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# The Borel Conjecture for hyperbolic and CAT(0)-groups

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## Abstract

We prove the Borel Conjecture for a class of groups containing word-hyperbolic groups and groups acting properly, isometrically and cocompactly on a finite-dimensional CAT(0)-space.

## Contents



## Introduction

*The Borel Conjecture.* A closed manifold $M$ is said to be *topologically rigid* if every homotopy equivalence to another closed manifold is homotopic to a homeomorphism. In particular, if $M$ is topologically rigid, then every manifold homotopy equivalent to $M$ is homeomorphic to $M$. The spheres $S^n$ are topologically rigid as predicted by the *Poincaré Conjecture.* We will focus on the *Borel Conjecture* which asserts:

*Closed aspherical manifolds are topologically rigid.*

An important result of Farrell-Jones is that this conjecture holds for manifolds of dimension $\geq 5$ which support a Riemannian metric of nonpositive sectional curvature [28]. In further work, Farrell-Jones extended this result to cover compact complete affine flat manifolds of dimension $\geq 5$ [29]. This is done by considering complete nonpositively curved manifolds that are not necessarily compact. Note that the universal cover is in these cases always homeomorphic to Euclidean space. We will go considerably beyond the world of Riemannian manifolds of nonpositive curvature. In particular, we prove the Borel Conjecture for closed aspherical manifolds of dimension $\geq 5$, whose fundamental group is hyperbolic in the sense of Gromov [14], [32] or is nonpositively curved in the sense, that it admits a cocompact isometric proper action on a finite-dimensional CAT(0)-space.

*Definition* (The class of groups $\mathcal{B}$). Let $\mathcal{B}$ be the smallest class of groups satisfying the following conditions:

(i) Hyperbolic groups belong to $\mathcal{B}$.
(ii) If $G$ acts properly cocompactly and isometrically on a finite-dimensional CAT(0)-space, then $G \in \mathcal{B}$.
(iii) The class $\mathcal{B}$ is closed under taking subgroups.
(iv) Let $\pi\colon G \to H$ be a group homomorphism. If $H \in \mathcal{B}$ and $\pi^{-1}(V) \in \mathcal{B}$ for all virtually cyclic subgroups $V$ of $H$, then $G \in \mathcal{B}$.
(v) $\mathcal{B}$ is closed under finite direct products.
(vi) $\mathcal{B}$ is closed under finite free products.
(vii) The class $\mathcal{B}$ is closed under directed colimits; i.e., if $\{G_i \mid i \in I\}$ is a directed system of groups (with not necessarily injective structure maps) such that $G_i \in \mathcal{B}$ for $i \in I$, then $\operatorname{colim}_{i\in I} G_i$ belongs to $\mathcal{B}$.

We refer to groups that admit an action on a CAT(0)-space as in (ii) as finite-dimensional CAT(0)-group. Notice that the underlying CAT(0)-space is automatically complete and proper (see [14, Ex. 8.4(1), p. 132]). If a group acts properly, cocompactly and isometrically on a CAT(0)-space, then the boundary of this CAT(0)-space is finite-dimensional [60, Th. 12]. It seems to be an open question whether the CAT(0)-space itself can be arranged to be finite-dimensional.

The following is our main theorem.

Theorem A. *Let $M$ be a closed aspherical manifold of dimension $\geq 5$. If $\pi_1(M) \in \mathcal{B}$, then $M$ is topologically rigid.*

We prove this result by establishing the Farrell-Jones Conjectures in algebraic $K$-theory and $L$-theory for this class of groups. (For finite-dimensional CAT(0)-groups this is not quite correct; the result does not cover higher $K$-theory, but suffices for the Borel Conjecture and the applications below.)

Provided that Thurston's Geometrization Conjecture is true, every closed 3-manifold with torsionfree fundamental group is topologically rigid and, in particular, the Borel Conjecture holds in dimension three (see [36, Th. 0.7]). Theorem A above remains true in dimension four if one additionally assumes that the fundamental group is good in the sense of Freedman [30]. In dimension $\leq 2$ the Borel Conjecture is known to be true by classical results. More information about topologically rigid (not necessarily aspherical) manifolds can be found in [36].

A number of further important applications of our results on the Farrell-Jones Conjecture can be summarized as follows. The *Novikov Conjecture* and the *Bass Conjecture* hold for all groups $G$ that belong to $\mathcal{B}$. If $G$ is torsion-free and belongs to $\mathcal{B}$, then the Whitehead group $\mathrm{Wh}(G)$ of $G$ is trivial, $\widetilde{K}_0(RG) = 0$ if $R$ is a principal ideal domain, and $K_n(RG) = 0$ for $n \leq -1$ if $R$ a regular ring. Furthermore the *Kaplansky Conjecture* holds for such $G$. These and further applications of the Farrell-Jones Conjectures are discussed in detail in [10] and [42]. We remark that Hu [34] proved that if $G$ is the fundamental group of a finite polyhedron with nonpositive curvature, then $\mathrm{Wh}(G) = 0$, $\widetilde{K}_0(\mathbb{Z}G) = 0$ and $K_n(\mathbb{Z}G) = 0$ for $n \leq -1$.

*The Farrell-Jones Conjectures.* According to the Farrell-Jones Conjectures [27] the algebraic $K$-theory and the $L$-theory of a group ring $\mathbb{Z}G$ can in a certain sense be computed in terms of the algebraic $K$-theory and the $L$-theory of $\mathbb{Z}V$ where $V$ runs over the family $\mathcal{VC}\mathrm{yc}$ of virtually cyclic subgroups of $G$. These conjectures are the key to the Borel Conjecture. See [42] for a survey on the Farrell-Jones Conjectures. Positive results on these conjectures for groups acting on nonpositively curved Riemannian manifolds are contained in [27]. The Farrell-Jones Conjectures have been generalized to include group rings $RG$ over arbitrary rings [5], and further to twisted group rings which are best treated using the language of actions of $G$ on additive categories [7], [12]. For a group $G$, the Farrell-Jones Conjectures with coefficients in the additive $G$-category $\mathcal{A}$ (with involution) assert that the assembly maps

$$H_m^G(E_{\mathcal{VC}\mathrm{yc}}(G); \mathbf{K}_{\mathcal{A}}) \to K_m(\textstyle\int_G \mathcal{A}), \tag{0.1}$$

$$H_m^G(E_{\mathcal{VC}\mathrm{yc}}(G); \mathbf{L}_{\mathcal{A}}^{-\infty}) \to L_m^{\langle -\infty \rangle}(\textstyle\int_G \mathcal{A}) \tag{0.2}$$

are isomorphisms. The $K$-theoretic Farrell-Jones Conjectures (with coefficients in an arbitrary additive category) for hyperbolic groups has been proven by Bartels-Lück-Reich in [9]. Here we extend this result to the $L$-theoretic Farrell-Jones Conjecture and (apart from higher $K$-theory) to CAT(0) groups.

Theorem B. *Let $G \in \mathcal{B}$.*

(i) *The $K$-theoretic assembly map* (0.1) *is bijective in degree $m \leq 0$ and surjective in degree $m = 1$ for any additive $G$-category $\mathcal{A}$.*

(ii) *The $L$-theoretic Farrell-Jones assembly map* (0.2) *with coefficients in any additive $G$-category $\mathcal{A}$ with involution is an isomorphism.*

We point out that the proof of Theorem B for CAT(0) groups depends on Proposition 2.2, which is proven in [6].

For virtually abelian groups Quinn [51] proved that (0.1) is an isomorphism for all $n$. (More precisely, in [51] only the untwisted case is considered: $\mathcal{A}$ is the category of finitely generated free $R$-modules for some ring $R$.) In Theorem 1.1 below we will give precise conditions under which our methods establish the assertions of Theorem B. In the proof homotopy actions play a prominent role. In $K$-theory, these are easier to treat for the groups $K_i$, $i \leq 1$ than for higher $K$-theory, and this is the reason for the restrictions in the $K$-theory statement in Theorem B. Wegner [61] extended the methods of this paper to higher $K$-theory. He showed in particular that CAT(0)-groups satisfy the full $K$-theoretic Farrell-Jones Conjecture; i.e., the assembly map (0.1) is for such groups an isomorphism for all $m$.

Next we explain the relation between Theorem B and Theorem A.

PROPOSITION 0.3. *Let $G$ be a torsion-free group. Suppose that the $K$-theoretic assembly map*

$$H_m^G(E_{\mathcal{VC}\mathrm{yc}}(G); \mathbf{K}_{\mathbb{Z}}) \to K_m(\mathbb{Z}G)$$

*is an isomorphism for $m \leq 0$ and surjective for $m = 1$ and that the $L$-theoretic assembly map*

$$H_m^G(E_{\mathcal{VC}\mathrm{yc}}(G); \mathbf{L}_{\mathbb{Z}}^{-\infty}) \to L_m^{-\infty}(\mathbb{Z}G)$$

*is an isomorphism for all $m \in \mathbb{Z}$, where we allow a twisting by any homomorphism $w\colon G \to \{\pm 1\}$. Then the following holds*:

(i) *The assembly map*

$$H_n(BG; \mathbf{L}_{\mathbb{Z}}^s) \to L_n^s(\mathbb{Z}G) \tag{0.4}$$

*is an isomorphism for all $n$.*

(ii) *The Borel Conjecture is true in dimension $\geq 5$, i.e., if $M$ and $N$ are closed aspherical manifolds of dimensions $\geq 5$ with $\pi_1(M) \cong \pi_1(N) \cong G$, then $M$ and $N$ are homeomorphic and any homotopy equivalence $M \to N$ is homotopic to a homeomorphism.* (*This is also true in dimension* 4 *if we assume that $G$ is good in the sense of Freedman.*)

(iii) *Let $X$ be a finitely dominated Poincaré complex of dimension $\geq 6$ with $\pi_1(X) \cong G$. Then $X$ is homotopy equivalent to a compact* ANR-*homology manifold.*

*Proof.* (i) Because $G$ is torsion-free and $\mathbb{Z}$ is regular, the above assembly maps are equivalent to the maps

$$H_m(BG; \mathbf{K}_{\mathbb{Z}}) \to K_m(\mathbb{Z}G), \tag{0.5}$$

$$H_m(BG; \mathbf{L}_{\mathbb{Z}}^{-\infty}) \to L_m^{-\infty}(\mathbb{Z}G) \tag{0.6}$$

(compare [42, Prop. 2.2, p. 685]). Because (0.5) is bijective for $m \leq 0$ and surjective for $m = 1$, we have $\mathrm{Wh}(G) = 0$, $\widetilde{K_0}(\mathbb{Z}G) = 0$ and $K_i(\mathbb{Z}G) = 0$ for $i < 0$; compare [42, Conj. 1.3, p. 653 and Rem. 2.5, p. 679]. This implies that (0.6) is equivalent to (0.4); compare [42, Prop. 1.5, p. 664].

(ii) We have to show that the geometric structure set of a closed aspherical manifold of dimension $\geq 5$ consists of precisely one element. This follows from (i) and the algebraic surgery exact sequence of Ranicki [52, Def. 15.19, p. 169] which agrees for an $n$-dimensional manifold for $n \geq 5$ with the Sullivan-Wall geometric exact surgery sequence (see [52, Th. 18.5, p. 198]).

(iii) See [15, Main Theorem, p. 439] and [52, Rem. 25.13, p. 297]. □

The assembly maps appearing in the proposition above are special cases of the assembly maps (0.1) and (0.2); compare [12, Cor. 6.17]. In particular, Theorem A follows from Theorem B and the above Proposition 0.3. In work with Shmuel Weinberger [11] we use Theorem B to show that if the boundary of a torsion-free hyperbolic group is a sphere of dimension $\geq 5$, then this hyperbolic group is the fundamental group of a closed aspherical manifold, not just of an ANR-homology manifold.

*Some groups from $\mathcal{B}$.* The class $\mathcal{B}$ contains in particular directed colimits of hyperbolic groups. The $K$-theory version of the Farrell-Jones Conjecture holds in all degrees for directed colimits of hyperbolic groups [4, Th. 0.8(i)]. Thus Theorem B implies that the Farrell-Jones Conjecture in $K$- and $L$-theory hold for directed colimits of hyperbolic groups. This class of groups contains a number of groups with unusual properties. Counterexamples to the Baum-Connes Conjecture with coefficients are groups with expanders [33]. The only known construction of such groups is as directed colimits of hyperbolic groups (see [2]). Thus the Farrell-Jones Conjecture in $K$- and $L$-theory holds for the only at present known counter examples to the Baum-Connes Conjecture with coefficients. (We remark that the formulation of the Farrell-Jones Conjecture we are considering allows for twisted group rings, so this includes the correct analog of the Baum-Connes Conjecture with coefficients.) The class of directed colimits of hyperbolic groups contains, for instance, a torsion-free noncyclic group all whose proper subgroups are cyclic constructed by Ol'shanskii [46]. Further examples are mentioned in [47, p. 5] and [58, §4]. These latter examples all lie in the class of lacunary groups. Lacunary groups can be characterized as certain colimits of hyperbolic groups.

A Coxeter system $(W, S)$ is a group $W$ together with a fundamental set $S$ of generators; see, for instance, [24, Def. 3.3.2]. Associated to the Coxeter system $(W, S)$ is a simplicial complex $\Sigma$ with a metric [24, Chap. 7] and a proper isometric $W$-action. Moussong [45] showed that $\Sigma$ is a CAT(0)-space; see also [24, Th. 12.3.3]. In particular, if $\Sigma$ is finite-dimensional and the action is cocompact, then $W$ is a finite-dimensional CAT(0)-group and belongs to $\mathcal{B}$. This is, in particular, the case if $S$ is finite. If $S$ is infinite, then any finite subset $S_0 \subset S$ generates a Coxeter group $W_0$; see [24, Th. 4.1.6]. Then $W_0$ belongs to $\mathcal{B}$ and so does $W$ as it is the colimit of the $W_0$. Therefore Coxeter groups belong to $\mathcal{B}$. Davis constructed, for every $n \geq 4$, closed aspherical manifolds whose universal cover is not homeomorphic to Euclidean space [23, Cor. 15.8]. In particular, these manifolds do not support metrics of nonpositive sectional curvature. The fundamental groups of these examples are finite index subgroups of Coxeter groups $W$. Thus these fundamental groups lie in $\mathcal{B}$ and Theorem A implies that Davis' examples are topological rigid (if the dimension is at least 5).

Davis and Januszkiewicz used Gromov's hyperbolization technique to construct further exotic aspherical manifolds. They showed that for every $n \geq 5$, there are closed aspherical $n$-dimensional manifolds whose universal cover is a CAT(0)-space whose fundamental group at infinite is nontrivial [25, Th. 5b.1]. In particular, these universal covers are not homeomorphic to Euclidean space. Because these examples are, in addition, nonpositively curved polyhedron, their fundamental groups are finite-dimensional CAT(0)-groups and belong to $\mathcal{B}$. There is a variation of this construction that uses the strict hyperbolization of Charney-Davis [20] and produces closed aspherical manifolds whose universal cover is not homeomorphic to Euclidean space and whose fundamental group is hyperbolic. All these examples are topologically rigid by Theorem A.

Limit groups, as they appear for instance in [59], have been in the focus of geometric group theory for the last years. Expositions about limit groups are, for instance, [19] and [48]. Alibegović-Bestvina have shown that limit groups are CAT(0)-groups [1]. A straightforward analysis of their argument shows that limit groups are finite-dimensional CAT(0)-groups and belong therefore to our class $\mathcal{B}$.

If a locally compact group $L$ acts properly cocompactly and isometrically on a finite-dimensional CAT(0)-space, then the same is true for any discrete cocompact subgroup of $L$. Such subgroups belong therefore to $\mathcal{B}$. For example, let $\mathbf{G}$ be a reductive algebraic group defined over a global field $k$ whose $k$-rank is 0. Let $S$ be a finite set of places of $k$ that contains the infinite places of $k$. The group $\mathbf{G}_S := \prod_{v \in S} \mathbf{G}(k_v)$ admits an isometric proper cocompact action on a finite-dimensional CAT(0)-space; see for example [35, p. 40]. Because $S$-arithmetic subgroups of $\mathbf{G}(k)$ can be realized (by the diagonal embedding) as discrete cocompact subgroups of $\mathbf{G}_S$ (see for example [35]), these $S$-arithmetic groups belong to $\mathcal{B}$.

Consider a (not necessarily cocompact) lattice in $\mathrm{SO}(n,1)$, e.g., a fundamental group of a hyperbolic Riemannian manifold with finite volume. By [14, II.11.28, p. 362] such groups are CAT(0)-groups and belong therefore to $\mathcal{B}$.

Finitely generated virtually abelian groups are finite-dimensional CAT(0)-groups and belong to $\mathcal{B}$. A simple induction shows that this implies that all virtually nilpotent groups belong to $\mathcal{B}$; compare the proof of [10, Lemma 1.13].

*Outline of the proof.* In Section 1 we formulate geometric conditions under which we can prove the Farrell-Jones Conjectures. These conditions are satisfied for hyperbolic groups and finite-dimensional CAT(0)-groups (see §2) and are similar to the conditions under which the $K$-theoretic Farrell-Jones Conjecture has been proven in [9]. Very roughly, these conditions assert the existence of a compact space $X$ with a homotopy $G$-action and the existence of a "long thin" $G$-equivariant cover of $G \times X$. New is the use of homotopy actions here; this is crucial for the application to finite-dimensional CAT(0)-groups. It suffices to have homotopy actions at hand since the transfer maps require only homotopy chain actions.

The general strategy of the proof is similar to the one employed in [9]. Controlled algebra is used to set up an obstruction category whose $K$- respectively $L$-theory gives the homotopy fiber of the assembly map in question; see Theorem 5.2. We will mostly study $K_1$ and $L_0$ of these categories. In $K$-theory we represent elements by automorphisms or, more generally, by self-chain homotopy equivalences. In $L$-theory we represent elements by quadratic forms or, more generally, by 0-dimensional ultra-quadratic Poincaré complexes; compare Section 4.5. For this outline it will be convenient to call these *representatives cycles*. In all cases these cycles come with a notion of size. More precisely, the obstruction category depends on a free $G$-space $Z$ (in the simplest case this space is $G$, but it is important to keep this space variable) and associated to any cycle is a subset (its support) of $Z \times Z$. If $Z$ is a metric space, then a cycle is said to be $\alpha$-controlled over $Z$ for some number $\alpha > 0$ if $d_Z(x,y) \leq \alpha$ for all $(x,y)$ in the support of the cycle. The Stability Theorem 5.3 for the obstruction category asserts (for a class of metric space) that there is $\varepsilon > 0$ such that the $K$-theory respectively $L$-theory class of every $\varepsilon$-controlled cycle is trivial. The strategy of the proof is then to prove that the $K$-theory respectively $L$-theory of the obstruction category is trivial by showing that every cycle is equivalent to an $\varepsilon$-controlled cycle.

This is achieved in two steps. Firstly, a transfer replacing $G$ by $G \times X$ for a suitable compact space $X$ is used. Secondly, the "long thin" cover of $G \times X$ is used to construct a contracting map from $G \times X$ to a $\mathcal{VC}$yc-CW-complex;

see Proposition 3.9. More precisely, this map is contracting with respect to the $G$-coordinate, but expanding with respect to the $X$-coordinate. Thus it is crucial that the output of the transfer is a cycle that is $\varepsilon$-controlled over $X$ for very small $\varepsilon$. To a significant extend, the argument in the $L$-theory case and the $K$-theory case are very similar. For example, the formalism of controlled algebra works for $L$-theory in the same way as for $K$-theory. This is because both functors have very similar properties; compare Theorem 5.1. However, the $L$-theory transfer is quite different from the $K$-theory transfer and requires new ideas.

*L-theory transfer.* The transfer is used to replace a cycle $a$ in the $K$- respectively $L$-theory of the obstruction category over $G$ by a cycle $\mathrm{tr}(a)$ over $G{\times}X$, such that $\mathrm{tr}(a)$ is $\varepsilon$-controlled (for very small $\varepsilon$) if control is measured over $X$ (using the canonical projection $G{\times}X \to X$). In $K$-theory the transfer is essentially obtained by taking a tensor product with the singular chain complex of $X$. More precisely, we use a chain complex $P$ chain homotopy equivalent to the singular complex such that, in addition, $P$ is $\varepsilon$-controlled over $X$; compare Proposition 7.2. (Roughly, this is the simplicial chain complex of a sufficiently fine triangulation of $X$.) The homotopy action on $X$ induces a corresponding action on $P$. This action is important as it is used to twist the tensor product. The homology of $P$ agrees with the homology of a point (because $X$ is contractible). This is important as it controls the effect of the transfer in $K$-theory; i.e., $\mathrm{tr}(a)$ projects to $a$ under the map induced by $G{\times}X \to G$. The datum needed for transfers in $L$-theory is a chain complex together with a symmetric form, i.e., a symmetric Poincaré complex. It is not hard, because $P$ has the homology of a point, to equip $P$ with a symmetric form. However, such a symmetric form on $P$ will not be $\varepsilon$-controlled over $X$ and is therefore not sufficient for the purpose of producing a cycle $\mathrm{tr}(a)$ that is $\varepsilon$-controlled over $X$.

In the case treated by Farrell-Jones, where $G$ is the fundamental group of a nonpositively curved manifold $M$, this problem is solved roughly as follows. In this situation the sphere bundle $SM \to M$ is considered. The fiber of this bundle is a manifold, and Poincaré duality yields an $\varepsilon$-controlled symmetric form on the simplicial chain complex of a sufficiently fine triangulation of the fiber. However, the signature of the fiber governs the effect of the transfer in $L$-theory, and since the signature of the sphere is trivial, the transfer is the zero map in $L$-theory in this case. This problem is overcome by considering the quotient of the fiber-wise product $SM{\times}_M SM$ by the involution that flips the two factors. The fiber of this bundle is a $\mathbb{Z}[1/2]$-homology manifold whose signature is 1 (if the dimension of $M$ is odd). In order to get a transfer over $\mathbb{Z}$ rather than $\mathbb{Z}[1/2]$ the singularities of this fiber have to be studied, and this leads to very technical arguments, but it can be done; see [26, §4]. The main problem here is that the normal bundle of the fixed point set of the flip (i.e., the diagonal sphere in the product) is in general not trivial.

For the groups considered here the space $X$ will in general not be a manifold, and we are forced to use a different approach to the $L$-theory transfer. Given the chain complex $P$, we use what we call the *multiplicative hyperbolic Poincaré chain complex* on $P$. As a chain complex this is $D := P^{-*} \otimes P$, and this chain complex carries a natural symmetric form given by the canonical isomorphism $(P^{-*} \otimes P)^{-*} \cong P \otimes P^{-*}$ followed by the flip $P \otimes P^{-*} \cong P^{-*} \otimes P$. The multiplicative hyperbolic Poincaré chain complex can naturally be considered as a complex over $X \times X$. Because of the appearance of the flip in the construction, it is not $\varepsilon$-controlled over $X \times X$. But this flip is the only problem, and the multiplicative hyperbolic Poincaré chain complex becomes $\varepsilon$-controlled over the quotient $P_2(X)$ of $X \times X$ by the flip $(x, y) \mapsto (y, x)$. This construction appears in the proof of Proposition 10.2. In the Appendix A to this paper, we review classical (i.e., uncontrolled) transfers in $K$-theory (for the Whitehead group) and $L$-theory. The reader is encouraged to refer to the appendix for motivation while reading Sections 6, 7 and 10. The appendix also contains a discussion of the multiplicative hyperbolic Poincaré chain complex in a purely algebraic context.

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## 1. **Axiomatic formulation**

*Summary.* In this section we describe conditions under which our arguments prove the Farrell-Jones Conjectures. If these conditions are satisfied for a group $G$ with respect to a family $\mathcal{F}$ of subgroups, then $G$ is said to be *transfer reducible* over $\mathcal{F}$; see Definition 1.8. Very roughly this means that there is a space $X$ satisfying suitable finiteness conditions such that $G \times X$ admits $G$-equivariant covers of uniformly bounded dimension that are very long in the $G$-direction, up to a twist described by a homotopy action on $G$; see Definition 1.4(iv). Theorem 1.1 is the most general statement about the Farrell-Jones Conjectures in this paper. It is conceivable that it applies to further interesting groups that do not belong to $\mathcal{B}$.

A *family* $\mathcal{F}$ *of subgroups of the group* $G$ is a set of subgroups of $G$ closed under conjugation and taking subgroups.

THEOREM 1.1 (Axiomatic Formulation). *Let* $\mathcal{F}$ *be a family of subgroups of the group* $G$.

*If* $G$ *is transfer reducible over* $\mathcal{F}$ (*see Definition* 1.8), *then the following holds*:

(i) *Let* $\mathcal{A}$ *be an additive* $G$*-category, i.e., an additive category with right* $G$*-action by functors of additive categories. Then the assembly map*

$$H_m^G(E_{\mathcal{F}}(G);\mathbf{K}_{\mathcal{A}}) \to K_m(\textstyle\int_G \mathcal{A}) \tag{1.2}$$

*is an isomorphism for* $m < 1$ *and surjective for* $m = 1$.

(ii) *Let* $\mathcal{A}$ *be an additive* $G$*-category with involution* (*in the sense of* [7, Def. 4.22]). *Then the assembly map*

$$H_m^G(E_{\mathcal{F}_2}(G);\mathbf{L}_{\mathcal{A}}^{-\infty}) \to L_m^{\langle -\infty\rangle}(\textstyle\int_G \mathcal{A}) \tag{1.3}$$

*is an isomorphism for all* $m \in \mathbb{Z}$. *Here* $\mathcal{F}_2$ *is the family of all subgroups* $V \subseteq G$ *for which there is* $F \subseteq V$ *such that* $F \in \mathcal{F}$ *and* $[V : F] \leq 2$.

The assembly maps appearing above have been introduced in [7] and [12], and the two slightly different approaches are identified in [7, Rem. 10.8]. If $\mathcal{F}$ is the family $\mathcal{VC}\mathrm{yc}$ of virtually cyclic groups, then these maps are the assembly maps (0.1) and (0.2) from the introduction. (Of course $\mathcal{VC}\mathrm{yc}_2 = \mathcal{VC}\mathrm{yc}$.)

In the following definition we weaken the notion of an action of a group $G$ on a space $X$ to a homotopy action that is only defined for a finite subset $S$ of $G$. Restriction of a $G$-action to such a finite subset $S$ yields a homotopy $S$-action. Other examples arise if we conjugate an honest action by a homotopy equivalence and restrict then to a finite subset $S$.

*Definition* 1.4 (Homotopy $S$-action). Let $S$ be a finite subset of a group $G$. Assume that $S$ contains the trivial element $e \in G$. Let $X$ be a space.

(i) A homotopy $S$-action on $X$ consists of continuous maps $\varphi_g \colon X \to X$ for $g \in S$ and homotopies $H_{g,h} \colon X \times [0,1] \to X$ for $g,h \in S$ with $gh \in S$ such that $H_{g,h}(-,0) = \varphi_g \circ \varphi_h$ and $H_{g,h}(-,1) = \varphi_{gh}$ holds for $g,h \in S$ with $gh \in S$. Moreover, we require that $H_{e,e}(-,t) = \varphi_e = \mathrm{id}_X$ for all $t \in [0,1]$.

(ii) Let $(\varphi, H)$ be a homotopy $S$-action on $X$. For $g \in S$, let $F_g(\varphi, H)$ be the set of all maps $X \to X$ of the form $x \mapsto H_{r,s}(x,t)$, where $t \in [0,1]$ and $r,s \in S$ with $rs = g$.

(iii) Let $(\varphi, H)$ be a homotopy $S$-action on $X$. For $(g,x) \in G \times X$ and $n \in \mathbb{N}$, let $S^n_{\varphi,H}(g,x)$ be the subset of $G \times X$ consisting of all $(h,y)$ with the following property: There are $x_0,\ldots,x_n \in X$, $a_1,b_1,\ldots,a_n,b_n \in S$,

$f_1, \widetilde{f}_1, \dots, f_n, \widetilde{f_n}\colon X \to X$ such that $x_0 = x$, $x_n = y$, $f_i \in F_{a_i}(\varphi, H)$, $\widetilde{f}_i \in F_{b_i}(\varphi, H)$, $f_i(x_{i-1}) = \widetilde{f}_i(x_i)$ and $h = ga_1^{-1}b_1 \dots a_n^{-1}b_n$.

(iv) Let $(\varphi, H)$ be a homotopy $S$-action on $X$ and $\mathcal{U}$ be an open cover of $G \times X$. We say that $\mathcal{U}$ is $S$-long with respect to $(\varphi, H)$ if, for every $(g, x) \in G \times X$, there is $U \in \mathcal{U}$ containing $S^{|S|}_{\varphi,H}(g, x)$ where $|S|$ is the cardinality of $S$.

If the homotopy action is the restriction of a $G$-action to $S$ and $S$ is symmetric with respect to $s \mapsto s^{-1}$, then $\varphi_g(x) = gx$, $H_{g,h}(x, t) = ghx$ for all $t$ and $S^n_{\varphi,H}(g, x) = \{(ga^{-1}, ax) \mid a = s_1 \dots s_{2|S|}, s_i \in S\}$. We will be able to restrict to a finite subset $S$ of $G$, because our cycles for elements in the algebraic $K$-theory or $L$-theory of the obstruction category will involve only a finite number of group elements. For example, if we are looking at an element in the $K$-theory of $RG$ given by an invertible matrix $A$ over $RG$, then the set $S$ consists of those group elements $g$ which can be written as a product $g_1g_2$ for which the coefficient of some entry in $A$ or $A^{-1}$ for $g_1$ and the coefficient of some entry in $A$ or $A^{-1}$ for $g_2$ are nontrivial.

*Definition* 1.5 ($N$-dominated space). Let $X$ be a metric space and $N \in \mathbb{N}$. We say that $X$ is controlled $N$-dominated if, for every $\varepsilon > 0$, there is a finite CW-complex $K$ of dimension at most $N$, maps $i\colon X \to K$, $p\colon K \to X$ and a homotopy $H\colon X \times [0, 1] \to X$ between $p \circ i$ and $\mathrm{id}_X$ such that for every $x \in X$, the diameter of $\{H(x, t) \mid t \in [0, 1]\}$ is at most $\varepsilon$.

*Remark* 1.6. For a hyperbolic group we will use the compactification of the Rips complex for $X$. This space is controlled $N$-dominated by finite subcomplexes of the Rips complex. The homotopy $S$-action on $X$ arises as the restriction to $S$ of the action of the hyperbolic group on $X$.

For a group $G$ that acts properly cocompactly and isometrically on a finite-dimensional CAT(0)-space $Z$, we will use a ball in $Z$ of sufficiently large radius for $X$. Projection along geodesics provides a homotopy inverse to the inclusion $X \to Z$. The homotopy $S$-action on $X$ is obtained by first restricting the $G$-action on $Z$ to $S$ and then conjugate it to $X$ using this homotopy equivalence. The controlled $N$-domination arises in this situation because $X$ is a Euclidean neighborhood retract.

We recall the following definition from [9, Def. 1.3].

*Definition* 1.7 (Open $\mathcal{F}$-cover). Let $Y$ be a $G$-space. Let $\mathcal{F}$ be a family of subgroups of $G$. A subset $U \subseteq Y$ is called an *$\mathcal{F}$-subset* if

(i) For $g \in G$ and $U \in \mathcal{U}$, we have $g(U) = U$ or $U \cap g(U) = \emptyset$, where $g(U) := \{gx \mid x \in U\}$.

(ii) The subgroup $G_U := \{g \in G \mid g(U) = U\}$ lies in $\mathcal{F}$.

An *open $\mathcal{F}$-cover* of $Y$ is a collection $\mathcal{U}$ of open $\mathcal{F}$-subsets of $Y$ such that the following conditions are satisfied:

(i) $Y = \bigcup_{U \in \mathcal{U}} U$.
(ii) For $g \in G$, $U \in \mathcal{U}$, the set $g(U)$ belongs to $\mathcal{U}$.

*Definition* 1.8 (Transfer reducible). Let $G$ be a group and $\mathcal{F}$ be a family of subgroups. We will say that $G$ is *transfer reducible* over $\mathcal{F}$ if there is a number $N$ with the following property.

For every finite subset $S$ of $G$, there are

- a contractible compact controlled $N$-dominated metric space $X$,
- a homotopy $S$-action $(\varphi, H)$ on $X$,
- a cover $\mathcal{U}$ of $G \times X$ by open sets

such that the following holds for the $G$-action on $G \times X$ given by $g \cdot (h, x) = (gh, x)$:

(i) $\dim \mathcal{U} \leq N$,
(ii) $\mathcal{U}$ is $S$-long with respect to $(\varphi, H)$,
(iii) $\mathcal{U}$ is an open $\mathcal{F}$-covering.

*Remark* 1.9. The role of the space $X$ appearing in Definition 1.8 is to yield enough space to be able to find the desired covering $\mathcal{U}$. On the first glance one might take $X = \{\mathrm{pt}\}$. But this is not a good choice by the following observation.

Suppose that the homotopy action actually comes from an honest $G$-action on $X$. Then for every $x \in X$ and every finitely generated subgroup $H \subseteq G_x$, we have $H \in \mathcal{F}$ by the following argument. Given a finite subset $S$ of $G_x$ with $e \in S$, we can find $U \in \mathcal{U}$ with $\{(s, x) \mid s \in S\} \subseteq U$ since $\mathcal{U}$ is $S$-long. Then $\{(e, x)\} \in s \cdot U \cap U$ for all $s \in S$. This implies $S \subseteq G_U$. Hence the subgroup of $G$ generated by $S$ belongs to $\mathcal{F}$ since $G_U$ belongs to $\mathcal{F}$ by assumption and a family is, by definition, closed under taking subgroups.

Of course we would like to arrange that we can choose $\mathcal{F}$ to be the family $\mathcal{VC}$yc. But this is only possible if all isotropy groups of $X$ are virtually cyclic.

The main difficulty in finding the desired covering appearing in Definition 1.8 is that the cardinality of $S$ can be arbitrarily large in comparison to the fixed number $N$.

## 2. **Proof of Theorem B modulo Theorem 1.1**

*Summary.* In this section we show that Theorem 1.1 implies Theorem B. To this end, in Lemma 2.3 we describe inheritance properties of the Farrell-Jones Conjectures and show that hyperbolic groups are transfer reducible over the family of virtually cyclic subgroups. The latter depends ultimately on work of Mineyev and Yu [43], [44]. We show in [6] that finite-dimensional

CAT(0)-groups are also transfer reducible over the family of virtually cyclic subgroups.

Proposition 2.1. *Every hyperbolic is transfer reducible over the family* $\mathcal{VC}$yc *of virtually cyclic subgroups.*

This will essentially follow from [8] and [13]; see also [9, Lemma 2.1]. However, the set-up in [8] is a little different; there $X$ is a $G$-space and the diagonal action $g \cdot (h, x) = (gh, gx)$ is considered on $G \times X$, whereas in this paper the $G$-action $g \cdot (h, x) = (gh, x)$ is used. The reason for this change is that we do not have a $G$-action on $X$ available in the more general setup of this paper; there is only a homotopy $G$-action.

*Proof of Proposition* 2.1. Let $d_G$ be a $\delta$-hyperbolic left-invariant word-metric on the hyperbolic group $G$. Let $P_d(G)$ be the associated *Rips complex* for $d > 4\delta + 6$. It is a finite-dimensional contractible locally finite simplicial complex. This space can be compactified to $X := P_d(G) \cup \partial G$, where $\partial G$ is the Gromov boundary of $G$ (see [14, III.H.3], [32]). Then $X$ is metrizable (see [14, III.H.3.18 (4), p. 433]). There is a simplicial action of $G$ on $P_d(G)$ which is proper and cocompact, and this action extends to $X$. According to [13, Th. 1.2] the subspace $\partial P_d(G) \subseteq X$ satisfies the Z-set condition. This implies the (weaker) [9, Assumption 1.2], which is a consequence of part (2) of the characterization of $Z$-sets before Theorem 1.2 in [13]. Thus there is a homotopy $H\colon X \times [0,1] \to X$ such that $H_0 = \mathrm{id}_X$ and $H_t(X) \subset P_d(G)$ for all $t > 0$. The compactness of $X$ implies that for $t > 0$, $H_t(X)$ is contained in a finite subcomplex of $P_d(G)$. Therefore $X$ is controlled $N'$-dominated, where $N'$ is the dimension of $P_d(G)$.

The main result of [8] asserts that there is a number $N$ such that for every $\alpha > 0$, there exists an open cover $\mathcal{U}_\alpha$ of $G \times X$ equipped with the diagonal $G$-action such that

- $\dim \mathcal{U}_\alpha \leq N$.
- For every $(g, x) \in G \times X$ there is $U \in \mathcal{U}_\alpha$ such that

$$g^\alpha \times \{x\} \subseteq U.$$

  (Here $g^\alpha$ denotes the open $\alpha$-ball in $G$ around $g$.)
- $\mathcal{U}_\alpha$ is a $\mathcal{VC}$yc-cover with respect to the diagonal $G$-action $g \cdot (h, x) = (gh, gx)$.

The map $(g, x) \mapsto (g, g^{-1}x)$ is a $G$-equivariant homeomorphism from $G \times X$ equipped with diagonal action to $G \times X$ equipped with the action $g \cdot (h, x) = (gh, x)$. Pushing the cover $\mathcal{U}_\alpha$ forward with this homeomorphism we obtain a new cover $\mathcal{V}_\alpha$ of $G \times X$ such that

- $\dim \mathcal{V}_\alpha \leq N$.

- For every $(g,y) \in G \times X$, there is $V \in \mathcal{V}_\alpha$ such that
$$\{(gh, h^{-1}y) \mid h \in e^\alpha\} \subseteq V.$$
(We denote by $e$ the unit element of $G$.)
- $\mathcal{V}_\alpha$ is a $\mathcal{VC}$yc-cover with respect to the left $G$-action $g \cdot (h,x) = (gh,x)$.

Consider a finite subset $S$ of $G$ containing $e$. Put $n = |S|$. Pick $\alpha > 0$ such that
$$\{l \in G \mid l = a_1^{-1}b_1 \ldots a_n^{-1}b_n \text{ for } a_i, b_i \in S\} \subseteq e^\alpha.$$
The $G$-action on $X$ induces a homotopy $S$-action $(\varphi, H)$ on $X$ where $\varphi_g$ is given by $l_g \colon X \to X$, $x \mapsto gx$ for $g \in S$, and $H_{g,h}(-,t) = l_{gh}$ for $g,h \in S$ with $gh \in S$ and $t \in [0,1]$. Notice that in this case
$$\begin{aligned} F_g(\varphi,H) &= \{l_g \colon X \to X\}, \\ S^n_{\varphi,H}(g,x) &= \{gl, l^{-1}x) \mid l = a_1^{-1}b_1 \ldots a_n^{-1}b_n \text{ for } a_i, b_i \in S\}. \end{aligned}$$
Hence $\mathcal{V}_\alpha$ is $S$-long with respect to $(\varphi, H)$. $\square$

Proposition 2.2. *Every finite-dimensional* CAT(0)*-group is transfer reducible to the family* $\mathcal{VC}$yc *of virtually cyclic subgroups.*

The proof of this result is postponed to [6].

Let $\mathcal{FJ}^K$ be the class of groups satisfying the $K$-theoretic Farrell-Jones Conjecture with coefficients in arbitrary additive $G$-categories $\mathcal{A}$, i.e., the class of groups for which the assembly map (0.1) is an isomorphism for all $\mathcal{A}$. By $\mathcal{FJ}^K_1$ we denote the class of groups for which this assembly map is bijective in degree $m \le 0$ and surjective in degree $m = 1$ for any $\mathcal{A}$. Let $\mathcal{FJ}^L$ be the class of groups satisfying the $L$-theoretic Farrell-Jones Conjecture with coefficients in arbitrary additive $G$-categories $\mathcal{A}$ with involutions, i.e., the class of groups for which the assembly map (0.1) is an isomorphism for all $\mathcal{A}$. (We could define $\mathcal{FJ}^L_1$, but because of the 4-periodicity of $L$-theory this is the same as $\mathcal{FJ}^L$.)

Lemma 2.3. *Let $\mathcal{C}$ be one of the classes $\mathcal{FJ}^K_1$, $\mathcal{FJ}^L$.*

(i) *If $H$ is a subgroup of $G$ and $G \in \mathcal{C}$, then $H \in \mathcal{C}$.*
(ii) *Let $\pi \colon G \to H$ be a group homomorphism. If $H \in \mathcal{C}$ and $\pi^{-1}(V) \in \mathcal{C}$ for all virtually cyclic subgroups $V$ of $H$, then $G \in \mathcal{C}$.*
(iii) *If $G_1$ and $G_2$ belong to $\mathcal{C}$, then $G_1 \times G_2$ belongs to $\mathcal{C}$.*
(iv) *If $G_1$ and $G_2$ belong to $\mathcal{C}$, then $G_1 * G_2$ belongs to $\mathcal{C}$.*
(v) *Let $\{G_i \mid i \in I\}$ be a directed system of groups (with not necessarily injective structure maps) such that $G_i \in \mathcal{C}$ for $i \in I$. Then $\operatorname{colim}_{i \in I} G_i$ belongs to $\mathcal{C}$.*

*Proof.* Note first that the product of two virtually cyclic groups acts properly, isometrically and cocompactly on a proper complete CAT(0)-space with finite covering dimension, namely $\mathbb{R}^2$. Thus such a product is a CAT(0) group.

It follows from Theorem 1.1 and Proposition 2.2 that such products belong to $\mathcal{FJ}_1^K \cap \mathcal{FJ}^L$. Note also that finitely generated virtually free groups are hyperbolic and belong $\mathcal{FJ}_1^K \cap \mathcal{FJ}^L$ by Theorem 1.1 and Proposition 2.2.

For $\mathcal{FJ}^L$, properties (i), (ii), (iii) and (v) follow from [7, Cors. 0.8, 0.9, 0.10, Rem. 0.11]. For (iv) we will use a trick from [57]. For $G_1$, $G_2 \in \mathcal{FJ}_1$, consider the canonical map $p\colon G_1 * G_2 \to G_1 \times G_2$. We have already shown that (0.2) is an isomorphism for $G_1 \times G_2$. By (ii) it suffices to show the same for $p^{-1}(V)$ for all virtually cyclic subgroups $V$ of $G_1 \times G_2$. By [57, Lemma 5.2] all such $p^{-1}(V)$ are virtually free. Such a virtually free group is the colimit of its finitely generated subgroups which are again virtually free. Thus (v) implies that virtually free groups belong to $\mathcal{FJ}^L$. The $K$-theoretic case can be proved completely analogously. One has to check that the argument works also for the statement that $K$-theoretic assembly map is bijective in degree $m \leq 0$ and surjective in degree $m = 1$. This follows from the fact that taking the colimit over a directed system is an exact functor. $\square$

The above arguments also show that $\mathcal{FJ}^K$ satisfies assertions (i), (ii) and (v) of Lemma 2.3. Assertions (iii) and (iv) follow once the $K$-theoretic Farrell-Jones Conjecture is established for groups of the form $V \times V'$, where $V$ and $V'$ are virtually cyclic. For arbitrary additive $G$-categories $\mathcal{A}$ this has not been carried out. See [51] for positive results in this direction.

*Proof of Theorem B.* In the language of this section, Theorem B can be rephrased to the statement that $\mathcal{B} \subseteq \mathcal{FJ}_1^K \cap \mathcal{F}^L$. Propositions 2.1 and 2.2 show that Theorem 1.1 applies to hyperbolic groups and finite-dimensional CAT(0)-groups. Thus all such groups are contained in $\mathcal{FJ}_1^K \cap \mathcal{F}^L$. Lemma 2.3 implies now that $\mathcal{B} \subseteq \mathcal{FJ}_1^K \cap \mathcal{F}^L$. $\square$

## 3. $S$-long covers yield contracting maps

*Summary.* The main result of this section is Proposition 3.9, in which we convert long covers of $G \times X$ in the sense of Definition 1.4(iv) to $G$-equivariant maps $G \times X \to \Sigma$, where $\Sigma$ is simplicial complex whose dimension is uniformly bounded and whose isotropy groups are not to large. Moreover, these maps have strong contracting property with respect to the metric $d_{S,\Lambda}$ from Definition 3.4. This metric scales (small) distances in the $X$-direction by $\Lambda$ (Lemma 3.5(iii)), while distances in the $G$-direction along the homotopy action are not scaled (Lemma 3.5(ii)).

Throughout this section we fix the following convention.

Convention 3.1. *Let*

- *$G$ be a group;*
- *$(X, d_X)$ be a compact metric space. We equip $G \times X$ with the $G$-action $g(h, x) = (gh, x)$;*

- *$S$ be a finite subset of $G$ (containing $e$);*
- *$(\varphi, H)$ be a homotopy $S$-action on $X$.*

3.1. *Homotopy $S$-actions and metrics.* For every number $\Lambda > 0$, we define a $G$-invariant (quasi-)metric $d_{S,\Lambda}$ on $G \times X$ as follows. For $(g,x),(h,y) \in G \times X$, consider $n \in \mathbb{Z}$, $n \geq 0$, elements $x_0, \dots, x_n \in X$, $z_0, \dots, z_n$ in $X$, elements $a_1, b_1, \dots, a_n, b_n$ in $S$ and maps $f_1, \widetilde{f}_1, \dots, f_n, \widetilde{f}_n \colon X \to X$ such that

$$\begin{aligned} & x = x_0, \quad z_n = y, \\ & f_i \in F_{a_i}(\varphi, H), \quad \widetilde{f}_i \in F_{b_i}(\varphi, H), \\ & f_i(z_{i-1}) = \widetilde{f}_i(x_i) \text{ for } i = 1, 2, \dots n, \\ & h = g a_1^{-1} b_1 \cdots a_n^{-1} b_n. \end{aligned} \tag{3.2}$$

(See Definition 1.4(ii) for the definition of $F_s(\varphi, H)$ for $s \in S$.) If $n = 0$, we just demand $x_0 = x$, $z_0 = y$, $g = h$ and no elements $a_i$, $b_i$, $f_i$ and $\widetilde{f}_i$ occur. To this data we associate the number

$$n + \sum_{i=0}^{n} \Lambda \cdot d_X(x_i, z_i). \tag{3.3}$$

*Definition* 3.4. For $(g,x),(h,y) \in G \times X$, define

$$d_{S,\Lambda}((g,x),(h,y)) \in [0, \infty]$$

as the infimum of (3.3) over all possible choices of $n$, $x_i$, $z_i$,$a_i$, $b_i$, $f_i$ and $\widetilde{f}_i$. If the set of possible choices is empty, then we put $d_{S,\Lambda}((g,x),(h,y)) := \infty$.

Of course, $d_{S,\Lambda}$ depends not only on $S$ and $\Lambda$, but also on $(X, d)$ and $(\varphi, H)$. That this is not reflected in the notation will hopefully not be a source of confusion. Recall that a quasi-metric is the same as a metric except that it may take also the value $\infty$.

LEMMA 3.5. (i) *For every $\Lambda > 0$, $d_{S,\Lambda}$ is a well defined $G$-invariant quasi-metric on $G \times X$. The set $S$ generates $G$ if and only if $d_{S,\Lambda}$ is a metric.*

(ii) *Let $(g,x),(h,y) \in G \times X$ and let $m \in \mathbb{Z}$, $m \geq 1$. If $d_{S,\Lambda}((g,x),(h,y)) \leq m$ for all $\Lambda$, then $(h,y) \in S^m_{\varphi,H}(g,x)$. (The set $S^m_{\varphi,H}(g,x)$ is defined in Definition* 1.4(iii).)

(iii) *For $x, y \in X$ and $g \in G$, we have $d_{S,\Lambda}((g,x),(h,y)) < 1$ if and only if $g = h$ and $\Lambda \cdot d_X(x,y) < 1$ hold. In this case we get*

$$d_{S,\Lambda}((g,x),(h,y)) = \Lambda \cdot d_X(x,y).$$

*The topology on $G \times X$ induced by $d_{S,\Lambda}$ is the product topology on $G \times X$ for the discrete topology on $G$ and the given one on $X$.*

*Proof.* (i) One easily checks that $d_{S,\Lambda}$ is symmetric and satisfies the triangle inequality. Obviously $d_{S,\Lambda}((g,x),(g,x)) = 0$. Suppose $d_{S,\Lambda}((g,x),(h,y)) = 0$. Given any real number $\epsilon$ with $0 < \epsilon < 1$, we can find $n$, $x_i$, $z_i$,$f_i$, $\widetilde{f}_i$,$a_i$ and $b_i$ as in (3.2) satisfying

$$n + \sum_{i=0}^{n} \Lambda \cdot d_X(x_i, z_i) \leq \epsilon.$$

We conclude that $n = 0$ and hence $\Lambda \cdot d_X(x,y) \leq \epsilon$. Since $\Lambda > 0$ and this holds for all $0 < \epsilon < 1$, we conclude that $d_X(x,y) = 0$ and hence $x = y$.

Obviously $d_{S,\Lambda}$ is $G$-invariant since for $k \in G$, we have $h = ga_1^{-1}b_1 \dots a_n^{-1}b_n$ if and only if $kh = kga_1^{-1}b_1 \dots a_n^{-1}b_n$ and $G$ acts on $G \times X$ by $k\cdot(h,x) = (kh,x)$.

The sets $F_g(\varphi, H)$ for $g \in S$ are never empty, and $F_e(\varphi, H)$ always contains $\mathrm{id}_X$. Hence the infimum in the definition of $d_{S,\Lambda}((g,x),(h,y))$ is finite if and only if we can find $n \in \mathbb{Z}, n \geq 0$ and elements $a_i, b_i \in S$ with $g^{-1}h = a_1^{-1}b_1 \cdots a_n^{-1}b_n$.

(ii) Let $(\Lambda^\nu)_{\nu \geq 1}$ be sequence of numbers such that $\lim_{\nu\to\infty} \Lambda^\nu = \infty$. The assumptions imply that there are $n^\nu$, $x_0^\nu, \dots, x_{n^\nu}^v$, $z_0^\nu, \dots, z_{n^\nu}^\nu$, $a_1^\nu, b_1^\nu, \dots, a_{n^\nu}^\nu$, $b_{n^\nu}^\nu \in S$ and $f_1^\nu, \widetilde{f}_1^\nu, \dots, f_{n^\nu}^\nu, \widetilde{f}_{n^\nu}^\nu$ such that (3.2) and

$$n^\nu + \sum_{i=0}^{n^\nu} \Lambda^\nu \cdot d_X(x_i^\nu, z_i^\nu) < m + 1/\nu \tag{3.6}$$

hold. In particular, $n^\nu \leq m$ for all $\nu$. For each $\nu$, we define $a_j^\nu = b_j^\nu = e$, $x_j^\nu = z_j^\nu = y$, $f_j^\nu = \widetilde{f}_j^\nu = \mathrm{id}_X$ for $j \in \{n^\nu + 1, \dots, m\}$. Hence we have now, for each $\nu$ and each $i \in \{1, 2, \dots, m\}$, elements $a_i^\nu$, $b_i^\nu$, $x_i^\nu$,$z_i^\nu$, $f_i^\nu$ and $\widetilde{f_i}^\nu$ and $x_0^\nu = x$ and $z_m^\nu = y$.

Because $X$ is compact, by passing to a subsequence of $(\Lambda^\nu)_{\nu\geq 1}$ we can arrange that for each $i \in \{0, 1, 2, \dots, m\}$ there are $x_i \in X$ with $\lim_{\nu\to\infty} x_i^\nu = x_i$ and $z_i \in X$ with $\lim_{\nu\to\infty} z_i^\nu \to z_i$. From (3.6), we deduce that

$$d_X(x_i^\nu, z_i^\nu) < \frac{m + 1/\nu}{\Lambda^\nu}$$

for $i \in \{0, 1, 2, \dots, m\}$. Since $\lim_{\nu\to\infty} \frac{m+1/\nu}{\Lambda^\nu} = 0$, we conclude that $d_X(x_i, z_i) = 0$ and therefore

$$x_i = z_i \quad \text{for } i \in \{0, 1, 2, \dots, m\}.$$

For $i \in \{0, 1, 2, \dots, m\}$, choose elements $t_i^\nu, \widetilde{t}_i^\nu \in [0,1]$, $r_i^\nu, s_i^\nu, \widetilde{r}_i^\nu, \widetilde{s}_i^\nu \in S$ with $r_i^\nu s_i^\nu = a_i^\nu$ and $\widetilde{r}_i^\nu \widetilde{s}_i^\nu = b_i^\nu$ such that $f_i^\nu = H_{r_i^\nu, s_i^\nu}(-, t_i^\nu)$ and $\widetilde{f}_i^\nu = H_{\widetilde{r}_i^\nu, \widetilde{s}_i^\nu}(-, \widetilde{t}_i^\nu)$ holds. Since $S$ is finite and $[0,1]$ is compact, by passing to a subsequence of $\{\Lambda^\nu\}$ we can arrange that there exist elements $r_i, s_i, \widetilde{r}_i, \widetilde{s}_i \in S_i$ and $t_i, \widetilde{t}_i \in [0,1]$ such that $r_i^\nu = r_i$, $s_i^\nu = s_i$, $\widetilde{r}_i^\nu = \widetilde{r}_i$ and $\widetilde{s}_i^\nu = \widetilde{s}_i$ holds for all $\nu$ and $\lim_{\nu\to\infty} t_i^\nu = t_i$

and $\lim_{\nu\to\infty} \widetilde{t}_i^\nu = \widetilde{t}_i$ is valid. Put $f_i = H_{r_i,s_i}(-,t_i)$ and $\widetilde{f}_i = H_{\widetilde{r}_i,\widetilde{s}_i}(-,\widetilde{t}_i)$. Then for $i \in \{0,1,2,\dots m\}$,

$$\begin{aligned} f_i &\in F_{a_i}(\varphi,H), \\ \widetilde{f}_i &\in F_{b_i}(\varphi,H), \\ \lim_{\nu\to\infty} f_i^\nu(x_{i-1}^\nu) &= f_i(x_{i-1}), \\ \lim_{\nu\to\infty} \widetilde{f}_i^\nu(z_i^\nu) &= \widetilde{f}_i(z_i). \end{aligned}$$

To summarize, we have constructed $x_0,\dots,x_m \in X$, $a_1,b_1,\dots,a_m,b_m \in S$, $f_1,\widetilde{f}_1,\dots f_m,\widetilde{f}_m \colon X \to X$ such that $x_0 = x$, $x_m = y$, $f_i \in F_{a_i}(\varphi,H)$, $\widetilde{f}_i \in F_{b_i}(\varphi,H)$, $f_i(x_{i-1}) = \widetilde{f}_i(x_i)$ for $i \in \{1,2,\dots,m\}$ and $h = ga_1^{-1}b_1\dots a_m^{-1}b_m$ holds. Thus $(h,y) \in S^m_{\varphi,H}(g,x)$.

(iii) Suppose that $d_{S,\Lambda}((g,x),(h,y)) < 1$. For every $\epsilon > 0$ with $\epsilon < 1 - d_{S,\Lambda}((g,x),(h,y))$, we can find appropriate $n$, $x_i$, $z_i$,$f_i$, $\widetilde{f}_i$,$a_i$ and $b_i$ with

$$n + \sum_{i=0}^{n} \Lambda \cdot d_X(x_i,z_i) < d_{S,\Lambda}((g,x),(h,y)) + \epsilon.$$

Since $d_{S,\Lambda}((g,x),(h,y)) + \epsilon < 1$, we conclude that $n = 0$ and hence $g = h$ and $\Lambda \cdot d_X(x,y) < d_{S,\Lambda}((g,x),(h,y)) + \epsilon$. Since this holds for all such $\epsilon$, we get $\Lambda \cdot d_X(x,y) \leq d_{S,\Lambda}((g,x),(h,y))$. Obviously $\Lambda \cdot d_X(x,y) \geq d_{S,\Lambda}((g,x),(h,y))$ because of $g = h$. This proves $d_{S,\Lambda}((g,x),(h,y)) = \Lambda \cdot d_X(x,y)$ and $g = h$ provided that $d_{S,\Lambda}((g,x),(h,y)) < 1$.

One easily checks that $g = h$ and $d_X(x,y) < 1$ implies $\Lambda \cdot d_{S,\Lambda}((g,x),(h,y)) < 1$. The claim about the topology is now obvious. □

3.2. *Contracting maps.*

PROPOSITION 3.7. *Let $\mathcal{U}$ be an $S$-long finite-dimensional $G$-equivariant cover of $G \times X$. Let $m$ be any number with $m \leq |S|$. Then there is $\Lambda > 0$ such that the Lebesgue number of $\mathcal{U}$ with respect to $d_{S,\Lambda}$ is at least $m/2$; i.e., for every $(g,x)$ there is $U \in \mathcal{U}$ containing the open $m/2$-ball $B_{m/2,\Lambda}(g,x)$ around $(g,x)$ with respect to the metric $d_{S,\Lambda}$.*

*Proof.* Fix $x \in X$. First we show the existence of $\Lambda_x > 0$ and $U_x \in \mathcal{U}$ such that the open $m$-ball $B_{m,\Lambda_x}(e,x)$ around $(e,x)$ with respect to $d_{S,\Lambda_x}$ lies in $U_x$.

Let $\mathcal{U}_x := \{U \in \mathcal{U} \mid (e,x) \in U\}$. Then $\mathcal{U}_x$ is finite, because $\mathcal{U}$ is finite-dimensional. We proceed by contradiction. So assume that for every $\Lambda > 0$, no $U \in \mathcal{U}_x$ contains $B_{m,\Lambda}(e,x)$. Thus we can find a monotone increasing sequence $(\Lambda_n)_{n\geq 1}$ of positive real numbers with $\lim_{n\to\infty} \Lambda_n = \infty$ such that for every $U \in \mathcal{U}_x$ and $n \geq 1$, there is $(h^{U,n},y^{U,n}) \in (G \times X) \setminus U$ satisfying

$$d_{\Lambda_n,S}((e,x),(h^{U,n},y^{U,n})) < m. \tag{3.8}$$

Because $X$ is compact, we can arrange by passing to a subsequence of $(\Lambda_n)_{n\geq 1}$ that for each $U \in \mathcal{U}_x$, there is $y^U \in X$ satisfying $\lim_{n\to\infty} y^{U,n} = y^U$. The definition of $d_{\Lambda_n,S}$ and (3.8) imply that each $h^{U,n}$ can be written as a product of at most $2m$ elements from $S \cup S^{-1}$. Therefore the $h^{U,n}$-s range over a finite subset of $G$. Thus we can arrange by passing to a subsequence of $(\Lambda_n)_{n\geq 1}$ that for each $U \in \mathcal{U}_x$, there is $h^U \in G$ such that $h^{U,n} = h^U$ holds for all $n$. As $(\Lambda_n)_{n\geq 1}$ is increasing, we obtain from (3.8) for $k \geq n$

$$\begin{aligned}
d_{S,\Lambda_n}\left((e,x),(h^U,y^U)\right) &\leq d_{S,\Lambda_n}\left((e,x),(h^U,y^{U,k})\right) + d_{S,\Lambda_n}\left((h^U,y^{U,k}),(h^U,y^U)\right)\\
&\leq d_{S,\Lambda_k}\left((e,x),(h^U,y^{U,k})\right) + d_{S,\Lambda_n}\left((h^U,y^{U,k}),(h^U,y^U)\right)\\
&< m + d_{S,\Lambda_n}\left((h^U,y^{U,k}),(h^U,y^U)\right).
\end{aligned}$$

Lemma 3.5(iii) implies $\lim_{k\to\infty} d_{S,\Lambda_n}\left((h^U,y^{U,k}),(h^U,y^U)\right) = 0$. We conclude that $d_{S,\Lambda_n}\left((e,x),(h^U,y^U)\right) \leq m$ for all $U \in \mathcal{U}_x$. By Lemma 3.5(ii) this implies $(h^U,y^U) \in S^m_{\varphi,H}(g,x)$ for all $U \in \mathcal{U}_x$. Because $\mathcal{U}$ is assumed to be $S$-long there is $U_0 \in \mathcal{U}_x$ such that $S^m_{\varphi,H}(g,x) \subseteq U_0$. Thus $(h^{U_0},y^{U_0}) \in U_0$. But this yields the desired contradiction:

$$\lim_{n\to\infty}(h^{U_0},y^{U_0,n}) = \lim_{n\to\infty}(h^{U_0,n},y^{U_0,n}) = (h^{U_0},y^{U_0})$$

together with the fact that $(h^{U_0,n},y^{U_0,n})$ lies in the closed subset $(G\times X)\setminus U_0$, implies $(h^{U_0},y^{U_0}) \in (G\times X)\setminus U_0$.

Now we can finish the proof of Proposition 3.7. For $x \in X$, the subset

$$B_{m/2,\Lambda_x}(e,x) \cap \{e\}\times X \subseteq \{e\}\times X = X$$

is open in $X$ because of Lemma 3.5(iii). Since $X$ is compact, we can find finitely many elements $x_1, x_2, \ldots, x_l$ such that

$$X = \{e\}\times X = \bigcup_{i=1}^{l}\left(B_{m/2,\Lambda_{x_i}}(e,x_i)\cap\{e\}\times X\right).$$

Put $\Lambda := \max\{\Lambda_{x_1},\ldots,\Lambda_{x_l}\}$. Consider $(g,x) \in G\times X$. Then we can find $i \in \{1,2,\ldots,l\}$ such that $(e,x) \in B_{m/2,\Lambda_{x_i}}(e,x_i)$. Hence

$$B_{m/2,\Lambda}(e,x) \subseteq B_{m/2,\Lambda_{x_i}}(e,x) \subseteq B_{m,\Lambda_{x_i}}(e,x_i).$$

We have already shown that there exists $U \in \mathcal{U}$ with $B_{m,\Lambda_{x_i}}(e,x_i) \subseteq U$. This implies

$$B_{m/2,\Lambda}(g,x) = g\left(B_{m/2,\Lambda}(e,x)\right) \subseteq g(U).$$

Since $\mathcal{U}$ is $G$-invariant, this finishes the proof of Proposition 3.7. □

In the following proposition, $d^1$ denotes the $l^1$-metric on simplicial complexes; compare [9, §4.2].

PROPOSITION 3.9. *Let $G$ be a finitely generated group that is transfer reducible over the family $\mathcal{F}$. Let $N$ be the number appearing in Definition* 1.8. *Let $S$ be a finite subset of $G$ (containing $e$) that generates $G$. Let $\varepsilon > 0$, $\beta > 0$. Then there are*

- *a compact contractible controlled $N$-dominated metric space $(X, d)$;*
- *a homotopy $S$-action $(\varphi, H)$ on $X$;*
- *a positive real number $\Lambda$;*
- *a simplicial complex $\Sigma$ of dimension $\leq N$ with a simplicial cell preserving $G$-action;*
- *a $G$-equivariant map $f \colon G \times X \to \Sigma$,*

*satisfying*

(i) *The isotropy groups of $\Sigma$ are members of $\mathcal{F}$.*
(ii) *If $(g, x), (h, y) \in G \times X$ and $d_{S,\Lambda}((g, x), (h, y)) \leq \beta$, then*

$$d^1(f(g, x), f(h, y)) \leq \epsilon.$$

*Proof.* Set $D := \frac{|S|}{2}$. Since for $S \subseteq T$ we have $d_{T,\Lambda} \leq d_{S,\Lambda}$, by possibly enlarging $S$ we can arrange

$$\beta \leq \frac{D}{4N} \quad \text{and} \quad \frac{16N^2\beta}{D} \leq \epsilon.$$

Because $G$ is transfer reducible over $\mathcal{F}$ there exists a contractible compact controlled $N$-dominated space $X$, a homotopy $S$-action $(\varphi, H)$ on $X$ and an $S$-long cover $\mathcal{U}$ of $G \times X$ such that $\mathcal{U}$ is an $N$-dimensional open $\mathcal{F}$-covering. Using Proposition 3.7 we find $\Lambda > 0$ such that the Lebesgue number of $\mathcal{U}$ with respect to $d_{S,\Lambda}$ is at least $D$. Let $\Sigma := |\mathcal{U}|$ be the realization of the nerve of $\mathcal{U}$. Since $\mathcal{U}$ is an open $\mathcal{F}$-cover, $\Sigma$ inherits a simplicial cell preserving $G$-action whose isotropy groups are members of $\mathcal{F}$. Let now $f \colon G \times X \to \Sigma$ be the map induced by $\mathcal{U}$, i.e.,

$$f(x) := \sum_{U \in \mathcal{U}} \frac{d_{S,\Lambda}(x, G \times X - U)}{\sum_{V \in \mathcal{U}} d_{S,\Lambda}(x, G \times X - V)} U.$$

This is a $G$-equivariant map since $d_{S,\Lambda}$ is $G$-invariant. From [9, Prop. 5.3], we get

$$d_{S,\Lambda}((g, x), (h, y)) \leq \frac{D}{4N} \implies d^1(f(g, x), f(h, y)) \leq \frac{16N^2}{D} d_{S,\Lambda}((g, x), (h, y)).$$

We conclude that

$$d_{S,\Lambda}((g, x), (h, y)) \leq \beta \implies d^1(f(g, x), f(h, y)) \leq \frac{16N^2\beta}{D} \leq \epsilon.$$

This finishes the proof of Proposition 3.9. □

## 4. **Controlled algebra with a view towards $L$-theory**

*Summary.* A crucial tool in the proof of Theorem 1.1 is controlled algebra. In this section we give a brief review of this theory where we emphasize the $L$-theory aspects. In Section 4.4 we define the obstruction categories whose $K$- respectively $L$-theory will appear as the homotopy groups of homotopy fibers of assembly maps. Elements in these $K$- and $L$-groups will be represented by chain homotopy equivalences in $K$-theory and by ultra-quadratic Poincaré complexes over these categories. (These are the cycles referred to in the introduction.)

4.1. *Additive $G$-categories with involution.* By an *additive category* $\mathcal{A}$ we will mean from now on a small additive category with a functorial strictly associative direct sum. For a group $G$, an *additive $G$-category* is by definition such an additive category together with a strict (right) $G$-action that is compatible with the direct sum. By an *additive $G$-category with involution* we will mean an additive $G$-category that carries, in addition, a strict involution inv (i.e., $\mathrm{inv} \circ \mathrm{inv} = \mathrm{id}_{\mathcal{A}}$ that is strictly compatible with the $G$-action (i.e., $\mathrm{inv} \circ g = g \circ \mathrm{inv}$) and the sum (i.e., $\mathrm{inv}(A \oplus B) = \mathrm{inv}(A) \oplus \mathrm{inv}(B)$); see [7, Def. 10.6]. The assembly maps (1.2) and (1.3) are defined for more general $\mathcal{A}$, but the assembly maps are isomorphisms for all such more general $\mathcal{A}$ if and only if they are isomorphism for all additive $G$-categories (with involution) as above; see [7, Th. 0.12].

4.2. *The category $\mathcal{C}^G(Y, \mathcal{E}, \mathcal{F}; \mathcal{A})$.* Let $G$ be a group, $Y$ a space and $\mathcal{A}$ be an additive category. Let $\mathcal{E} \subseteq \{E \mid E \subseteq Y \times Y\}$ and $\mathcal{F} \subseteq \{F \mid F \subseteq Y\}$ be collections satisfying the conditions from [5, p. 167]. (These conditions are designed to ensure that we indeed obtain an additive category (with involution) and are satisfied in all cases of interest.) The category $\mathcal{C}(Y; \mathcal{E}, \mathcal{F}; \mathcal{A})$ is defined as follows. Objects are given by sequences $(M_y)_{y \in Y}$ of objects in $\mathcal{A}$ such that

(4.1) $M$ is $\mathcal{F}$-controlled: there is $F$ in $\mathcal{F}$ such that the *support* $\operatorname{supp} M := \{y \mid M_y \neq 0\}$ is contained in $F$.

(4.2) $M$ has locally finite support: for every $y \in Y$, there is an open neighborhood $U$ of $y$ such that $U \cap \operatorname{supp} M$ is finite.

A morphism $\psi$ from $M = (M_y)_{y \in Y}$ to $M' = (M'_y)_{y \in Y}$ is given by a collection $(\psi_{y',y} \colon M_y \to M'_{y'})_{(y',y) \in Y \times Y}$ of morphisms in $\mathcal{A}$ such that

(4.3) $\psi$ is $\mathcal{E}$-controlled: there is $E \in \mathcal{E}$ such that the *support* $\operatorname{supp}(\psi) := \{(y', y) \mid \psi_{y',y} \neq 0\}$ is contained in $E$.

(4.4) $\psi$ is row and column finite: for every $y \in Y$, the sets $\{y' \in Y \mid (y, y') \in \operatorname{supp} \psi\}$ and $\{y' \in Y \mid (y', y) \in \operatorname{supp} \psi\}$ are finite.

Composition of morphisms is given by matrix multiplication; i.e., $(\psi' \circ \psi)_{y'',y} = \sum_{y' \in Y} \psi_{y'',y'} \circ \psi_{y',y}$. If $\mathrm{inv} \colon \mathcal{A} \to \mathcal{A}$ is a strict involution, then $\mathcal{C}(Y; \mathcal{F}, \mathcal{E}; \mathcal{A})$

inherits a strict involution. For objects it is defined by $(\mathrm{inv}(M))_y = \mathrm{inv}(M_y)$, and for morphisms it is defined by $(\mathrm{inv}(\psi))_{y',y} = \mathrm{inv}(\psi_{y,y'})$. Let now $Y$ be a (left) $G$-space and assume that $\mathcal{A}$ is equipped with a (strict) right $G$-action; i.e., $\mathcal{A}$ is an additive $G$-category. Assume that the $G$-action on $Y$ preserves both $\mathcal{F}$ and $\mathcal{E}$. Then $\mathcal{C}(Y, \mathcal{E}, \mathcal{F}; \mathcal{A})$ inherits a (right) $G$-action making it an additive $G$-category. For an object $M$ and $g \in G$, the action is given by $(Mg)_y = (M_{gy})g$. If the action on $\mathcal{A}$ is compatible with a (strict) involution inv on $\mathcal{A}$, i.e., if $\mathcal{A}$ is an additive $G$-category with involution, then $\mathcal{C}(Y; \mathcal{E}, \mathcal{F}; \mathcal{A})$ is also an additive $G$-category with involution under the induced action and involution. We will denote by $\mathcal{C}^G(Y; \mathcal{E}, \mathcal{F}; \mathcal{A})$ the subcategory of $\mathcal{C}(Y; \mathcal{E}, \mathcal{F}; \mathcal{A})$ that is (strictly) fixed by $G$.

4.3. *Metric control*: *the category* $\mathcal{C}(Z, d; \mathcal{A})$. Let $(Z, d)$ be a metric space. Let $\mathcal{E}(Z, d) := \{E_\alpha \mid \alpha > 0\}$ where $E_\alpha := \{(z, z') \mid d(z, z') \leq \alpha\}$. For an additive category $\mathcal{A}$ (with or without involution), we define $\mathcal{C}(Z, d; \mathcal{A}) := \mathcal{C}(Z; \mathcal{E}(Z, d), \{Z\}; \mathcal{A})$. Let $\varepsilon > 0$. A morphism $\psi$ in $\mathcal{C}(Z, d; \mathcal{A})$ is said to be *$\varepsilon$-controlled* if $\mathrm{supp}(\psi) \subseteq E_\varepsilon$.

The *idempotent completion* $\mathrm{Idem}(\mathcal{A})$ of an additive category $\mathcal{A}$ is the following additive category. Objects are morphisms $p\colon M \to M$ in $\mathcal{A}$ satisfying $p^2 = p$. A morphism $f\colon (M, p) \to (N, q)$ in $\mathrm{Idem}(\mathcal{A})$ is a morphism $f\colon M \to N$ satisfying $q \circ f \circ p = f$. Composition and the additive structure are inherited from $\mathcal{A}$ in the obvious way. Recall that for us an additive category is always understood to be small; i.e., the objects form a set. If $\mathcal{A}$ is an additive category which is equivalent to the category of finitely generated free $R$-modules, then $\mathrm{Idem}(\mathcal{A})$ is equivalent to the category of finitely generated projective $R$-modules.

An object $A = (M, p) \in \mathrm{Idem}(\mathcal{C}(Z, d; \mathcal{A}))$ (where $p\colon M \to M$ is an idempotent in $\mathcal{C}(Z, d; \mathcal{A})$) is called *$\varepsilon$-controlled* if $p$ is $\varepsilon$-controlled. A morphism $\psi\colon (M, p) \to (M', p')$ in $\mathrm{Idem}(\mathcal{C}(Z, d; \mathcal{A}))$ is called *$\varepsilon$-controlled* if $\psi\colon M \to M'$ is $\varepsilon$-controlled as a morphism in $\mathcal{C}(Z, d; \mathcal{A})$. A chain complex $P$ over $\mathrm{Idem}(\mathcal{C}(Z, d; \mathcal{A}))$ is called *$\varepsilon$-controlled* if $P_n$ is $\varepsilon$-controlled for all $n$, and the differential $\partial_n\colon P_n \to P_{n-1}$ is $\varepsilon$-controlled for all $n$. A graded map $P \to Q$ of chain complex over $\mathrm{Idem}(\mathcal{C}(Z, d; \mathcal{A}))$ is said to be *$\varepsilon$-controlled* if it consists of morphisms in $\mathrm{Idem}(\mathcal{C}(Z, d; \mathcal{A}))$ that are $\varepsilon$-controlled. A chain homotopy equivalence $\psi\colon P \to Q$ of chain complexes over $\mathrm{Idem}(\mathcal{C}(Z, d; \mathcal{A}))$ is said to be an *$\varepsilon$-chain homotopy equivalence over* $\mathrm{Idem}(\mathcal{C}(Z, d; \mathcal{A}))$ if there is a chain homotopy inverse $\varphi$ for $\psi$ and chain homotopies $H$ from $\varphi \circ \psi$ to $\mathrm{id}_P$ and $K$ from $\psi \circ \varphi$ to $\mathrm{id}_Q$ such that $P$, $Q$, $\varphi$, $\psi$, $H$ and $K$ are $\varepsilon$-controlled.

By $\mathcal{F}(\mathbb{Z})$ we denote the following small model for the category of finitely generated free $\mathbb{Z}$-modules. Objects are $\mathbb{Z}^n$ with $n \in \mathbb{N} \cup \{0\}$. Morphisms are given by matrices over $\mathbb{Z}$. Composition is given by matrix multiplication.

The category $\mathcal{F}(\mathbb{Z})$ is an additive category by taking sums of matrices and has a (strictly associative functorial) direct sum which is given on objects by $\mathbb{Z}^m \oplus \mathbb{Z}^n = \mathbb{Z}^{m+n}$. We will use the (strict) involution of additive categories on $\mathcal{F}(\mathbb{Z})$ which acts as the identity on objects and by transposition of matrices on morphisms. We write $\mathcal{C}(Z, d; \mathbb{Z}) := \mathcal{C}(Z, d; \mathcal{F}(\mathbb{Z}))$.

4.4. *The obstruction category* $\mathcal{O}^G(Y, Z, d; \mathcal{A})$. Let $Y$ be a $G$-space and let $(Z, d)$ be a metric space with isometric $G$-action. Let $\mathcal{A}$ be an additive $G$-category (with or without involution). In [5, Def. 2.7] (see also [9, §3.2]), the *equivariant continuous control condition* $\mathcal{E}^Y_{Gcc} \subseteq \{E \subseteq (Y \times [1, \infty))^{\times 2}\}$ has been introduced. Define $\mathcal{E}(Y, Z, d)$ as the collection of all $E \subseteq (G \times Z \times Y \times [1, \infty))^{\times 2}$ that satisfy the following conditions:

(4.5) $E$ is $\mathcal{E}^Y_{Gcc}$-controlled: there exists an element $E' \in \mathcal{E}^Y_{Gcc}$ with the property that $((g, z, y, t), (g', z', y', t')) \in E$ implies $((y, t), (y', t')) \in E'$.

(4.6) $E$ is bounded over $G$: there is a finite subset $S$ of $G$ with the property that $((g, z, y, t), (g', z', y', t')) \in E$ implies $g^{-1}g' \in S$.

(4.7) $E$ is bounded over $Z$: there is $\alpha > 0$ such that $((g, z, y, t), (g', z', y', t')) \in E$ implies $d(z, z') \leq \alpha$.

We define $\mathcal{F}(Y, Z, d)$ to be the collection of all $F \subseteq G \times Z \times Y \times [1, \infty)$ for which there is a compact subset $K$ of $G \times Z \times Y$ such that for $(g, z, y, t) \in F$, there is $h \in G$ satisfying $(hg, hz, hy) \in K$. Then we define

$$\mathcal{O}^G(Y, Z, d; \mathcal{A}) := \mathcal{C}^G(G \times Z \times Y \times [1, \infty); \mathcal{E}(Y, Z, d), \mathcal{F}(Y, Z, d); \mathcal{A}), \tag{4.8}$$

where we use the $G$-action on $G \times Z \times Y \times [0, \infty)$ given by $g(h, z, y, t) := (gh, gz, gy, t)$. We will also use the case where $Z$ is trivial, i.e., a point; in this case we write $\mathcal{O}^G(Y; \mathcal{A})$ and drop the point from the notation.

We remark that all our constructions on this category will happen in the $G \times Z$ factor of $G \times Z \times Y \times [1, \infty)$; in particular, it will not be important for the reader to know the precise definition of the equivariant continuous control condition $\mathcal{E}^Y_{Gcc}$. (We will, on the other hand, use results from [9] that depend very much on the precise definition of $\mathcal{E}^Y_{Gcc}$.)

Let $S \subseteq G$ and $\varepsilon > 0$. A morphism $\psi$ in $\mathcal{O}^G(Y, Z, d; \mathcal{A})$ is said to be $(\varepsilon, S)$-*controlled* if $((g, z, y, t), (g', z', y', t')) \in \operatorname{supp}(\psi)$ implies $d(z, z') \leq \varepsilon$ and $g^{-1}g' \in S$. If $\psi$ is an isomorphism such that both $\psi$ and $\psi^{-1}$ are $(\varepsilon, S)$-controlled, then $\psi$ is said to be an $(\varepsilon, S)$-*isomorphism.* An object $A = (M, p) \in \operatorname{Idem}(\mathcal{O}^G(Y, Z, d; \mathcal{A}))$ (where $p\colon M \to M$ is an idempotent in $\mathcal{C}^G(Y, Z, d; \mathcal{A})$) is called $(\varepsilon, S)$-*controlled* if $p$ is $(\varepsilon, S)$-controlled. A morphism $\psi\colon (M, p) \to (M', p')$ in $\operatorname{Idem}(\mathcal{O}^G(Y, Z, d; \mathcal{A}))$ is called $(\varepsilon, S)$-*controlled* if $\psi\colon M \to M'$ is $(\varepsilon, S)$-controlled as a morphism in $\mathcal{O}^G(Y, Z, d; \mathcal{A})$. A chain complex $P$ over $\operatorname{Idem}(\mathcal{O}^G(Y, Z, d; \mathcal{A}))$ is called $(\varepsilon, S)$-*controlled* if $P_n$ is $(\varepsilon, S)$-controlled for all $n$ and the differential $\partial_n\colon P_n \to P_{n-1}$ is $(\varepsilon, S)$-controlled for all $n$. A graded map

$P \to Q$ of chain complexes over $\mathrm{Idem}(\mathcal{O}^G(Y,Z,d;\mathcal{A}))$ is called $(\varepsilon,S)$-*controlled* if it consists of $(\varepsilon,S)$-controlled morphisms in $\mathrm{Idem}(\mathcal{O}^G(Y,Z,d;\mathcal{A}))$. A chain homotopy equivalence $\psi\colon P \to Q$ of chain complexes over $\mathrm{Idem}(\mathcal{O}^G(Y,Z,d;\mathcal{A}))$ is said to be an $(\varepsilon,S)$-*chain homotopy equivalence over* $\mathrm{Idem}(\mathcal{O}^G(Y,Z,d;\mathcal{A}))$ if there is a chain homotopy inverse $\varphi$ for $\psi$ and chain homotopies $H$ from $\varphi\circ\psi$ to $\mathrm{id}_P$ and $K$ from $\psi\circ\varphi$ to $\mathrm{id}_Q$ such that $P$, $Q$, $\varphi$, $\psi$, $H$ and $K$ are $(\varepsilon,S)$-controlled. We write $\varepsilon$-*controlled* for $(\varepsilon,G)$-controlled and $S$-*controlled* for $(\infty,S)$-controlled.

Note that every $S$-controlled morphism has a unique decomposition

$$\psi = \sum_{a\in S}\psi_a, \tag{4.9}$$

where $\psi_a$ is $\{a\}$-controlled. Namely, put $(\psi_a)_{(g,z,y,t),(g',z',y',t')} = \psi_{(g,z,y,t),(g',z',y',t')}$ if $g^{-1}g' = a$, and $(\psi_a)_{(g,z,y,t),(g',z',y',t')} = 0$ otherwise.

*Remark* 4.10. If $G$ is finitely generated, then (4.6) can be expressed using a word-metric $d_G$ as it is done in [9, §3.4]. However, the notation there is slightly different: the category (4.8) is denoted in [9] by $\mathcal{O}^G(Y, G\times Z, d_G\times d;\mathcal{A})$.

4.5. *Controlled algebraic Poincaré complexes.* We give a very brief review of the part of Ranicki's algebraic $L$-theory that we will need. We will follow [56, §17]. Let $\mathcal{A}$ be an additive category with involution inv. For such a category, Ranicki defines $L$-groups $L_n^{\langle j\rangle}(\mathcal{A})$, where $n\in\mathbb{Z}$ and $j\in\{1,0,-1,\dots,-\infty\}$ (see [56, Def. 17.1, p. 145 and Def. 17.7, p. 148]). If $R$ is a ring with involution and we take $\mathcal{A}$ to be the additive category of finitely generated free $R$-modules, then $L_n^{\langle 1\rangle}(\mathcal{A})$ agrees with $L_n^h(R)$ and $L_n^{\langle 1\rangle}(\mathcal{A})$ agrees with $L_n^p(R)$ (see [56, Example 17.4, p. 147]).

For a chain complex $C$ over $\mathcal{A}$, we write $C^{-*}$ for the chain complex over $\mathcal{A}$ with $(C^{-*})_n := \mathrm{inv}(C_{-n})$ and differential $\partial_n := (-1)^n\,\mathrm{inv}(d_{-n+1})$, where $d_n\colon C_n\to C_{n-1}$ is the $n$-th differential of $C$. For a map $f\colon C\to D$ of degree $k$, the map $f^{-*}$ of degree $k$ is defined by $(f^{-*})_n := (-1)^{nk}\,\mathrm{inv}(f_{-n})\colon (D^{-*})_n \to (C^{-*})_{n+k}$. Note that if $f=\sum_{a\in S} f_a$ and $f^{-*}=\sum_{a\in S}(f^{-*})_a$ where $f_a$ and $(f^{-*})_a$ are $\{a\}$-controlled, then $(f^{-*})_a = (f_{a^{-1}})^{-*}$. A 0-*dimensional ultra-quadratic Poincaré complex* $(C,\psi)$ over $\mathcal{A}$ is a finite-dimensional chain complex $C$ over $\mathcal{A}$ together with a chain map $\psi\colon C^{-*}\to C$ (of degree 0), such that $\psi+\psi^{-*}$ is a chain homotopy equivalence. If $(C,\psi)$ is concentrated in degree 0, then it is a quadratic form over $\mathcal{A}$.

For us the following facts will be important.

(4.11) Every 0-dimensional ultra-quadratic Poincaré complex $(C,\psi)$ over $\mathcal{A}$ yields an element $[(C,\psi)]\in L_0^{\langle 1\rangle}(\mathcal{A})$.

(4.12) If $(C,\psi)$ and $(D,\varphi)$ are both 0-dimensional ultra-quadratic Poincaré complexes over $\mathcal{A}$ and $f\colon C\to D$ is chain homotopy equivalence such that $f \circ \psi \circ f^{-*}$ is chain homotopic to $\varphi$, then $[(C,\psi)] = [(D,\varphi)] \in L_0^{\langle 1\rangle}(\mathcal{A})$.

(4.13) Every element in $L_0^{\langle 1\rangle}(\mathcal{A})$ can be realized by a quadratic form.

(4.14) If $K_n(\mathcal{A}) = 0$ for $n \leq 1$, then the natural map $L_0^{\langle 1\rangle}(\mathcal{A}) \to L_0^{\langle -\infty\rangle}(\mathcal{A}) = L_0^{\langle -\infty\rangle}(\operatorname{Idem}\mathcal{A})$ is an isomorphism (see [56, Th. 17.2, p. 146]).

*Definition* 4.15 (0-dimensional ultra-quadratic $(\varepsilon, S)$-Poincaré complex). Let $Y$ be a $G$-space. Let $(Z,d)$ be a metric space equipped with an isometric $G$-action. Let $\mathcal{A}$ be an additive $G$-category with involution. Consider $S \subseteq G$ and $\varepsilon > 0$. A 0-*dimensional ultra-quadratic* $(\varepsilon, S)$-*Poincaré complex* over $\operatorname{Idem}(\mathcal{O}^G(Y,Z,d;\mathcal{A}))$ is a 0-dimensional ultra-quadratic Poincaré complex $(C,\psi)$ over $\operatorname{Idem}(\mathcal{O}^G(Y,Z,d;\mathcal{A}))$ such that $\psi$ and $C$ are $(\varepsilon,S)$-controlled and $\psi + \psi^{-*}$ is an $(\varepsilon,S)$-chain homotopy equivalence.

## 5. **Stability and the assembly map**

*Summary*. Theorem 5.2 asserts that the vanishing of the algebraic $K$- and $L$-theory of the obstruction categories yields isomorphism statements for the corresponding assembly maps. Theorem 5.3 shows that sufficiently controlled chain homotopy equivalences represent the trivial element the algebraic $K$-theory of the obstruction category. Similarly this result shows that sufficiently controlled ultra-quadratic Poincaré complexes represent the trivial element in the $L$-theory of the obstruction category.

Let $\mathcal{A}$ be an additive category with involution. Its $L$-groups $L_n^{\langle -\infty\rangle}(\mathcal{A})$, $n \in \mathbb{Z}$, can be constructed as the homotopy groups of a (nonconnective) spectrum $\mathbf{L}^{\langle -\infty\rangle}(\mathcal{A})$ which is constructed in [18, Def. 4.16] following ideas of Ranicki. Similarly, the $K$-groups $K_n(\mathcal{A})$, $n \in \mathbb{Z}$ of an additive category $\mathcal{A}$ are defined as the homotopy groups of a (nonconnective) spectrum $\mathbf{K}(\mathcal{A})$ which has been constructed in [49]. (See [5, §§2.1 and 2.5] for a brief review.) If $R$ is a ring (with involution) and one takes $\mathcal{A}$ to be the category of finitely generated free $R$-modules, then $L_n^{\langle -\infty\rangle}(\mathcal{A})$ and $K_n(\mathcal{A})$ reduce to the classical groups $L_n^{\langle -\infty\rangle}(R)$ and $K_n(R)$ for all $n \in \mathbb{Z}$

It is of course well known that the functors $\mathbf{L}^{\langle -\infty\rangle}$ and $\mathbf{K}$ have very similar properties. To emphasize this, we recall the following important properties of these functors. Recall that an additive category (with involution) is called *flasque* if there is a functor of such categories $\Sigma^\infty\colon \mathcal{A} \to \mathcal{A}$ together with a natural equivalence of functors of such categories $\mathrm{id}_{\mathcal{A}} \oplus \Sigma^\infty \xrightarrow{\cong} \Sigma^\infty$. A functor $F\colon \mathcal{A} \to \mathcal{B}$ of additive categories (with or without involutions) is called an *equivalence* if for any object $B \in \mathcal{B}$ there is an object $A \in \mathcal{A}$ such that $F(A)$ and $B$ are isomorphic in $\mathcal{B}$ and for any two objects $A_0, A_1 \in \mathcal{A}$, the map

$\operatorname{mor}_{\mathcal{A}}(A_0, A_1) \to \operatorname{mor}_{\mathcal{B}}(F(A_0), F(A_1))$ sending $f$ to $F(f)$ is bijective. For the notion of a *Karoubi filtration* we refer, for instance, to [17].

THEOREM 5.1.

(i) *If $\mathcal{A}$ is a flasque additive category, then $\mathbf{K}(\mathcal{A})$ is weakly contractible. If $\mathcal{A}$ is a flasque additive category with involution, then $\mathbf{L}^{\langle -\infty \rangle}(\mathcal{A})$ is weakly contractible.*

(ii) *If $\mathcal{A} \subseteq \mathcal{U}$ is a Karoubi filtration of additive categories, then*

$$\mathbf{K}(\mathcal{A}) \to \mathbf{K}(\mathcal{U}) \to \mathbf{K}(\mathcal{U}/\mathcal{A})$$

*is a homotopy fibration sequence of spectra. If $\mathcal{A} \subseteq \mathcal{U}$ is a Karoubi filtration of additive categories with involutions, then*

$$\mathbf{L}^{\langle -\infty \rangle}(\mathcal{A}) \to \mathbf{L}^{\langle -\infty \rangle}(\mathcal{U}) \to \mathbf{L}^{\langle -\infty \rangle}(\mathcal{U}/\mathcal{A})$$

*is a homotopy fibration sequence of spectra.*

(iii) *If $\varphi\colon \mathcal{A} \to \mathcal{B}$ is an equivalence of additive categories, then $\mathbf{K}(\varphi)$ is a weak equivalence of spectra. If $\varphi\colon \mathcal{A} \to \mathcal{B}$ is an equivalence of additive categories with involution, then $\mathbf{L}^{\langle -\infty \rangle}(\varphi)$ is a weak equivalence of spectra.*

(iv) *If $\mathcal{A} = \operatorname{colim}_i \mathcal{A}_i$ is a colimit of additive categories over a directed system, then the natural map $\operatorname{colim}_i \mathbf{K}(\mathcal{A}_i) \to \mathbf{K}(\mathcal{A})$ is a weak equivalence. If $\mathcal{A} = \operatorname{colim}_i \mathcal{A}_i$ is a colimit of additive categories with involution over a directed system, then the natural map $\operatorname{colim}_i \mathbf{L}^{\langle -\infty \rangle}(\mathcal{A}_i) \to \mathbf{L}^{\langle -\infty \rangle}(\mathcal{A})$ is a weak equivalence.*

*Proof.* (i) This is the well-known *Eilenberg-swindle*. See, for instance, [18, Lemma 4.12].

(ii) See [17] and [18, Th. 4.2].

(iii) See, for instance, [18, Lemma 4.17].

(iv) For $K$-theory this follows from [50, (9), p. 20]. The proof for $L$-theory in [4, Lemma 5.2] for rings carries over to additive categories. $\square$

Many $K$-theory results in controlled algebra depend only on the properties of $K$-theory listed in Theorem 5.1, and therefore there are corresponding results in $L$-theory. This applies in particular to Proposition 3.8 and Theorem 7.2 in [9]. In the following we give minor variations of these results.

THEOREM 5.2. *Let $G$ be a group. Let $\mathcal{F}$ be a family of subgroups of $G$.*

(i) *Suppose that there is $m_0 \in \mathbb{Z}$ such that*

$$K_{m_0}(\mathcal{O}^G(E_{\mathcal{F}}G; \mathcal{A})) = 0$$

*holds for all additive $G$-categories $\mathcal{A}$.*

*Then the assembly map* (1.2) *is an isomorphism for* $m < m_0$ *and surjective for* $m = m_0$ *for all such* $\mathcal{A}$.

(ii) *Suppose that there is* $m_0 \in \mathbb{Z}$ *such that*

$$L_{m_0}^{\langle -\infty \rangle}(\mathcal{O}^G(E_{\mathcal{F}_2}(G); \mathcal{A})) = 0$$

*holds for all additive* $G$*-categories* $\mathcal{A}$ *with involution.*

*Then the assembly map* (1.3) *is an isomorphism for all* $m$ *and such* $\mathcal{A}$.

*Proof.* For $K$-theory, the statement is almost that same as [9, Prop. 3.8]. The only difference from the present statement is that in the above reference the vanishing of the $K$-group is assumed for all $m \geq m_0$ and the conclusion is an isomorphism for all $m$. A straightforward modification of the proof from [9] yields the proof of our present $K$-theory statement. This proof uses, in fact, only the properties of $K$-theory listed in Theorem 5.1 and carries therefore over to $L$-theory. Because $L$-theory is periodic, we get in this case the stronger statement stated above. □

In order to formulate the next result, we quickly recall that for an additive category $\mathcal{B}$, elements of $K_1(\mathcal{B})$ can be thought of as self-chain homotopy equivalences over $\mathcal{B}$. If $f\colon C \to C$ is a self-chain homotopy equivalence of a finite chain complex over $\mathcal{A}$, then the *self-torsion* is an element

$$[(C, f)] \in K_1(\mathcal{B}).$$

It depends only on the chain homotopy class of $f$. If $f\colon C \to D$ and $g\colon D \to C$ are chain homotopy equivalences of finite $\mathcal{B}$-chain complexes, then we obtain $[(C, g \circ f)] = [(D, f \circ g)]$. In particular, we get $[(C, f)] = [(D, g)]$ for self-chain homotopy equivalences of finite $\mathcal{B}$-chain complexes $f\colon C \to C$ and $g\colon D \to D$ provided that there is a chain homotopy equivalence $u\colon C \to D$ with $u \circ f \simeq g \circ u$. If $v\colon B \to B$ is an automorphism in $\mathcal{B}$ and $0(v)\colon 0(B) \to 0(B)$ is the obvious automorphism of the $\mathcal{B}$-chain complex $0(B)$ which is concentrated in dimension 0 and given there by $B$, then the class $[v]$ in $K_1(\mathcal{A})$ coming from the definition of $K_1(\mathcal{A})$ agrees with the self-torsion $[0(v)]$ (see [31], [39, Example 12.17, p. 246], [55]).

In the following theorem, $d^1$ denotes the $l^1$-metric on simplicial complexes; compare [9, §4.2].

THEOREM 5.3. *Let* $N \in \mathbb{N}$. *Let* $\mathcal{F}$ *be a family of subgroups of a group* $G$. *Let* $S$ *be a finite subset of* $G$.

(i) *Let* $\mathcal{A}$ *be an additive* $G$*-category. Then there exists a positive real number* $\varepsilon = \varepsilon(N, \mathcal{A}, G, \mathcal{F}, S)$ *with the following property. If* $\Sigma$ *is a simplicial complex of dimension* $\leq N$ *equipped with a simplicial action of* $G$ *all whose isotropy groups are members of* $\mathcal{F}$ *and* $\alpha\colon C \to C$ *is* *an $(\varepsilon, S)$-chain homotopy equivalence over $\mathcal{O}^G(E_{\mathcal{F}}G, \Sigma, d^1; \mathcal{A})$ where $C$ is concentrated in degrees $0, \dots, N$, then*

$$[(C, \alpha)] = 0 \in K_1(\mathcal{O}^G(E_{\mathcal{F}}G, \Sigma, d^1; \mathcal{A})).$$

(ii) *Let $\mathcal{A}$ be an additive $G$-category with involution. Then there exists a positive real number $\varepsilon = \varepsilon(N, \mathcal{A}, G, \mathcal{F}, S)$ with the following property. If $\Sigma$ is a simplicial complex of dimension $\leq N$ equipped with a simplicial action of $G$ all whose isotropy groups are members of $\mathcal{F}$ and $(C, \psi)$ is a $0$-dimensional ultra-quadratic $(\varepsilon, S)$-Poincaré complex over* $\operatorname{Idem}(\mathcal{O}^G(E_{\mathcal{F}}G, \Sigma, d^1; \mathcal{A}))$ *concentrated in degrees $-N, \dots, N$, then*

$$[(C, \psi)] = 0 \in L_0^{\langle -\infty \rangle}(\mathcal{O}^G(E_{\mathcal{F}}G, \Sigma, d^1; \mathcal{A})).$$

*Proof.* The $K$-theory statement can be deduced from [9, Th. 7.2] in roughly the same way as [3, Cor. 4.6] is deduced from [3, Prop. 4.1].

For the convenience of the reader we give more details. We will proceed by contradiction and assume that there is no such $\varepsilon = \varepsilon(N, \mathcal{A}, G, \mathcal{F}, S)$. Then for every $n \in \mathbb{N}$, there are

- a simplicial complex $\Sigma_n$ of dimension $\leq N$ equipped with a simplicial action of $G$ all whose isotropy groups are members of $\mathcal{F}$,
- an $(1/n, S)$-chain homotopy equivalence $\alpha^n \colon C^n \to C^n$ over the additive category $\mathcal{O}^G(E_{\mathcal{F}}G, \Sigma_n, d^1; \mathcal{A})$ where $C^n$ is concentrated in degrees $0, \dots, N$,

such that

$$[(C^n, \alpha^n)] \neq 0 \in K_1(\mathcal{O}^G(E_{\mathcal{F}}G, \Sigma_n, d^1; \mathcal{A})).$$

Now consider the product category

$$\prod_{n \in \mathbb{N}} \mathcal{O}^G(E_{\mathcal{F}}G, \Sigma_n, d^1; \mathcal{A}).$$

Objects of this category are given by sequences $(M_n)_{n \in \mathbb{N}}$ where each $M_n$ is an object in $\mathcal{O}^G(E_{\mathcal{F}}G, \Sigma_n, d^1; \mathcal{A})$; morphisms $(M_n)_{n \in \mathbb{N}} \to (N_n)_{n \in \mathbb{N}}$ are given by sequences $(\psi_n \colon M_n \to N_n)_{n \in \mathbb{N}}$ where each $\psi_n$ is a morphism in $\mathcal{O}^G(E_{\mathcal{F}}G, \Sigma_n, d^1; \mathcal{A})$. We will use the subcategory $\mathcal{L}$ of this product category that has the same objects as the product category but has fewer morphisms: a morphism $(\psi_n)_{n \in \mathbb{N}}$ is a morphism in $\mathcal{L}$ if and only if there are a finite subset $T \subset G$ and a number $A > 0$ such that $\psi^n$ is $(A/n, T)$-controlled for each $n \in \mathbb{N}$. Observe that $(\alpha^n)_{n \in \mathbb{N}} \colon (C^n)_{n \in \mathbb{N}} \to (C^n)_{n \in \mathbb{N}}$ is a chain homotopy equivalence in this category $\mathcal{L}$. We denote by $[(C^n, \alpha^n)_{n \in \mathbb{N}}] \in K_1(\mathcal{L})$ its $K$-theory class. Let

$$\mathcal{L}_{\oplus} := \bigoplus_{n \in \mathbb{N}} \mathcal{O}^G(E_{\mathcal{F}}G, \Sigma_n, d^1; \mathcal{A}).$$

This is, in a canonical way, a subcategory of $\mathcal{L}$. It is proven in [9, Th. 7.2] that this inclusion $\iota \colon \mathcal{L}_{\oplus} \to \mathcal{L}$ induces an isomorphism in $K$-theory. Consider an element $a \in K_1(\mathcal{L}_\oplus)$ satisfying $\iota_*(a) = [(C^n, \alpha^n)_{n\in\mathbb{N}}] \in K_1(\mathcal{L})$. Denote by $p_n\colon \mathcal{L} \to \mathcal{O}^G(E_\mathcal{F}G, \Sigma_n, d^1; \mathcal{A})$ the canonical projection. The definition of $\mathcal{L}_\oplus$ as a direct sum implies that there is $m_0 \in \mathbb{N}$ such that $(p_m \circ \iota)_*(a) = 0$ for all $m \geq m_0$. Thus we obtain the desired contradiction

$$[(C^m, \alpha^m)] = (p_m)_*([(C^n, \alpha^n)_{n\in\mathbb{N}}]) = (p_m \circ \iota)_*(a) = 0$$

for $m \geq m_0$.

In [9, Th. 7.2] it is assumed that the action of $G$ on $\Sigma_n$ is, in addition, cell preserving. This assumption makes no real difference to our result here: we can always replace $\Sigma$ by it first barycentric subdivision and obtain a cell preserving action. The subdivision changes the metric only in a uniformly controlled way. (On the other hand, the proof of [9, Th. 7.2] does not really use the assumption cell preserving.)

This proof carries over to $L$-theory in a straightforward fashion, because it only depends on the properties of $K$-theory that are listed in Theorem 5.1 and also hold in $L$-theory. □

## 6. **Transfer up to homotopy**

*Summary.* In this section we transfer morphisms $\psi$ in $\mathcal{C}^G(Y;\mathcal{A})$ to chain maps $\operatorname{tr}^{\mathbf{P}} \psi$ over $\mathcal{C}^G(Y,Z;\mathcal{A})$. This transfer depends on the choice of a chain complex $\mathbf{P}$ over $\mathcal{C}(Z;\mathbb{Z})$ equipped with a homotopy action; see Definition 6.2. It is functorial up to homotopy; see Lemma 6.4.

Throughout this section we fix the following convention.

Convention 6.1. *Let*

- *$G$ be a group,*
- *$Y$ be a $G$-space,*
- *$(Z, d)$ be a compact metric space,*
- *$\mathcal{A}$ be an additive $G$-category.*

*We will use the following $G$-actions: $g \in G$ acts trivially on $Z$, on $G \times Y \times [1,\infty)$ by $g \cdot (h, y, t) = (gh, gy, t)$ and on $G \times Z \times Y \times [1,\infty)$ by $g \cdot (h, z, y, t) = (gh, z, gy, t)$.*

We will use the following chain complex analogue of homotopy $S$-actions.

*Definition* 6.2 (Chain homotopy $S$-action). Let $S$ be a finite subset of $G$ (containing $e$).

(i) Let $P$ be a chain complex over $\operatorname{Idem}(\mathcal{C}(Z, d;\mathbb{Z}))$. A *homotopy $S$-action* on $P$ consists of chain maps $\varphi_g\colon P \to P$ for $g \in S$ and chain homotopies $H_{g,h}$ for $g, h, \in S$ with $gh \in S$ from $\varphi_g \circ \varphi_h$ to $\varphi_{gh}$. Moreover, we require $\varphi_e = \mathrm{id}$ and $H_{e,e} = 0$. In this situation we will also say that $(P, \varphi, H)$ is a *homotopy $S$-chain complex over* $\operatorname{Idem}(\mathcal{C}(Z, d;\mathbb{Z}))$.

(ii) Let $\mathbf{P} = (P, \varphi^P, H^P)$ and $\mathbf{Q} = (Q, \varphi^Q, H^Q)$ be homotopy $S$-chain complexes over $\operatorname{Idem}(\mathcal{C}(Z,d;\mathbb{Z}))$. A homotopy $S$-chain map $\mathbf{P} \to \mathbf{Q}$ is a chain map $f\colon P \to Q$ such that $f \circ \varphi^P_g$ and $\varphi^Q_g \circ f$ are chain homotopic for all $g \in S$. It is called a homotopy $S$-chain equivalence if $f$ is, in addition, a chain homotopy equivalence.

(iii) Let $z_0 \in Z$. The *trivial homotopy $S$-chain complex $\mathcal{C}(Z,d;\mathbb{Z}))$ at $z_0$*, which we will denote by $\mathbf{T} = (T, \varphi^T, H^T)$, is defined by $(T_0)_{z_0} = \mathbb{Z}$, $(T_n)_z = 0$, unless $n = 0$, $z = z_0$, $\varphi^T_a = \mathrm{id}_T$ and $H^T_{a,b} = 0$.

Let $\mathcal{F}(\mathbb{Z})$ denote our choice of a small model for the category of finitely generated free $\mathbb{Z}$-modules (see §4.3). Recall from Section 4.1 that $\mathcal{A}$ comes with a strictly associative functorial sum $\oplus$. We define a bi-linear functor of additive categories called the *tensor product functor*

$$\otimes\colon \mathcal{A} \times \mathcal{F}(\mathbb{Z}) \to \mathcal{A}$$

as follows. On objects put $A \otimes \mathbb{Z}^n = \bigoplus_{i=1}^n A$. Given a morphism $f\colon A \to B$ in $\mathcal{A}$ and a morphism $U\colon \mathbb{Z}^m \to \mathbb{Z}^n$ defined by a matrix $U = (u_{i,j})$, let

$$f \otimes U\colon A \otimes \mathbb{Z}^m \to B \otimes \mathbb{Z}^n$$

be the morphism $\bigoplus_{i=1}^m A \to \bigoplus_{j=1}^n B$ which is given by the matrix $(u_{i,j} \cdot f)$ of morphisms in $\mathcal{A}$. This construction is functorial in $\mathcal{A}$. For objects $M \in \mathcal{O}^G(Y;\mathcal{A}) = \mathcal{C}^G(G \times Y \times [1,\infty); \mathcal{E}(Y), \mathcal{F}(Y); \mathcal{A})$ and $F \in \mathcal{C}(Z,d;\mathbb{Z})$, we define

$$M \otimes F \in \mathcal{O}^G(Y,Z,d;\mathcal{A}) = \mathcal{C}^G(G \times Z \times Y \times [1,\infty); \mathcal{E}(Y,Z,d), \mathcal{F}(Y,Z,d); \mathcal{A})$$

by putting

$$(M \otimes F)_{g,z,y,t} := M_{g,y,t} \otimes F_z.$$

This construction is clearly functorial in $F$ and $M$. It is easy to check that also the control conditions $\mathcal{E}(Y,Z,d)$ are satisfied because they are implemented by projections to one of the spaces $Z$, $Y \times [1,\infty)$ or $G$. Thus we obtain a tensor product functor

$$\otimes\colon \mathcal{O}^G(Y;\mathcal{A}) \otimes \mathcal{C}(Z,d;\mathbb{Z}) \to \mathcal{O}^G(Y,Z,d;\mathcal{A}). \tag{6.3}$$

This functor can, in particular, be applied to an object $M \in \mathcal{O}^G(Y;\mathcal{A})$ and a chain complex $P$ over $\operatorname{Idem}(\mathcal{C}(Z,d;\mathbb{Z}))$ to produce a chain complex $M \otimes P$ over $\operatorname{Idem}(\mathcal{O}^G(Y,Z,d;\mathcal{A}))$.

Next we will consider homotopy $S$-actions on $P$ to twist the functoriality in $M$. Let $S$ be a finite subset of $G$ and $\mathbf{P} = (P, \varphi^P, H^P)$ be a homotopy $S$-chain complex over $\operatorname{Idem}(\mathcal{C}(Z,d;\mathbb{Z}))$. For an $S$-morphism $\psi\colon M \to N$ in $\mathcal{O}^G(Y;\mathcal{A})$, we define a chain map $\operatorname{tr}^{\mathbf{P}} \psi\colon M \otimes P \to N \otimes P$ by putting

$$(\operatorname{tr}^{\mathbf{P}} \psi)_{(g,z,y,t),(g',z',y',t')} := \psi_{(g,y,t),(g',y',t')} \otimes \left(\varphi^P_{g^{-1}g'}\right)_{z,z'}.$$

If we write $\psi = \sum_{a \in S} \psi_a$ as in (4.9), then $\operatorname{tr}^{\mathbf{P}} \psi = \sum_{a \in S} \psi_a \otimes \varphi_a^P$. This is not strictly functorial in $M$; see Lemma 6.4 below. (The definition of $\operatorname{tr}^{\mathbf{P}} \psi$ is very much in the spirit of the classical $K$-theory transfer; compare Section A.1 and in particular (A.1).)

Let $f\colon \mathbf{P} \to \mathbf{Q}$ be a map of homotopy $S$-chain complexes over $\mathcal{C}(Z, d; \mathbb{Z})$, where $\mathbf{Q} = (Q, \varphi^Q, H^Q)$. It induces a chain map $\operatorname{id}_M \otimes f\colon M \otimes P \to M \otimes Q$ over $\mathcal{O}^G(Y, Z, d; \mathcal{A})$. If $f\colon \mathbf{P} \to \mathbf{Q}$ is a homotopy $S$-chain equivalence over $\mathcal{C}(Z, d; \mathbb{Z})$, then $\operatorname{id}_M \otimes f$ is a chain homotopy equivalence over $\mathcal{O}^G(Y, Z, d; \mathcal{A})$. If $\psi\colon M \to N$ is an $S$-morphism, then $(\operatorname{id}_N \otimes f) \circ \operatorname{tr}^{\mathbf{P}} \psi$ and $\operatorname{tr}^{\mathbf{Q}} \psi \circ (\operatorname{id}_M \otimes f)$ are homotopic as chain maps over $\mathcal{O}^G(Y, Z, d; \mathcal{A})$.

LEMMA 6.4. *Let $S$ be finite subset of $G$ (containing $e$) and $\mathbf{P} = (P, \varphi, H)$ be a homotopy $S$-chain complex over $\mathcal{C}(Z, d; \mathbb{Z})$. Let $T$ be a subset of $S$ (also containing $e$) such that $a, b \in T$ implies $ab \in S$. Let $\psi = \sum_{a \in T} \psi_a\colon M \to M'$ and $\psi' = \sum_{a \in T} \psi'_a\colon M' \to M''$ be $T$-morphisms in $\mathcal{C}^G(Y, G; \mathcal{A})$, where $\psi_a$ and $\psi'_a$ are $\{a\}$-morphisms. Then*

$$\sum_{a,b \in T} (\psi'_a \circ \psi_b) \otimes H_{a,b}$$

*is a chain homotopy over $\mathcal{O}^G(Y, Z, d; \mathcal{A})$ from $\operatorname{tr}^{\mathbf{P}} \psi' \circ \operatorname{tr}^{\mathbf{P}} \psi$ to $\operatorname{tr}^{\mathbf{P}}(\psi' \circ \psi)$.*

*Proof.* This is a straightforward calculation. □

## 7. The transfer in $K$-theory

*Summary.* In this section we construct a controlled $K$-theory transfer. The construction from the previous section is applied to lift a given automorphism $\alpha$ in the obstruction category $\mathcal{O}^G(Y; \mathcal{A})$ to chain homotopy selfequivalences $\hat{\alpha}$ over the idempotent completion of $\mathcal{O}^G(Y, G \times X, d_{S,\Lambda}; \mathcal{A})$. The contractibility of $X$ is used to show that this transfer lifts $K$-theory elements. In addition, the control properties of $\hat{\alpha}$ are important. This construction is a variation of [5, §12]. A review of the classical (uncontrolled) $K$-theory transfer can be found in Section A.1 of the appendix.

Throughout this section we fix the following convention.

CONVENTION 7.1. *Let*

- *$G$ be a group,*
- *$N \in \mathbb{N}$,*
- *$(X, d) = (X, d_X)$ be a compact contractible controlled $N$-dominated metric space,*
- *$Y$ be a $G$-space,*
- *$\mathcal{A}$ be an additive $G$-category.*

Proposition 7.2. *Let $S$ be a finite subset of $G$ (containing $e$) and $(\varphi, H)$ be a homotopy $S$-action on $X$. For every $\varepsilon > 0$, there exists a homotopy $S$-chain complex $\mathbf{P} = (P, \varphi^P, H^P)$ over $\operatorname{Idem}(\mathcal{C}(X, d; \mathbb{Z}))$ satisfying*:

(i) *$P$ is concentrated in degrees $0, \ldots, N$;*
(ii) *$P$ is $\varepsilon$-controlled;*
(iii) *there is a homotopy $S$-chain equivalence $f \colon \mathbf{P} \to \mathbf{T}_{x_0}$ to the trivial homotopy $S$-chain complex at $x_0 \in X$ for some (and hence all) $x_0 \in X$;*
(iv) *if $g \in S$ and $(x, y) \in \operatorname{supp} \varphi_g^P$, then $d(x, \varphi_g(y)) \leq \varepsilon$;*
(v) *if $g, h \in S$ with $gh \in S$ and $(x, y) \in \operatorname{supp} H_{g,h}^P$, then there is $t \in [0, 1]$ such that $d(x, H_{g,h}(y, t)) \leq \varepsilon$.*

The idea of the proof of Proposition 7.2 is not complicated. Consider the subcomplex $C^{\mathrm{sing},\varepsilon}(X)$ of the singular chain complex of $X$ spanned by singular simplices of diameter bounded by an appropriate small constant. This chain complex is in an $\varepsilon$-controlled way finitely dominated, because $X$ is controlled $N$-dominated, and can therefore up to controlled homotopy be replaced by finite projective chain complex $P$. The homotopy $S$-action on $X$ induces through this homotopy equivalence the chain homotopy $S$-action on $P$. The details of this proof are somewhat involved and postponed to the next section.

Proposition 7.3. *Let $T \subseteq S$ be finite subsets of $G$ (both containing $e$) such that for $g, h \in T$, we have $gh \in S$. Let $\alpha \colon M \to M$ be a $T$-automorphism in $\mathcal{O}^G(Y; \mathcal{A})$. Let $\Lambda > 0$. Then there is an $(S, 2)$-chain homotopy equivalence $\hat{\alpha} \colon C \to C$ over $\operatorname{Idem}(\mathcal{O}^G(Y, G \times X, d_{S,\Lambda}; \mathcal{A}))$, where $C$ is concentrated in degrees $0, \ldots, N$, such that*

$$[p(C, \hat{\alpha})] = [(M, \alpha)] \in K_1(\operatorname{Idem}(\mathcal{O}^G(Y; \mathcal{A}))),$$

*where $p \colon \operatorname{Idem}(\mathcal{O}^G(Y, G \times X, d_{S,\Lambda}; \mathcal{A})) \to \operatorname{Idem}(\mathcal{O}^G(Y; \mathcal{A}))$ is the functor induced by the projection $G \times X \to \mathrm{pt}$.*

*Proof.* Let $\varepsilon := 1/\Lambda$. Let $\mathbf{P} = (P, \varphi^P, H^P)$ be a homotopy $S$-chain complex over $\operatorname{Idem}(\mathcal{C}(X, d; \mathbb{Z}))$ that satisfies the assertion of Proposition 7.2. It follows from Lemma 6.4 that $\operatorname{tr}^{\mathbf{P}}(\alpha) \colon M \otimes P \to M \otimes P$ is an $S$-chain homotopy equivalence over $\mathcal{O}^G(Y, X, d; \mathcal{A})$.

Let $f \colon \mathbf{P} \to \mathbf{T}_{x_0}$ be the weak equivalence from assertion (iii) in Lemma 7.2. Let $q \colon \mathcal{O}^G(Y, X, d; \mathcal{A}) \to \mathcal{O}^G(Y; \mathcal{A})$ be the functor induced by $X \to \mathrm{pt}$. Then $\mathrm{id}_M \otimes f$ is a chain homotopy equivalence, $\operatorname{tr}^P \alpha \circ (\mathrm{id}_M \otimes f)$ is chain homotopic to $(\mathrm{id}_M \otimes f) \circ \operatorname{tr}^{\mathbf{T}_{x_0}} \alpha$ and $q(\operatorname{tr}^{\mathbf{T}_{x_0}} \alpha) = \alpha$ (up to a canonical isomorphism $q(M \otimes T) \cong M$). Therefore $q[(M \otimes P, \operatorname{tr}^{\mathbf{P}} \alpha)] = q[(M \otimes T, \operatorname{tr}^{\mathbf{T}_{x_0}} \alpha)] = [(M, \alpha)] \in K_1(\operatorname{Idem}(\mathcal{O}^G(Y; \mathcal{A})))$. Let $F \colon \mathcal{O}^G(Y, X, d; \mathcal{A}) \to \mathcal{O}^G(Y, G \times X, d_{S,\Lambda}; \mathcal{A})$ be the functor induced by the map $(g, x, y, t) \mapsto (g, g, x, y, t)$ and set $(C, \hat{\alpha}) := F(M \otimes P, \operatorname{tr}^{\mathbf{P}} \alpha)$. Since $p \circ F = q$, we have $[p(C, \hat{\alpha})] = [\alpha]$. That $\hat{\alpha}$ is a $(S, 2)$-chain homotopy equivalence follows from our choice of $\varepsilon$, Definition 3.4 of the metric $d_{S,\Lambda}$ and the concrete formula for $\mathrm{tr}^{\mathbf{P}}(\alpha)$. The key observation is that for $t \in T$ and $(x, y) \in \mathrm{supp}(\varphi_t^P)$, we have

$$d_{S,\Lambda}((e, x), (t, y)) \leq 1 + \Lambda \cdot d_X(x, \phi_g(x)) \leq 1 + \Lambda \cdot \epsilon = 2. \qquad \square$$

## 8. **Proof of Proposition 7.2**

Throughout this section we use Convention 7.1.

Let $Z$ be a metric space. If $q\colon B \to Z$ is a map, then a homotopy $H\colon A \times [0, 1] \to B$ is called $\varepsilon$-controlled over $q$ if for every $a \in A$, the set $\{q(H(a, t)) \mid t \in [0, 1]\}$ has diameter at most $\varepsilon$ in $Z$. The following lemma shows that we can replace the CW-complexes appearing in the definition of controlled $N$-dominated metric spaces by simplicial complexes.

Lemma 8.1. *Let $q\colon K \to Z$ be a map from an $N$-dimensional finite* CW*-complex to a metric space. Let $\varepsilon > 0$ be given. Then there is an $N$-dimensional finite simplicial complex $L$, maps $i\colon K \to L$, $p\colon L \to K$ and a homotopy $h\colon p \circ i \simeq \mathrm{id}_K$ that is $\varepsilon$-controlled over $q$.*

*Proof.* We proceed by induction over the skeleta of $K$. For $K^{(0)}$ the claim is obviously true. Assume for the induction step that there is $0 < \delta < \varepsilon$, a finite $n$-dimensional simplicial complex $L$, maps $i\colon K^{(n)} \to L$, $p\colon L \to K^{(n)}$ and a homotopy $h\colon p \circ i \simeq \mathrm{id}_{K^{(n)}}$ that is $\delta$-controlled over $q$. For a given $\delta'$ with $\delta < \delta' < \varepsilon$, we will construct a finite $(n+1)$-dimensional simplicial complex $L'$, maps $i'\colon K^{(n+1)} \to L'$, $p'\colon L \to K^{(n+1)}$ and a homotopy $h'\colon p' \circ i' \simeq \mathrm{id}_{K^{(n+1)}}$ that is $\delta'$-controlled over $q$.

Let $\varphi\colon \amalg_I S^n \to K^{(n)}$ for some finite index set $I$ be the attaching map of the $(n+1)$-skeleton, i.e., $K^{(n+1)} = D^{n+1} \cup_\varphi \amalg_I K^{(n)}$. Pick $\alpha > 0$ such that $\delta + \alpha < \delta'$. By subdividing $L$ we can assume that the image under $q \circ p$ of each simplex in $L$ has diameter at most $\alpha$ in $Z$. Let $\psi$ be a simplicial approximation of $i \circ \varphi$; i.e., $\amalg_I S^n$ is a simplicial complex and $\psi$ is a simplicial map such that for any $x \in \amalg_I S^n$, the point, $\psi(x)$ is contained in the smallest simplex of $L$ that contains $i(\varphi(x))$. In particular, there is a homotopy $k\colon \psi \simeq i \circ \varphi$ that is $\alpha$-controlled over $q \circ p$. Define $L' := \amalg_I D^{n+1} \cup_\psi L$. Since $\psi$ is a simplicial map and $L'$ is the mapping cone of $\psi$, we can extend the simplicial structure of $L$ to a simplicial structure on $L'$. Pick $\alpha' > 0$ such that $\delta + \alpha + \alpha' < \delta'$ and choose $\beta > 0$ such that for any $x \in \amalg_I S^n$, the diameter of $\{q(rx) \mid r \in [1 - 2\beta, 1]\}$ is at most $\alpha'$. In order to extend $i$ to a map $i'\colon K^{(n+1)} \to L'$ it suffices to specify a map $\amalg_I D^{n+1} \to L'$ such that its restriction to the boundary is $i \circ \varphi$. We use polar coordinates on $\amalg_I D^{(n+1)}$. Define the desired extension $i'$ by setting

$$i'(rx) := \begin{cases} rx & \text{if} \quad r \in [0, 1 - 2\beta], \\ (2r - 1 + 2\beta)x & \text{if} \quad r \in [1 - 2\beta, 1 - \beta], \\ k(x, (r - 1 + \beta)/\beta) & \text{if} \quad r \in [1 - \beta, 1] \end{cases}$$

for $r \in [0,1]$ and $x \in \amalg_I S^n$, where $rx$ and $(2r-1+2\beta)x$ are understood to be the images of these points in $\amalg_I D^{n+1}$ under the canonical map $\amalg_I D^{n+1} \to L' := \amalg_I D^{n+1} \cup_\psi L$. Notice that the map $\amalg_I D^{n+1} \to L'$ above is the identity on $\amalg_I D^{n+1}$ except for a neighborhood of the boundary, where we use the homotopy $k$, and this neighborhood is smaller the smaller $\beta$ is.

By a similar formula we extend $p$ to a map $p' \colon L' \to K^{(n+1)}$, where we use the homotopy $(h^- \circ \varphi \times \mathrm{id}_{[0,1]}) * (p \circ k^-) \colon \varphi \simeq p \circ \psi$ in place of $k$. (Here $*$ denotes concatenation of homotopies and $h^-$ and $k^-$ are $h$ and $k$ run backwards.) Then $p' \circ i'$ is an extension of $p \circ i$ such that

$$(p' \circ i')(rx) = \begin{cases} rx & \text{if } \; r \in [0, 1-2\beta], \\ (4r-3+6\beta)x & \text{if } \; r \in [1-2\beta, 1-3\beta/2], \\ h^-(\varphi(x), (4r-4+6\beta)/\beta) & \text{if } \; r \in [1-3\beta/2, 1-5\beta/4], \\ (p \circ k^-)(x, (4r-4+5\beta)/\beta) & \text{if } \; r \in [1-5\beta/4, 1-\beta)], \\ (p \circ k)(x, (r-1+\beta)/\beta) & \text{if } \; r \in [1-\beta, 1] \end{cases}$$

for $r \in [0,1]$ and $x \in \amalg_I S^n$.

In the next step we cancel $p \circ k$ and $p \circ \bar{k}$ appearing above. The constant homotopy $(y,t) \mapsto (p \circ i)(y)$ on $K$ has a canonical extension to a homotopy $h_0'$ from $p' \circ i'$ to an extension $f$ of $p \circ i$ such that

$$f(rx) = \begin{cases} rx & \text{if } \; r \in [0, 1-2\beta], \\ (4r-3+6\beta)x & \text{if } \; r \in [1-2\beta, 1-3\beta/2], \\ h^-(x, (4r-4+6\beta)/\beta) & \text{if } \; r \in [1-3\beta/2, 1-5\beta/4], \\ (p \circ i \circ \varphi)(x) & \text{if } \; r \in [1-5\beta/4, 1] \end{cases}$$

for $r \in [0,1]$ and $x \in \amalg_I S^n$. This homotopy $h_0'$ is $\alpha$-controlled over $q$ because $p \circ k$ is $\alpha$-controlled over $q$.

In the final step we use the appearance of $h$ in the above formula for $f$. This (and reparametrization in $r \in [1-2\beta, 1]$) yields an extension of the homotopy $h$ to a homotopy $h_1'$ from $f$ to $\mathrm{id}_{K^{(n+1)}}$, and this homotopy is $\varepsilon + \alpha'$-controlled over $q$. (This comes from the control of $h$ and the control of reparametrizations in $r \in [1-2\beta, 1]$ by our choice of $\beta$.) Then $h' := h_0' * h_1' \colon p' \circ i' \simeq \mathrm{id}_{K^{(n+1)}}$ is $\delta'$-controlled over $q$ because $\delta + \alpha + \alpha' < \delta'$. This finishes the construction of $L'$, $i'$, $p'$ and $h'$ and concludes the induction step. $\square$

*Remark* 8.2. If $q \colon K \to X$ is a map from a finite simplicial complex to $X$, then the simplicial complex $C(K)$ of $K$ is in a natural way (using the images of barycenters under $q$) a chain complex over $\mathcal{C}(X, d; \mathbb{Z})$. We will also need to use the subcomplex $C^{\mathrm{sing},\varepsilon}(X)$ of singular chain complex of $X$ spanned by singular simplices of diameter $\leq \varepsilon$ in $X$. This is not naturally a chain complex over $\mathcal{C}(X, d; \mathbb{Z})$ because it fails the locally finiteness condition. However, if we drop this conditions (and allow a large class of $\mathbb{Z}$-modules at every point), then we get an additive category $\overline{\mathcal{C}}(X, d; \mathbb{Z})$. There is an obvious inclusion $\mathcal{C}(X, d; \mathbb{Z}) \subset$

$\overline{\mathcal{C}}(X,d;\mathbb{Z})$ of additive categories that is full and faithful on morphism sets. Moreover, $C^{\text{sing},\varepsilon}(X)$ is naturally a chain complex over $\overline{\mathcal{C}}(X,d;\mathbb{Z})$. Namely, a singular simplex $\sigma\colon \Delta \to X$ defines a point in $X$, the image of the barycenter of $\Delta$ under $\sigma$. In particular, the notion of $\varepsilon$-control is defined for maps involving $C^{\text{sing},\varepsilon}(X)$. It will be important to carry out certain construction in the larger category $\overline{\mathcal{C}}(X,d;\mathbb{Z})$ and then go back to $\mathcal{C}(X,d;\mathbb{Z})$. The latter step is described in the next remark.

*Remark* 8.3. Let $\mathcal{C} \subset \overline{\mathcal{C}}$ be an inclusion of additive categories that is full and faithful on morphism sets. Let $C$ be a chain complex over $\overline{\mathcal{C}}$ and $D$ be a chain complex over $\mathcal{C}$, where $C$ is concentrated in nonnegative degree and $D$ is concentrated in degree $0,\dots,N$. Let $i\colon C \to D$, $r\colon D \to C$, be chain maps (over $\overline{\mathcal{C}}$) and let $h\colon r\circ i \simeq \mathrm{id}_C$ be a chain homotopy. In the following we recall explicit formulas from [54] that allow to construct from this data a chain complex $P$ over $\mathrm{Idem}(\mathcal{C})$, chain maps $f\colon C \to P$, $g\colon P \to C$ over $\mathrm{Idem}(\overline{\mathcal{C}})$ and chain homotopies $k\colon f\circ g \simeq \mathrm{id}_P$, $l\colon g\circ f \simeq \mathrm{id}_C$.

Define the chain complex $C'$ over $\mathcal{C}$ by defining its $m$-th chain object to be

$$C'_m = \bigoplus_{j=0}^{m} D_j$$

and its $m$-th differential to be

$$c'_m\colon C'_m = \bigoplus_{j=0}^{m} D_j \to C'_{m-1} = \bigoplus_{k=0}^{m-1} D_k,$$

where the $(j,k)$-entry $d_{j,k}\colon D_j \to D_k$ for $j \in \{0,1,2\dots,m\}$ and $k \in \{0,1,2\dots, m-1\}$ is given by

$$d_{j,k} := \begin{cases} 0 & \text{if } j \geq k+2,\\ (-1)^{m+k}\cdot d_j & \text{if } j = k+1,\\ \mathrm{id} - r_j \circ i_j & \text{if } j = k, j \equiv m \mod 2,\\ r_j \circ i_j & \text{if } j = k, j \equiv m+1 \mod 2,\\ (-1)^{m+k+1}\cdot i_k \circ h_{k-1} \circ \dots \circ h_j \circ r_j & \text{if } j \leq k-1. \end{cases}$$

Define chain maps $f'\colon C \to C'$ and $g'\colon C' \to C$ by

$$f'_m\colon C_m \to C'_m = D_0 \oplus D_1 \oplus \dots \oplus D_m, \quad x \mapsto (0,0,\dots,i_m(x))$$

and

$$g'_m\colon C'_m = D_0 \oplus D_1 \oplus \dots \oplus D_m \to C_m,$$
$$(x_0,x_1,\cdots x_m) \mapsto \sum_{j=0}^{m} h_{m-1}\circ \dots \circ h_j \circ r_j(x_j).$$

We have $g' \circ f' = r \circ i$ and hence $h$ is a chain homotopy $g' \circ f' \simeq \mathrm{id}_C$. We obtain a chain homotopy $k' \colon f' \circ g' \simeq \mathrm{id}_{C'}$ if

$$k'_m \colon C'_m = D_0 \oplus D_1 \oplus \cdots \oplus D_m \to C'_{m+1} = D_0 \oplus D_1 \oplus \cdots \oplus D_m \oplus D_{m+1}$$

is the obvious inclusion.

Recall that $D$ is $N$-dimensional. Thus we get $C'_m = C'_N$ for $m \geq N$ and $c'_{m+1} = \mathrm{id} - c'_m$ for $m \geq N+1$. Since $c'_{m+1} \circ c'_m = 0$ for all $m$, we conclude that $c'_m \circ c'_m = c'_m$ for $m \geq N+1$. Hence $C'$ has the form

$$\cdots \to C'_N \xrightarrow{c'_{N+1}} C'_N \xrightarrow{\mathrm{id} - c'_{N+1}} C'_N \xrightarrow{c'_{N+1}} C'_N \xrightarrow{c'_N} C'_{N-1} \xrightarrow{c'_{N-1}} \cdots \xrightarrow{c'_1} C'_0 \to 0 \to \cdots .$$

Define an $N$-dimensional chain complex $D'$ over $\mathrm{Idem}(\mathcal{C})$ by

$$0 \to 0 \to (C'_N, \mathrm{id} - c_{N+1}) \xrightarrow{c'_N \circ i} C'_{N-1} \xrightarrow{c'_{N-1}} \cdots \xrightarrow{c'_1} C'_0 \to 0 \to \cdots ,$$

where $i \colon (C'_N, \mathrm{id} - c_{N+1}) \to C'_N$ is the obvious morphism in $\mathrm{Idem}(\mathcal{C})$ which is given by $\mathrm{id} - c_{N+1} \colon C'_N \to C'_N$. Let

$$u \colon D' \to C'$$

be the chain map for which $u_m$ is the identity for $m \leq N-1$, $u_N$ is

$$i \colon (C'_N, \mathrm{id} - c_{N+1}) \to C'_N,$$

and $u_m \colon 0 \to C_m$ is the canonical map for $m \geq N+1$. Let

$$v \colon C' \to D'$$

be the chain map which is given by the identity for $m \leq N-1$, by the canonical projection $C'_m \to 0$ for $m \geq N+1$ and for $m = N$ by the morphism $C_N \to (C'_N, \mathrm{id} - c_{N+1})$ defined by $\mathrm{id} - c_{N+1} \colon C'_N \to C'_N$. Obviously $v \circ u = \mathrm{id}_{D'}$. We obtain a chain homotopy $l' \colon \mathrm{id}_{C'} \sim u \circ v$ if we take $l_m = 0$ for $m \leq N$, $l_m = c'_{N+1}$ for $m \geq N, m - N \equiv 0 \mod 2$ and $l_m = 1 - c'_{N+1}$ for $m \geq N, m - N \equiv 1 \mod 2$.

Define the desired chain complex $P$ by $P := D'$. Define

$$f \colon C \to P$$

to be the composite $v \circ f'$. Define

$$g \colon P \to C$$

to be the composite $g' \circ u$. We obtain chain homotopies

$$k = v \circ h \circ u \colon f \circ g \simeq \mathrm{id}_P$$

and

$$l = h - g' \circ l' \circ f' \colon g \circ f \simeq \mathrm{id}_C .$$

LEMMA 8.4. *Let $\varepsilon > 0$ be given. Then there is an $N$-dimensional $\varepsilon$-controlled chain complex $D$ over $\mathcal{C}(X, d; \mathbb{Z})$, $\varepsilon$-controlled chain maps $i\colon C^{\mathrm{sing}\,\varepsilon}(X) \to D$, $r\colon D \to C^{\mathrm{sing},\varepsilon}(X)$ and an $\varepsilon$-controlled chain homotopy $h\colon r \circ i \simeq \mathrm{id}_C$.*

*Proof.* Since $(X, d)$ is a compact contractible $N$-dominated metric space, we can find a finite CW-complex $K$ of dimension $\leq N$ and maps $j\colon X \to K$ and $q\colon K \to X$ and an $\varepsilon$-controlled homotopy $H\colon q \circ j \simeq \mathrm{id}_X$. Because of Lemma 8.1 we can assume without loss of generality that $K$ is a finite simplicial complex of dimension $\leq N$.

Subdividing $K$, if necessary, we can assume that the diameter of the images of simplices of $K$ under $q$ are at most $\varepsilon$. Using $q$ we consider the simplicial chain complex $C(K)$ of $K$ as a chain complex over $\mathcal{C}(X, d; \mathbb{Z})$. Similarly, the subcomplex $C^{\mathrm{sing},\varepsilon}(K)$ of the singular chain complex spanned by singular simplices in $K$ whose image under $q$ has diameter $\leq \varepsilon$ is a chain complex over $\overline{\mathcal{C}}(X, d; \mathbb{Z})$. Analogously to the proof of [9, Lemma 6.9], one shows that

$$C^{\mathrm{sing},\epsilon}(X) \xrightarrow{j_*} C_*^{\mathrm{sing},2\epsilon}(K) \xrightarrow{q_*} C_*^{\mathrm{sing},2\epsilon}(X)$$

is well defined and that the composition is homotopic to the inclusion

$$\mathrm{inc}_*\colon C_*^{\mathrm{sing},\epsilon}(X) \to C_*^{\mathrm{sing},2\epsilon}(X)$$

by a chain homotopy that is $2\epsilon$-controlled. A slight modification of the proof of [9, Lemma 6.7(iii)] shows that the canonical chain map

$$a\colon C(K) \to C^{\mathrm{sing},2\epsilon}(K)$$

is a $2\varepsilon$-chain homotopy equivalence over $\mathcal{C}(X, d; \mathbb{Z})$. Let $b\colon C^{\mathrm{sing},2\epsilon}(K) \to C(K)$ be a $2\epsilon$-controlled chain homotopy inverse of $a$. Put $D := C(K)$. Define

$$i\colon C^{\mathrm{sing},\epsilon}(X) \to D$$

to be $b \circ j_*$. Define

$$r\colon D \to C^{\mathrm{sing},\epsilon}(X)$$

to be the composite of an $2\epsilon$-controlled inverse of $\mathrm{inc}_*$, $q_*$ and $a$. Then $i$ resp. $r$ are $3\varepsilon$ resp. $4\varepsilon$-controlled over $(X, d)$ and there exists a chain homotopy $h\colon r \circ i \simeq \mathrm{id}_C$ which is $5\varepsilon$-controlled over $(X, d)$. This finishes the proof since $\varepsilon$ is arbitrary. □

LEMMA 8.5. *Let $\varepsilon, \delta > 0$. Let $\varphi, \varphi'\colon X \to X$ be maps such that satisfying the following growth condition. If $d(x, y) \leq \varepsilon$, then $d(\varphi(x), \varphi(y)), d(\varphi'(x), \varphi'(y)) \leq \delta$. If $H\colon \varphi \simeq \varphi'$ is a homotopy, then there is a chain homotopy $H_*\colon \varphi_* \simeq \varphi'_*$ over $\overline{\mathcal{C}}(X, d; \mathbb{Z})$ such that*

$$\operatorname{supp} H_* \subseteq \{(H(x, t), y) \mid t \in [0, 1], d(x, y) \leq \varepsilon\}.$$

*(Here $\varphi_*, \varphi'_*\colon C^{\mathrm{sing},\varepsilon}(X) \to C^{\mathrm{sing},\delta}(X)$ denote the induced chain maps.)*

*Proof.* The usual construction of a chain homotopy associated to a homotopy $H$ uses suitable simplicial structures on $\Delta\times[0,1]$, but in general this yields only a chain homotopy between chain maps $C^{\mathrm{sing},\varepsilon}(X) \to C^{\mathrm{sing},\delta}(X)$ because we do control not the diameter of images of simplices in $\Delta\times[0,1]$, under $H\circ(\sigma\times\mathrm{id}_{[0,1]})$, where $\sigma\colon \Delta^n \to X$ is a singular simplex in $X$ (whose image has diameter $\leq \varepsilon$). This can be fixed using subdivisions. It is not hard to construct (by induction on $n$) for every such $\sigma$ a (possibly degenerate) simplicial structure $\tau_\sigma$ on $\Delta^n\times[0,1]$ with the following properties: the image of every simplex of $\tau_\sigma$ under $H\circ(\sigma\times\mathrm{id}_{[0,1]})$ has diameter $\leq\delta$, $\tau_\sigma$ is natural with respect to restriction to faces of $\sigma$, and $\tau_\sigma$ yields the standard simplicial structure on $\Delta^n\times\{0,1\}$. Degenerated simplices may appear for the following reason: in the induction step we need to extend a given simplicial structure on the boundary of $\Delta^n\times[0,1]$ to all of $\Delta^n\times[0,1]$. In order to arrange for the diameters of images of simplices to be small we may need to use barycentric subdivision, and this changes the given simplicial structure on the boundary. However, using degenerated simplices, we can interpolate between a simplex and its barycentric subdivision. □

LEMMA 8.6. *Let $S$ be finite subset of $G$ (containing $e$) and $(\varphi,H)$ be a homotopy $S$-action on $X$. Then there are maps $\alpha,\beta\colon(0,\infty)\to(0,\infty)$ such that the following hold*:

(i) *If $d(x,y)\leq\varepsilon$, $g\in S$, then $d(\varphi_g(x),\varphi_g(y))\leq\beta(\varepsilon)$; if $d(x,y)\leq\varepsilon$, $g,h\in S$ with $gh\in S$ and $t\in[0,1]$, then $d(H_{g,h}(x,t),H_{g,h}(y,t))\leq\beta(\varepsilon)$.*

(ii) $\lim_{\varepsilon\to 0}\beta(\varepsilon)=0$.

(iii) *If $d(x,y)\leq\alpha(\varepsilon)$, then $d(\varphi_g(x),\varphi_g(y))\leq\varepsilon$ for all $g\in S$.*

*Proof.* This is an easy consequence of the compactness of $X$ and $X\times[0,1]$. □

*Proof of Proposition* 7.2. Consider any $\epsilon>0$. Applying the construction from Remark 8.3 to $C:=C^{\mathrm{sing},\varepsilon}(X)$ and $D$, $i$, $r$ and $h$ as in the assertion of Lemma 8.4, we obtain a chain complex $P$ over $\mathrm{Idem}(\mathcal{C}(X,d;\mathbb{Z}))$, chain maps $f\colon C\to P$, $g\colon P\to C$ and chain homotopies $k\colon f\circ g\simeq\mathrm{id}_P$ $l\colon g\circ f\simeq\mathrm{id}_C$. By inspecting the formulas from Remark 8.3 we see that $P$, $f$, $g$, $k$ and $l$ are $(N+2)\epsilon$-controlled. (Here we use that control is additive under composition and that the sum of $\varepsilon$-controlled maps is again $\varepsilon$-controlled.) In particular this takes care of assertions (i) and (ii) since $\epsilon>0$ is arbitrary.

Next we define the desired homotopy $S$-action on $P$. Let $\alpha$ be the function from Lemma 8.6. Put

$$\delta:=\alpha(\varepsilon)\quad\text{ and }\quad\gamma:=\alpha(\delta)=\alpha\circ\alpha(\varepsilon).$$

In the sequel we abbreviate $C^\epsilon:=C^{\mathrm{sing},\epsilon}(X)$, $C^\delta:=C^{\mathrm{sing},\delta}(X)$, and $C^\gamma:=C^{\mathrm{sing},\gamma}(X)$. Let $(\varphi_h)_*$ be the chain map $C^\gamma\to C^\delta$, $C^\delta\to C^\epsilon$ or $C^\gamma\to C^\epsilon$

respectively induced by $\varphi_h\colon X \to X$. Let $r\colon C^\epsilon \to C^\gamma$, $r\colon C^\epsilon \to C^\delta$ and $r\colon C^\delta \to C^\gamma$ respectively be an $\epsilon$-controlled chain homotopy inverse of the inclusion $C^\gamma \to C^\epsilon$, $C^\delta \to C^\epsilon$ and $C^\gamma \to C^\delta$ respectively. For their existence, see [9, Lemma 6.7(i)]). For $h \in S$, define

$$\varphi_h^P\colon P \to P$$

to be the composite $P \xrightarrow{g} C^\epsilon \xrightarrow{r} C^\gamma \xrightarrow{(\varphi_h)_*} C^\epsilon \xrightarrow{f} P$, if $h \neq e$ and $\varphi_h^P = \mathrm{id}_P$ if $h = e$.

Recall that $r$, $g$ and $f$ are $\epsilon$-controlled. We have $(x,y) \in \operatorname{supp}((\varphi_h)_*)$ if and only if $y = \varphi_h(x)$. Consider $(x,y) \in \operatorname{supp}(\varphi_h^P)$. Then there exists $x_1$, $x_2$, $x_3$, $x_4$ and $x_5$ with $x = x_1$, $y = x_5$, $(x_1,x_2) \in \operatorname{supp}(g)$, $(x_2,x_3) \in \operatorname{supp}(r)$, $(x_3,x_4) \in \operatorname{supp}((\varphi_h)_*)$ and $(x_4,x_5) \in \operatorname{supp}(f)$. This implies $d(x,x_2) \leq \epsilon$, $d(x_2,x_3) \leq \epsilon$, $x_4 = \varphi_h(x_3)$ and $d(x_4,x_5) \leq \epsilon$. Using the function $\beta$ appearing in Lemma 8.6, we conclude that $d(\varphi_h(x),\varphi(x_4)) \leq \beta(2\epsilon)$ and hence $d(\varphi_h(x),y) \leq \beta(2\epsilon) + \varepsilon$. Thus we have shown $d(\varphi_h(x),y) \leq \beta(2\epsilon) + \varepsilon$ for $(x,y) \in \operatorname{supp}((\varphi_h)_*)$ if $h \in S$ and $(x,y) \in \operatorname{supp}(\varphi_h^P)$. Since $\epsilon > 0$ is arbitrary, and because of Lemma 8.6(ii), this takes care of assertion (iv).

Consider $h,k \in S$ with $hk \in S$. Consider the following diagram of chain maps of chain complexes over $\operatorname{Idem}(\mathcal{C}(X,d;\mathbb{Z}))$:

$$\begin{tikzcd}
P \arrow[r, "g"] & C^\epsilon \arrow[r, "r"] & C^\gamma \arrow[r, "(\varphi_k)_*"] \arrow[dr, "(\varphi_k)_*"'] \arrow[ddrr, bend right, "(\varphi_{hk})_*"'] & C^\epsilon \arrow[r, "f"] \arrow[d, "r"] \arrow[dr, "\mathrm{id}"] & P \arrow[d, "g"] \\
 & & & C^\delta \arrow[dr, "r"] \arrow[ddr, "(\varphi_h)_*"'] & C^\epsilon \arrow[l, "r"'] \arrow[d, "r"] \\
 & & & & C^\gamma \arrow[d, "(\varphi_h)_*"] \\
 & & & & C^\epsilon \arrow[d, "f"] \\
 & & & & P.
\end{tikzcd}$$

The chain maps $f$, $r$ and $g$ are $\epsilon$-controlled. For all triangles appearing in the above diagram we have explicit chain homotopies which make them commute up to homotopy: The homotopy for the triangle involving $(\varphi_k)_*$, $(\varphi_h)_*$ and $(\varphi_{hk})_*$ is induced by $H_{h,k}$; see Lemma 8.5. In particular, $\operatorname{supp}(H_{h,k})_* \subseteq \{(H(x,t),y) \mid t \in [0,1], d(x,y) \leq \varepsilon\}$. The chain homotopy for the triangle involving $f$, $g$ and id is the chain homotopy $l$ which is $\epsilon$-controlled. The chain homotopy for the triangle involving $r$, $r$ and id is the trivial one. The chain homotopy $K$ for the triangle involving $(\varphi_k)_*$, $(\varphi_k)_*$ and $r$ comes from a $\epsilon$-controlled chain homotopy from the composite $C^\delta \xrightarrow{i} C^\epsilon \xrightarrow{r} C^\delta$ for $i$ the inclusion to $\mathrm{id}\colon C^\delta \to C^\delta$. We have $\operatorname{supp} K \subseteq \{(x,y) \mid d(\varphi(y),x) \le \varepsilon\}$. We obtain $\epsilon$-controlled chain homotopies for the remaining triangles analogously. The composite obtained by going first horizontally from the left upper corner to the right upper corner and then vertically to the right lower corner is by definition $\varphi_h^P \circ \varphi_k^P$. The composite obtained by going diagonally from the left upper corner to the right lower corner is by definition $\varphi_{hk}^P$. Putting all these chain homotopies together yields a chain homotopy

$$H_{h,k}^P\colon \varphi_h^P \circ \varphi_k^G \simeq \varphi_{hk}^P$$

such that

$$\operatorname{supp} H_{h,k}^P \subseteq \{(x,y) \mid \exists t \in [0,1] : d(x, H(t,y)) \le \varepsilon'\},$$

where $\varepsilon' := (2\beta(\beta(\varepsilon)) + 3\beta(\varepsilon) + \varepsilon)$. (We leave the verification of this precise formula to the interested reader; note however that the precise formula is not important for us.) Since $\epsilon > 0$ is arbitrary and because of Lemma 8.6(ii), this takes care of assertion (v).

It remains to deal with assertion (iii). The inclusion $i\colon C^{\mathrm{sing},\epsilon}(X) \to C^{\mathrm{sing}}(X)$ is a chain homotopy equivalence (see [9, Lemma 6.7(i)]). Hence the composition $a\colon P \xrightarrow{g} C^{\mathrm{sing},\epsilon}(X) \xrightarrow{i} C^{\mathrm{sing}}(X)$ is a chain homotopy equivalence. One easily checks that $C^{\mathrm{sing}}(\varphi_g) \circ a \simeq a \circ \varphi_g^P$ holds for all $g \in G$. The inclusion $\{x_0\} \to X$ and augmentation induce chain maps $j\colon T_{x_0} \to C^{\mathrm{sing}}(X)$ and $q\colon C^{\mathrm{sing}}(X) \to T_{x_0}$. Obviously $q \circ j = \mathrm{id}_{T_{x_0}}$. Since $X$ is contractible, there is also a chain homotopy from $j \circ q$ to $\mathrm{id}_{C^{\mathrm{sing}}(X)}$. Obviously $q \circ C^{\mathrm{sing}}(\phi_g) = \phi_g^T \circ q$ for all $g \in S$. Hence the composite $q \circ a\colon P \to T$ is a chain homotopy equivalence of homotopy $S$-chain complexes over $\mathrm{Idem}(\mathcal{C}(X,d;\mathbb{Z}))$. This finishes the proof of Proposition 7.2. □

## 9. **The space** $P_2(X)$

*Summary.* In this section we introduce the space $P_2(X)$. As explained in the introduction, this space will be the fiber for the $L$-theory transfer. We also prove a number of estimates for specific metrics on $P_2(X)$, $G \times P_2(X)$ and $P_2(G \times X)$. These will be used later to produce a contracting map defined on $G \times P_2(X)$ using the contracting map defined on $G \times X$ from Proposition 3.9.

*Definition* 9.1 (The space $P_2(X)$). Let $X$ be a space.

(i) Let $P_2(X)$ denote the space of unordered pairs of points in $X$, i.e., $P_2(X) = X \times X/\sim$, where $(x,y) \sim (y,x)$ for all $x,y \in X$. We will use the notation $(x:y)$ for unordered pairs. Note that $X \mapsto P_2(X)$ is a functor.

(ii) If $d$ is a metric on $X$, then

$$d_{P_2(X)}((x:y),(x':y')) := \min\{d(x,x') + d(y,y'), d(x,y') + d(y,x')\}$$

defines a metric on $P_2(X)$.

LEMMA 9.2. *Let $\mathcal{F}$ be a family of subgroups of a group $G$. Denote by $\mathcal{F}_2$ the family of subgroups of $G$ which are contained in $\mathcal{F}$ or contain a member of $\mathcal{F}$ as subgroup of index two. Let $G$ act on a space $X$ such that all isotropy groups belong to $\mathcal{F}$. Then the isotropy groups for the induced action on $P_2(X)$ are all members of $\mathcal{F}_2$.*

*Proof.* Let $(x : y) \in P_2(X)$ and $g \in G_{(x:y)}$. Then either ($gx = x$ and $gy = y$), or ($gx = y$ and $gy = x$). Obviously $G_x \cap G_y \subseteq G_{(x:y)}$ and $G_x \cap G_y \in \mathcal{F}$. Hence it remains to show that the index of $G_x \cap G_y$ in $G_{(x:y)}$ is two if $G_x \cap G_y \neq G_{(x:y)}$. Choose $g_0 \in G_{(x:y)} \setminus G_x \cap G_y$. Then for every $g \in G_{(x:y)} \setminus G_x \cap G_y$, we have $gg_0 \in G_x \cap G_y$. □

*Remark* 9.3 (The role of $\mathcal{F}_2$). In $K$-theory one can replace the family $\mathcal{VC}\mathrm{yc}$ by the family $\mathcal{VC}\mathrm{yc}_I$ of subgroups which are either finite or virtually cyclic of type I, see [21], [22]. The corresponding result does not hold for $L$-theory: in the calculation of the $L$-theory of the infinite dihedral group nontrivial UNil-terms appear (see [16]). Hence in the proof of the $L$-theory case there must be an argument in the proof that does not appear in the $K$-theory case and where one in contrast to the $K$-Theory case needs to consider virtually cyclic groups of type II as well. This actually happens in Lemma 9.2, which forces us to replace $\mathcal{F}$ by $\mathcal{F}_2$.

In the $L$-theory case the situation is just the other way around. It turns out that one can ignore the virtually cyclic groups of type I (see [40, Lemma 4.2]) but not the ones of type II.

LEMMA 9.4. *Let $\Sigma$ be a finite-dimensional simplicial complex. Then $P_2(\Sigma)$ can be equipped with the structure of a simplicial complex such that*

(i) *For every simplicial automorphism $f$ of $\Sigma$, the induced automorphism $P_2(f)$ of $P_2(\Sigma)$ is simplicial.*

(ii) *For $\varepsilon > 0$, there is $\delta > 0$ depending only on $\varepsilon$ and the dimension of $\Sigma$ such that for $z, z' \in P_2(\Sigma)$,*

$$d_{P_2(\Sigma,d^1)}(z, z') \leq \delta \quad \Longrightarrow \quad d^1_{P_2(\Sigma)}(z, z') \leq \varepsilon,$$

*where $d^1_{P_2(\Sigma)}$ is the $l^1$-metric for the simplicial complex $P_2(\Sigma)$ and $d_{P_2(\Sigma,d^1)}$ is the metric induced from the $l^1$-metric on $\Sigma$; see Definition* 9.1(ii).

*Proof.* Let $\Sigma^1$ denote the first barycentric subdivision of $\Sigma$. The vertices of each simplex in $\Sigma^1$ are canonically ordered. Then $\Sigma \times \Sigma$ can be given a simplicial structure as follows. The set of vertices is $\Sigma^1(0) \times \Sigma^1(0)$, where $\Sigma^1(0)$ denotes the set of vertices of $\Sigma^1$. The simplices are of the form $\{(e_0, f_0), \ldots, (e_n, f_n)\}$ where

- $e_i, f_i \in \Sigma^1(0)$;
- $\Delta := \{e_0, \ldots, e_n\}$ and $\Delta' := \{f_0, \ldots, f_n\}$ are simplices of $\Sigma^1$;

- for $i = 1, \ldots, n$, we have $e_{i-1} \leq e_i$ and $f_{i-1} \leq f_i$ with respect to the order of the simplices of $\Delta$ and $\Delta'$.

The flip map $\Sigma \times \Sigma \to \Sigma \times \Sigma$, $(x, y) \mapsto (y, x)$, is a simplicial map. If for a simplex $\tau$ the interior of $\tau$ and the image of the interior of $\tau$ under the flip map have a nonempty intersection, then the flip map is already the identity on $\tau$. Thus we obtain an induced simplicial structure on $P_2(\Sigma)$. It is now easy to see that this simplicial structure has the required properties mentioned in (i).

It remains to prove assertion (ii). Fix $\epsilon > 0$. Let $\Delta_{4(\dim(\Sigma)+1)-1}$ be the simplicial complex given by the standard $(4(\dim(\Sigma)+1)-1)$-simplex. *A priori* we have four topologies on $P_2(\Delta_{4(\dim(\Sigma)+1)-1})$. The first one comes from the topology on $\Delta_{4(\dim(\Sigma)+1)-1}$, the second one from the simplicial structure on $P_2(\Delta_{4(\dim(\Sigma)+1)-1})$ constructed above, and the third and fourth come from the metrics $d^1_{P_2(\Delta_{4(\dim(\Sigma)+1)-1})}$ and $d_{P_2(\Delta_{4(\dim(\Sigma)+1)-1},d^1)}$. Since $P_2(\Delta_{4(\dim(\Sigma)+1)-1})$ is compact and hence locally finite, one easily checks that all these topologies agree. Since $P_2(\Delta_{4(\dim(\Sigma)+1)-1})$ is compact, we can find $\delta > 0$ such that for all $z, z' \in P_2(\Delta_{4(\dim(\Sigma)+1)-1})$,

$$d_{P_2(\Delta_{4(\dim(\Sigma)+1)-1},d^1)}(z, z') \leq \delta \implies d^1_{P_2(\Delta_{4(\dim(\Sigma)+1)-1})}(z, z') \leq \epsilon. \tag{9.5}$$

For the general case we make the following three observations. Firstly, the $l^1$-metric is preserved under inclusions of subcomplexes. Secondly, the construction of the simplicial structure on the product is natural with respect to inclusions of subcomplexes. Thirdly, for every choice of four points in $\Sigma$, there is a subcomplex with at most $4(\dim \Sigma + 1)$ vertices containing these four points. Since (9.5) holds for $\Delta_{4(\dim(\Sigma)+1)-1}$, it holds for $\Sigma$. $\square$

LEMMA 9.6. *Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces. Let $f\colon X \to Y$ be a map. Suppose that for $\delta, \epsilon > 0$, $d_Y(f(x), f(x')) \leq \epsilon/2$ holds for all $x, x' \in X$ which satisfy $d_X(x, x') \leq \delta$.*

*Then $d_{P_2(Y)}(P_2(f)(z), P_2(f)(z')) \leq \epsilon$ holds for all $z, z' \in P_2(X)$ which satisfy $d_{P_2(X)}(z, z') \leq \delta$.*

*Proof.* Suppose for $(x : x'), (y : y') \in P_2(X)$ that $d_{P_2(X)}((x : x'), (y : y')) \leq \delta$ holds. By definition we get that $d_X(x, y) + d_X(x', y') \leq \delta$ or $d_X(x, y') + d_X(x', y) \leq \delta$ holds. This implies that $d_X(x, y), d_X(x', y') \leq \delta$ or $d_X(x, y'), d_X(x', y) \leq \delta$ is valid. We conclude from the assumptions that $d_Y(f(x), f(y)) \leq \epsilon/2$ and $d_Y(f(x'), f(y')) \leq \epsilon/2$ hold or that $d_Y(f(x), f(y')) \leq \epsilon/2$ and $d_Y(f(x'), f(y)) \leq \epsilon/2$ hold. This implies that $d_Y(f(x), f(y)) + d_Y(f(x'), f(y')) \leq \epsilon$ or $d_Y(f(x), f(y')) + d_Y(f(x'), f(y)) \leq \epsilon$ is true. Hence

$$\begin{aligned} &d_{P_2(Y)}(P_2(f)(x : x'), P_2(f)(y : y')) = d_{P_2(Y)}(f(x) : f(x')), (f(y) : f(y'))) \\ &= \min\{d_Y(f(x), f(y)) + d_Y(f(x'), f(y')), d_Y(f(x), f(y')) + d_Y(f(y), f(x'))\} \\ &\leq \epsilon. \end{aligned}$$

$\square$

Let $S \subseteq G$ be a finite subset and $\Lambda > 0$. Let $(X,d)$ be a metric space with a homotopy $S$-action $(\phi, H)$. Since $P_2(X)$ is functorial in $X$ and there is a natural map $P_2(X) \times [0,1] \to P_2(X \times [0,1])$, we obtain an induced homotopy $S$-action $(P_2(\phi), P_2(H))$ on $P_2(X)$. Let $d_{S,\Lambda,G\times P_2(X)}$ be the metric on $G \times P_2(X)$ associated in Definition 3.4 to $(P_2(X), d_{P_2(X)})$ and the homotopy $S$-action $(P_2(\phi), P_2(H))$, where $d_{P_2(X)}$ has been introduced in Definition 9.1(ii) with respect to the given metric $d$ on $X$. Let $d_{S,\Lambda,G\times X}$ be the metric on $G\times X$ associated in Definition 3.4 to the given metric $d$ and homotopy $S$-action $(\phi, H)$ on $X$. Let $d_{S,\Lambda,P_2(G\times X)}$ be the metric on $P_2(G \times X)$ introduced in Definition 9.1(ii) with respect to the metric $d_{S,\Lambda,G\times X}$.

Lemma 9.7. *The map*

$$\omega\colon G \times P_2(X) \to P_2(G \times X), \quad (g,(x:y)) \mapsto ((g,x):(g,y))$$

*is well defined. For $(g,(x:x'))$ and $(h,(y:y'))$ in $G \times P_2(X)$, we have*

$$\begin{aligned} d_{S,\Lambda,P_2(G\times X)}&\left(\omega((g,(x:x'))), \omega((h,(y:y')))\right) \\ &\leq 2 \cdot d_{S,\Lambda,G\times P_2(X)}((g,(x:x')),(h,(y:y'))). \end{aligned}$$

*Proof.* Consider $(g,(x:x'))$ and $(h,(y:y'))$ in $G\times P_2(X)$. Consider $\epsilon > 0$. By definition we find $n \in \mathbb{Z}$, $n \geq 0$, elements $x_0, \dots, x_n$, $x'_0, \dots, x'_n$, $z_0, \dots, z_n$ and $z'_0, \dots, z'_n$ in $X$, elements $a_1, b_1, \dots, a_n, b_n$ in $S$ and maps $f_1, \widetilde{f}_1, \dots, f_n$, $\widetilde{f}_n\colon X \to X$ such that

$$\begin{aligned} &(x:x') = (x_0 : x'_0), (z_n : z'_n) = (y:y'),\\ &f_i \in F_{a_i}(\varphi, H), \widetilde{f}_i \in F_{b_i}(\varphi,H), P_2(f_i)(z_{i-1}:z'_{i-1}) = P_2(\widetilde{f}_i)(x_i:x'_i),\\ &h = g a_1^{-1} b_1 \dots a_n^{-1} b_n,\\ &n + \sum_{i=0}^{n} \Lambda \cdot d_{P_2(X)}((x_i:x'_i),(z_i:z'_i))\\ &\qquad \leq d_{S,\Lambda,G\times P_2(X)}((g,(x:x')),(h,(y:y'))) + \epsilon. \end{aligned}$$

Next we construct sequences of elements $x''_0, \dots, x''_n$, $z''_0, \dots, z''_n$, $x'''_0, \dots, x'''_n$ and $z'''_0, \dots, z'''_n$ in $X$ such that

$$\begin{gathered} (x''_i : x'''_i) = (x_i : x'_i), \quad (z''_i : z'''_i) = (z_i : z'_i),\\ d(x''_i, z''_i) + d(x'''_i, z'''_i) = d_{P_2(X)}\left((x_i:x'_i),(z_i:z'_i)\right),\\ f_i(z''_{i-1}) = \widetilde{f}_i(x''_i), \quad f_i(z'''_{i-1}) = \widetilde{f}_i(x'''_i). \end{gathered}$$

The construction is done inductively. Put $x''_0 := x$ and $x'''_0 := x'_0$.

Suppose that we have defined $x''_0$, $z''_0$, $\cdots$ ,$z''_{i-1}$, $x''_i$ and $x'''_0$, $z'''_0$, $\cdots$ ,$z'''_{i-1}$, $x'''_i$. We have to specify $z''_i$ and $z'''_i$. By definition,

$$d_{P_2(X)}\left((x_i:x'_i),(z_i:z'_i)\right) = \min\{d(x_i,z_i) + d(x'_i,z'_i), d(x_i,z'_i) + d(x'_i,z_i)\}.$$

If $x''_i = x_i$ and $d_{P_2(X)}\left((x_i : x'_i), (z_i : z'_i)\right) = d(x_i, z_i) + d(x'_i, z'_i)$ hold or if $x''_i = x'_i$ and $d_{P_2(X)}\left((x_i : x'_i), (z_i : z'_i)\right) = d(x_i, z'_i) + d(x'_i, z_i)$ hold, then put $z''_i := z_i$ and $z'''_i := z'_i$. If $x''_i = x_i$ and $d_{P_2(X)}\left((x_i : x'_i), (z_i : z'_i)\right) = d(x_i, z'_i) + d(x'_i, z_i)$ hold or if $x''_i = x'_i$ and $d_{P_2(X)}\left((x_i : x'_i), (z_i : z'_i)\right) = d(x_i, z_i) + d(x'_i, z'_i)$ hold, then put $z''_i := z'_i$ and $z'''_i := z_i$.

Suppose that we have defined $x''_0$, $z''_0$, $\cdots$, $x''_{i-1}$, $z''_{i-1}$ and $x'''_0$, $z'''_0$, $\cdots$, $x'''_{i-1}$, $z'''_{i-1}$. Then we have to specify $x''_i$ and $x'''_i$. Since $P_2(f_i)(z_{i-1} : z'_{i-1}) = P_2(\widetilde{f}_i)(x_i : x'_i)$, we have $f_i(z_{i-1}) = \widetilde{f}_i(x_i)$ and $f_i(z'_{i-1}) = \widetilde{f}_i(x'_i)$ or we have $f_i(z_{i-1}) = \widetilde{f}_i(x'_i)$ and $f_i(z'_{i-1}) = \widetilde{f}_i(x_i)$. In the first case, put $x''_i := x_i$ and $x'''_i := x'_i$ if $z''_{i-1} = z_{i-1}$ and put $x''_i = x'_i$ and $x'''_i = x_i$ if $z''_{i-1} = z'_{i-1}$. In the second case, put $x''_i := x'_i$ and $x'''_i := x_i$ if $z''_{i-1} = z_{i-1}$ and put $x''_i := x_i$ and $x'''_i := x'_i$ if $z''_{i-1} = z'_{i-1}$. This finishes the construction of the elements $x''_i$, $x'''_i$, $z''_i$ and $z'''_i$. One easily checks that the desired properties hold.

Put $y'' := z''_n$, $y''' := z'''_n$, $x'' := x''_0$ and $x''' := x'''_0$. From Definition 3.4 we conclude that

$$d_{S,\Lambda,G\times X}((g, x''), (h, y'')) \leq n + \sum_{i=0}^{n} \Lambda \cdot d(x''_i, z''_i),$$
$$d_{S,\Lambda,G\times X}(g, x'''), (h, y''')) \leq n + \sum_{i=0}^{n} \Lambda \cdot d(x'''_i, z'''_i).$$

This implies

$$\begin{aligned}
&d_{S,\Lambda,G\times X}((g, x''), (h, y'')) + d_{S,\Lambda,G\times X}(g, x'''), (h, y'''))\\
&\leq 2n + \sum_{i=0}^{n} \Lambda \cdot \left(d(x''_i, z''_i) + d(x'''_i, z'''_i)\right)\\
&\leq 2 \cdot \left(n + \sum_{i=0}^{n} \Lambda \cdot \left(d(x''_i, z''_i) + d(x'''_i, z'''_i)\right)\right).\\
&\leq 2 \cdot \left(n + \sum_{i=0}^{n} \Lambda \cdot d_{P_2(X)}\left((x_i : x'_i), (z_i : z'_i)\right)\right).\\
&\leq 2 \cdot \left(d_{S,\Lambda,G\times P_2(X)}((g, (x : x')), (h, (y : y'))) + \epsilon\right).
\end{aligned}$$

Since $\epsilon > 0$ was arbitrary, we conclude that

$$\begin{aligned}
d_{S,\Lambda,G\times X}((g, x''), (h, y'')) + d_{S,\Lambda,G\times X}(g, x'''), (h, y'''))&\\
\leq 2 \cdot d_{S,\Lambda,G\times P_2(X)}((g, (x : x')), (h, (y : y'))).&
\end{aligned}$$

This implies

$$\begin{aligned}
&d_{S,\Lambda,P_2(G\times X)}\left(\omega((g, (x : x'))), \omega((h, (y : y')))\right)\\
&\qquad = d_{S,\Lambda,P_2(G\times X)}\left(((g, x) : (g, x')), ((h, y) : (h, y'))\right)
\end{aligned}$$

$$\begin{aligned}
&= \min\{d_{S,\Lambda,G\times X}((g,x),(h,y)) + d_{S,\Lambda,G\times X}((g,x'),(h,y')),\\
&\qquad\qquad d_{S,\Lambda,G\times X}((g,x),(h,y')) + d_{S,\Lambda,G\times X}((g,x'),(h,y))\}\\
&\le d_{S,\Lambda,G\times X}((g,x''),(h,y'')) + d_{S,\Lambda,G\times X}(g,x'''),(h,y'''))\\
&\le 2\cdot d_{S,\Lambda,G\times P_2(X)}((g,(x:x')),(h,(y:y'))). \qquad \square
\end{aligned}$$

## 10. **The transfer in $L$-theory**

*Summary.* In this section we construct a controlled $L$-theory transfer. Its formal properties are similar to the $K$-theory case (see Proposition 10.3), but its construction is more complicated and uses the multiplicative hyperbolic Poincaré chain complex already mentioned in the introduction. This yields a suitable controlled symmetric form on the fiber $P_2(X)$ for the transfer, see Proposition 10.2. Here we make crucial use of the flexibility of algebraic $L$-theory. There are many more 0-dimensional Poincaré chain complexes than there are 0-dimensional manifolds.

Throughout this section we fix the following convention.

Convention 10.1. *Let*

- *$G$ be a group,*
- *$N \in \mathbb{N}$,*
- *$(X,d)$ be a compact contractible $N$-dominated metric space,*
- *$Y$ be a $G$-space,*
- *$\mathcal{A}$ be an additive $G$-category with involution.*

*We equip $X \times X$ with the metric $d_{X\times X}$, defined by*

$$d_{X\times X}((x_0,y_0),(x_1,y_1)) = d(x_0,y_0) + d(x_1,y_1) \quad \textit{for } (x_0,y_0),(x_1,y_1) \in X \times X.$$

Similar to the tensor product constructed in Section 6 there is a tensor product

$$\mathcal{C}(X,d;\mathbb{Z}) \otimes \mathcal{C}(X,d;\mathbb{Z}) \to \mathcal{C}(X\times X, d_{X\times X};\mathbb{Z})$$

induced by the canonical tensor product $\mathcal{F}(\mathbb{Z})\otimes\mathcal{F}(\mathbb{Z}) \to \mathcal{F}(\mathbb{Z})$. This tensor product is strictly compatible with the involution: we have $\operatorname{inv}(M\otimes N) = \operatorname{inv}(M)\otimes\operatorname{inv}(N)$ for objects $M$ and $N$, and similar $\operatorname{inv}(f\otimes g) = \operatorname{inv}(f)\otimes\operatorname{inv}(g)$ for morphisms $f$ and $g$. This tensor product is symmetric in the following sense. For objects $M$ and $N$ there is a canonical isomorphism $\operatorname{flip}_{M,N}\colon M\otimes N \to N\otimes M$. This tensor product has a canonical extension to the idempotent completions.

We fix sign conventions for the induced tensor product of chain complexes. If $C$ and $D$ are chain complexes (over $\mathcal{C}(X,d;\mathbb{Z})$) with differentials $d^C$ and $d^D$, then the differential $d^{C\otimes D}$ of the chain complex $C\otimes D$ (over $\mathcal{C}(X\times X, d_{X\times X};\mathbb{Z})$) is defined by $d^{C\otimes D}|_{C_p\otimes D_q} = d^C\otimes \mathrm{id}_{D_q} + (-1)^p \mathrm{id}_{C_p}\otimes d^D$. If $f\colon A\to C$ and

$g\colon B \to D$ are maps of chain complexes, then we define $f \otimes g\colon A \otimes B \to C \otimes D$ by setting $(f \otimes g)|_{A_p \otimes B_q} := (-1)^{|g| \cdot p} f|_{A_p} \otimes g|_{B_q}$ (where $|g|$ is the degree of $g$). The following flip will be important. For chain complexes $C$ and $D$ we define an isomorphism $\mathrm{flip}_{C,D}\colon C \otimes D \to D \otimes C$ by $(\mathrm{flip}_{C,D})|_{C_p \otimes C_q} := (-1)^{pq}\,\mathrm{flip}_{C_p,C_q}$.

Proposition 10.2. *Let $S$ be a finite subset of $G$ (containing $e$) such that $S = S^{-1}$; i.e., if $g \in S$, then $g^{-1} \in S$. Let $(\varphi, H)$ be a homotopy $S$-action on $X$. For every $\varepsilon > 0$, there exists a homotopy $S$-chain complex $\mathbf{D} = (D, \varphi^D, H^D)$ over $\mathrm{Idem}(\mathcal{C}(P_2(X), d_{P_2(X)}; \mathbb{Z}))$ together with a chain isomorphism $\mu = \mu^D\colon D^{-*} \to D$ over $\mathrm{Idem}(\mathcal{C}(P_2(X), d_{P_2(X)}; \mathbb{Z}))$ such that*

(i) *$D$ is concentrated in degrees $-N, \ldots, N$.*
(ii) *$D$ is $\varepsilon$-controlled.*
(iii) *There is a homotopy $S$-chain equivalence $f\colon \mathbf{P} \to \mathbf{T}_{x_0}$ to the trivial homotopy $S$-chain complex at $x_0 \in X$ such that $f \circ \mu^D \circ f^{-*}$ is the canonical identification of $T^{-*}$ with $T$.*
(iv) *If $g \in S$ and $(x, y) \in \operatorname{supp} \varphi^D_a$, then $d(x, P_2(\varphi_g)(y)) \le \varepsilon$.*
(v) *If $g, h \in S$ with $gh \in S$, and $(x, y) \in \operatorname{supp} H^P_{a,b}$, then there is $t \in [0, 1]$ such that $d(x, P_2(H_g(-, t))(y)) \le \varepsilon$.*
(vi) *$\operatorname{supp} \mu \subseteq \{(z, z) \mid z \in P_2(X)\}$, $\mu^{-*} = \mu$, $\mu \circ (\varphi^D_{g^{-1}})^{-*} = \varphi^D_g \circ \mu$ for all $g \in S$.*

The idea of the construction of $(\mathbf{D}, \mu)$ is very simple. We take $\mathbf{P}$ from Lemma 7.2 and define $(\mathbf{D}, \mu)$ as the multiplicative hyperbolic Poincaré chain complex on $\mathbf{P}$ viewed over $P_2(X)$.

*Proof of Proposition* 10.2. Let $\mathrm{pr}\colon X \times X \to P_2(X)$ be the obvious projection. Then

$$d_{P_2(X)}(\mathrm{pr}(x_0, y_0), \mathrm{pr}(x_1, y_1)) \le d_{X \times X}((x_0, y_0), (x_1, y_1))$$

holds for all $(x_0, y_0), (x_1, y_1) \in X \times X$.

Let $\mathbf{P} = (P, \varphi^P, H^P)$ be a homotopy $S$-chain complex over $\mathrm{Idem}(\mathcal{C}(X, d; \mathbb{Z}))$ fulfilling the assertions of Lemma 7.2 with respect to $\varepsilon' := \varepsilon/2$ in place of $\varepsilon$. We obtain a chain complex $P^{-*} \otimes P$ over $\mathrm{Idem}(\mathcal{C}(X \times X, d_{X \times X}; \mathbb{Z}))$. (At $(x, y) \in X \times X$, we have $(P^{-*} \otimes P)_{(x,y)} = (P^{-*})_x \otimes P_y$.) Define the chain complex $D$ over $\mathrm{Idem}(\mathcal{C}(P_2(X), d_{P_2(X)}; \mathbb{Z}))$ to be the image of $(P^{-*}) \otimes P$ under the functor $\mathrm{pr}_*$ from chain complexes over $\mathrm{Idem}(\mathcal{C}(X \times X, d_{X \times X}; \mathbb{Z}))$ to chain complexes over $\mathrm{Idem}(\mathcal{C}(P_2(X), d_{P_2(X)}; \mathbb{Z}))$ induced by pr. Hence for $(x : y) \in P_1(X)$, we have

$$D_{(x:y)} = \bigoplus_{\substack{(x', y') \in X \times X, \\ \mathrm{pr}(x', y') = (x:y)}} (P^{-*})_{x'} \otimes P_{y'}.$$

One easily checks that assertions (i) and (ii) are satisfied.

In the sequel we will define certain chain maps and homotopies on the level of $X \times X$. It is to be understood that we will apply the functor $\mathrm{pr}_*$ to obtain constructions over $P_2(X)$. We define the homotopy $S$-action by putting $\varphi_g^D := (\varphi_{g^{-1}}^P)^{-*} \otimes \varphi_g^P$ and $H_{g,h}^D := (H_{h^{-1},g^{-1}}^P)^{-*} \otimes (\varphi_g^P \circ \varphi_h^P) + (\varphi_{(gh)^{-1}}^P)^{-*} \otimes H_{g,h}^P$. We define $\mu \colon D^{-*} \to D$ as flip: $(P^{-*} \otimes P)^{-*} = P \otimes P^{-*} \to P^{-*} \otimes P$.

One easily checks that because of Proposition 7.2, assertions (iii), (iv) and (v) are satisfied.

Notice that the support of $\mu$ is contained in the subset $\Xi = \{((x,y),(x',y')) \mid x = y', x' = y\}$ of $(X \times X) \times (X \times X)$ and that the image of $\Xi$ under $\mathrm{pr} \times \mathrm{pr} \colon (X \times X) \times (X \times X) \to P_2(X) \times P_2(X)$ is contained in the diagonal $\{(z,z) \mid z \in P_2(X)\}$ of $P_2(X) \times P_2(X)$. Straightforward calculation shows that $\mu = \mu^{-*}$ and $\mu \circ (\varphi_g^D)^{-*} = (\varphi_g^D) \circ \mu$. This implies assertion (vi). $\square$

PROPOSITION 10.3. *Let $T \subset S$ be finite subsets of $G$ (both containing $e$) such that for $g, h \in T$, we have $gh \in S$. Assume $T = T^{-1}$; i.e., if $g \in T$, then $g^{-1} \in T$. Let $\alpha \colon M^{-*} \to M$ be a quadratic form such that $\alpha$ is a $T$-morphism in $\mathcal{O}^G(Y;\mathcal{A})$ and $\alpha + \alpha^*$ is an $T$-isomorphism in $\mathcal{O}^G(Y;\mathcal{A})$. Let $\Lambda > 0$. Then there is a 0-dimensional ultra-quadratic $(S,2)$-Poincaré complex $(C,\psi)$ over $\mathrm{Idem}(\mathcal{O}^G(Y, G \times P_2(X), d_{S,\Lambda,G\times P_2(X)};\mathcal{A}))$ which is concentrated in degrees $N, \dots, -N$ such that*

$$[p(C,\psi)] = [(M,\alpha)] \in L_0^{\langle 1 \rangle}(\mathrm{Idem}(\mathcal{O}^G(Y;\mathcal{A}))),$$

*where $p \colon \mathrm{Idem}(\mathcal{O}^G(Y, G \times P_2(X), d_{S,\Lambda,G\times P_2(X)};\mathcal{A})) \to \mathrm{Idem}(\mathcal{O}^G(Y;\mathcal{A}))$ is the functor induced by the projection $G \times P_2(X) \to \mathrm{pt}$.*

Recall from Section 9 that we use the metric $d_{P_2(X)}$ on $P_2(X)$ (see Definition 9.1(ii)) in order to construct the metric $d_{S,\Lambda,G\times P_2(X)}$ as in Definition 3.4. The proof will use a controlled version of the classical $L$-theory transfer; see A.2.

*Proof of Proposition* 10.3. Let $\varepsilon := 1/\Lambda$. Let $\mathbf{D} = (D, \varphi^D, H^D)$, $\mu^D \colon D^{-*} \to D$ satisfy the assertions of Proposition 10.2. We define $\widetilde{C} := M \otimes D$ and $\widetilde{\psi} := \mathrm{tr}^{\mathbf{D}} \alpha \circ (\mathrm{id}_{M^{-*}} \otimes \mu)$. Using Proposition 10.2(vi) we compute

$$\begin{aligned}
\widetilde{\psi} + \widetilde{\psi}^{-*} &= \sum_{a \in T} \alpha_a \otimes (\varphi_a^D \circ \mu^D) + (\alpha^*)_a \otimes ((\mu^D)^{-*} \circ ((\varphi^D)^{-*})_a) \\
&= \sum_{a \in T} \alpha_a \otimes (\varphi_a^D \circ \mu^D) + (\alpha^*)_a \otimes (\mu^D \circ (\varphi_{a^{-1}}^D)^{-*}) \\
&= \sum_{a \in T} \alpha \otimes (\varphi_a^D \circ \mu^D) + (\alpha^*)_a \otimes (\varphi_a^D \circ \mu^D) \\
&= \sum_{a \in T} (\alpha + \alpha^*)_a \otimes (\varphi_a^D \circ \mu^D) \\
&= \mathrm{tr}^{\mathbf{D}} (\alpha + \alpha^*)_a \circ (\mathrm{id} \otimes \mu^D).
\end{aligned}$$

Lemma 6.4 implies that $(\mathrm{id}\otimes(\mu^D)^{-1})\circ\mathrm{tr}^{\mathbf{D}}((\alpha+\alpha^*)^{-1})$ is a chain homotopy inverse for $\widetilde{\psi}+\widetilde{\psi}^{-*}$ with homotopies given by

$$\sum_{a,b\in T}((\alpha+\alpha^*)_a\circ((\alpha+\alpha^*)^{-1})_b)\otimes H^D_{a,b}$$

and

$$\sum_{a,b\in T}(((\alpha+\alpha^*)^{-1})_a\circ(\alpha+\alpha^*)_b)\otimes H^D_{a,b}.$$

We conclude from Proposition 10.2 that the pair $(\widetilde{C},\widetilde{\psi})$ is a 0-dimensional ultra-quadratic $(S,\varepsilon)$-Poincaré complex over $\mathrm{Idem}\,\mathcal{O}^G(Y,P_2(X),d_{P_2(X)};\mathcal{A})$.

Let $f\colon\mathbf{P}\to\mathbf{T}_{x_0}$ be the weak equivalence from assertion (iii) in Proposition 10.2. Let $q\colon\mathcal{O}^G(Y,P_2(X),d_{P_2(X)};\mathcal{A})\to\mathcal{O}^G(Y;\mathcal{A})$ be the functor induced by $P_2(X)\to\mathrm{pt}$. Then $\mathrm{id}_M\otimes f$ is a chain homotopy equivalence. Using assertion (iii) in Proposition 10.2, it is not hard to check that $(\mathrm{id}_M\otimes f)\circ\psi\circ(\mathrm{id}_M\otimes f)^{-*}$ is chain homotopic to $\mathrm{tr}^{\mathbf{T}_{x_0}}\alpha$. Note that $q(\mathrm{tr}^{\mathbf{T}_{x_0}}\alpha)=\alpha$ (up to a canonical isomorphism $q(M\otimes T)\cong M$). Using (4.12), we conclude that $q[(\widetilde{C},\widetilde{\psi})]=[(M,\alpha)]\in L_0^{\langle 1\rangle}(\mathrm{Idem}(\mathcal{O}^G(Y;\mathcal{A})))$.

Let $F\colon\mathcal{O}^G(Y,P_2(X),d_{P_2(X)};\mathcal{A})\to\mathcal{O}^G(Y,G\times P_2(X),d_{S,\Lambda};\mathcal{A})$ be the functor induced by the map $(g,x,e,t)\mapsto(g,g,x,e,t)$ and set $(C,\psi):=f(C,\psi)$. Since $p\circ F=q$, we have $[p(C,\psi)]=[(M,\alpha)]$. From the Definition 3.4 of $d_{S,\Lambda}$ and our choice of $\varepsilon$ it follows that $(C,\psi)$ is a 0-dimensional ultra-quadratic $(S,2)$-Poincaré complex over $\mathrm{Idem}\,\mathcal{O}^G(Y,G\times P_2(X),d_{S,\Lambda};\mathcal{A})$. □

## 11. Proof of Theorem 1.1

*Proof of Theorem* 1.1(i). Because of Lemma 2.3(v) we can assume without loss of generality that $G$ is finitely generated. Let $N$ be the number appearing in Definition 1.8. According to Theorem 5.2(i) it suffices to show $K_1(\mathcal{O}^G(E_{\mathcal{F}}G;\mathcal{A}))=0$ for every additive $G$-category $\mathcal{A}$. Fix such an $\mathcal{A}$. Consider $a\in K_1(\mathcal{O}^G(E_{\mathcal{F}}G;\mathcal{A}))$. Pick an automorphism $\alpha\colon M\to M$ in $\mathcal{O}^G(E_{\mathcal{F}}G;\mathcal{A}))$ such that $[(M,\alpha)]=a$. By definition $\alpha$ is an $T$-automorphism for some finite subset $T$ of $G$ (containing $e$). We can assume without loss of generality that $T$ generates $G$, otherwise we enlarge $T$ by a finite set of generators. Set $S:=\{ab\mid a,b\in T\}$. Let $\varepsilon=\varepsilon(N,\mathcal{A},G,\mathcal{F},S)$ be the number appearing in Theorem 5.3(i). Set $\beta:=2$. Let $(X,d)$,$(\varphi,H)$, $\Lambda$, $\Sigma$ and $f$ be as in Proposition 3.9. Consider the following commuting diagram of functors:

$$\begin{tikzcd}
\mathcal{O}^G(E_{\mathcal{F}}G,G\times X,d_{S,\Lambda};\mathcal{A}) \arrow[rr, "F"] \arrow[dr, "p"'] & & \mathcal{O}^G(E_{\mathcal{F}}G,\Sigma,d^1;\mathcal{A}) \arrow[dl, "q"] \\
& \mathcal{O}^G(E_{\mathcal{F}}G;\mathcal{A}), &
\end{tikzcd}$$

where $p$ resp. $q$ are induced by projecting $G \times X$ resp. $\Sigma$ to a point and $F$ is induced by $f$. By Proposition 7.3, there is a $(\beta, S)$-chain homotopy equivalence $[(C, \hat{\alpha})]$ over $\mathrm{Idem}(\mathcal{O}^G(E_{\mathcal{F}}, G \times X, d_{S,\Lambda}; \mathcal{A}))$ such that $[p(C, \hat{\alpha})] = a$. Proposition 3.9(ii) implies that $F(\hat{\alpha})$ is an $(\varepsilon, S)$-chain homotopy equivalence over $\mathrm{Idem}(\mathcal{O}^G(E_{\mathcal{F}}G, \Sigma, d^1; \mathcal{A}))$. By Theorem 5.3(i), $[F(C, \hat{\alpha})] = 0$. Therefore $a = [q \circ F(C, \hat{\alpha})] = 0$. $\square$

We record the following corollary to Theorem 1.1(i).

Corollary 11.1. *Let $G$ be a finitely generated group that is transfer reducible over $\mathcal{F}$. Let $\mathcal{A}$ be an additive $G$-category. Then for $i \leq 1$, we have*

$$K_i(\mathcal{O}^G(E_{\mathcal{F}_2}(G); \mathcal{A})) = 0.$$

*Proof.* Clearly $G$ is also transfer reducible over $\mathcal{F}_2$. Therefore, by Theorem 1.1(i), the assembly map (1.2) is an isomorphism for $n < 1$ and surjective for $n = 1$. The result follows because the $K$-theory of $\mathcal{O}^G(E_{\mathcal{F}_2}(G); \mathcal{A})$ is the cofiber of this assembly map; compare [9, §3.3]. $\square$

*Proof of Theorem* 1.1(ii). Because of Lemma 2.3(v), we can assume without loss of generality that $G$ is finitely generated. Let $N$ be the number appearing in Definition 1.8. According to Theorem 5.2(ii) it suffices to show $L_0^{\langle -\infty \rangle}(\mathcal{O}^G(E_{\mathcal{F}_2}G; \mathcal{A})) = 0$ for every additive $G$-category $\mathcal{A}$ with involution. Fix such an $\mathcal{A}$. By (4.14) and Corollary 11.1, we know that

$$L_0^{\langle 1 \rangle}(\mathcal{O}^G(E_{\mathcal{F}_2}G; \mathcal{A})) \to L_0^{\langle -\infty \rangle}(\mathcal{O}^G(E_{\mathcal{F}_2}G; \mathcal{A}))$$

is an isomorphism. It suffices to show that this map is also the zero map.

Consider $a \in L^{\langle 1 \rangle}(\mathcal{O}^G(E_{\mathcal{F}_2}G; \mathcal{A}))$. Pick a quadratic form $(M, \alpha)$ over the category $\mathcal{O}^G(E_{\mathcal{F}_2}G; \mathcal{A}))$ such that $[(M, \alpha)] = a$. By definition there is a finite subset $T$ of $G$ (containing $e$) such that $\alpha$ is $T$-controlled and $\alpha + \alpha^*$ is a $T$-isomorphism. We can assume without loss of generality that $T = T^{-1}$ and that $T$ generates $G$. Set $S := \{ab \mid a, b \in T\}$. Let $\epsilon = \epsilon(2N, \mathcal{A}, G, \mathcal{F}_2, S)$ be the number appearing in Theorem 5.3(ii). By Lemma 9.4(ii) there is $\delta$ such that for every simplicial complex $\Sigma'$ of dimension $\leq N$, we have

$$d_{P_2(\Sigma', d^1)}(x, y) \leq \delta \quad \Longrightarrow \quad d^1_{P_2(\Sigma')}(x, y) \leq \varepsilon$$

for $x, y \in P_2(\Sigma')$. Set $\beta := 2$. Let $(X, d)$,$(\varphi, H)$, $C$, $\Sigma$ and $f$ be as in Proposition 3.9, but with respect to $\delta/2$ in place of $\epsilon$ and $2\beta$ instead of $\beta$. In particular, we have

$$d_{S,\Lambda,G \times X}((g, x), (h, y) \leq 2\beta \quad \Longrightarrow \quad d^1_{\Sigma}(f(g, x), f(h, y)) \leq \delta/2$$

for $(g, x), (h, y) \in G \times X$. We conclude from Lemma 9.6 that for $z, z' \in P_2(G \times X)$, we have

$$d_{S,\Lambda,P_2(G \times X)}(z, z') \leq 2\beta \quad \Longrightarrow \quad d_{P_2(\Sigma, d^1)}(P_2(f)(z), P_2(f)(z') \leq \delta.$$

Let $\widehat{f}\colon G \times P_2(X) \to P_2(\Sigma)$ be the composite of $P_2(f)$ with the map $\omega\colon G \times P_2(X) \to P_2(G \times X)$ defined in Lemma 9.7. Because of Lemma 9.7, we have

$$\begin{aligned} &d_{S,\Lambda,G\times P_2(X)}((g,(x:x')),(h,(y:y'))) \le \beta \\ &\qquad\qquad\Longrightarrow \quad d_{S,\Lambda,P_2(G\times X)}(\omega(g,(x:x')),\omega(h,(y:y'))) \le 2\beta \end{aligned}$$

for $(g,(x:x')),(h,(y:y')) \in G \times P_2(X)$. We conclude that for $(g,(x:x'),$ $(h,(y:y') \in G \times P_2(X)$,

$$\begin{aligned} (11.2)\quad &d_{S,\Lambda,G\times P_2(X)}((g,(x:x')),(h,(y:y'))) \le \beta \\ &\qquad\qquad\Longrightarrow \quad d^1_{P_2(\Sigma)}(\widehat{f}(g,(x:x'),\widehat{f}(h,(y:y')) \le \epsilon. \end{aligned}$$

Consider the following commuting diagram of functors:

$$\begin{array}{ccc} \mathcal{O}^G(E_{\mathcal{F}_2}(G), G\times P_2(X), d_{S,\Lambda,G\times P_2(X)};\mathcal{A}) & \xrightarrow{F} & \mathcal{O}^G(E_{\mathcal{F}_2}(G), P_2(\Sigma), d^1_{P_2(\Sigma)};\mathcal{A}) \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & \mathcal{O}^G(E_{\mathcal{F}_2}(G);\mathcal{A}), & \end{array}$$

where $p$ resp. $q$ are induced by projecting $G\times P_2(X)$ resp. $P_2(\Sigma)$ to a point and $F$ is induced by $\widehat{f}$. By Proposition 10.3 there is a 0-dimensional ultra-quadratic $(S,\beta)$-Poincaré complex $(C,\psi)$ over

$$\operatorname{Idem}(\mathcal{O}^G(E_{\mathcal{F}_2}(G), G\times P_2(X), d_{S,\Lambda,G\times P_2(X)};\mathcal{A}))$$

concentrated in degrees $-N,\dots,N$ such that

$$[p(C,\psi)] = [(M,\alpha)] \in L_0^{\langle 1\rangle}(\operatorname{Idem}(\mathcal{O}^G(E_{\mathcal{F}_2}(G);\mathcal{A}))).$$

We conclude from (11.2) that $F(C,\psi)$ is an $(S,\varepsilon)$-Poincaré complex. From Theorem 5.3(ii) and Lemma 9.2 we deduce that

$$[F(C,\psi)] = 0 \in L_0^{\langle -\infty\rangle}(\mathcal{O}^G(E_{\mathcal{F}_2}(G), P_2(\Sigma), d^1;\mathcal{A})).$$

Therefore $a = [q\circ F(C,\psi)] = 0$ holds in $L_0^{\langle -\infty\rangle}(\mathcal{O}^G(E_{\mathcal{F}_2}(G);\mathcal{A}))$. □

## Appendix A. Classical transfers and the multiplicative hyperbolic form

*Summary.* In this appendix we review classical (uncontrolled) transfers. We also discuss the multiplicative hyperbolic form in an uncontrolled context and show that this construction yields a homomorphism from $K_0(\Lambda)$ to $L^0_p(\Lambda)$.

A.1. *Transfer for the Whitehead group.* We briefly review the transfer for the Whitehead group for a fibration $F \to E \xrightarrow{p} B$ of connected finite CW-complexes. For simplicity we will assume that $\pi_1(p)\colon \pi_1(E) \to \pi_1(B)$ is bijective and we will identify in the sequel $G := \pi_1(E) = \pi_1(B)$.

Recall that the fiber transport gives a homomorphism of monoids $G \to [F, F]$. Thus we obtain a finite-free $\mathbb{Z}$-chain complex $C = C_*(F)$; namely, the cellular $\mathbb{Z}$-chain complex of $F$, together with an operation of $G$ up to chain homotopy, i.e., a homomorphism of monoids $\rho\colon G \to [C, C]_{\mathbb{Z}}$ to the monoid of chain homotopy classes of $\mathbb{Z}$-chain maps $C \to C$ (compare [38, §5]). An algebraic transfer map $p^*\colon \mathrm{Wh}(B) \to \mathrm{Wh}(E)$ in terms of chain complexes is given in [37, §4]. We recall its definition in the special case, where $\pi_1(p)$ is bijective.

Given an element $a = \sum_{g\in G} \lambda_g g \in \mathbb{Z}G$, define a $\mathbb{Z}G$-chain map of finitely generated free $\mathbb{Z}G$-chain complexes, unique up to $\mathbb{Z}G$-chain homotopy, by

$$a \otimes_t C\colon \mathbb{Z}G \otimes_{\mathbb{Z}} C \to \mathbb{Z}G \otimes_{\mathbb{Z}} C, \quad g' \otimes x \mapsto \sum_{g\in G} \lambda_g \cdot g'g^{-1} \otimes r(g)(x), \tag{A.1}$$

where $r(g)\colon C \to C$ is some representative of $\rho(g)$ (compare [38, §5]). Thus we obtain a ring homomorphism $\mathbb{Z}G \to [\mathbb{Z}G \otimes_{\mathbb{Z}} C, \mathbb{Z}G \otimes_{\mathbb{Z}} C]_{\mathbb{Z}G}$ to the ring of $\mathbb{Z}G$-chain homotopy classes of $\mathbb{Z}G$-chain maps $\mathbb{Z}G \otimes_{\mathbb{Z}} C \to \mathbb{Z}G \otimes_{\mathbb{Z}} C$. It extends in the obvious way to matrices over $\mathbb{Z}G$; namely, for a matrix $A \in M_{m,n}(\mathbb{Z}G)$ we obtain a $\mathbb{Z}G$-chain map, unique up to $G$-homotopy,

$$A \otimes_t C\colon \mathbb{Z}G^m \otimes_{\mathbb{Z}} C \to \mathbb{Z}G^n \otimes_{\mathbb{Z}} C.$$

The algebraic transfer $p^*\colon \mathrm{Wh}(G) \to \mathrm{Wh}(G)$ sends the class of an invertible matrix $A \in GL_n(\mathbb{Z}G)$ to the Whitehead torsion of the $\mathbb{Z}G$-self-chain homotopy equivalence $A \otimes_t C\colon \mathbb{Z}G^n \otimes_{\mathbb{Z}} C \to \mathbb{Z}G^n \otimes_{\mathbb{Z}} C$.

A.2. *Classical $L$-theory transfer.* To obtain an $L$-theory transfer, we have additionally to assume that $F$ is a finite $n$-dimensional Poincaré complex. For simplicity we assume that $F$ is an oriented $n$-dimensional Poincaré complex and the fiber transport $G \to [F, F]$ takes values in homotopy classes of orientation preserving self-homotopy equivalences and — as before — that $\pi_1(p)$ is bijective. We review the algebraically defined transfer maps $p^*\colon L_m(\mathbb{Z}G) \to L_{m+n}(\mathbb{Z}G)$ (see [41]). Because $F$ is a Poincaré complex, there is a symmetric form $\varphi\colon C^{-*} \to C$, where $C^{-*}$ denotes the dual of the cellular chain complex of $F$, i.e., $(C^{-*})_n = (C_{-n})^*$. If $\psi\colon M^* \to M$ is a quadratic form over $\mathbb{Z}[G]$, then the composition

$$\psi \otimes_t (C, \varphi)\colon (M \otimes C)^{-*} \cong M^* \otimes C^{-*} \xrightarrow{\mathrm{id} \otimes \varphi} M^* \otimes C \xrightarrow{\psi \otimes_t C} M \otimes C$$

defines an ultra-quadratic form on $M \otimes C$. The $L$-theory transfer sends the class of $(M, \psi) \in L_0(\mathbb{Z}G)$ to the class of $(M \otimes C, \psi \otimes_t (C, \varphi))$.

A.3. *The multiplicative hyperbolic form.* Let $\Lambda$ be a commutative ring. Let $P$ be a finitely generated projective $\Lambda$-module. Since $\Lambda$ is commutative, the dual $\Lambda$-module $P^* = \hom_\Lambda(P;\Lambda)$ and the tensor product $P \otimes_\Lambda P^*$ are finitely generated projective $\Lambda$-modules. Define the $\Lambda$-isomorphism

$$\psi_P \colon P^* \otimes_\Lambda P \to (P^* \otimes_\Lambda P)^* \tag{A.2}$$

by sending $\alpha \otimes x \in P^* \otimes_\Lambda P$ to the $\Lambda$-homomorphisms $P^* \otimes_\Lambda P \to \Lambda$, $\beta \otimes y \mapsto \alpha(y) \cdot \beta(x)$. The composite

$$P^* \otimes_\Lambda P \cong ((P^* \otimes_\Lambda P)^*)^* \xrightarrow{\psi_P^*} (P^* \otimes_\Lambda P)^*$$

agrees with $\psi_P$. Hence $(P^* \otimes_\Lambda P, \psi_P)$ is a nonsingular symmetric $\Lambda$-form. We call it the *multiplicative hyperbolic symmetric* $\Lambda$*-form* associated to the finitely generated projective $\Lambda$-module $P$ and denote it by $H_\otimes(P)$. Ian Hambleton pointed out that under the identification $P^* \otimes_\Lambda P \cong \mathrm{End}_\Lambda(P, P)$ this form corresponds to the trace form $(A, B) \mapsto \mathrm{tr}(AB)$. The name multiplicative hyperbolic form comes from the fact that it is the obvious multiplicative version of the standard hyperbolic symmetric form $H(P)$ which is given by the $\Lambda$-isomorphism $P^* \oplus P \to (P^* \oplus P)^*$ sending $(\alpha, x) \in P^* \oplus_\Lambda P$ to the $\Lambda$-homomorphisms $P^* \oplus P \to \Lambda$, $(\beta, y) \mapsto \alpha(y) + \beta(x)$; just replace $\oplus$ by $\otimes$ and $+$ by $\cdot$.

The hyperbolic symmetric form of a finitely generated projective $\Lambda$-module $P$ represents zero in the symmetric $L$-group $L^0_p(\Lambda)$. This is not true for the multiplicative version. Namely, define a homomorphism

$$H^\Lambda_\otimes \colon K_0(\Lambda) \to L^0_p(\Lambda) \tag{A.3}$$

by sending the class $[P]$ of a finitely generated projective $\Lambda$-module $P$ to the class $[H_\otimes(P)]$ of the nonsingular symmetric $\Lambda$-form $H_\otimes(P)$. We have to show that this is well defined; i.e., we must prove

$$[H_\otimes(P \oplus Q)] = [H_\otimes(P)] + [H_\otimes(P)] \in L^0_p(\Lambda)$$

for two finitely generated projective $\Lambda$-modules $P$ and $Q$. This follows from the fact that we have an isomorphism of $\Lambda$-modules

$$\begin{aligned}(P \oplus Q)^* \otimes_\Lambda P \oplus Q &\cong P^* \otimes_\Lambda P \oplus Q^* \otimes_\Lambda Q \oplus Q^* \otimes_\Lambda P \oplus P^* \otimes_\Lambda Q \\ &\cong P^* \otimes_\Lambda P \oplus Q^* \otimes_\Lambda Q \oplus (Q^* \otimes_\Lambda P \oplus (Q^* \otimes_\Lambda P)^*)\end{aligned}$$

which induces an isomorphism of nonsingular symmetric $\Lambda$-forms

$$H_\otimes(P \oplus Q) \cong H_\otimes(P) \oplus H_\otimes(Q) \oplus H(Q^* \otimes_\Lambda P).$$

Since $\Lambda$ is commutative, the tensor product $\otimes_\Lambda$ induces the structure of a commutative ring on $K_0(\Lambda)$ and $L^0_p(\Lambda)$. One easily checks that the map $H^\Lambda_\otimes$ of (A.3) is a ring homomorphism.

*Example* A.4 ($\Lambda = \mathbb{Z}$). If we take, for instance, $\Lambda = \mathbb{Z}$, we obtain isomorphisms

$$\mathrm{rk}\colon K_0(\mathbb{Z}) \xrightarrow{\cong} \mathbb{Z},$$
$$\mathrm{sign}\colon L_p^0(\mathbb{Z}) \xrightarrow{\cong} \mathbb{Z}$$

by taking the rank of a finitely generated free abelian group and the signature (see [53, Prop. 4.3.1, p. 419]). Obviously

$$H_\otimes^{\mathbb{Z}}\colon K_0(\mathbb{Z}) \to L_p^0(\mathbb{Z})$$

sends $[\mathbb{Z}]$ to the class of the symmetric form $\mathbb{Z} \to \mathbb{Z}^*$ sending $1 \in \mathbb{Z}$ to the identity $\mathrm{id}_{\mathbb{Z}} \in \mathbb{Z}^*$. Hence $H_\otimes^{\mathbb{Z}}$ is a bijection.

A.4. *The chain complex version of* $H_\otimes$. Next we give a chain complex version of this construction. Let $C$ be a finite projective $\Lambda$-chain complex, i.e., a $\Lambda$-chain complex such that each $\Lambda$-module $C_i$ is finitely generated projective and $C_i$ is nontrivial for only finitely many $i \in \mathbb{Z}$.

Given two $\Lambda$-chain complexes $C$ and $D$, define their *tensor product*

$$(C \otimes_\Lambda D, c \otimes_\Lambda d)$$

to be the $\Lambda$-chain complex whose $n$-th-chain module is $\bigoplus_{i,j,i+j=n} C_i \otimes_\Lambda D_j$. The differential is given by the formula

$$(c \otimes d)(x \otimes y) := c(x) \otimes y + (-1)^{|x|} \cdot x \otimes d(y).$$

We need $\Lambda$ to be commutative to ensure that $C \otimes_\Lambda D$ is indeed a $\Lambda$-chain complex. If $C$ and $D$ are finite projective $\Lambda$-chain complexes, then $C \otimes_\Lambda D$ is a finite projective $\Lambda$-chain complex.

If $f\colon A \to C$ and $g\colon B \to D$ are maps of chain complexes of degree $|f|$ and $|g|$, then we define $f \otimes g\colon A \otimes B \to C \otimes D$ by $(f \otimes g)(x \otimes y) := (-1)^{|g|\cdot|x|} f(x) \otimes g(y)$. The flip isomorphism

$$\mathrm{flip}\colon C \otimes_\Lambda D \xrightarrow{\cong} D \otimes_\Lambda C$$

is given by $\mathrm{flip}(x \otimes y) := (-1)^{|x|\cdot|y|} \cdot y \otimes x$. For chain complexes $C$ and $D$, we define a chain map

$$\mu_{C,D}\colon C^{-*} \otimes D^{-*} \to (C \otimes D)^{-*}$$

by $\mu_{C,D}(\alpha \otimes \beta)(a \otimes b) := (-1)^{|\beta|\cdot|a|}\alpha(a)\beta(b)$. Then $\otimes$ and ${}^{-*}$ are compatible in the following sense. If $f\colon A \to C$ and $g\colon B \to D$ are maps of chain complexes, then $\mu_{A,B} \circ (f^{-*} \otimes g^{-*}) = (f \otimes g)^{-*} \circ \mu_{C,D}$. If $C$ and $D$ are finite projective $\Lambda$-chain complexes, then $\mu_{C,D}$ is an isomorphism and yields a canonical identification. Suppressing this identification, the formula reads $f^{-*} \otimes g^{-*} = (f \otimes g)^{-*}$. Define

$$\mu_C\colon C \otimes_\Lambda C^{-*} \xrightarrow{\cong} (C^{-*} \otimes_\Lambda C)^{-*}$$

to be the composite

$$C \otimes_\Lambda C^{-*} \xrightarrow{\cong} (C^{-*})^{-*} \otimes_\Lambda C^{-*} \xrightarrow{\mu_{C^{-*},C}} (C^{-*} \otimes_\Lambda C)^{-*}.$$

Explicitly $\mu_C$ sends $x \otimes \alpha \in C_i \otimes_\Lambda (C_{-j})^*$ to the $\Lambda$-map $C_i^* \otimes C_{-j} \to \Lambda$, $\beta \otimes y \mapsto \beta(x) \cdot \alpha(y)$, where we think of $\hom_\Lambda(C_i^* \otimes C_{-j}; \Lambda)$ as a submodule of the $(i+j)$-th chain module of $(C^{-*} \otimes_\Lambda C)^{-*}$ in the obvious way. Define an isomorphism of $\Lambda$-chain complexes

$$\psi_C \colon (C^{-*} \otimes_\Lambda C)^{-*} \xrightarrow{\cong} C^{-*} \otimes_\Lambda C \tag{A.5}$$

by the composition of the inverse of $\mu_C$ with the flip isomorphism flip: $C \otimes_\Lambda C^{-*} \xrightarrow{\cong} C^{-*} \otimes_\Lambda C$. It is straightforward to check that $(C^{-*} \otimes_\Lambda C, \psi_C)$ is a 0-dimensional symmetric Poincaré $\Lambda$-chain complex. We call it the *multiplicative hyperbolic symmetric Poincaré $\Lambda$-chain complex* associated to the finite projective $\Lambda$-chain complex $C$ and denote it by $H_\otimes(C)$.

Given a finite projective $\Lambda$-chain complex $C$, define its (*unreduced*) *finiteness obstruction* to be

$$o(C) = \sum_{n \in \mathbb{Z}} (-1)^n \cdot [C_n] \quad \in K_0(R).$$

Lemma A.6. *Let $C$ be a finite projective $\Lambda$-chain complex. Then the homomorphism defined in* (A.3),

$$H_\otimes^\Lambda \colon K_0(\Lambda) \to L_p^0(\Lambda),$$

*sends $o(C)$ to the class $[H_\otimes(C)]$ of the symmetric 0-dimensional Poincaré $\Lambda$-chain complex $H_\otimes(C)$.*

*Proof.* Let $(C^{-*} \otimes_\Lambda C)_{\geq 1}$ be the $\Lambda$-chain complex which has in dimension $n \geq 1$ the same chain modules as $C^{-*} \otimes_\Lambda C$, whose differentials in dimensions $n \geq 2$ are the same as the one for $C^{-*} \otimes_\Lambda C$, and whose chain modules in dimensions $\leq 0$ are trivial. Let $p \colon C^{-*} \otimes_\Lambda C \to (C^{-*} \otimes_\Lambda C)_{\geq 1}$ be the obvious surjective $\Lambda$-chain map. Since the composite $p \circ \psi_C \circ p^{-*}$ is trivial, we can perform algebraic surgery in the sense of Ranicki (see [53, §1.5]) on $p$ to obtain a new symmetric 0-dimensional Poincaré chain complex $(D, \psi)$ such that $H_\otimes(C)$ and $(D, \psi)$ are algebraically bordant and hence represent the same class in $L_p^0(\Lambda)$. Since $\psi_C$ is a chain isomorphism, one easily checks that $(D, \psi)$ is $\Lambda$-chain homotopy equivalent to the 0-dimensional Poincaré complex whose underlying $\Lambda$-chain complex is concentrated in dimension zero and given there by the $\Lambda$-module $(C^{-*} \otimes_\Lambda C)_0$ and whose Poincaré $\Lambda$-chain homotopy equivalence is the inverse of the $\Lambda$-isomorphism $\psi_{(C^{-*} \otimes_\Lambda C)_0}$. Hence the class $[H_\otimes(C)] \in L_p^0(\Lambda)$ of $H_\otimes(C)$ corresponds to the nonsingular symmetric $\Lambda$-form

$$\psi_{(C^{-*} \otimes_\Lambda C)_0} \colon (C^{-*} \otimes_\Lambda C)_0 \xrightarrow{\cong} \left((C^{-*} \otimes_\Lambda C)_0\right)^*.$$

Recall that

$$(C^{-*} \otimes_\Lambda C)_0 = \bigoplus_{i \in \mathbb{Z}} C_i^* \otimes_\Lambda C_i$$

and that under this decomposition $\psi_{(C^{-*} \otimes_\Lambda C)_0}$ decomposes as the direct sum over $i \in \mathbb{Z}$ of the inverses of the $\Lambda$-isomorphisms (see (A.2)):

$$(-1)^i \cdot \psi_{C_i} \colon C_i^* \otimes_\Lambda C_i \to (C_i^* \otimes_\Lambda C_i)^* .$$

Notice that we pick a sign $(-1)^i$ since in the definition of the flip isomorphism for chain complexes a sign appears. This implies in $L_p^0(\Lambda)$ that

$$\begin{aligned}
[H_\otimes(C)] &= [(C^{-*} \otimes_\Lambda C)_0, \psi_{(C^{-*} \otimes_\Lambda C)_0}] \\
&= \sum_{i \in \mathbb{Z}} [(C_i^* \otimes_\Lambda C_i, (-1)^i \cdot \psi_{C_i})] \\
&= \sum_{i \in \mathbb{Z}} (-1)^i \cdot [(C_i^* \otimes_\Lambda C_i, \psi_{C_i})] \\
&= \sum_{i \in \mathbb{Z}} (-1)^i \cdot H_\otimes^\Lambda([C_i]) \\
&= H_\otimes^\Lambda\left( \sum_{i \in \mathbb{Z}} (-1)^i \cdot [C_i] \right) \\
&= H_\otimes^\Lambda(o(C)).
\end{aligned}$$

□

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