \mathbf{R}^{n+1}$, $a \in \mathbf{R}$, and $b \in \mathbf{R} \setminus \{0\}$, the second derivative of $G(sy, 1 + as, bs)$ at $s = 0$ is just $b^2 \partial_{ss}\big|_{s=0} F_{x_s, t_s}(\Sigma_s)$ with $x_s = s\frac{y}{b}$ and $t_s = 1 + \frac{a}{b} s$. In particular, it is negative by (V2). On the other hand, by Theorem 4.14, the second derivative of $G(sy, 1 + as, 0)$ at $s = 0$ is given by

$$(7.37) \qquad (4\pi)^{-n/2} \int_\Sigma \left(-a^2 H^2 - \frac{\left|y^\perp\right|^2}{2} \right) e^{\frac{-|x|^2}{4}} \, d\mu \,.$$

Since Σ does not split off a line, it is not a hyperplane and, thus, H cannot be zero everywhere; similarly, $|y^\perp|$ can only vanish identically when $y = 0$. We conclude that the Hessian of G at $(0, 1, 0)$ is negative definite. It follows that G has a strict local maximum at $(0, 1, 0)$, so there exists $\varepsilon_2 \in (0, \varepsilon)$ so that

$$(7.38) \qquad G(x_0, t_0, s) < G(0, 1, 0) \text{ if } 0 < x_0^2 + (\log t_0)^2 + s^2 < \varepsilon_2^2.$$

Step (2). If we restrict G to Σ_0, then Lemma 7.10 gives that $G(\cdot, \cdot, 0)$ has a strict global maximum at $(0, 1, 0)$. It follows that $\lambda(\Sigma) = G(0, 1, 0)$, and there exists $\delta > 0$ so that

$$(7.39) \qquad G(x_0, t_0, 0) < G(0, 1, 0) - \delta \text{ for all } x_0, t_0 \text{ with } \frac{\varepsilon_2^2}{4} < x_0^2 + (\log t_0)^2.$$

Step (3). Using the first variation formula (Lemma 3.1), we get that
(7.40)
$$\partial_s G(x_0, t_0, s) = (4\pi t_0)^{-\frac{n}{2}} \int_{\Sigma_s} f_{\Sigma_s} \left(H_{\Sigma_s} - \frac{\langle x - x_0, \mathbf{n}_{\Sigma_s} \rangle}{2t_0} \right) e^{\frac{-|x - x_0|^2}{4t_0}} \, d\mu_{\Sigma_s},$$

where H_{Σ_s} is the mean curvature of Σ_s and $d\mu_{\Sigma_s}$ is the volume measure on Σ_s. Since f_{Σ_s} is smooth and compactly supported and Σ has polynomial volume growth, $\partial_s G$ is evidently continuous in all three variables x_0, t_0, and s. We conclude that $\partial_s G$ is uniformly bounded on compact sets.

Step (4). Since Σ_s is a C^2 family of surfaces that vary only in a compact set and s is in the compact interval $[-\varepsilon, \varepsilon]$, there are uniform upper bounds

$$(7.41) \qquad \mathrm{Vol}(\Sigma_s \setminus \Sigma) \le C_V \text{ and } \max_{\Sigma_s \setminus \Sigma} |x| \le C_x.$$

It follows that for $|x_0| > C_x + R$ with $R > 0$, we have

$$(7.42) \qquad G(x_0, t_0, s) \leq (4\pi\, t_0)^{-\frac{n}{2}} \left(\int_{\Sigma_s \setminus \Sigma} e^{-\frac{|x-x_0|^2}{4t_0}} + \int_{\Sigma} e^{-\frac{|x-x_0|^2}{4t_0}} \right)$$

$$\leq C_V (4\pi\, t_0)^{-\frac{n}{2}} e^{-\frac{R^2}{4t_0}} + G(x_0, t_0, 0).$$

Therefore, if we define functions $u_R(r) = r^{-\frac{n}{2}} e^{-\frac{R^2}{4r}}$ for $r > 0$, then for $|x_0| > C_x + R$, we have

$$(7.43) \qquad G(x_0, t_0, s) \leq C_V (4\pi)^{-\frac{n}{2}} u_R(t_0) + G(x_0, t_0, 0).$$

Thus step (4) follows from step (2) and showing that $\sup_{r>0} u_R(r)$ converges to zero as $R \to \infty$. To see this, note that $u_R(r) = R^{-n} u_1\left(\frac{r}{R^2}\right)$ and u_1 is bounded since it is continuous and goes to zero as r goes to either zero or infinity.

Step (5). For the easy case when t_0 is large, step (2) gives $\alpha_0 > 0$ and $\bar{T} < \infty$ with

$$(7.44) \qquad \sup \{G(x_0, t_0, 0) | |\log t_0| \geq \bar{T}\} < \lambda(\Sigma) - 3\,\alpha_0,$$

and we have the trivial bound
$$(7.45)$$
$$G(x_0, t_0, s) \leq (4\pi\, t_0)^{-\frac{n}{2}} \operatorname{Vol}(\Sigma_s \setminus \Sigma) + G(x_0, t_0, 0) \leq C_V (4\pi\, t_0)^{-\frac{n}{2}} + G(x_0, t_0, 0).$$

This bound also immediately gives a uniform bound for $G(x_0, t_0, s)$ for all $t_0 \geq 1$:

$$(7.46) \qquad \sup_{t_0 \geq 1} G(x_0, t_0, s) \leq C_V (4\pi)^{-\frac{n}{2}} + \lambda(\Sigma).$$

To get the case where t_0 is close to zero, first use step (4) to get $\bar{R}$ and $\alpha_1 > 0$ with

$$(7.47) \qquad \sup \{G(x_0, t_0, s) | |x_0| \geq \bar{R}\} < \lambda(\Sigma) - \alpha_1.$$

Next, by compactness, there is a uniform constant C_H with

$$(7.48) \qquad \sup_{|s| \leq \varepsilon} \left(\sup_{\Sigma_s \setminus \Sigma} |H_{\Sigma_s}| \right) \leq C_H.$$

Arguing as in (7.5), we have
$$(7.49)$$
$$\partial_{t_0} G(x_0, t_0, s) \geq -(4\pi t_0)^{-\frac{n}{2}} \int_{\Sigma} \frac{H_{\Sigma}^2}{4} e^{-\frac{|x-x_0|^2}{4t_0}} - (4\pi t_0)^{-\frac{n}{2}} \int_{\Sigma_s \setminus \Sigma} \frac{H_{\Sigma_s}^2}{4} e^{-\frac{|x-x_0|^2}{4t_0}}$$

$$\geq -\lambda(\Sigma) \left(|x_0|^2 + C_n\, t_0 \right) - \frac{C_H^2}{4} G(x_0, t_0, s),$$

where C_n is a constant depending only on n, the second inequality used Lemma 7.28 and the self-shrinker equation to bound the first term, and the second inequality used (7.48) to bound the second term. Note that (7.49) gives a constant C_0 so that if $|x_0| \leq \bar{R}$ and $t_0 \leq 1$, then

$$(7.50) \qquad \partial_{t_0} \left[\mathrm{e}^{\frac{C_H^2 t_0}{4}} G(x_0, t_0, s) \right] \geq -C_0.$$

The point is that C_0 does not depend on x_0, s, or t_0. One immediate consequence of this together with (7.46) is that there is a uniform upper bound $\lambda(\Sigma_s) \leq \bar{\lambda}$. A second consequence is that if $|x_0| \leq \bar{R}$ and $0 < t_1 < \bar{t} \leq 1$, then

$$(7.51) \qquad G(x_0, t_1, s) \leq \mathrm{e}^{\frac{C_H^2 (\bar{t} - t_1)}{4}} G(x_0, \bar{t}, s) + \mathrm{e}^{\frac{-C_H^2 t_1}{4}} C_0 \left(\bar{t} - t_1 \right)$$

$$\leq \mathrm{e}^{\frac{C_H^2 \bar{t}}{4}} G(x_0, \bar{t}, s) + C_0 \bar{t}.$$

Fix a small positive constant $\bar{t}$ with $\log \bar{t} < -\bar{T}$ so that

$$(7.52) \qquad \mathrm{e}^{\frac{C_H^2 \bar{t}}{4}} \left(\lambda(\Sigma) - 2\alpha_0 \right) + C_0 \bar{t} < \lambda(\Sigma) - \alpha_0.$$

Step (3) gives a uniform bound

$$(7.53) \qquad \sup \left\{ |\partial_s G(x_0, \bar{t}, s)| \,|\, |x_0| \leq \bar{R} \right\} \leq C,$$

where C depends on $\bar{R}$ and $\bar{t}$, but not on s. After possibly shrinking ε so that $2C\varepsilon < \alpha_0$, we see that (7.44) and (7.53) together give that

$$(7.54) \qquad \sup \left\{ G(x_0, \bar{t}, s) | |x_0| \leq \bar{R} \right\} < \lambda(\Sigma) - 2\alpha_0.$$

Combining (7.54), (7.51), and (7.52) gives that

$$(7.55) \qquad \sup \left\{ G(x_0, t_0, s) | |x_0| \leq \bar{R} \text{ and } t_0 \leq \bar{t} \right\} < \lambda(\Sigma) - \alpha_0.$$

Together with (7.47), this completes the proof. $\square$

Combining Theorem 4.30 and Theorem 0.15, we get that the sphere is the only closed entropy-stable self-shrinker.

COROLLARY 7.56. *Suppose that $\Sigma \subset \mathbf{R}^{n+1}$ is a smooth closed self-shrinker, but is not a round sphere. Then Σ can be perturbed to an arbitrarily close hypersurface $\tilde{\Sigma}$ where the entropy is strictly less. In particular, Σ (or a translation of it) cannot arise as a tangent flow to the MCF starting from $\tilde{\Sigma}$.*

Proof. Since Σ is a closed self-shrinker, but is not the round sphere, Theorem 4.30 implies that Σ is F-unstable. Clearly, Σ does not split off a line, so Theorem 0.15 gives that it is entropy-unstable. $\square$

8. **Self-shrinkers with bounded diameter**

In this section, we will give a short direct proof of Theorem 0.10. This theorem will follow immediately from the main theorem in [CMa], but the direct proof in this section serves to illustrate the key ideas in a simpler case. The results in this section are for surfaces in $\mathbf{R}^3$; most of the ingredients in the proof work for all dimensions, but the following compactness theorem from [CMb] is proven only for surfaces in $\mathbf{R}^3$.

THEOREM 8.1 ([CMb]). *Given a nonnegative integer g and a positive constant V, the space of smooth complete embedded self-shrinkers Σ with genus at most g, $\partial \Sigma = \emptyset$, and the scale-invariant area bound* $\mathrm{Area}\,(B_R(x_0) \cap \Sigma) \leq V R^2$ *for all $x_0 \in \mathbf{R}^3$ and all $R > 0$ is compact. Namely, any sequence has a subsequence that converges in the topology of C^k convergence on compact subsets for any $k \geq 2$.*

Throughout this section, we will fix a nonnegative integer g and a positive constant $\bar{\lambda}$; these will be bounds for the genus and entropy which will in applications come from bounds on the initial surface in a MCF. We will not say so explicitly below, but all constants in this section will be allowed to depend on g and $\bar{\lambda}$.

Given a constant $D > 0$, let $S_D = S_{g,\bar{\lambda},D}$ denote the space of all smooth closed embedded self-shrinkers in $\mathbf{R}^3$ with genus at most g, entropy at most $\bar{\lambda}$, and diameter at most D.

As a corollary of Theorem 8.1, we get a smooth compactness theorem for S_D.

COROLLARY 8.2. *For each fixed D, the space S_D is compact. Namely, any sequence in S_D has a subsequence that converges uniformly in the C^k topology (any fixed k) to a surface in S_D.*

Proof. Given $\Sigma \in S_D$, the bound on the entropy gives the scale-invariant area bound

$$(8.3) \qquad (4\pi)^{-1}\,e^{-\frac{1}{4}}\,\frac{\mathrm{Area}\,(B_R(x_0) \cap \Sigma)}{R^2} \leq (4\pi\,R^2)^{-1} \int_{B_R(x_0)\cap\Sigma} e^{-\frac{|x-x_0|^2}{4R^2}}$$
$$\leq F_{x_0,R^2}(\Sigma) \leq \bar{\lambda}.$$

Therefore, Corollary 8.1 gives a subsequence that converges on compact subsets to a (possibly noncompact) limiting self-shrinker. However, the limit must have diameter at most D (if not, one of the surfaces sufficiently far out in the subsequence would have diameter greater than D), so it is closed and also in S_D. $\qquad\square$

The next result is a corollary of the compactness theorem for self-shrinkers together with the result that the round sphere is the only closed entropy-stable self-shrinker.

COROLLARY 8.4. *Given $D > 0$, there exists $\varepsilon > 0$ so that if $\Sigma \in S_D$ is not the round sphere, then there is a graph Γ over Σ with $\lambda(\Gamma) < \lambda(\Sigma) - \varepsilon$.*

Proof. Given any $\Sigma_1 \in S_D$ that is not the round sphere, Corollary 7.56 gives a graph $\tilde{\Sigma}_1$ over Σ_1 and a constant $\varepsilon_1 > 0$ so that

$$(8.5) \qquad \lambda(\tilde{\Sigma}_1) \leq \lambda(\Sigma_1) - \varepsilon_1.$$

It follows from the first variation formula (Lemma 3.1) that $F_{0,1}(\Sigma_s)$ is continuous in s when Σ_s varies continuously in a C^2 neighborhood of Σ_1. We can therefore choose a neighborhood U_1 of Σ_1 in the C^2 topology so that

$$(8.6) \qquad \min_{\Sigma \in U_1} \lambda(\Sigma) \geq \min_{\Sigma \in U_1} F_{0,1}(\Sigma) \geq F_{0,1}(\Sigma_1) - \frac{\varepsilon_1}{2} = \lambda(\Sigma_1) - \frac{\varepsilon_1}{2},$$

where the last equality used Lemma 7.10. After possibly shrinking the neighborhood U_1 of Σ_1, we can assume that each pair of surfaces in U_1 are graphs over each other.

We observe next that $\mathbf{S}^2$ is an isolated point in S_D with respect to the C^2 topology since any sufficiently close self-shinker is still closed and mean convex and, thus, a round $\mathbf{S}^2$ by Huisken's classification, [Hui90]. In particular, since S_D is compact in the C^2 topology by Corollary 8.2, so is $S_D \setminus \{\mathbf{S}^2\}$. Therefore, we can choose a finite set of self-shrinkers $\Sigma_i \in S_D \setminus \{\mathbf{S}^2\}$, graphs $\tilde{\Sigma}_i$ over Σ_i, neighborhoods U_i of Σ_i, and constants $\varepsilon_i > 0$ so that

- $S_D \setminus \{\mathbf{S}^2\} \subset \cup U_i$.
- If $\Sigma \in U_i \cap S_D$, then $\lambda(\tilde{\Sigma}_i) \leq \lambda(\Sigma) - \varepsilon_1/2$ and $\tilde{\Sigma}_i$ is a graph over Σ.

The corollary follows with $\varepsilon = \frac{1}{2} \min \varepsilon_i$. □

8.1. *MCF's that disappear in compact points generically do so in round points.* We will need the following consequence of Brakke's theorem, [Bra78], or White, [Whi05].

LEMMA 8.7. *Suppose that M_t is a MCF of smooth closed surfaces in $\mathbf{R}^3$ for $t < 0$ and Σ_0 is a closed smooth embedded self-shrinker equal to the $t = -1$ time-slice of a multiplicity one tangent flow to M_t at $(0,0)$. Then we can choose a sequence $s_j > 0$ with $s_j \to 0$ so that*

$$(8.8) \qquad \frac{1}{\sqrt{s_j}} M_{-s_j} \text{ is a graph over } \Sigma_0 \text{ of a function } u_j \text{ with } ||u_j||_{C^2} \to 0.$$

Proof. The main point is to show that Huisken's density is sufficiently close to one on all sufficiently small scales for the rescalings of M_t. Once we

have this, then Brakke's theorem will give uniform estimates and the lemma will follow easily.

Fix $\varepsilon > 0$ small (to be given by Theorem 3.1 in [Whi05]). Since Σ_0 is a smooth closed embedded surface, (2) and (3) in Lemma 7.2 give some $\bar{r} > 0$ so that

$$(8.9) \qquad \sup_{t_0 \leq \bar{r}} \left(\sup_{x_0 \in \mathbf{R}^3} F_{x_0,t_0}(\Sigma_0) \right) < 1 + \varepsilon.$$

The definition of tangent flow (see, e.g., [Ilm95b, Lemma 8] or [Whi97]) gives a sequence $s_j > 0$ with $s_j \to 0$ so that the rescaled flows $M_t^j = \frac{1}{\sqrt{s_j}} M_{\frac{t}{s_j}}$ converge to the multiplicity-one flow $\sqrt{-t}\,\Sigma_0$. Let $M_{-1}^j = \frac{1}{\sqrt{s_j}} M_{\frac{-1}{s_j}}$ be the time $t = -1$ slice of the j-th rescaled flow. We can assume that the M_{-1}^j's converge to Σ_0 as Radon measures and with respect to Hausdorff distance (see 7.1 in [Ilm94]).

We will use the convergence together with (8.9) to get uniform bounds for the F functionals on the M_{-1}^j's. To do this, define a sequence of functions g_j by

$$(8.10) \qquad g_j(x_0, t_0) = F_{x_0,t_0}(M_{-1}^j).$$

We will only consider the g_j's on the region $\overline{B} \times [\bar{r}/3, \bar{r}]$ where $B \subset \mathbf{R}^3$ is a fixed ball that contains Σ_0 and all of the M_{-1}^j's. It follows from the first variation formula (Lemma 3.1) that the g_j's are uniformly Lipschitz in this region with

$$(8.11) \qquad \sup_{\overline{B} \times [\bar{r}/3, \bar{r}]} |\nabla_{x_0, t_0}\, g_j| \leq C,$$

where C depends on $\bar{r}$, the radius of the ball B, and the scale-invariant local area bounds for the M_{-1}^j's which are uniformly bounded by Lemma 2.9. Since the M_{-1}^j's converge to Σ_0 as Radon measures and Σ_0 satisfies (8.9), it follows from (8.9) that

$$(8.12) \qquad \lim_{j \to \infty} g_j(x_0, t_0) < 1 + \varepsilon \text{ for each fixed } (x_0, t_0) \in B \times [\bar{r}/3, \bar{r}].$$

Combining this with (8.11) and the compactness of $\overline{B} \times [\bar{r}/3, \bar{r}]$, there exists some $\bar{j}$ sufficiently large so that for all $j > \bar{j}$, we have

$$(8.13) \qquad \sup_{\overline{B} \times [\bar{r}/3, \bar{r}]} F_{x_0,t_0}(M_{-1}^j) = \sup_{\overline{B} \times [\bar{r}/3, \bar{r}]} g_j(x_0, t_0) < 1 + 2\varepsilon.$$

By (1.9) (which used Huisken's monotonicity formula), we get, for every $t > -1$, that

$$(8.14) \qquad F_{x_0,t_0}(M_t^j) \leq F_{x_0,t_0+(t+1)}(M_{-1}^j).$$

Hence, if $t \in (-1 + \frac{\bar{r}}{3}, -1 + \frac{2\bar{r}}{3})$, $t_0 \leq \frac{\bar{r}}{3}$, and $j > \bar{j}$, then (8.13) and (8.14) together yield

$$(8.15) \qquad\qquad F_{x_0,t_0}(M_t^j) < 1 + 2\varepsilon.$$

This is precisely what is needed to apply Theorem 3.1 in [Whi05] to get uniform $C^{2,\alpha}$ bounds on M_t^j for all $t \in (-1 + \frac{4\bar{r}}{9}, -1 + \frac{5\bar{r}}{9})$ for some $\alpha \in (0,1)$. We can slightly change the s_j's so that we instead have uniform $C^{2,\alpha}$ bounds on M_{-1}^j.

Finally, observe that if Σ_j is a sequence of closed surfaces converging to a closed surface Σ_0 as Radon measures and both the Σ_j's and Σ_0 satisfy uniform $C^{2,\alpha}$ bounds, then the Σ_j's must converge to Σ_0 uniformly in C^2.   □

Proof of Theorem 0.10. We will construct a piece-wise MCF with a finite number of discontinuities that eventually becomes extinct in a round point. This comes from doing a smooth jump just before a (nonround) singular time, where we replace a time slice of the flow by a graph over it, and the crucial point is to show that there is a positive constant $\varepsilon > 0$ so that the entropy decreases by at least ε after each replacement. This will come from Corollary 8.4. We repeat this until we get to a singular point where every tangent flow consists of shrinking spheres.

Let M_t^0 denote the MCF with initial surface M_0 and let t_0^{sing} be the first singular time. By assumption, all of the tangent flows at t_0^{sing} are smooth, have multiplicity one, and correspond to compact self-shrinkers with diameter at most D. In particular, since the tangent flows are compact, there is only one singular point $x_0 \in \mathbf{R}^3$. Let Σ_0 be a self-shrinker equal to the $t = -1$ time-slice of a multiplicity one tangent flow at the singularity. Since Σ_0 is assumed to be smooth, closed, embedded, and multiplicity one, Lemma 8.7 gives a sequence $s_j > 0$ with $s_j \to 0$ so that

$$(8.16)$$
$$\frac{1}{\sqrt{s_j}}\left(M_{t_0^{\mathrm{sing}}-s_j}^0 - x_0\right) \text{ is a graph over } \Sigma_0 \text{ of a function } u_j \text{ with } \|u_j\|_{C^2} \to 0.$$

There are two possibilities. First, if Σ_0 is the round sphere for every tangent flow at x_0, then (8.16) implies that $M_{t_0^{\mathrm{sing}}-s_j}^0$ is converging to a round sphere for every sequence $s_j \to 0$.

Suppose instead that there is at least one tangent flow so that Σ_0 is not the round sphere. Then, we can apply Corollary 8.4 to get a graph Γ_0 over Σ_0 with $\lambda(\Gamma_0) < \lambda(\Sigma_0) - \varepsilon$ where $\varepsilon > 0$ is a fixed constant given by the corollary. When j is sufficiently large, $\left(\sqrt{s_j}\Gamma_0\right) + x_0$ is a graph over $M_{t_0^{\mathrm{sing}}-s_j}^0$ and

$$(8.17) \qquad \lambda((\sqrt{s_j}\,\Gamma_0) + x_0) = \lambda(\Gamma_0) < \lambda(\Sigma_0) - \varepsilon \leq \lambda(M_{t_0^{\mathrm{sing}}-s_j}^0) - \varepsilon,$$

where the first equality used the scale invariance of entropy (by (1.8)) and the last inequality used the monotonicity of entropy under MCF (Lemma 1.11).

We then replace $M^0_{t_0^{\mathrm{sing}}-s_j}$ with $\left(\sqrt{s_j}\Gamma_0\right) + x_0$ and restart the flow. Since the entropy goes down by a uniform constant $\varepsilon > 0$ at each replacement and is nonincreasing under MCF, this can only occur a finite number of times and the flow eventually reaches a singularity where all tangent flows are round. $\qquad\square$

9. Noncompact self-shrinkers

In this section, suppose that $\Sigma \subset \mathbf{R}^{n+1}$ is a complete noncompact hypersurface without boundary that satisfies $H = \frac{\langle x,\mathbf{n}\rangle}{2}$ and has polynomial volume growth. Throughout, $L = \Delta + |A|^2 + \frac{1}{2} - \frac{1}{2}x \cdot \nabla$ will be operator from the second variation formula for the $F_{0,1}$ functional.

9.1. *The spectrum of L when Σ is noncompact.* Since Σ is noncompact, there may not be a lowest eigenvalue for L. However, we can still define the bottom of the spectrum (which we still call μ_1) by

$$
(9.1) \qquad \mu_1 = \inf_f \frac{\int_\Sigma \left(|\nabla f|^2 - |A|^2 f^2 - \frac{1}{2} f^2\right) \mathrm{e}^{-\frac{|x|^2}{4}}}{\int_\Sigma f^2 \mathrm{e}^{-\frac{|x|^2}{4}}},
$$

where the infimum is taken over smooth functions f with compact support. Since Σ is noncompact, we must allow the possibility that $\mu_1 = -\infty$. Clearly, we have that $\mu_1 \leq \mu_1(\Omega)$ for any subdomain $\Omega \subset \Sigma$.

We show next that μ_1 is always negative; this result will not be used here, but it is used in [CMb].

THEOREM 9.2. *If the hypersurface $\Sigma \subset \mathbf{R}^{n+1}$ is a smooth complete embedded self-shrinker without boundary and with polynomial volume growth, then $\mu_1 \leq -\frac{1}{2}$.*

Proof of Theorem 9.2. Fix a point p in Σ and define a function v on Σ by

$$
(9.3) \qquad\qquad v(x) = \langle \mathbf{n}(p), \mathbf{n}(x)\rangle.
$$

It follows that $v(p) = 1$, $|v| \leq 1$, and, by (5.3),

$$
(9.4) \qquad\qquad Lv = \frac{1}{2}\, v.
$$

When Σ is closed, it follows immediately that $\mu_1 \leq -\frac{1}{2}$. However, if Σ is open, then we cannot use v as a test function to get an upper bound for μ_1. To deal with this, given any fixed $\mu < -\frac{1}{2}$, we will use v to construct a compactly supported function u with

$$
(9.5) \qquad\qquad -\int \left(uLu + \mu u^2\right) \mathrm{e}^{\frac{-|x|^2}{4}} < 0.
$$

Given any smooth function η, we have

$$(9.6) \qquad L(\eta\, v) = \eta\, Lv + v\mathcal{L}\,\eta + 2\langle\nabla\eta, \nabla v\rangle = \frac{1}{2}\,\eta\, v + v\,\mathcal{L}\eta + 2\langle\nabla\eta, \nabla v\rangle,$$

where $\mathcal{L} = \Delta - \frac{1}{2}x\cdot\nabla$ is the operator from Lemma 3.8. Taking η to have compact support, we get that

$$(9.7) \quad -\int \eta\, v\, L(\eta\, v)\mathrm{e}^{\frac{-|x|^2}{4}} \leq -\int\left[\frac{1}{2}\,\eta^2 v^2 + \eta\, v^2\mathcal{L}\,\eta + \frac{1}{2}\langle\nabla\eta^2, \nabla v^2\rangle\right]\mathrm{e}^{\frac{-|x|^2}{4}}$$

$$= \int\left[-\frac{1}{2}\,\eta^2 v^2 + v^2\,|\nabla\eta|^2\right]\mathrm{e}^{\frac{-|x|^2}{4}},$$

where the equality came from applying Lemma 3.8 to get

$$(9.8)$$
$$\int\frac{1}{2}\langle\nabla\eta^2, \nabla v^2\rangle\,\mathrm{e}^{\frac{-|x|^2}{4}} = -\frac{1}{2}\int\left(v^2\mathcal{L}\eta^2\right)\mathrm{e}^{\frac{-|x|^2}{4}} = -\int v^2(\eta\mathcal{L}\eta + |\nabla\eta|^2)\mathrm{e}^{\frac{-|x|^2}{4}}.$$

If we take η to be identically one on B_R and cut it off linearly to zero on $B_{R+1}\setminus B_R$, then (9.7) gives

$$(9.9) \qquad -\int\left[\eta\, v\, L(\eta\, v) + \mu\eta^2 v^2\right]\mathrm{e}^{\frac{-|x|^2}{4}}$$

$$\leq (1 - \mu)\int_{\Sigma\setminus B_R} v^2\mathrm{e}^{\frac{-|x|^2}{4}} - \left(\frac{1}{2} + \mu\right)\int_{B_R\cap\Sigma} v^2\mathrm{e}^{\frac{-|x|^2}{4}}.$$

However, since $|v| \leq 1$ and Σ has polynomial volume growth, we know that

$$(9.10) \qquad\qquad \lim_{R\to\infty}\int_{\Sigma\setminus B_R} v^2\mathrm{e}^{\frac{-|x|^2}{4}} = 0.$$

Combining this and $\left(\frac{1}{2} + \mu\right) > 0$, we see that the right-hand side of (9.9) must be negative for all sufficiently large R's. In particular, when R is large, the function $u = \eta\, v$ satisfies (9.5). This completes the proof. $\qquad\square$

The main result of this subsection is Theorem 9.36 below which shows that the bottom of the spectrum of L is strictly less than -1 if the mean curvature changes sign.

The following standard lemma shows that we get the same μ_1 if we take the infimum over a broader class of functions.

LEMMA 9.11. *When Σ is complete and noncompact, we get the same μ_1 by taking the infimum over Lipschitz functions f satisfying*

$$(9.12) \qquad\qquad \int_\Sigma\left(f^2 + |\nabla f|^2 + |A|^2 f^2\right)\mathrm{e}^{-\frac{|x|^2}{4}} < \infty.$$

Proof. By standard density arguments, we get the same μ_1 if we take the infimum over Lipschitz functions with compact support. Let ϕ_j be a Lipschitz cutoff function that is identically one on B_j and cuts off to zero linearly between ∂B_j and ∂B_{j+1}. Given any C^1 function f (not identically zero) that satisfies

(9.12), we set $f_j = \phi_j f$. It follows from (9.12) and the monotone convergence theorem that

(9.13)
$$\int_\Sigma \left(|\nabla f_j|^2 - |A|^2 f_j^2 - \frac{1}{2} f_j^2\right) \mathrm{e}^{-\frac{|x|^2}{4}} \to \int_\Sigma \left(|\nabla f|^2 - |A|^2 f^2 - \frac{1}{2} f^2\right) \mathrm{e}^{-\frac{|x|^2}{4}},$$

(9.14)
$$\int_\Sigma f_j^2 \mathrm{e}^{-\frac{|x|^2}{4}} \to \int_\Sigma f^2 \mathrm{e}^{-\frac{|x|^2}{4}},$$

and the lemma follows from this. $\qquad\square$

The next lemma records integral estimates for eigenfunctions that are either in the weighted $W^{1,2}$ space or are positive.

LEMMA 9.15. *Suppose that f is a C^2 function with $Lf = -\mu f$ for $\mu \in \mathbf{R}$.*
(1) *If f is in the weighted $W^{1,2}$ space, then $|A| f$ is in the weighted L^2 space and*

(9.16)
$$2\int_\Sigma |A|^2 f^2 \, \mathrm{e}^{-\frac{|x|^2}{4}} \leq \int_\Sigma \left((1 - 2\mu)f^2 + 4|\nabla f|^2\right) \mathrm{e}^{-\frac{|x|^2}{4}}.$$

(2) *If $f > 0$ and ϕ is in the weighted $W^{1,2}$ space, then*

(9.17)
$$\int_\Sigma \phi^2 \left(|A|^2 + |\nabla \log f|^2\right) \mathrm{e}^{-\frac{|x|^2}{4}} \leq \int_\Sigma \left(4|\nabla \phi|^2 - 2\mu\phi^2\right) \mathrm{e}^{-\frac{|x|^2}{4}}.$$

Proof. Within this proof, we will use square brackets $[\cdot]$ to denote the weighted integrals $[f] = \int_\Sigma f \mathrm{e}^{-\frac{|x|^2}{4}} \, d\mu$. We will also use the operator $\mathcal{L} = \Delta - \frac{1}{2}x \cdot \nabla$ from Lemma 3.8.

Proof of (1). If ϕ is a smooth function with compact support, then self-adjointness of $\mathcal{L} = \Delta - \frac{1}{2}x \cdot \nabla$ (Lemma 3.8) gives

(9.18) $\quad \left[\langle \nabla \phi^2, \nabla f^2 \rangle\right] = -\left[\phi^2 \mathcal{L} f^2\right] = -2\left[\phi^2 \left(|\nabla f|^2 - \left(|A|^2 + \frac{1}{2} + \mu\right) f^2\right)\right],$

where the last equality used that $\mathcal{L} f^2 = 2|\nabla f|^2 + 2f \mathcal{L} f$ and

(9.19)
$$\mathcal{L} f = \left(L - |A|^2 - \frac{1}{2}\right) f = -\left(\mu + |A|^2 + \frac{1}{2}\right) f.$$

Assume now that $\phi \leq 1$ and $|\nabla \phi| \leq 1$. Rearranging the terms in (9.18) and using Cauchy-Schwarz gives

(9.20)
$$\left[\phi^2(1 + 2\mu + 2|A|^2)f^2\right] = 4\left[\phi f \langle \nabla \phi, \nabla f \rangle\right] + 2\left[\phi^2 |\nabla f|^2\right]$$
$$\leq 2\left[f^2\right] + 4\left[|\nabla f|^2\right].$$

Finally, applying this with $\phi = \phi_j$ where ϕ_j is one on B_j and cuts off linearly to zero from ∂B_j to ∂B_{j+1}, letting $j \to \infty$, and using the monotone convergence theorem gives (9.16).

Proof of (2). Since $f > 0$, $\log f$ is well defined and we have

$$(9.21) \quad \mathcal{L} \log f = -|\nabla \log f|^2 + \frac{\Delta f - \frac{1}{2}\langle x, \nabla f \rangle}{f} = -\mu - |A|^2 - \frac{1}{2} - |\nabla \log f|^2.$$

Suppose that η is a function with compact support. As in (1), self-adjointness of $\mathcal{L}$ (Lemma 3.8) gives

$$(9.22) \quad \left[\langle \nabla \eta^2, \nabla \log f \rangle \right] = -\left[\eta^2 \mathcal{L} \log f \right] = \left[\eta^2 \left(\mu + |A|^2 + \frac{1}{2} + |\nabla \log f|^2 \right) \right].$$

Combining this with the absorbing inequality

$$(9.23) \qquad\qquad \langle \nabla \eta^2, \nabla \log f \rangle \le 2|\nabla \eta|^2 + \frac{1}{2} \eta^2 |\nabla \log f|^2$$

gives that

$$(9.24) \qquad\qquad \left[\eta^2 \left(|A|^2 + |\nabla \log f|^2 \right) \right] \le \left[4|\nabla \eta|^2 - 2\mu \eta^2 \right].$$

Let η_j be one on B_j and cut off linearly to zero from ∂B_j to ∂B_{j+1}. Since ϕ is in the weighted $W^{1,2}$ space, applying (9.24) with $\eta = \eta_j \phi$, letting $j \to \infty$, and using the monotone convergence theorem gives that (9.24) also holds with $\eta = \phi$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

We will need the following characterization of the bottom of the spectrum when Σ is noncompact that generalizes (4) in Theorem 5.15.

LEMMA 9.25. *If $\mu_1 \ne -\infty$, then there is a positive function u on Σ with $Lu = -\mu_1 u$. Furthermore, if v is in the weighted $W^{1,2}$ space and $Lv = -\mu_1 v$, then $v = Cu$ for $C \in \mathbf{R}$.*

Proof. Within this proof, square brackets $[\cdot]$ denote the weighted integrals $[f] = \int_\Sigma f e^{-\frac{|x|^2}{4}} \, d\mu$. We will use the operator $\mathcal{L} = \Delta - \frac{1}{2} x \cdot \nabla$ from Lemma 3.8 and Corollary 3.10.

Part 1: *Existence of u.* Fix a point $p \in \Sigma$ and let B_k denote the intrinsic ball in Σ with radius k and center p. For each k, Theorem 5.15 implies that the lowest Dirichlet eigenvalue $\mu_1(B_k)$ is a real number and can be represented by a positive Dirichlet eigenfunction u_k on B_k with

$$(9.26) \qquad\qquad\qquad Lu_k = -\mu_1(B_k)u_k$$

and, after multiplying u_k by a constant, we can assume that $u_k(p) = 1$. By the domain monotonicity of eigenvalues,[13] we know that $\mu_1(B_k)$ is a decreasing

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[13]This follows immediately from the variational characterization (2) and uniqueness (4) in Theorem 5.15.

sequence of real numbers with $\mu_1(B_k) > \mu_1$ for each k. Furthermore, it is easy to see that

$$(9.27) \qquad \lim_{k\to\infty} \mu_1(B_k) = \mu_1 > -\infty,$$

where the last inequality is by assumption. The Harnack inequality gives uniform upper and lower bounds for the u_k's on compact sets. Combining this with elliptic estimates, we get uniform $C^{2,\alpha}$ bounds on the u_k's on each compact set so that Arzela-Ascoli gives a subsequence that converges uniformly in C^2 to a nonnegative entire solution u of $Lu = -\mu_1 u$ with $u(p) = 1$. Another application of the Harnack inequality gives that u is positive.

Part 2: *Weighted integral estimates.* Since v is in the weighted $W^{1,2}$ space, we can apply part (1) of Lemma 9.15 to v to get that $|A|\,v$ is in the weighted L^2 space. Using the equation for $\mathcal{L}v$

$$(9.28) \qquad \mathcal{L}v = \left(L - |A|^2 - \frac{1}{2}\right)v = -\left(\mu_1 + |A|^2 + \frac{1}{2}\right)v,$$

it follows that

$$(9.29) \qquad \left[|v|^2 + |\nabla v|^2 + |v\mathcal{L}v|\right] < \infty.$$

Since v is in the weighted $W^{1,2}$ space, we can apply (2) in Lemma 9.15 with $f = u$ and $\phi = v$ to get that $|A|\,v$ and $|\nabla \log u|\,v$ are in the weighted L^2 space. We conclude that

$$(9.30) \qquad \left[v^2|\mathcal{L}\log u|\right] = \left[v^2\left|\mu_1 + \frac{1}{2} + |A|^2 + |\nabla \log u|^2\right|\right] < \infty.$$

Since $2v^2|\nabla \log u| \le v^2 + v^2|\nabla \log u|^2$ and $|\nabla v^2|\,|\nabla \log u| \le |\nabla v|^2 + v^2|\nabla \log u|^2$, we conclude

$$(9.31) \qquad \left[v^2|\nabla \log u| + |\nabla v^2|\,|\nabla \log u| + v^2|\mathcal{L}\log u|\right] < \infty.$$

Part 3: *Uniqueness.* Since v^2 and $\log u$ satisfy (9.31), Corollary 3.10 gives

$$(9.32) \quad \left[\langle \nabla v^2, \nabla \log u\rangle\right] = -\left[v^2\mathcal{L}\log u\right] = \left[v^2\left(\mu_1 + |A|^2 + \frac{1}{2} + |\nabla \log u|^2\right)\right],$$

where the second equality used the equation for $\mathcal{L}\log u$. Similarly, (9.29) allows us to apply Corollary 3.10 to two copies of v to get

$$(9.33) \qquad \left[|\nabla v|^2\right] = -[v\mathcal{L}v] = \left[v^2\left(\mu_1 + |A|^2 + \frac{1}{2}\right)\right],$$

where the second equality used the equation (9.28) for $\mathcal{L}v$. Substituting (9.33) into (9.32) and subtracting the left-hand side gives

$$(9.34) \quad 0 = \left[v^2|\nabla \log u|^2 - 2\langle v\nabla \log u, \nabla v\rangle + |\nabla v|^2\right] = \left[|v\nabla \log u - \nabla v|^2\right].$$

We conclude that

$$(9.35) \qquad v\nabla \log u - \nabla v = 0,$$

which implies that $\frac{v}{u}$ is constant. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

We have already seen that the mean curvature H is an eigenfunction of L with eigenvalue -1. The next theorem shows that if H changes, then the bottom of the spectrum μ_1 is strictly less than -1.

THEOREM 9.36. *If the mean curvature H changes sign, then $\mu_1 < -1$.*

Proof. Clearly, we may assume that $\mu_1 \neq -\infty$.

Step 1: *H, $|\nabla H|$, and $|A||H|$ are in the weighted L^2 space.* The polynomial volume growth of Σ and the self-shrinker equation $H = \frac{\langle x, \mathbf{n} \rangle}{2}$ immediately yield that H is in the weighted L^2 space, so we must show that $|\nabla H|$ and $|H||A|$ are also in the weighted L^2 space.

Since $\mu_1 \neq -\infty$, the first part of Lemma 9.25 gives a positive function u with $Lu = -\mu_1 u$. Using this u in Part (2) of Lemma 9.15 gives

$$(9.37) \qquad \int_\Sigma \phi^2 |A|^2 \, \mathrm{e}^{-\frac{|x|^2}{4}} \leq \int_\Sigma \left(4|\nabla \phi|^2 - 2\mu_1 \phi^2 \right) \mathrm{e}^{-\frac{|x|^2}{4}},$$

where ϕ is any Lipschitz function with compact support. If we take ϕ to be $|x|$ on B_R and cut it off linearly between ∂B_R and ∂B_{2R}, so that $|\nabla \phi| \leq 1$ and $\phi^2 \leq |x|^2$, then we get

$$(9.38) \qquad \int_{B_R \cap \Sigma} |A|^2 \, |x|^2 \, \mathrm{e}^{-\frac{|x|^2}{4}} \leq \int_\Sigma \left(4 + 2|\mu_1| \, |x|^2 \right) \mathrm{e}^{-\frac{|x|^2}{4}}.$$

The right-hand side of (9.38) is independent of R and is finite since Σ has polynomial volume growth. Thus, taking $R \to \infty$ and applying the monotone convergence theorem implies that

$$(9.39) \qquad \int_\Sigma |A|^2 \, |x|^2 \, \mathrm{e}^{-\frac{|x|^2}{4}} \leq \int_\Sigma \left(4 + 2|\mu_1| \, |x|^2 \right) \mathrm{e}^{-\frac{|x|^2}{4}} < \infty.$$

Using the self-shrinker equation $H = \frac{\langle x, \mathbf{n} \rangle}{2}$, we showed in (5.6) that $2\nabla_{e_i} H = -a_{ij} \langle x, e_j \rangle$ so that

$$(9.40) \qquad 4|\nabla H|^2 \leq |A|^2 |x|^2 \text{ and } |A|^2 H^2 \leq |A|^2 |x|^2.$$

Consequently, (9.39) implies that $|\nabla H|$ and $|A||H|$ are in the weighted L^2 space as desired.

Step 2: *$\mu_1 \leq -1$.* We will show this by using H as a test function in the definition of μ_1; this is allowed by Lemma 9.11 since we showed in Step 1 that H, $|\nabla H|$, and $|A||H|$ are in the weighted L^2 space. We will need that, by Theorem 5.2,

$$(9.41) \qquad \mathcal{L}H + \left(|A|^2 + \frac{1}{2} \right) H = LH = H,$$

so that $H\mathcal{L}H$ is in the weighted L^1 space. Consequently, we can apply Corollary 3.10 to two copies of H to get

$$(9.42) \quad \int_\Sigma |\nabla H|^2 \, \mathrm{e}^{-\frac{|x|^2}{4}} = -\int_\Sigma (H\mathcal{L}H)\mathrm{e}^{-\frac{|x|^2}{4}} = -\int_\Sigma H^2 \left(\frac{1}{2} - |A|^2\right) \mathrm{e}^{-\frac{|x|^2}{4}},$$

so we get that $\mu_1 \leq -1$ as claimed.

Step 3: $\mu_1 \neq -1$. If we did have that $\mu_1 = -1$, then the positive function u in the first step and H would both be eigenfunctions for L with the same eigenvalue $\mu_1 = -1$. Since we have shown that H is in the weighted $W^{1,2}$ space, the second part of Lemma 9.25 gives that H must be a constant multiple of u. However, this is impossible since u is positive and H changes sign. $\square$

9.2. *Constructing unstable variations when* $\mu_1 < -1$. We have shown that if H changes sign, then Σ has $\mu_1 < -1$. When Σ was closed, it followed immediately from this and the orthogonality of eigenfunctions with different eigenvalues that Σ was unstable. This orthogonality uses an integration by parts which is not justified when Σ is open. Instead, we will show that the lowest eigenfunction on a sufficiently large ball is *almost* orthogonal to H and the translations. This will be enough to prove instability.

In this subsection, we will use square brackets $[\cdot]_X$ to denote a weighted integral over a set $X \subset \mathbf{R}^{n+1}$:

$$(9.43) \qquad\qquad [f]_X = \int_{X\cap\Sigma} f\mathrm{e}^{-\frac{|x|^2}{4}} \, d\mu_{X\cap\Sigma}.$$

We will use this notation when X is a ball B_r, the boundary ∂B_r of a ball, or an annulus $A_{r_1,r_2} = B_{r_2} \setminus B_{r_1}$.

The next lemma shows that two eigenfunctions of L with different eigenvalues are almost orthogonal when one of them vanishes on the boundary of a ball, the other is small in a neighborhood of the boundary, and the radius is large.

LEMMA 9.44. *Suppose that* $\mu_1(B_R) \in [-3/2, -1)$ *for some* $R > 3$, $Lu = -\mu_1(B_R)u$ *on* B_R, u *is zero on* ∂B_R, *and* u *is not identically zero. If* $Lv = \mu v$ *and* $\mu \geq -1$, *then*

$$(9.45)$$

$$\frac{\left|[vu]_{B_R}\right|}{[u^2]_{B_R}^{1/2}} \leq \frac{-4}{\mu_1(B_R) + 1} \left(\frac{\sqrt{7 \, \mathrm{Vol}(B_R \cap \Sigma)}}{\mathrm{e}^{\frac{(R-3)^2}{8}}} \max_{B_R} |v| + \left[|\nabla v|^2\right]_{A_{R-2,R-1}}^{1/2}\right)$$

$$+ \left[v^2\right]_{A_{R-2,R}}^{1/2}.$$

Proof. If v also vanished on ∂B_R, then an integration by parts argument would show that u and v are orthogonal. The key for the integration by parts

is that if $\mathcal{L} = \Delta - \frac{1}{2}x \cdot \nabla$ is the operator from Lemma 3.8, then

(9.46) $v\mathcal{L}u - u\mathcal{L}v = vLu - uLv = (-\mu_1(B_R) + \mu)vu.$

Since $\operatorname{div}\left((v\nabla u - u\nabla v)\mathrm{e}^{-\frac{|x|^2}{4}}\right) = \mathrm{e}^{-\frac{|x|^2}{4}}(v\mathcal{L}u - u\mathcal{L}v)$, Stokes' theorem gives for $r \leq R$

(9.47) $(-\mu_1(B_R) + \mu)\,[vu]_{B_r} = [(v\partial_r u - u\partial_r v)]_{\partial B_r}.$

We cannot control the boundary term when $r = R$, but we will be able to use the co-area formula to bound it for r near R (this is where the term in the big brackets in (9.45) comes from); the remainder is then bounded by the last term in (9.45).

   We will need a L^1 bound on $|\nabla u|$ on annuli. To get this, choose a cutoff function ϕ which is one on $A_{R-2,R-1}$ and cuts off linearly to be zero on ∂B_R and ∂B_{R-3} and then apply (2) in Lemma 9.15 to get

(9.48) $\left[\dfrac{|\nabla u|^2}{u^2}\right]_{A_{R-2,R-1}} \leq [7]_{A_{R-3,R}} \leq 7\,\mathrm{Vol}(B_R \cap \Sigma)\mathrm{e}^{-\frac{(R-3)^2}{4}}.$

Combining this with the Cauchy-Schwarz inequality gives

(9.49) $[|\nabla u|]_{A_{R-2,R-1}}^2 = \left[\dfrac{|\nabla u|}{u}\,u\right]_{A_{R-2,R-1}}^2 \leq 7\,\mathrm{Vol}(B_R \cap \Sigma)\mathrm{e}^{-\frac{(R-3)^2}{4}}\left[u^2\right]_{B_R}.$

If we let α denote the positive number $(-\mu_1(B_R) - 1)$, then (9.47) gives

(9.50) $\alpha\left|[vu]_{B_r}\right| \leq \max_{\partial B_r}|v|\,[|\nabla u|]_{\partial B_r} + \left[u^2\right]_{\partial B_r}^{1/2}\left[|\nabla v|^2\right]_{\partial B_r}^{1/2}.$

Given a nonnegative L^1 function f on $A_{R-2,R-1} \cap \Sigma$, it follows from the coarea formula that

(9.51) $\displaystyle\int_{R-2}^{R-1}[f]_{\partial B_r}\,dr = [f|\nabla|x||]_{A_{R-2,R-1}} \leq [f]_{A_{R-2,R-1}},$

so we conclude that there is a measurable set $I_f \subset [R-2, R-1]$ with Lebesgue measure at most $1/4$ so that if $r \notin I_f$, then

(9.52) $[f]_{\partial B_r} \leq 4\,[f]_{A_{R-2,R-1}}.$

Applying this with f equal to $|\nabla u|$, f equal to $|\nabla v|^2$, and f equal to u^2, we conclude that at least one quarter of the r's in $[R-2, R-1]$ simultaneously satisfy

(9.53) $[|\nabla u|]_{\partial B_r} \leq 4\,[|\nabla u|]_{A_{R-2,R-1}},$

(9.54) $\left[|\nabla v|^2\right]_{\partial B_r} \leq 4\,\left[|\nabla v|^2\right]_{A_{R-2,R-1}},$

(9.55) $\left[u^2\right]_{\partial B_r} \leq 4\,\left[u^2\right]_{A_{R-2,R-1}} \leq 4\,\left[u^2\right]_{B_R}.$

Fix one such r. Substituting these into (9.50) and using (9.49) on the $|\nabla u|$ integral gives

(9.56)

$$\alpha \left| [vu]_{B_r} \right| \leq 4 \max_{B_R} |v| \, [|\nabla u|]_{A_{R-2,R-1}} + 4 \left[u^2 \right]_{B_R}^{1/2} \left[|\nabla v|^2 \right]_{A_{R-2,R-1}}^{1/2}$$

$$\leq 4 \left(\sqrt{7 \operatorname{Vol}(B_R \cap \Sigma)} \, \mathrm{e}^{-\frac{(R-3)^2}{8}} \max_{B_R} |v| + \left[|\nabla v|^2 \right]_{A_{R-2,R-1}}^{1/2} \right) \left[u^2 \right]_{B_R}^{1/2}.$$

The lemma follows from this since

$$(9.57) \qquad \left| [vu]_{B_R} \right| \leq \left| [vu]_{B_r} \right| + [|vu|]_{A_{R-2,R}} \leq \left| [vu]_{B_r} \right| + \left[u^2 \right]_{B_r}^{1/2} \left[v^2 \right]_{A_{R-2,R}}^{1/2}.$$

$\square$

The next lemma gives the desired instability.

LEMMA 9.58. *If $\mu_1 < -1$, then there exists $\bar{R}$ so that if $R \geq \bar{R}$ and u is a Dirichlet eigenfunction for $\mu_1(B_R)$, then for any $h \in \mathbf{R}$ and any $y \in \mathbf{R}^{n+1}$, we have*

$$(9.59) \qquad \left[-uLu + 2uhH + u\langle y, \mathbf{n} \rangle - h^2 H^2 - \frac{\langle y, \mathbf{n} \rangle^2}{2} \right]_{B_R} < 0.$$

Proof. Using the Cauchy-Schwartz inequality $ab \leq \frac{1}{2}(a^2 + b^2)$ on the cross-term $u\langle y, \mathbf{n} \rangle$, the left-hand side of (9.59) is bounded from above by

$$(9.60) \qquad \left[\left(\frac{1}{2} + \mu_1(B_R) \right) u^2 + 2uhH - h^2 H^2 \right]_{B_R}.$$

We will show that this is negative when R is large. There are two cases depending on μ_1.

Suppose first that $\mu_1 < -\frac{3}{2}$ and choose $\bar{R}$ so that $\mu_1(B_{\bar{R}}) < -\frac{3}{2}$. Given any $R \geq \bar{R}$, then (9.60) is strictly less than

$$(9.61) \qquad \left[-u^2 + 2uhH - h^2 H^2 \right]_{B_R} = - \left[(u - hH)^2 \right]_{B_R},$$

which gives (9.59) in this case.

Suppose now that $-\frac{3}{2} \leq \mu_1 < -1$. In this case, we will take $\bar{R}$ to be the maximum of $\bar{R}_1$ and $\bar{R}_2$, where $\bar{R}_1$ is chosen so that $\mu(B_R) < -1$ for all $R \geq \bar{R}_1$ and $\bar{R}_2$ will be chosen to handle the cross-term. Precisely, we will prove that there is some $\bar{R}_2$ so that for all $R \geq \bar{R}_2$,

$$(9.62) \qquad [uH]_{B_R}^2 \leq \frac{1}{2} \left[H^2 \right]_{B_R} \left[u^2 \right]_{B_R}.$$

It is then easy to see that (9.60) is negative when $R \geq \max\{\bar{R}_1, \bar{R}_2\}$. Namely, using $\mu_1(B_R) < -1$ and (9.62) in (9.60) gives

(9.63)
$$\left[\left(\frac{1}{2} + \mu_1(B_R)\right) u^2 + 2uhH - h^2 H^2\right]_{B_R}$$
$$< -\frac{1}{2} \left[u^2\right]_{B_R} + \sqrt{2}\, h \left[u^2\right]_{B_R}^{1/2} \left[H^2\right]_{B_R}^{1/2} - h^2 \left[H^2\right]_{B_R}$$
$$= -\left(\frac{1}{\sqrt{2}} \left[u^2\right]_{B_R}^{1/2} - h \left[H^2\right]_{B_R}^{1/2}\right)^2 .$$

It remains to prove the existence of $\bar{R}_2$ so that the claim (9.62) holds for all $R \geq \bar{R}_2$. Clearly, we may assume that H is not identically zero, so there is some $H_0 > 0$ with

(9.64)
$$\left[H^2\right]_{B_R} \geq H_0 \text{ for all } R > H_0^{-1}.$$

Applying Lemma 9.44 with $v = H$ and $\mu = -1$ gives that

(9.65)
$$\frac{[H\,u]_{B_R}^2}{[u^2]_{B_R}} \leq 2 \left(\frac{-4}{\mu_1(B_R) + 1}\right)^2$$
$$\cdot \left(\frac{(7CR^d)^{1/2}\, R}{e^{\frac{(R-3)^2}{8}}} + R \left[|A|^2\right]_{A_{R-2,R-1}}^{1/2}\right)^2 + 2 \left[R^2\right]_{A_{R-2,R}},$$

where we substituted in the bound $|H| \leq |x|$, the gradient bound $|\nabla H| \leq |A|\,|x|$ which follows from (5.6), and the polynomial volume bound $\text{Vol}(B_R \cap \Sigma) \leq CR^d$ (which holds for some C and d and all $R \geq 1$). To bound the $|A|$ term, choose a cutoff function ϕ which is one on $A_{R-2,R-1}$ and cuts off linearly to be zero on ∂B_R and ∂B_{R-3} and then apply (2) in Lemma 9.15 to get

(9.66) $\left[|A|^2\right]_{A_{R-2,R-1}} \leq [7]_{A_{R-3,R}} \leq 7\,\text{Vol}(B_R \cap \Sigma) e^{-\frac{(R-3)^2}{4}} \leq 7\,CR^d\, e^{-\frac{(R-3)^2}{4}}.$

It follows that all of the terms in (9.65) decay exponentially and, thus, that $\frac{[H\,u]_{B_R}^2}{[u^2]_{B_R}}$ can be made as small as we like by taking $\bar{R}_2$ sufficiently large. Since $\left[H^2\right]_{B_R}$ has a uniform positive lower bound by (9.64), we can choose $\bar{R}_2$ so that (9.62) holds for all $R \geq \bar{R}_2$.   □

10. The classification of self-shrinkers with $H \geq 0$

In [Hui90], Huisken showed that the only smooth closed self-shrinkers with nonnegative mean curvature in $\mathbf{R}^{n+1}$ (for $n > 1$) are round spheres (i.e., $\mathbf{S}^n$). In the remaining case $n = 1$, Abresch and Langer, [AL86], classified all smooth closed self-shrinking curves in $\mathbf{R}^2$ and showed that the *embedded* ones are round circles.

In the noncompact case, Huisken showed in [Hui93] that the only smooth open embedded self-shrinkers in $\mathbf{R}^{n+1}$ with $H \geq 0$, polynomial volume growth,

and $|A|$ bounded are isometric products of a round sphere and a linear subspace (i.e., $\mathbf{S}^k \times \mathbf{R}^{n-k} \subset \mathbf{R}^{n+1}$). The assumption of a bound on $|A|$ was natural in [Hui93] since Huisken was classifying tangent flows associated to type I singularities and these automatically satisfy such a bound. However, for our applications to generic flows we do not have an *a priori* bound on $|A|$ and, thus, we are led to show that Huisken's classification holds even without the $|A|$ bound. This is the next theorem (Theorem 0.17 from the introduction).

THEOREM 10.1. $\mathbf{S}^k \times \mathbf{R}^{n-k}$ *are the only smooth complete embedded self-shrinkers without boundary, with polynomial volume growth, and $H \geq 0$ in* $\mathbf{R}^{n+1}$.

The $\mathbf{S}^k$ factor in Theorem 10.1 is round and has radius $\sqrt{2k}$; we allow the possibilities of a hyperplane (i.e., $k = 0$) or a sphere ($n - k = 0$).

10.1. *Two general inequalities bounding $|\nabla|A||$ by $|\nabla A|$.* The next lemma holds for all hypersurfaces.

LEMMA 10.2. *If we fix a point p and choose a frame e_i, $i = 1, \ldots, n$, so that a_{ij} is in diagonal form at p, that is, $a_{ij} = \lambda_i\, \delta_{ij}$, then we have at p that*

$$(10.3) \qquad |\nabla|A||^2 \leq \sum_{i,k} a_{ii,k}^2 \leq \sum_{i,j,k} a_{ij,k}^2 = |\nabla A|^2.$$

Moreover,

$$(10.4) \qquad \left(1 + \frac{2}{n+1}\right)|\nabla|A||^2 \leq |\nabla A|^2 + \frac{2n}{n+1}\,|\nabla H|^2.$$

Proof. Observe first that, since a_{ij} is symmetric, we may choose e_i, $i = 1, \ldots, n$, so that a_{ij} is in diagonal form at p. Since $\nabla|A|^2 = 2|A|\nabla|A|$, we have at p

(10.5)
$$4\,|A|^2|\nabla|A||^2 = \sum_{k=1}^n \left((\sum_{i,j=1}^n a_{ij}^2)_k\right)^2 = 4\sum_{k=1}^n \left(\sum_{i=1}^n a_{ii,k}\,\lambda_i\right)^2 \leq 4\,|A|^2 \sum_{i,k=1}^n a_{ii,k}^2,$$

where the second equality used that a_{ij} is in diagonal form at p and the inequality used the Cauchy-Schwarz inequality on the vectors $(a_{11,k}, \ldots, a_{nn,k})$ and $(\lambda_1, \ldots, \lambda_n)$. This proves the first inequality in (10.3); the second inequality follows trivially.

To prove (10.4), observe that by (10.3) and the definition $H = -\sum a_{jj}$, we have

(10.6)
$$|\nabla|A||^2 \leq \sum_{i,k=1}^n a_{ii,k}^2 = \sum_{i\neq k} a_{ii,k}^2 + \sum_{i=1}^n a_{ii,i}^2 = \sum_{i\neq k} a_{ii,k}^2 + \sum_{i=1}^n \left(H_i + \sum_{\{j\,|\,i\neq j\}} a_{jj,i}\right)^2$$

$$\leq \sum_{i \neq k} a_{ii,k}^2 + n \sum_{i=1}^n \left(H_i^2 + \sum_{\{j \mid i \neq j\}} a_{jj,i}^2 \right) = n \left| \nabla H \right|^2 + (n+1) \sum_{i \neq k} a_{ii,k}^2$$

$$= n \left| \nabla H \right|^2 + (n+1) \sum_{i \neq k} a_{ik,i}^2 = n \left| \nabla H \right|^2 + \frac{n+1}{2} \left(\sum_{i \neq k} a_{ik,i}^2 + \sum_{i \neq k} a_{ki,i}^2 \right).$$

The second inequality in (10.6) used the algebraic fact $\left(\sum_{j=1}^n b_j \right)^2 \leq n \sum_{j=1}^n b_j^2$, and the last two equalities used the Codazzi equations (i.e., $a_{ik,j} = a_{ij,k}$; see, e.g., (B.5) in [Sim83]). Multiplying (10.6) by $\frac{2}{n+1}$ and adding this to the first inequality in (10.3) gives

$$(10.7) \quad \left(1 + \frac{2}{n+1} \right) \left| \nabla \left| A \right| \right|^2 \leq \sum_{i,k=1}^n a_{ii,k}^2 + \sum_{i \neq k} a_{ik,i}^2 + \sum_{i \neq k} a_{ki,i}^2 + \frac{2n}{n+1} \left| \nabla H \right|^2$$

$$\leq \sum_{i,j,k=1}^n a_{ij,k}^2 + \frac{2n}{n+1} \left| \nabla H \right|^2,$$

which completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

10.2. *Simons' inequality for $|A|$.* We will assume throughout the remainder of this section that $\Sigma \subset \mathbf{R}^{n+1}$ is a smooth complete embedded self-shrinker without boundary and with polynomial volume growth.

The next lemma computes the operator $L = \Delta + |A|^2 + \frac{1}{2} - \frac{1}{2}x \cdot \nabla$ on the second fundamental form; this is analogous to Simons' equation for minimal hypersurfaces (see, e.g., [CM99, §2.1]; cf. [CM11, eq. (2.16)]).

LEMMA 10.8. *If we extend the operator L to tensors, then $LA = A$. In particular, if $|A|$ does not vanish, then*

$$(10.9) \qquad\qquad L|A| = |A| + \frac{|\nabla A|^2 - |\nabla |A||^2}{|A|} \geq |A|.$$

Proof. For a general hypersurface (without using the self-shrinker equation), the Laplacian of A is (see, e.g., Lemma B.8 in [Sim83] where the sign convention for the a_{ij}'s is reversed)

$$(10.10) \qquad\qquad (\Delta A)_{ij} = -|A|^2 a_{ij} - H \, a_{ik} \, a_{jk} - H_{ij}.$$

Here H_{ij} is the ij component of the Hessian of H. Working at a point p and choosing the frame e_i so that $\nabla_{e_i}^T e_j(p) = 0$, it follows from (5.7) and (5.8) that

$$(10.11) \qquad\qquad 2H_{ij} = -a_{ik,j}\langle x, e_k \rangle - a_{ij} - 2Ha_{ik}a_{kj}.$$

Substituting this into (10.10) and using Codazzi's equation (i.e., $a_{ik,j} = a_{ij,k}$), we get at p

$$(10.12) \qquad\qquad (\Delta A)_{ij} = -|A|^2 a_{ij} + a_{ij,k} \left\langle \frac{x}{2}, e_k \right\rangle + \frac{1}{2} a_{ij},$$

so that at p, we have $LA = A$. Since p is arbitrary, this holds everywhere on Σ.

To get the second claim, first compute (without using the above equation for A) that

(10.13)

$$L|A| = L\left(|A|^2\right)^{1/2} = \frac{\Delta|A|^2}{2|A|} - \frac{|\nabla|A|^2|^2}{4|A|^3} + \left(|A|^2 + \frac{1}{2}\right)|A| - \frac{1}{4}\left\langle x, \frac{\nabla|A|^2}{|A|}\right\rangle$$

$$= \frac{\langle A, \Delta A\rangle + |\nabla A|^2}{|A|} - \frac{|\nabla|A|^2|^2}{4|A|^3} + \left(|A|^2 + \frac{1}{2}\right)|A| - \frac{\langle A, \nabla_{\frac{x}{2}} A\rangle}{|A|}$$

$$= \frac{\langle A, LA\rangle}{|A|} + \frac{|\nabla A|^2 - |\nabla|A||^2}{|A|}.$$

This is well defined and smooth since $|A| \neq 0$. Substituting $LA = A$ into this and using (10.3) gives (10.9). $\qquad\square$

10.3. *Weighted estimates for* $|A|$ *and a uniqueness result.* In this subsection, square brackets $[\cdot]$ denote the weighted integrals $[f] = \int_\Sigma f e^{-\frac{|x|^2}{4}}\, d\mu$. We will use the operator $\mathcal{L} = \Delta - \frac{1}{2}x\cdot\nabla$ from Lemma 3.8 and Corollary 3.10.

The next proposition adapts an estimate of Schoen, Simon, and Yau, [SSY75], for stable minimal hypersurfaces to self-shrinkers with positive mean curvature. The idea is to carefully play two inequalities off of each other: a stability type inequality with a cutoff function in terms of $|A|$ and a Simons' type inequality (see [CM99, §2.3] or [CM11, Th. 2.21] for the minimal case). The stability type inequality in this case will come from that $H > 0$ is an eigenfunction for L.

PROPOSITION 10.14. *If* $H > 0$, *then* $[|A|^2 + |A|^4 + |\nabla|A||^2 + |\nabla A|^2] < \infty$.

Proof. First, since $H > 0$, $\log H$ is well defined and Theorem 5.2 gives

(10.15) $\quad \mathcal{L}\log H = -|\nabla\log H|^2 + \frac{\Delta H - \frac{1}{2}\langle x, \nabla H\rangle}{H} = \frac{1}{2} - |A|^2 - |\nabla\log H|^2.$

Given any compactly supported function ϕ, self-adjointness of $\mathcal{L}$ (Lemma 3.8) gives

(10.16) $\quad \left[\langle\nabla\phi^2, \nabla\log H\rangle\right] = -\left[\phi^2\mathcal{L}\log H\right] = \left[\phi^2\left(|A|^2 - \frac{1}{2} + |\nabla\log H|^2\right)\right].$

Combining this with the inequality $|\langle\nabla\phi^2, \nabla\log H\rangle| \leq |\nabla\phi|^2 + \phi^2|\nabla\log H|^2$ gives the "stability inequality"

(10.17) $$\left[\phi^2|A|^2\right] \leq \left[|\nabla\phi|^2 + \frac{1}{2}\phi^2\right].$$

We will apply this with $\phi = \eta\,|A|$ where $\eta \geq 0$ has compact support to get

$$(10.18) \quad \left[\eta^2 |A|^4\right] \leq \left[\eta^2 |\nabla|A||^2 + 2\eta\,|A|\,|\nabla|A||\,|\nabla\eta| + |A|^2 |\nabla\eta|^2 + \frac{1}{2}\,\eta^2 |A|^2\right]$$

$$\leq (1+\varepsilon)\left[\eta^2 |\nabla|A||^2\right] + \left[|A|^2\left(\left(1+\frac{1}{\varepsilon}\right)|\nabla\eta|^2 + \frac{1}{2}\,\eta^2\right)\right],$$

where $\varepsilon > 0$ is arbitrary and the last inequality used the absorbing inequality $2ab \leq \varepsilon a^2 + \frac{b^2}{\varepsilon}$.

Second, combining the Leibniz rule $\mathcal{L}f^2 = 2|\nabla f|^2 + 2f\mathcal{L}f$ for a general function f, the equation $\mathcal{L} = L - |A|^2 - \frac{1}{2}$, the first equality in (10.9), and (10.4) gives the inequality

(10.19)

$$\mathcal{L}|A|^2 = 2|\nabla|A||^2 + 2|A|\mathcal{L}|A| = 2|\nabla|A||^2 + 2|A|\left(L|A| - |A|^3 - \frac{1}{2}|A|\right)$$

$$= 2|\nabla A|^2 + |A|^2 - 2|A|^4$$

$$\geq 2\left(1 + \frac{2}{n+1}\right)|\nabla|A||^2 - \frac{4n}{n+1}\,|\nabla H|^2 + |A|^2 - 2|A|^4.$$

Integrating this against $\frac{1}{2}\,\eta^2$ and using the self-adjointness of $\mathcal{L}$ (Lemma 3.8) gives

(10.20)

$$-2\left[\langle \eta\,|A|\,\nabla\eta, \nabla|A|\rangle\right] \geq \left[\eta^2\left(1 + \frac{2}{n+1}\right)|\nabla|A||^2 - \frac{2n}{n+1}\,\eta^2 |\nabla H|^2 - \eta^2 |A|^4\right],$$

where we dropped the $|A|^2\eta^2$ term that only helped us. Using the absorbing inequality $2ab \leq \varepsilon a^2 + \frac{b^2}{\varepsilon}$ gives

(10.21)

$$\left[\eta^2 |A|^4\right] + \left[\frac{2n}{n+1}\,\eta^2 |\nabla H|^2 + \frac{1}{\varepsilon}\,|A|^2 |\nabla\eta|^2\right] \geq \left(1 + \frac{2}{n+1} - \varepsilon\right)\left[\eta^2 |\nabla|A||^2\right].$$

We will assume that $|\eta| \leq 1$ and $|\nabla\eta| \leq 1$. Combining (10.18) and (10.21) gives

$$(10.22) \qquad \left[\eta^2 |A|^4\right] \leq \frac{1+\varepsilon}{1 + \frac{2}{n+1} - \varepsilon}\,\left[\eta^2 |A|^4\right] + C_\varepsilon \left[|\nabla H|^2 + |A|^2\right],$$

where the constant C_ε depends only on ε. Choosing $\varepsilon > 0$ small so that $\frac{1+\varepsilon}{1 + \frac{2}{n+1} - \varepsilon} < 1$ (i.e., $\varepsilon < \frac{1}{n+1}$), we can absorb the $\left[\eta^2 |A|^4\right]$ term on the right to get

$$(10.23) \qquad \left[\eta^2 |A|^4\right] \leq C\left[|\nabla H|^2 + |A|^2\right] \leq C\left[|A|^2(1 + |x|^2)\right]$$

for some constant C depending only on n, where the last inequality used the gradient bound $|\nabla H| \leq |A|\,|x|$ which follows from (5.6). Since $H > 0$, it follows from Part (2) of Lemma 9.15 and the polynomial volume growth that $\left[|A|^2(1 + |x|^2)\right] < \infty$, so (10.23) and the monotone convergence theorem give that $\left[|A|^4\right] < \infty$.

The bound $\left[|\nabla|A||^2\right] < \infty$ follows immediately from (10.21), $\left[|A|^4\right] < \infty$, and the monotone convergence theorem, so it only remains to show that $\left[|\nabla A|^2\right] < \infty$. To do this, integrate the equality in (10.19) against $\frac{1}{2}\eta^2$ and use the self-adjointness of $\mathcal{L}$ (Lemma 3.8) to get

$$(10.24) \qquad \left[\eta^2(|\nabla A|^2 - |A|^4)\right] \le 2\left[|A|\,|\nabla|A||\right] \le \left[|A|^2 + |\nabla|A||^2\right].$$

Here we used again that $|\eta| \le 1$ and $|\nabla\eta| \le 1$. Since $\left[|A|^2 + |\nabla|A||^2 + |A|^4\right] < \infty$, the monotone convergence theorem gives that $\left[|\nabla A|^2\right] < \infty$, completing the proof. $\qquad\square$

Proof of Theorem 0.17 = *Theorem* 10.1. Since $H \ge 0$ and $LH = H$ by Theorem 5.2, the Harnack inequality implies that either $H \equiv 0$ or $H > 0$ everywhere. In the trivial case where $H \equiv 0$, the self-similar equation implies that Σ is a smooth minimal cone and, hence, a hyperplane through 0. Therefore, we will assume below that $H > 0$.

In the first step, we will prove weighted integral estimates that will be needed to justify various integrations by parts in the second step. The second step uses $LH = H$ and $L|A| \ge |A|$ (and the estimates from the first step) to show that $|A| = \beta H$ for a constant $\beta > 0$. As in Huisken, [Hui93], this geometric identity is the key for proving the classification; we do this in the third step.

Step 1: *Weighted estimates.* Since $H > 0$, Proposition 10.14 gives

$$(10.25) \qquad \left[|A|^2 + |A|^4 + |\nabla|A||^2 + |\nabla A|^2\right] < \infty.$$

Therefore, $|A|$ is in the weighted $W^{1,2}$ space, and so (2) in Lemma 9.15 gives $\left[|A|^2|\nabla\log H|^2\right] < \infty$. Furthermore, using that $\mathcal{L}\log H = \frac{1}{2} - |A|^2 - |\nabla\log H|^2$, we also get that $\left[|A|^2|\mathcal{L}\log H|\right] < \infty$. This gives by the trivial inequality $2ab \le a^2 + b^2$ the integral estimate

$$(10.26) \qquad \left[|A|^2|\nabla\log H| + |\nabla|A|^2|\,|\nabla\log H| + |A|^2|\mathcal{L}\log H|\right] < \infty.$$

The definitions of $\mathcal{L}$ and L and the equality in (10.9) give
$$(10.27)$$
$$|A|\mathcal{L}|A| = |A|\left(L|A| - |A|^2|A| - \frac{1}{2}|A|\right) = \frac{1}{2}|A|^2 - |A|^4 + |\nabla A|^2 - |\nabla|A||^2,$$

and combining this with (10.25) and the trivial inequality $2ab \le a^2 + b^2$ gives

$$(10.28) \qquad \left[|A||\nabla|A|| + |\nabla|A||^2 + |A|\,|\mathcal{L}|A||\right] < \infty.$$

Step 2: *Geometric identities.* By (10.26), we can apply Corollary 3.10 to $|A|^2$ and $\log H$ to get that
$$(10.29)$$
$$\left[\langle\nabla|A|^2, \nabla\log H\rangle\right] = -\left[|A|^2\mathcal{L}\log H\right] = \left[|A|^2\left(|A|^2 - \frac{1}{2} + |\nabla\log H|^2\right)\right],$$

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where the second equality used the equation for $\mathcal{L} \log H$. Similarly, (10.28) allows us to apply Corollary 3.10 to two copies of $|A|$ and get

(10.30)
$$\left[|\nabla|A||^2\right] = -\left[|A|\mathcal{L}\,|A|\right] = -\left[|A|\left(L|A| - |A|^2|A| - \frac{1}{2}\,|A|\right)\right] \le \left[|A|^4 - \frac{1}{2}\,|A|^2\right],$$

where the inequality used that $L|A| \ge |A|$ by Lemma 10.8. Substituting (10.30) into (10.29) and subtracting the left-hand side gives

(10.31)
$$0 \ge \left[|A|^2|\nabla \log H|^2 - 2\langle|A|\,\nabla \log H, \nabla|A|\rangle + |\nabla|A||^2\right] = \left[\big||A|\,\nabla \log H - \nabla|A|\big|^2\right].$$

We conclude that $|A|\,\nabla \log H - \nabla|A| \equiv 0$ and, thus, that $|A| = \beta H$ for a constant $\beta > 0$. In particular, $|A|$ satisfies the equation $L|A| = |A|$ and, again by Lemma 10.8, we get

(10.32)
$$|\nabla A|^2 = |\nabla|A||^2.$$

Step 3: *Classifying* Σ *using the geometric identities.* Now that we have obtained the key geometric identities (10.32) and $|A| = \beta H$, the rest of the argument follows Huisken's argument on pages 187 and 188 of [Hui93].

As in Lemma 10.2, fix a point $p \in \Sigma$ and choose e_i, $i = 1, \ldots, n$, so that at p we have $a_{ij} = \lambda_i\,\delta_{ij}$. We showed in (10.5) that

(10.33)
$$|A|^2|\nabla|A||^2 = \sum_k \left(\sum_i a_{ii,k}\,\lambda_i\right)^2 \le |A|^2 \sum_{i,k} a_{ii,k}^2 \le |A|^2 \sum_{i,j,k} a_{ij,k}^2 = |A|^2|\nabla A|^2.$$

By (10.32), we must have equality in the two inequalities in (10.33) so we conclude that

(1) For each k, there is a constant α_k so that $a_{ii,k} = \alpha_k\,\lambda_i$ for every i.
(2) If $i \ne j$, then $a_{ij,k} = 0$.

Since $a_{ij,k}$ is fully symmetric in i, j, and k by the Codazzi equations, (2) implies

(2′) $a_{ij,k} = 0$ unless $i = j = k$.

If $\lambda_i \ne 0$ and $j \ne i$, then $0 = a_{ii,j} = \alpha_j\,\lambda_i$ so we must have $\alpha_j = 0$. In particular, if the rank of A is at least two at p, then every $\alpha_j = 0$ and, thus, (1) implies that $\nabla A(p) = 0$.

We now consider two separate cases depending on the rank of A.

Case 1: *The rank is greater than one.* Suppose first that there is some point p where the rank of A is at least two. We will show that the rank of A is at least two everywhere. To see this, for each $q \in \Sigma$, let $\lambda_1(q)$ and $\lambda_2(q)$ be the two eigenvalues of $A(q)$ that are largest in absolute value and define the set

(10.34)
$$\Sigma_2 = \{q \in \Sigma \,|\, \lambda_1(q) = \lambda_1(p) \text{ and } \lambda_2(q) = \lambda_2(p)\}.$$

The subset Σ_2 contains p and, since the λ_i's are continuous in q, it is automatically closed. Given any point $q \in \Sigma_2$, it follows that the rank of A is at least 2 at q. Since this is an open condition, there is an open set Ω containing q where the rank of A is at least two. However, we have already shown that $\nabla A = 0$ on this set, and this implies that the eigenvalues of A are constant on Ω. Thus, $\Omega \subset \Sigma_2$ and we conclude that Σ_2 is open. Since Σ is connected, we conclude that $\Sigma = \Sigma_2$ and, thus, also that $\nabla A = 0$ everywhere on Σ.

In Theorem 4 of [Law69], Lawson showed that every smooth hypersurface with $\nabla A = 0$ splits isometrically as a product of a sphere and a linear space. Finally, such a product satisfies the self-shrinker equation only if the sphere is centered at 0 and has the appropriate radius.

Case 2: *The rank is one.* Since $H > 0$, the remaining case is where the rank of A is exactly one at every point. It follows that there is a unit vector field V so that

(10.35) $$A(v, w) = H \langle v, V \rangle \langle w, V \rangle.$$

The vector field V is well defined in a neighborhood of each point, but *a priori* is globally well defined only up to ± 1. (It is really the direction that is globally well defined.)

Working at a point p and using the notation of (10.33), suppose that $\lambda_1 > 0$ (i.e, that $e_1 = V$ at p). It follows from (1) and (2$'$) that $a_{ij,k} = 0$ except possibly when $i = j = k = 1$. This means that at p,

(10.36) $$\nabla_u A(v, w) = \frac{\nabla_V A(V, V)}{H} \langle u, V \rangle A(v, w).$$

Since ∇A and $\frac{\nabla_V A(V,V)}{H} \langle \cdot, V \rangle A$ are both $(0, 3)$-tensors and p is arbitrary, this equality is independent of the choice of frame and, thus, (10.36) holds at every point. Hence, if $\gamma(t)$ is a unit speed geodesic on Σ and $v(t)$ is a parallel vector field tangent to Σ along γ, then

(10.37) $$\partial_t A(v, v) = \nabla_{\gamma'} A(v, v) = \frac{\nabla_V A(V, V)}{H} \langle \gamma', V \rangle A(v, v).$$

In particular, if $v(0)$ is in the kernel of A at $\gamma(0)$, then $A(v, v)$ vanishes along $\gamma(t)$. (The proof of this is sometimes called "Gronwall's Lemma.") In other words, the kernel of A is preserved under parallel transport. Therefore, if $v(0)$ is in the kernel of A at $\gamma(0)$, then (10.36) gives

(10.38) $$\langle \nabla_{\gamma'}^{\mathbf{R}^{n+1}} v, \mathbf{n} \rangle = A_{\gamma(t)} \left(\gamma'(t), v(t) \right) \equiv 0,$$

and so the vector field $v(t)$ is not just parallel along the curve, but it is actually *constant*. It follows that there are $(n - 1)$ constant vectors $e_2, \ldots, e_n$ tangent to Σ that give a global orthonormal frame for the kernel of A. Consequently, Σ is invariant under the (isometric) translations in the $(n - 1)$-plane spanned

by $e_2, \ldots, e_n$. Therefore, Σ is a product of a curve $\tilde{\gamma} \subset \mathbf{R}^2$ and this $(n-1)$-dimensional subspace. The curve $\tilde{\gamma}$ is a smooth embedded self-shrinking curve in $\mathbf{R}^2$ with positive curvature (i.e., it is convex) and with "polynomial length growth." Finally, by Corollary 10.45 below, $\tilde{\gamma}$ must be a round circle. $\square$

The next lemma, which was used in the last paragraph of the preceding proof, shows that any complete self-shrinking curve in $\mathbf{R}^2$ is either a straight line or it lies in a bounded set. In particular, if it has "polynomial length growth," then it is either a straight line or it is closed. We were unable to find a proof of this in the literature, so we will include one here.

LEMMA 10.39. *If $\gamma : \mathbf{R} \to \mathbf{R}^2$ satisfies $|\gamma'| = 1$ and $H = \frac{1}{2}\langle x, \mathbf{n} \rangle$, then either γ is a straight line through the origin or H^2 is positive and $|x|^2$ is bounded.*

Proof. The geodesic curvature is $H = -\langle \nabla_{\gamma'} \gamma', \mathbf{n} \rangle$. Since $\nabla_{\gamma'} \mathbf{n} = H\gamma'$, we get

$$(10.40) \qquad 2H' = \nabla_{\gamma'}\langle x, \mathbf{n} \rangle = \langle \gamma', \mathbf{n} \rangle + \langle x, \nabla_{\gamma'} \mathbf{n} \rangle = H\langle x, \gamma' \rangle.$$

Similarly, differentiating $|x|^2$ gives

$$(10.41) \qquad \left(|x|^2 \right)' = \nabla_{\gamma'}\langle x, x \rangle = 2\langle \gamma', x \rangle.$$

Combining these two equations, we see that

$$(10.42) \quad \mathrm{e}^{\frac{|x|^2}{2}} \left(H^2 \mathrm{e}^{-\frac{|x|^2}{2}} \right)' = 2HH' - \frac{1}{2} H^2 \left(|x|^2 \right)' = H^2\langle x, \gamma' \rangle - H^2\langle x, \gamma' \rangle = 0,$$

so the quantity $H^2 \mathrm{e}^{-\frac{|x|^2}{2}}$ is constant on γ; call this constant $E = E(\gamma)$.

To see why this implies that H^2 is bounded, we use that $H^2 \leq \frac{|x|^2}{4}$ and, thus,

$$(10.43) \qquad H^2 \mathrm{e}^{-2H^2} \geq H^2 \mathrm{e}^{-\frac{|x|^2}{2}} = E(\gamma).$$

In particular, since $\lim_{u \to \infty}(u^2 \mathrm{e}^{-2u^2}) = 0$ and we may assume that $E(\gamma) > 0$ (otherwise $H \equiv 0$), we see that H is bounded.

Suppose that γ is not a straight line through the origin. It follows that $E(\gamma) > 0$ and, thus, that

$$(10.44) \qquad \mathrm{e}^{\frac{|x|^2}{2}} = \frac{H^2}{E(\gamma)}.$$

It follows that $H^2 > 0$ and, since H^2 is bounded by (1), that $|x|^2$ is bounded. $\square$

It follows from Lemma 10.39 that any simple complete self-shrinking curve in $\mathbf{R}^2$ without boundary and with polynomial length growth must either be

a straight line or must be closed and convex. Combining this with Gage-Hamilton, [GH86], gives the following corollary.[14]

COROLLARY 10.45. *Any simple complete self-shrinking curve in* $\mathbf{R}^2$ *without boundary and with polynomial length growth must either be a straight line or a round circle.*

Corollary 10.45 has the following immediate consequences for self-shrinkers in $\mathbf{R}^3$.

COROLLARY 10.46. *Any smooth complete embedded self-shrinker in* $\mathbf{R}^3$ *without boundary and with polynomial area growth that splits off a line must either be a plane or a round cylinder.*

Remark 10.47. Lemma 10.39 holds also for immersed self-shrinking curves.

11. The classification of stable noncompact self-shrinkers

In this section, suppose that $\Sigma \subset \mathbf{R}^{n+1}$ is a complete noncompact hypersurface without boundary that satisfies $H = \frac{\langle x, \mathbf{n} \rangle}{2}$ and has polynomial volume growth. Throughout, $L = \Delta + |A|^2 + \frac{1}{2} - \frac{1}{2} x \cdot \nabla$ will be operator from the second variation formula for the $F_{0,1}$ functional.

11.1. *The classification of F-stable noncompact self-shrinkers.* We can now prove that hyperplanes are the only open F-stable self-shrinkers with polynomial volume growth. To do this, we will show that if Σ is not a hyperplane, then there is a compactly supported variation of Σ for which F'' is negative no matter which values of y and h that we choose.

Proof of Theorem 4.31. We will consider three separate cases.

Case 1: *H vanishes identically.* It follows that Σ is minimal and, since $H = \frac{\langle x, \mathbf{n} \rangle}{2}$, Σ is also a cone. However, the only (smooth) embedded minimal cones are hyperplanes.

Case 2: *H does not vanish anywhere.* In this case, Theorem 0.17 (our extension of Huisken's classification of mean convex self-shrinkers from [Hui93]) gives that Σ is a product $\mathbf{R}^k \times \mathbf{S}^{n-k}$ where the $\mathbf{S}^{n-k}$ has radius $\sqrt{2(n-k)}$

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[14]Gage-Hamilton showed that the curve shortening flow of any simple closed convex curve becomes extinct at a round point; in particular, the round circle is the only simple closed convex self-shrinker. This can also be seen by using Hamilton, [Ham95], or Huisken, [Hui98]. Finally, one could alternatively use the Abresch and Langer, [AL86], classification of simple self-shrinkers; see Andrews, [And03], for yet another alternative proof that, unlike [AL86], is not computer-assisted.

and $0 < k < n$. We will see that these are all F-unstable.[15] Observe that $H = \sqrt{(n-k)}/\sqrt{2}$ and $|A|^2 = \frac{1}{2}$ are constant. Let x_1 be the coordinate function corresponding to the first coordinate in the $\mathbf{R}^k$. Since x_1 is harmonic on Σ and $\nabla x_1 = \partial_1$, we get that

$$(11.1) \qquad \mathcal{L}x_1 = \Delta x_1 - \frac{1}{2}\langle x, \nabla x_1 \rangle = -\frac{1}{2}x_1,$$

$$(11.2) \qquad L x_1 = \mathcal{L}x_1 + \frac{1}{2}x_1 + |A|^2 x_1 = \frac{1}{2}x_1.$$

It follows that $|x_1|$, $|\nabla x_1|$, and $|\mathcal{L}x_1|$ are all in the weighted L^2 space since Σ has polynomial volume growth. Since the constant functions are also in the weighted $W^{1,2}$ space, Corollary 3.10 with $u = 1$ and $v = x_1$ gives

$$(11.3) \qquad 0 = \int_\Sigma [\mathcal{L}x_1]\, \mathrm{e}^{\frac{-|x|^2}{4}} = -\int_\Sigma \left[\frac{x_1}{2}\right]\, \mathrm{e}^{\frac{-|x|^2}{4}}.$$

Suppose next that $y \in \mathbf{R}^{n+1}$ is fixed and ϕ is a bounded function on $\mathbf{R}^k$, so that $\phi(x_1, \ldots, x_k)x_1$ is in the weighted L^1 space on Σ. It follows from Fubini's theorem that

$$(11.4) \quad \int_\Sigma \phi(x_1, \ldots, x_k)x_1 \langle y, \mathbf{n} \rangle \mathrm{e}^{\frac{-|x|^2}{4}}$$
$$= \int_{(x_1,\ldots,x_k)\in\mathbf{R}^k} \left[\phi(x_1, \ldots, x_k)x_1\, \mathrm{e}^{\frac{-(x_1^2+\cdots+x_k^2+2(n-k))}{4}} \int_{\mathbf{S}^{n-k}} \langle y, \mathbf{n} \rangle \right].$$

It is easy to see that $\langle y, \mathbf{n} \rangle$ is independent of $(x_1, \ldots, x_k)$ and is an eigenfunction of the Laplacian $\Delta_{\mathbf{S}^{n-k}}$ on the $\mathbf{S}^{n-k}$ factor. In particular, $\langle y, \mathbf{n} \rangle$ integrates to zero on each slice where $(x_1, \ldots, x_k)$ is fixed, so we conclude that

$$(11.5) \qquad \int_\Sigma \phi(x_1, \ldots, x_k)x_1 \langle y, \mathbf{n} \rangle \mathrm{e}^{\frac{-|x|^2}{4}} = 0.$$

We would like to use x_1 as the variation, but it does not have compact support. To deal with this, let ϕ_j be a C^2 cutoff function on $\mathbf{R}^k$ that is identically one on B_j, cuts off to zero between ∂B_j and ∂B_{j+2}, has $|\phi_j| \leq 1$, $|\nabla \phi_j| \leq 2$, and $|\Delta \phi_j| \leq C$ for a fixed constant C. Therefore, using $f_j = \phi_j x_1$ as a variation, Theorem 4.14 gives

(11.6)
$$F''_{f_j} = (4\pi)^{-n/2} \int_\Sigma \left[-f_j\, Lf_j + 2\frac{\sqrt{n-k}}{\sqrt{2}}\, h\, f_j - \frac{(n-k)h^2}{2} - \frac{\langle y, \mathbf{n} \rangle^2}{2} \right] \mathrm{e}^{\frac{-|x|^2}{4}},$$

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[15]If we do not assume embeddedness, then there is the additional possibility that Σ is a product of a Euclidean factor and one of the immersed locally convex Abresch-Langer examples. However, it follows from the linear analysis in Section 2 of [EW87] that the number of negative eigenvalues of the linearized operator is greater than can be accounted for by H and the translations. Hence, these are F-unstable.

where we used (11.5) to get rid of the $f_j \langle y, \mathbf{n} \rangle$ term. Using (11.2) and (11.3), it follows easily from the monotone convergence theorem that

$$(11.7) \quad \lim_{j \to \infty} \int_\Sigma \left[-f_j \, L f_j \right] \mathrm{e}^{\frac{-|x|^2}{4}} = - \int_\Sigma \left[\frac{x_1^2}{2} \right] \mathrm{e}^{\frac{-|x|^2}{4}} \text{ and } \lim_{j \to \infty} \int_\Sigma f_j \, \mathrm{e}^{\frac{-|x|^2}{4}} = 0.$$

Hence, for all j sufficiently large, we get

$$(11.8) \qquad\qquad F''_{f_j} \leq \frac{1}{2} \, (4\pi)^{-n/2} \int_\Sigma \left[-\frac{x_1^2}{2} \right] \mathrm{e}^{\frac{-|x|^2}{4}}.$$

Since this is negative for every h and y, we get that Σ is unstable as claimed.

Case 3: *H vanishes somewhere, but not identically.* By Theorem 9.36, we know that $\mu_1 < -1$. Therefore, we can apply Lemma 9.58 to get some $\bar{R}$ so that if $R \geq \bar{R}$ and u is a Dirichlet eigenfunction for $\mu_1(B_R)$, then for any $h \in \mathbf{R}$ and any $y \in \mathbf{R}^{n+1}$, we have

$$(11.9) \qquad \int_{B_R \cap \Sigma} \left[-uLu + 2uhH + u \langle y, \mathbf{n} \rangle - h^2 H^2 - \frac{\langle y, \mathbf{n} \rangle^2}{2} \right] \mathrm{e}^{-\frac{|x|^2}{4}} < 0.$$

If we fix any $R \geq \bar{R}$ and then use the corresponding u in the second variation formula from Theorem 4.14, then we get that F'' is negative for every h and y, and so we get that Σ is unstable as claimed. $\qquad\square$

11.2. *Noncompact entropy-stable self-shrinkers.* In this section, we will use our characterization of entropy-stable self-shinkers together with our classification of F-stable self-shinkers to prove Theorems 0.7 and 0.12. By Corollary 10.46, the only embedded self-shrinkers in $\mathbf{R}^3$ that split off a line are cylinders and planes, so Theorem 0.7 is an immediate consequence of part (2) of Theorem 0.12.

Throughout this subsection, $\Sigma \subset \mathbf{R}^{n+1}$ will be a smooth complete embedded self-shrinker without boundary and with $\lambda(\Sigma) < \infty$.

We now prove Theorem 0.12, which classifies entropy-stable self-shrinkers; as mentioned, this immediately implies Theorem 0.7.

Proof of Theorem 0.12. We have already shown that the sphere is the only smooth *closed* entropy-stable self-shrinker, so we may assume that Σ is open.

We will first show part (2) of the theorem.

> If Σ is not a round $\mathbf{S}^n$ and does not split off a line, then Σ can be perturbed to a graph $\tilde{\Sigma}$ of a compactly supported function u over Σ with $\lambda(\tilde{\Sigma}) < \lambda(\Sigma)$. In fact, we can take u to have arbitrarily small C^m norm for any fixed m.

Since Σ does not split off a line, this follows immediately from combining Theorem 0.15 and Theorem 4.31.

Now that we have part (2), the key for proving part (1) is the following observation.

Claim. If $\Sigma = \mathbf{R}^{n-k} \times \Sigma_0$ where $\Sigma_0 \subset \mathbf{R}^{k+1}$, then

$$(11.10) \qquad F_{x,t_0}^{\mathbf{R}^{n+1}}(\Sigma) = F_{x_0,t_0}^{\mathbf{R}^{k+1}}(\Sigma_0),$$

where x_0 is the projection of x to $\mathbf{R}^{k+1}$.

Proof of the claim. By induction, it suffices to do the case where $(n-k)=1$ and $k+1 = n$. Given a vector $y \in \mathbf{R}^{n+1}$, set $y = (y_1, y_0)$ with y_1 and y_0 the projections of y to $\mathbf{R}^1$ and $\mathbf{R}^n$, respectively. Since

$$(11.11) \qquad \int_{\mathbf{R}} e^{\frac{-|x_1 - y_1|^2}{4t_0}} \, dy_1 = (4t_0)^{\frac{1}{2}} \int_{\mathbf{R}} e^{-s^2} \, ds = (4\pi t_0)^{\frac{1}{2}},$$

Fubini's theorem gives

$$
\begin{aligned}
(11.12) \qquad F_{x,t_0}^{\mathbf{R}^{n+1}}(\Sigma) &= (4\pi t_0)^{-n/2} \int_{\mathbf{R}\times\Sigma_0} e^{\frac{-|x-y|^2}{4t_0}} \\
&= (4\pi t_0)^{-n/2} \int_{\Sigma_0} \left(\int_{\mathbf{R}} e^{\frac{-|x_1-y_1|^2}{4t_0}} \, dy_1 \right) e^{\frac{-|x_0-y_0|^2}{4t_0}} \\
&= (4\pi t_0)^{-(n-1)/2} \int_{\Sigma_0} e^{\frac{-|x_0-y_0|^2}{4t_0}} = F_{x_0,t_0}^{\mathbf{R}^n}(\Sigma_0).
\end{aligned}
$$

This gives the claim.

To prove (1), suppose that $\Sigma = \mathbf{R}^{n-k} \times \Sigma_0$, where Σ_0 is a self-shrinking hypersurface in $\mathbf{R}^{k+1}$ and is *not* a k-dimensional round sphere and does not split off another line. Part (2) that we just proved gives a graph $\tilde{\Sigma}_0$ of a compactly supported function u on Σ_0 so that $\lambda_{\mathbf{R}^{k+1}}(\tilde{\Sigma}_0) < \lambda_{\mathbf{R}^{k+1}}(\Sigma_0)$. Part (1) follows from combining this with the claim. □

12. **Regularity of F-stable self-shrinkers**

The combined work of Brakke (Allard), [Bra78], [All72], Huisken, [Hui90], Ilmanen, [Ilm95b], and White, [Whi97], yields that, for a MCF in $\mathbf{R}^{n+1}$ starting at a smooth closed embedded hypersurface, any tangent flow (also past the first singular time) has the property that the time -1 slice is an n-dimensional F-stationary integral varifold[16] with Euclidean volume growth. The main result of this section, Theorem 12.1 below, shows that if the regular part of such an n-dimensional F-stationary integral varifold is orientable and F-stable and the singular set has finite $(n-2)$-dimensional Hausdorff measure, then it is smooth (at least for $n \leq 6$) and thus Theorem 0.16 applies and shows that such a self-shrinker is either a round sphere or a hyperplane; see Corollary 12.2 below.

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[16]An F-stationary integral varifold is a weakly defined minimal hypersurface in the conformally changed metric on $\mathbf{R}^{n+1}$.

THEOREM 12.1. *Let $\Sigma \subset \mathbf{R}^{n+1}$ be an n-dimensional integral varifold that is a time slice of a tangent flow of a MCF starting at a smooth closed embedded hypersurface in $\mathbf{R}^{n+1}$. If the regular part of Σ is orientable and F-stable and the singular set has finite $(n-2)$-dimensional Hausdorff measure, then it corresponds to an embedded, analytic hypersurface away from a closed set of singularities of Hausdorff dimension at most $n-7$, which is absent if $n \leq 6$ and is discrete if $n = 7$.*

Recall that, for a MCF, at any point in space-time tangent flows exist and are Brakke flows. In particular, the time slices of such a tangent flow are F-stationary integral varifolds.

As an immediate corollary of Theorem 12.1 combined with Theorem 0.16, we get

COROLLARY 12.2. *Let $\{\mathcal{T}_t\}_{t<0}$ be a tangent flow of a MCF starting at a smooth closed embedded hypersurface in $\mathbf{R}^{n+1}$. If the regular part of $\mathcal{T}_{-1}$ is orientable and F-stable and the singular set has finite $(n-2)$-dimensional Hausdorff measure and $n \leq 6$, then $\mathcal{T}_{-1}$ is either a round sphere or a hyperplane.*

Recall that we defined the operator L in Section 4 by

$$(12.3) \qquad L\,v = \mathrm{e}^{\frac{|x|^2}{4}} \operatorname{div} \left(\mathrm{e}^{-\frac{|x|^2}{4}} \nabla v \right) + |A|^2 v + \frac{1}{2} v.$$

LEMMA 12.4. *Suppose that $\Sigma \subset \mathbf{R}^{n+1}$ is a F-stationary n-dimensional integral varifold and $\Omega \subset \Sigma$ is an open subset of the regular part of Σ. If $\mu_1(L, \Omega) < -\frac{3}{2}$, then Ω is F-unstable.*

Proof. If $\mu_1(L, \Omega) < -\frac{3}{2}$, then we get a function f_0 with compact support in Ω satisfying

$$(12.5) \qquad -\int_\Omega (f_0\,Lf_0)\,\mathrm{e}^{-\frac{|x|^2}{4}} < -\frac{3}{2} \int_\Omega f_0^2 \mathrm{e}^{-\frac{|x|^2}{4}}.$$

Substituting f_0 into the second variation formula from Theorem 4.14 gives

(12.6)

$$F'' < (4\pi)^{-n/2} \int_\Omega \left[-\frac{3}{2}\,f_0^2 + 2f_0\,hH + f_0\langle y, \mathbf{n}\rangle - h^2 H^2 - \frac{\langle y, \mathbf{n}\rangle^2}{2} \right] \mathrm{e}^{\frac{-|x|^2}{4}}$$

$$= (4\pi)^{-n/2} \int_\Omega \left[-\frac{1}{2}\,(f_0 - \langle y, \mathbf{n}\rangle)^2 - (f_0 - hH)^2 \right] \mathrm{e}^{\frac{-|x|^2}{4}}.$$

Since $F'' < 0$ no matter which values of h and y that we use, it follows that Ω is F-unstable. $\qquad\square$

In the next lemma, $\operatorname{reg}(\Sigma)$ is the regular part of an integral varifold Σ. Moreover, in this next lemma, we will assume that $\operatorname{Vol}(B_r(x)) \leq V r^n$ for all

$0 < r < 1$ and all x. As we have noted before, this is automatically satisfied for tangent flows.

LEMMA 12.7. *Given $\varepsilon > 0$, an integer n, and R, $V > 0$, there exists $r_0 = r_0(\varepsilon, n, R, V) > 0$ such that the following holds.*

Suppose that $\Sigma \subset \mathbf{R}^{n+1}$ is an n-dimensional F-stationary integral varifold, F-stable on the regular part and $\mathrm{Vol}(B_r(x) \cap \Sigma) \leq V r^n$ for all $0 < r < 1$ and $x \in \mathbf{R}^{n+1}$. Tthen

- Σ *is stationary with respect to the metric* $g_{ij} = \mathrm{e}^{-\frac{|x|^2}{4}} \delta_{ij}$ *on* $\mathbf{R}^{n+1}$.
- *For all $x_0 \in B_R(0)$ and for all smooth functions ϕ with compact support in the regular part of $B_{r_0}(x_0) \cap \Sigma$, we have the stability-type inequality*

$$(12.8) \qquad \int_\Sigma |A_g|_g^2 \phi^2 d\,\mathrm{Vol}_g \leq (1 + \varepsilon) \int_\Sigma |\nabla_g \phi|_g^2 d\,\mathrm{Vol}_g \,.$$

Here A_g, ∇_g, $|\cdot|_g$, and $d\,\mathrm{Vol}_g$ are with respect to the metric $g = g_{ij}$ on $\mathbf{R}^{n+1}$.

Proof. By inspection, the argument in Section 3 for smooth self-shrinkers easily generalizes to show that Σ is a stationary varifold in the metric g_{ij} on $\mathbf{R}^{n+1}$. We need to show that, in the g_{ij} metric on $\mathbf{R}^{n+1}$ where the varifold is stationary, the usual second variational operator (or stability operator) satisfies a stability-type inequality, i.e., (12.8), on sufficiently small balls. This will follow from $\mu_1(L, \mathrm{reg}(\Sigma)) \geq -\frac{3}{2}$ together with the Sobolev and Cauchy-Schwarz inequalities.

The bulk of the lemma is to establish (12.8) for A, $\nabla = \nabla_{\mathbf{R}^{n+1}}$, and with respect to the volume on Σ coming from thinking of Σ as a subset of Euclidean space. This is because the lemma will follow from this with a slightly worse ε because of the following.

For a metric of the form $g_{ij} = f^2 \delta_{ij}$, the Christoffel symbols are

$$(12.9) \qquad \Gamma_{k\ell}^i = \frac{1}{2} g^{im}(g_{mk,\ell} + g_{m\ell,k} - g_{k\ell,m}) = \frac{f_\ell\,\delta_{ik} + f_k\,\delta_{i\ell} - f_i\,\delta_{k\ell}}{f}.$$

Hence, for vector fields X and Y,

(12.10)

$$\nabla_X^g Y = \nabla_X^{\mathbf{R}^{n+1}} Y + X_k Y_\ell \Gamma_{k\ell}^i \partial_i = \nabla_X^{\mathbf{R}^{n+1}} Y + X_k Y_\ell \frac{f_\ell\,\delta_{ik} + f_k\,\delta_{i\ell} - f_i\,\delta_{k\ell}}{f} \partial_i$$

$$= \nabla_X Y + \frac{\langle Y, \nabla f \rangle X + \langle X, \nabla f \rangle Y - \langle X, Y \rangle \nabla f}{f},$$

where all quantities in the second line are computed in the Euclidean metric. If the vector fields X and Y are tangent to Σ whose (Euclidean) unit normal is $\mathbf{n}$, then the second fundamental form A^g in the g metric is given by

(12.11)

$$A^g(X, Y) = f^2 \langle \nabla_X^g Y, f^{-1}\mathbf{n} \rangle = f \langle \nabla_X^g Y, \mathbf{n} \rangle = f\,A(X, Y) - \langle X, Y \rangle \langle \nabla f, \mathbf{n} \rangle,$$

where the last equality used that X and Y are tangential while $\mathbf{n}$ is normal. If e_i is a δ_{ij}-orthonormal frame for Σ, then $f^{-1} e_i$ is a frame in the new metric and, thus,

$$(12.12) \qquad a^g_{ij} \equiv f^{-2} A^g(e_i, e_j) = f^{-1} a_{ij} - f^{-2}\langle e_i, e_j\rangle\langle\nabla f, \mathbf{n}\rangle.$$

Squaring and using an absorbing inequality yields, for all $\delta > 0$,

$$(12.13) \qquad |A_g|^2_g \leq (1+\delta)\frac{|A|^2}{f^2} + n\left(1 + \frac{1}{\delta}\right)\frac{|\nabla f|^2}{f^4}.$$

Moreover, $d\operatorname{Vol}_g = f^n\, d\operatorname{Vol}$, and $f\,|\nabla_g\phi|_g = |\nabla\phi|$. Integrating, we get

$$(12.14) \qquad \int_\Sigma |A_g|^2_g \phi^2 d\operatorname{Vol}_g \leq (1+\delta)\sup_{|\phi|>0} f^{n-2} \int_\Sigma |A|^2\phi^2 d\operatorname{Vol}$$

$$+ n\left(1 + \frac{1}{\delta}\right)\sup_{|\phi|>0}(|\nabla f|^2 f^{n-4})\int_\Sigma \phi^2 d\operatorname{Vol}.$$

Since $f = e^{-\frac{|x|^2}{4n}}$ and the support of ϕ is contained in $B_{r_0}(x_0)$, it follows that

$$(12.15) \qquad \sup_{|\phi|>0} f^{n-2} \leq (1 + O_{R,n}(r_0)) \inf_{|\phi|>0} f^{n-2},$$

where $O_{R,n}(r_0) \to 0$ as $r_0 \to 0$. The lemma now easily follows from (12.14) and (12.15) provided we can show (12.8) for A, ∇, $|\cdot|$, $d\operatorname{Vol}$ and we can show that $\int_\Sigma |\phi|^2 d\operatorname{Vol}$ can be bounded by a small constant times $\int_\Sigma |\nabla\phi|^2 d\operatorname{Vol}$. (The last will follow, after choosing r_0 sufficiently small, from the Dirichlet-Poincaré inequality that we show below.)

In the remainder of the lemma, gradients, second fundamental forms, and volumes are all with respect to the Euclidean metric δ_{ij} and the metric on Σ that it induces.

Observe first that by the Sobolev inequality (Simon, [Sim83, Th. 18.6, p. 93]; cf. also [All72], [MS73]) for any smooth nonnegative function $\psi \in C_0^\infty(B_{r_0}(x_0) \cap \Sigma)$ with compact support on the regular part of the varifold (here we also use that since it is F-stationary $H = \frac{\langle x, \mathbf{n}\rangle}{2}$ and that $|x_0| \leq R$),

$$(12.16) \qquad \left(\int_\Sigma \psi^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}} \leq C\int_\Sigma(|\nabla\psi| + |H|\,\psi) \leq C\int_\Sigma(|\nabla\psi| + |x|\,\psi)$$

$$\leq C\int_\Sigma(|\nabla\psi| + (R + r_0)\,\psi).$$

On the other hand, by the Hölder inequality,

$$(12.17) \qquad \int_\Sigma \psi \leq \operatorname{Vol}(B_{r_0}(x_0) \cap \Sigma)^{\frac{1}{n}}\left(\int_\Sigma \psi^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}}.$$

Combining these two inequalities and setting $\psi = |\phi|$ yields

$$(12.18) \qquad \int_\Sigma |\phi| \leq C\operatorname{Vol}(B_{r_0}(x_0) \cap \Sigma)^{\frac{1}{n}}\left(\int_\Sigma |\nabla\phi| + (R + r_0)\int_\Sigma |\phi|\right).$$

Combining this with the bound $\mathrm{Vol}(B_{r_0}(x_0) \cap \Sigma) \leq V r_0^n$ yields

$$(12.19) \qquad \left(1 - CV^{\frac{1}{n}} r_0(R + r_0)\right) \int_\Sigma |\phi| \leq CV^{\frac{1}{n}} r_0 \int_\Sigma |\nabla \phi|.$$

Finally, choosing r_0 sufficiently small and applying the Cauchy-Schwarz inequality gives the Dirichlet-Poincaré inequality for a constant C (here, as elsewhere in this lemma, C denotes a constant, though the actual constant may change from line to line):

$$(12.20) \qquad \int_\Sigma |\phi|^2 \leq C r_0^2 \int_\Sigma |\nabla \phi|^2.$$

Combining (12.3) and Lemma 12.4 yields

$$(12.21) \quad \int_\Sigma |A|^2 \phi^2 \mathrm{e}^{-\frac{|x|^2}{4}} = \int_\Sigma \phi\, L\, \phi\, \mathrm{e}^{-\frac{|x|^2}{4}} + \int_\Sigma |\nabla\phi|^2 \mathrm{e}^{\frac{-|x|^2}{4}} - \frac{1}{2}\int_\Sigma \phi^2 \mathrm{e}^{-\frac{|x|^2}{4}}$$

$$\leq -\left(\mu_1 + \frac{1}{2}\right) \int_\Sigma \phi^2 \mathrm{e}^{-\frac{|x|^2}{4}} + \int_\Sigma |\nabla\phi|^2 \mathrm{e}^{\frac{-|x|^2}{4}}$$

$$\leq \int_\Sigma \phi^2 \mathrm{e}^{-\frac{|x|^2}{4}} + \int_\Sigma |\nabla\phi|^2 \mathrm{e}^{\frac{-|x|^2}{4}}.$$

Multiplying both sides by $\mathrm{e}^{\frac{|x_0|^2}{4}}$ and using that $||x_0|^2 - |x|^2| \leq r_0(2R + r_0)$ on $B_{r_0}(x_0)$ yields

$$\mathrm{e}^{-r_0(R+r_0)} \int_\Sigma |A|^2 \phi^2 \leq \int_\Sigma |A|^2 \phi^2 \mathrm{e}^{\frac{|x_0|^2 - |x|^2}{4}}$$

$$\leq \int_\Sigma \phi^2 \mathrm{e}^{\frac{|x_0|^2 - |x|^2}{4}} + \int_\Sigma |\nabla\phi|^2 \mathrm{e}^{\frac{|x_0|^2 - |x|^2}{4}}$$

$$\leq \mathrm{e}^{r_0(R+r_0)} \int_\Sigma \phi^2 + \mathrm{e}^{r_0(R+r_0)} \int_\Sigma |\nabla\phi|^2.$$

Hence,

$$\int_\Sigma |A|^2 \phi^2 \leq \mathrm{e}^{2r_0(R+r_0)} \int_\Sigma \phi^2 + \mathrm{e}^{2r_0(R+r_0)} \int_\Sigma |\nabla\phi|^2.$$

The lemma now easily follows from the Dirichlet-Poincaré inequality after choosing r_0 sufficiently small. □

Proof of Theorem 12.1. This is an immediate consequence of [SS81] and Lemma 12.7. Note that the argument in [SS81] goes through with the slightly weaker stability inequality (12.8). □

Using these regularity results, we can now extend the entropy-stability theorem, i.e., Theorem 0.12, to this setting.

Proof of Theorem 0.14. The proof of Theorem 0.12 combined the classification of smooth F-stable self-shrinkers (Theorem 0.16) with the splitting theorem (Theorem 0.15). The classification of F-stable self-shrinkers was extended to this setting in Corollary 12.2. However, the splitting theorem needs

to be modified slightly. To explain why, recall that the proof of Theorem 0.15 used the following four ingredients:

- the first variation formula,
- integration by parts for the drift Laplacians,
- the easy case of the second variation formula (when only the parameters x_0 and t_0 change),
- the fact that any (smooth) minimal cone must split (in fact, must be $\mathbf{R}^n$).

The first three ingredients all generalize to this setting, but the last does not. (Consider, for example, the cone on the Clifford torus $\mathbf{S}^1 \times \mathbf{S}^1$.) Thus, we must deal with the new possibility of a minimal cone that does not split.

Suppose now that the self-shrinker Σ is a n-dimensional minimal cone in $\mathbf{R}^{n+1}$ whose singular set has locally finite codimension-two measure and Σ does not split. We will consider variations Σ_s so that each Σ_s is a cone with vertex at the origin. As in the proof of the splitting theorem, we define a function

$$G(x_0, t_0, s) = F_{x_0, t_0}(\Sigma_s).$$

Because of the dilation invariance of Σ_s, we have for any $\lambda > 0$ that

$$G(x_0, t_0, s) = G(x_0/\lambda, t_0/\lambda^2, s).$$

Thus, it suffices to find a variation Σ_s so that for every $s \neq 0$, we have

(12.22) $$\sup_{x_0} G(x_0, 1, s) < \lambda(\Sigma).$$

We will next choose the variation vector field. Let $\Gamma = \partial B_1 \cap \Sigma$ denote the "link" of the cone Σ. It follows that Γ is a minimal hypersurface in the n-sphere ∂B_1 and the singular set of Γ has finite codimension-two measure. In order to preserve the cone structure, we will work with variations of the form $f\mathbf{n}$ with

$$f(x) = |x| \, g(x/|x|),$$

where g is an eigenfunction for $\Delta_\Gamma + |A|^2$ on Γ with

$$(\Delta_\Gamma + |A|^2)g = -\mu g.$$

(Initially, we make no assumptions on the eigenvalue μ.) Here A is the second fundamental form of the whole cone and A_Γ is the second fundamental form of the link Γ as a submanifold of the unit sphere.

Using that Σ is the n-dimensional minimal cone over Γ and f is homogeneous of degree one, we compute that

$$\frac{\partial f}{\partial r}(x) = \frac{f(x)}{|x|} \text{ and } \frac{\partial^2 f}{\partial r^2} = 0,$$

so we get that

$$Lf = \left(|x|^{-2}\,\Delta_\Gamma f + \frac{n-1}{|x|}\frac{\partial f}{\partial r} + \frac{\partial^2 f}{\partial r^2}\right) + \frac{1}{2}\,f + |A|^2 f - \frac{|x|}{2}\frac{\partial f}{\partial r}$$

$$= |x|^{-1}\left[(\Delta_\Gamma + |A|^2)g\right] + \frac{n-1}{|x|^2}\,f = \frac{(-\mu + n - 1)g}{|x|}.$$

Using $f\mathbf{n}$ to generate the variation Σ_s (this is homogeneous, and so preserves the cone structure), the second variation formula from Theorem 4.14 gives

(12.23)

$$\partial_{ss}\Big|_{s=0}G(ys,1,s) = (4\pi)^{-\frac{n}{2}}\int_\Sigma \left(-fLf + f\langle y,\mathbf{n}\rangle - \frac{1}{2}\langle y,\mathbf{n}\rangle^2\right)e^{-\frac{|x|^2}{4}}$$

$$= (4\pi)^{-\frac{n}{2}}\int_\Sigma \left((\mu + 1 - n)\,g^2 + |x|\,g\langle y,\mathbf{n}\rangle - \frac{1}{2}\langle y,\mathbf{n}\rangle^2\right)e^{-\frac{|x|^2}{4}}.$$

We claim next that we can choose a fixed function g to make (12.23) negative for all possible choices of $y \in \mathbf{R}^{n+1}$. To prove this, we will consider two separate cases depending on the regularity of the link Γ.

Case 1. Suppose first that Γ is smooth. Standard calculations (see, e.g., [CM99] or [CM11, eq. (1.180)]) show that if $y \in \mathbf{R}^{n+1}$, then $\langle y,\mathbf{n}\rangle$ is in the kernel of $\Delta + |A|^2$ on Σ and, using that Σ is the cone on Γ, satisfies

$$(\Delta_\Gamma + |A|^2)\langle y,\mathbf{n}\rangle = 0 \text{ on } \Gamma.$$

Since Σ does not split, we know that $\langle y,\mathbf{n}\rangle$ is not identically zero unless $y = 0$. It follows that the multiplicity of the kernel is at least $(n+1)$. Since this multiplicity is greater than one, 0 is not the lowest eigenvalue for the operator $\Delta_\Gamma + |A|^2$. Let g be the lowest eigenfunction for $\Delta_\Gamma + |A|^2$ on Γ, so that

$$(\Delta_\Gamma + |A|^2)g = -\mu g \text{ and } \mu < 0.$$

Since g and $\langle y,\mathbf{n}\rangle$ are eigenfunctions, with different eigenvalues, of a symmetric operator, it follows that they are L^2-orthogonal on Γ. Therefore, the cross-term integrates to zero in (12.23) on each level set of $|x|$ and we get

$$\partial_{ss}\Big|_{s=0}G(ys,1,s) = (4\pi)^{-\frac{n}{2}}\int_\Sigma \left((\mu + 1 - n)\,g^2 - \frac{1}{2}\langle y,\mathbf{n}\rangle^2\right)e^{-\frac{|x|^2}{4}},$$

which is strictly negative since $\mu < 0$.

Case 2. Γ has nonempty singular set. Let μ_1 be the bottom of the spectrum of $\Delta_\Gamma + |A|^2$ on the regular part of Γ. The second variation operator for Γ as a minimal hypersurface of the n-sphere ∂B_1 is given by

$$\Delta_\Gamma + |A_\Gamma|^2 + \mathrm{Ric}_{\partial B_1}(\mathbf{n},\mathbf{n}) = \Delta_\Gamma + |A_\Gamma|^2 + (n-1).$$

Therefore, if $\mu_1 \neq -\infty$, then there is a uniform lower bound for the spectrum of the second variation operator of Γ and, by using the Sobolev inequality as

in the proof of Lemma 12.7, it follows that all sufficiently small balls in Γ have the slightly weaker stability inequality (12.8). Since the singular set of Γ has finite codimension-two measure and the argument in [SS81] goes through with the slightly weaker stability inequality (12.8), we conclude that Γ is smooth (putting us back in Case 1). Thus, we may assume that $\mu_1 = -\infty$ and we may choose a compact subset Ω contained in the regular part of Γ so that the lowest Dirichlet eigenfunction g of $\Delta_\Gamma + |A|^2$ on Ω satisfies

$$(\Delta_\Gamma + |A|^2)g = -\mu g \text{ where } \mu < -1.$$

Using the Cauchy-Schwarz inequality $2|x|\,g\langle y, \mathbf{n}\rangle \leq \langle y, \mathbf{n}\rangle^2 + |x|^2 g^2$, (12.23) gives

$$\partial_{ss}\Big|_{s=0} G(ys, 1, s) \leq (4\pi)^{-\frac{n}{2}} \int_\Sigma \left((\mu + 1 - n)\, g^2 + \frac{1}{2}|x|^2 g^2 \right) \mathrm{e}^{-\frac{|x|^2}{4}}$$

$$= (4\pi)^{-\frac{n}{2}} \int_\Sigma (\mu + 1)\, g^2\, \mathrm{e}^{-\frac{|x|^2}{4}},$$

where the second equality used that g is independent of $|x|$, Σ is an n-dimensional cone, and

$$\frac{1}{2} \int_0^\infty r^{n+1}\, \mathrm{e}^{-\frac{r^2}{4}}\, dr = n \int_0^\infty r^{n-1}\, \mathrm{e}^{-\frac{r^2}{4}}\, dr.$$

Since $\mu < -1$, we get that (12.23) is negative as claimed.

Now that we have established that (12.23) is negative, we can argue as in the proof of the splitting theorem. First, we argue as in Lemma 7.10 (with the constant a there equal to zero) to see that

(1) $G(x_0, 1, 0)$ has a strict global maximum at $x_0 = 0$.

Second, by Proposition 3.6, $\nabla_{x_0, s} G(0, 1, 0) = 0$ and the negativity of (12.23) gives that the hessian of $G(x_0, 1, s)$ is negative definite at $x_0 = 0$ and $s = 0$. It follows that

(2) $G(x_0, 1, s)$ has a strict local maximum at $x_0 = 0$ and $s = 0$.

Third, using that the links of the cones are varying smoothly, the first variation formula easily gives that

(3) $|\partial_s G(x_0, t_0, s)|$ is uniformly bounded when $|x_0|$ and $|\log t_0|$ are uniformly bounded.

Fourth, arguing as in (7.5) and (7.49), we get

$$\partial_{t_0} G(x_0, t_0, s) \geq -(4\pi t_0)^{-\frac{n}{2}} \int_\Sigma \frac{H_\Sigma^2}{4}\, \mathrm{e}^{-\frac{|x - x_0|^2}{4t_0}} - (4\pi t_0)^{-\frac{n}{2}} \int_{\Sigma_s \backslash \Sigma} \frac{H_{\Sigma_s}^2}{4}\, \mathrm{e}^{-\frac{|x - x_0|^2}{4t_0}}$$

$$= -(4\pi t_0)^{-\frac{n}{2}} \int_{\Sigma_s \backslash \Sigma} \frac{H_{\Sigma_s}^2}{4}\, \mathrm{e}^{-\frac{|x - x_0|^2}{4t_0}}$$

$$\geq -C_H (4\pi t_0)^{-\frac{n}{2}} \int_{\Sigma_s \backslash \Sigma} |x|^{-2}\, \mathrm{e}^{-\frac{|x - x_0|^2}{4t_0}},$$

where the equality used that Σ is minimal and the last inequality used that $|H_{\Sigma_s}|^2 \leq C_H |x|^{-2}$ since $\Sigma_s \setminus \Sigma$ is a cone that is smooth (with boundary) away from 0. We will show that this integral has a uniform bound when $|x_0| = 1$ and $t_0 \leq 1$. To do this, we divide $\Sigma_s \setminus \Sigma$ into three subsets:

$$
\begin{aligned}
S_1 &= B_{\frac{1}{2}}(x_0) \cap \Sigma_s \setminus \Sigma, \\
S_2 &= B_1 \cap \Sigma_s \setminus (S_1 \cup \Sigma), \\
S_3 &= \Sigma_s \setminus (S_1 \cup S_2 \cup \Sigma).
\end{aligned}
$$

Since the triangle inequality gives that $|x| \geq \frac{1}{2}$ on S_1, we get

$$
(4\pi t_0)^{-\frac{n}{2}} \int_{S_1} |x|^{-2} \, e^{-\frac{|x-x_0|^2}{4t_0}} \leq 4(4\pi t_0)^{-\frac{n}{2}} \int_{S_1} e^{-\frac{|x-x_0|^2}{4t_0}}.
$$

To bound this integral, we use that S_1 is given as the graph of a smooth function over a smooth subset of Σ to get a constant C_V so that for every $r \leq \frac{1}{2}$,

$$
\mathrm{Vol}(B_r(x_0) \cap S_1) \leq C_V r^n.
$$

We will divide the integral into annuli

$$
A_j = B_{\sqrt{j\,t_0}}(x_0) \cap S_1 \setminus B_{\sqrt{(j-1)t_0}}(x_0).
$$

The volume bound gives that $\mathrm{Vol}(A_j) \leq C_V j^{\frac{n}{2}} t_0^{\frac{n}{2}}$; we also have that

$$
\sup_{A_j} e^{-\frac{|x-x_0|^2}{4t_0}} = e^{\frac{1-j}{4}}.
$$

We can now estimate the integral over S_1 by

$$
\begin{aligned}
4(4\pi t_0)^{-\frac{n}{2}} \int_{S_1} e^{-\frac{|x-x_0|^2}{4t_0}} &\leq 4(4\pi t_0)^{-\frac{n}{2}} \int_{B_{\sqrt{t_0}}(x_0) \cap S_1} 1 \\
&\quad + 4(4\pi t_0)^{-\frac{n}{2}} \sum_{j \geq 2} \int_{A_j} e^{-\frac{|x-x_0|^2}{4t_0}} \\
&\leq C_V 4(4\pi)^{-\frac{n}{2}} + C_V 4(4\pi)^{-\frac{n}{2}} \sum_{j \geq 2} \left(j^{\frac{n}{2}} e^{\frac{1-j}{4}} \right) \leq c_1,
\end{aligned}
$$

where the last inequality used that the series converges and that the expression does not depend on t_0 since the powers of t_0 cancel.

To get the bound on S_2, use the triangle inequality to get that $|x - x_0| \geq \frac{1}{2}$ on S_2, and then use the cone structure to see that $\mathrm{Vol}(\partial B_r \cap \Sigma_s) = C r^{n-1}$. Thus we can bound the integral by

$$
\begin{aligned}
(4\pi t_0)^{-\frac{n}{2}} \int_{S_2} |x|^{-2} \, e^{-\frac{|x-x_0|^2}{4t_0}} &\leq (4\pi t_0)^{-\frac{n}{2}} \int_0^1 (C r^{n-1}) r^{-2} \, e^{-\frac{1}{8t_0}} \, dr \\
&\leq C(4\pi t_0)^{-\frac{n}{2}} \, e^{-\frac{1}{8t_0}} \int_0^1 r^{n-3} \, dr.
\end{aligned}
$$

Since n is at least three (the only minimal cones in $\mathbf{R}^3$ with locally finite codimension-two measure are planes), the integral $\int_0^1 r^{n-3} \, dr$ is finite. Moreover, the function

$$h(t_0) = (4\pi t_0)^{-\frac{n}{2}} \, \mathrm{e}^{-\frac{1}{8t_0}}$$

is smooth on $(0, 1]$ and has $\lim_{t_0 \to 0} h(t_0) = 0$, so it is uniformly bounded on $(0, 1]$. We conclude that the integral over S_2 is also bounded.

Finally, on S_3, we use the triangle inequality to see that $|x - x_0| \geq \frac{|x|}{2}$ and then use the cone structure to get

$$
\begin{aligned}
(4\pi t_0)^{-\frac{n}{2}} \int_{S_3} |x|^{-2} \, \mathrm{e}^{-\frac{|x-x_0|^2}{4t_0}} &\leq (4\pi t_0)^{-\frac{n}{2}} \int_1^\infty (Cr^{n-1}) r^{-2} \, \mathrm{e}^{-\frac{r^2}{8t_0}} \, dr \\
&\leq C(4\pi t_0)^{-\frac{n}{2}} \int_1^\infty r^{n-1} \, \mathrm{e}^{-\frac{r^2}{8t_0}} \, dr \leq c_3,
\end{aligned}
$$

where we used that $r \geq 1$ in the second inequality and the last inequality used that this integral is bounded by (a constant times)

$$1 = \int_{\mathbf{R}^n} H(0, x, 2t_0) dx = c_n (4\pi 2t_0)^{-\frac{n}{2}} \int_0^\infty r^{n-1} \, \mathrm{e}^{-\frac{r^2}{8t_0}} \, dr,$$

where H is the heat kernel on $\mathbf{R}^n$. Now that we have gotten uniform bounds on the integrals on all three regions (independent of $t_0 \in (0, 1]$), it follows that

(4) If $|x_0| = 1$ and $t_0 \leq 1$, then $\partial_{t_0} G(x_0, t_0, s) \geq -C_1$ for a fixed constant C_1.

Fifth, (1) gives $\alpha > 0$ and $\bar{R}$ so that

$$\sup \{ G(x_0, 1, 0) | \, |x_0| \geq \bar{R} \} < \lambda(\Sigma) - 3\alpha.$$

Using, in order, the dilation invariance of Σ_s, (4), (3), dilation invariance again, and then the above inequality gives

$$
\begin{aligned}
G(x_0, 1, s) = G(x_0/|x_0|, |x_0|^{-2}, s) &\leq G(x_0/|x_0|, \bar{R}^{-2}, s) + C_1 \bar{R}^{-2} \\
&\leq G(x_0/|x_0|, \bar{R}^{-2}, 0) + \bar{C} s + C_1 \bar{R}^{-2} \\
&= G(\bar{R} x_0/|x_0|, 1, 0) + \bar{C} s + C_1 \bar{R}^{-2} \\
&\leq \lambda(\Sigma) - 3\alpha + \bar{C} s + C_1 \bar{R}^{-2},
\end{aligned}
$$

where C_1 is given by (4) and $\bar{C} = \bar{C}(\bar{R})$ is given by (3). Thus, after possibly increasing $\bar{R}$ to make $C_1 \bar{R}^{-2} < \alpha$, thus determining $\bar{C}$, and then setting $\bar{\varepsilon} = \alpha/(2\bar{C})$, we get that

(5) There exist $\alpha > 0$, $\bar{R}$, and $\bar{\varepsilon}$ so that

$$\sup \{ G(x_0, 1, s) | \, |x_0| \geq \bar{R} \text{ and } |s| \leq 2\bar{\varepsilon} \} < \lambda(\Sigma) - \alpha.$$

Proving (12.22) *using* (1)–(5). The case of (12.22) where $|x_0| \geq \bar{R}$ is given in (5). Likewise, (2) gives (12.22) when $|x_0| \leq \bar{r}$ for some $\bar{r} > 0$. Finally, the remaining case where $\bar{r} < |x_0| < \bar{R}$ follows by combining (1) and (3).   □

The proof of Theorem 12.1 relied on the result of Schoen-Simon, [SS81]. This result was recently significantly sharpened by Wickramasekera, [Wic09]. Thus, if we use Wickramasekera's result, then we get a sharper regularity result that we describe below. Before recalling the theorem of Wickramasekera, we need the following definition of his.

Fix any $\alpha \in (0,1)$. We say that an n-dimensional varifold $\Sigma \subset \mathbf{R}^{n+1}$ satisfies the α-*structural hypothesis* if no singular point of Σ has a neighborhood in which the support of Σ is the union of three or more embedded $C^{1,\alpha}$ hypersurfaces with boundary meeting (only) along an $(n-1)$-dimensional embedded $C^{1,\alpha}$ submanifold.

PROPOSITION 12.24. *Let* $\Sigma \subset \mathbf{R}^{n+1}$ *be an* n-*dimensional* F-*stationary integral varifold having Euclidean volume growth and orientable,* F-*stable regular part. Fix any* $\alpha \in (0,1)$, *and suppose that* Σ *satisfies the* α-*structural hypothesis.*

The varifold then corresponds to an embedded, analytic hypersurface away from a closed set of singularities of Hausdorff dimension at most $n-7$, *which is absent if* $n \leq 6$ *and is discrete if* $n = 7$.

As mentioned, the proof of this proposition will use a very recent result of Wickramasekera, [Wic09], in place of the result of Schoen-Simon. For convenience of the reader, we state this next. (The proposition is stated for a unit ball in Euclidean space, but holds for all sufficiently small balls in a fixed Riemannian manifold.)

PROPOSITION 12.25 (Wickramasekera [Wic09]). *Consider an* n-*dimensional stationary integral varifold* Σ *in an open ball in* $\mathbf{R}^{n+1}$ *having finite mass and orientable regular part. Suppose also that for all smooth functions with compact support contained in the regular part of* Σ, *we have the stability-type inequality* $\int_{\Sigma} |A|^2 \phi^2 \leq (1 + \varepsilon) \int_{\Sigma} |\nabla \phi|^2$ *for some sufficiently small* $\varepsilon > 0$. *Fix any* $\alpha \in (0,1)$, *and suppose that* Σ *satisfies the* α-*structural hypothesis.*

The varifold then corresponds to an embedded, analytic hypersurface away from a closed set of singularities of Hausdorff dimension at most $n-7$, *which is absent if* $n \leq 6$ *and is discrete if* $n = 7$.

We can now prove the sharper regularity result of Proposition 12.24.

Proof of Proposition 12.24. Using that the regular part is orientable and F-stable, we can apply Lemma 12.7 to get the stability-type inequality.

For all smooth functions ϕ with compact support contained in
the intersection of a sufficiently small ball with the regular set

$$\int_\Sigma |A|^2 \phi^2 \leq (1+\varepsilon) \int_\Sigma |\nabla \phi|^2$$

for any small $\varepsilon > 0$ (taking ε smaller forces us to reduce the
size of the balls).

Therefore, Proposition 12.25 applies in these small balls to give the desired local regularity. The proposition follows by piecing together the local regularity. □

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