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Special L-values of Drinfeld modules

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Abstract

We state and prove a formula for a certain value of the Goss L-function of a Drinfeld module. This gives characteristic-p-valued function field analogues of the class number formula and of the Birch and Swinnerton-Dyer conjecture. The formula and its proof are presented in an entirely self-contained fashion.

Contents

1. Introduction and statement of the theorem

Let k be a finite field of q elements. For a finite $k[t]$-module M we define $|M| \in k[T]$ to be the characteristic polynomial of the endomorphism t of the k-vector space M, so

$$|M| = \det_{k[T]}\left(T - t \,\middle|\, M\right).$$

In other words, if $M \cong \oplus_i k[t]/f_i(t)$ with the f_i monic then $|M| = \prod_i f_i(T)$. We will treat the invariant $|M|$ of the finite $k[t]$-module M as an analogue of the cardinality $\#A$ of a finite abelian group A.

Let R be the integral closure of $k[t]$ in a finite extension K of the field $k(t)$ of rational functions. Consider in the Laurent series field $k((T^{-1}))$ the infinite sum

$$\sum_I \frac{1}{|R/I|},$$

where I ranges over all the nonzero ideals of R. Since there are only finitely many ideals of given index, the sum converges to an element of $1+T^{-1}k[[T^{-1}]]$

which we denote by $\zeta(R,1)$. By unique factorization into prime ideals, we also have

$$\zeta(R,1) = \prod_{\mathfrak{m}} \left(1 - \frac{1}{|R/\mathfrak{m}|}\right)^{-1},$$

where $\mathfrak{m}$ ranges over all the maximal ideals of R. We stress that $\zeta(R,1)$ depends not only on the ring structure of R but also on its $k[t]$-algebra structure.

A particular case of our main result will be a formula for $\zeta(R,1)$ analogous to the class number formula for the residue at $s = 1$ of the Dedekind zeta function of a number field. Whereas the class number formula is essentially a statement about the multiplicative group $\mathbf{G}_m$, our formula for $\zeta(R,1)$ will essentially be a statement about the *Carlitz module*:

Definition 1. The *Carlitz module* is the functor

$$C\colon \{k[t]\text{-algebras}\} \to \{k[t]\text{-modules}\}$$

that associates with a $k[t]$-algebra B the $k[t]$-module $C(B)$ whose underlying k-vector space is B and whose $k[t]$-module structure is given by the homomorphism of k-algebras

$$\varphi_C\colon k[t] \to \mathrm{End}_k(B)\colon t \mapsto t + \tau,$$

where τ denotes the Frobenius endomorphism $b \mapsto b^q$.

This definition might appear rather ad hoc, yet the functor C is in many ways analogous to the functor

$$\mathbf{G}_m\colon \{\mathbf{Z}\text{-algebras}\} \to \{\mathbf{Z}\text{-modules}\}.$$

For example, in analogy with $\#\mathbf{G}_m(F) = \#F - 1$ for a finite field F, we have

PROPOSITION 1. *If ℓ is a $k[t]$-algebra that is a finite field, then* $|C(\ell)| = |\ell| - 1$.

In particular, this proposition allows us to re-express the infinite product defining $\zeta(R,1)$ as follows:

$$\zeta(R,1) = \prod_{\mathfrak{m}} \frac{|R/\mathfrak{m}|}{|C(R/\mathfrak{m})|} \in 1 + T^{-1}k[[T^{-1}]].$$

Proof of Proposition 1. We need to show that for every finite extension ℓ/k and for every element $t \in \ell$, we have

$$\det_{k[T]}\Big(T - t - \tau \,\Big|\, \ell\Big) = \det_{k[T]}\Big(T - t \,\Big|\, \ell\Big) - 1.$$

We will do so by extending scalars from k to an algebraic closure $\bar{k}$. Let S be the set of k-algebra embeddings of ℓ into $\bar{k}$. We have an isomorphism of $\bar{k}$-algebras

$$\ell \otimes_k \bar{k} \to \bar{k}^S\colon x \otimes y \mapsto (\sigma(x)y)_{\sigma \in S}.$$

Under this isomorphism the action of $t\otimes 1$ on $\ell\otimes\bar{k}$ corresponds to the diagonal action of $(\sigma(t))_{\sigma\in S}$ on $\bar{k}^S$ and that of $\tau\otimes 1$ to a cyclic permutation of S. From this description it is easy to verify that

$$\det_{\bar{k}[T]}\Big(T - t\otimes 1 - \tau\otimes 1 \,\Big|\, \ell\otimes_k \bar{k}\Big) = \det_{\bar{k}[T]}\Big(T - t\otimes 1 \,\Big|\, \ell\otimes_k \bar{k}\Big) - 1,$$

as desired. □

We will also consider objects more general than the Carlitz module:

Definition 2. Let n be a nonnegative integer. A *Drinfeld module of rank* n over R is a functor

$$E\colon \{R\text{-algebras}\} \to \{k[t]\text{-modules}\}$$

that associates with a $k[t]$-algebra B the $k[t]$-module $E(B)$ whose underlying k-vector space is B and whose $k[t]$-module structure is given by

$$\varphi_E\colon k[t] \to \operatorname{End}_k(B)\colon t \mapsto t + r_1\tau + \cdots + r_n\tau^n$$

for fixed $r_i \in R$, depending only on E, and with $r_n \neq 0$.

Remark 1. Equivalently a Drinfeld module over R is a $k[t]$-module scheme E over R whose underlying k-vector space scheme is $\mathbf{G}_{a,R}$, and where the induced action of $k[t]$ on $\operatorname{Lie}(E) = R$ (the tangent space at zero) is the tautological one.

Remark 2. The above definition is more general than the standard one in that it allows primes of bad reduction (r_n is not required to be a unit) and that it allows the trivial case $n = 0$.

For a Drinfeld module E over R, we define $L(E/R)$ as follows:

$$L(E/R) := \prod_{\mathfrak{m}} \frac{|R/\mathfrak{m}|}{|E(R/\mathfrak{m})|} \in 1 + T^{-1}k[[T^{-1}]].$$

It is not hard to show that this infinite product indeed converges, but in any case it will follow from the proof of the main result below.

Example 1. The Carlitz module C is a Drinfeld module of rank 1 and $\zeta(R,1) = L(C/R)$.

Let K be the field of fractions of R and denote the tensor product $R\otimes_{k[t]} k((t^{-1}))$ by K_∞. Note that K_∞ is isomorphic to a product of finite extensions of the field $k((t^{-1}))$ and that R is discrete and co-compact in K_∞.

Proposition 2. *For every Drinfeld module E, there exists a unique power series*

$$\exp_E X = X + e_1X^q + e_2X^{q^2} + e_3X^{q^3} + \cdots \in K_\infty[[X]]$$

such that

$$\exp_E(tX) = \varphi_E(t)(\exp_E X). \tag{1}$$

The power series $\exp_E$ *has infinite radius of convergence and defines a map*

$$\exp_E : K_\infty \to E(K_\infty)$$

which is $k[t]$*-linear, continuous and open.*

Proof (see also [9, §3]). The proof is straightforward. The second condition gives a recursion relation for the coefficients e_i which determines them uniquely, and a closer look at the recursion relation reveals that the e_i tend to zero fast enough for $\exp_E$ to have infinite radius of convergence. Clearly $\exp_E$ defines a continuous k-linear function on K_∞ and the functional equation (1) guarantees that it is $k[t]$-linear when seen as a map from K_∞ to $E(K_\infty)$. The non-archimedean implicit function theorem (see, for example, [12, 2.2]) implies that $\exp_E$ is open. □

PROPOSITION 3. $\exp_E^{-1} E(R) \subset K_\infty$ *is discrete and co-compact.*

Remark 3. In particular $\exp_E^{-1} E(R)$ is a finitely generated $k[t]$-module. The module $E(R)$, however, is only finitely generated in case E is the Drinfeld module of rank 0 (see [15]).

Proof of Proposition 3 (see also [17]). This follows from the openness of $\exp_E$ and the fact that $E(R)$ is discrete and co-compact in $E(K_\infty)$. □

Let V be a finite-dimensional $k((t^{-1}))$-vector space. A *lattice* in V is by definition a discrete and co-compact sub-$k[t]$-module. The set $\mathcal{L}(V)$ of lattices in V carries a natural topology.

PROPOSITION 4. *There is a unique function*

$$[- : -] : \mathcal{L}(V) \times \mathcal{L}(V) \to k((T^{-1}))$$

that is continuous in both arguments and with the property that if Λ_1 *and* Λ_2 *are sub-lattices of a lattice* Λ *in* V, *then*

$$[\Lambda_1 : \Lambda_2] = \frac{|\Lambda/\Lambda_2|}{|\Lambda/\Lambda_1|}.$$

Proof. Let Λ_1 be a lattice in V. The set of lattices Λ_2 such that there is a lattice Λ containing both Λ_1 and Λ_2 is dense in $\mathcal{L}(V)$. It follows that if the function exists, it is necessarily unique. Let $\imath$ denote the obvious isomorphism

$$k((t^{-1})) \to k((T^{-1})) : \sum a_i t^i \mapsto \sum a_i T^i.$$

Given lattices Λ_1 and Λ_2 there exists a $\sigma \in \mathrm{GL}(V)$ so that $\sigma(\Lambda_1) = \Lambda_2$. If we define $[\Lambda_1 : \Lambda_2]$ to be the unique monic representative of $k^\times \imath(\det(\sigma))$ (which does not depend on the choice of σ), then it is easy to verify that this defines a function $[- : -]$ with the required properties. $\square$

We define $H(E/R)$ to be the $k[t]$-module

$$H(E/R) := \frac{E(K_\infty)}{E(R) + \exp_E K_\infty}.$$

PROPOSITION 5. *$H(E/R)$ is finite.*

Proof (see also [17]). The quotient $H(E/R)$ is compact because $E(R)$ is co-compact in $E(K_\infty)$ and discrete because $\exp_E K_\infty$ is open in $E(K_\infty)$. $\square$

In this paper we shall prove the following result.

THEOREM 1. *For every Drinfeld module E over R, we have*

$$L(E/R) = \left[\, R : \exp_E^{-1} E(R) \,\right] \cdot \left| H(E/R) \right|.$$

Example 2. If $n = 0$, then $\exp_E$ is the identity map and both left- and right-hand side of the formula equal 1.

Example 3. The smallest nontrivial example is for $E = C$, with $R = k[t]$. One can verify directly that $H(C/k[t]) = 0$ and $\exp^{-1} C(k[t])$ is of rank 1, generated by $\exp^{-1}(1)$. One thus recovers the formula

$$\zeta(k[t], 1) = \exp^{-1}(1)$$

that was already known to Carlitz [6, p. 160], [2, p. 199].

Remark 4. The quantity $[R : \exp_E^{-1} E(R)] \in k((T^{-1}))$ is typically transcendental over $k(T)$; see [14], [8], [7].

Remark 5. Although we will not need this, we briefly indicate how $L(E/R)$ is the value at some integer s of a *Goss L-function* (see [11] for the theory of such L-functions).

If the leading coefficient r_n of $\varphi_E(t)$ is a unit in R (in other words, if E has everywhere good reduction), then $L(E/R)$ is the value at $s = 0$ of the Goss L-function $L(E_K^\vee, s)$ defined through the dual of the Tate module of E_K. Using Gardeyn's Euler factors at places of bad reduction [10], one can define $L(E_K^\vee, s)$ even when r_n is not a unit. Our $L(E/R)$ however depends on the chosen integral model, and will in general differ from $L(E_K^\vee, 0)$ by a factor in $k(T)^\times$.

Remark 6. The statement of Theorem 1 is surprisingly similar to the statement of the class number formula. Let F be a number field and $\mathcal{O}_F$ be the ring of integers in F. Consider the exponential map

$$\exp\colon (\mathcal{O}_F/\mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{R} \to (\mathcal{O}_F \otimes_{\mathbf{Z}} \mathbf{R})^\times / \mathbf{R}_{>0}^\times.$$

Both $\mathcal{O}_F/\mathbf{Z}$ and $\exp^{-1}\mathcal{O}_F^\times$ are discrete and co-compact subgroups of $V := (\mathcal{O}_F/\mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{R}$. For the latter, this follows from the Dirichlet unit theorem. The class number formula, usually printed as

$$\lim_{s\to 1}(s-1)\zeta_F(s) = \frac{2^{r_1}(2\pi)^{r_2}R_F h_F}{w_F\sqrt{|\Delta_F|}},$$

implies that

$$\lim_{s\to 1}(s-1)\zeta_F(s) = \lambda \cdot \frac{\operatorname{covol}(\exp^{-1}(\mathcal{O}_F^\times))}{\operatorname{covol}(\mathcal{O}_F/\mathbf{Z})}$$

for some $\lambda \in \mathbf{Q}^\times$, if both co-volumes are computed with respect to the same Haar measure on V (see, for example, [21, I.§8]).

Remark 7. If E is a Drinfeld module of rank 2, then the Euler factors in the product defining the left-hand side in Theorem 1 share many properties with the Euler factors $L_v(A,1)$ for an elliptic curve A over a number field. In this case the theorem gives a Drinfeld module analogue of the Birch and Swinnerton-Dyer conjecture. In fact, it is heuristics for the growth of the infinite product

$$\prod_p \frac{p}{\#A(\mathbf{F}_p)}$$

that led Birch and Swinnerton-Dyer to the statement of their conjecture; see [16].

Remark 8. Let $\mathbf{A}_K$ be the ring of adèles of K. With a reasonable theory of $k((T^{-1}))$-valued "measures" on topological $k[t]$-modules (extending our notion of "cardinality" $|M|$ of finite $k[t]$-modules M) one should be able to express Theorem 1 as a kind of Tamagawa number statement, namely as stating that the "volume" of the compact $k[t]$-module $E(\mathbf{A}_K)/E(K)$ equals the "volume" of the $k[t]$-module $\mathbf{A}_K/K$. (See also Remark 14.)

Remark 9. A generalization of Theorem 1 from Drinfeld modules to Anderson's abelian t-modules [1] should give, among others, a formula for the zeta values

$$\zeta(R,n) := \sum_I \frac{1}{|R/I|^n}$$

for $n > 1$, generalizing results of Anderson and Thakur [4] for $R = k[t]$.

Remark 10. Going even further, it should be possible to formulate and prove an equivariant version of Theorem 1 for abelian t-modules equipped with an action of a group (or an associative algebra). This should lead to Stark "units" in extensions of function fields.

Remark 11. We have only considered "∞-adic" special values, that is, values in $k((T^{-1}))$. In analogy with classical p-adic special values, one could ask for v-adic special values associated with Drinfeld modules, with v a finite prime of $k(T)$. Vincent Lafforgue has expressed those in terms of extensions of shtukas and the equal characteristic analogue of Fontaine theory [13]. In the special case of the Carlitz module, it is shown in [18] that $\exp_E^{-1} E(R)$ and $H(E/R)$ can be computed as an Ext^1 respectively Ext^2 in the category of shtukas considered by Lafforgue.

Finally, we end this introduction with a short overview of the proof of Theorem 1.

Consider the power series

$$\Theta := \frac{1-\varphi_E(t)T^{-1}}{1-tT^{-1}} - 1 = \sum_{n=1}^{\infty} (t-\varphi_E(t))t^{n-1}T^{-n}$$

in T^{-1} and τ. For all maximal ideals $\mathfrak{m}$ of R, we have

$$\frac{|R/\mathfrak{m}|}{|E(R/\mathfrak{m})|} = \frac{\det_{k[T]}\Big(T-t\,\Big|\,R/\mathfrak{m}\Big)}{\det_{k[T]}\Big(T-\varphi_E(t)\,\Big|\,R/\mathfrak{m}\Big)} = \det_{k[[T^{-1}]]}\Big(1+\Theta\,\Big|\,R/\mathfrak{m}\Big)^{-1}.$$

Using the trace formula of Section 3 (essentially a variation on the Woods Hole fixed-point theorem) we obtain

$$\begin{aligned} L(E/R) &= \prod_{\mathfrak{m}} \det_{k[[T^{-1}]]}\Big(1+\Theta\,\Big|\,R/\mathfrak{m}\Big)^{-1} \\ &= \det_{k[[T^{-1}]]}\Big(1+\Theta\,\Big|\,\frac{K_\infty}{R}\Big), \end{aligned}$$

where the last determinant is an infinite-dimensional determinant which "converges" in the sense of Section 2. (One should think of Θ as a trace-class operator and of the determinant as a Fredholm determinant.) So we have replaced the infinite product of L-factors by a single determinant, at the expense of having to deal with k-vector spaces of infinite dimension.

Now enters the exponential map. For simplicity, assume that $H(E/R)$ vanishes. Then the exponential power series defines a k-linear isomorphism

$$\exp_E\colon \frac{K_\infty}{\exp^{-1} E(R)} \to \frac{K_\infty}{R},$$

and by the functional equation (1), we have

$$1+\Theta = \frac{1-\exp_E t \exp_E^{-1} T^{-1}}{1-tT^{-1}}$$

as endomorphisms of $\frac{K_\infty[[T^{-1}]]}{R[[T^{-1}]]}$. If K_∞/R and $E(K_\infty)/E(R)$ were not just compact but actually *finite*, we would conclude that

$$\det_{k[[T^{-1}]]}\Big(1+\Theta\,\Big|\,\frac{K_\infty}{R}\Big) = \frac{|K_\infty/\exp_E^{-1} E(R)|}{|K_\infty/R|},$$

but of course they are not finite and the right-hand side does not make any sense. However we do show in Section 4 that in some sense the determinant can be interpreted as the ratio of the "volumes" of $K_\infty/\exp_E^{-1} E(R)$ and K_∞/R. More precisely, we show that

$$\det_{k[[T^{-1}]]}\left(1+\Theta \,\middle|\, \frac{K_\infty}{R}\right) = [R : \exp_E^{-1} E(R)].$$

Since we assumed $|H(E/R)| = 1$, this then yields the claimed identity $L(E/R) = |H(E/R)| \cdot [R : \exp_E^{-1} E(R)]$.

Notation and conventions. k is a finite field of q elements, fixed throughout the text.

If M is a finite $k[t]$-module, then $|M|$ is by definition the characteristic polynomial of the action of t on M, in the variable T. Namely,

$$|M| := \det_{k[T]}\left(T - t \,\middle|\, M\right) \in k[T].$$

The cardinality of a set S will be denoted by $\#S$. (For example, $\#M = q^{\deg |M|}$.)

If α and β are endomorphisms of some abelian group, and if β is invertible, then

$$\frac{\alpha}{\beta}$$

will mean $\alpha\beta^{-1}$.

If F is a field with a non-archimedean absolute value $\|\cdot\|\colon F \to \mathbf{R}_{\geq 0}$, then a (non-archimedean) *norm* $\|\cdot\|$ on an F-vector space V is a map

$$\|\cdot\|\colon V \to \mathbf{R}_{\geq 0}$$

satisfying

(1) $\|v\| = 0$ if and only if $v = 0$,
(2) $\|\lambda v\| = \|\lambda\| \cdot \|v\|$ for all $v \in V$ and $\lambda \in F$,
(3) $\|v + w\| \leq \max(\|v\|, \|w\|)$.

A norm defines a metric and a topology on V. We equip the finite field k with the trivial absolute value ($\|\lambda\| = 1$ for all $\lambda \neq 0$) and the field $k((t^{-1}))$ with an absolute value determining the natural topology on $k((t^{-1}))$. In particular, we have $\|t^{-1}\| < 1$.

2. Nuclear operators and determinants

We will make use of determinants of certain class of endomorphisms of infinite-dimensional vector spaces and will use some terminology inspired by functional analysis for this. It must be stressed, however, that this is mostly a bookkeeping tool; all "functional analysis" here is merely a reflection of linear algebra on finite-dimensional quotients of the spaces considered.

The reader that is familiar with Anderson's "trace calculus" [3, §2] (or Tate's traces of "finite potent" endomorphisms [20, §1]) will easily recognize that the contents of this section is mostly an adaptation of their work. We should warn the same reader, however, that our notion of nucleus is dual to that of Anderson. (See also Remark 13 in §3.)

Let $V = (V, \|\cdot\|)$ be a normed k-vector space and let φ be a continuous endomorphism of V.

Definition 3. We say that φ is *locally contracting* if there is an open subspace $U \subset V$ and a real number $0 < c < 1$ so that

$$\|\varphi(v)\| \leq c\|v\|$$

for all v in U. Any such open subspace U which moreover satisfies $\varphi(U) \subset U$ is called a *nucleus* for φ. An open subspace $U \subset V$ with $\varphi(U) \subset U$ and such that $\varphi^n(U)$ is a nucleus for some $n > 0$ is called a *pre-nucleus*.

Example 4. Let F be a local field containing k and let $\|\cdot\|$ be an absolute value for F. Let $\mathcal{O}_F$ denote the ring of integers. Then $\|\cdot\|$ is a norm on the k-vector space $\mathcal{O}_F$ and the continuous endomorphism

$$\tau\colon \mathcal{O}_F \to \mathcal{O}_F\colon f \mapsto f^q$$

is locally contracting. The maximal ideal $\mathfrak{m}_F \subset \mathcal{O}_F$ is a nucleus for τ.

Example 5. We use the notation of the previous example. Let $R \subset F$ be a discrete and co-compact sub-k-algebra. Then $\|\cdot\|$ induces a norm on the k-vector space F/R. The continuous endomorphism

$$\tau\colon \frac{F}{R} \to \frac{F}{R}\colon f \mapsto f^q$$

is locally contracting and the image of $\mathfrak{m}_F$ in F/R is a nucleus.

Proposition 6. *Any finite collection of locally contracting endomorphisms of V has a common nucleus.*

Proof. It suffices to observe that if φ is a locally contracting endomorphism, then there is an $\epsilon > 0$ so that the open ball of radius ϵ around the origin is a nucleus for φ. □

Proposition 7. *If φ and ψ are locally contracting, then so are the sum $\varphi + \psi$ and composition $\varphi\psi$.*

Proof. Clear. □

Let V be a normed k-vector space. For every positive integer N, we denote by $V[[Z]]/Z^N$ the $k[[Z]]/Z^N$-module$V \otimes_k k[[Z]]/Z^N$ and by $V[[Z]]$ the $k[[Z]]$-module $V[[Z]] := \varprojlim_n V[[Z]]/Z^N$ equipped with the limit topology.

Any continuous $k[[Z]]$-linear endomorphism

$$\Phi\colon V[[Z]] \to V[[Z]]$$

is of the form

$$\Phi = \sum_{n=0}^{\infty} \varphi_n Z^n,$$

where the φ_i are continuous endomorphisms of V. Similarly, any continuous $k[[Z]]/Z^n$-linear endomorphism of $V[[Z]]/Z^N$ is of the form

$$\Phi = \sum_{n=0}^{N-1} \varphi_n Z^n.$$

From now one, we assume that V is *compact*. In particular every open subspace $U \subset V$ is of finite co-dimension.

Definition 4. We say that the continuous $k[[Z]]$-linear endomorphism Φ of $V[[Z]]$ (resp. of $V[[Z]]/Z^N$) is *nuclear* if for all n (resp. for all $n < N$), the endomorphism φ_n of V is locally contracting.

PROPOSITION 8. *Assume that $\Phi\colon V[[Z]]/Z^N \to V[[Z]]/Z^N$ is nuclear. Let U_1 and U_2 be common nuclei for the φ_n with $n < N$. Then*

$$\det_{k[[Z]]/Z^N}\left(1 + \Phi \,\middle|\, (V/U_i)[[Z]]/Z^N\right) \in k[[Z]]/Z^N$$

is independent of $i \in \{1, 2\}$.

Proof. It suffices to show that if $U' \subset U$ are common nuclei for the φ_n with $n < N$, then

$$\det_{k[[Z]]/Z^N}\left(1 + \Phi \,\middle|\, (U/U') \otimes_k k[[Z]]/Z^N\right) = 1.$$

Intersecting U with open balls of varying radius, we can find

$$U = U_0 \supset U_1 \supset \cdots \supset U_m = U'$$

with $\varphi_n(U_i) \subset U_{i+1}$ for all $n < N$ and $i < m$. Clearly, for every i, we have

$$\det_{k[[Z]]/Z^N}\left(1 + \Phi \,\middle|\, (U_i/U_{i+1}) \otimes_k k[[Z]]/Z^N\right) = 1,$$

and the proposition follows. □

Definition 5. If Φ is a nuclear endomorphism of $V[[Z]]/Z^N$, then we denote the determinant in Proposition 8 by

$$\det_{k[[Z]]/Z^N}\left(1 + \Phi \,\middle|\, V\right).$$

If Φ is a nuclear endomorphism of $V[[Z]]$, then we denote by

$$\det_{k[[Z]]}\left(1 + \Phi \,\middle|\, V\right) \in k[[Z]]$$

the unique power series that reduces to

$$\det_{k[[Z]]/Z^N}\left(1+\Phi \,\middle|\, V\right)$$

modulo Z^N for every N.

Remark 12. It would be more correct to denote the above determinants by

$$\det_{k[[Z]]/Z^N}\left(1+\Phi \,\middle|\, V[[Z]]/Z^N\right)$$

and

$$\det_{k[[Z]]}\left(1+\Phi \,\middle|\, V[[Z]]\right),$$

but we will generally drop the "$[[Z]]$" and "$[[Z]]/Z^N$" from V in order not to overload the notation.

Example 6. If V is finite-dimensional over k, then any continuous $k[[Z]]$-linear endomorphism Φ of $V[[Z]]$ is nuclear with nucleus $0 \subset V$, and

$$\det_{k[[Z]]}\left(1+\Phi \,\middle|\, V\right)$$

coincides with the determinant in the usual sense.

Example 7. If $\varphi\colon V \to V$ is locally contracting, then we call

$$\det_{k[[Z]]}\left(1-\varphi Z \,\middle|\, V\right) \in k[[Z]]$$

the *characteristic power series* of φ (which is in fact a polynomial). We have

$$\det_{k[[Z]]}\left(1-\varphi Z \,\middle|\, V\right) = \det_{k[[Z]]}\left(1-\varphi Z \,\middle|\, V/U\right)$$

for any pre-nucleus U.

PROPOSITION 9 (Multiplicativity in short exact sequences). *Let Φ be a nuclear endomorphism of $V[[Z]]$. Let $W \subset V$ be a closed subspace so that*

$$\Phi(W[[Z]]) \subset W[[Z]].$$

Then Φ is nuclear on $W[[Z]]$ and $(V/W)[[Z]]$, and

$$\det_{k[[Z]]}\left(1+\Phi \,\middle|\, V\right) = \det_{k[[Z]]}\left(1+\Phi \,\middle|\, W\right)\det_{k[[Z]]}\left(1+\Phi \,\middle|\, V/W\right).$$

Proof. Clear from the multiplicativity of finite determinants. □

PROPOSITION 10 (Multiplicativity for composition). *Let Φ and Ψ be nuclear endomorphisms of $V[[Z]]$. Then $(1+\Phi)(1+\Psi)-1$ is nuclear, and*

$$\det_{k[[Z]]}\left((1+\Phi)(1+\Psi) \,\middle|\, V\right) = \det_{k[[Z]]}\left(1+\Phi \,\middle|\, V\right)\det_{k[[Z]]}\left(1+\Psi \,\middle|\, V\right).$$

Proof. Clear from the multiplicativity of finite determinants. □

Theorem 2. *If φ, $\varphi\alpha$, and $\alpha\varphi$ are locally contracting, then*

$$\det_{k[[Z]]}\Big(1-\varphi\alpha Z\,\Big|\,V\Big) = \det_{k[[Z]]}\Big(1-\alpha\varphi Z\,\Big|\,V\Big).$$

Proof (see also Anderson [3, Prop. 8], Tate [20, (T_5)]). Let U be a common nucleus for $\alpha\varphi$, $\varphi\alpha$ and φ. Consider the iterated inverse images

$$U_n := (\alpha\varphi)^{-n}(U) = \{u \in V \,\big|\, (\alpha\varphi)^n(u) \in U\}.$$

Clearly the U_n are pre-nuclei for $\alpha\varphi$. We claim that for every nonnegative integer n, the subspace $\alpha^{-1}(U_n)$ is a pre-nucleus for $\varphi\alpha$. Indeed, since $\alpha\varphi(U_n) \subset U_n$, we have the implication

$$\alpha(v) \in U_n \implies \alpha\varphi\alpha(v) \in U_n;$$

hence

$$v \in \alpha^{-1}(U_n) \implies \varphi\alpha(v) \in \alpha^{-1}(U_n).$$

Moreover, we have

$$(\varphi\alpha)^{n+1}(\alpha^{-1}(U_n)) \subset \varphi U \subset U.$$

This proves the claim.

Now consider the the sequence of maps

$$\cdots \xrightarrow{\alpha} V/U_2 \xrightarrow{\varphi} V/\alpha^{-1}(U_1) \xrightarrow{\alpha} V/U_1 \xrightarrow{\varphi} V/\alpha^{-1}(U_0) \xrightarrow{\alpha} V/U_0.$$

Since these are injective maps between finite-dimensional vector spaces, all but finitely many of them are isomorphisms. So for n large enough, the top and bottom maps in the commutative square

$$\begin{array}{ccc} V/\alpha^{-1}(U_n) & \xrightarrow{\alpha} & V/U_n \\ {\scriptstyle\varphi\alpha}\big\downarrow & & {\scriptstyle\alpha\varphi}\big\downarrow \\ V/\alpha^{-1}(U_n) & \xrightarrow{\alpha} & V/U_n \end{array}$$

are isomorphisms, and it follows that $\varphi\alpha$ and $\alpha\varphi$ have the same characteristic power series. $\square$

Corollary 1. *Let $N > 0$ be an integer. Assume that all compositions*

$$\varphi, \varphi\alpha, \alpha\varphi, \varphi^2, \ldots$$

of endomorphisms in $\{\varphi, \alpha\}$, containing at least one endomorphism φ and at most $N-1$ endomorphisms α, are locally contracting. Let γ denote $1+\varphi$. Then

$$\frac{1-\gamma\alpha Z}{1-\alpha\gamma Z} \mod Z^N$$

is a nuclear endomorphism of $V[[Z]]/Z^N$, and

$$\det_{k[[Z]]/Z^N}\Big(\frac{1-\gamma\alpha Z}{1-\alpha\gamma Z}\,\Big|\,V\Big) = 1.$$

Proof. We can inductively find uniquely determined endomorphisms φ_n so that

$$\frac{1-\gamma\alpha Z}{1-\alpha Z} = \prod_{n=1}^{\infty}(1-\varphi_n\alpha Z^n)$$

and

$$\frac{1-\alpha\gamma Z}{1-\alpha Z} = \prod_{n=1}^{\infty}(1-\alpha\varphi_n Z^n).$$

Each φ_n is a (noncommutative) polynomial in φ and α whose constituting monomials contain at least one endomorphism φ and at most $n-1$ endomorphisms α. So using the preceding Theorem 2 we conclude:

$$\det{}_{k[[Z]]/Z^N}\Big(\frac{1-\gamma\alpha Z}{1-\alpha\gamma Z}\,\Big|\, V\Big) = \prod_{n=1}^{N-1}\det{}_{k[[Z]]/Z^N}\Big(\frac{1-\varphi_n\alpha Z^n}{1-\alpha\varphi_n Z^n}\,\Big|\, V\Big) = 1. \qquad \square$$

3. A trace formula

In this section we establish a trace formula which will enable us to express $L(E/R)$ as the determinant of $1+\Theta$ for some nuclear endomorphism Θ of some compact $k[[T^{-1}]]$-module.

Similar techniques have been used by Taguchi and Wan [19], Anderson [3], Böckle and Pink [5], and by Lafforgue [13] to study L-functions of φ-sheaves. Our approach differs in that by allowing more general operators, we are able to compute the L-*value* directly without first computing the L-*function*. Also, we will work directly with the Drinfeld module E, and not with its associated φ-sheaf or t-motive.

Let X be a connected projective scheme, smooth of dimension 1 over k. Let $Y \subset X$ be a nonempty affine open sub-scheme, and R the ring of regular functions on Y. Let K be the function field of X and denote by K_∞ the product of the completions K_x for all $x \in X\backslash Y$. Let $\|\cdot\|\colon K_\infty \to \mathbf{R}$ be the absolute value defined as the maximum of the normalized absolute values on the K_x.

Let M be a finitely generated projective R-module and $\tau_M\colon M \to M$ a k-linear endomorphism satisfying

$$\tau_M(rm) = r^q\tau_M(m)$$

for all $r \in R$ and $m \in M$.

Let $\|\cdot\|$ be a norm for the free K_∞-module $M\otimes_R K_\infty$. This norm induces a norm $\|\cdot\|$ on the compact k-vector space $M\otimes_R \frac{K_\infty}{R}$.

Denote by $R\{\tau\}$ the twisted polynomial ring whose elements are polynomials

$$r_0 + r_1\tau + \cdots + r_d\tau^d$$

in τ, and where multiplication is defined by the commutation rule $\tau r = r^q \tau$ for all $r \in R$. By sending τ to τ_M, we have k-algebra homomorphisms

$$R\{\tau\} \to \mathrm{End}_k(\star)$$

with $\star$ either M, $M \otimes_R \frac{K_\infty}{R}$, or $M/\mathfrak{m}M$ for some maximal ideal $\mathfrak{m} \subset R$.

PROPOSITION 11. *For every $\varphi \in R\{\tau\}\tau$, the induced endomorphism*

$$\varphi\colon M \otimes_R \frac{K_\infty}{R} \to M \otimes_R \frac{K_\infty}{R}$$

is locally contracting.

Proof. Clear. In fact any c with $0 < c < 1$ will do. □

Denote by $R\{\tau\}[[Z]]$ the ring of formal power series

$$\varphi_0 + \varphi_1 Z + \varphi_2 Z^2 + \cdots$$

with $\varphi_i \in R\{\tau\}$, where the variable Z is central. By Proposition 11 any $\Theta \in R\{\tau\}[[Z]]\tau$ defines a nuclear endomorphism of $\left(M \otimes_R \frac{K_\infty}{R}\right)[[Z]]$.

The main goal of this section is to establish the following theorem.

THEOREM 3 (Trace formula). *For all $\Theta \in R\{\tau\}[[Z]]\tau Z$, the infinite product*

$$\prod_{\mathfrak{m}} \det{}_{k[[Z]]}\left(1 + \Theta \,\middle|\, M/\mathfrak{m}M\right)$$

converges to

$$\det{}_{k[[Z]]}\left(1 + \Theta \,\middle|\, M \otimes_R \frac{K_\infty}{R}\right)^{-1}$$

in $k[[Z]]$.

This theorem is a variation on Anderson's trace formula; see Remark 13 at the end of this section.

We will first show that the validity of Theorem 3 is invariant under localization. Let $\mathfrak{p}$ be a maximal ideal in R and denote by $R[\mathfrak{p}^{-1}]$ and $M[\mathfrak{p}^{-1}]$ the localizations of R and M at the multiplicative system $\mathfrak{p} \cup \{1\} \subset R$. The endomorphism Θ of $M[[Z]]$ extends naturally to an endomorphism of $M[\mathfrak{p}^{-1}][[Z]]$. Let $K_\mathfrak{p}$ be the completion of K at $\mathfrak{p}$.

LEMMA 1 (Localization). *For all Θ as in Theorem 3, we have*

$$\det{}_{k[[Z]]}\left(1 + \Theta \,\middle|\, M/\mathfrak{p}M\right) = \frac{\det_{k[[Z]]}\left(1 + \Theta \,\middle|\, M[\mathfrak{p}^{-1}] \otimes_{R[\mathfrak{p}^{-1}]} \frac{K_\infty \times K_\mathfrak{p}}{R[\mathfrak{p}^{-1}]}\right)}{\det_{k[[Z]]}\left(1 + \Theta \,\middle|\, M \otimes_R \frac{K_\infty}{R}\right)}. \tag{2}$$

Proof. Denote the completion of R at $\mathfrak{p}$ by $R_\mathfrak{p}$. Since $K_\mathfrak{p} = R_\mathfrak{p} + R[\mathfrak{p}^{-1}]$ and $R = R_\mathfrak{p} \cap R[\mathfrak{p}^{-1}]$, we have a natural short exact sequence

$$0 \to R_\mathfrak{p} \to \frac{K_\infty \times K_\mathfrak{p}}{R[\mathfrak{p}^{-1}]} \to \frac{K_\infty}{R} \to 0$$

inducing a short exact sequence

$$0 \to M \otimes_R R_\mathfrak{p} \to M[\mathfrak{p}^{-1}] \otimes_{R[\mathfrak{p}^{-1}]} \frac{K_\infty \times K_\mathfrak{p}}{R[\mathfrak{p}^{-1}]} \to M \otimes_R \frac{K_\infty}{R} \to 0$$

which is respected by all coefficients θ_n of Θ. It follows that

$$\det{}_{k[[Z]]}\Big(1+\Theta \,\Big|\, M \otimes_R R_\mathfrak{p}\Big) = \frac{\det_{k[[Z]]}\Big(1+\Theta \,\Big|\, M[\mathfrak{p}^{-1}] \otimes_{R[\mathfrak{p}^{-1}]} \frac{K_\infty \times K_\mathfrak{p}}{R[\mathfrak{p}^{-1}]}\Big)}{\det_{k[[Z]]}\Big(1+\Theta \,\Big|\, M \otimes_R \frac{K_\infty}{R}\Big)}.$$

But since the subspace $M \otimes_R \mathfrak{p}R_\mathfrak{p} \subset M \otimes_R R_\mathfrak{p}$ is a common nucleus for the coefficients θ_n of Θ, we also have

$$\det{}_{k[[Z]]}\Big(1+\Theta \,\Big|\, M \otimes_R R_\mathfrak{p}\Big) = \det{}_{k[[Z]]}\Big(1+\Theta \,\Big|\, M/\mathfrak{p}M\Big),$$

which proves the lemma. □

Proof of Theorem 3. Fix positive integers D and N and consider the subset $\mathcal{S}_{D,N} \subset R\{\tau\}[[Z]]/Z^N$ defined by

$$\mathcal{S}_{D,N} := \left\{1 + \sum_{n=1}^{N-1} \varphi_n Z^n \,\Big|\, \deg(\varphi_n) < \frac{nD}{N} \text{ for all } n < N\right\}.$$

The set $\mathcal{S}_{D,N}$ is a group under multiplication.

We claim that for all $1+\Theta \in \mathcal{S}_{D,N}$, the product

$$\prod_{\mathfrak{m}} \det{}_{k[[Z]]/Z^N}\Big(1+\Theta \,\Big|\, M/\mathfrak{m}M\Big)$$

converges to

$$\det{}_{k[[Z]]/Z^N}\Big(1+\Theta \,\Big|\, M \otimes_R \frac{K_\infty}{R}\Big)^{-1}.$$

Since D and N are arbitrary, this claim implies Theorem 3.

We now apply a variation on a trick of Anderson [3, Prop. 9]. By Lemma 1 we may and will assume that R has no residue fields of degree $d < D$ over k. For every $d < D$, we can then find a finite collection of f_{dj} and a_{dj} in R so that

$$1 = \sum_j f_{dj}(a_{dj}^{q^d} - a_{dj}).$$

In particular, for every $r \in R$, $n < N$ and $d < D$, we have

$$1 - r\tau^d Z^n \equiv \prod_j \frac{1-(rf_{dj}\tau^d)a_{dj}Z^n}{1-a_{dj}(rf_{dj}\tau^d)Z^n} \mod Z^{n+1}.$$

Using this congruence, it is easy to show that the group $\mathcal{S}_{N,D}$ is generated by elements of the form

$$\frac{1-(s\tau^d)aZ^n}{1-a(s\tau^d)Z^n}$$

with $a, s \in R$ and d and n positive integers. For these elements, the claim holds thanks to Theorem 2, and by the multiplicativity of determinants (Proposition 10) we may conclude that it holds for all $1+\Theta$ in $\mathcal{S}_{D,N}$. $\square$

Remark 13. The formulation and proof of Theorem 3 grew out of an attempt to understand and simplify the proof of Anderson's trace formula [3, Thm. 1]. We sketch how to deduce Anderson's result from Theorem 3.

Let $\Omega_{R/k}$ denote the R-module of Kähler differentials of R over k. We have a pairing

$$\frac{K_\infty}{R} \times \Omega_{R/k} \to k$$

given by

$$(f, \omega) \mapsto \sum_{x \in X \setminus Y} \mathrm{Tr}_{k(x)/k} \mathrm{Res}_x(f\omega).$$

This pairing is perfect in the sense that it identifies either factor with the space of continuous linear forms on the other. The transpose of the continuous endomorphism

$$\tau\colon \frac{K_\infty}{R} \to \frac{K_\infty}{R} : x \to x^q$$

is the q-Cartier operator $c\colon \Omega_{R/k} \to \Omega_{R/k}$. Similarly, for any finitely generated projective R-module M and any $\tau_M\colon M \to M$ as above, the induced homomorphism

$$\tau_M\colon M \otimes_R \frac{K_\infty}{R} \to M \otimes_R \frac{K_\infty}{R}$$

has a "Cartier-linear" transpose endomorphism

$$c_M\colon \mathrm{Hom}_R(M, \Omega_{R/k}) \to \mathrm{Hom}_R(M, \Omega_{R/k}).$$

Using this one deduces Anderson's trace formula from Theorem 3 applied to $\Theta = 1 - \tau_M Z$.

4. **Ratio of co-volumes**

Let V be a finite-dimensional $k((t^{-1}))$-vector space. For lattices Λ_1 and Λ_2 in V, we have defined in Section 1, Proposition 4 the quantity $[\Lambda_1 : \Lambda_2] \in k((T^{-1}))$. In the present section we will express $[\Lambda_1 : \Lambda_2]$ as a determinant in the sense of Section 2.

Let $\|\cdot\|\colon V \to \mathbf{R}_{\geq 0}$ be a norm on the $k((t^{-1}))$-vector space V and let Λ_1 and Λ_2 be lattices in V.

Definition 6. Let N be a positive integer. We say that a continuous k-linear map

$$\gamma\colon V/\Lambda_1 \to V/\Lambda_2$$

is *N-tangent to the identity* (on V) if there is an open $U \subset V$ such that

(1) $U \cap \Lambda_1 = U \cap \Lambda_2 = 0$;
(2) γ restricts to an isometry between the images of U;
(3) for all $v \in U$, we have $\|\gamma(v) - v\| \le \|t^{-N}\| \cdot \|v\|$.

Definition 7. We say that γ is *infinitely tangent to the identity* (on V) if it is N-tangent to the identity for every positive integer N.

PROPOSITION 12. *Let F be a finite extension of $k((t^{-1}))$ and let $\|\cdot\|$ be an extension of the absolute value on $\|\cdot\|$ on $k((t^{-1}))$. Let Λ_1 and Λ_2 be lattices in the $k((t^{-1}))$-vector space F. Let $\gamma\colon F \to F$ be a k-linear map defined by an everywhere convergent power series*

$$\gamma(x) = x + \gamma_1 x^q + \gamma_2 x^{q^2} + \cdots$$

and such that $\gamma(\Lambda_1) \subset \Lambda_2$. Then the induced map

$$\gamma\colon F/\Lambda_1 \to F/\Lambda_2$$

is infinitely tangent to the identity on F.

Proof. Let N be a positive integer. By the convergence of the power series the coefficients γ_i are bounded, and so there exists an $\epsilon > 0$ such that

$$\|\gamma(x) - x\| < \|t^{-N}\| \cdot \|x\|$$

for all $x \in F$ with $\|x\| < \epsilon$. Shrinking ϵ if necessary, we may assume that $\|\lambda\| \ge \epsilon$ for all nonzero λ in Λ_1 or Λ_2. Then by the non-archimedean inverse function theorem [12, 2.2] the open subset $U = \{x \in F\colon \|x\| < \epsilon\}$ satisfies the requirements. □

Let H_1 and H_2 be finite $k[t]$-modules and denote $V/\Lambda_i \times H_i$ by M_i. Let

$$\gamma\colon M_1 \to M_2$$

be a continuous k-linear map. We say that γ is N-tangent to the identity on V if the composition

$$V/\Lambda_1 \hookrightarrow M_1 \xrightarrow{\gamma} M_2 \twoheadrightarrow V/\Lambda_2$$

is N-tangent to the identity, and we say that γ is infinitely tangent to the identity if it is N-tangent to the identity on V for all positive integers N.

For every k-linear isomorphism $\gamma\colon M_1 \to M_2$, we define an endomorphism endomorphism

$$\Delta_\gamma = \frac{1 - (\gamma^{-1} t \gamma) T^{-1}}{1 - tT^{-1}} - 1$$

of $M_1[[T^{-1}]]$. We have $\Delta_\gamma = \sum_{n=1}^{\infty} \delta_n T^{-n}$ with

$$\delta_n = (t - \gamma^{-1} t \gamma) t^{n-1}.$$

THEOREM 4. *If γ is infinitely tangent to the identity on V, then Δ_γ is nuclear and*

$$\det_{k[[T^{-1}]]}\Big(1 + \Delta_\gamma \,\Big|\, M_1\Big) = [\Lambda_1 : \Lambda_2] \frac{|H_2|}{|H_1|}. \tag{3}$$

Remark 14. It is natural to think of

$$[\Lambda_1 : \Lambda_2] \frac{|H_2|}{|H_1|}$$

as the ratio of the "volumes" of M_2 and M_1. It seems plausible that one could work out a theory of $k((T^{-1}))$-valued "Haar measures" on topological $k[t]$-modules in which such a statement would have a precise meaning. (See also Remark 8.)

Remark 15. If $V = 0$, then $M_1 = H_1$ and $M_2 = H_2$ are finite, and the theorem trivially holds since

$$\det_{k[[T^{-1}]]}\Big(1 + \Delta_\gamma \,\Big|\, M_1\Big) = \frac{\det_{k[[T^{-1}]]}\Big(1 - tT^{-1} \,\Big|\, H_2\Big)}{\det_{k[[T^{-1}]]}\Big(1 - tT^{-1} \,\Big|\, H_1\Big)} = \frac{|H_2|}{|H_1|}.$$

Remark 16. By Theorem 4 the existence of a k-linear isomorphism $\gamma \colon M_1 \to M_2$ which is infinitely tangent to the identity implies that $[\Lambda_1 : \Lambda_2]\frac{|H_2|}{|H_1|}$, which is *a priori* an element of $k((T^{-1}))^\times$, has valuation zero. It is not hard to show that the converse is also true.

The proof of Theorem 4 makes up the rest of this section.

LEMMA 2. *If γ is N-tangent to the identity, then Δ_γ modulo T^{-N} is a nuclear endomorphism of $M_1[[T^{-1}]]/T^{-N}$.*

Proof. For all $n < N$ and for all $m \in M_1$ sufficiently small, we have

$$\begin{aligned} \|\delta_n(m)\| &= \|\gamma^{-1}\Big(\gamma(t^n m) - t^n m\Big) + \gamma^{-1}\Big(t^n m - t\gamma(t^{n-1}m)\Big)\| \\ &\leq \|t^{-1}\| \cdot \|m\|. \end{aligned}$$

Thus δ_n is locally contracting. □

LEMMA 3 (Independence of γ). *Assume that $\gamma_1 \colon M_1 \to M_2$ and $\gamma_2 \colon M_1 \to M_2$ are N-tangent to the identity. Then*

$$\det_{k[[T^{-1}]]/T^{-N}}\Big(1 + \Delta_{\gamma_1} \,\Big|\, M_1\Big) = \det_{k[[T^{-1}]]/T^{-N}}\Big(1 + \Delta_{\gamma_2} \,\Big|\, M_1\Big)$$

in $k[[T^{-1}]]/T^{-N}$.

Proof. By Proposition 10 it suffices to show that

$$\det_{k[[T^{-1}]]/T^{-N}}\Big(1+\Delta_\gamma\,\Big|\,M\Big)=1$$

when γ is an automorphism of $M=M_1$ which is N-tangent to the identity. This follows from Corollary 1, with $\alpha=t\gamma^{-1}$ and $\varphi=\gamma-1$. □

Lemma 4 (Torsion). *If both Λ_1 and Λ_2 are contained in a common $k[t]$-lattice Λ, then Theorem 4 holds.*

Proof. By the previous lemma we are free to choose γ. Let $U\subset V$ be an open subspace that is a complement of $\Lambda\subset V$. We have decompositions

$$M_i=U\times\Lambda/\Lambda_i\times H_i\qquad(i=1,2).$$

Choose a k-isomorphism

$$\gamma: M_1\to M_2$$

that is the identity on U and that maps $\Lambda/\Lambda_1\times H_1$ isomorphically to $\Lambda/\Lambda_2\times H_2$. Clearly γ is tangent to the identity on V. The finite subspace

$$\Lambda/\Lambda_1\times H_1\subset M_1$$

is preserved under the coefficients δ_n of Δ_γ and they induce the zero map on the quotient; hence

$$\begin{aligned}\det_{k[[T^{-1}]]}\Big(1+\Delta_\gamma\,\Big|\,M_1\Big)&=\det_{k[[T^{-1}]]}\Big(1+\Delta_\gamma\,\Big|\,\Lambda/\Lambda_1\times H_1\Big)\\&=\frac{|\Lambda/\Lambda_2\times H_2|}{|\Lambda/\Lambda_1\times H_1|}\\&=[\Lambda_1:\Lambda_2]\frac{|H_2|}{|H_1|},\end{aligned}$$

as desired. □

Lemma 5 (Approximation). *Let N be a positive integer, $U\subset V$ an open sub-$k[[t^{-1}]]$-module and σ an element of $\mathrm{GL}_{k[[t^{-1}]]}(U)$ inducing the identity on $U/t^{-N}U$. Then for every lattice Λ in V, the isomorphism*

$$\sigma\colon V/\Lambda\to V/\sigma(\Lambda)$$

is N-tangent to the identity, and we have

$$\det_{k[[T^{-1}]]/T^{-N}}\Big(1+\Delta_\sigma\,\Big|\,V/\Lambda\Big)=1\bmod T^{-N}.$$

Proof. Clearly σ is N-tangent to the identity, and since σ is $k[t]$-linear, we have $\Delta_\sigma=0$. □

Proof of Theorem 4. Let N be an arbitrary positive integer. It suffices to prove (3) modulo T^{-N}. But by Lemmas 5 and 3 we may replace Λ_1 with a lattice Λ_1' which is contained in a common over-lattice with Λ_2, and such that

$$\frac{[\Lambda_1 : \Lambda_2]}{[\Lambda_1' : \Lambda_2]} \in 1 + T^{-N}k[[T^{-1}]]$$

and

$$\frac{\det_{k[[T^{-1}]]}\Big(1+\Delta_\gamma \,\Big|\, M_1\Big)}{\det_{k[[T^{-1}]]}\Big(1+\Delta_{\gamma'} \,\Big|\, M_1'\Big)} \in 1 + T^{-N}k[[T^{-1}]]$$

for every $\gamma' \colon M_1' \to M_1$ infinitely tangent to the identity. Together with Lemma 4 this proves the theorem modulo T^{-N}. □

5. **Proof of the main result**

We now have at our disposal all the ingredients necessary to prove the main result, Theorem 1.

We recall the set-up. The ring R is the integral closure of $k[t]$ in a finite extension K of $k(t)$, and K_∞ denotes the $k((t^{-1}))$-algebra $R \otimes_{k[t]} k((t^{-1}))$. Let E be a Drinfeld module over R defined by

$$\varphi_E(t) = t + r_1\tau + \cdots + r_n\tau^n \in R\{\tau\},$$

where τ is the q-th power Frobenius endomorphism of the additive group.

Consider the power series

$$\Theta := \frac{1-\varphi_E(t)T^{-1}}{1-tT^{-1}} - 1 = \sum_{n=1}^{\infty} (t-\varphi_E(t))t^{n-1}T^{-n}$$

in $R\{\tau\}[[T^{-1}]]\tau T^{-1}$. We can express the special value $L(E/R)$ using Θ as follows:

$$\begin{aligned} L(E/R) &= \prod_{\mathfrak{m}} \frac{|R/\mathfrak{m}|}{|E(R/\mathfrak{m})|} \\ &= \prod_{\mathfrak{m}} \det_{k[[T^{-1}]]}\Big(1+\Theta \,\Big|\, R/\mathfrak{m}\Big)^{-1}. \end{aligned}$$

Theorem 3 applied to Θ acting on $R[[T^{-1}]]$ then gives

$$L(E/R) = \det_{k[[T^{-1}]]}\Big(1+\Theta \,\Big|\, \frac{K_\infty}{R}\Big).$$

Now consider the short exact sequence of $k[t]$-modules

$$0 \longrightarrow \frac{K_\infty}{\exp_E^{-1} E(R)} \xrightarrow{\exp_E} \frac{E(K_\infty)}{E(R)} \longrightarrow H(E/R) \longrightarrow 0.$$

Since the $k[t]$-module on the left is divisible, this sequence splits. The choice of a section gives an isomorphism

$$\frac{K_\infty}{\exp^{-1} E(R)} \times H(E/R) \xrightarrow{\sim} \frac{E(K_\infty)}{E(R)}.$$

Denote by γ the composition

$$\frac{K_\infty}{\exp_E^{-1}(E(R))} \times H(E/R) \xrightarrow{\sim} \frac{E(K_\infty)}{E(R)} \xrightarrow{\sim} \frac{K_\infty}{R}$$

with the tautological map. Then γ is a k-linear isomorphism (but in general it is not $k[t]$-linear).

Let $\|\cdot\|\colon K_\infty \to \mathbf{R}$ be an absolute value extending the given absolute value on $k((t^{-1}))$.

LEMMA 6. *γ is infinitely tangent to the identity on K_∞.*

Proof. This boils down to the statement that

$$\exp_E\colon \frac{K_\infty}{\exp^{-1} E(R)} \to \frac{K_\infty}{R}$$

is infinitely tangent to the identity, which follows from Proposition 12. □

Since we have an equality

$$1 + \Theta = \frac{1 - \gamma t \gamma^{-1} T^{-1}}{1 - tT^{-1}}$$

of endomorphisms of $\frac{K_\infty}{R}[[T^{-1}]]$, we conclude using Theorem 4:

$$\begin{aligned} L(E/R) &= \det_{k[[T^{-1}]]}\left(1 + \Theta \,\middle|\, \frac{K_\infty}{R}\right) \\ &= \left[\, R : \exp_E^{-1} E(R) \,\right] \cdot |H(E/R)| \,. \end{aligned}$$

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